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ON THE EMBEDDING OF UNIVERSAL ALGEBRAS IN GROUPOIDS HOLDING THE LAW $XY*ZU**=XZ*YU**$

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Summary. It has been proved the following result[1]): Any Ω-algebra may be embedded in a semigroup. G. Čupona[2]) raised the problem: Is it possible to embed any *entropic* algebra in an *entropic* groupoid.

In this paper this problem is solved but not only for entropic algebra. Namely, we shall prove the following result:

Any Ω-algebra is embeddable in an entropic groupoid.

The main role in the proof is played by the term $xa*ay**$ which is unchangeable under the law

(*E*) $$xy*zu**=xz*yu**.$$

Similarly, it may be proved that any Ω-algebra may be embedded in a groupoid holding some law Z if there exists a term unchangeable under Z. For example, for the law $xy*zu**=uy*zx**$, a corresponding unchangeable term is $ax*ya**$.

I. Our main result is the following

Theorem 1. *If Q is an arbitrary Ω-algebra*[1]) *there exists an entropic*[3]) *groupoid* $(G, \mathbf{o})$ *having the following properties:*

1° Q is a subset of G;

2° If $\omega\in\Omega$ is an n-ary operation, then there exists in the set G an element $\bar{\omega}$ such that

(1) $$x_1x_2\cdots x_n\omega=x_1x_2\cdots x_n\underbrace{\mathrm{o}_\omega\mathrm{o}_\omega\cdots\mathrm{o}_\omega}_{(n-1)-\text{times}}\quad (n\geqslant 2)$$

(2) $$x\omega=x\bar{\omega}\,\mathrm{o}_\omega\quad (n=1)$$

(3) $$\omega=\bar{\omega}\bar{\omega}\,\mathrm{o}_\omega\quad (n=0)$$

where

$$xy\mathrm{o}_\omega\overset{\mathrm{def}}{=}x\bar{\omega}\,\mathrm{o}\,\bar{\omega}y\mathrm{oo}.$$

[1] C o h n P. M., *Universal algebra*, Tokyo, 1965, 184—186.

[2] *On some primitive classes of universal algebras*, Mat. vesnik, 3(18) pp. 105—108, 1966.

[3] That means: the groupoid (G, o) holds: $xyozuoo=xzoyuoo$.

At first, we introduce some definitions and prove one lemma. Let **G** be the *minimal set* satisfying the following two conditions:

The set $X \stackrel{\text{def}}{=} Q \cup \Omega$ *is a subset of* **G**;

If $u, v \in \mathbf{G}$ *then* $uv\bullet \in \mathbf{G}$.

It is clear that $(\mathbf{G}, \mathbf{o})$ si a groupoid, where $uv\mathbf{o} \stackrel{\text{def}}{=} uv\bullet$.

In the set **G** we consider the following *minimal subset* O:

X *is a subset of* O;

If $u, v \in O$ *and* $\omega \in \Omega$ *then* $u\omega\bullet\omega v\bullet\bullet \in O$.

Further, in the set O, we define some *operations* which are necessary in what follows. Let $\omega \in \Omega(n)$, then

(1′) $$u_1 u_2 \cdots u_n \boldsymbol{\omega} \stackrel{\text{def}}{=} u_1 u_2 \cdots u_n \underbrace{\mathrm{o}_\omega \mathrm{o}_\omega \cdots \mathrm{o}_\omega}_{(n-1)-\text{times}} \quad (n \geqslant 2)$$

(2′) $$u\boldsymbol{\omega} \stackrel{\text{def}}{=} u\omega\mathbf{o}_\omega \quad (n=1)$$

(3′) $$\boldsymbol{\omega} \stackrel{\text{def}}{=} \omega\omega\,\mathbf{o}_\omega \quad (n=0)$$

where

$$uv\mathbf{o}_\omega \stackrel{\text{def}}{=} u\omega\bullet\omega v\bullet\bullet.$$

Next, we introduce binary relations $\vdash$ and $\sim$.

Definition. *We say that* $t_1 \vdash t_2$ *where* $t_1, t_2 \in G$ *if and only if:*

The term t_2 *may be obtained from the term* t_1 *by substituting one subterm* α *of* t_1 *by the term* β, *where* α *and* β *may be:*

I. α *is of the form* $uv\bullet u_1 v_1\bullet\bullet$, *then* β *is* $uu_1\bullet vv_2\bullet\bullet$;

II. $\alpha = x_1 x_1 \cdots x_n \boldsymbol{\omega}$ $(x_2 \in Q)$, *then* $\beta = x_1 x_2 \cdots x_n \omega$ *and* $\beta \in Q$;

III. *Let* $x_1, x_2, \ldots, x_n \in Q$ *and* $\alpha = x_1 x_2 \cdots x_n \omega$ $(\alpha \in Q)$, *than* β *is the term (the word)* $x_1 x_2 \cdots x_n \boldsymbol{\omega}$.

In the cases I, II, III *we write, respectively*

$$t_1 \underset{\text{I}}{\vdash} t_2, \qquad t_1 \underset{\text{II}}{\vdash} t_2, \qquad t_1 \underset{\text{III}}{\vdash} t_2.$$

Let $\sim$ *be the minimal equivalence relation in the set* **G** *which prolongs the relation* $\vdash$ (*t. i.* $\vdash \subseteq \sim$).

The relation $\sim$ is a congruence in the groupoid $(\mathbf{G}, \mathbf{o})$.

Lemma. *If* $u \vdash v$ *and* $u \in O$, *then* $v \in O$.

Proof. Let $u \in O$. We will distinguish three cases:

$$1.\ u \underset{\text{I}}{\vdash} v; \qquad 2.\ u \underset{\text{II}}{\vdash} v; \qquad 3.\ u \underset{\text{III}}{\vdash} v,$$

and we will prove that in each of them $v \in O$ holds.

Ad 1. In this case, at first, we deduce:

P. *If* t *is an element of* O, *then each of its subterm of the form* $uv\bullet u_1 v_1\bullet\bullet$ *is an element of the set* O *and consequently, sastisfies* $v = u_1 = \omega$, *where* $\omega \in \Omega$.

We prove that by induction on $\sigma(t)$, the lenght of the word t. For instance, $\sigma(x) = 1$, $\sigma(xy\bullet) = 3$. **P.** is true if $\sigma(t) = 1$, since t has no subterm of the form $uv\bullet u_1 v_1\bullet\bullet$.

Let

$$t = t_1 \omega \bullet \omega t_2 \bullet\bullet, \qquad \sigma(t) = n (n > 1)$$

and let $\alpha = uv \bullet u_1 v_1 \bullet\bullet$ be a subterm of t. If α is a subterm of t_1 or of t_2 we apply induction hypothesis. In the opposite case the term t must be of the form $u\omega \bullet \omega v_1 \bullet\bullet$ where u, v_1 are certain elements of the set O and $\omega \in \Omega$. That completes the inductive proof.

By the property **P.**, each subterm of t $(t \in O)$ of the form $uv \bullet u_1 v_1 \bullet\bullet$ (t. i. the subterm which may be 'changend' under the law (E)) is an element of the set O and, consequently, is 'unchangeable' under (E). Accordingly, the following condition holds:

(4) $$u \underset{\mathrm{I}}{\vdash} v \Rightarrow u = v \quad (u \in O)$$

By (4) we conclude if $u \in O$ and $u \underset{\mathrm{I}}{\vdash} v$ than $v \in O$.

The definitions of $\underset{\mathrm{II}}{\vdash}$, $\underset{\mathrm{III}}{\vdash}$ imply the proof in the cases 2. and 3.||

2. Proof *of the theorem.* At first, let us prove the following:

If $x, y \in Q$ and $\bar{x} = \bar{y}$ then $x = y$, where $\bar{x}$, $\bar{y}$ are the equivalence classes of x, y with respect to $\sim$.

Let $\bar{x} = \bar{y}$. It means that there exists a natural number n and elements $u_1, u_2, \ldots, u_n$ $(u_i \in \mathbf{G})$ such that

$$u_1 \vdash u_2, \; u_2 \vdash u_3, \; \ldots, \; u_{n-1} \vdash u_n; \; x = u_1, \; y = u_n.$$

Because of $x, y \in O$ we have (by Lemma) $u_1, u_2, \ldots, n_n \in O$, too. Now we introduce the following *interpretation Int*:

$$Int(x) \overset{\mathrm{def}}{=} x \text{ if } x \in Q$$

$$Int(t_1 t_2 \cdots t_k \omega) \overset{\mathrm{def}}{=} Int(t_1)\, Int(t_2) \cdots Int(t_k)\, \omega \quad (\omega \in \Omega(k))$$

Obviously, *Int* is a function which carries a subset of the set O into Q and the image of the sequence $u_1, u_2, \ldots, u_n$ is the following sequence of the algebra Q:

(5) $$Int(u_1),\; Int(u_1),\; Int(u_2),\; \ldots,\; Int(u_n).$$

If $u_i \underset{\mathrm{I}}{\vdash} u_{i+1}$ then (4) implies $u_i = u_{i+1}$ and, consequently, $Int(u_i) = Int(u_{i+1})$.

In the case $u_i \underset{\mathrm{II}}{\vdash} u_{i+1}$ or $u_i \underset{\mathrm{III}}{\vdash} u_{i+1}$, by the definitions of the relations $\underset{\mathrm{II}}{\vdash}$ and $\underset{\mathrm{III}}{\vdash}$, we also obtain $Int(u_i) = Int(u_{i+1})$. That implies the following property of (5):

(6) $$Int(u_1) = Int(u_2) = \cdots = Int(u_n).$$

By definition of *Int* it follows

(7) $$Int(u_1) = x, \quad Int(u_n) = y,$$

because of $x = u_1$ and $y = u_n$. The equalities (6) and (7) imply $x = y$, as asserted.

In order to conclude the proof of the theorem we *introduce*

$$G \overset{\text{def}}{=} \mathbf{G}_{/\sim}; \quad \bar{x}\bar{y}\mathbf{o} \overset{\text{def}}{=} \overline{xy\mathbf{o}}. \tag{8}$$

By definition of the relation $\sim$ (t. i. $\underset{I}{\vdash}$) it is easily seen that (G, o) is an entropic groupoid.

Further, in the set

$$\tilde{Q} \overset{\text{def}}{=} \{\bar{x} / \ x \in Q\}$$

for each $\omega \in \Omega(n)$, we define a *binary operation* $\mathbf{o}_\omega$ and *n-ary operation* $\tilde{\omega}$ as follows

$$\bar{x}\bar{y}\mathbf{o}_\omega \overset{\text{def}}{=} \overline{xy\mathbf{o}_\omega}; \ \bar{x}_1\bar{x}_2 \ \ldots \ \bar{x}_n\tilde{\omega} \overset{\text{def}}{=} \overline{x_1x_2\cdots x_n\omega} \tag{9}$$

By (8) and (9), and by definition of the operation ω we obtain the following equalities

$$\bar{x}_1\bar{x}_2\cdots\bar{x}_n\tilde{\omega} = \bar{x}_1\bar{x}_2\cdots\bar{x}_n\underbrace{\mathrm{o}_\omega\mathrm{o}_\omega\cdots\mathrm{o}_\omega}_{(n-1)-\text{times}} \quad (n \geqslant 2) \tag{1''}$$

(2'') $\bar{x}\tilde{\omega} = \bar{x}\bar{\omega}\,\mathrm{o}_\omega \quad (n=1)$

(3'') $\tilde{\omega} = \bar{\omega}\bar{\omega}\,\mathrm{o}_\omega \quad (n=0)$

It is clear that $\tilde{Q}$ is an Ω-algebra ($\tilde{\omega}$ is corresponding to ω).

By the first part of that proof it follows that Q and $\tilde{Q}$ are isomorphic algebras (an isomorphism is $f: x \to \bar{x}$). That completes the proof of the theorem. ||

3. The proof of the Theorem 1. suggests the following

Theorem 2. *If the groupoid law*

$$F_1(x_1, x_2, \ldots, x_n, x, y) = F_2(x_1, x_2, \ldots, x_n, x, y) \tag{Z}$$

has the following property:

C. *The term $F_1(a, a, \ldots, a, x, y)$ (a is a constant) is unchangeable under (Z), that means that the terms $F_1(a, a, \ldots, a, x, y)$ and $F_2(a, a \ldots, a, x, y)$ coincide.;*

then for each Ω-algebra Q there exists a groupoid (G, o) holding the law (Z) and satisfying the following two conditions:

1° *Q is a subset of G;*

2° *If $\omega \in \Omega(n)$ then there exists an element $\bar{\omega} \in G$ such that (I) — (3) holds, where $xy\mathbf{o}_\omega \overset{\text{def}}{=} F_1(\bar{\omega}, \bar{\omega}, \ldots, \bar{\omega}, x, y)$.*

The proof is analogous as in the Theorem 1. ||

For example, the law $xy*zu** = uy*zx**$ satisfies the condition **C.** and thus the operation $\mathbf{o}_\omega$ is defined as follows: $xy\mathbf{o}_\omega \overset{\text{def}}{=} \bar{\omega}x*y\bar{\omega}**$.

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