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ORDERS OF INFINITY

THE ‘INFINITÄRCALCÜL’ OF PAUL DU BOIS-REYMOND

BY

G. H. HARDY

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PREFACE TO THE SECOND EDITION

THE present edition of this tract embodies a large number of alterations and additions. In particular I have rewritten Section VI completely, and hope it may now be useful as an introduction, from a special point of view, to a large field of modern research.

I should like to add a few words concerning the motives of Sections III—V, which form the most characteristic part of the tract, and I can make my point best by reference to a particular problem. Suppose that the problem is that of determining the behaviour of the power series $\Sigma\, \phi(n)\, x^n$ when x tends to unity. It is usual to delimit the problem in one or other of two ways. One is to restrict $\phi(n)$ by 'conditions of inequality', to suppose, for example, that $\phi(n)$ and a certain number of its derivatives or differences are monotonic functions of specified signs. The other is to confine our attention to special forms of $\phi(n)$, such as $n^{\alpha} (\log n)^{\beta} (\log\log n)^{\gamma} \ldots$, sufficiently general to illustrate the principal questions at issue.

There is, however, a third point of view which is often advantageous in the discussion of problems of this character. We may suppose that $\phi(n)$ is any function of some standard corpus whose rate of increase is not too large; and the natural corpus to select is the corpus of 'L-functions', that is to say of functions finitely definable by logarithms and exponentials. Thus, in the particular problem which I have mentioned, we may suppose $\phi(n)$ to be any L-function whose increase does not exceed that of all powers of n. In this way we may hope to prove theorems, not of course exhaustive, but including all the standard examples as particular cases. This point of view is adopted by implication in much of du Bois-Reymond's work, and it is that which is usually adopted here. It is, however, obviously necessary to begin by an exact and general investigation of the properties of L-functions, and this du Bois-Reymond omitted. The first essential theorem, for example, is that which appears here as Theorem 13. This theorem may be verified immediately in any particular case, but du Bois-Reymond never proves it and, so far as I know, no general proof had been given before the publication of this tract.

I am much indebted to Mr E. C. Titchmarsh and Mr A. Oppenheim for suggestions made in the course of correction of the proofs.

G. H. H.

20 *February*, 1924.

CONTENTS

I

INTRODUCTION

1.1. THE notions of the 'order of greatness' or 'order of smallness' of a function $f(n)$ of a positive integral variable n, when n is 'large', or of a function $f(x)$ of a continuous variable x, when x is 'large' or 'small' or 'nearly equal to a', are important even in the most elementary stages of mathematical analysis*. We learn there that x^2 tends to infinity with x, and moreover that x^2 tends to infinity *more rapidly than* x, *i.e.* that the ratio x^2/x tends to infinity also; and that x^3 tends to infinity more rapidly than x^2, and so on indefinitely. We are thus led to the idea of a 'scale of infinity' (x^n) formed by the functions $x, x^2, x^3, \ldots, x^n, \ldots$. This scale may be supplemented and to some extent completed by the interpolation of non-integral powers of x. But there are functions whose rates of increase cannot be measured by any of the functions of our scale, even when thus completed. Thus $\log x$ tends to infinity more slowly, and e^x more rapidly, than *any* power of x; and $x/(\log x)$ tends to infinity more slowly than x, but more rapidly than any power of x less than the first.

As we proceed further in analysis, and come into contact with its modern developments, such as the theory of Fourier's series, the theory of integral functions, or the theory of singular points of analytic functions in general, the importance of these ideas becomes greater and greater. It is the systematic study of them, the investigation of general theorems concerning them and ready methods of handling them, that is the subject of Paul du Bois-Reymond's *Infinitärcalcül* or 'calculus of infinities'.

1.2. Let us suppose that f and ϕ are two functions of the continuous variable x, defined for all values of x from a certain value x_0 onwards. Further, let us suppose that f and ϕ are positive, continuous, and steadily increasing, and tend to infinity with x; and let us consider the behaviour of the ratio f/ϕ when $x \to \infty$. We can distinguish four cases.

(i) If $f/\phi \to \infty$, we shall say that the *order*, or the *rate of increase*, or simply the *increase*, of f is greater than that of ϕ, and write

$$f \succ \phi.$$

(ii) If $f/\phi \to 0$, we shall say that the increase of f is less than that of ϕ, and write

$$f \prec \phi.$$

* See, for instance, Hardy, 1, 360.

(iii) If f/ϕ remains, for all values of x from a certain value x_1 onwards*, between two positive numbers δ and Δ, so that $0 < \delta < f/\phi < \Delta$, we shall say that the increase of f is equal to that of ϕ, and write

$$f \asymp \phi.$$

It may happen, in this case, that f/ϕ tends to a definite limit. If this is so, we shall write

$$f \mathrel{\overline{\asymp}} \phi.$$

Finally, if this limit is *unity*, we shall write

$$f \sim \phi.$$

When we can compare the increase of f with that of some standard function ϕ by means of a relation of the type $f \asymp \phi$, we shall say that ϕ *measures*, or simply *is*, the increase of f. Thus we shall say that the increase of $2x^2 + x + 3$ is x^2.

It often happens that f/ϕ is monotonic (*i.e.* steadily increasing or steadily decreasing) as well as f and ϕ themselves. In this case f/ϕ must tend to infinity, or to zero, or to a positive limit: so that $f \succ \phi$ or $f \prec \phi$ or $f \mathrel{\overline{\asymp}} \phi$. We shall see in a moment that this is not true in general.

(iv) It may happen that f/ϕ neither tends to infinity nor to zero, nor remains between positive bounds.

Suppose, for example, that ϕ_1, ϕ_2 are two continuous and increasing functions such that $\phi_1 \succ \phi_2$. A glance at the figure (Fig. 1) will probably show with sufficient clearness how we can construct, by means of a 'staircase' of straight or curved lines, running backwards and forwards between the graphs of ϕ_1 and ϕ_2, the graph of a steadily increasing function f such that $f = \phi_1$ for $x = x_1, x_3, \ldots$ and $f = \phi_2$ for $x = x_2, x_4, \ldots$. Then $f/\phi_1 = 1$ for

$$x = x_1, x_3, \ldots,$$

but assumes for $x = x_2, x_4, \ldots$ values which decrease beyond all limit; while $f/\phi_2 = 1$ for $x = x_2, x_4, \ldots$, but assumes for $x = x_1, x_3, \ldots$ values which increase beyond all limit; and f/ϕ, where ϕ is a function, such as $\sqrt{(\phi_1\phi_2)}$, for which $\phi_1 \succ \phi \succ \phi_2$, assumes both values which increase beyond all limit and values which decrease beyond all limit.

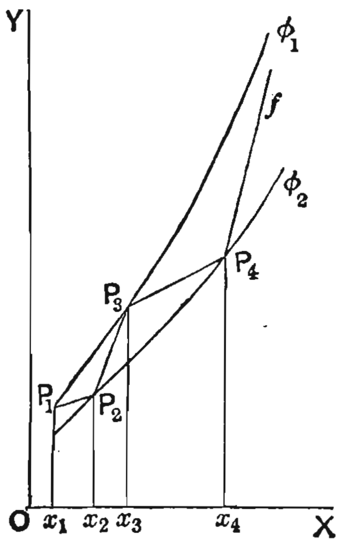

Fig. 1

* No mention of x_1 is really necessary when (as is supposed in the text) f and ϕ are positive and continuous. There are then numbers δ_1 and Δ_1 such that $0 < \delta_1 < f/\phi < \Delta_1$ for $x_0 \leqq x < x_1$, and $0 < \delta < f/\phi < \Delta$ for $x \geqq x_1$ implies $0 < \delta_2 < f/\phi < \Delta_2$, where $\delta_2 = \mathrm{Min}\,(\delta, \delta_1)$, $\Delta_2 = \mathrm{Max}\,(\Delta, \Delta_1)$, for $x \geqq x_0$.

It is however often convenient to extend our definitions to more general cases in which this argument would be invalid, and we retain the unnecessary words in order that the definitions may be more immediately adaptable.

Later on (§ 4.43) we shall meet with cases of this kind in which the functions are defined by explicit analytical formulae.

1.3. If a positive constant δ can be found such that $f > \delta\phi$ for all sufficiently large values of x, we shall write

$$f \succcurlyeq \phi;$$

and if a positive constant Δ can be found such that $f < \Delta\phi$ for all sufficiently large values of x, we shall write

$$f \preccurlyeq \phi.$$

If $f \succcurlyeq \phi$ and $f \preccurlyeq \phi$, then $f \asymp \phi$.

It is however important to observe that $f \succcurlyeq \phi$ is not logically equivalent to the negation of $f \prec \phi$. The relations $f \succcurlyeq \phi$, $f \prec \phi$ are mutually exclusive, but not exhaustive; the first implies the negation of the second, but the converse is not true. Again, $f \succcurlyeq \phi$ is not equivalent to the alternative '$f \succ \phi$ *or* $f \asymp \phi$'. Each of these points may be illustrated by the example at the end of § 1.2. Here $f \succcurlyeq \phi_1$ and $f \prec \phi_1$ are both false; and $f \succcurlyeq \phi_2$, but neither $f \succ \phi_2$ nor $f \asymp \phi_2$ is true. In the language of upper and lower limits, $f \succcurlyeq \phi$ means

$$\lambda = \underline{\lim} \frac{f}{\phi} > 0$$

and '$f \prec \phi$ is false' means

$$\Lambda = \overline{\lim} \frac{f}{\phi} > 0;$$

while to assert '$f \succ \phi$ or $f \asymp \phi$' is to assert that $\lambda > 0$ and that, if λ is finite, Λ is also finite.

The reader will have no difficulty in proving the following theorems. There are many other simple theorems of the same character, but these seem the most important.

(*a*) *If* $f \succ \phi$, $\phi \succcurlyeq \psi$, *then* $f \succ \psi$.
(*b*) *If* $f \succcurlyeq \phi$, $\phi \succ \psi$, *then* $f \succ \psi$.
(*c*) *If* $f \succcurlyeq \phi$, $\phi \succcurlyeq \psi$, *then* $f \succcurlyeq \psi$.
(*d*) *If* $f \asymp \phi$, $\phi \asymp \psi$, *then* $f \asymp \psi$.
(*e*) *If* $f \succcurlyeq \phi$, *then* $f + \phi \asymp f$.
(*f*) *If* $f \succ \phi$, *then* $f - \phi \asymp f$.
(*g*) *If* $f \succ \phi$, $f_1 \succ \phi_1$, *then* $f + f_1 \succ \phi + \phi_1$.
(*h*) *If* $f \succ \phi$, $f_1 \asymp \phi_1$, *then* $f + f_1 \succcurlyeq \phi + \phi_1$.
(*i*) *If* $f \asymp \phi$, $f_1 \asymp \phi_1$, *then* $f + f_1 \asymp \phi + \phi_1$.
(*j*) *If* $f \succ \phi$, $f_1 \succcurlyeq \phi_1$, *then* $ff_1 \succ \phi\phi_1$.
(*k*) *If* $f \asymp \phi$, $f_1 \asymp \phi_1$, *then* $ff_1 \asymp \phi\phi_1$.

He will also find it instructive to state for himself a series of similar theorems involving also the symbols $\succcurlyeq\!\!\preccurlyeq$ and $\sim$.

1.4. So far we have supposed that the functions considered all tend to infinity with x. There is nothing to prevent us from including cases in which f or ϕ tends steadily to zero, or to a limit other than zero; thus we may write $x \succ 1$, or $x \succ 1/x$, or $1/x \succ 1/x^2$. Bearing this in mind, the reader should frame a series of theorems similar to those of § 1.3 but involving quotients instead of sums or products.

It is also convenient to extend our definitions so as to apply to *negative* functions which tend steadily to $-\infty$, or to 0 or to some other limit. In such cases we make no distinction, when using the symbols $\succ$, $\prec$, $\asymp$, $\succcurlyeq$, between the function and its modulus: thus we write $-x \prec -x^2$ or $-1/x \prec 1$, meaning thereby exactly the same as by $x \prec x^2$ or $1/x \prec 1$. But $f \sim \phi$ is to be interpreted as a statement about the actual functions and not about their moduli.

It will be well now to lay down the principle that functions referred to in this tract, from this point onwards, are to be understood, unless the contrary is expressly stated or obviously implied, to be positive, continuous, and monotonic, increasing if they tend to infinity, and decreasing if they tend to zero. But it is sometimes convenient to depart from these conventions. We may abandon the restriction to continuous functions, writing, for example,

$$[x] \sim x, \quad \pi(x) \prec x,$$

where $[x]$ is the integral part of x and $\pi(x)$ the number of primes which do not exceed x. Or we may write

$$1 + \sin x \prec x, \; x^2 \succ x \sin x,$$

meaning by the first formula, for example, that $(1 + \sin x)/x \to 0$. We may even apply our notation to complex functions, writing $e^{ix} \prec x$ or $e^{ix} \asymp 1$. The reader will find no difficulty in modifying the definitions in the appropriate manner.

There are other possibilities to be considered. We have so far confined our attention to functions of a continuous variable x which tends to $+\infty$. This case may be held to include one which is perhaps even more important in applications, viz. that of functions of the positive integral variable n. We have only to disregard non-integral values of x. Thus $n! \succ n^2$, $-1/n \prec n$.

Finally, by putting $x = -y$, $x = 1/y$, or $x = 1/(y-a)$, we are led to consider functions of a continuous variable y which tends to $-\infty$ or 0 or a. The reader will easily supply the necessary modifications of detail.

In what follows we shall generally state and prove our theorems only for the case with which we started, that of continuous and increasing functions of a continuous variable which tends to infinity, and shall leave to the reader the task of formulating the corresponding theorems for the other cases.

1.5. There are some other symbols which we shall sometimes find it convenient to use in special senses. By

$$O(\phi)$$

we shall denote a function f, otherwise unspecified, but such that

$$|f| < K\phi,$$

where K is a constant and ϕ a positive function of x. This notation was first used by Bachmann*, though its general adoption is due to the influence of Landau. Thus

$$x+1 = O(x), \quad x = O(x^2), \quad \sin x = O(1).$$

It is clear that the three assertions

$$f = O(\phi), \quad |f| < K\phi, \quad f \preccurlyeq \phi$$

are equivalent to one another. By

$$o(\phi)$$

we shall, again following Landau †, denote a function f such that $f/\phi \rightarrow 0$. Thus

$$x = o(x^2), \quad 1 = o(x), \quad \sin x = o(x)$$

and

$$f = o(\phi), \quad f/\phi \rightarrow 0, \quad f \prec \phi$$

are equivalent.

We shall follow Borel ‡ in using the same letter K in a whole series of inequalities to denote a positive number, independent of the variable under consideration, but not necessarily the same in all inequalities where it occurs. Thus

$$\sin x < K, \quad 2x+1 < Kx, \quad x^m < Ke^x \quad (x \geqq 1).$$

If we use K thus in any finite number of inequalities which (like the first two above) do not involve any variables other than x, or whatever other variable we are considering, then all the values of K lie between two numbers K_1 and K_2: thus K_1 might be 10^{-10} and K_2 be 10^{10}. In this case all the K's satisfy $0 < K_1 < K < K_2$, and every relation $f < K\phi$ might be replaced by $f < K_2\phi$, and every relation $f > K\phi$ by $f > K_1\phi$. But we shall also have occasion to use K in equalities which (like the third above) involve a parameter (here m). In this case K, though independent of x, is a function of m. Suppose that a finite number of parameters $\alpha, \beta, \ldots$ occur in this way in this tract. Then if we give any

* Bachmann, **1**, 401. † Landau, **1**, 61. ‡ Borel, **6** and **2**, 105.

special system of values to $\alpha, \beta, \ldots$, we can determine K_1, K_2 as above. Thus all our K's satisfy

$$0 < K_1(\alpha, \beta, \ldots) < K < K_2(\alpha, \beta, \ldots),$$

where K_1, K_2 are positive functions of $\alpha, \beta, \ldots$ defined for any permissible set of values of those parameters. But K_1 may have the lower bound zero, and K_2 may be unbounded. We can then, by choosing $\alpha, \beta, \ldots$ appropriately, make K_1 as small and K_2 as large as we please.

When a function f possesses a property for all values of x greater than some definite value, this value of course depending on the function and the property, we shall say that f possesses the property for $x > x_0$. Thus

$$x > 100 \ (x > x_0), \quad e^x > 100x^2 \ (x > x_0).$$

We shall use δ and Δ to denote arbitrary but fixed positive numbers, using δ when we wish to emphasize the possible smallness of the number, and Δ when we wish to emphasize its possible largeness. Thus

$$f < \delta\phi \quad (x > x_0)$$

means 'however small δ, we can find x_0 so that $f < \delta\phi$ for $x > x_0$', *i.e.* means the same as $f \prec \phi$; and

$$(\log x)^\Delta \prec x^\delta$$

means 'any power of $\log x$ (however great) tends to infinity more slowly than any positive power of x (however small)'.

Finally, we denote by ϵ a function (of a variable or variables indicated by the context or by a suffix) whose limit is zero when the variable or variables are made to tend to infinity or to their limits in the way we happen to be considering. Thus ϵ means the same as $o(1)$, and

$$f = \phi(1+\epsilon), \ f \sim \phi, \ f = \phi + o(\phi)$$

are equivalent to one another.

1.6. In order to become familiar with the use of the symbols defined in the preceding sections the reader is advised to verify the following relations, in which $P_m(x)$, $Q_n(x)$ denote polynomials whose degrees are m and n and whose leading coefficients are positive:

$$P_m(x) \succ Q_n(x) \ (m > n), \quad P_m(x) \asymp Q_n(x) \ (m = n),$$

$$P_m(x) \asymp x^m, \quad P_m(x)/Q_n(x) \asymp x^{m-n},$$

$$\sqrt{(ax^2+2bx+c)} \asymp x \ (a>0), \quad \sqrt{(x+a)} \sim \sqrt{x}, \quad \sqrt{(x+a)} - \sqrt{x} \sim \tfrac{1}{2}ax^{-\frac{1}{2}},$$

$$e^x \succ x^\Delta, \quad e^{x^2} \succ e^{\Delta x}, \quad e^{e^x} \succ e^{x^\Delta}, \quad \log x \prec x^\delta, \quad \log\log x \prec (\log x)^\delta,$$

$$\log P_m(x) \asymp \log Q_n(x), \quad \log\log P_m(x) \sim \log\log Q_n(x),$$

$$x + a\sin x \sim x, \quad x(a+\sin x) \asymp x \ (a > 1),$$

$$e^{a+\sin x} \asymp 1, \quad \cosh x \sim \sinh x \sim \tfrac{1}{2}e^x, \quad \cosh(x+a) \asymp \cosh x,$$

$$x^\Delta = o(e^{\delta x}), \quad (\log x)^\Delta = o\{e^{(\log x)^\delta}\}, \quad x^\Delta = o\{e^{(\log x)^{1+\delta}}\},$$

$$1+\frac{1}{2}+\ldots+\frac{1}{n} \sim \log n, \quad 1+\frac{1}{2}+\ldots+\frac{1}{n}-\log n \asymp 1,$$

$$n! \prec n^n, \quad n! \succ e^{\Delta n}, \quad n! = n^{n^{1+\epsilon}} = n^{n(1+\epsilon)},$$

$$n! \sim n^{n+\frac{1}{2}}e^{-n}\sqrt{(2\pi)}, \quad n!\,(e/n)^n = (1+\epsilon)\sqrt{(2\pi n)},$$

$$\int_2^x \frac{dt}{\log t} \sim \frac{x}{\log x}, \quad \int_2^x \frac{dt}{\log t} = \frac{x}{\log x} + O\left\{\frac{x}{(\log x)^2}\right\}, \quad \int_3^x \frac{dt}{\log\log t} \sim \frac{x}{\log\log x}.$$

II

SCALES OF INFINITY IN GENERAL

2.1. If we start from a function ϕ, such that $\phi \succ 1$, we can, in a variety of ways, form a series of functions

$$\phi_1 = \phi, \quad \phi_2, \quad \phi_3, \ldots, \quad \phi_n, \ldots$$

such that the increase of each function is greater than that of its predecessor. Such a sequence of functions we shall denote for shortness by (ϕ_n).

One obvious method is to take $\phi_n = \phi^n$. Another is as follows: If $\phi \succ x$, it is clear that

$$\phi\{\phi(x)\}/\phi(x) \to \infty,$$

and so $\phi_2(x) = \phi\phi(x) \succ \phi(x)$; similarly $\phi_3(x) = \phi\phi_2(x) \succ \phi_2(x)$, and so on.

Thus the first method, with $\phi = x$, gives the scale x, x^2, x^3, ... or (x^n); the second, with $\phi = x^2$, gives the scale x^2, x^4, x^8, ... or (x^{2^n}). In this case the second scale is merely a selection from the terms of the first. With $\phi = e^x$, the two methods give the scales e^x, e^{2x}, e^{3x}, ... and

$$e^x, \quad e^{e^x}, \quad e^{e^{e^x}}, \quad \ldots.$$

Here the second term of the second scale is of greater increase than any term of the first.

These scales are *enumerable* scales, formed by a simple progression of functions. We can also, of course, by replacing the integral parameter n

by a continuous parameter a, define scales containing a non-enumerable multiplicity of functions: the simplest is (x^a), where a is any positive number. But such scales play a subordinate part in the theory.

It is obvious that we can always insert a new term (and therefore, of course, any number of new terms) in a scale at the beginning or between any two terms: thus $\sqrt{\phi}$ (or ϕ^a, where a is any positive number less than unity) has an increase less than that of any term of the scale, and $\sqrt{(\phi_n \phi_{n+1})}$ or $\phi_n^a \phi_{n+1}^{1-a}$ has an increase intermediate between those of ϕ_n and ϕ_{n+1}. A less obvious and more important theorem is the following.

Theorem 1*. *Given any ascending scale of increasing functions* ϕ_n, *i.e. a series of functions such that* $\phi_1 \prec \phi_2 \prec \phi_3 \prec \dots$, *we can always find a function* f *which increases more rapidly than any function of the scale, i.e. which satisfies the relation* $\phi_n \prec f$ *for all values of* n.

In view of the fundamental importance of this theorem we shall give two entirely different proofs.

2.21. We know that $\phi_{n+1} \succ \phi_n$ for all values of n, but this, of course, does not necessarily imply that $\phi_{n+1} \geqslant \phi_n$ for all values of x and n in question†. We can, however, construct a new scale of functions ψ_n such that

(*a*) ψ_n is identical with ϕ_n for all values of x from a certain value x_n onwards (x_n, of course, depending upon n);

(*b*) $\psi_{n+1} \geqslant \psi_n$ for all values of x and n.

For suppose that we have constructed such a scale up to its nth term ψ_n. Then it is easy to see how to construct ψ_{n+1}. Since $\phi_{n+1} \succ \phi_n$, $\phi_n \sim \psi_n$, it follows that $\phi_{n+1} \succ \psi_n$, and so $\phi_{n+1} \geqq \psi_n$ from a certain value of x (say x_{n+1}) onwards. For $x \geqslant x_{n+1}$ we take $\psi_{n+1} = \phi_{n+1}$. For $x < x_{n+1}$ we give ψ_{n+1} a value equal to the greater of the values of ϕ_{n+1}, ψ_n. Then it is obvious that ψ_{n+1} satisfies the conditions (*a*) and (*b*).

Now let $$f(n) = \psi_n(n).$$

* This is the theorem usually called the 'Theorem of Paul du Bois-Reymond'; see for example Borel, **1**, 113. Actually the theorem first proved explicitly by du Bois-Reymond was the corresponding theorem for descending scales (Theorem 3, § 2.4). See du Bois-Reymond, **4**, 365.

† $\phi_{n+1} \succ \phi_n$ implies $\phi_{n+1} > \phi_n$ for sufficiently large values of x, say for $x > x_n$. But x_n may tend to infinity with n. Thus $x_n = n+1$ if $\phi_n = x^n/n!$

From $f(n)$ we can deduce a continuous and increasing function $f(x)$, such that

$$\psi_n(x) < f(x) < \psi_{n+1}(x)$$

for $n < x < n+1$, by joining the points $(n, \psi_n(n))$ by straight lines or suitably chosen arcs of curves. Then

$$f/\psi_n > \psi_{n+1}/\psi_n$$

for $x > n+1$, and so $f \succ \psi_n$; therefore $f \succ \phi_n$, and the theorem is proved.

It is perhaps worth while to call attention explicitly to a small point that has sometimes been overlooked*. It is not always the case that the use of straight lines will ensure

$$f(x) > \psi_n(x)$$

for $x > n$ (see, for example, Fig. 2, where the dotted line represents an appropriate arc).

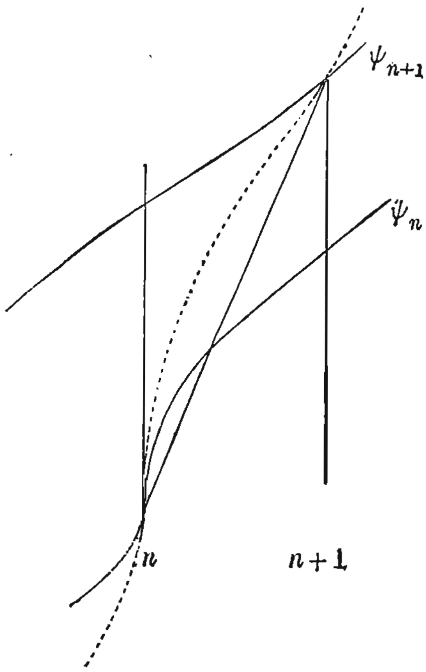

FIG. 2

The proof which precedes may be made more general by taking $f(n) = \psi_\nu(n)$, where ν is an integer depending upon n and tending steadily to infinity with n.

2.22. The second proof of du Bois-Reymond's Theorem proceeds on entirely different lines. We can always choose positive coefficients a_n so that

$$f(x) = \sum_1^\infty a_n \psi_n(x)$$

is convergent for all values of x. This will certainly be the case, for instance, if

$$1/a_n = \psi_1(1)\,\psi_2(2)\,\dots\,\psi_n(n).$$

For then, if ν is any integer greater than x, $\psi_n(x) < \psi_n(n)$ for $n \geqq \nu$, and the series will certainly be convergent if

$$\sum_\nu^\infty \frac{1}{\psi_1(1)\,\psi_2(2)\,\dots\,\psi_{n-1}(n-1)}$$

is convergent, as is obvious.

Also $$f(x)/\psi_n(x) > a_{n+1}\psi_{n+1}(x)/\psi_n(x) \to \infty,$$

so that $f \succ \phi_n$ for all values of n.

2.31. Suppose, *e.g.*, that $\phi_n = x^n$. If we restrict ourselves to values of x greater than 1, we may take $\psi_n = \phi_n = x^n$. The first method of construction would naturally lead to

$$f = n^n = e^{n \log n},$$

* Borel, **1**, 114; **5**, 25.

or $f=\nu^n$, where ν is defined as at the end of § 2.21, and each of these functions has an increase greater than that of any power of n. The second method gives

$$f(x)=\sum_1^\infty \frac{x^n}{1^1 2^2 3^3 \dots n^n}.$$

It is known* that when x is large the order of magnitude of this function is roughly the same as that of

$$e^{\frac{1}{2}(\log x)^2/\log\log x}.$$

As a matter of fact it is by no means necessary, in general, in order to ensure the convergence of the series by which $f(x)$ is defined, to suppose that a_n decreases so rapidly. It is very generally sufficient to suppose $1/a_n=\phi_n(n)$: this is always the case, for example, if $\phi_n(x)=\{\phi(x)\}^n$, as the series

$$\sum \left\{\frac{\phi(x)}{\phi(n)}\right\}^n$$

is always convergent. This choice of a_n would, when $\phi=x$, lead to

$$f(x)=\sum\left(\frac{x}{n}\right)^n \sim \sqrt{\left(\frac{2\pi x}{e}\right)}\, e^{x/e}\dagger.$$

But the simplest choice here is $1/a_n=n!$, when

$$f(x)=\sum\frac{x^n}{n!}=e^x-1\sim e^x;$$

it is naturally convenient to disregard the irrelevant term -1.

2.32. We can always suppose, if we please, that $f(x)$ is defined by a power series $\sum a_n x^n$ convergent for all values of x, in virtue of a theorem of Poincaré's‡ which is of sufficient intrinsic interest to deserve a formal statement and proof.

Theorem 2. *Given any continuous increasing function $\phi(x)$, we can always find an integral function $f(x)$ (i.e. a function $f(x)$ defined by a power series $\sum a_n x^n$ convergent for all values of x) such that $f(x)\succ\phi(x)$.*

The following simple proof is due to Borel §.

Let $\Phi(x)$ be any function (such as the square of ϕ) such that $\Phi\succ\phi$. Take an increasing sequence of positive numbers a_n such that $a_n\to\infty$, and another sequence of numbers b_n such that

$$a_1<b_2<a_2<b_3<a_3<\dots\dots$$

We can then choose a sequence of positive integers ν_n so that (i) $\nu_{n+1}>\nu_n$ and (ii)

$$\left(\frac{a_n}{b_n}\right)^{\nu_n}>\Phi(a_{n+1}).$$

Now let
$$f(x)=\sum\left(\frac{x}{b_n}\right)^{\nu_n}.$$

* Hardy, **6**. See also § 6.3.

† See Lindelöf, **2**, 41 and **3**; le Roy, **1**; and § 6.3.

‡ Poincaré, **1**, 214.

§ Borel, **4**, 27.

This series is convergent for all values of x; for the nth root of the nth term is not greater (when $b_n > x$) than x/b_n, and so tends to zero. Also

$$f(x) > \left(\frac{a_n}{b_n}\right)^{\nu_n} > \Phi(a_{n+1}) > \Phi(x),$$

if $a_n \leqq x < a_{n+1}$, and so for all values of x greater than a_1; so that $f \succ \phi$.

2.4. So far we have confined our attention to ascending scales, such as $x, x^2, x^3, \ldots, x^n, \ldots$ or (x^n); but it is obvious that we may consider in a similar manner *descending* scales such as $x, \sqrt{x}, \sqrt[3]{x}, \ldots, \sqrt[n]{x}, \ldots$ or $(\sqrt[n]{x})$. It is very generally (though not always) true that if (ϕ_n) is an ascending scale, and ψ denotes the function inverse to ϕ, then (ψ_n) is a descending scale.

If $\phi > \bar{\phi}$ for all values of x (or all values greater than some definite value), then a glance at Fig. 3 is enough to show that, if ψ and $\bar{\psi}$ are the functions inverse to ϕ and $\bar{\phi}$, then $\psi < \bar{\psi}$ for all values of x (or all values greater than some definite value). We have only to remember that the graph of ψ may be obtained from that of ϕ by looking at the latter from a different point of view (interchanging the parts of x and y). But it is not true that $\phi \succ \bar{\phi}$ involves $\psi \prec \bar{\psi}$. Thus $e^x \succ e^x/x$. The function inverse to e^x is $\log x$: the function inverse to e^x/x is obtained by solving the equation $x = e^y/y$ with respect to y. This equation gives

$$y = \log x + \log y,$$

and it is easy to see that $y \sim \log x$.

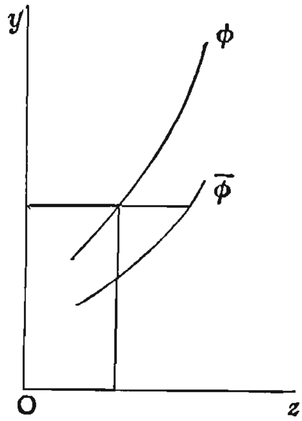

FIG. 3

Theorem 3. *Given a scale of increasing functions ϕ_n such that*

$$\phi_1 \succ \phi_2 \succ \phi_3 \succ \ldots \succ 1,$$

we can find an increasing function f such that $\phi_n \succ f \succ 1$ for all values of n.

The proof of this theorem, which is in principle the same as the first proof (§ 2.21) of Theorem 1, may be left to the reader.

2.5. The following extensions of Theorems 1 and 3 are due to du Bois-Reymond, Pincherle, and Hadamard*.

Theorem 4. *Given* $\phi_1 \prec \phi_2 \prec \phi_3 \prec \ldots \prec \phi_n \prec \ldots \prec \Phi$, *we can find f so that $\phi_n \prec f \prec \Phi$ for all values of n.*

* du Bois-Reymond, **7**; Hadamard, **2**; Pincherle, **1**.

Theorem 5. *Given* $\psi_1 \succ \psi_2 \succ \psi_3 \succ \dots \succ \psi_n \succ \dots \succ \Psi$, *we can find* f *so that* $\psi_n \succ f \succ \Psi$ *for all values of* n.

Theorem 6. *Given an ascending sequence* (ϕ_n) *and a descending sequence* (ψ_p) *such that* $\phi_n \prec \psi_p$ *for all values of* n *and* p, *we can find* f *so that*

$$\phi_n \prec f \prec \psi_p$$

for all values of n *and* p.

To prove Theorem 4 we have only to observe that

$$\Phi/\phi_1 \succ \Phi/\phi_2 \succ \dots \succ \Phi/\phi_n \succ \dots \succ 1,$$

and to construct (as we can in virtue of Theorem 3) a function F which tends to infinity more slowly than any of the functions Φ/ϕ_n. Then

$$f = \Phi/F$$

is a function such as is required. Similarly for Theorem 5. Theorem 6 requires a little more attention.

In the first place, we may suppose that $\phi_{n+1} > \phi_n$ for all values of x and n: for if this is not so we can modify the definitions of the functions ϕ_n as in § 2.21. Similarly we may suppose $\psi_{p+1} < \psi_p$ for all values of x and p.

Secondly, we may suppose that, if x is fixed, $\phi_n \to \infty$ as $n \to \infty$, and $\psi_p \to 0$ as $p \to \infty$. For if this is not true of the functions given, even when their definitions are modified as above, we may replace the modified ϕ_n and ψ_p by $\Phi_n = 2^n \phi_n$ and $\Psi_p = 2^{-p}\psi_p$; and then $\Phi_n > 2^n \phi_1$, $\Psi_p < 2^{-p}\psi_1$, so that $\Phi_n \to \infty$ when $n \to \infty$ and $\Psi_p \to 0$ when $p \to \infty$.

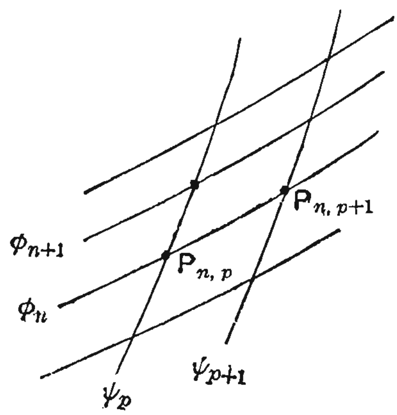

FIG. 4

Since $\psi_p \succ \phi_n$ but, for any given x, $\psi_p < \phi_n$ for sufficiently large values of n, it is clear that the curve $y = \psi_p$ intersects the curve $y = \phi_n$ for all sufficiently large values of n (say for $n > n_p$). The curves being continuous, their intersections form a closed set of points; and they have therefore a last point of intersection, which we denote by $P_{n,p}$.

If p is fixed, $P_{n,p}$ exists for $n > n_p$; similarly, if n is fixed, $P_{n,p}$ exists for $p > p_n$. And as either n or p increases, so do both the ordinate and the abscissa of $P_{n,p}$. The curve $y = \psi_p$ contains all the points $P_{n,p}$ for which p has a fixed value, and $y = \phi_n$ contains all the points for which n has a fixed value.

It is clear that, in order to define a function f which tends to infinity more rapidly than any ϕ_n and less rapidly than any ψ_p, all that we have to do is to draw a curve, making everywhere a positive acute angle with each of the axes of coordinates, and crossing all the curves $y = \phi_n$ from below to above, and all the curves $y = \psi_p$ from above to below.

Choose a positive integer N_p, corresponding to each value of p, such that (i) $N_p > n_p$ and (ii) $N_p \to \infty$ as $p \to \infty$. Then $P_{N_p, p}$ exists for each value of p. And it is clear that we have only to join the points $P_{N_1,1}$, $P_{N_2,2}$, $P_{N_3,3}$, ... by straight lines or other suitably chosen arcs of curves in order to obtain a curve which fulfils our purpose. The theorem is therefore established.

2.61. There are further interesting developments concerning scales of infinity due to Pincherle*.

We have defined $f \succ \phi$ to mean $f/\phi \to \infty$, or, what is the same thing,

(2.611) $$\log f - \log \phi \to \infty.$$

We might equally well have defined $f \succ \phi$ to mean

(2.612) $$F(f) - F(\phi) \to \infty,$$

where $F(x)$ is any function which tends steadily to infinity with x (*e.g.* x, e^x). Let us say that if (2.612) holds then

(2.613) $$f \succ \phi \ (F),$$

so that $f \succ \phi$ is equivalent to $f \succ \phi \ (\log x)$. Similarly we define $f \prec \phi \ (F)$ to mean that $F(f) - F(\phi) \to -\infty$, and $f \asymp \phi \ (F)$ to mean that $F(f) - F(\phi)$ is bounded. Thus

$$x + \log x \asymp x, \quad x + \log x \succ x \ (x),$$
$$x + 1 \asymp x \ (x), \quad x + 1 \succ x \ (e^x),$$

since $e^{x+1} - e^x = (e-1)e^x \to \infty$.

It is clear that, the more rapid the increase of F, the more likely is it to discriminate between the rates of increase of two given functions f and ϕ. More precisely, *if*

$$f \succ \phi \ (F),$$

and $\bar{F} = FF_1$, *where* F_1 *is any increasing function, then*

$$f \succ \phi \ (\bar{F}).$$

For

$$\bar{F}(f) - \bar{F}(\phi) = F(f)\,F_1(f) - F(\phi)\,F_1(\phi) > \{F(f) - F(\phi)\}\,F_1(\phi) \to \infty.$$

2.62. The substance of the following theorems is due in part to du Bois-Reymond and in part to Pincherle†.

Theorem 7. *However rapid the increase of f, as compared with that of ϕ, we can choose F so that $f \asymp \phi \ (F)$.*

Theorem 8. *If $f - \phi$ is positive for $x > x_0$, we can choose F so that $f \succ \phi \ (F)$.*

* Pincherle, **1**.

† du Bois-Reymond, **4**; Pincherle, **1**.

Theorem 9. *If $f \geqq \phi$ and $f - \phi$ is monotonic for $x > x_0$, and $f \preceq \phi\ (F)$, however great be the increase of F, then $f = \phi$ from a certain value of x onwards.*

(1) If $f \succ \phi$, we may regard f as an increasing function of ϕ, say

$$f = \theta(\phi),$$

where $\theta(x) \succ x$. We can choose a constant g greater than 1, and then choose X so that $\theta(x) > gx$ for $x > X$. Let a be any number greater than X, and let

$$a_1 = \theta(a),\ a_2 = \theta(a_1),\ a_3 = \theta(a_2), \ldots.$$

Then (a_n) is an increasing sequence, and $a_n \to \infty$, since $a_n > g^n a$.

We can now construct an increasing function F such that

$$F(a_n) = \tfrac{1}{2} nK,$$

where K is a constant. Then if $a_{\nu-1} \leqq x \leqq a_\nu$, $a_\nu \leqq \theta(x) \leqq a_{\nu+1}$, and

$$F\{\theta(x)\} - F(x) \leqq F(a_{\nu+1}) - F(a_{\nu-1}) = K.$$

Thus $F(f) - F(\phi)$ remains less than a constant, and Theorem 7 is established.

(2) Let $f - \phi = \lambda$, so that $\lambda > 0$. If λ, as x increases, remains greater than a constant K, then

$$e^f - e^\phi > (e^K - 1)\, e^\phi \to \infty,$$

so that we may take $F(x) = e^x$.

If it is not true that $\lambda \geqq K$, the lower bound of λ is zero. Let $\lambda_1(x)$ be defined as the lower bound of $\lambda(\xi)$ for $\xi \leqq x$. Then λ_1 tends steadily to zero as $x \to \infty$, and $\lambda_1 \leqq \lambda$. We may also regard λ_1 as a steadily decreasing function of ϕ, say $\lambda_1 = \mu(\phi)$.

Let $\varpi(\phi)$ be an increasing function of ϕ such that $\mu\varpi \succ 1$. Then if

$$F(\phi) = \int^\phi \varpi(t)\, dt,$$

$$F(f) - F(\phi) = \int_\phi^{\phi+\lambda} \varpi\, dt \geqq \int_\phi^{\phi+\mu(\phi)} \varpi\, dt > \mu(\phi)\, \varpi(\phi) \succ 1,$$

and $F(x)$ fulfils the requirement of Theorem 8. Finally, Theorem 9 is an obvious corollary of Theorem 8.

The three theorems which follow are of the same character as those which we have just proved. The reader will find it instructive to deduce them, or prove them independently.

Theorem 10. *However great be the increase of f, as compared with that of ϕ, we can determine an increasing function F such that $F(f) \preceq F(\phi)$.*

Theorem 11. *If $f - \phi$ is positive for $x > x_0$, we can determine an increasing function F such that $F(f) \succ F(\phi)$.*

Theorem 12. *If $f \geqq \phi$ and $f-\phi$ is monotonic for $x > x_0$, and $F(f) \asymp F(\phi)$ however great the increase of F, then $f=\phi$ from a certain value of x onwards.*

To these he may add the theorem (analogous to that proved at the end of § 2.61) that *$f \succ \phi$ involves $F(f) \succ F(\phi)$ if $\log F(x)/\log x$ is an increasing function* (a condition which is roughly equivalent to $F \succ x$).

2.63. Let us consider some examples of the theorems of the last paragraph.

(i) Let $f=x^m$ $(m>1)$ and $\phi=x$. Then, following the argument of § 2.62 (1), we have $\theta(\phi)=\phi^m$. We may take

$$a=e,\ a_1=e^m,\ a_2=e^{m^2},\ \dots,\ a_n=e^{m^n},\ \dots,$$

and we have to define F so that

$$F(e^{m^n})=\tfrac{1}{2}nK.$$

The most natural solution of this equation is

$$F(x)=\frac{K\log\log x}{2\log m}.$$

And in fact

$$F(x^m)-F(x)=\frac{K}{2\log m}\{\log(m\log x)-\log\log x\}=\tfrac{1}{2}K,$$

so that $x^m \asymp x$ (F).

(ii) Let $f=e^x+e^{-x}$, $\phi=e^x$. Following the argument of § 2.62 (2), we find

$$\lambda=e^{-x}=\lambda_1,\quad \mu(\phi)=1/\phi,$$

and we may take $\varpi(\phi)=\phi^{1+a}$ $(a>0)$. This makes $F(\phi)$ a constant multiple of ϕ^{2+a}, and it is easy to verify that

$$(e^x+e^{-x})^k-e^{kx}\to\infty,$$

if $k>2$.

(iii) The relation $F(f)\asymp F(\phi)$ is equivalent to $f\asymp\phi$ $(\log F)$. Using the result of (i), we see that $F(x^m)\asymp F(x)$ if $1\prec F\preccurlyeq\log x$. Similarly, using the result of (ii), we see that $F(e^x+e^{-x})\succ F(e^x)$ if $F\succcurlyeq e^{x^k}$ $(k>2)$.

2.7. Before leaving this part of our subject, let us observe that the substance of §§ 2.1—2.5 may be extended to the case in which our symbols $\succ$, etc. are defined by reference to an arbitrary increasing function F. We leave it as an exercise to the reader to effect these extensions.

III

LOGARITHMICO-EXPONENTIAL SCALES: THE FUNDAMENTAL THEOREMS

3.1. THE scales of infinity that are of most practical importance in analysis are those which may be constructed by means of the logarithmic and exponential functions.

We have already seen (§ 1.1) that

$$e^x \succ x^n$$

for any value of n; and from this it follows that

$$\log x \prec x^{1/n}$$

for any value of n*. It is easy to deduce that

$$e^{e^x} \succ e^{x^n},\quad e^{e^{e^x}} \succ e^{e^{x^n}},\ \dots,$$

$$\log\log x \prec (\log x)^{1/n},\quad \log\log\log x \prec (\log\log x)^{1/n},\ \dots.$$

The repeated logarithmic and exponential functions are so important in this subject that it is worth while to adopt a notation for them of a less cumbrous character. We shall write

$$l_1 x = lx = \log x,\quad l_2 x = llx,\quad l_3 x = ll_2 x,\ \dots\dagger,$$

$$e_1 x = ex = e^x,\quad e_2 x = eex,\quad e_3 x = ee_2 x,\ \dots.$$

It is easy, with the aid of these functions, to write down any number of ascending scales, each containing only functions whose increase is greater than that of any function in any preceding scale: for example

$$x,\ x^2,\ \dots,\ x^n,\ \dots;\ e^x,\ e^{2x},\ \dots,\ e^{nx},\ \dots;\ e^{x^2},\ e^{x^3},\ \dots,\ e^{x^n},\ \dots.$$

Among the functions of these scales we can interpolate new functions as freely as we like, using, for instance, such functions as

$$x^\alpha e^{\beta x^\gamma e^{\delta x^\epsilon}},$$

where α, β, γ, δ, ϵ are any positive numbers; and we can construct non-enumerable as well as enumerable scales‡. Similarly we can construct any number of descending scales, each composed of functions whose

* It was pointed out in § 2.4 that $\phi \succ \bar{\phi}$ does not necessarily involve $\psi \prec \bar{\psi}$ (ψ, $\bar{\psi}$ being the functions inverse to ϕ, $\bar{\phi}$). But it does involve $\psi < \bar{\psi}$ for sufficiently large values of x, and therefore $\psi \preccurlyeq \bar{\psi}$. Hence $\phi \succ \phi_n$ (for any n) involves $\psi \preccurlyeq \psi_n$ (for any n) and therefore, if (ψ_n) is a descending scale, as is in this case obvious, $\psi \prec \psi_n$ for any n.

† lx is defined for $x > 0$, $l_2 x$ for $x > 1$, $l_3 x$ for $x > e$, $l_4 x$ for $x > e^e$, and so on.

‡ See § 2.1.

increase is less than that of any functions in any preceding scale: for example

$$lx,\ (lx)^{1/2},\ \dots,\ (lx)^{1/n},\ \dots;\ l_2x,\ (l_2x)^{1/2},\ \dots,\ (l_2x)^{1/n},\ \dots.$$

Two special scales are of fundamental importance; the ascending scale

$$(E) \qquad x,\ ex,\ e_2x,\ e_3x,\ \dots,$$

and the descending scale

$$(L) \qquad x,\ lx,\ l_2x,\ l_3x,\ \dots.$$

These scales mark the *limits* of all logarithmic and exponential scales. It is of course possible, in virtue of the general theorems of §§ 2.1—2.5, to define functions whose increase is more rapid than that of any e_nx or slower than that of any l_nx; but, as we shall see in a moment, this is not possible if we confine ourselves to functions defined by finite and explicit formulae involving only the ordinary functional symbols of elementary analysis.

3.2. We define a *logarithmico-exponential function* (shortly, an *L-function*) as a real one-valued function defined, for all values of x greater than some definite value, by a finite combination of the ordinary algebraic symbols (viz. $+$, $-$, $\times$, $\div$, $\sqrt[n]{}$) and the functional symbols $\log(\dots)$ and $e^{(\dots)}$, operating on the variable x and on real constants.

It is to be observed that the result of working out the value of the function, by substituting x in the formula defining it, is to be real at all stages of the work. It is important to exclude such a function as

$$\tfrac{1}{2}\{e^{\sqrt{(-x^2)}}+e^{-\sqrt{(-x^2)}}\},$$

which, with a suitable interpretation of the roots, is equal to $\cos x$.

We might generalize our definition by admitting implicit algebraic functions, including, for example, such functions as $e_2\sqrt{(ly)}$, where $y^5+y-x=0$; but this generalization is not particularly interesting.

Theorem 13. *Any L-function is ultimately continuous, of constant sign, and monotonic, and tends, as $x\to\infty$, to infinity, or to zero or to some other definite limit. Further, if f and ϕ are L-functions, one or other of the relations*

$$f\succ\phi,\quad f\asymp\phi,\quad f\prec\phi$$

holds between them.

We may classify L-functions as follows, by a method similar to that by which Liouville* classified the 'elementary' functions. An L-function is of order 0 if it is purely algebraic; of order 1 if the functional

* Liouville, **1**; Watson, **1**, 111. See also Hardy, **2**; this tract contains fuller references to Liouville's memoirs.

symbols $l(\dots)$ and $e(\dots)$ which occur in it operate only on algebraic functions; of order 2 if they operate only on algebraic functions or L-functions of order 1; and so on. Thus

$$e^{x^2}+\sqrt{(\log x)},\quad x^{\sqrt{2}}=e^{\sqrt{2}\log x},\quad x^{x^x}=e^{\log x\, e^{x\log x}}$$

are of orders 1, 2, and 3 respectively. As the results stated in the theorem are true of algebraic functions, it is sufficient to prove that, if true of L-functions of order $n-1$, they are true of L-functions of order n.

It should be observed that an L-function of order n may always be expressed as a function of any higher order; thus $x=e(lx)=e_2(l_2x)=\dots$. We need not suppose that our functions are always expressed in the simplest possible form. In Liouville's work it is essential to assume that an 'elementary function of order n' cannot be expressed as a function of lower order; but no such hypothesis is necessary here.

The following additional definitions will be found useful. We shall say that f_n, an L-function of order n, is *integral* if it is of the form

$$\Sigma\rho_{n-1}e\sigma_{n-1}(l\tau_{n-1}^{(1)})^{\kappa_1}(l\tau_{n-1}^{(2)})^{\kappa_2}\dots(l\tau_{n-1}^{(h)})^{\kappa_h},$$

where the functions with suffix $n-1$ are L-functions of order $n-1$ and $\kappa_1, \kappa_2, \dots, \kappa_h$ are positive integers. We call $\kappa_1+\kappa_2+\dots+\kappa_h$ the *logarithmic degree*, or, simply, the *degree*, of the typical term of f_n; and, if λ is the greatest value of $\kappa_1+\kappa_2+\dots+\kappa_h$, we say that f_n is *of logarithmic degree* λ. If the number of terms of degree λ in f_n is μ, we say that f_n is *of logarithmic type* (λ, μ). We denote integral L-functions by the letter M, with or without suffixes, indices, etc.

If an integral L-function is of degree 0, *i.e.* of the form

$$\Sigma\rho_{n-1}e\sigma_{n-1},$$

we shall say that it is *exponential*. If an integral exponential L-function contains ϖ terms, we shall say that it is *of type* ϖ; if $\varpi=1$, we shall say that it is *simply exponential*. Thus $(lx)^2e(e^x lx)$ is a simply exponential L-function of order 2, while $(lx)^2(l_2x)^2e(e^x lx)$ is an integral function of order 2, of type (2, 1). We shall in general denote integral exponential L-functions by the letter N.

If f_n is the quotient of two integral functions, *i.e.* of the form M_1/M_2, we shall say that it is *rational*. If M_1 and M_2 are exponential, *i.e.* if f_n is of the form N_1/N_2, we shall say that f_n is a *rational exponential* L-function.

It may be verified immediately that:

(i) The derivative of an L-function of order n is an L-function of order n. In exceptional cases the derivative may be of order $n-1$.

(ii) The derivative of a simply exponential function is a simply exponential function, with the same exponential factor.

(iii) The derivative of an integral exponential function of type ϖ is in general an integral exponential function of type ϖ. If one of the terms of the original function is a constant, the derivative is of type $\varpi - 1$.

(iv) The derivative of an integral function of logarithmic type (λ, μ) is in general a function of the same type. If the exponential factor $\rho_{n-1} e \sigma_{n-1}$ of one of the terms of degree λ is a constant, then the derivative is of type $(\lambda, \mu - 1)$; if $\mu = 1$, the derivative is of degree $\lambda - 1$.

3.3. We can simplify the induction required for the proof of Theorem 13 by two preliminary remarks.

(i) If f is an L-function of order n, so is its derivative f'. Hence, if every such function is ultimately continuous and of constant sign, every such function is ultimately monotonic.

(ii) If f and ϕ are L-functions of order n, so is f/ϕ. Hence, if every such function is ultimately monotonic, f/ϕ must tend to infinity or a limit, and one of the relations $\succ$, $\asymp$, $\prec$ holds between the functions.

It is therefore sufficient to prove that *if Theorem 13 is true for functions of order $n-1$, then any function of order n is ultimately continuous and of constant sign.*

We prove this first when f_n is an integral exponential function. The result is obvious when f_n is of type 1 (*i.e.* when f_n is a simply exponential function $\rho_{n-1} e \sigma_{n-1}$). Let us then assume it true for functions of type $\varpi - 1$; and let

$$f_n = \Sigma \rho_{n-1} e \sigma_{n-1}$$

be of type ϖ. If $\rho_{n-1} e \sigma_{n-1}$ is any one of the terms of f_n, the function

$$F_n = f_n / (\rho_{n-1} e \sigma_{n-1})$$

is of type ϖ, with one term a constant (unity); and so, by § 3.2 (iii), F_n' is of type $\varpi - 1$. Hence F_n' is ultimately continuous and of constant sign; and so the same is true of F_n, and therefore of f_n.

We prove next that the result is true when f_n is any integral function of order n. Suppose that

$$f_n = \Sigma \rho_{n-1} e \sigma_{n-1} (l \tau_{n-1}^{(1)})^{\kappa_1} (l \tau_{n-1}^{(2)})^{\kappa_2} \dots (l \tau_{n-1}^{(h)})^{\kappa_h}$$

is of logarithmic type (λ, μ). The result has been proved true when $\lambda = 0$. Hence it is enough to prove

(i) that, if true for functions of logarithmic degree $\lambda - 1$, it is true for functions of degree λ and type $(\lambda, 1)$;

(ii) that, if true for functions of type $(\lambda, \mu-1)$, it is true for functions of type (λ, μ).

Suppose that the typical term written above in the expression of f_n is one of the terms of degree λ, and let $F_n = f_n/(\rho_{n-1} e\sigma_{n-1})$ as before. Then, by § 3.2 (iv), F_n' is of type $(\lambda, \mu-1)$, unless $\mu=1$, when it is of degree $\lambda-1$. Hence, whichever of the inductions (i), (ii) we are engaged in proving, F_n' is ultimately continuous and of constant sign; and we deduce as before that the same is true of f_n.

We are now in a position to complete the proof of the theorem. Any L-function f_n is of the form

$$f_n = A\,\{e\phi_{n-1}^{(1)},\ e\phi_{n-1}^{(2)},\ \dots,\ e\phi_{n-1}^{(r)},\ l\psi_{n-1}^{(1)},\ \dots,\ l\psi_{n-1}^{(s)},\ \chi_{n-1}^{(1)},\ \dots,\ \chi_{n-1}^{(t)}\}$$
$$= A\,(z_1, z_2, \dots, z_q),$$

say, where $q = r+s+t$ and A is an algebraic function of its arguments. There is therefore an identical relation

$$F(x, y) = M_0 y^p + M_1 y^{p-1} + \dots + M_p = 0,$$

where $y = f_n$ and the coefficients $M_1, M_2, \dots, M_p$ are integral L-functions of order n. The derivatives of these coefficients are also integral. It therefore follows from what has already been proved that

$$F_x = \frac{\partial F}{\partial x} = \Sigma \frac{dM_i}{dx} y^{p-i}, \quad F_y = \frac{\partial F}{\partial y} = \Sigma\,(p-i) M_i y^{p-i-1},$$

considered as functions of the two variables x, y, are continuous for all sufficiently large values of x and for all values of y.

Let ξ, η be a pair of values of x and y satisfying the equation $F=0$. Then, if only ξ is large enough, F_y cannot vanish for $x=\xi$, $y=\eta$. For, if F and F_y both vanish for $x=\xi$, $y=\eta$, the eliminant of y between $F=0$ and $F_y=0$ vanishes for $x=\xi$. But this eliminant is plainly an integral L-function of order n, and so cannot vanish for values of x surpassing all limit. It follows, by the fundamental theorem concerning implicit functions*, that f_n is an ultimately continuous function of x. Finally, f_n is ultimately of constant sign. For $f_n = 0$ involves $M_p = 0$, and we have already seen that it is impossible that this equation should be satisfied for values of x surpassing all limit. This completes the proof of the theorem.

3.4. The limits of the increase of L-functions. The increase of an increasing L-function is subject to limitations of rapidity or slowness. We may say roughly that *an L-function cannot increase more rapidly than any exponential or more slowly than any logarithm;* if f is

* Goursat, **1** (1), 81, 94; Hardy, **1**, 192; Young, **1**.

any L-function, we can determine k so that $f \prec e_k x$, and if f is any L-function which tends to infinity, we can determine k so that $f \succ l_k x$.

More precisely, we have the two following theorems:

Theorem 14. *An L-function of order n cannot satisfy*

$$f_n \succ e_n(x^\Delta).$$

Theorem 15. *An L-function of order n cannot satisfy*

$$1 \prec f_n \prec (l_n x)^\delta.$$

Theorem 14 is very easy to prove. It is plainly sufficient to establish an induction from $n-1$ to n.

Any function of order n is an algebraical function of certain arguments $e\phi_{n-1}, \ldots, l\psi_{n-1}, \ldots, \chi_{n-1}, \ldots$, the increase of any one of which is *ex hypothesi* less than that of

$$e(e_{n-1}x^K) = e_n(x^K)$$

for some value of K. Hence the increase of the function is less than that of

$$(e_n x^K)^{K_1}$$

for some values of K and K_1; and so less than that of $e_n(x^{K_2})$ for some value of K_2. Thus the theorem is established.

The proof of Theorem 15 is considerably more troublesome, and, though it presents no particular difficulty of principle, is too long to be inserted here*. It is included in a more precise theorem, viz.

Theorem 16. *If f is an L-function of order n, and*

$$1 \prec f \prec (l_{n-1}x)^\delta,$$

then

$$f \asymp (l_n x)^h,$$

where h is rational.

3.5. Let f and ϕ be any two L-functions which tend to infinity with x, and let a be any positive number. Then one of the three relations

$$f \succ \phi^a, \quad f \asymp \phi^a, \quad f \prec \phi^a$$

must hold between f and ϕ; and the second can hold for at most one value of a. If the first holds for any a it holds for any smaller a; and if the last holds for any a it holds for any greater a.

Then there are three possibilities. Either the first relation holds for every a; then

$$f \succ \phi^\Delta.$$

Or the third holds for every a; then

$$f \prec \phi^\delta.$$

* The details of the proof will be found in Hardy, **9**, 65—72.

Or the first holds for some values of α and the third for others; and then there is a value a of α which divides the two classes of values of α, and we may write

$$f = \phi^a f_1,$$

where $\phi^{-\delta} \prec f_1 \prec \phi^\delta$. We shall find this result very useful in the sequel.

3.6. It is possible to classify the possible modes of increase of L-functions of given order much more precisely. Thus we have:

Theorem 17. *An L-function of order* 1, *which tends to infinity with x, is of one of the forms*

$$e^{Ax^s(1+\epsilon)}, \quad Ax^s(\log x)^t(1+\epsilon),$$

where s and t are rational.

Theorem 18. *An L-function of order* 2, *which tends to infinity with x, is of one of the forms*

$$e^{e^{Ax^s(1+\epsilon)}}, \quad e^{Ax^s(lx)^t(1+\epsilon)}, \quad x^s e^{A(lx)^t(1+\epsilon)}, \quad Ax^s(lx)^t(l_2x)^u(1+\epsilon),$$

where s, t, and u are rational.

The functions

$$e_2x, \quad l_2x, \quad x^x, \quad x^{\sqrt{2}}, \quad e^{\sqrt{(lx)}}$$

each increase in a manner which differs from either of those of Theorem 17. They are therefore not expressible as L-functions of order lower than 2. Similarly l_3x, $(lx)^{\sqrt{2}}$, or $e^{\sqrt{(l_2x)}}$ are not expressible as L-functions of order lower than 3. No L-function of order 1 can satisfy

$$x^\Delta \prec f \prec e^{x^\delta},$$

and no L-function of order 2 can satisfy either of

$$e^{x^\Delta} \prec f \prec e_2 x^\delta, \quad e^{(lx)^\Delta} \prec f \prec e^{x^\delta}.$$

The reader will find detailed proofs of these theorems in a memoir by the author*.

* Hardy, **9**.

IV

SPECIAL PROBLEMS CONNECTED WITH LOGARITHMICO-EXPONENTIAL SCALES

4.1. The functions $e_r(l_s x)^\mu$. We have agreed to express the fact that, however large be a and however small be b, x^a has an increase less than that of e^{x^b}, by

(4.11) $$x^\Delta \prec e^{x^\delta}.$$

Let us endeavour to find a function f such that

(4.12) $$x^\Delta \prec f \prec e^{x^\delta}.$$

If $\phi_1 \succ \phi_2$, $e^{\phi_1} \succ e^{\phi_2}$. Thus (4.12) will certainly be satisfied if

$$\log x \prec \log f \prec x^\delta.$$

Hence a solution of our problem is given by

$$f = e^{(\log x)^\beta} \quad (\beta > 1).$$

Similarly we can prove that

$$f = e^{(\log x)^a} \quad (0 < a < 1)$$

satisfies $$(\log x)^\Delta \prec f \prec x^\delta.$$

It will be convenient to write

$$e_0 x \equiv l_0 x \equiv x,$$

a for a positive number less than 1, β for a positive number greater than 1, and γ for any positive number; and then we have the relations

(4.13) $$e_0(l_1 x)^\gamma \prec e_1(l_1 x)^a \prec e_0(l_0 x)^\gamma \prec e_1(l_1 x)^\beta \prec e_1(l_0 x)^\gamma.$$

Let us now consider the functions

$$f = e_r(l_s x)^\mu, \quad f' = e_{r'}(l_{s'} x)^{\mu'},$$

where μ, μ' are positive and not equal to 1. If $r = r'$, $f \succ f'$ or $f \prec f'$ according as $s < s'$ or $s > s'$. If $s = s'$, the same relations hold according as $r > r'$ or $r < r'$. If $r = r'$ and $s = s'$, then $f \succ f'$ or $f \prec f'$ according as $\mu > \mu'$ or $\mu < \mu'$. Leaving these cases aside, suppose $s > s'$, $s - s' = \sigma > 0$. Putting $l_{s'} x = y$, we obtain

$$f = e_r(l_\sigma y)^\mu, \quad f' = e_{r'} y^{\mu'}.$$

If $r \leqq r'$, it is clear that $f \prec \phi$. If $r > r'$, let $r - r' = \rho$; then

$$l_r f = (l_\sigma y)^\mu, \quad l_r f' = l_\rho y^{\mu'} \asymp l_\rho y:$$

if $\rho > 1$ the symbol $\asymp$ may be replaced by $\sim$. If $\sigma > \rho$, $l_r f \prec l_r f'$ and so $f \prec f'$. If $\sigma < \rho$, $f \succ f'$. If $\sigma = \rho$, $f \succ f'$ or $f \prec f'$ according as $\mu > 1$ or $\mu < 1$. Thus

$$f \succ f' \ (r - s > r' - s'), \quad f \prec f' \ (r - s < r' - s'),$$

while if $r - s = r' - s'$, $f \succ f'$ or $f \prec f'$ according as $\mu > 1$ or $\mu < 1$, μ being the exponent of the logarithm of higher order which occurs in f or f'.

From this it follows that

$$\ldots e_1(l_2x)^\alpha \prec e_0(l_1x)^\gamma = (lx)^\gamma \prec e_1(l_2x)^\beta \prec e_2(l_3x)^\beta \prec \ldots,$$

$$\ldots \prec e_2(l_2x)^\alpha \prec e_1(l_1x)^\alpha \prec e_0(l_0x)^\gamma = x^\gamma \prec e_1(l_1x)^\beta \prec \ldots,$$

$$\ldots \prec e_3(l_2x)^\alpha \prec e_2(l_1x)^\alpha \prec e_1(l_0x)^\gamma = ex^\gamma \prec e_2(l_1x)^\beta \prec \ldots.$$

These relations enable us to interpolate to any extent among what we may call the fundamental logarithmico-exponential orders of infinity, viz. $(l_kx)^\gamma$, x^γ, e_kx^γ. Thus

$$e^{(lx)^\beta},\quad e^{e^{(llx)^\beta}},\ \ldots\quad (\beta > 1),$$

and

$$e^{e^{(lx)^\alpha}},\quad e^{e^{e^{(llx)^\alpha}}},\ \ldots\quad (0 < \alpha < 1),$$

are two scales, the first rising from above x^γ, the second falling from below ex^γ, and never overlapping.

These scales, and the analogous scales which can be interpolated between other pairs of the fundamental logarithmico-exponential orders, possess another interesting property. The two scales written above *cover up* (to put it roughly) *the whole interval between* x^γ *and* ex^γ, *so far as L-functions are concerned*: that is to say, it is impossible that an L-function f should satisfy

$$f \succ e_r(l_rx)^\beta, \qquad \textit{(every } r\textit{)},$$

$$f \prec e_{r+1}(l_rx)^\alpha, \qquad \textit{(every } r\textit{)};$$

and the corresponding pairs of scales lying between $(l_{k+1}x)^\gamma$ and $(l_kx)^\gamma$, or between e_kx^γ and $e_{k+1}x^\gamma$, possess a similar property. This property is analogous to that possessed (Theorems 14 and 15) by the scales (l_rx), (e_rx); viz. that no L-function f can satisfy $f \succ e_rx$, or $1 \prec f \prec l_rx$, for all values of r. A little consideration is all that is needed to render the theorem plausible: for a formal proof we must refer to the memoir quoted on p. 21.

4.21. Successive approximations to a logarithmico-exponential function. Consider such a function as

$$f = \sqrt{x}\,(lx)^2\, e^{\sqrt{(lx)}\,(l_2x)^2\, e^{\sqrt{(l_2x)}\,(l_3x)^2}}.$$

If we omit one or more of the parts of the expression of f, we obtain another function whose increase differs more or less widely from that of f. The question arises which parts are of the greatest and which of the least importance, *i.e.* which are the parts whose omission affects the increase of f most or least fundamentally.

Taking logarithms we find

$$(4.211)\qquad lf = \tfrac{1}{2}lx + \sqrt{(lx)}\,(l_2x)^2\, e^{\sqrt{(l_2x)}\,(l_3x)^2} + 2l_2x,$$

the three terms being arranged in order of importance. Again

$$l_2f = l_2x - l2 + \epsilon,\quad l_3f = l_3x + \epsilon.$$

If we neglect the ϵ's in these equations, we deduce the approximations

$$(1)\ f = x,\qquad (2)\ f = \sqrt{x}.$$

By neglecting the last term in the equation (4.211) we obtain the much closer approximation

$$(6)\quad f=\sqrt{x}e^{\sqrt{(lx)}(l_2x)^2}e^{\sqrt{(l_2x)(l_3x)^2}}.$$

In order to obtain a more complete series of approximations, we must replace the equation (4.211) by a series of approximate equations. Now if

$$\phi=\sqrt{(lx)}\,(l_2x)^2\,e^{\sqrt{(l_2x)(l_3x)^2}},$$

we have

$$l\phi=\tfrac{1}{2}l_2x+\sqrt{(l_2x)}\,(l_3x)^2+2l_3x,$$

$$l_2\phi=l_3x-l2+\epsilon,\quad l_3\phi=l_4x+\epsilon.$$

Hence we obtain (0) $\phi=lx$, (3) $\phi=\sqrt{(lx)}$, and (5) $\phi=\sqrt{(lx)}\,e^{\sqrt{(l_2x)(l_3x)^2}}$ as approximations to the increase of ϕ: of these, however, the first is valueless, inasmuch as it would make ϕ preponderate over the first term on the right hand side of (4.211).

A similar argument, applied to the function $e^{\sqrt{(l_2x)(l_3x)^2}}$, leads us to interpolate (4) $\phi=\sqrt{(lx)}\,e^{\sqrt{(l_2x)}}$ between (3) and (5). We can now, by substituting these approximations to ϕ in (4.211), deduce a complete system of closer and closer approximations to the increase of f, viz.

$$(1)\ x,\quad (2)\ \sqrt{x},\quad (3)\ \sqrt{x}\,e^{\sqrt{(lx)}},\quad (4)\ \sqrt{x}\,e^{\sqrt{(lx)}\,e^{\sqrt{(l_2x)}}},$$

$$(5)\ \sqrt{x}\,e^{\sqrt{(lx)}\,e^{\sqrt{(l_2x)(l_3x)^2}}},\quad (6)\ \sqrt{x}\,e^{\sqrt{(lx)}\,(l_2x)^2\,e^{\sqrt{(l_2x)(l_3x)^2}}}.$$

This order corresponds to the order of importance of the various parts of the expression of f.

4.22. Legitimate and illegitimate forms of approximation to a logarithmico-exponential function. In applications of this theory, such as occur, for instance, in the theory of integral functions, we are continually meeting such equations as

$$(4.221)\qquad f=(1+\epsilon)\,e^{x^a},\quad f=e^{(1+\epsilon)\,x^a},\quad f=e^{x^{a+\epsilon}}\quad (a>0).$$

It is important to have clear ideas as to the degree of accuracy of such representations of f. The simplest method is to take logarithms repeatedly, as in § 4.21.

In the first example the term ϵ does not affect the increase of f: we have $f\sim ex^a$. This is not true in the second; but $lf\sim x^a$, so that the term ϵ does not affect the increase of lf; while in the third this is not true, though $llf\sim a$. Of the three formulae the first gives the most, and the last the least, information as to the increase of f.

Such a formula as

$$(4.222)\qquad f=xe^{(1+\epsilon)\,x^a}$$

would not be a legitimate form of approximation at all, since the factor $e\,(\epsilon x^a)$ may well be far more important than the accurate factor x, and (4.222) conveys no more information than the second equation (4.221).

4.3. Attempts to represent orders of infinity by symbols. It is natural to try to devise some simple method of representing orders of infinity by symbols which can be manipulated according to laws resembling as far as possible those of ordinary algebra. Thus Thomae* has proposed to represent the order of infinity of $f=x^a(lx)^{a_1}(l_2x)^{a_2}\dots$ by

$$Of=a+a_1l_1+a_2l_2+\dots\dagger,$$

where the symbols l_1, l_2, ... are to be regarded as new units. It is clear that these units cannot, in relation to one another, obey the Axiom of Archimedes ‡: however great n, nl_2 cannot be as great as l_1, nor nl_1 as great as 1.

The consideration of a few simple cases is enough to show that any such notation, if it is to be useful, must obey the following laws:

$$\text{(i)}\quad \text{if } f\succcurlyeq\phi,\quad O(f+\phi)=Of;$$

$$\text{(ii)}\quad O(f\phi)=Of+O\phi;$$

$$\text{(iii)}\quad O\{f(\phi)\}=Of\times O\phi.$$

And Pincherle § has pointed out that these laws are in any case inconsistent with the maintenance of the laws of algebra in their entirety. Thus if

$$Ox=1,\quad Olx=\lambda,$$

we have, by (iii), $Ollx=\lambda^2$, and by (iii) and (ii),

$$Ol(xlx)=\lambda(1+\lambda)=\lambda+\lambda^2;$$

and on the other hand, by (i),

$$Ol(xlx)=O(lx+llx)=\lambda.$$

Pincherle has suggested another system of notation; but the best yet formulated is Borel's ‖. Borel preserves the three laws (i), (ii), (iii), the commutative law of addition, and the associative law of multiplication. But multiplication is no longer commutative, and distributive on one side only ¶. He would denote the orders of

$$e^x x^n,\quad x^n(lx)^p,\quad e^{2x},\quad e^{x^2},\quad e^{e^x},\quad e^{\sqrt{lx}},\quad \tfrac{1}{2}x,$$

by

$$\omega+n,\quad n+\frac{p}{\omega},\quad 2.\omega,\quad \omega.2,\quad \omega^2,\quad \omega.\frac{1}{2}.\frac{1}{\omega},\quad \frac{1}{\omega}.\frac{1}{2}.\omega.$$

But little application, however, has yet been found for any such system of notation; and the whole matter appears to be rather of the nature of a mathematical curiosity.

4.41. Functions which do not conform to any logarithmico-exponential scale. We saw in § 1.2 that, given two increasing functions ϕ and ψ such that $\phi\succ\psi$, we can always construct an increasing function f which is, for an infinity of values of x increasing beyond all limit, of

* Thomae, **1**, 144.

† The reader will not confuse this use of the symbol O (which does not extend beyond this paragraph) with that explained in § 1.5.

‡ 'If $x>y>0$, we can find an integer n such that $ny>x$'.

§ Pincherle, **1**.

‖ Borel, **4**, 35 and **5**, 14.

¶ $(a+b)c=ac+bc$, but in general $a(b+c)\neq ab+ac$.

the order of ϕ, and for another infinity of values of x of the order of ψ. The actual construction of such functions by means of explicit formulae we left till later. We shall now consider the matter more in detail, with special reference to the case in which ϕ and ψ are L-functions.

We shall say that f is an *irregularly increasing* function (*fonction à croissance irrégulière*) if we can find two L-functions ϕ and ψ $(\phi \succ \psi)$ such that

$$f \geqslant \phi \quad (x = x_1, x_2, \ldots), \quad f \leqslant \psi \quad (x = x_1', x_2', \ldots),$$

$x_1, x_2, \ldots$ and $x_1', x_2', \ldots$ being any two indefinitely increasing sequences of values of x. We shall also say that 'the increase of f is irregular' and that 'the logarithmico-exponential scales are *inapplicable to* f'.

The phrase '*fonction à croissance irrégulière*' has been defined by various writers in various senses. Borel* originally defined f to be *à croissance régulière* if

$$e^{x^{a-\delta}} < f < e^{x^{a+\delta}} \quad (x > x_0),$$

or in other words if $llf \asymp lx$.

This definition was of course designed to meet the particular needs of the theory of integral functions; and has been made more precise by Boutroux and Lindelöf†, who use inequalities of the form

$$e^{x^{a}(lx)^{a_1}\ldots(l_k x)^{a_k-\delta}} < f < e^{x^{a}(lx)^{a_1}\ldots(l_k x)^{a_k+\delta}}.$$

All functions which are not *à croissance régulière* for these writers are included in our class of irregularly increasing functions.

4.42. The logarithmico-exponential scales may fail to give a complete account of the increase of a function in two different ways. The function may be of irregular increase, as explained above, and the scales *inapplicable*, or on the other hand they may be, not inapplicable, but *insufficient* (*en défaut*). That is to say, although the increase of the function does not oscillate from that of one L-function to that of another, there may be no L-function capable of measuring it. That such functions exist follows at once from the general theorems of § 3.4. Thus we can define a function which tends to infinity more rapidly than any $e_r x$, or more slowly than any $l_r x$; and the increase of such a function is more rapid or slower than that of any L-function. Or again (§ 2.5, Theorem 6) we can define a function whose increase is, for every r, greater than that of $e_r(l_r x)^{\beta}$ and less than that of $e_{r+1}(l_r x)^{a}$, if $0 < a < 1 < \beta$; and (§ 4.1) the increase of such a function cannot be equal to that of any L-function.

We shall now discuss some actual examples of functions for which the logarithmico-exponential scales are inapplicable or insufficient.

* Borel, **2**, 108. † Boutroux, **1**; Lindelöf, **2**. See also Blumenthal, **1**, 7.

4.43. Irregularly increasing functions. Functions whose increase is irregular may be constructed in a variety of ways.

(i) Pringsheim* has used, in connection with the theory of the convergence of series, functions of an integral variable n whose increase is irregular. A simple example of such a function is

$$f(n)=10^{[(\log_{10} n)^{1/\tau}]^\tau} \quad (\tau>1),$$

where $[x]$ denotes the integral part of x. It is easily verified, for instance, when $\tau=2$, that the increase of $f(n)$ varies between that of n and that of $n \,.\, 10^{-2\sqrt{(\log_{10} n)}}$.

(ii) A more natural type of function is given by

$$f=\phi \cos^2 \theta+\psi \sin^2 \theta,$$

where ϕ, ψ, θ are increasing L-functions and $\phi \succ \psi \succ 1$. We have to consider what conditions ϕ, ψ, θ must satisfy in order that f may increase steadily with x. That its increase oscillates between that of ϕ and that of ψ is obvious.

Differentiating, we obtain

$$f'=\phi' \cos^2 \theta+\psi' \sin^2 \theta+2(\psi-\phi)\,\theta' \cos \theta \sin \theta.$$

We shall now assume that (as will be proved in the next chapter) relations between L-functions, involving the symbols $\succ, \ldots$, may be differentiated and integrated. The condition that f' should always be positive is that

$$\phi'\psi' > (\phi-\psi)^2\,\theta'^2,$$

or $\phi'\psi' > \phi^2\theta'^2$. Since $\phi' \succ \psi'$, this involves $\phi' \succ \phi\theta'$, or $\log \phi \succ \theta$; and f will certainly be monotonic if

$$\log \phi \succ \theta, \quad \psi' \succ \phi^2\theta'^2/\phi'.$$

These conditions are satisfied, for example, if $\phi=x^\alpha ex^\rho$, $\psi=x^\beta ex^\rho$, $\theta=x$, and $\alpha-2\rho+2<\beta<\alpha$. Changing our notation a little we see that

$$f=(x^{\gamma+\delta} \cos^2 x+x^{\gamma-\delta} \sin^2 x)\, e^{x^\rho}$$

is monotonic if $0<\delta<\rho-1$; and the increase of f oscillates between that of $x^{\gamma+\delta} ex^\rho$ and that of $x^{\gamma-\delta} ex^\rho$. Similarly it may be shown that

$$f=(e^{\mu x} \cos^2 x+e^{\nu x} \sin^2 x) e^{e^x}$$

is monotonic if $\nu<\mu<\nu+2$†; and again the increase of f is irregular.

(iii) Borel‡ has shown how, by means of power series, we may define functions which increase steadily with x, while their increase oscillates to an arbitrary extent.

Let $$\phi(x)=\Sigma a_n x^n, \quad \psi(x)=\Sigma b_n x^n$$
be two integral functions of x with positive coefficients. The increase

* Pringsheim, **5** and **1**, 373. † Hardy, **3** (1).

‡ Borel, **2**, 120 and **4**, 32. Borel considers the cases only in which $\psi=ex$, $\phi=ex^2$ or e_2x, but his method is obviously general. The proof given here is however more general and simpler.

of ϕ and ψ may be as large as we like (§ 2.32, Theorem 2). We suppose that $\phi \succ \psi \succ x^{\Delta}$. Then we can define a function

$$f(x) = \Sigma c_n x^n,$$

where every c_n is equal either to a_n or to b_n, in such a way that $f \sim \phi$ for an infinity of values x_ν whose limit is infinity, and $f \sim \psi$ for a similar infinity of values x_ν'*.

Let (η_ν) be a sequence of decreasing positive numbers whose limit is zero. Take a positive number x_0 such that $\phi(x_0) > 1$, $\psi(x_0) > 1$, and a number x_1 greater than x_0. When x_1 is fixed, we can choose n_1 so that

$$\sum_{n_1}^{\infty} a_n x_1^n < \tfrac{1}{3}\eta_1, \quad \sum_{n_1}^{\infty} b_n x_1^n < \tfrac{1}{3}\eta_1,$$

and so, however c_n be selected for different values of n,

$$\sum_{n_1}^{\infty} c_n x_1^n < \sum_{n_1}^{\infty} (a_n + b_n) x_1^n < \tfrac{2}{3}\eta_1.$$

We take $c_n = a_n$ for $0 \leqslant n < n_1$. Then

$$|f(x_1) - \phi(x_1)| < \sum_{n_1}^{\infty} (a_n + c_n) x_1^n < \eta_1,$$

and so, since $\phi(x_1) > 1$,

$$\left| \frac{f(x_1)}{\phi(x_1)} - 1 \right| < \eta_1.$$

Now let x_2 be a number greater than x_1; we can suppose x_2 chosen so that

$$\left(\sum_{0}^{n_1-1} a_n x_2^n\right) \Big/ \psi(x_2) < \tfrac{1}{5}\eta_2, \quad \left(\sum_{0}^{n_1-1} b_n x_2^n\right) \Big/ \psi(x_2) < \tfrac{1}{5}\eta_2.$$

When x_2 is fixed we can choose n_2, greater than n_1, so that

$$\sum_{n_2}^{\infty} a_n x_2^n < \tfrac{1}{5}\eta_2, \quad \sum_{n_2}^{\infty} b_n x_2^n < \tfrac{1}{5}\eta_2.$$

We take $c_n = b_n$ for $n_1 \leqq n < n_2$; and, however c_n be chosen for $n \geqq n_2$, we have

$$\sum_{n_2}^{\infty} c_n x_2^n < \sum_{n_2}^{\infty} (a_n + b_n) x_2^n < \tfrac{2}{5}\eta_2.$$

Also

$$|f(x_2) - \psi(x_2)| < \sum_{0}^{n_1-1} a_n x_2^n + \sum_{0}^{n_1-1} b_n x_2^n + \sum_{n_2}^{\infty} c_n x_2^n + \sum_{n_2}^{\infty} b_n x_2^n$$
$$< \tfrac{2}{5}\eta_2 \psi(x_2) + \tfrac{3}{5}\eta_2 < \eta_2 \psi(x_2),$$

and so

$$\left| \frac{f(x_2)}{\psi(x_2)} - 1 \right| < \eta_2.$$

* By '$f \sim \phi$ for an infinity of values x_ν' we mean of course that $f/\phi \to 1$ when $x \to \infty$ through this particular sequence of values.

It is plain that, by a repetition of this process, we can find a sequence $x_1, x_2, x_3, \ldots$ whose limit is infinity, so that

$$\left|\frac{f(x_3)}{\phi(x_3)}-1\right|<\eta_3, \quad \left|\frac{f(x_4)}{\psi(x_4)}-1\right|<\eta_4, \ \ldots\ldots,$$

and our conclusion is established.

We may remark that not only f itself, but all its derivatives also, are increasing and continuous. It is clear also that, if we were given any number of integral functions $\phi_1, \phi_2, \ldots, \phi_k$, with positive coefficients, we could define f so that $f \sim \phi_s$, as $x \to \infty$ through a suitably chosen sequence of values, for each of the functions ϕ_s.

Another interesting method for the construction of irregularly increasing functions by means of power series will be explained in § 6.34.

4.44. Functions which transcend the logarithmico-exponential scales. We turn our attention now to functions for which the logarithmico-exponential scales are not inapplicable but *insufficient* (§ 4.42). Of the existence of such functions we are already assured; thus a function which assumes the values $e_1(1), e_2(2), \ldots, e_\nu(\nu), \ldots$ for $x=1, 2, \ldots, \nu, \ldots$ has certainly an increase greater than that of any logarithmico-exponential function.

(i) The series

$$\Sigma \frac{e_\nu(x)}{e_\nu(\nu)},$$

if convergent for all values of x, has a sum $f(x)$ whose increase is plainly greater than that of any $e_\nu(x)$. Suppose that $k-1 \leqq x < k$. Then

$$\frac{e_k(x)}{e_k(k)}<1, \quad \frac{e_{k+\nu}(x)}{e_{k+\nu}(k+\nu)}<\frac{e_{k+\nu}(k)}{e_{k+\nu}(k+\nu)}<\frac{e_{k+\nu}(k)}{e_{k+\nu}(k+1)} \quad (\nu \geqq 1).$$

But, by the Mean Value Theorem,

$$e_{k+\nu}(k+1)=e_{k+\nu}(k)+e_{k+\nu}(y)\, e_{k+\nu-1}(y) \ldots e_2(y)\, e_1(y),$$

where y is some number between k and $k+1$; and so

$$e_{k+\nu}(k+1)>e_{k+\nu}(k)\, e_{k+\nu-1}(k) \ldots e_1(k).$$

It follows that the terms of the series

$$\sum_{\nu=k}^{\infty} \frac{e_\nu(x)}{e_\nu(\nu)}$$

are less than those of the series

$$1+\sum_{\nu=1}^{\infty} \frac{1}{e_1(k)\, e_2(k) \ldots e_{k+\nu-1}(k)},$$

which is plainly convergent, so that the original series is convergent. It is obvious that we can in this way construct any number of functions $f(x)$ of the kind required.

(ii) Let $\phi(x)$ be an increasing function such that $\phi(0) > 0$, $\phi \succ x$. We can define an increasing function f, which satisfies the equation

(4.441) $$f_2(x) = f\{f(x)\} = \phi(x),$$

as follows.

Draw the curves $y = x$, $y = \phi(x)$ (Fig. 5). Take Q_0 arbitrarily on OP_0; draw Q_0R_1 parallel to OX and complete the rectangle Q_0Q_1. Join Q_0, Q_1 by any continuous curve inclined everywhere at an acute angle to the axes. On this curve take any point Q; draw QP, QR parallel to the axes, and complete the rectangle QQ'. When Q moves from Q_0 to Q_1, Q' moves from Q_1 to Q_2, say; and as we constructed Q' from Q, so we can construct Q'' from Q'. Proceeding thus, we define a continuous curve $Q_0Q_1Q_2Q_3 \ldots$ corresponding to a continuous and increasing function $f(x)$. Then $f(x)$ satisfies (4.441). For if $y = f(x)$ is the ordinate of Q, it is clear that $f_2(x)$ is the ordinate of Q', which is equal to $\phi(x)$, the ordinate of P.

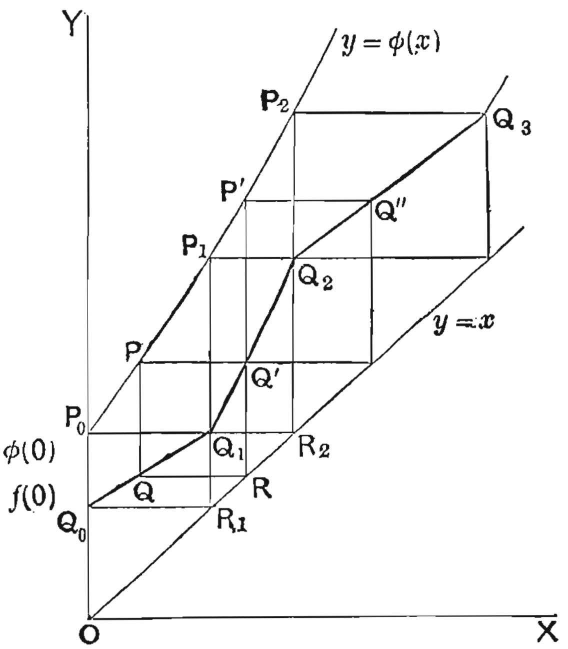

FIG. 5

Let us write $f(x) = f_1(x)$ and $f\{f_n(x)\} = f_{n+1}(x)$, so that Q_n is the point $f_n(0)$, $f_{n+1}(0)$. Further, let us suppose that ψ is the function inverse to ϕ, that $\psi(x) = \psi_1(x)$, $\psi\{\psi(x)\} = \psi_2(x)$, …; and that the equation of Q_0Q_1 is $\theta(x, y) = 0$. Then it is easy to see that the equations of $Q_{2n}Q_{2n+1}$ and of $Q_{2n+1}Q_{2n+2}$ are respectively

$$\theta\{\psi_n(x), \psi_n(y)\} = 0, \quad \theta\{\psi_{n+1}(y), \psi_n(x)\} = 0.$$

Suppose for example that $\phi(x) = e^x$, $OQ_0 = \frac{1}{2}$, and that Q_0Q_1 is the straight line $y = \frac{1}{2} + x$. Then the equations of $Q_{2n}Q_{2n+1}$ and of $Q_{2n+1}Q_{2n+2}$ are

$$l_n y = \tfrac{1}{2} + l_n x, \quad l_n x = \tfrac{1}{2} + l_{n+1} y,$$

or $$y = e_{n-1}\{\sqrt{e}\,(l_{n-1}x)\} = e_{n-2}\{(l_{n-2}x)^{\sqrt{e}}\} = \lambda_n(x),$$

and $$y = e_n\{(l_{n-1}x)/\sqrt{e}\} = e_{n-1}\{(l_{n-2}x)^{1/\sqrt{e}}\} = \mu_n(x),$$

say. Now (§ 4.1)

$$x^\gamma \prec \lambda_3 \prec \ldots \prec \lambda_n \prec \ldots \prec \mu_n \prec \ldots \prec \mu_3 \prec e^{x^\gamma},$$

and a function f, such that $\lambda_n \prec f \prec \mu_n$ for all values of n, transcends the logarithmico-exponential scales. Our function f is plainly an example*.

It is easily verified that $\lambda_n\lambda_n x \prec e^x$ and $\mu_n\mu_n x \succ e^x$ for all values of n. Hence it is clear *a priori* that any increasing solution of (1) must satisfy $\lambda_n \prec f \prec \mu_n$ for all values of n.

* For fuller details see Hardy, **9**.

This graphical method may also be employed to define functions whose increase is slower than that of any logarithm or more rapid than that of any exponential. It can be employed, for example, to solve the equation

$$\phi(2^x)=2\phi(x);$$

and it is easily proved that the increase of a function such that $\phi(2^x) \asymp \phi(x)$ is slower than that of any logarithm.

4.5. The importance of the logarithmico-exponential scales. We have seen that it is possible, in a variety of ways, to construct functions whose increase cannot be measured by any L-function. It is none the less true that no one yet has succeeded in defining a mode of increase which is genuinely independent of all logarithmico-exponential modes. Our irregularly increasing functions oscillate, according to a logarithmico-exponential law of oscillation, between two logarithmico-exponential functions; and the function of § 4.44 (ii) was constructed expressly to fill a gap between two logarithmico-exponential scales. No function has yet presented itself in analysis the laws of whose increase, in so far as they can be stated at all, cannot be stated, so to say, in logarithmico-exponential terms.

It would be natural to expect that the arithmetical functions which occur in the theory of numbers might give rise to genuinely new modes of increase; but, so far as analysis has gone, the evidence is the other way. See § 6.26.

V

DIFFERENTIATION AND INTEGRATION

5.1. Integration. It is important to know when relations of the types $f(x) \succ \phi(x)$, etc., can be differentiated or integrated*. For brevity we denote

$$\int_a^x f(t)\,dt, \quad \int_a^x \phi(t)\,dt$$

(where a is a constant) by $F(x)$ and $\Phi(x)$.

It may be well to repeat that f and ϕ are supposed to be positive, continuous, and monotonic (at any rate for $x > x_0$), unless the contrary is stated or clearly implied. Some of our conclusions are valid under more general conditions; but the case thus defined, and the corresponding cases in which f or ϕ or both of them are negative, are the cases of most importance.

* The problem was first considered generally by du Bois-Reymond, **1**, **2**, **4**.

Lemma. *If* $\Phi \succ 1$, *and* $f > H\phi$ *for* $x > x_0 \geqq a$, *then* x_1 *can be found so that* $F > (H - \delta)\,\Phi$ *for* $x > x_1$: *similarly* $f < h\phi$ *for* $x > x_0$ *involves* $F < (h + \delta)\,\Phi$ *for* $x > x_1$.

For, if $f > H\phi$ for $x > x_0$, we have

$$F = \int_a^x f\,dt > \int_a^{x_0} f\,dt + H\int_{x_0}^x \phi\,dt = H\Phi + \int_a^{x_0} f\,dt - H\int_a^{x_0} \phi\,dt.$$

Since $\Phi \succ 1$, we can choose x_1 so that

$$\frac{1}{\Phi}\left(\int_a^{x_0} f\,dt + H\int_a^{x_0} \phi\,dt\right) < \delta$$

for $x > x_1$; and the first part of the lemma follows. The second part may be proved similarly. From the lemma we can at once deduce

Theorem 19. *If either* $F \succ 1$ *or* $\Phi \succ 1$, *then any one of the relations*

$$f \succ \phi,\ f \prec \phi,\ f \asymp \phi,\ f \stackrel{\asymp}{=} \phi,\ f \sim \phi$$

involves the corresponding one of the relations

$$F \succ \Phi,\ F \prec \Phi,\ F \asymp \Phi,\ F \stackrel{\asymp}{=} \Phi,\ F \sim \Phi.$$

To this we may add

Theorem 20. *If both* $\int^\infty f\,dt$, $\int^\infty \phi\,dt$ *are convergent, then*

$$f \succ \phi,\ f \prec \phi,\ f \asymp \phi,\ f \stackrel{\asymp}{=} \phi,\ f \sim \phi$$

involve corresponding relations between

$$F_1 = \int_x^\infty f\,dt,\quad \Phi_1 = \int_x^\infty \phi\,dt.$$

The proof we may leave to the reader.

5.21. Differentiation. From Theorems 19 and 20 we deduce

Theorem 21. *If* $f \succ 1$ *or* $\phi \succ 1$, *or* $f \prec 1$ *and* $\phi \prec 1$, *and if some one of the relations* $\succ$, $\prec$, $\asymp$, $\stackrel{\asymp}{=}$, $\sim$ *must hold between* f' *and* ϕ', *then* $f \succ \phi$ *involves* $f' \succ \phi'$; *and there are corresponding results for the other relations.*

In interpreting this theorem regard must be paid to the conventions laid down in § 1.4. Thus if $f \succ \phi \succ 1$, f' and ϕ' are positive, and $f' \succ \phi'$. But if $f \succ 1 \succ \phi$, ϕ is a decreasing function and $\phi' < 0$. In this case $f' \succ -\phi'$, a relation which we have agreed to denote by $f' \succ \phi'$. If $1 \succ f \succ \phi$, both f' and ϕ' are negative: the relation $-f' \prec -\phi'$ would involve

$$-\int_x^\infty f'\,dt \prec -\int_x^\infty \phi'\,dt$$

or $f \prec \phi$, and is therefore impossible; and similarly $-f' \asymp -\phi'$ is impossible. We must therefore have $-f' \succ -\phi'$, a relation which we have agreed to denote also by $f' \succ \phi'$. The case in which $f \asymp 1$, $\phi \preccurlyeq 1$, is exceptional; any one of the relations $f' \succ \phi'$, etc. may hold in this case. Thus if $f = 1 + e^{-x}$, $\phi = 1/x$, we have $f \succ \phi$, $f' \prec \phi'$. The fact is that in this case f, regarded as the integral of f', is dominated by the constant of integration.

It is to be observed that the assumption that one of the relations holds between f' and ϕ' is essential. We cannot *infer* that one of them holds; we cannot even infer that f' or ϕ' is a steadily increasing or decreasing function. Thus if $f = e^x$, $\phi = e^x + \sin e^x$, we have $f' = e^x$ and $\phi' = e^x(1 + \cos e^x)$. Here f and ϕ increase steadily and $f' \sim f \sim \phi$; but ϕ' does not tend to infinity, and in fact vanishes for an infinity of values of x. Again if

$$\phi = e^x(\sqrt{2} + \sin x) + \tfrac{1}{2}x^2,$$

we have

$$\phi' = e^x(\sqrt{2} + \sin x + \cos x) + x$$

and $\phi \asymp e^x$, while ϕ' oscillates between the orders of e^x and x. It is possible, though less easy, to obtain examples of this character in which ϕ' also is monotonic.

5.22. Differentiation of *L*-functions. If f and ϕ are L-functions, so are f' and ϕ', and one of the relations $f' \succ \phi'$, $f' \succeq\!\preceq \phi'$, $f' \prec \phi'$ certainly holds (§ 3.2, Theorem 13). Thus in this case *both differentiation and integration are always legitimate* except when* $f \asymp 1$, $\phi \preccurlyeq 1$, *or* $f \preccurlyeq 1$, $\phi \asymp 1$.

In what follows we shall suppose that all the functions concerned are L-functions, or at any rate resemble L-functions in so far that one of the relations $f \succ \phi$, $f \succeq\!\preceq \phi$, $f \prec \phi$ is bound to hold between any pair of functions, and that differentiation and integration are permissible†.

Theorem 22. *If f is an increasing function, and $f' \succ f$, then $f \succ e^{\Delta x}$. If $f' \prec f$, then $f \prec e^{\delta x}$. Similarly, if f is a decreasing function, $f' \succ f$ and $f' \prec f$ involve $f \prec e^{-\Delta x}$ and $f \succ e^{-\delta x}$ respectively. If $f' \succeq\!\preceq f$, then we can find a number μ such that $f = e^{\mu x} f_1$, where $e^{-\delta x} \prec f_1 \prec e^{\delta x}$.*

The proofs of these propositions are almost obvious. Thus if f is an increasing function, and $f' \succ f$, we have

$$f'/f \succ 1, \quad \log f \succ x,$$

and so $\log f > \Delta x$ for $x > x_0$, *i.e.* $f > e^{\Delta x}$, or, what is the same thing, $f \succ e^{\Delta x}$. The last clause of the theorem follows at once from § 3.5.

Theorem 23. *More generally, if v is any increasing function, $f'/f \succ v'/v$ involves $f \succ v^{\Delta}$ or $f \prec v^{-\Delta}$, according as f is an increasing or a decreasing function; and $f'/f \prec v'/v$ involves $f \prec v^{\delta}$ or $f \succ v^{-\delta}$. If $f'/f \succeq\!\preceq v'/v$, we can find a number μ such that $f = v^{\mu} f_1$, where $v^{-\delta} \prec f_1 \prec v^{\delta}$.*

* A tacit assumption to this effect underlies much of du Bois-Reymond's work.

† The results which follow are all in substance due to du Bois-Reymond.

When f is an increasing function, which tends to infinity with x, we shall call f'/f the *type* t of f*: it being understood that t may be replaced by any simpler function τ such that $t \asymp \tau$. The type of a *decreasing* function f we define to be the same as that of the increasing function $1/f$. The following table shows the types of some standard functions:

Function ...	llx	lx	x^a	e^x	e^{ax^β}	e_2x	e_3x	...
Type ...	$\dfrac{1}{x\,lx\,llx}$	$\dfrac{1}{x\,lx}$	$\dfrac{1}{x}$	1	$x^{\beta-1}$	ex	$e_2x\,ex$	...

If $f \succ \phi$, then $f'/f \succcurlyeq \phi'/\phi$. By making the increase of f large enough we can make the increase of $t=f'/f$ as large as we please. The reader will find it instructive to write out formal proofs of these propositions, and also of the following.

1. As the increase of f becomes smaller and smaller, f'/f tends to zero more and more rapidly, but, so long as $f \to \infty$, we cannot have

$$\frac{f'}{f} \prec \phi(x), \quad \int^\infty \phi\, dx \text{ convergent.}$$

If on the other hand the last integral is divergent, we can always find f so that $f \succ 1$, $f'/f \prec \phi$.

2. Although we can find f so that f'/f shall have an increase larger than that of any given function of x, we cannot have

$$\frac{f'}{f} \succ \phi(f), \quad \int^\infty \frac{dx}{x\phi(x)} \text{ convergent.}$$

If on the other hand the last integral is divergent, we can always find f so that $f'/f \succ \phi(f)$.

Thus we cannot find a function f which tends to infinity so slowly that $f'/f \prec 1/x^a$ $(a > 1)$. But we can find f so that $f'/f \prec 1/x lx llx$ (*e.g.* $f = l_3x$). We cannot find f so that $f'/f \succ f^a$ or $f' \succ f^{1+a}$ $(a > 0)$. But we can find f so that $f'/f \succ lf$ (*e.g.* $f = e_3x$).

3. If $f \succ e_kx$ for all values of k, f'/f satisfies the same condition, and

$$f' \succ f lf l_2f \ldots l_kf\text{†}.$$

There are of course corresponding theorems about functions of a positive variable x which tends to zero.

5.23. Successive differentiation. du Bois-Reymond‡ has given the following general theorem, which enables us to write down the increase of any derivative of any logarithmico-exponential function. We write t for f'/f, as in the last section.

* du Bois-Reymond (**1**, **2**) calls f/f' the type; the notation here adopted seems slightly more convenient.

† In this case f cannot be an L-function (§ 3.4, Theorem 14). It is however supposed to possess the properties stated at the beginning of this section.

‡ du Bois-Reymond, **2**.

Theorem 24. (i) *If* $t \succ 1/x$ (*so that* $f \succ x^{\Delta}$ *or* $f \prec x^{-\Delta}$) *then*

$$f \asymp f'/t \asymp f''/t^2 \asymp f'''/t^3 \ldots \asymp f^{(n)}/t^n \ldots.$$

(ii) *If* $t \prec 1/x$ (*so that* $1 \prec f \prec x^{\delta}$ *or* $x^{-\delta} \prec f \prec 1$) *then*

$$f \asymp f'/t \asymp xf''/t \asymp x^2 f'''/t \ldots \asymp x^{n-1} f^{(n)}/t \ldots.$$

(iii) *If* $t \asymp 1/x$ (*so that* $f = x^{\mu} f_1$, *where* $x^{-\delta} \prec f_1 \prec x^{\delta}$), *then if* μ *is not integral either set of formulae is valid. If* μ *is integral then*

$$f \asymp xf' \asymp x^2 f'' \ldots \asymp x^{\mu} f^{(\mu)} \asymp x^{\mu} f^{(\mu+1)}/t_1 \asymp x^{\mu+1} f^{(\mu+2)}/t_1 \ldots,$$

where t_1 *is the type of* f_1, *unless* $f_1 \asymp 1$.

(i) If $t \succ 1/x$, $1/t \prec x$ and so $t'/t^2 \prec 1$; hence $t'/t \prec t = f'/f$ or

$$ft' \prec f't.$$

Differentiating the relation $f' \asymp ft$, and using the relation just established, we obtain

$$f'' \asymp f't + ft' \asymp f't.$$

Thus the type of f' is the same as that of f. The argument may therefore be repeated, and the first part of the theorem follows.

(ii) If $t \prec 1/x$, $xf' \prec f$ and so

$$xf'' + f' \prec f',$$

which cannot be true unless $xf'' \asymp f'$. Differentiating again we infer

$$xf''' + 2f'' \prec f'',$$

whence $xf''' \asymp f''$; and so on generally*. Thus the second part follows.

(iii) If $t \asymp 1/x$, $f = x^{\mu} f_1$ and t_1, the type of f_1, satisfies $t_1 \prec 1/x$. Then

$$f' = \mu x^{\mu-1} f_1 + x^{\mu} f_1' \asymp x^{\mu-1} f_1 (\mu + x t_1) \asymp x^{\mu-1} f_1.$$

Similarly $f'' \asymp x^{\mu-2} f_1$, and so on. We can proceed indefinitely in this way unless μ is integral: in this case $f^{(\mu)} \asymp f_1$, and from this point we proceed as in case (ii).

If μ is an integer n, and $f_1 \asymp 1$, then $f^{(n)} \asymp 1$, but the theorem fails for the higher derivatives. In this case $f = Ax^n + o(x^n) = Ax^n + \phi$, say, and we must begin our analysis again with ϕ in place of f.

Examples. (i) If $f = e^{\sqrt{x}}$, then $t = x^{-\frac{1}{2}} \succ 1/x$, and $f^{(n)} \asymp x^{-\frac{1}{2}n} e^{\sqrt{x}}$. If $f = e^{(\log x)^2}$, then $t = (\log x)/x \succ 1/x$, and $f^{(n)} \asymp e^{(\log x)^2} (\log x)^n / x^n$.

(ii) If $f = (\log x)^m$, then $t = 1/(x \log x) \prec 1/x$, and

$$f^{(n)} \asymp t x^{-(n-1)} f \asymp (\log x)^{m-1}/x^n.$$

(iii) If $f = x^2 llx$, $t \asymp 1/x$. Here

$$f' \asymp x\, llx,\ f'' \asymp llx,\ f''' \asymp 1/x\, lx,\ f'''' \asymp 1/x^2 lx, \ldots.$$

* More precisely $xf'' \sim -f'$, $xf''' \sim -2f''$, and so on.

(iv) The results of the theorem, in the first two cases, can be stated more precisely as follows. If $t \succ 1/x$, then

$$f^{(n)} \sim (f'/f)^n f.$$

If $t \prec 1/x$, then

$$f^{(n)} \sim (-1)^{n-1}(n-1)!\, f'/x^{n-1}.$$

If f is a positive increasing function, and $t \succ 1/x$, then all the derivatives are ultimately positive. If $t \prec 1/x$, they are alternately positive and negative.

5.3. Further theorems on integration. It is possible to give a finite system of rules which enable us to determine the asymptotic behaviour of the integral of any L-function. The results are naturally not essentially different from those of § 5.23. We write

$$F(x) = \int_a^x f(t)\,dt, \quad F(x) = \int_x^\infty f(t)\,dt$$

according as the latter integral is divergent or convergent.

Theorem 25. *If* $f \succ x^\Delta$ *or* $f \prec x^{-\Delta}$, *then*

$$F \sim f^2/f'.$$

If $f = x^a f_1$, *where* $x^{-\delta} \prec f_1 \prec x^\delta$, *then*

$$F \sim \frac{x^{a+1}}{a+1} f_1,$$

unless $a = -1$, *in which case further rules are required.*

(1) If $f \succ x^\Delta$, the integral up to infinity is divergent, and

$$F = \int_a^x f\,dt = \int_a^x f' \frac{f}{f'}\,dt = \frac{\{f(x)\}^2}{f'(x)} - \frac{\{f(a)\}^2}{f'(a)} - \int_a^x f \frac{d}{dt}\left(\frac{f}{f'}\right) dt.$$

Now $\log f \succ \log x$, and so

$$\frac{f'}{f} \succ \frac{1}{x}, \quad x \succ \frac{f}{f'}, \quad 1 \succ \frac{d}{dx}\left(\frac{f}{f'}\right), \quad \int_a^x f\,dt \succ \int_a^x f \frac{d}{dt}\left(\frac{f}{f'}\right) dt,$$

so that $F \sim f^2/f'$. The case in which $f \prec x^{-\Delta}$ may be disposed of similarly.

(2) If $f = x^a f_1$, where $a > -1$, the integral up to infinity is again divergent; and

$$F = \int_a^x t^a f_1\,dt = \frac{x^{a+1}}{a+1} f_1(x) - \frac{a^{a+1}}{a+1} f_1(a) - \frac{1}{a+1}\int_a^x t^{a+1} f_1'\,dt.$$

But

$$\log f_1 \prec \log x, \quad \frac{f_1'}{f_1} \prec \frac{1}{x}, \quad x^{a+1} f_1' \prec x^a f_1, \quad \int_a^x t^{a+1} f_1'\,dt \prec \int_a^x t^a f_1\,dt;$$

whence the result. The case in which $a < -1$ is not essentially different.

When $a = -1$, further analysis is required, which will be found in the author's paper quoted on p. 21.

Another interesting problem is that of the behaviour of F when $f = \phi e^{i\psi}$, ϕ and ψ being L-functions. Let us suppose that $\psi \succ 1$ and $\phi \succ \psi'$, so that the integral does not converge up to infinity*. Then the problem is solved by

* If $\psi \preccurlyeq 1$, $e^{i\psi}$ tends to a limit, and the oscillating factor introduces no new feature. If $\phi \prec \psi'$, the integral up to infinity is convergent.

Theorem 26. *If* $\psi \succ 1$, $\phi \succ \psi'$, *then* F *is asymptotically equivalent to*

$$\Phi e^{i\psi}, \quad \frac{\Phi e^{i\psi}}{1+Ai}, \quad \frac{\phi}{i\psi'} e^{i\psi},$$

where

$$\Phi \sim \int^x \phi \, dt,$$

according as $\psi \prec l\Phi$, $\psi \sim Al\Phi$, *or* $\psi \succ l\Phi$.

The details of the proof will be found in a note by the author*.

5.4. Some 'Tauberian' theorems. We pointed out in § 5.21 that inferences from the order of magnitude of a function to that of its derivative are essentially more difficult than inferences in the opposite direction, and that special conditions are always required in order that any such inference should be possible. The hypothesis of §§ 5.22—5.23, that the functions concerned are L-functions, is of course an assumption of a very drastic kind. In this section we abandon this hypothesis, and prove some theorems of a more general and much more subtle type. These theorems belong to the class which Mr Littlewood and the author have called 'Tauberian'.

Theorem 27†. *If* $xf(x)$ *is continuous and increasing for* $x > a$, *and*

$$F(x) = \int_a^x f\,dt \sim Ax^m \quad (m > 0),$$

then

$$f(x) \sim mAx^{m-1}.$$

The converse inference would be an immediate corollary of Theorem 19.

We may suppose $A = 1$, so that $F = x^m + o(x^m)$. Hence, if η is positive and less than 1, we have

$$F(x+\eta x) - F(x) = \int_x^{x+\eta x} f\,dt = \{(1+\eta)^m - 1\}\, x^m + o(x^m)$$
$$= m\eta x^m + O(\eta^2 x^m) + o(x^m),$$

where $O(\eta^2 x^m)$ is a function whose modulus is less than a constant multiple of $\eta^2 x^m$ for all values of x and η in question. But

$$\int_x^{x+\eta x} f\,dt \geqq \frac{\eta x f(x)}{1+\eta},$$

since tf increases throughout the range of integration. Combining this inequality with the preceding equation, and dividing by $\eta x^m/(1+\eta)$, we obtain

$$\frac{f(x)}{x^{m-1}} \leqq m(1+\eta) + H\eta + o(1),$$

* Hardy, **3** (6). † Landau, **2**, 218 and **3**, 116.

where H is independent of m and of η. If now we make $x \to \infty$, we find

$$\overline{\lim_{x \to \infty}} \frac{f(x)}{x^{m-1}} \leqq m(1+\eta) + H\eta;$$

and this involves

$$\overline{\lim_{x \to \infty}} \frac{f(x)}{x^{m-1}} \leqq m,$$

since η may be as small as we please. Arguing in the same way with the interval $(x - \eta x, x)$, we obtain

$$\underline{\lim_{x \to \infty}} \frac{f(x)}{x^{m-1}} \geqq m;$$

and these two inequalities embody the result of the theorem.

Theorem 28. *If* $(1-x)f'(x)$ *is continuous and increasing for* $0 < x < 1$, *and*

(5.41) $$f(x) \sim \frac{A}{(1-x)^m} \quad (m > 0)$$

when $x \to 1$, *then*

(5.42) $$f'(x) \sim \frac{mA}{(1-x)^{m+1}}.$$

We have only to write

$$x = \frac{1}{1-y}, \quad f(x) = g(y)$$

in Theorem 27, and then replace y by x.

Theorem 29*. *If* $f(x) = \Sigma a_n x^n$ *is a power series with positive coefficients, convergent for* $0 \leqq x < 1$, *then* (5.41) *involves* (5.42).

Let $a_0 + a_1 + \dots + a_n = A_n$, so that

$$g(x) = \sum_0^\infty A_n x^n = \frac{f(x)}{1-x} \sim \frac{A}{(1-x)^{m+1}}.$$

Then $\quad (1-x)g'(x) = A_1 + (2A_2 - A_1)x + (3A_3 - 2A_2)x^2 + \dots$

increases steadily, since the coefficients are positive. Hence, by Theorem 28,

$$g'(x) \sim \frac{(m+1)A}{(1-x)^{m+2}},$$

$$f'(x) = (1-x)g'(x) - g(x) \sim \frac{mA}{(1-x)^{m+1}}.$$

Theorem 30. *If* $f(x)$ *possesses a second derivative* $f''(x)$, *and* $f = O(x^\alpha)$, $f'' = O(x^\beta)$, *where* $\beta > -1$, *when* $x \to \infty$, *then*

$$f' = O\{x^{\frac{1}{2}(\alpha+\beta)}\}.$$

If $\alpha \geqq \beta + 2$ the result is trivial, since $f' = O(x^{\beta+1})$, by Theorem 19,

* Hardy and Littlewood, **2**.

and $\beta+1 \leqq \frac{1}{2}(\alpha+\beta)$. We may therefore suppose that $\alpha<\beta+2$. If $0<\eta<1$ we have, by Taylor's Theorem,

$$f(x+\eta x)-f(x)=\eta x f'(x)+\tfrac{1}{2}\eta^2 x^2 f''(x+\theta\eta x),$$

where $0<\theta<1$; and so

$$|f'(x)| \leqq \frac{|f(x+\eta x)|+|f(x)|}{\eta x}+\tfrac{1}{2}\eta x\,|f''(x+\theta\eta x)|<H\left(\frac{x^{\alpha-1}}{\eta}+\eta x^{\beta+1}\right),$$

where H is independent of x and η. We may take $\eta^2=x^{\alpha-\beta-2}$, since this is certainly less than 1 when x is large; and then

$$|f'(x)|<2Hx^{\frac{1}{2}(\alpha+\beta)},$$

which proves the theorem.

The theorem is not true if $\beta \leqq -1$: consider, for example, the case $f=x+\log x$. It is one of a system of theorems important in the theory of infinite series*.

5.5. Functions of an integral variable. There are theorems for functions of an integral variable n, corresponding to those of §§ 5.1—5.4, but involving sums

$$A_n=a_1+a_2+\ldots+a_n$$

instead of integrals, and differences

$$\Delta a_n=a_n-a_{n+1}$$

instead of differential coefficients. The reader will be able to formulate and prove for himself the theorems which correspond to those of the preceding paragraphs. Thus

'$a_n \succ b_n$, $a_n \prec b_n$, $a_n \asymp b_n$, $a_n \succcurlyeq\!\!\preccurlyeq b_n$, $a_n \sim b_n$ *involve the corresponding equations for* A_n *and* B_n, *if one at least of* A_n *and* B_n *tends to infinity with* n';

and so on†.

5.6. Further developments of the Infinitärcalcül. The functions $f(x+a)$, $f(ax)$, etc. It is often necessary to obtain approximations to such functions as

$$\frac{f(x+a)}{f(x)},\quad \frac{f(ax)}{f(x)},\quad f(x+a)-f(x),$$

where a is itself a function of x‡. We shall assume that all the functions which occur are L-functions, or at any rate that the theorems of §§ 5.2—5.3 may be applied to them as if they were.

* See Hardy and Littlewood, **1**, **2**.

† This is a well known theorem of Cauchy and Stolz: see Bromwich, **1**, 377; Knopp, **1**, 72.

‡ du Bois-Reymond, **4**. The substance of the theorems which follow is in the main due to du Bois-Reymond; but his presentation of them is inconclusive.

Theorem 31. *If $a \prec f/f'$ then*

$$\frac{f(x+a)}{f(x)} \sim 1.$$

We may suppose that $f \succ 1$ and $a > 0$. If, in the first place, $t = f'/f \preccurlyeq 1$, we have

$$\frac{f(x+a)}{f(x)} = e^{lf(x+a)-lf(x)} = e\left\{a\frac{f'(x+\alpha)}{f(x+\alpha)}\right\} = e\{a\,t(x+\alpha)\},$$

and $t(x+\alpha) < Kt(x)$, so that $a\,t(x+\alpha) \to 0$, which proves the theorem.

If $t \succ 1$, and $T = 1/t$, we have

$$a\,t(x+\alpha) = a\,t(x)\frac{T(x)}{T(x+\alpha)} = a\,t(x)\Big/\left\{1+\alpha\frac{T'(x+\alpha_1)}{T(x)}\right\},$$

where $0 < \alpha_1 < \alpha < a$. But $at \prec 1$, $\alpha/T \prec 1$, and $T' \prec 1$ (since $T \prec 1$). Hence $a\,t(x+\alpha) \to 0$, which again proves the theorem.

In particular the conditions are satisfied if (i) $x^{-\Delta} \prec f \prec x^{\Delta}$ and $a \prec x$ or (ii) $e^{-\Delta x} \prec f \prec e^{\Delta x}$ and $a \prec 1$.

Theorem 32. *If $la \prec f/xf'$ then*

$$\frac{f(ax)}{f(x)} \sim 1.$$

This is a corollary of Theorem 31: we have only to write $lx = y$, $la = b$, and $f(x) = \phi(y)$.

In particular the conditions are satisfied if $(lx)^{-\Delta} \prec f \prec (lx)^{\Delta}$ and $x^{-\delta} \prec a \prec x^{\delta}$ or if $x^{-\Delta} \prec f \prec x^{\Delta}$ and $a \sim 1$.

We add some further results.

(1) *If $a \prec 1/f'$ then $f(x+a) - f(x) \prec 1$.*

(2) *If $a \prec f'/f''$ then $f(x+a) - f(x) \sim af'(x)$.*

These results follow from Theorem 31 and the formula

$$f(x+a) - f(x) = af'(x)\frac{f'(x+\alpha)}{f'(x)} \quad (0 < \alpha < a).$$

The second result is true in particular if $1 \prec f \prec x^{\delta}$ and $a \prec x$, or if $f \succ x^{\Delta}$ and $a \prec f/f'$; the forms of the conditions to be imposed on a may be deduced from Theorem 24.

(3) *If $e^{-\Delta\sqrt{(lx)}} \prec f \prec e^{\Delta\sqrt{(lx)}}$, then*

$$\frac{f\{xf(x)\}}{f(x)} \asymp 1, \quad e\left\{\frac{xlf(x)f'(x)}{f(x)}\right\} \asymp 1;$$

and the limits of the two functions are the same: and if $e^{-\delta\sqrt{(lx)}} \prec f \prec e^{\delta\sqrt{(lx)}}$ this limit is unity.

Suppose that $f \succ 1$, and let $f(x)=\phi(lx)$, $a=f(x)$. Then

$$\frac{f(ax)}{f(x)}=e^{l\phi(lx+la)-l\phi(lx)}=e^{la\phi'(lx+la_1)/\phi(lx+la_1)},$$

where $1<a_1<a$. The exponent is

$$l\phi(lx+la_1)\frac{\phi'(lx+la_1)}{\phi(lx+la_1)}\frac{l\phi(lx)}{l\phi(lx+la_1)}.$$

Now $a=f(x)\prec x^{\delta}$ and therefore $la_1 \preccurlyeq la \prec lx$, and so, by Theorem 31,

$$l\phi(lx+la_1)\sim l\phi(lx)$$

if $l\phi \prec x^{\Delta}$ or if $f \prec e^{(lx)^{\Delta}}$, which is certainly the case. Hence the exponent is asymptotically equivalent to

$$l\phi(u)\,\phi'(u)/\phi(u),$$

where $u=lx+la_1$. And $l\phi(\phi'/\phi)\preccurlyeq 1$ if $(l\phi)^2\preccurlyeq u$, *i.e.* if $\phi \preccurlyeq e^{\Delta\sqrt{u}}$ or $f\preccurlyeq e^{\Delta\sqrt{(lx)}}$. In this case $f(ax)\asymp f(x)$; and it is easy to see that if $f\preccurlyeq e^{\delta\sqrt{(lx)}}$ the symbol $\asymp$ may be replaced by $\sim$.

(4) *If* $f(x)=x\phi(x)$, *and* $e^{-\delta\sqrt{(lx)}}\prec\phi\prec e^{\delta\sqrt{(lx)}}$, *then*

$$f_2(x)\equiv ff(x)\sim x\phi^2,\ \dots,\ f_n\sim x\phi^n,\ \dots$$

5.7. Approximate solution of equations. We may say that

$$y=\psi(x,u)$$

is an 'approximate form' of y if ψ is a known function and u an unknown function whose increase is subject to known limitations. Thus

$$e^{x^u}\ (u\sim 1),\quad e^{(1+u)x}\ (u\prec 1),\quad x^{1+u}e^x\ (u\prec 1)$$

are approximate forms of $y=xe^x/lx$, and represent the increase of y with increasing accuracies. Another example of an approximation is given by the formula

$$\frac{f(x+a)}{f(x)}=e^{\left\{u\frac{f'(x)}{f(x)}\right\}}\quad(u\sim a),$$

valid if $a\prec f/f'\prec 1$.

It is often important to obtain an asymptotic solution of an equation $f(x,y)=0$, *i.e.* to find a function whose increase gives an approximation to that of y. No very general methods of procedure can be given, but the kind of methods which may be pursued are worth illustrating by a few examples.

Suppose that the equation is

(5.71) $$x=y\kappa(y),$$

where $y^{-\delta}\prec\kappa\prec y^{\delta}$. If the increase of κ is so slow that $\kappa\{y\kappa(y)\}\asymp\kappa(y)$ it is clear that

$$y\asymp x/\kappa(y)\asymp x/\kappa(x):$$

and if the increase of κ is slow enough we may have $y\sim x/\kappa(x)$.

The conditions

$$e^{-\Delta\sqrt{(ly)}}\prec\kappa(y)\prec e^{\Delta\sqrt{(ly)}},\quad e^{-\delta\sqrt{(ly)}}\prec\kappa(y)\prec e^{\delta\sqrt{(ly)}}$$

are, by (3) of § 5.6, enough to ensure the truth of these hypotheses; and then $y=ux/\kappa(x)$, where $u\asymp 1$ (or $u\sim 1$), is an approximate solution of our equation.

du Bois-Reymond has proved that more elaborate approximations, such as

$$y = \frac{ux}{\kappa\,(x/\kappa)},$$

have a wider range of validity. The more general equation $x = y^m \kappa\,(y)$ can clearly be reduced to the form considered above by writing x^m for x and κ^m for κ.

In general, if $x = \phi\,(y)$, the more rapid the increase of ϕ the more precisely can we determine the increase of y as a function of x. Thus if $x = ye^y$ we have $lx = y + ly$ and

$$y = lx - ly = lx\,(1 - u),$$

where $u \sim ly/lx \sim llx/lx$. This is a solution of a much more precise kind than those considered above.

The reader will find it instructive to verify the following examples.

(1) If $x = ye^{(ly)^{\frac{3}{8}}}$, then $y \sim xe^{-(lx)^{\frac{3}{8}}}$.

(2) If $x = ye^{(ly)^{\frac{5}{8}}}$, then

$$y \sim xe\,\{-(lx)^{\frac{5}{8}} + \tfrac{5}{8}\,(lx)^{\frac{1}{4}}\}.$$

(3) If $x = y^m\,(ly)^{m_1}\,(l_2y)^{m_2} \ldots (l_ry)^{m_r}$, then

$$y \sim m^{m_1/m}\,x^{1/m}\,(lx)^{-m_1/m} \ldots (l_rx)^{-m_r/m}.$$

(4) If $x = y/ly$, then

$$y = x\left(lx + l_2x + \frac{l_2x}{lx}\right) + O\left\{x\,\frac{(l_2x)^2}{(lx)^2}\right\}.$$

The last example is of interest in the theory of primes.

VI

APPLICATIONS

6.1. In this chapter we give a brief sketch of certain regions of analysis in which the ideas of which we have given an account are of dominating importance.

6.21. Convergence and divergence of series and integrals. The logarithmic tests. A series Σu_n of positive terms is convergent if

$$u_n \preccurlyeq (n\,ln \ldots l_{k-1}n)^{-1}\,(l_kn)^{-1-\alpha},$$

where $\alpha > 0$, and divergent if

$$u_n \succcurlyeq (n\,ln \ldots l_kn)^{-1}.$$

Here $k \geqq 0$ and $l_0n = n$.

An integral $\int^{\infty} f(x)\,dx$, with positive integrand, is convergent if

$$f \preccurlyeq (x\,lx \ldots l_{k-1}x)^{-1}\,(l_kx)^{-1-\alpha},$$

where $\alpha > 0$, and divergent if

$$f \succcurlyeq (x\,lx \ldots l_k x)^{-1}.$$

Similarly the integral $\int_0 f(x)\,dx$ is convergent if

$$f \preccurlyeq (1/x)\{l(1/x) \ldots l_{k-1}(1/x)\}^{-1}\{l_k(1/x)\}^{-1-\alpha},$$

where $\alpha > 0$, and divergent if

$$f \succcurlyeq (1/x)\{l(1/x) \ldots l_k(1/x)\}^{-1}.$$

These results are classical. The first general statement of the 'logarithmic criteria', so far as series are concerned, appears to have been made by De Morgan, **1**, 325. The essentials, however, appear in a posthumous memoir of Abel (**2**) also first published in 1839: see also Abel, **1**. The case of $k=1$ had been dealt with by Cauchy, **2**. Bertrand (**1**) arrived at De Morgan's results independently, and the criteria are very commonly attributed to him. The first general and explicit statement of the criteria for integrals seems to be due to Bonnet, **1**.

For further information concerning the logarithmic tests, and the corresponding 'ratio-tests' for the convergence of series, see Bromwich, **1**, 29; du Bois-Reymond, **3**; Goursat, **1** (1), 403; Hardy, **1**, 374; Knopp, **1**, 117; Pringsheim, **1** (310), **2** (77), **3**; Riemann, **1**.

6.22. Theorems analogous to du Bois-Reymond's Theorem. We should mention also certain theorems of a negative character, analogous to du Bois-Reymond's theorem of § 2.1.

Given any divergent series Σu_n of positive terms, we can find a function v_n such that $v_n \prec u_n$ and Σv_n is divergent; *i.e.* given any divergent series we can find one more slowly divergent.

Given any convergent series Σu_n of positive terms, we can find v_n so that $v_n \succ u_n$ and Σv_n is convergent; *i.e.* given any convergent series we can find one more slowly convergent.

Given any function $\phi(n)$ tending to infinity, however slowly, we can find a convergent series Σu_n and a divergent series Σv_n such that $v_n/u_n = \phi(n)$.

Given an infinite sequence of series, each converging (diverging) more slowly than its predecessor, we can find a series which converges (diverges) more slowly than any of them.

There is no function $\phi(n)$ such that $u_n\phi(n) \succcurlyeq 1$ is a necessary condition for the divergence of Σu_n, and no function $\phi(n)$ such that $\phi(n) \succ 1$ and $u_n\phi(n) \preccurlyeq 1$ is a necessary condition for the convergence of Σu_n.

If u_n is a steadily decreasing function of n, then $nu_n \prec 1$ is a necessary condition for convergence; but there is no function $\phi(n)$ such that $\phi(n) \succ 1$ and $n\phi(n)\,u_n \prec 1$ is a necessary condition.

If however nu_n decreases steadily, then $n \log n u_n \to 0$ is a necessary condition; and if $n\psi(n) u_n$, where $n\psi(n) \succ 1$ and $\int \frac{dn}{n\psi(n)} \succ 1$, decreases steadily, then

$$\left(n\psi(n) \int \frac{dn}{n\psi(n)}\right) u_n \to 0$$

is a necessary condition.

If Σu_n is divergent, and $U_n = u_1 + u_2 + \ldots + u_n$, then $\Sigma (u_n / U_n)$ is also divergent; and if also $u_n \prec U_n$ then

$$\frac{u_1}{U_1} + \frac{u_2}{U_2} + \ldots + \frac{u_n}{U_n} \sim \log U_n.$$

See Abel, **1**, **2**; Bromwich, **1**, 40; Dini, **1**; Hadamard, **2**; Littlewood, **5**; Pringsheim, **1** (353, 939), **2** (89), **3**, **4**.

For examples of series and integrals which converge or diverge so slowly as not to answer to any of the logarithmic criteria, so that the logarithmic tests are insufficient (§ 4.42), or to which the logarithmic tests are inapplicable, see Borel, **4**, **5**; du Bois-Reymond, **3**, **7**, **8**; Gilbert, **1**; Goursat, **1** (1), 219; Hardy, **3**, (1), (2), (3), (5); Pringsheim, **1** (353), **3** (343), **5**, **6**; Thomae, **2**.

6.23. Asymptotic formulae for finite sums. A closely connected problem is that of the determination of asymptotic formulae for

$$A_n = a_1 + a_2 + \ldots + a_n$$

when the behaviour of a_n for large values of n is known. The principal weapons for dealing with this problem are (i) the theorem of Cauchy and Stolz, that $A_n \sim CB_n$ if Σb_n is a divergent series of positive terms and $a_n \sim Cb_n$, (ii) the 'Euler-Maclaurin sum formula'

$$\sum_1^n f(\nu) = \int^n f(x)\,dx + C + \tfrac{1}{2} f(n) + \frac{B_1}{2!} f'(n) - \frac{B_2}{4!} f'''(n) + \ldots,$$

and in particular (iii) the theorem of Maclaurin and Cauchy that

$$f(1) + f(2) + \ldots + f(n) - \int_1^n f(x)\,dx,$$

where $f(x)$ is a positive decreasing function of x, tends to a limit when $n \to \infty$.

For (i) see Cauchy, **1**, 59; Jensen, **1**; Stolz, **1**; and for (iii) Cauchy, **2**; Maclaurin, **1** (1), 289. Proofs of either theorem will be found in any modern text book of analysis or the theory of series; see Bromwich, **1**, 29, 377; Knopp, **1**, 68, 286. For further developments see Bromwich, **2**; Dahlgren, **1**; Hardy, **3** (4), **8**; Nörlund, **1**. The literature of the general Euler-Maclaurin sum formula is too extensive to be summarized here; see Bromwich, **1**, 238, 324; Nörlund, **1**, **2**; Pringsheim, **2**, 102; Seliwanoff, **1**, 929.

Among the most important results which follow from these theorems are

$$1^s+2^s+\dots+n^s \sim \frac{n^{s+1}}{s+1} \quad (s>-1),$$

$$1^s+2^s+\dots+n^s-\frac{n^{s+1}}{s+1} \sim \zeta(-s) \quad (-1<s<0),$$

and generally

$$\sum_1^n \nu^s - \frac{n^{s+1}}{s+1} - \tfrac{1}{2}n^s - \sum_1 (-1)^{i-1}\binom{s}{2i-1}\frac{B_i}{2i}n^{s-2i+1} \sim \zeta(-s).$$

Here s is positive and not integral, $\zeta(-s)$ is the Zeta-function of Riemann, and the summation with respect to i is continued till we come to a negative power of n. Again

$$1+\frac{1}{2}+\dots+\frac{1}{n}-\log n \sim A,$$

$$1+\frac{\alpha.\beta}{1.\gamma}+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}+\dots \text{ to } n \text{ terms},$$

$$\sim \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\frac{n^{\alpha+\beta-\gamma}}{\alpha+\beta-\gamma} \quad (\alpha+\beta>\gamma),$$

or

$$\sim \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\log n \quad (\alpha+\beta=\gamma).$$

In connection with the last result see Bromwich, **4**; in the earlier formula A is Euler's constant.

The most important formula of this kind is

$$\log 1+\log 2+\dots+\log n-(n+\tfrac{1}{2})\log n+n \sim \tfrac{1}{2}\log(2\pi),$$

which, in the form

$$n! \sim n^{n+\frac{1}{2}}e^{-n}\sqrt{(2\pi)},$$

constitutes *Stirling's Theorem*. Another formula of the same kind is

$$1^1 2^2 3^3 \dots n^n \sim Bn^{\frac{1}{2}n^2+\frac{1}{2}n+\frac{1}{12}}e^{-\frac{1}{4}n^2},$$

where B is a constant defined by the equation

$$\log B = \tfrac{1}{12}\log 2\pi + \tfrac{1}{12}\gamma + \frac{1}{2\pi^2}\sum_1^\infty \frac{\log \nu}{\nu^2}.$$

The literature of Stirling's Theorem is also very extensive; see Bromwich, **1**, 461; Brunel, **1**; Nielsen, **1**, 92; Whittaker and Watson, **1**, 251, 276. As regards the constant B see Barnes, **1**; Glaisher, **1**, **2**; Kinkelin, **1**.

6.24. A proof of Stirling's Theorem. Stirling's Theorem, as stated in § 6.23, may be proved in an almost elementary manner*; but

* For such a proof see, *e.g.*, Cesàro, **1**, 221, 395; Jolliffe, **1**. The principal difficulty of an elementary proof is naturally the determination of the constant $\sqrt{(2\pi)}$.

it will be more instructive here to give a proof depending on the representation of $\Gamma(n+1)$ as a definite integral. The method employed, the principle of which may be traced back to Laplace*, is that which, when extended to the field of the complex variable, is known as the 'Methode der Sattelpunkte' or 'method of steepest descents'†.

We suppose n positive and large, but not necessarily integral. In the integral

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^n \, dx$$

the maximum of the integrand occurs for $x = n$. We therefore write

(6.241)

$$\Gamma(n+1) = J = \int_0^\infty = \int_0^{(1-\eta)n} + \int_{(1-\eta)n}^{(1+\eta)n} + \int_{(1+\eta)n}^{2n} + \int_{2n}^\infty = J_1 + J_2 + J_3 + J_4,$$

say, where $0 < \eta < 1$.

In J_1 and J_3 we write $x = n(1-y)$ and $x = n(1+y)$ respectively. Observing that the functions $e^y(1-y)$ and $e^{-y}(1+y)$ each decrease steadily as y increases from 0 to 1, we obtain

(6.242)

$$J_1 = n^{n+1}e^{-n} \int_\eta^1 e^{ny}(1-y)^n \, dy < n^{n+1}e^{-n}\{e^\eta(1-\eta)\}^n = n^{n+1}e^{-n}E_1^{\,n},$$

(6.243)

$$J_3 = n^{n+1}e^{-n} \int_\eta^1 e^{-ny}(1+y)^n \, dy < n^{n+1}e^{-n}\{e^{-\eta}(1+\eta)\}^n = n^{n+1}e^{-n}E_3^{\,n},$$

say; here E_1 and E_3 are less than 1. And if we apply the same transformation to J_4 as to J_3, and observe that $e^{-y}(1+y)$ also decreases from $y = 1$ onwards, we find

(6.244)

$$J_4 = n^{n+1}e^{-n} \int_1^\infty e^{-ny}(1+y)^n dy < n^{n+1}e^{-n}\left(\frac{2}{e}\right)^{n-1} \int_1^\infty e^{-y}(1+y)\, dy$$
$$= \frac{3}{2} e n^{n+1} e^{-n} \left(\frac{2}{e}\right)^n.$$

From (6.242), (6.243), and (6.244) it follows that

(6.245) $$\lim_{n\to\infty} n^{-n-\frac{1}{2}} e^n (J_1 + J_3 + J_4) = 0.$$

In J_2 we write again $x = n(1+y)$. We have then

$$n \log x - x = n \log n - n - ny + n \log(1+y) = n \log n - n - \frac{ny^2}{2(1+\theta y)^2},$$

where $-1 < \theta < 1$. Hence J_2 lies between

$$n^{n+1}e^{-n} \int_{-\eta}^{\eta} e\left\{-\frac{ny^2}{2(1-\eta)^2}\right\} dy = (1-\eta)\, n^{n+\frac{1}{2}} e^{-n} \sqrt{2} \int_{-\zeta}^{\zeta} e^{-w^2} dw,$$

* Laplace, **1**, 88. † Watson, **1**, 235.

where
$$\zeta = \frac{\eta}{1-\eta}\sqrt{\left(\frac{n}{2}\right)},$$
and the corresponding expression in which $1-\eta$ is replaced by $1+\eta$. The limit of the integral when n, and therefore ζ, tends to infinity is $\sqrt{\pi}$. Hence

(6.246)
$$\varliminf_{n\to\infty} n^{-n-\frac{1}{2}} e^n J_2 \geqq (1-\eta)\sqrt{(2\pi)}, \quad \varlimsup_{n\to\infty} n^{-n-\frac{1}{2}} e^n J_2 \leqq (1+\eta)\sqrt{(2\pi)}.$$

From (6.245) and (6.246) it follows that
$$(1-\eta)\sqrt{(2\pi)} \leqq \varliminf n^{-n-\frac{1}{2}} e^n J \leqq \varlimsup n^{-n-\frac{1}{2}} e^n J \leqq (1+\eta)\sqrt{(2\pi)}.$$
But η is arbitrary, and J is independent of η. Hence

(6.247)
$$\lim n^{-n-\frac{1}{2}} e^n J = \sqrt{(2\pi)},$$
which is Stirling's Theorem.

As a corollary we note that

(6.248)
$$\frac{\Gamma(n+a)}{\Gamma(n+b)} \sim \frac{(n+a)^{n+a-\frac{1}{2}}}{(n+b)^{n+b-\frac{1}{2}}} e^{b-a} \sim n^{a-b}.$$

6.25. A general result for L-functions. The results of Section 5 enable us to obtain a general formula for A_n whenever a_n is an L-function of n and Σa_n is divergent.

Theorem 33. *If a_n is the value for $x=n$ of an L-function $a(x)$, and Σa_n is divergent, then*

(6.251)
$$A_n \sim a(n)$$
if $a(x) \succ e^{\Delta x}$,

(6.252)
$$A_n \sim \int_1^n a(x)\,dx$$
if $a(x) \prec e^{\delta x}$, and

(6.253)
$$A_n \sim \frac{\alpha}{1-e^{-\alpha}} \int_1^n a(x)\,dx$$
if $a(x) = e^{\alpha x} b(x)$, where $e^{-\delta x} \prec b(x) \prec e^{\delta x}$.

Suppose first that $a(x) \succ e^{\Delta x}$, so that $a' \succ a$. Then, if we suppose, as we may do without loss of generality, that $a(x)$ increases from $x=1$, we have
$$A_{n-1} = \sum_1^{n-1} a(\nu) < \int_1^n a(x)\,dx \sim \frac{\{a(n)\}^2}{a'(n)} \prec a(n),$$
by Theorem 25. Hence $A_n \sim a(n)$.

Next, suppose $a(x) \prec e^{\delta x}$, so that $a' \prec a$. Then
$$a_\nu - c_\nu = a(\nu) - \int_{\nu-1}^{\nu} a(x)\,dx = \int_{\nu-1}^{\nu} \{a(\nu) - a(x)\}\,dx = \int_{\nu-1}^{\nu} (\nu - x) a'(r)\,dx,$$

where $\nu - 1 < r < \nu$. But $a'(r) \sim a'(\nu)$, by Theorem 31; and so

$$a_\nu - c_\nu \preccurlyeq a'(\nu) \prec a(\nu) = a_\nu.$$

It follows that $a_\nu \sim c_\nu$ and $A_n \sim C_n$, which is (6.252).

Finally suppose $a(x) = e^{\alpha x} b(x)$. Then

$$\int_{\nu-1}^{\nu} \{a(\nu) - a(x)\}\, dx = b(\nu) \int_{\nu-1}^{\nu} (e^{\alpha\nu} - e^{\alpha x})\, dx + \int_{\nu-1}^{\nu} e^{\alpha x} \{b(\nu) - b(x)\}\, dx$$

$$= \beta_\nu + \gamma_\nu,$$

say. Here $b(\nu) - b(x) = (\nu - x)\, b'(r) \preccurlyeq b'(\nu)$, and so

$$\gamma_\nu \preccurlyeq e^{\alpha\nu} b'(\nu) \prec e^{\alpha\nu} b(\nu) = a_\nu,$$

while
$$\beta_\nu = \left(1 - \frac{1 - e^{-\alpha}}{\alpha}\right) a(\nu).$$

Hence

$$a_\nu - c_\nu \sim 1 - \frac{1 - e^{-\alpha}}{\alpha} a_\nu, \quad a_\nu \sim \frac{\alpha}{1 - e^{-\alpha}} c_\nu;$$

and (6.253) follows.

It is also possible, by using Theorem 26, to obtain formulae for A_n when $a_n = f_n e^{i\phi_n}$, where f and ϕ are L-functions subject to certain limitations; but the results are more complicated and less general. It is easy to see that comprehensive results are not to be expected here. The series $\Sigma e^{\alpha i n^2}$, for example, behaves in a very intricate manner, depending on the arithmetic nature of the number α*. But, if the increase of f_n and ϕ_n is sufficiently slow, A_n will behave like the integral $\int^n f(u)\, e^{i\phi(u)}\, du$, and the series $\Sigma\, a_n$ will be convergent if $f \prec \phi'$.

6.26. Formulae involving prime numbers and arithmetical functions. It is known that, if $\pi(n)$ is the number of prime numbers not exceeding n, and p_n is the nth prime, so that $\pi(n)$ and p_n are inverse functions, then

(6.261) $$\pi(n) \sim \frac{n}{\log n}, \quad p_n \sim n \log n.$$

More precisely

(6.262) $$\pi(n) = \int_2^n \frac{dt}{\log t} + O\left(ne^{-A\sqrt{(ln\, lln)}}\right) = \mathrm{Li}\, n + \bar{O}\left(ne^{-A\sqrt{(ln\, lln)}}\right)†,$$

where $A > 0$. If the hypothesis of Riemann concerning the zeros of the Zeta-function $\zeta(s)$ is true, the error term may be replaced by $O(n^{\frac{1}{2}+\delta})$ and indeed by $O(\sqrt{n}\, ln)$. On the other hand the order of the error is certainly not less than

$$O\left(\frac{\sqrt{n}\, l_3 n}{ln}\right) ‡.$$

* Hardy and Littlewood, **3** (2). For a discussion of the series $\Sigma n^{-b} e^{Ain^{a}}$, where $0 < a < 1$, see Hardy, **8**.

† The classical formula has an error term $O\left\{ne^{-A\sqrt{(ln)}}\right\}$. For the more precise result stated here see Landau, **5**, **6**; Littlewood, **7**.

‡ Littlewood, **6**; Hardy and Littlewood, **4**.

It is easily proved by partial integration that

$$(6.263)\qquad \int_2^n \frac{dt}{\log t} = \frac{n}{ln} + \frac{n}{(ln)^2} + \frac{2!n}{(ln)^3} + \ldots + \frac{(k-1)!n}{(ln)^k} + O\left\{\frac{n}{(ln)^{k+1}}\right\}$$

for every value of k; while $e^{-A\sqrt{(ln\, lln)}}$ tends to zero more rapidly than any power of ln. Hence the right hand side of (6.263) is a genuine approximation to $\pi(n)$ for any value of k.

The order of magnitude of a sum of the form

$$\sum_{p<n} f(p)$$

may, with certain reservations, be found by replacing the nth prime by $n \log n$. Thus

$$\sum_{p<x} \frac{1}{p} \sim l_2 x, \quad \sum_{p<x} \frac{lp}{p} \sim lx, \quad \sum_{p<x} lp \sim x,$$

while $\Sigma \dfrac{1}{plp}$ is convergent. For a comprehensive account of the theory see Landau, 1.

We quote some additional examples of asymptotic formulae for arithmetical functions. We write $\pi_\nu(x)$ for the number of numbers, less than x, composed of just ν factors (repeated or not); $Q(x)$ for the number of numbers with no repeated factor; $R(x)$ for the number of numbers of the form $2^{a_2} 3^{a_3} \ldots p^{a_p}$, where $a_2 \geqq a_3 \geqq \ldots$; $p(n)$ for the number of partitions of n; and $p_r(n)$ for the number of partitions of n into perfect rth powers. Then

$$\pi_\nu(x) \sim \frac{1}{(\nu-1)!} \frac{x(llx)^{\nu-1}}{lx} *, \quad Q(x) \sim \frac{6x}{\pi^2} \dagger,$$

$$\log R(x) \sim \frac{2\pi}{\sqrt{3}} \sqrt{\left(\frac{lx}{llx}\right)} \ddagger, \quad p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{(\frac{2}{3}n)}} \S,$$

$$p_r(n) \sim (2\pi)^{-\frac{1}{2}(r+1)} \sqrt{\left(\frac{r}{r+1}\right)}\, k n^{\frac{1}{r+1} - \frac{3}{2}} e^{\left\{(r+1)\, k n^{\frac{1}{r+1}}\right\}},$$

where

$$k = \left\{\frac{1}{r} \Gamma\left(1 + \frac{1}{r}\right) \zeta\left(1 + \frac{1}{r}\right)\right\}^{\frac{r}{r+1}}$$

and $\zeta(s)$ is Riemann's Zeta-function.

6.31. Power-series. The theory of integral functions. The radius of convergence R of a power-series

$$(6.311)\qquad f(x) = \Sigma a_n x^n$$

is given || by

$$\frac{1}{R} = \overline{\lim_{n\to\infty}} |a_n|^{\frac{1}{n}}.$$

The series is convergent for all values of x if $\sqrt[n]{|a_n|} \to 0$, *i.e.* if $|a_n| \prec e^{-\Delta n}$. In this case $f(x)$ is called an *integral function.*

* Landau, **1**, 208, 211.

† Landau, **1**, 582.

‡ Hardy and Ramanujan, **1**.

§ Hardy and Ramanujan, **2**.

|| See, *e.g.*, Goursat, **1** (1), 443.

The three most important characteristics of an integral function are measured by

(i) $\alpha_n = |a_n|$, the modulus of the nth coefficient;

(ii) $M(r)$, the maximum of $|f(x)|$ on the circle $|x| = r$;

(iii) $\gamma_n = |c_n|$, the modulus of the nth zero, in order of absolute magnitude.

It is known that $M(r)$ is a steadily increasing function of r, and that $M(r) \succ r^{\Delta}$, except in the trivial case (which we ignore) in which $f(x)$ is a polynomial*. A function for which $M(r) \prec e^{r^{\Delta}}$ is called a function *of finite order*, and we shall consider such functions only.

The principal problem of the theory is to determine the relations between the rates of increase of $1/\alpha_n$, $M(r)$, and γ_n. Those which hold between the first two functions are the simplest, and we shall confine our attention to them. The theory of γ_n is complicated by the 'Picard case of exception', arising from functions which, like e^x, have no zeros, or whose zeros are scattered abnormally over the plane. The increases of the three functions may be measured by 'indices' defined as follows†.

The μ-index μ of $f(x)$ is the greatest number ξ such that

$$\sqrt[n]{\alpha_n} < n^{-\xi+\epsilon} \tag{6.312}$$

for every positive ϵ and all sufficiently large values of n. It is plain that $\mu \geqq 0$, since $\alpha_n \to 0$. It may happen that (6.312) is true for all values of μ; in this case we say that the μ-index is infinite. The ν-index ν is the least number ξ such that

$$M(r) < e^{r^{\xi+\epsilon}} \tag{6.313}$$

for every positive ϵ and all sufficiently large values of r. The ρ-index ρ is the least number ξ such that

$$\Sigma \frac{1}{\gamma_n^{\xi+\epsilon}}$$

is convergent for every positive ϵ. In particular these conditions are satisfied if

$$n^{-\mu-\delta} \prec \sqrt[n]{\alpha_n} \prec n^{-\mu+\delta},\quad e^{r^{\nu-\delta}} \prec M(r) \prec e^{r^{\nu+\delta}},\quad n^{\frac{1}{\rho}-\delta} \prec \gamma_n \prec n^{\frac{1}{\rho}+\delta},$$

or if

$$l\left(\frac{1}{\alpha_n}\right) \sim \mu n\, ln,\quad l_2 M(r) \sim \nu\, lr,\quad l\gamma_n \sim \frac{ln}{\rho}.$$

* For the second proposition see Goursat, **1** (2), 92. It is curiously difficult to give a reference to a direct and explicit proof of the first. It is included implicitly in one of the classical proofs of the fundamental theorem of algebra (see, *e.g.*, Hardy, **1**, 433) and in the familiar theorem that a potential function cannot have a maximum at a point of regularity.

† Vivanti, **1**, 228.

The fundamental theorems of the subject are (i) that $\nu = 1/\mu$, it being understood that this means $\nu = 0$ when the μ-index is infinite, and (ii) that, with certain reservations, $\rho = \nu$.

6.32. Proof* that $\mu = 1/\nu$. (i) We suppose $\mu > 0$, and we prove first that $\nu \geqslant 1/\mu$. We denote by $\mathbf{M}(r)$ the maximum of $a_n r^n$ for $n=0, 1, 2, \ldots$. Then† $M(r) > \mathbf{M}(r)$ for all values of r. It follows from the definition of μ that

(6.321) $$\sqrt[n]{a_n} > n^{-\mu-\epsilon}$$

for every positive ϵ and an indefinitely increasing sequence (n_j) of values of n. If n has one of these values,

(6.322) $$a_n r^n > (rn^{-\mu-\epsilon})^n.$$

The right hand side, considered as a function of a continuous variable n, attains a maximum

$$e^{\left(\frac{\mu+\epsilon}{e} r^{\mathbf{m}}\right)},$$

where $\mathbf{m} = 1/(\mu + \epsilon)$, for

(6.323) $$n = r^{\mathbf{m}}/e.$$

If r has such a value that (6.323) is one of the integers n_j, then

$$M(r) > \mathbf{M}(r) > e^{\left(\frac{\mu+\epsilon}{e} r^{\mathbf{m}}\right)}.$$

This is true for a sequence of values of r surpassing all limit, and $\mathbf{m}$ is any number less than $1/\mu$. It follows that $\nu \geqq 1/\mu$.

(ii) To obtain an upper bound for $M(r)$, we observe that

$$r\sqrt[n]{a_n} < rn^{-\mu+\epsilon} < \tfrac{1}{2}$$

if

(6.324) $$n \geqq n_r = (2r)^{\mathbf{m}'},$$

where $\mathbf{m}' = 1/(\mu - \epsilon)$, and r is large enough. Thus

(6.325) $$M(r) \leqq \sum_{0}^{n_r-1} a_n r^n + \sum_{n_r}^{\infty} a_n r^n < n_r \mathbf{M}(r) + \sum_{0}^{\infty} 2^{-n}$$
$$= n_r \mathbf{M}(r) + 2 < 2n_r \mathbf{M}(r),$$

if r is large enough. But

(6.326) $$\mathbf{M}(r) = \operatorname{Max} a_n r^n \leqq \operatorname{Max}(rn^{-\mu+\epsilon})^n = e^{\left(\frac{\mu-\epsilon}{e} r^{\mathbf{m}'}\right)}.$$

From (6.324), (6.325), and (6.326) it follows that

$$M(r) < 2\,(2r)^{\mathbf{m}'} e^{\left(\frac{\mu-\epsilon}{e} r^{\mathbf{m}'}\right)}$$

for all sufficiently large values of r. Here $\mathbf{m}'$ is any number greater than $1/\mu$. Hence $\nu \leqq 1/\mu$, and so $\nu = 1/\mu$.

* The proof is modelled on that given by Lindelöf, **3**. † Goursat, **1** (2), 92.

The fundamental idea of the proof which precedes is that *the increase of $f(x)$ is measured*, with sufficient accuracy for the determination of the indices, *by that of its greatest term*. In the exponential series, for example, the greatest term is that for which $n=[x]$, and the increase of this term is $e^x/\sqrt{x}$.

We have assumed μ positive and finite. A slight variation of the argument shows (*a*) that $\nu=0$ when μ is infinite, and (*b*) that $f(x)$ is not of finite order when $\mu=0$.

6.33. Special results. If we make more drastic assumptions about the coefficients a_n, we can naturally obtain more precise results about $f(x)$. Thus if

$$\{n\,(ln)^{-b_1}\ldots(l_k n)^{-b_k+\delta}\}^{-1/\nu} \prec \sqrt[n]{a_n} \prec \{n\,(ln)^{-b_1}\ldots(l_k n)^{-b_k-\delta}\}^{-1/\nu},$$

then
$$e\,\{r^\nu\,(lr)^{b_1}\ldots(l_k r)^{b_k-\delta}\} \prec M(r) \prec e\,\{r^\nu\,(lr)^{b_1}\ldots(l_k r)^{b_k+\delta}\},$$
and conversely. If

$$\sqrt[n]{a_n}=n^{-1/\nu}\,\lambda(n),$$

where
$$e^{-\delta\sqrt{ln}} \prec \lambda(n) \prec e^{\delta\sqrt{ln}},$$

then
$$\log f(x) \sim \frac{1}{\nu e}\,\{x\lambda(x^\nu)\}^\nu.$$

As examples of still more accurate and special results we may quote the following:

$$\Sigma\,\frac{x^n}{n^{an}} \sim \sqrt{\left(\frac{2\pi}{ea}\right)}\,x^{1/2a}\,e^{(a/e)\,x^{1/a}},$$

$$\Sigma\,\frac{x^n}{(n\,!)^a} \sim \frac{1}{\sqrt{a}}\,(2\pi)^{(1-a)/2}\,x^{(1-a)/2a}\,e^{ax^{1/a}}, \quad \Sigma\,\frac{x^n}{\Gamma(an+1)} \sim \frac{1}{a}\,e^{x^{1/a}},$$

$$\Sigma\,e^{-n^p}\,x^n \sim \sqrt{\left\{\frac{2\pi}{p\,(p-1)}\right\}}\left(\frac{\log x}{p}\right)^{\frac{2-p}{2p-2}} e^{(p-1)\left(\frac{\log x}{p}\right)^{p/(p-1)}},$$

where $a>0$ and in the last formula $1<p<2$, and $x\to\infty$ by positive values. These results may of course be used to give an upper limit for the modulus of the particular function considered when x is not necessarily real, and so for $M(r)$.

General accounts of the theory of integral functions are given by Borel, **2**; Vivanti, **1**; Bieberbach, **1**; Valiron, **1**. The second edition of the first work contains a very valuable note by Valiron on the latest developments of the theory, and the second work a very complete bibliography up to 1906. Particularly important memoirs (beyond those on which Borel's account of the theory is based) are those of Boutroux, **1**; Lindelöf, **2**; Pringsheim, **7**; Valiron, **2**, **3**; and Wiman, **1**, **2**, **3**. For more precise and special developments, such as those quoted at the beginning of this section, see in particular Le Roy, **1**; Lindelöf, **3**; Littlewood, **1**, **2**, **3**, **4**; and Mellin, **1**. For the theory of integral functions of infinite order, see Blumenthal, **1**.

6.34. Irregularly increasing functions defined by power series. Power series with gaps. The theory of integral functions suggests a method of much interest for the construction of irregularly increasing functions.

Suppose that $\phi(x)=\Sigma a_n x^n$ is an integral function with positive and decreasing coefficients, and that, for a given x, $\varpi(x)=a_\nu x^\nu$ is the greatest term of the series. In general one term will be the greatest, but for particular values of x, say $\xi_1, \xi_2, \ldots$, two consecutive terms will be equal*.

As x increases, the index ν of $\varpi(x)$ increases, and tends to infinity with n: it thus defines a function $\nu(x)$ such that

$$\nu(x)=i \quad (\xi_i < x < \xi_{i+1}).$$

At the point of discontinuity ξ_i, where $\nu(x)$ jumps from $i-1$ to i, we may assign to it the value i. When ν is thus defined for all values of x, $\varpi(x)$ defines a function of x which tends continuously and steadily to infinity with x; and it may be expected that the increase of ϖ will give a fair approximation to that of ϕ.

Now let
$$f(x)=\Sigma a_{\chi(n)} x^{\chi(n)},$$
where $\chi(n)\succ n$; and let $p(x)$ be the function related to f as $\varpi(x)$ is to ϕ. The laws of increase of $\varpi(x)$ and of $p(x)$ may be expected to be very much the same, for $p(x)$ is defined by a selection from *some* of the terms from *all* of which $\varpi(x)$ was selected. The increase of $f(x)$ clearly cannot be greater, and may be expected to be less, than that of $\phi(x)$; but it cannot be less than that of $p(x)$. Hence we may expect relations of the type

$$p \asymp \varpi \prec f \prec \phi.$$

The more rapidly we suppose $\chi(n)$ to increase, the lower in the gap between ϖ and ϕ will f sink, and, if we suppose χ to increase with sufficient rapidity, we may expect to find that $\varpi \asymp f$, so that the increase of f is completely dominated by that of one variable term. We shall then have

$$f(x) \asymp a_{N(x)} x^{N(x)},$$

where $N(x)$ is a function of x which assumes successively each of a series of integral values N_i, so that

$$N(x)=N_i, \quad (x_i \leqq x < x_{i+1}).$$

But, as x increases from x_i to x_{i+1}, the order of $a_{N_i} x^{N_i}$, considered as a function of x, may vary considerably, since N_i, though depending on the interval (x_i, x_{i+1}), does not depend on the particular position of x in that interval. We are thus likely to be led to functions whose increase is irregular in the sense explained in § 4.41.

Suppose, for example, that $a_n=n^{-n}$, so that (§ 6.33)

$$\phi(x)=\Sigma\left(\frac{x}{n}\right)^n \sim \sqrt{\left(\frac{2\pi x}{e}\right)}\, e^{x/e}.$$

Here
$$\xi_i=i\left(1+\frac{1}{i}\right)^{i+1} \sim ei,$$

and it is easily shown that $\varpi(x) \asymp e^{x/e}$.

Now let $\chi(n)=2^n$, so that

$$f(x)=\Sigma \frac{x^{2^n}}{2^{n2^n}}=\Sigma v_n,$$

* We ignore the possibility of more than two terms being equal.

say Then $v_{i-1}=v_i$ if $x=2^{i+1}$, so that $x_i=2^{i+1}$ and $N_i=2^i$ for

$$2^{i+1} \leqq x < 2^{i+2}.$$

For this range of values of x, v_i is the greatest term; when $x=2^{i+2}$, $v_i=v_{i+1}$. Further, it is not difficult to show that $f(x) \asymp p(x)=v_i$, the behaviour of $f(x)$ being dominated by that of its greatest term*. If we put $x=2^{i+1+\theta}$, where $0<\theta<1$, we find

$$f(x) \asymp v_i = 2^{(1+\theta)\,2^i} = 2^{ax},$$

where $a=(1+\theta)\,2^{-1-\theta}$. This is a maximum when $1+\theta=1/(\log 2)$, when it is equal to $1/(e\log 2)=\cdot 53\ldots$. Hence the increase of $f(x)$ oscillates (roughly) between those of $2^{\cdot 53\ldots x}$ and $2^{\frac{1}{2}x}$

Another example of an irregularly increasing function defined in a similar manner is

$$f(x)=\Sigma \frac{x^{n^3}}{(n^3)!},$$

the increase of which oscillates between the increases of $e^x/\sqrt{x}$ and

$$x^{-\frac{1}{2}} e^{x-\frac{9}{8}x^{1/3}}\dagger.$$

These examples are of course typical of a large class of functions.

6.35. Power-series with a finite radius of convergence. When the radius of convergence of the power-series (6.311) is finite, it may be supposed, without loss of generality, to be 1. The necessary and sufficient condition for this is that $\overline{\lim}\sqrt[n]{|a_n|}=1$; this is true in particular if a_n is positive and $e^{-\delta n} \prec a_n \prec e^{\delta n}$.

Suppose in particular that a_n is positive and that Σa_n is divergent, so that $f(x)\to\infty$ when $x\to 1$‡. Then a large number of important theorems have been proved which embody relations between (*a*) the increase of $A_n=a_0+a_1+\ldots+a_n$ as $n\to\infty$ and (*b*) the increase of $f(x)$ as $x\to 1$.

The most fundamental theorem is

Theorem 34. *If a_n and b_n are positive, and $A_n \sim B_n$, then*

(6.351) $$f(x)=\Sigma a_n x^n \sim g(x)=\Sigma b_n x^n.$$

In particular this is so if $a_n \sim b_n$§.

* We may say roughly that *in general* $f\sim p$, that is to say, $f/p\to 1$ as $x\to\infty$ through any sequence of values not falling inside any of certain intervals, as small as we please, surrounding the values ξ_i. At a point ξ_i, f/p is nearly equal to 2.

† Hardy, **3** (3).

‡ Bromwich, **1**, 130

§ Bromwich, **1**, 132. The theorem is due to Cesàro, **2**.

We have

$$F(x)=\frac{f(x)}{1-x}=\Sigma A_n x^n, \quad G(x)=\frac{g(x)}{1-x}=\Sigma B_n x^n,$$

and it is enough to prove that $F(x)\sim G(x)$.

Given any positive ϵ, we have $B_n(1-\epsilon)<A_n<B_n(1+\epsilon)$ for $n\geqq N(\epsilon)$, say; and

$$F(x)=\sum_0^{N-1} A_n x^n+\sum_N^{\infty} A_n x^n=F_N(x)+\sum_N^{\infty} A_n x^n$$

lies between

$$F_N(x)+(1-\epsilon)\sum_N^{\infty} B_n x^n, \quad F_N(x)+(1+\epsilon)\sum_N^{\infty} B_n x^n;$$

and therefore between

$$-B_N+(1-\epsilon)G(x), \quad A_N+(1+\epsilon)G(x).$$

Hence

$$1-\epsilon\leqq \varliminf_{x\to 1}\frac{F(x)}{G(x)}\leqq \varlimsup_{x\to 1}\frac{F(x)}{G(x)}\leqq 1+\epsilon$$

for every positive ϵ, which proves the theorem.

We have, for example,

$$\frac{\Gamma(1-p)}{(1-x)^{1-p}}=\sum_{n=0}^{\infty}\frac{\Gamma(n+1-p)}{\Gamma(n+1)}x^n=\Sigma b_n x^n,$$

say, and $b_n\sim n^{-p}=a_n$, by (6.248)*. Hence

$$\Sigma\frac{x^n}{n^p}\sim\frac{\Gamma(1-p)}{(1-x)^{1-p}} \qquad (p<1).$$

Similarly

$$F(a,\beta,\gamma,x)\sim\frac{\Gamma(\gamma)\Gamma(a+\beta-\gamma)}{\Gamma(a)\Gamma(\beta)}\frac{1}{(1-x)^{a+\beta-\gamma}} \quad (a+\beta>\gamma),$$

$$F(a,\beta,a+\beta,x)\sim\frac{\Gamma(a+\beta)}{\Gamma(a)\Gamma(\beta)}\,l\left(\frac{1}{1-x}\right).$$

Of further results the following is typical: if

$$a_n\sim n^{-p}\{ln\ldots l_{m-1}n\,(l_m n)^q\ldots(l_{m+k}n)^{q_k}\}^{-1},$$

then

$$f(x)\sim\frac{\Gamma(1-p)}{(1-x)^{1-p}}\left\{l\frac{1}{1-x}\ldots l_{m-1}\frac{1}{1-x}\left(l_m\frac{1}{1-x}\right)^q\ldots\left(l_{m+k}\frac{1}{1-x}\right)^{q_k}\right\}^{-1}$$

if $p<1$, $q\neq 1$: but

$$f(x)\sim\frac{1}{1-q}\left(l_m\frac{1}{1-x}\right)^{1-q}\left\{\left(l_{m+1}\frac{1}{1-x}\right)^q\ldots\left(l_{m+k}\frac{1}{1-x}\right)^{q_k}\right\}^{-1}$$

if $p=1$, $q<1$. Thus

$$\Sigma\frac{x^n}{n^p(lx)^q}\sim\frac{\Gamma(1-p)}{(1-x)^{1-p}}\left(l\frac{1}{1-x}\right)^{-q} \quad (p<1).$$

* Appell, 1.

As specimens of further results of this character we may quote

$$x+x^4+x^9+\ldots \sim \tfrac{1}{2}\sqrt{\left(\frac{\pi}{1-x}\right)},$$

$$x+x^a+x^{a^2}+\ldots \sim \frac{1}{la}\, l\left(\frac{1}{1-x}\right) \qquad (a>1),$$

$$\Sigma a^n x^{n^2} \sim e\left\{\tfrac{1}{4}\frac{(la)^2}{l(1/x)}\right\} \qquad (a>1),$$

$$\Sigma\, e^{n/ln}\, x^n = e_2\{u/(1-x)\} \qquad (u \sim 1).$$

Many similar results have been established about series other than power series: thus

$$\Sigma \frac{x^n}{n(1+x^n)} \sim \tfrac{1}{2} l\left(\frac{1}{1-x}\right),$$

$$\Sigma \frac{x^n}{1-x^n} \sim \frac{1}{1-x}\, l\left(\frac{1}{1-x}\right).$$

As an example of a more precise result we may quote the formula

$$\Sigma \frac{x^n}{1+x^{2n}} = \tfrac{1}{4}\left\{\frac{\pi}{l(1/x)} - 1\right\} + O\{(1-x)^\Delta\}.$$

For accounts of these results, and extensions in various directions, see Barnes, **2**; Borel, **4**; Bromwich, **2**; Hardy, **12**; Knopp, **2**, **3**, **4**; Landau, **4**; Lasker, **1**; Le Roy, **1**; Pringsheim, **8**.

6.41. The increase of real solutions of algebraic differential equations. Suppose that the differential equation

$$(6.411) \qquad f(x, y, y') \equiv \Sigma A x^m y^n y'^p = 0$$

possesses a solution $y = y(x)$ which is real and continuous for $x > x_0$. The problem is to specify as completely as possible the various ways in which y may behave as $x \to \infty$.

This problem was first attacked by Borel (**7**), who proved that the equation cannot have a solution y such that

$$y > e^{e^x} = e_2(x)$$

for values of x surpassing all limit. Borel also stated the corresponding theorem for equations of the second order, viz. that no continuous solution can exceed $e_3(x)$ for values of x surpassing all limit; but his proof is incomplete, and no rigorous proof has yet been found, though there can be little doubt of the truth either of this or the corresponding general theorem for equations of any order.

Later Lindelöf (**1**) returned to the questions raised by Borel, and proved a much more precise result, viz.: *if the equation* (6.411) *is of degree m in x, then*

$$y < e^{\Delta x^{m+1}}$$

for some Δ *and for* $x > x_0$. Further, he proved that either $|y| < e^{x^\delta}$ for

$x > x_0$, or $e^{x^{\rho-\delta}} < |y| < e^{x^{\rho+\delta}}$ for a positive ρ and for $x > x_0$. The solutions of the first class may oscillate, but those of the second are ultimately monotonic, together with all their derivatives.

It is possible to prove a good deal more than this about the equation (6.411)*. Here however we consider only the special equation

$$(6.412) \qquad y' = P(x, y)/Q(x, y),$$

where P and Q are polynomials. We prove first that y' is ultimately of constant sign, so that *every solution is ultimately monotonic.*

Suppose the contrary. Then the curves $y = y(x)$, $P = 0$ intersect at points corresponding to an infinity of values of x surpassing all limit. But $P = 0$ consists of a finite number of branches, and so $y = y(x)$ must intersect at least one of these infinitely often.

Now the branches of $P = 0$, which extend to infinity in the direction of the axis of x, consist of (i) a finite number of straight lines $y = c_s$, and (ii) a finite number of branches $y = Y_t(s)$ along which y ultimately increases or decreases. And, in the first place, $y = y(x)$ cannot cut $y = Y_t(x)$ infinitely often. For suppose, for example, that Y_t is ultimately increasing, and that R and S are two successive points of intersection†. Then $y = y(x)$ crosses $y = Y_t(x)$ at R and S, and in each case from above to below, and this is plainly impossible.

We have next to consider the possible intersections of $y = y(x)$ with the straight lines (i), and we may suppose x so large that all intersections with branches (ii) have already been exhausted, so that y' can vanish only at the intersections we are considering. Then y cannot have a maximum or minimum; for at such a point y' would change sign, while P would not, since the line (i) through the point would be the tangent to the point. Hence $y = y(x)$ crosses the tangent and, having crossed it, it cannot return to it without passing through a maximum or minimum. It follows that there is at most a finite number of the intersections in question. Thus y is ultimately monotonic.

6.42. We can go further and prove the following lemma.

Lemma. *Any rational function*

$$H(x, y) = K(x, y)/L(x, y)$$

is ultimately monotonic along the curve $y = y(x)$, *unless* $L = 0$ *is a solution of the equation* (6.411).

We have

$$\frac{dH}{dx} = \frac{\partial H}{\partial x} + \frac{P}{Q}\frac{\partial H}{\partial y} = \frac{U}{W},$$

* Hardy, **10**. See also Boutroux, **1**, 217.

† There must be *successive* points, for all intersections are isolated: see Hardy, **10**.

where U and W are polynomials, and d/dx implies differentiation along the curve $y = y(x)$. If dH/dx is not ultimately of constant sign on the curve, it must vanish or become infinite infinitely often on it. In the first case the curve must have an infinity of intersections with at least one of the finite number of branches of $U = 0$. This branch C may, for sufficiently large values of x, be represented in the form

$$(6.421) \qquad y = A_0 x^{\alpha_0} + A_1 x^{\alpha_1} + \dots,$$

a convergent series of (not generally integral) descending powers of x; and, if $\delta/\delta x$ refers to differentiation along C, then

$$(6.422) \qquad y_1 = \frac{\delta y}{\delta x} = A_0 \alpha_0 x^{\alpha_0 - 1} + A_1 \alpha_1 x^{\alpha_1 - 1} + \dots$$

Again, along C, $R(x, y)$ is an algebraic function of x, which may, for sufficiently large values of x, be expressed in the form

$$(6.423) \qquad R = B_0 x^{\beta_0} + B_1 x^{\beta_1} + \dots,$$

another series of descending powers; and, unless the series (6.422), (6.423) are identical, we shall have $y_1 > R$ or $y_1 < R$ at all points of C from some definite point onwards. From this it follows that, at the points of intersection, C always crosses $y = y(x)$ from one and the same side to the other and the same side, which is plainly impossible.

On the other hand, if the series (6.422) and (6.423) are identical, we have $y_1 = R$, and $U = 0$ is a solution of (6.411). In other words, H is constant along $y = y(x)$.

There remains only the possibility that

$$\frac{dH}{dx} = \left(L\frac{dK}{dx} - K\frac{dL}{dx}\right) \Big/ L^2$$

should become infinite infinitely often, as we describe $y = y(x)$. This cannot be true owing to K or L or

$$\frac{dK}{dx} = \frac{\partial K}{\partial x} + \frac{\partial K}{\partial y}\frac{dy}{dx}$$

or dL/dx becoming infinite, and so can only occur if L vanishes infinitely often. But then we can show as above that $L = 0$ is a solution of the equation (6.411). Thus the proof of the lemma is completed. As a corollary we see that *any rational function $H(x, y, y')$ is ultimately monotonic, unless its denominator vanishes identically in virtue of* (6.411).

6.43. We can now obtain very accurate information concerning the increase of the solutions of (6.411). We write (6.411) in the form $Qy' - P = 0$. The ratio of any two terms is of one of the forms

$$Ax^m y^n, \quad Ax^m y^n y',$$

where A is a constant, and is ultimately monotonic; and so, between any two terms X_i, X_j, there subsists one of the relations

$$X_i \succ X_j, \quad X_i \asymp X_j, \quad X_i \prec X_j.$$

It follows that there must be one pair of terms at any rate such that $X_i \asymp X_j$. If both or neither of X_i, X_j contain y', we obtain at once

$$(6.431) \qquad y \sim Ax^s,$$

where s is rational. If one only contains y', we obtain a relation

$$(6.432) \qquad y^m y' \sim Ax^n.$$

Here four cases present themselves. If $m \neq -1$, $n \neq -1$, we obtain a relation of the type (6.431). If $m \neq -1$, $n = -1$, we obtain

$$(6.433) \qquad y \sim A (\log x)^{1/p},$$

where p is an integer. If $m = -1$, $n \neq -1$, we obtain a relation

$$\log y \sim Ax^p,$$

$$(6.434) \qquad y = e^{Ax^p(1+\epsilon)}.$$

Here p may be supposed a positive integer, as $y \asymp 1$ if p is negative or zero*. Finally, if $m = -1$, $n = -1$, we obtain

$$\log y \sim A \log x,$$

$$(6.435) \qquad y = x^{A+\epsilon}.$$

This last form of y includes both (6.431) and (6.433) as special cases, since in the latter case $y = x^\epsilon$. We have thus proved

Theorem 35. *Any continuous solution of* (6.411) *is ultimately monotonic, and of one of the forms*

$$e^{Ax^p(1+\epsilon)}, \quad x^{A+\epsilon},$$

where p is a positive integer.

It is possible to go a good deal further. All derivatives of y are ultimately monotonic, and y satisfies one of the relations

$$y \sim Ax^a e^{\Pi(x)}, \quad y \sim A (x^p \log x)^{1/q},$$

where $\Pi(x)$ is a polynomial and p and q are integers.

For fuller developments see Hardy, **10**. For analogous investigations of equations of the second order, for which the possibilities are much more complex, see Fowler, **1**, **2**. These memoirs contain many additional references to the literature of the subject.

6.5. Oscillating Dirichlet's Integrals. The theory of Fourier series, when developed according to the ideas initiated by Dirichlet,

* p is clearly at most equal to $r+1$, where r is the degree of (6.411) in x: this, of course, agrees with Lindelöf's result quoted in § 6.41.

depends on Dirichlet's integral

$$(6.51) \qquad J(\lambda) = \int_0^{\xi} \frac{\sin \lambda x}{x} f(x)\, dx \quad (\xi > 0),$$

which has, under appropriate conditions, the limit $\frac{1}{2}\pi f(+0)$ when $\lambda \to \infty$.

A very interesting problem in the theory is that of finding asymptotic formulae for $J(\lambda)$ when

$$f(x) = \rho(x)\, e^{i\sigma(x)},$$

ρ and σ are L-functions, and $\sigma \to \infty$ when $x \to 0$. This problem was first attacked by du Bois-Reymond (**6**), who enunciated a number of striking theorems, but whose analysis is very inconclusive, and so obscure that it is almost impossible to distinguish between what he proved and what he did not*. The problem was reconsidered more recently by the author†, who obtained more definite results, and these results were afterwards completed in various respects by Kuniyeda‡.

In stating these results we assume throughout that $\rho \prec \sigma'$, this being the necessary and sufficient condition for the existence of the integral $J(\lambda)$. There are three cases which have to be distinguished, those in which

$$(A)\ \sigma \prec l\left(\frac{1}{x}\right), \qquad (B)\ \sigma \asymp l\left(\frac{1}{x}\right), \qquad (C)\ \sigma \succ l\left(\frac{1}{x}\right);$$

and the main theorems are as follows. The proofs are too elaborate for reproduction here.

Theorem 36. *If $\sigma \prec l(1/x)$ and $\rho = x^{-a}\Theta(x)$, where $x^{\delta} \prec \Theta \prec x^{-\delta}$, so that $a \leqq 1$, then*

$$J(\lambda) = O(\lambda^{-1+\delta}) \quad (a \leqq -1),$$

$$J(\lambda) \sim -\Gamma(-a) \sin \tfrac{1}{2} a\pi\, \rho\left(\frac{1}{\lambda}\right) e^{i\sigma(1/\lambda)} \quad (-1 < a < 1),$$

$$J(\lambda) \sim \lambda T\left(\frac{1}{\lambda}\right) \quad (a = 1),$$

where

$$T(x) = \int_0^{\lambda} \rho(t)\, e^{i\sigma(t)}\, dt$$

and $-\Gamma(-a) \sin \frac{1}{2} a\pi$ is to be replaced by $\frac{1}{2}\pi$ when $a = 0$.

Theorem 37. *If $\sigma \sim bl(1/x)$ then*

$$J(\lambda) = O(\lambda^{-1+\delta}) \quad (a \leqq -1),$$

$$J(\lambda) \sim -\Gamma(-a - bi) \sin \tfrac{1}{2}(a + bi)\pi\, \rho\left(\frac{1}{\lambda}\right) e^{i\sigma(1/\lambda)} \quad (-1 < a \leqq 1).$$

* du Bois-Reymond asks only whether $J(\lambda)$ does or does not tend to a limit, and does not attempt to find asymptotic formulae in the case of oscillation.

† Hardy, **11**. ‡ Kuniyeda, **1**.

Theorem 38. *If* $l(1/x) \prec \sigma \prec (1/x)^{\Delta}$ *and* $\rho \prec x\sigma'$, *then*

$$J(\lambda) = O\left(\frac{1}{\lambda}\right)$$

if $\rho \preccurlyeq x\sqrt{\sigma''/\sigma'}$, *and*

$$J(\lambda) \sim \sqrt{(\tfrac{1}{2}\pi)}\, e^{(\beta - \frac{1}{4}\pi)i} \frac{\rho(\theta)}{\theta\sqrt{\{\sigma''(\theta)\}}}$$

if $x\sqrt{\sigma''/\sigma'} \prec \rho \prec x\sigma'$. *Here* $\beta = \lambda\theta + \sigma(\theta)$, *and* θ *is determined as a function of* λ *by* $\sigma'(\theta) + \lambda = 0$.

These theorems are stated in the form finally given to them by Kuniyeda. It should be observed that they are still not quite complete. No asymptotic formula has been obtained when $\sigma \succ (1/x)^{\Delta}$, and no account is taken, in Theorem 38, of the range $x\sigma' \preccurlyeq \rho \prec \sigma'$*. There is also room for a more accurate determination of the first formula of Theorems 36 and 37.

Kuniyeda has also investigated the integral $K(\lambda)$ in which $\cos \lambda x$ appears instead of $\sin \lambda x$, the results being of the same character. This integral appears in the theory of the trigonometrical series conjugate to the Fourier series of $f(x)$, and in the theory of power-series on the circle of convergence.

Apart from the work of du Bois-Reymond, special cases of the problem had already been considered by Darboux, Hamy, and Fejér†. In particular Fejér determined the asymptotic formula

$$a_n \sim \frac{1}{\sqrt{(e\pi)}} n^{-\frac{3}{4} + \frac{1}{2}p} \sin(2\sqrt{n} + \tfrac{3}{4}\pi - \tfrac{1}{2}p\pi)$$

for the coefficients in the power series

$$f(x) = (1-x)^{-p} e^{-1/(1-x)} = \Sigma a_n x^n.$$

6.61. Arithmetic applications. The classification of irrational numbers. We conclude with a brief sketch of some of the most important applications of the theory in arithmetical directions. These applications bear primarily on problems connected with the classification of irrational numbers.

An *algebraic number of degree k* is a root of an irreducible equation

$$(6.611) \qquad f(x) = a_0 x^k + a_1 x^{k-1} + \dots + a_k = 0,$$

in which the coefficients are rational integers without common factor. If $a_0 = 1$, x is an *integer*. A number which is not algebraic is *transcendental.*

In what follows we confine our attention to real numbers. The aggregate of algebraic numbers is enumerable, and there are therefore transcendental numbers in every interval of the continuum‡. The

* See Kuniyeda, **1**, 35.

† Darboux, **1**; Hamy, **1**; Fejér, **1**, **2**.

‡ Cantor, **1**. For accounts of the relevant parts of Cantor's theory see Borel, **1**; Hausdorff, **1**; Hobson, **2**; Jourdain, **1**.

theory of aggregates establishes in this manner the existence of transcendental numbers, but does not suggest directly any method for constructing them. Such a construction was first effected by Liouville*, by means of the following theorem.

Theorem 39. *If x is an algebraic number of degree k, and p/q is a rational number not equal to x†, then there is a number $M = M(x)$, independent of q, such that*

$$\left|x - \frac{p}{q}\right| > \frac{1}{Mq^k}. \tag{6.612}$$

Suppose, as plainly we may, that there is no better approximation to x, with denominator q, than p/q. Then p/q differs from x by less than $1/q$, and $|f'(y)|$ has, in the interval $|y - x| \leqq 1/q$, an upper bound $\frac{1}{2}M$ independent of q‡. But

$$f\left(\frac{p}{q}\right) = f\left(\frac{p}{q}\right) - f(x) = \left(\frac{p}{q} - x\right) f'(y),$$

where y lies between x and p/q, and so

$$\left|x - \frac{p}{q}\right| > \frac{1}{M}\left|f\left(\frac{p}{q}\right)\right|.$$

As $|f(p/q)|$ is a rational number whose denominator is q^k and whose numerator is at least 1, the theorem follows. It is plain that

$$\left|x - \frac{p}{q}\right| > \frac{1}{q^{k+1}} \tag{6.613}$$

for all sufficiently large values of q.

Liouville's theorem shows in effect that *it is impossible to approximate to an algebraic number by rationals with more than a certain accuracy.* On the other hand it is easy to write down particular irrationals which possess rational approximations of any degree of accuracy whatever. Suppose for example that ϕ_n is an increasing function of n, integral for every integral n, and let

$$x = 10^{-\phi_1} + 10^{-\phi_2} + \dots + 10^{-\phi_n} + \dots.$$

If p_n/q_n is the sum of the first n terms of the series, so that $q_n = 10^{-\phi_n}$, then

$$0 < x - \frac{p_n}{q_n} = 10^{-\phi_{n+1}} + 10^{-\phi_{n+2}} + \dots \leqq \frac{10}{9} 10^{-\phi_{n+1}}$$

* Liouville, **2**.

† This provision is naturally only necessary when $k = 1$.

‡ We have certainly, for example,

$$\tfrac{1}{2}M \leqq k\,|a_0|\,(|x| + 1)^{k-1} + \dots + |a_{k-1}|.$$

and

$$(6.614) \qquad \left| x - \frac{p_n}{q_n} \right| \leqq \frac{10}{9} q_n^{-\chi_n},$$

where $\chi_n = \phi_{n+1}/\phi_n$. If $\chi_n \to \infty$, (6.613) and (6.614) are contradictory, so that x is transcendental. We may for example take $\phi_n = n!$.

Cantor's theory shows that transcendental numbers exist, and Liouville's theorem enables us to produce examples of them. To prove that a particular number, arising independently in analysis, is transcendental, or even irrational, is in general a far more difficult problem. It has never been proved, for example, that $2^{\sqrt{2}}$, e^{π}, or Euler's constant γ are irrational.

There are a few classes of numbers, such as $\sqrt{2}$, $\sqrt{n}$, $\sqrt[3]{2}$, e, $\log_{10} 2$, ..., whose irrationality is classical: see for example Hardy, **1**, 6, 380, 387. For the irrationality of π, first proved by Lambert (**1**), and π^2, see Perron, **1**, 254; Vahlen, **1**, 319. The problem of proving that $\sqrt[3]{2}$ is not expressible by any finite combination of quadratic surds is famous historically: see Enriques, **1**; Hudson, **1**; Klein, **1**. For an elementary proof that e is not quadratic, see Vahlen, **1**, 325. The transcendentality of e was first proved by Hermite (**1**), and that of π by Lindemann (**1**); full accounts of these problems are given by Enriques and Klein, and also by Hessenberg, **1**; Hobson, **3**; Perron, **2**. See also Maillet, **1**.

6.62. In the preceding construction, there is naturally no special merit in the number 10. We may use any other scale; and we may also employ other representations of irrationals, for example by continued fractions. The number

$$x = \frac{1}{a_1 +} \frac{1}{a_2 +} \cdots$$

will certainly be transcendental if a_n increases with sufficient rapidity, for, if p_n/q_n is the nth convergent, a_n' the complete quotient corresponding to a_n, and $q_n' = a_n' q_{n-1} + q_{n-2}$, we have

$$\left| x - \frac{p_n}{q_n} \right| = \frac{1}{q_n q'_{n+1}} < \frac{1}{a_{n+1} q_n^2};$$

and, in order to obtain a contradiction with (6.613), it is only necessary to suppose that $a_{n+1} \succ q_n^{\Delta}$ or, what is equivalent, that $q_{n+1} \succ q_n^{\Delta}$. It is easily proved that this is so whenever $a_{n+1} \succ a_n^{\Delta}$. Thus we might take

$$a_1 = 1, \ a_2 = 2^{a_1} = 2, \ \ldots, \ a_{n+1} = 2^{a_n}, \ \ldots.$$

When $k=1$ or $k=2$, Liouville's theorem is, in a sense, final: it is not possible to replace the q^k on the right hand side by any lower power of q. When $k > 2$, more is true: thus Thue (**1**) proved that

$$\left| x - \frac{p}{q} \right| > \frac{1}{M q^{\frac{1}{2}k+1+\epsilon}},$$

where $M=M(x, \epsilon)$, and Siegel (1) that

$$\left|x-\frac{p}{q}\right|>\frac{1}{Mq^{2\sqrt{k}}},$$

where $M=M(x)$. The index assigned by Siegel's theorem is better if $k>11$. The problem of finding the best possible index is unsolved, except when k is 1 or 2.

When $k=2$, the continued fraction for x is periodic, so that $a_n=O(1)$. It is natural to ask whether anything can be said about the order of a_n when x is an algebraic number of higher degree. It is easy to deduce from Liouville's theorem that $a_n \prec e_2(an)$, where a is a number depending on k, and similar deductions can be drawn from Thue's and Siegel's theorems. What can be proved in this way amounts to very little, and it is very unlikely that it is anywhere near the ultimate truth.

6.63. Although so little is known about the order of magnitude of a_n for particular classes of irrationals, very interesting results have been found concerning what may be called its 'usual' order of magnitude. We may say that x has *usually* the property P, or that P is usually true, if the set of values of x for which P is false has measure zero. If then ϕ_n is an increasing function of n, and we write $\phi_n=k_n$ or $\phi_n=d_n$, according as $\Sigma(1/\phi_n)$ is convergent or divergent, then

$$a_n<k_n \qquad (n>n_0)$$

is usually true, and

$$a_n<d_n \qquad (n>n_0)$$

is usually false*. Thus $a_n<n(ln)^2$ is usually true, and $a_n<n\,ln$ is usually false†.

It is easily proved that, if

$$\left|x-\frac{p}{q}\right|<\frac{1}{q\phi_q} \tag{6.631}$$

for an infinity of values of q, and $\phi_q=k_q$, then $a_n>k_n$ for an infinity of values of n‡. Hence (6.631) is in this case usually false.

We may ask generally for what irrationals (6.631) is infinitely often true. The results known in this direction are as follows. If ϕ_q is a constant C, and $C \leqq \sqrt{5}$, then (6.631) is *always* true (for an infinity of values of q). If

$$\sqrt{5}<C\leqq 2\sqrt{2},$$

then (6.631) is true except for irrationals equivalent§ to

$$a=\frac{1}{1+}\,\frac{1}{1+}\cdots.$$

If $2\sqrt{2}<C<3$, then (6.631) is true except for the numbers equivalent to one or other of a finite number of quadratic surds. If $C\geqq 3$, it is usually true, but the exceptions are non-enumerable. It is still usually true if ϕ_q is an increasing function whose increase is sufficiently slow; but it is usually false when

* That is to say, it is usually true that $a_n>d_n$ for an infinity of values of n.

† Borel, **9**; Bernstein, **1**.

‡ Here k_n is some function of n such that $\Sigma(1/k_n)$ is convergent. It is not the same function of n that k_q is of q.

§ *I.e.* numbers $(a\alpha+b)/(c\alpha+d)$, where a, b, c, d are integers and $ad-bc=1$.

$\phi_q = k_q$. For fuller information see Borel, **5**; Bohr and Cramér, **1**; Grace, **1**; Heawood, 1; Hermite, 1; Hurwitz, 1; Markoff, **1**; Minkowski, **1**; Perron, 2.

Yet another closely allied problem is that of the distribution of the numbers (nx), where x is irrational, and $(u)=u-[u]$, in the interval (0, 1). The fundamental theorem, due to Kronecker, is that the numbers lie everywhere dense in the interval. There are many memoirs concerned with this theorem and its extensions. See Behnke, **1**; Bohr, **1**; Bohr and Cramér, **1**; Hardy and Littlewood, **3**; Hecke, **1**; Kronecker, **1**, **2**; Lettenmeyer, **1**; Minkowski, **1**; Ostrowski, **1**; Weyl, **1**.

6.64. Applications to the theory of convergence. Liouville's theorem, and the other theorems of which we have spoken, have many interesting applications to the theory of convergence of series.

The typical problem is that of the convergence of the series

$$\Sigma \frac{\phi_n}{|\sin n\pi x|}, \tag{6.641}$$

where ϕ_n is a decreasing function of n and x is irrational. If p_ν/q_ν is a convergent to x, then

$$|\sin q_\nu \pi x| < \frac{A}{q_{\nu+1}} < \frac{A}{a_{\nu+1} q_\nu},$$

where the A's are constants, and the increase of $a_{\nu+1}$, regarded as a function of q_ν, may be as rapid as we please. It follows that (6.641) *is divergent, for appropriate values of x, however rapid the decrease of ϕ_n may be.*

If x is an algebraic number of degree k, then, by (6.612),

$$|\sin n\pi x| > \frac{B}{n^{k-1}}$$

where B is a positive function of x only, for all values of n. Hence (6.641) is convergent whenever $\phi_n < n^{-a}$ and $a > k$: this result can naturally be improved upon by the use of Thue's and Siegel's theorems. Thus

$$\Sigma \frac{n^{-2-\delta}}{|\sin n\pi x|}$$

is convergent for all quadratic x, and

$$\Sigma \frac{e^{-\delta n}}{|\sin n\pi x|}$$

is convergent for all algebraic x. The 2 in the first of these results may in fact be replaced by 1, but a more elaborate proof is needed. It also follows from the results of § 6.63 that (6.641) is usually convergent if $\Sigma k_q \phi_q$ is convergent, as for example if $\phi_q = q^{-2}(\log q)^{-4}$, when we may take $k_q = q(\log q)^2$.

The series $\Sigma z^n \operatorname{cosec} n\pi x$ may, according to the arithmetic nature of x, represent an integral function of z, or a function regular inside a circle which is a line of singularities of the function; or again it may diverge for all values of z.

The theory of the non-absolute convergence of such a series as $\Sigma \phi_n \operatorname{cosec} n\pi x$ is naturally more intricate.

For fuller information see Hardy, **5**; Hardy and Littlewood, **3** (3); Lerch, **1**; Riemann, **2**; Smith, **1**. Analogous questions concerning integrals are discussed by Hardy, **3** (5).

APPENDIX

SOME NUMERICAL ILLUSTRATIONS*

1. *Table of the functions* $\log x$, $\log\log x$, $\log\log\log x$, *etc.*

x	$\log x$	$\log_2 x$	$\log_3 x$	$\log_4 x$	$\log_5 x$
10	2·30	0·834	−0·182	—	—
10^3	6·91	1·933	0·659	−0·417	—
10^6	13·82	2·626	0·966	−0·035	—
10^{10}	23·03	3·137	1·143	0·134	−2·011
10^{15}	34·54	3·542	1·265	0·235	−1·449
10^{20}	46·05	3·830	1·343	0·295	−1·221
10^{30}	69·08	4·235	1·443	0·367	−1·003
10^{60}	138·15	4·928	1·595	0·467	−0·762
10^{100}	230·26	5·439	1·693	0·527	−0·641
10^{1000}	2302·58	7·742	2·047	0·716	−0·334
10^{10^6}	2303×10^3	14·650	2·685	0·987	−0·013
$10^{10^{10}}$	2303×10^7	23·860	3·172	1·154	0·144

2. *Table of the functions* e^x, e^{e^x}, $e^{e^{e^x}}$, *etc.*

x	ex	$e_2 x$	$e_3 x$	$e_4 x$
1	2·718	15·154	3,814,260	$10^{1,656,510}$
2	7·389	1618·2	$5·85 \times 10^{702}$	—
3	20·085	$5·28 \times 10^8$	$10^{2·295 \times 10^8}$	—
5	148·413	$2·85 \times 10^{64}$	$10^{1·21 \times 10^{64}}$	—
10	22026	$9·44 \times 10^{9565}$	—	—

The function $\log x$ is defined only for $x > 0$, $\log_2 x$ for $x > 1$, $\log_3 x$ for $x > e$, $\log_4 x$ for $x > e^e = e_2$, and so on. The values of the first few numbers e, e_2, e_3, ... are given above, viz. $e = 2·718$, $e_2 = 15·154$, $e_3 = 3,814,260$, $e_4 = 10^{1,656,510}$.

* The tables in this appendix were calculated by Mr J. Jackson.

3. *Table to illustrate the convergence of the series*

(1) $\sum_{3}^{\infty} \frac{1}{n\, ln\, (lln)^2}$. (2) $\sum_{2}^{\infty} \frac{1}{n\, (ln)^2}$. (3) $\sum_{1}^{\infty} \frac{1}{n^s}$. (4) $\sum_{0}^{\infty} x^n$.

(5) $\sum_{0}^{\infty} \frac{1}{n!}$. (6) $\sum_{1}^{\infty} \frac{1}{n^n}$ (7) $\sum_{0}^{\infty} x^{n^2}$. (8) $\sum_{1}^{\infty} n^{-n^n}$.

Series	Sum	Number of terms required to calculate the sum correctly to 2	10	100	1000
			decimal places		
1	38·43	$10^{3\cdot14\times10^{86}}$	——	——	——
2	2·11	$7\cdot23\times10^{86}$	$10^{8\cdot6\times10^{9}}$	——	——
$3(s=1\cdot1)$	10·58	10^{33}	10^{113}	10^{1013}	10^{10013}
$3(s=1\cdot5)$	2·612	160,000	16×10^{20}	16×10^{200}	16×10^{2000}
$3(s=2)$	$\frac{1}{6}\pi^2=1\cdot64493$	200	2×10^{10}	2×10^{100}	2×10^{1000}
$3(s=10)$	1·0009846	1	11	$1\cdot093\times10^{11}$	$1\cdot093\times10^{111}$
$3(s=100)$	$1+(1\cdot27\times10^{-30})$	1	1	10	$1\cdot213\times10^{10}$
$4(x=\cdot9)$	10	73	247	2214	21883
$4(x=\cdot3)$	2	9	36	336	3325
$4(x=\cdot1)$	10/9	3	11	101	1001
5	$e-1=1\cdot718282$	5	13	70	440
6	1·291286	3	10	57	386
$7(x=\cdot9)$	3·234989	8	15	46	148
$7(x=\cdot5)$	1·564468	3	6	19	58
$7(x=\cdot1)$	1·100100	2	4	11	32
8	1·062500	2	2	3	4

The phrase 'calculate the sum correctly to m decimal places' is used as equivalent to 'calculate with an error less than $\frac{1}{2}\times10^{-m}$'. In the case of a very slowly convergent series the interpretation affects the numbers to a considerable extent. The numbers would be considerably more difficult to calculate were the phrase interpreted in its literal sense.

Such a series as $3\,(s=100)$ is of course exceedingly rapidly convergent *at first*, *i.e.* a very few terms suffice to give the sum correctly to a considerable number of places; but if the sums are wanted to a very large number of places, even the series $4\,(x=\cdot9)$ proves to be far more practicable.

4. *Table to illustrate the divergence of the series*

(1) $\frac{1}{\log\log 3}+\frac{1}{\log\log 4}+\dots$ (2) $\frac{1}{\log 2}+\frac{1}{\log 3}+\dots$

(3) $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\dots$ (4) $1+\frac{1}{2}+\frac{1}{3}+\dots$

(5) $\frac{1}{2\log 2}+\frac{1}{3\log 3}+\dots$ (6) $\frac{1}{3\log 3\log\log 3}+\frac{1}{4\log 4\log\log 4}$.

Series	Number of terms required to make the sum greater than					
	3	5	10	100	1000	10^6
1	1	1	1	116	1800	$2{\cdot}6\times10^6$
2	3	7	20	440	7600	$1{\cdot}5\times10^7$
3	5	10	33	2500	$2{\cdot}5\times10^5$	$2{\cdot}5\times10^{11}$
4	11	82	12390	10^{43}	$10^{{\cdot}43\times10^3}$	$10^{{\cdot}43\times10^6}$
5	8690	$1{\cdot}3\times10^{29}$	10^{4300}	$10^{5\times10^{42}}$	——	——
6	1	60 *to* 70	$10^{10^{100}}$	——	——	——

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