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FRACTAL NOISES IN THE PROBLEMS OF IMAGE RESTORATION

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On practice instead of a true signal which one can obtain from an ideal instrument we usually observe some distorted signal. The most popular distortions elimination way is signal restoration (which is usually called a postriori). Under image (signal) restoration we understand such treatment of response on device output which allows to obtain a function more close to the true signal. As a rule real phenomena provoking distortions are substituted by mathematical models. Image restoration is a typical inverse problem and belongs to the class of ill-posed problems, general formulation of which may be represented by the following way

$$Ks + n = f, \tag{1}$$

where f is an output signal distorted by a random noise n, s is an input signal, K is a linear operator modelling the measuring instrument.

For analysis of ill-posed problems usually one considers two principally distinguishable models: deterministic and random ones. The random model represents the most interest, since it is based on introduction of an additional a priori information about character of errors and it allows to obtain in mean more accurate results than at the deterministic model. We would like to lay stress on that algorithms of solution of ill-posed problems in the framework of the random model have the essential limitation - assumption about Gaussian model of the random noise [1], [2].

Statistical properties of a random noise one can investigate by using of the normalized range method suggested by Hurst [3]. In his paper it has been shown that temporal random series may be characterized by the exponent H $(0 < H < 1)$. Theoretical analysis shows that for temporal series which represent itself a random process with independent values and finite dispersion the Hurst exponent H equals 1/2 [3]. Statistical data collected by Hurst show that for many natural processes $H \neq 1/2$ [4].

At investigation of statistical properties of a random process two limiting cases are usually considered: when the memory (dependence a realization in some moment of time t on realizations in the more late moments of time $t' \in [0,t]$) is absence (for instance, Guassian white noise) and when the memory is ideal one (brownian process) [5].

In general case when the memory exists and is not ideal one a random process is considered over fractal Cantor set with a dimension $0 < D_f < 1$ [4]. As it has been shown in the paper [6] the such approach leads to the concept of fractional integral D_0^{ν}, therefore we can perform the random process $X(t)$ by the following way

$$X(t) \propto D_0^{-\nu}[f(t)] = \frac{1}{\Gamma(\nu)} \int_0^t (t-t')^{\nu-1} f(t')dt' = \frac{t^{\nu-1} * f(t)}{\Gamma(\nu)}, \quad (2)$$

here * - convolution sigh, $\Gamma(\nu)$ - Gamma function, ν - fractional integral exponent, coinciding with fractal dimension D_f, $f(t)$ - probability density function. From the expression (2) it follows that at $\nu = 1$ (ideal memory) $X(t)$ represent itself usual Gaussian process. Taking into account that for gaussian process $H = 1/2$, we can write the following relationship $\nu = H + 1/2$. Based on the expression (2) and a sequence of Gaussian random values a fractal noise with the Hurst exponent H is simulated in such a way

$$n_H(t) = \Delta X_H(t) = X_H(t) - X_H(t-1). \quad (3)$$

The purpose of the given paper is image restoration at the supposition that a random noise is fractal one and investigation the dependence quality of image restoration on the Hurst exponent H. In the paper the most popular procedures of signal restoration are considered such as: Least Squares Polynomial Smoothing (LSPS), Fourier Transform Smoothing (FTS), Optimal Linear Filtering (OFL). For estimation restoration quality we used the mean quadratic error criterion and namely: mean square (dispersion) of deviation of restored signal $\hat{s}$ from true one s, i.e. $\sigma_1^2 = E[\hat{s} - s]$ (E denotes mathematical expectation); dispersion of deviation of initial data f from true signal s $\sigma_2^2 = E[f - s]$; deviation of mean squares of error in the presence of restoration and without one $\Delta\sigma^2 = \sigma_1^2 - \sigma_2^2$ [7].

The spectrum treated (a volume of sampling equals 1000) is the superposition of the Gaussian contour and the fractal noise with unit dispersion. The signal amplitude is 10% from noise level. As fractal noises we used ones with the Hurst exponents H equals 0.2, 0.5, 0.8 respectively.

1. Least Squares Polynomial Smoothing (Savitzky-Golay method [8]). According to this method for chosen n points it is constructed a polynomial $a_m t^m + a_{m-1} t^{m-1} + \ldots + a_1 t + a_0$ of power m, coefficients of which are found on the basis of Least-Squares Procedure

$$\min\|f - Va\|_2, \qquad (4)$$

f - vector of length n, a - vector of length $m+1$, $V_{ij} = i^{j-1}$ - Vandermond matrix. The smoothed estimation may be written in the matrix representation by the following way

$$\hat{s} = V\left(V^T V\right)^{-1} V^T f. \qquad (5)$$

To realize this procedure we used the following parameters: $n = 9$, $m = 3$. The results of smoothing at the assumption about fractal nature of the random noise are given in the tab. 1.

2. Fourier Transform Smoothing [9]. This is a traditional procedure of filtering of initial data in frequency domain. Corresponding to this algorithm for the signal observed $f(t)$ is found the Fourier image $f(\omega)$, which is multiplied by the transmission function $h(\omega)$. Inverse Fourier transform of the resulting signal is a spectrum smoothed. If we denote by means of F and F^{-1} direct and inverse Fourier transform then the smoothed estimation may be written in the form

$$\hat{s}(t) = F^{-1}\left\{h(\omega) F\left\{f(t)\right\}\right\}. \qquad (6)$$

As $h(\omega)$ we used the transmission function of ideal filter. The results of smoothing of the spectrum by that method are given in the tab. 1.

3. Optimal Linear Filtering [7]. When the additional information about some statistical characteristics of signal and noise exists this procedure is usually used. If for the exact solution of equation (1) s and for the random noise n (it is supposed that s and n is referred to class of stationary random processes) their power spectral densities P_s and P_n respectively are known then the restored signal on output has in the matrix representation the following form

$$\hat{s} = \left(K^T K + P_n P_s^{-1}\right)^{-1} K^T f. \qquad (7)$$

where K is a matrix of weight coefficients. The results of restoration are represented in the tab. 1.

From the tab. 1 it follows that at $H < 1/2$ quality of signal restoration for the methods considered is essentially improved, but at $H > 1/2$ one becomes worse. In such a way, finding the Hurst exponent H for spectrum not containing a useful signal by Hurst method we obtain the additional information about character of a random noise. If the value of the Hurst exponent lays into the internal $0 < H < 1/2$ (in this case in the noise we observe the extension of high-frequency components with small amplitudes) then the random noise preferably to perform in the form of fractional integral (2), since in this case (as it follows from table 1) quality of image restoration is considerably improved. The result given may be used for construction of image restoration algorithms taking into account fractal nature of noises. We note that the correction stipulated by self-similar correlations of a random noise is connected mainly with the modification (as it follows from the integral representation (2) of fractal noises) of the transform operator of the problem.

Table 1. Dependence mean squares (dispersions) of errors σ_1^2, σ_2^2, $\Delta\sigma^2$ on the Hurst exponent H

method	LSPS			FTS			OFV		
H	0.2	0.5	0.8	0.2	0.5	0.8	0.2	0.5	0.8
σ_1	0.108	0.255	0.789	$7\cdot10^{-4}$	0.008	0.133	$5\cdot10^{-4}$	0.008	0.102
σ_2	1.550	1.013	1.532	1.553	1.016	1.533	1.516	0.992	1.497
$\Delta\sigma$	1.442	0.758	0.743	1.553	1.008	1.400	1.516	0.983	1.395

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