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ON CERTAIN FORMULAS FOR EQUIVALENCE AND ORDER RELATIONS

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Summary

In this paper we give formulas for obtaining all equivalence and order relations on a given set S. These formulas are

— For equivalence relations: $x\,\alpha\,y \overset{\text{def}}{\Leftrightarrow} (\forall u)(u\,\pi_1\,x \Leftrightarrow u\,\pi_1\,y)$

— For order relations: $x\,\beta\,y \overset{\text{def}}{\Leftrightarrow} (\forall u)\,(((u\,\pi_2\,x \Rightarrow x\,\pi_2\,u) \Rightarrow u=x) \Rightarrow ((u\,\pi_2\,y \Rightarrow y\,\pi_2\,u) \Rightarrow u=y))$

(π_1, π_2 are arbitrary relations on S).

1. Proposition 1. *Let S be a nonempty set. By formula*

$$x\,\alpha\,y \overset{\text{def}}{\Leftrightarrow} (\forall u)(u\,\pi\,x \Leftrightarrow u\,\pi\,y) \tag{1}$$

(π-an arbitrary binary relation of S) are determined all equivalence relations of the set S.

Proof. At first, we prove that each relalation α defined by the formula (1) is an equivalence relation of the set S. The relation α is reflexive, symmetric and transitive because the formulas

$$(\forall u)(u\,\pi\,x \Leftrightarrow u\,\pi\,x)$$

$$(\forall u)(u\,\pi\,x \Leftrightarrow u\,\pi\,y) \Rightarrow (\forall u)(u\,\pi\,y \Leftrightarrow u\,\pi\,x)$$

$$((\forall u)(u\,\pi\,x \Leftrightarrow u\,\pi\,y) \wedge (\forall u)(u\,\pi\,y \Leftrightarrow u\,\pi\,z)) \Rightarrow (\forall u)(u\,\pi\,x \Leftrightarrow u\,\pi\,z)$$

are, obviously, valid (for each binary relation π of S).

We next prove if α is an equivalence relation of S, then α may be obtained by the formula (1) by means of some π. Precisely, we prove if α is an equivalence, then the formula

$$x\,\alpha\,y \Leftrightarrow (\forall u)(u\,\alpha\,x \Leftrightarrow u\,\alpha\,y) \tag{2}$$

is valid. Namely, according to transitivity of α, the following formulas are true

$$u\,\alpha\,x \wedge x\,\alpha\,y \Rightarrow u\,\alpha\,y, \quad u\,\alpha\,y \wedge y\,\alpha\,x \Rightarrow u\,\alpha\,x.$$

Since α is symmetric, then applying the tautology $(p \wedge q \Rightarrow r) \Leftrightarrow (p \Rightarrow (q \Rightarrow r))$ the previous two formulas become

$$x \alpha y \Rightarrow (u \alpha x \Rightarrow u \alpha y), \quad x \alpha y \Rightarrow (u \alpha y \Rightarrow u \alpha x)$$

whence we obtain

$$x \alpha y \Rightarrow (u \alpha x \Leftrightarrow u \alpha y)$$

(applying the tautology $(q \Rightarrow q) \wedge (p \Rightarrow q_1) \Rightarrow (p \Rightarrow q \wedge q_1)$).

At last, by the valid formula $(\forall u)(A \Rightarrow B) \Rightarrow (A \Rightarrow (\forall u) B)$ (u is not free in formula A), we obtain

(3) $$x \alpha y \Rightarrow (\forall u)(u \alpha x \Leftrightarrow u \alpha y).$$

It remains to prove the following implication

(4) $$(\forall u)(u \alpha x \Leftrightarrow u \alpha y) \Rightarrow x \alpha y.$$

Suppose that the antecedent $(\forall u)(u \alpha x \Leftrightarrow u \alpha y)$ is true. Setting $u = x$, we obtain $x \alpha x \Leftrightarrow x \alpha y$. Since $x \alpha x$ is true, we conclude that $x \alpha y$ is true.

Consequently, the formula (4) is proved. By (3) and (4) follows the equivalence (2). This means that the equivalence relation α may be obtained from formula (1), on choosing $\pi = \alpha$.

The proof is finished.

Corollary. *If π is a reflexive and antisymmetric relation, then by formula* (1) *is determined the relation "equality" in the set S.*

Examples. The following formulas are valid

$$x = y \Leftrightarrow (\forall u)(u = x \Leftrightarrow u = y)$$

$$x = y \Leftrightarrow (\forall u)(u \subset x \Leftrightarrow u \subset y),$$

$$x = y \Leftrightarrow (\forall u)(x \subset u \Leftrightarrow y \subset u),$$

(x, y, u — sets),

$$x = y \Leftrightarrow (\forall u)(u \leqslant x \Leftrightarrow u \leqslant y),$$

$$x = y \Leftrightarrow (\forall u)(x \leqslant u \Leftrightarrow y \leqslant u)$$

(x, y, u — real numbers),

$$x = y \Leftrightarrow (\forall u)(u \mid x \Leftrightarrow u \mid y),$$

$$x = y \Leftrightarrow (\forall u)(x \mid u \Leftrightarrow y \mid u)$$

(x, y, u — natural numbers).

2. Proposition 2. *All order relations of the set S are determined by the formula*

(6) $$x \alpha y \overset{\text{def}}{\Leftrightarrow} (\forall u)(u \pi x \Rightarrow u \pi y)$$

where π is an arbitrary reflexive and antisymmetric relation of S.

Proof. Let α be a relation defined by (6), providing π is reflexive and antisymmetric. The relation α is reflexive and transitive because the formulas

$$(\forall u)(u\,\pi\,x \Rightarrow u\,\pi\,x)$$

$$(\forall u)(u\,\pi\,x \Rightarrow u\,\pi\,y) \wedge (\forall u)(u\,\pi\,y \Rightarrow u\,\pi\,z) \Rightarrow (\forall u)(u\,\pi\,x \Rightarrow u\,\pi\,z)$$

are valid.

To prove antisymmetry, i.e. the implication

$$(\forall u)(u\,\pi\,x \Rightarrow u\,\pi\,y) \wedge (\forall u)(u\,\pi\,y \Rightarrow u\,\pi\,x) \Rightarrow x = y$$

suppose that its antecedent is true. Then replacing u in the formulas

$$(\forall u)(u\,\pi\,x \Rightarrow u\,\pi\,y), \quad (\forall u)(u\,\pi\,y \Rightarrow u\,\pi\,x)$$

respectively by x and y, we obtain $(x\,\pi\,x \Rightarrow x\,\pi\,y) \wedge (y\,\pi\,y \Rightarrow y\,\pi\,x)$. Since π is symmetric the last formula becomes $x\,\pi\,y \wedge y\,\pi\,x$, whence we obtain $x = y$.

Let, now, α be an order relation of the set S. We prove that the equivalence

$$x\,\alpha\,y \Leftrightarrow (\forall u)(u\,\alpha\,x \Rightarrow u\,\alpha\,y) \tag{7}$$

is true. Similarly as in the proposition 1, the formula

$$x\,\alpha\,y \Rightarrow (\forall u)(u\,\alpha\,x \Rightarrow u\,\alpha\,y) \tag{8}$$

is true. It follows from transitivity of α.

To establish the inverse implication of (8) suppose that $(\forall u)(u\,\alpha\,x \Rightarrow u\,\alpha\,y)$ is true. Replacing u by x we conclude that $x\,\alpha\,y$ is true. The proof of formula (7) is finished. From formula (7) we conclude that α may be obtained by (6) setting α instead of π. The proposition 2 is proved.

We next determine formula of reflexive and antisymmetric relations.

Proposition 3. *By the formula*

$$x\,\alpha\,y \overset{\text{def}}{\Leftrightarrow} ((x\,\pi\,y \Rightarrow y\,\pi\,x) \Rightarrow x = y) \tag{9}$$

(π — *an arbitrary relation of* S) *are determined all reflexive and antisymmetric relations of the set* S.

Proof. Obviously, a relation α defined by (9) is reflexive. To prove antisymmetry of α, suppose $x\,\alpha\,y \wedge y\,\alpha\,x$, i.e.

$$((x\,\pi\,y \Rightarrow y\,\pi\,x) \Rightarrow x = y) \wedge ((y\,\pi\,x \Rightarrow x\,\pi\,y) \Rightarrow y = x).$$

Hence it is easy to conclude that $x = y$.

Thus the formula (9) determines, for any relation π, reflexive and antisymmetric relation.

Let now α be a reflexive and antisymmetric relation of S. This relation may be obtained by formula (9) setting α instead of π, becuse the formula $x\,\alpha\,y \Leftrightarrow ((x\,\alpha\,y \Rightarrow y\,\alpha\,x) \Rightarrow x = y)$ is true.

The proof is finished.

Corollary. *The formula*

$$(10)\qquad x\,\alpha\,y \overset{\text{def}}{\Leftrightarrow} (\forall u)\,(((u\,\pi\,x \Rightarrow x\,\pi\,u) \Rightarrow u=x) \Rightarrow ((u\,\pi\,y \Rightarrow y\,\pi\,u) \Rightarrow u=y))$$

(π — *an arbitrary relation of* S) *defines all order relation of the set* S.

Examples. By proposition 2 the following formulas are true:

$$x \subset y \Leftrightarrow (\forall u)\,(u \subset x \Rightarrow u \subset y)$$

$$x \subset y \Leftrightarrow (\forall u)\,(y \subset u \Rightarrow x \subset u)^{1)}$$

(x, y, u — sets)

$$x \leqslant y \Leftrightarrow (\forall u)\,(u \leqslant x \Rightarrow u \leqslant y)$$

$$x \leqslant y \Leftrightarrow (\forall u)\,(y \leqslant u \Rightarrow x \leqslant u)$$

(x, y, u — real numbers)

$$x \mid y \Leftrightarrow (\forall u)\,(u \mid x \Rightarrow u \mid y)$$

$$x \mid y \Leftrightarrow (\forall u)\,(y \mid u \Rightarrow x \mid u)$$

(x, y, u — natural numbers)

[1] It is easy to prove that all order relations are also determined by the formula: $x\alpha y \overset{\text{def}}{\Leftrightarrow} (\forall u)(y\pi u \Rightarrow x\,\pi\,u)$, where π is an arbitrary reflexive and antisymmetric relation.

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