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Priredili Žarko Mijajlović i Slaviša Milisavljević

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SADRŽAJ

ON AN INEQUALITY CONCERNING CARTESIAN MULTIPLICATION

D. KUREPA

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1. For a family F of sets let DF be the supremum of the cardinal numbers of disjointed subfamilies of F. Let F^{I2} be the set of all the cartesian products $X \times Y$ with $X, Y \in F$. Analogously, for any ordinal number r let Ir be the interval of the ordinals $< r$ and let F^{Ir} be the system of the cartesian products of all r-sequences of members of F.

2. For a space S let GS be the system of all the open sets of G; we put $DS = D(GS)$; $DS^{Ir} = D(G(S^{Ir}))$; the number DS is called the cellularity or disjunction degree of the space S.

The question arises to find the relations between the numbers DF^{Ir} $(r = 1, 2, \ldots)$ for any set family F and particularly for $F = GS$, S being any given topological space.

3. Let (G, ϱ) be a binary graph i. e. G is a set and ϱ is a binary reflexive and symmetrical relation in G. Let I be a non void set and for every $i \in I$ let (G_i, ϱ_i) be a binary graph; we define the product of the graphs (G_i, ϱ_i) as (G, ϱ), where $G = \prod G_i$ and where for $x, y \in G$ the relation $x \varrho y$ means $\bigwedge_i x_i \varrho_i y_i$, i. e. for every $i \in I$ one has $x_i \varrho_i y_i$ (let us remind that $x \in \prod G_i$ means that x is a mapping of I such that $x_i \in G_i$ for every $i \in I$). Let $k_c(G, \varrho)$ (resp. $k_{\bar{c}}(G, \varrho)$ or $k_a(G, \varrho)$) be the supremum of the cardinal numbers of chains (resp. antichains) of (G, ϱ).

The problem arises to find the connections between the numbers $k_a G^{Ir}$ and $k_a G$.

4. Theorem. *For any set system F with infinite DF one has: $(DF)^n \leqq DF^{In} \leqq 2^{DF}$ for any natural number n.* (II) *For any ordinal α there is a system F_α of sets such that $DF_\alpha = \aleph_\alpha$, $DF^{I2} = 2^{\aleph_\alpha}$, and consequently $DF_\alpha < D(F_\alpha^{I2})$.*

5. Theorem. *For any binary graph (G, ϱ) one has $(k_a G)^n \leqq k_a G^{I2} \leqq 2^{k_a G}$; if $k_a G \geqq \aleph_0$, then $k_a G^{In} \leqq 2^{k_a G}$ for every natural number n.*

6. Theorem. *For any metrical infinite space S and any positive integer n one has $k_a S = k_a S^{In}$.*

7. Theorem. *For totally ordered sets O the relation (1) $k_a O = k_a O^{I2}$ is equivalent to the following reduction principle: Every infinite ramified set R of regular cardinality kR contains a degenerated subset D of cardinality kR (any ordered set O*

in which every principal ideal $O(.,x) = \{y; y < x; y \in O\}$ *is a chain is said to be ramified; O is degenerated if both: principal ideals and dual principal ideals of O are chains). The relation* (1) *is connected to the well-known Suslin problem.*

8. Problem. As yet one does not know any topological infinite space S satisfying $DS < DS^{I2}$; the problem is to exhibit such a space.

Dragan Žarku Repovšu
S najboljim željama
6.6.1973. Kurepa

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THE CARTESIAN MULTIPLICATION AND THE CELLULARITY NUMBER

Đuro Kurepa

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1. Introduction

There are many questions in connexion with the cartesian multiplications of sets, structures etc. In particular, the question is to find how some property of the cartesian product is induced by the analogous property of the factors. Some classical facts show that big differences may occur between the factors and the product. E. g. the problem of measure on the line R and in the square (plane) R^2 are of a different kind then the problem of the measure in the space R^3 or in spaces R^n for $n>2$. The problem whether the cardinality $k\,E^2$ of infinite sets E equals $k\,E$ is equivalent to the choice axiom.

In this article we shall examine a particular number — the cellularity number $c\,X$ ($=\text{cel}\,X$) where X is any family of sets, any topological space or structure in order to see how the cellularity of product depends on the cellularity of the factors. At the same time, we shall become aware how the complete answer to the problem is connected with the tree hypothesis, and with the general continuum hypothesis. At this opportunity it is interesting to observe that the chain $\times$ antichain hypothesis holds for every square or hypersquare of every tree or ramified set.

In the present first part of the paper main results are contained in theorems 3.4; 3.7; 4.10; 5.4; 5.6; 5.8 and 6.4.

2. Cellularity of a system of sets

2.1. For any system S of sets we define the cellularity $c = c\,S = cel\,S$ of S by the relation

$$c\,S = \sup_{\Phi} k\,\Phi, \tag{1}$$

Φ consisting of pairwise disjoint sets belonging to S; $k\,\Phi$ means the cardinality of Φ.

2.2. For any space E the cellularity $c\,E$ of E is defined as the cellular number of the family of all open sets of the space E.

2.3. For any totally ordered set O the cellularity $c\,O$ of O is defined as the cellular number of the system of sets of the form

$$O\,[x,\,.\,)=\{y;\ x\leqslant y;\ \{x,y\}\subseteq O\}$$

$$O\,(\,.\,,x]=\{y;\ y\leqslant x;\ \{x,y\}\subseteq O\}$$

$$O\,(x,y)=\{z;\ x<z<y \text{ or } x<z<y;\ \{x,y,z\}\subseteq O\}.$$

3. Cartesian multiplication of sets and of families of sets, respectively

3.1 Definition. Let I be any set of >1 elements (the members of I shall serve as indices) ; for any $i\in I$ let X_i be a nonvoid set; *the cartesian product of the sets* X_i *is the set* $Y=\prod_{i\in I} X_i$ *consisting of all the single-valued functions* f *on* I *such that for every* $i\in I$ *one has* $f(i)\in X_i$;
i. e.

$$Y=\prod_{i\in I} X_i=\{f;\ f:I\to\bigcup_{i\in I} X_i,\ f(i)\in X_i\};$$

for $f,g\in Y$ one defines $f=g\Leftrightarrow f(i)=g(i)\ (i\in I)$.

In particular, for any ordinal number α one defines the hypercube X^α as the set of all the α-sequences of members in X; X^0 means the empty set.

3.2. Let I be a non void set and F a mapping on I such that, for every $i\in I$, F_i be a nonempty family of non void sets; the cartesian product of the families F_i is the family $F^{\cdot I}$ of the products $\prod_{i\in I} X_i$ where $X_i\in F_i\ (i\in I)$. In particular, for any ordinal α and any family F of sets one defines $F^{\cdot I}=F^{\cdot I\alpha}=\{\prod_i X_i;\ X_i\in F,\ i<\alpha;\ I_\alpha$ means the set of all the ordinal numbers $<\alpha\}$.

$F^{\cdot 2}$ consists of all the products $X_0\times X_1$ where X_0, X_1 run independently through F. One proves readily the following lemma.

3. Lemma. For every disjoint (antidisjoint) family F of sets one has $(c\,F)^{kI\alpha}=c\,(F^{\cdot I\alpha})$.

3.3. Definition. A family of sets is called *disjoint (antidisjoint)*, provided its members are pairwise disjoint (nondisjoint).

3.4. Main theorem. *(I) For any family* F *of sets and any natural number* n *the relation* $c\,F\geqslant\aleph_0$ *implies*

(1) $$(c\,F)^n\leqslant c\,(F^{\cdot n})\leqslant 2^{cF};$$

(II) For any ordinal number α *there exists a system* F *of sets such that*

(2) $$c\,F_\alpha=\aleph_\alpha,\quad (c\,F^{\cdot 2})=2^{\aleph_\alpha}$$

Therefore the evaluation in (1) *is a best one.*

3.5. Proof of the theorem 3.4. (I). 1. The first relation (1) is obvious because for every disjoint system d of sets in F we have the system $d^{\cdot r}$ in $F^{\cdot r}$ that is disjoint and of a cardinality $\geqslant kd$. Therefore, we have still to prove the second relation in (1). The proof will be carried out by induction relative to n.

First of all the relation (1) holds for $n=2$. The proof of this fact is quite characteristic. We have to prove that every disjoint system D of sets in $F^{\cdot I_2}$ is of a cardinality $\leqslant 2^{cF}$.

2. Now, let D be any disjoint system of the family $F^{\cdot I_2}$; this means that $X, Y \in D \Rightarrow X \cap Y = \varnothing$ or $X = Y$.

Now, $X = X_0 \times X_1$, $Y = Y_0 \times Y_1$, X_0, X_1, Y_0, Y_1 being elements of F. The relation

$$(X_0 \times X_1) \cap (Y_0 \times Y_1) = \varnothing$$

is equivalent to the disjunction

$$X_0 \cap Y_0 = \varnothing \vee X_1 \cap Y_1 = \varnothing .$$

Let $(D; \rho)$ be the binary graph supported by D and where the relation $X \rho Y$ means that $X_0 \cap Y_0 = \varnothing$ holds. Thus if C is a ρ-chain in $(D; \rho)$, then $C_0 = \{X_0;\ X \in C\}$ is a disjointed system of F and therefore $k C_0 \leqslant cF$. If $\bar{C}$ is an antichain in (D, ρ), then $\{X, Y\} \in \bar{C}$ implies the negation $X \rho' Y$ of $X \rho Y$ i. e. that $X_0 \cap Y_0 \neq \varnothing$ and consequently $X_1 \cap Y_1 = \varnothing$; this means that again $\bar{C}_1 = \{X_1;\ X \in \bar{C}\}$ is a disjointed system in F. Consequently, every chain as well as every antichain of (D, ρ) is $\leqslant cF$. In virtue of our graph theorem we have

$kD \leqslant 2^{cF}$ (cf. [4]. 3 Theorem 0.1 p. 82 and [4]. 4 Theorem 6.2.2).

This holding for every disjointed system D in $F^{\cdot 2}$ one has

$$\sup_D kD \leqslant 2^{cF} \quad \text{i. e.} \quad cF^{\cdot 2} \leqslant 2^{cF}.$$

Consequently, the theorem holds for $n=2$.

3. Now, suppose that r be any natural number >2 and the relation (1) holds for any natural number $n<r$; let us prove that (1) holds for $n=r$ too. Now, let D be any disjoint system in $F^{\cdot n}$; then for $X = (X_0 \times X_1 \times \ldots \times X_{r-1}) \in F$ and $Y = (Y_0 \times Y_1 \times \ldots \times Y_{r-1}) \in F$ the disjonction $X \cap Y = \varnothing$ means

$$(3) \qquad (X_0 \times X_1 \times \ldots \times X_{r-2}) \cap (Y_0 \times Y_1 \times \ldots \times Y_{r-2}) = \varnothing$$

or

$$X_{r-1} \cap Y_{r-1} = \varnothing .$$

With respect to the relation (3) the subset D of $F^{\cdot r}$ is a binary graph; by an argument like in 2 one proves that the induction hypothesis implies that every chain of this graph is $\leqslant 2^{cF}$ and that every antichain of the graph is $\leqslant cF$; in virtue of the graph theorem we infer that

$kD \leqslant (2^{cD})^{cD} = 2^{cD}$; this holding for every D, the operator sup yields (1). And this was to be shown.

3.6. Proof of the theorem 3.4. (II). Let α be any ordinal number and let

$$M = Q(\omega_\alpha)$$

be the system of all the ω_α — sequences of rational numbers ordered by the principle of the first differences: for any 2 different such sequences a, b let $i = i(a, b)$ be the ordinal such $a_{i'} = b_{i'}$ for every ordinal $i' < i$ and $a_i \neq b_i$; we put $a < b$ if and only if $a_i < b_i$.

3.6.1. Lemma. $kQ(\omega_\alpha) = \aleph_0^{\aleph_\alpha} (= 2^{\aleph_\alpha})$.

3.6.2. M is a chain with respect to the relation $<$ and every interval of M has $2^{\aleph_\alpha}$ points.

In fact let $a = (a_\nu)_\nu$, $b = (b_\nu)_\nu$ be 2 distinct elements of M; hence $i = i(a, b) < \omega_\alpha$ and either $a_i < b_i$ or $a_i > b_i$; if then c is any element of M such that $c_i \in Q(a_i, b_i)$, $i(a, c) = i(b, c)$, one has $c \in M(a, b)$; in particular, the ω_α-sequence $c_{i+1}, \ldots, c_{i+\omega'\alpha}$, might be any ω_α-sequence of rational numbers.

3.6.3. Lemma. *Any increasing (decreasing) sequence in M is of a cardinality* $\leqslant \aleph_\alpha$.

First of all the set M contains a ω_α-sequence as well as an $\omega^*{}_\alpha$-sequence; such are e. g. the sequences:

$$a^\zeta = \{1\}_\zeta \dot{+} \{0\}_{-\zeta+\omega_\alpha} \quad (\zeta < \omega_\alpha)$$

$$b^\zeta = \{0\}_\zeta \dot{+} \{1\}_{-\zeta+\omega_\alpha} \quad (\zeta < \omega_\alpha).$$

Further let us suppose that $(M, <)$ contains a well-ordered subset W of cardinality $> \aleph_\alpha$. In particular we might suppose that the type of W be $\omega_{\alpha+1}$. Now, every member x of W is a ω_α-sequence (x_ζ) with $x_\zeta \in Q$; for any pair x, y of distinct members of M let $i(x, y)$ be the first ordinal ν such that $x_\nu \neq y_\nu$. The ordinal $i(x, y)$ is like *a proximity degree (or dual distance)* between x, y and one proves readily that

$$(1) \qquad x < y < z \Rightarrow i(x, z) = \inf \{i(x, y),\ i(y, z)\}.$$

This relation is like triangular relation.

Consequently, for every member $x \in W$ we have the non decreasing monotone sequence

$$(2) \qquad i(x, y),\ (y \in W(x, .))$$

of ordinal numbers $< \omega_\alpha$; let $g(x)$ be the first $y > x$ in W such that

$i(x, gx)$ equals the infimum of the numbers (2).

In other words

$$(3) \qquad i(x, g(x)) = \inf\ i(x, y), \quad (y \in W(x, .)).$$

The relations (1) and (3) yield the following relation

$$(4) \qquad i(x, y) = i(x, g(x)),\ (y \in W(g(x), .)).$$

Geometrically, the relation (4) means that the terminating interval $W(gx, .)$ of W is located on the „sphere" $S(x, r^*)$, the center and the dual radius r^* of which are x and $r^* = i(x, gx)$ respectively; at the same time, gx is the first point of $W(x, .)$ located on this sphere.

Now by induction procedure we shall prove that the space $(W; i)$ (or ordered set $(W, <)$) would contain a subset $K = (k_0 < k_1 < \ldots)$ of cardinality $\aleph_{\alpha+1}$ of points with a *constant mutual proximity* δ or there would be a decreasing sequence of cardinality $\aleph_{\alpha+1}$ of „spheres" (or terminating intervals of $(W, <)$) having no point in commun. None of these possibilities might occur in the present case. For the last eventuality the thing is obvious; as to the first eventuality, the set K would be a well-ordered subset of $(W, <)$ and for every 2-point-set $\{x, y\} \subset K$ one would have $i(x, y) = \delta$; the set K of all the δ^{th} coordinates x_δ of members x of K would be a subset of $(W, <)$ isomorphic with $(K, <)$—absurdity.

To start with, let $k_0 = W_0$; put $k_1 = W_{i(k_0, gk_0)}$ (cf. (3)); suppose that ν be an ordinal $< \omega_{\alpha+1}$ and that the decreasing „spheres" $S(k_\zeta; \overset{*}{r}_\zeta) = W(g k_\zeta, .)$ $(\xi < \nu)$ with $\overset{*}{r}_\zeta = i(k_\zeta, gk_\zeta)$ are defined; we put $k_\nu = W_{i(k_{\nu-1}, gk_{\nu-1})}$ or $k_\nu = \sup_{\zeta<\nu} k_\zeta$, according as $\nu - 1$ is limit or non limit ordinal. The construction of k_ν is well determined for every $\nu < \omega_{\alpha+1}$ and one sees by induction argument that really

(5) $\quad S(k_\nu, \overset{*}{r}_\nu) = W(gk, .)$ for every $\nu < \omega_{\alpha+1}$; in other words

(6) $\quad i(k_\nu, y) = \overset{*}{r}_\nu = i(k_\nu, gk_\nu)$ for every $y \in W(g k_\nu, .)$.

The function $\nu \rightarrow \overset{*}{r}_\nu$ is a monotone non decreasing function of $I\omega_{\alpha+1}$ into $I\omega_\alpha$. Let r^* be the supremum of the ordinals $\overset{*}{r}_\nu$. One has

(7)
$$r^* \leqslant \omega_\alpha .$$

Now, the relation $r^* = \omega_\alpha$ would imply that some ω_α-sequence of segments $W(g k^\zeta, .)_\zeta$ would have a void intersection (take e. g. k^ζ as the first k_ν satisfying $i(k_\nu, gk_\nu) = \overset{*}{r}_\zeta$); in other words the ω_α-sequence gk^ζ would be cofinal with the $\omega_{\alpha+1}$-sequence W— absurdity.

The relation

(8)
$$r^* < \omega_\alpha$$

does not hold neither. Namely, if the number r^* is isolated, there would be $r^* = r^*_\mu = i(k_\mu, gk_\mu)$ for a $\mu < \omega_{\alpha+1}$; if r^* is non isolated, then for some strictly increasing sequence $r^*_{n_\zeta} = i(k^\zeta, g(k^\zeta))$ of cardinality $\leqslant \aleph_\alpha$ there would be $r^* = \sup r^*_{n_\zeta}$; in either case one concludes that $r^* = r^*_\nu$ for every ν of the final section $S = K(z, .)$, where $z = k_\mu$ or $z = \sup k^\zeta$. According to (6) this means that $i(k_\nu, y) = r^* = i(k_\nu, gk_\nu)$ for every $y \in W[z, .)$. Therefore by (3) we infer that all distinct points in $W[z, .)$ have a same mutual proximity — the number r^*. This fact implies that the set $W[z, .)$ we defined above is a subset of Q isomorphic to $W[z, .)$ and W — absurdity.[1]

3.6.4. *A partial order associated to the linear order* $(M; <)$.

Let $x \rightarrow ux$ $(x \in M)$ be a normal well-ordering uM of M i. e. such that uM be nonequivalent to any of its proper initial portions; in other words let $u = ux$ be a one-to-one mapping of M onto the segment of ordinal numbers corresponding to an initial ordinal ω_β. Let then the partial order $\prec$ in M be defined as superposition of the orders $<$ and u:

$$a \preccurlyeq b \text{ means } a \leqslant b \text{ and } ua \leqslant ub.$$

1. *Every chain* C *in* $(M \prec)$ *is of a cardinality* $\leqslant \aleph_\alpha$.

In fact C is a well-ordered subset in $(M; <)$ and in virtue of 3.6.3 C is $\leqslant \aleph_\alpha$.

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[1] The foregoing proof of Lemma 3.6.1 represents a space-theoretical wording (using abstract distance or abstract proximity) of the theorem XIV in Hausdorff [1].

2. *Every antichain A in $(M; \prec)$ is of a cardinality $\leqslant \aleph_\alpha$.*

As a matter of fact, A is a decreasing sequence in $(M; <)$ and in virtue of 3.6.3 A is $\leqslant \aleph_\alpha$.

3.6.5. Let $x \in M$ and (cf. Sierpiński [5])

$$E_x = \{\{x, x'\};\ x' \in M,\ x' \prec x \text{ or } x \prec x'\}.$$

The mapping $x \to E_x$ is biunique.

For if $y \in M$ and e. g. $x < y$ let then $z \in M(x, y)$ and $uz > \sup\{ux, uy;\}$ one has $x \prec z$ and thus $\{x, z\} \in E_x$; on the contrary $\{x, z\}$ non $\in E_y$, because y non $\in \{x\,z\}$.

3.6.6. Let $F = \{E_x;\ x \in M\}$.

Then $kF = kM = 2^{\aleph_\alpha}$.

3.6.7. We consider the graph $(F; D)$, D being the disjunction relation. *Every antichain as well as every chain of the graph $(F; D)$ is $\leqslant \aleph_\alpha$.*

In fact, let A be an antichain in $(F; D)$; let E_x, E_y be two distinct elements of A; then $\{x, y\}_{\neq} \subset M$ and $E_x \cap E_y \neq \varnothing$; let $\{x, x'\} = \{y, y'\}$ be an element of E_x and E_y; then $x' = y$, $y' = x$; consequently, the points x, y are $\prec$—comparable in M; and vice versa, if x, y are 2 distinct $\prec$—comparable points of $(M; \prec)$, then $E_x \cap E_y \neq \varnothing$. If $E_x \cap E_y = \varnothing$, then x, y are not $\prec$—comparable:

$$x \not\equiv y\ (\cdot \prec) \Leftrightarrow E_x \cap E_y = \varnothing$$

$$x \equiv y\ (\cdot \prec) \Leftrightarrow E_x \cap E_y \neq \varnothing.$$

Consequently, to every $\prec$—chain C in $(M; \prec)$ corresponds the I—chain consisting of the elements $E_x\,(x \in C)$; to every $\prec$—antichain A corresponds the disjointed system $E_x\,(x \in A)$.

As a consequence of 3.6.3. one has therefore 3.6.7.

3.6.8. The system G of sets.

For any $x \in M$ let $G_x = \{\{x, y\}_{\neq};\ y$ is $\prec$— incomparable to $x\}$ i. e. $(x < y) \wedge (ux > uy) \vee (x > y \wedge (ux < uy)$. Let $G = \{G_x;\ x \in M\}$.

1. Every chain and every antichain in $(G; D)$ is $\leqslant \aleph_\alpha$. Again, $x \not\equiv y\,(\cdot \prec) \Leftrightarrow G_x \cap G_y \neq \varnothing$ i. e. x comp. $y \Leftrightarrow G_x \cap Gy = \varnothing$.

3.6.9. $F \cdot \cap \cdot G = \varnothing$ i. e. $x \in F \wedge y \in G \Rightarrow x \cap y = \varnothing$.

3.6.10. *Family H. Let $H = F \cup G$; the family $(H; D)$ is the required family: every D-chain and every D-antichain is $\leqslant \aleph_\alpha$.*

Now $H^{\cdot I2}$ contains a disjointed system of kM elements because the sets

$$H_i = E_i \times G_i\,(i \in M)$$

are pairwise disjoint. As a matter of fact, let $x \neq y$ and $x, y \in M$; then either x, y are comparable or incomparable in M; if x, y are comparable, then G_x, G_y are disjoint and so are the sets H_x, H_y; if x, y are incomparable, the sets E_x, E_y are disjoint and so are also the sets H_x, H_y. The theorem 3. 3. (II) is proved.

3.7. Theorem *For every family F of sets and every ordinal number α we have:* (1) $cF = cF^{\cdot I2}$ *provided* $cF = 1$

(2) $(cF)^{k\alpha} \leqslant cF^{\cdot I\alpha} \leqslant kF^{k\alpha}$; *if* $cF > 1$, *then for some ordinal* α_0 *of cardinality* $\leqslant kF$ *we have*

(3) $(cF)^{k\alpha} = cF^{\cdot I\alpha}$ *for every ordinal* $\alpha \geqslant \alpha_0$.

The relation (1) is obvious; the first relation in (2) is a consequence of the fact that the cartesian product of a disjoint system of sets is again a disjoint system of sets. The second relation in (2) is obvious because the cellularity of any family of sets is less than or equals to the cardinality of the same family; on the other hand, the cardinality of $F^{\cdot I\alpha}$ equals $(kF)^{k\alpha}$. Therefore, the relations (2) hold. Finally, if $kF \leqslant k\alpha$, then $kF^{k\alpha} = 2^{k\alpha}$, and therefore according to (2) we have $cF^{I\alpha} \leqslant 2^{k\alpha}$; this relation joint with the relation $2^{k\alpha} \leqslant (cF)^{k\alpha}$ and the first relation in (2) yields the requested equality (3).

3.8. Theorem. *For any ordered pair* (a,b) *of cardinal numbers* a, b *there exists a family* F *of sets and some ordinal number* α *such that* $a = cF$ *and* $cF^{I\alpha} \geqslant b$.

As a matter of fact, we can consider any disjoint family F *of cardinality* a; *then for some* α *we have* $a^{k\alpha} \geqslant b$ *and consequently* $(cF)^{k\alpha} \geqslant b$.

3.9. Theorem. *For any* F *and any sets* A, B *we have*

$$kA = kB \Rightarrow cF^{\cdot A} = c(F^{\cdot B}) \text{ and}$$

$$kA \leqslant kB \Rightarrow cF^{\cdot A} \leqslant c(F^{\cdot B}).$$

As a matter of fact let t be a one-to-one mapping of A into B; and let D be a disjoint system in $F^{\cdot A}$; for $f \in D$ we define $t = t(f)$ in this way

$$f: A^{\cdot} \to fA^{\cdot}, \text{ where } f\,A^{\cdot} \in F;$$

the antidomain tA of t is a part of B; to every mapping $f: A \to F$ we define the mapping $v_f: B \to F$ as the one which equals ft^{-1} in tA and which, in $B \setminus tA$, equals a constant $b \in B \setminus tA$. Then $v_f \in F^{\cdot B}$.

To every disjoint set D in $F^{\cdot A}$ corresponds an equivalent system $v_D = \{v_f : f \in D\}$ in $F^{\cdot B}$. If $tA = B$, then the mapping $f \to v_f$ is an isomorphism from $F^{\cdot A}$ onto $F^{\cdot B}$.

3.10. Theorem. *Let* A,B *be any sets and* F *a family of sets; then* $cF^{\cdot(A\cup B)} \leqslant s^i$, *where* $s = \sup\{a, b\}$, $i = \inf\{a, b\}$, $a = cF^{\cdot A}$, $b = cF^{\cdot B}$. *If the product* ab *is infinite, then* $cF^{(A\cup B)} \leqslant 2^{\sup\{a,b\}}$.

We shall consider the case that A, B are non empty disjoint sets. Then every member $x \in F^{\cdot(A\cup B)}$ is the set of the functions $g \mid (A \cup B)$ where for $i \in A \cup B$ one has $g_i \in x_i$, x_i being a member of F. Let g_A be the corresponding subfunction in A and let x_A be the set of all these subfunctions g_A; analogously one has g_B and x_B. For any set $S \subseteq F^{\cdot(A\cup B)}$ one has the „projections"

$$S_A = \{x_A;\ x \in F^{\cdot(A\cup B)}\},\ S_B = \{x_B;\ x \in F^{\cdot(A\cup B)}\}.$$

In particular, for every disjoint set D in $F^{\cdot(A\cup B)}$ we have D_A, D_B. For any members X, Y of $F^{\cdot(A\cup B)}$ the relation $X \cap Y = \varnothing$ means

(1) $$X_A \cap Y_A = \varnothing \text{ or } x_B \cap y_B = \varnothing. \quad (2)$$

Let $X \rho Y$ mean (1) i. e. that $X_A \cap Y_A = \varnothing$. Then D is a graph relative to the relation ρ. Let L be a ρ-chain in D; this means that

(3) $$\{X, Y\}_{\neq} \subseteq L \Rightarrow \{X_A, Y_A\} \subseteq L_A \text{ and } X_A \cap Y_A = \varnothing$$

and that L_A is a disjoint system in D_A. Now, the family D_A is isomorphic to a subsystem of the family $F^{\cdot A}$, therefore

(4) $$k\,L_A \leqslant a\,(=c\,F^{\cdot A}).$$

Because of the relations (3) the correspondence $X \in L \to X_A \in L_A$ is onto and one-to-one:

$kL = kL_A$ what jointly with (4) yields

$kL \leqslant c\,A$ for every ρ-chain L in the graph $(A;\,\rho)$.

Analogously, one proves that every antichain M of $(D,\,\rho)$ yields disjointed system M_B of cardinality kM; since $k\,M_B \leqslant c\,F^{\cdot B}$ this means that every antichain in $(D,\,\rho)$ is of a cardinality $\leqslant b = cF^{\cdot B}$. Consequently, by the graph -chain-antichain-theorem we have the requested relation.

3.11. Theorem. *If r is a natural number and F a set family then*

(1) $$cF^{\cdot Ir} = 2^{cF} \Rightarrow c\,F^{\cdot I\,(r+n)} = c\,F^{\cdot Ir} \qquad (2)$$

for every integer n.

Proof. The proof is carried out by induction relative to n. Let D be a disjointed system in $F^{\cdot I\,(r+1)}$; then

$$\{X,\ Y\}_{\neq} \subseteq D \Leftrightarrow$$

(3) $$(X_0 \times X_1 \times \ldots \times X_{r-1}) \cap (Y_0 \times Y_1 \times \ldots \times Y_{r-1}) = \varnothing \ \vee$$

$$\vee\ X_r \cap Y_r = \varnothing\,. \qquad (4)$$

Let $X \rho Y$ mean that (3) occurs; then to every ρ-chain $L \subseteq D$ corresponds the disjointed chain L_{Ir}-projection of L into the product $F^{\cdot Ir}$; consequently $k\,L_{Ir} \leqslant c\,F^{\cdot Ir}$ and according to the assumption (1) we have $k\,L_{Ir} \leqslant 2^{cF}$; again $kL = k\,L_{Ir}$ and thus $kL \leqslant 2^{cF}$. Consequently, every chain of the graph $(D,\,\rho)$ is $\leqslant 2^{cF}$. Analogously, one proves that every antichain of $(D,\,\rho)$ is $\leqslant cF$. Hence $k\,D \leqslant (2^{cF})^{cF} = 2^{cF}$.

The implication (1)$\Rightarrow$(2) is thus proved for every r and $n=1$; writing in particular $r+1, r+2, \ldots$ instead of r, the implication (1)$\Rightarrow$(2) is proved for $n = 1, 2, 3, \ldots$ i. e. for every n.

3.12. Problem. *Let F be a system of sets and n a natural number satisfying $\aleph_0 \leqslant c\,F^{\cdot In} = c\,F^{\cdot (n+1)}$; is there one-to one mapping of $F^{\cdot I\,(n+1)}$ into $F^{\cdot In}$ which conserves both disjointness and jointness of sets? In other words, is then the disjonction graph $(F^{\cdot I\,(n+1)};\,D)$ isomorph to a subgraph of $(F^{\cdot I\,(n)};\,D)$?*

§ 4. Disjoint systems in $F_1 \cdot \times \cdot F_2$.

4.1. F_1, F_2 being set families let Δ be a disjoint system (or D-chain) in the product

(1) $$F_1 \cdot \times \cdot F_2 = \{x_1 \times x_2;\ x_1 \in F_1 \wedge x_2 \in F_2\}.$$

Let $pr_1 \Delta = p_1 \Delta$ and $p_2 \Delta$ be the first and the second projection of Δ respectively. For any $a_1 \in pr_1 \Delta$ we have the following antiprojection of a_1 into Δ:

$p_1^{-1}\Delta(a,.)=\{(a,y);\ y\in F_2\ (a,y)\in\Delta\}$; for any subset $A\subseteq F_1$ we have the corresponding first antiprojection of A into Δ defined by

$$p_1^{-1}\Delta(A,.)=\bigcup_{a\in A}\{p_1^{-1}\Delta(a,.)\}.$$

Analogously, one defines the second antiprojection of any $B\subseteq F_2$ in this way:

$$p_2^{-1}\Delta(.\,,B)=\bigcup_{b\in B}\{p_2^{-1}\Delta(.\,,b)\},\ \text{where}$$

$$\{p_2^{-1}\Delta(.\,,b)\}=\{(x,b);\ x\in F_1,(x,b)\in\Delta\}.$$

By an argument we used in section 3 one proves readily the following items.

4.2. Lemma. For every $a_1\in F_1$ the first antiprojection $p_1^{-1}\Delta(a_1,.)$ in Δ yields the disjoint second projection $p_2p_1^{-1}\Delta(a_1,.)$; therefore the cardinality of this set as well as that of $p_1^{-1}a_1$ is $\leqslant cF_2$. The first antiprojection in Δ of any jointed system C in F_1 is a disjoint system in F_1; the p_2 — projection of $p_1^{-1}C$ is a one-to-one mapping yielding a disjoint system of cardinality $\leqslant cF_2$ in F_2.

4.3. Lemma. Let $T=T(\Delta_1)$ be any tree or ramified table of the family $(F,\supseteq)$; then every D-chain in T is $\leqslant cF_1$ and every J-chain of T is $\leqslant cF_2$. If the number $s=\sup\{cF_1,\ cF_2\}$ is infinite, then one knows that

(1) $kT\leqslant s^{\varepsilon}$; where $s^{\varepsilon}\in\{s,\ s^+\}$; in particular the tree hypothesis yields $s^{\varepsilon}=s=cF_1\cdot cF_2$ and therefore

(2) $kT\leqslant cF_1\cdot cF_2$.

4.4. For every $x_1\in\Delta_1$ let

(3) $\Delta_1(.,\ x_1]_{\geqslant q}$; denote the system of all the members of Δ_1, each joint with x_1 and none contained as a proper part of x_1; then one has the star number $S\Delta_1(.,\ x_1]$ as the minimal number of chains in (3) exhausting (3). Each J-chain in (3) being $\leqslant cF_2$ (Lemma 4.2), we infer that

(3) $k\Delta_1(.,\ x_1)_q\leqslant cF_2\cdot S\Delta_1(..\ x_1)_q,\ (x_1\in\Delta_1)$ and hence

(4) $k\Delta_1(.,\ x_1]_q\leqslant cF_2\cdot s_1F_1$, where

(5) $s_1F_1=\sup\limits_{x\in F_1}S(.,\ x]_q$; the number s_1F_1 is called the *left local star number of the family* $(F_1;\supseteq)$.

4.5. Now as consequence of the choice axiom it is easy to prove the existence of a subtree $T=T(\Delta_1)$ in Δ_1 that is *quasi-cofinal* with Δ_1 in the sense that[1]

$$\Delta_1=\bigcup_{t\in T}\Delta_1(.,\ t)_{\geqslant q}; \tag{6}$$

this means that to every $x\in\Delta_1$ corresponds some $t\in T$ such that x meets t but is not a proper part of t^1. By induction the rows $T_0, T_1,\ldots$ of such a T are defined in this way; T_0 is any maximal disjoined system in Δ_1; for every $t_0\in T_0$ let ft_0 be any *maximal* disjoint system in Δ_1, each member of ft_0 being a proper subset of t_0; one puts

$$T_1=\cup ft_0,(t_0\in T_0)\ \text{etc.}$$

Putting for every ordinal α

$$T^{\alpha}=\cup_{\xi}T_{\xi}(\xi<\alpha)$$

[1] This fact was found also by S. Mardešić.

one sees that T^α is a tree; if T is quasi-cofinal with Δ_1, we put $T^\alpha = T$; if T^α is not quasi-cofinal to Δ_1, we construct T_α as $\cup f t_{\alpha-1} (t_{\alpha-1} \in T_{\alpha-1})$ if the ordinal α is of the first kind; if α is a limit ordinal >0, we consider every jointed decreasing $\alpha-$ sequence C^α of members of T^α and consider any maximal disjoint system $f C^\alpha$ of $\Delta_1 \setminus \cup \Delta_1 (. , t)_{\geqslant q}$, each being a proper part of every member of C^α. One puts then

$$T = \bigcup_{C^\alpha} f C^\alpha$$

(one sees that the construction for non limit α is reducible to this construction using C^α, because for every $t_{\alpha-1} \in T_{\alpha-1}$ the $\alpha-$sequence of oversets of $t_{\alpha-1}$ in T^α is such that $f t_{\alpha-1}$ serves as $f C^\alpha$).

4.6. This being done let $T(\Delta_1)$ be any quasi-cofinal subtree of Δ_1. The decomposition (6) yields jointly with (4)

$$k \Delta_1 \leqslant k T \cdot c F_2 \cdot s_1 F_1. \tag{7}$$

The relation (7) by (1) yields

$$k \Delta_1 \leqslant c^\varepsilon \cdot c F_2 \cdot s_1 F_1. \tag{8}$$

Going back from Δ_1 to Δ the relation (8) in virtue of Lemma 4.2. gives

$$k \Delta \leqslant c F_1 \cdot s^\varepsilon \cdot c F_2 \cdot s_1 F_1. \tag{9}$$

This holding for every $D-$chain Δ of $F = F_1 \cdot \times \cdot F_2$ one concludes that

$$c F (= \sup_{\Delta \subset F} k \Delta) \leqslant c F_1 \cdot s^\varepsilon \cdot c F_2 \cdot s_1 F_1. \tag{10_1}$$

4.7. Analogously, considering the second projection Δ_2 of Δ one proves that

$$c (F_1 \cdot \times \cdot F_2) \leqslant c F_2 \cdot s^\varepsilon \cdot c F_1 \cdot s_1 F_2. \tag{10_2}$$

4.8. The relations (10_1), (10_2)yield by multiplication:

$$(c F)^2 \leqslant (c F_1 \cdot c F_2)^2 \cdot (s^\varepsilon)^2 \cdot z_1 \ F_1 \cdot s_1 F_2. \tag{11}$$

4.9. If s is infinite, then cF is infinite also and $(cF)^2 = cF$ and the exponents 2 in (11) could be dropped; we obtain

$$c F \leqslant s \cdot s^\varepsilon \cdot s_1, \text{ where } s_1 = \sup \{s_1 F_1, \ s_1 F_2\}.$$

Since $s \leqslant s^\varepsilon$ we have $s \cdot s = s^\varepsilon$ and consequently

$$c \leqslant s \cdot s_1.$$

Since obviously c_1, $c_2 \leqslant c$ thus $s \leqslant c$ and we have proved the following relation

$$s \leqslant c F \leqslant s \cdot s_1. \tag{12}$$

4.10. Theorem. (I) *Let I be a finite index set (e.g. the interval In of ordinals $< n$, where n is a given finite ordinal); let F_i, $(i \in I)$ be a finite sequence of non void set systems such that at least one of the cellular numbers cF_i be infinite; then*

$$s \leqslant c \prod_i F_i \leqslant s^\varepsilon \cdot s_1, \text{ where } s = \sup_i c F, \ s_1 = \sup_i s_1 F_i, \text{ and } s^\varepsilon \in \{s, s^+\}.$$

In particular for any set system G and any natural number n one has

$$c\,G \leqslant c\,(G^n) \leqslant (c\,G)^{\varepsilon} \cdot s_1.$$

(II) One has $s^{\varepsilon} = s$ if and only if the tree hypothesis

$$k_c\,T \leqslant s \text{ and } k_c\,T \leqslant s \Rightarrow k\,T \leqslant k_c\,T \cdot k_{c'}\,T$$

is true or false.

Since the theorem (II) was proved else (G. Kurepa, [1] p. 106 theor. 1), let us prove the theorem (1).

We just proved that the theorem (I) holds if the index set I has 2 members; by induction argument one sees that the same conclusion holds for any finite set I.

Let us prove the theorem (I) if I has 3 membres 1,2,3. Let Δ be any D-chain in $F\,(=F_1 \times F_2 \times F_3)$ and Δ_{12} and Δ_3 its projections into $F_1 \cdot \times \cdot F_2$ and F_3 respectively. For any tree T in Δ_{12} that is quasi—cofinal with Δ_{12} we have (like in (7)):

$$k\,\Delta_{12} \leqslant k\,T \cdot c\,F_3 \cdot s_1\,(F_1 \cdot \times \cdot F_2). \tag{13}$$

We have to evaluate the factors $k\,T$, s_1 in (13). First of all,

$$s_1\,(F_1 \cdot \times \cdot F_2) \leqslant s_1\,F_1 \cdot s_1\,F_2. \tag{14}$$

As a matter of fact, for any $x_1 \in F_1$ let A_1 be a system of J-chains of sets $\in F_1$ exhausting $F_1\,(.\,,\,x_1)\supseteq_q$; analogously, for $x_2 \in F_2$ one has a family A_2 of jointed systems of sets-members in $F_2\,(.\,,\,d_2)\supseteq_q$ exhausting this family. Then we have the element $x_1 \times x_2 \in F_1 \cdot \times \cdot F_2$ and the system $A_1 \cdot \times \cdot A_2$; all elements of this system are J-chains, each quasi-containing $x_1 \times x_2$ and the system exhausts $F_0 = F\,(.\,,\,x_1 \times x_2)\supseteq_q$; because if $M_1 \times M_2$ is any member of F_0 q-containing $x_1 \times x_2$, then M_i q-contains x_i and for some $J_i \in A_i$ we have $M_i \in J_i$ and hence $M_1 \times M_2 \in J_1 \times J_2 \in A_1 \cdot \times \cdot A_2$. In this way we proved that

$$s\,(F_1 \times F_2)\,(.\,,\,x_1 \times x_2)\supseteq_q \leqslant s\,F_1\,(.\,,\,x_1)\supseteq_q \cdot s\,F_2\,(.\,,\,x_2)\supseteq_q;$$

from here allowing x_1, x_2 to vary in F_1, F_2 respectively and taking sup we have the requested relation (14).

The relations (13), (14) yield

$$k\,\Delta_{12} \leqslant kT \cdot cF_3 \cdot s_1\,F_1 \cdot s_1\,F_2. \tag{15}$$

4.11. Lemma. Let $s_{12} = \sup\,\{cF_1,\ cF_2\}$; every chain and every antichain of every tree $T \subset F_1 \cdot \times \cdot F_2$ is $\leqslant s_{12}^{\varepsilon}$; also $k\,T \leqslant s_{12}^{\varepsilon}$.

In opposite case there would be a tree T_a in F of cardinality $\geqslant s_{12}^{\varepsilon+}$; now obviously, $w_d\,(F_1 \cdot \times \cdot F_2) \leqslant w_d\,F_1 \cdot w_d\,F_2$, where $w_d\ A$ for any family A of sets denotes the supremum of cardinalities of strictly decreasing sequences of sets in A. Since kT_a is greater than the cardinality of any tree in F_1 or in F_2, and since this fact is not due to J-subchains of trees, it should be due to D-subchains, and the tree should contain an antichain $\geqslant {}_{12}^{\varepsilon+}$, in contradiction with 4.3.

The relation (15) and the Lemma 4.11. imply

$$k\,\Delta_{12} \leqslant s_{12}^{\varepsilon} \cdot cF_3 \cdot s_1\,F_1 \cdot s_1\,F_2.$$

From here, going back to Δ:

(16) $$k\Delta \leqslant (s_{12}^{\varepsilon}\, cF_3 \cdot s_1 F_1 \cdot s_1 F_2) \cdot cF_3.$$

Now, let $s = \sup_i cF_i$ and $s_1 = \sup_i s_1 F_1$; then $s_{12} \leqslant s$, $cF_3 \leqslant s$ and therefore $s_{12}^{\varepsilon}(cF_3)^2 = s^{\varepsilon}$; again $s_1 F_1 \cdot s_1 F_2 \leqslant s_1$ and the relation (16) yields

$$k\Delta \leqslant s^{\varepsilon} s_1.$$

This proves the theorem for $I = 1, 2, 3$. By induction argument one proves the theorem for every finite index set I. *Q. E. D.*

The foregoing theorem, by particularization implyes the following.

4.12. Theorem. *Let I be a finite index set and $F_i (i \in I)$ a sequence of non void set systems with sup $cF_i = s = \infty$; if $s_1 F_i \leqslant s$, then*

$$s \leqslant c \prod_i F_i \leqslant s^{\varepsilon}.$$

Such a case holds particularly if F_i is a system of intervals of a totally ordered set $O_i (i \in I)$; in this case one has $s_1 F_i \leqslant 2$.

As a matter of fact any system S of intervals overlapping a given interval x of a given totally ordered set equals $S_1 \cup S_2$, where S_1 denotes all the members of S containing the left extremity of x and where S_2 denotes all the members of S each containing the right extremity of x; obviously, S_i is jointed.

§ 5. Cartesian multiplication of topological spaces

5.1. Definition of cartesian multiplication of spaces[1]. For every $i \in I$ let X_i be a topological space; the cartesian product X of sets X_i of points of X_i will be called the *topological product of spaces* X_i provided for every point $x \in X$ the neighbourhoods are defined in the following way; let I_0 be a *finite part of I;* for every $i_0 \in I_0$ let $O(i_0)$ be a neighbourhood of the point x_{i_0} in the space X_0; for every $i \in I$ let X_i^+ be $O(i_i)$ or X_i, according as $i \in I_0$ or $i \in I \setminus I_0$; the cartesian product of all the sets X_i^+ is called neighbourhood of the point x. This neighbourhood depends on finite set $I_0 \subseteq I$ and on the neigbourhoods $O(i_0)$ in X_{i_0} for $i_0 \in I_0$. The stress in the foregoing definition is the finiteness of subsets I_0 of I.

5.2. The neighbourhoods could be defined in this way also. For a point x_i on the i^{th} coordinate axis let $p_i^{-1}(x_i)$ be the i^{th} antiprojection of x_i into the space i. e. the set of all the points x of the space, the i^{th} coordinate of which is just the point x_i of the space X_i. For a subset S_i of the space X_i we define the i^{th} antiprojection $p_i^{-1} S_i$ as the union of all the sets $p_i^{-1} x_i (x_i \in S_i)$. In other words, the set $p_i^{-1} S_i$ is the anti-projection of S_i in the direction of the X_i-axis. Then the foregoing neighbourhood is the intersection of the open sets like this

(1) $$\cap p_{i_0}^{-1}(0x_{i_0})\ (i_0 \in I_0).$$

[1] We shall consider topological T_2-spaces i. e. Fréchet's V-spaces satisfying the Hausdorff's T_2-condition of separation. The T_2-separation condition means that for any 2-point set $\{a, b\}$ there is a neighbourhood $V(a)$ of a and a neighbourhood $V(b)$ of b such that $V(a) \cap V(b) = \varnothing$.

Marczewski-Szpilrajn [1] proved that if the topological spaces X_i are of a countable weight each, then the weight of the cartesian product X is countable too. The phenomenon is a general one and we have the following.

5.3. Theorem. *For every topological T_2-spase S and every non void index set I the cellularity of the cartesian hyper-cube S^I equals $(cS)^{kI}$ for $w.\ kI < \aleph_0$; if $kI \cdot w \geqslant \aleph_0$, then $\sup\{\aleph_0, cS\} \leqslant cS^I \leqslant w$, where the weight $w\ (=wS)$ of the space S is defined as the infimum of cardinal numbers of neighbourhood bases of the space S.*

The theorem 5.3. is a special case of the following theorem (in the wording of the theorem put X_i = fixed space S for every $i \in I$).

5.4. Theorem. *Let I be a non void set and X_i, for every $i \in I$, a topological space. Let wX_i denote the weight number of the space X_i and $w = \sup wX_i$; then for the cellularity number cX of the cartesian product $X = \prod_i X_i$ one has:*

$$kI \cdot w < \aleph_0 \Rightarrow \prod_i cX_i = cX \tag{2}$$

$$kI \cdot w \geqslant \aleph_0 \Rightarrow \sup\{\aleph_0, \sup_i cX_i\} \leqslant cX \leqslant w. \tag{2'}$$

5.5. Proof of the theorem 5.4.

1. First case: The number of factors X_i is finite and every X_i is finite. In this case, obviously $cX_i = kX_i = wX_i$ and $cX = kX = wX$; since $kX = \prod_i kX_i$, the preceding relations yield the requested implication (2).

Second case: $kI \cdot w \geqslant \aleph_0$. Now, it is obvious that if kI is infinite and every factor has at least 2 points, then the number $c\ (=cX)$ can not be finite. Therefore we have still to consider the case that the weight of every factor is infinite, irrespective what happens with kI.

Let c denote the number cX. First of all, $c \geqslant cX_i$ for every $i \in I$ and hence $c \geqslant c_s\ (=\sup cX_i)$. As a matter of fact, let D_i be any disjoint system of open sets of the space X_i; putting $\{i\} = I_0$ and taking $O_i \in D_i$, $O_j = X_j$ for $j \in I \setminus \{i\}$, one gets a system of cardinality kD_i of open sets $\prod_{x \in I} O_x$ of the space X; therefore $c \geqslant kD_i$ and $c \geqslant cX_i\ (= \sup_{D_i \subset X_i} kD_i)$ for every $i \in I$. The relations $c \geqslant cX_i$ imply $c \geqslant \sup c_i$ i. e. $c \geqslant c_s$. Therefore the requested relation (2′) will result of the impossibility of the relation $c > w$. Obviously we can suppose also that $w \geqslant \aleph_0$.

5.5.2. Now, suppose on the contrary that $c > w$ and that there exists a *disjoint* system D of open sets of the space X such that

$$kD > w \geqslant \aleph_0. \tag{2}$$

One might suppose that the members of D are of the form (1), where I_0 is a (variable) finite subset of I (it is sufficient to choose an element of the form (1) in each member of D and consider the system of the selected elements). For every $i \in I$ let B_i be a basis of neighbourhoods of the space X_i such that

$$kB_i = wX_i\ (:= w_i). \tag{3}$$

This being done, let n be any natural number and D_n the system of all the members (1) in D such that $kI_0 = n$. Obviously $D = \cup_n D_n\ (n < \omega)$ and since, by (2), the system D is non countable there exists an integer m such that also

$$kD_m > w. \tag{4}$$

5.5.3. Let $x = \prod X_i^+$ $(i \in I)$ be a particular member of D_m; then the set I_{om} of the points $i \in I$ such that $X_i^+ \neq X_i$ is a well determined *finite* subset of m points of I. The members of D_m being pairwise disjoint one has in particular $x \cap y = \varnothing$ for every $y \in D_m \setminus \{x\}$; this means that for every such y one has $x_i \cap y_i = \varnothing$ for some $i = i(y) \in I_{om}$ because $x_i \cap y_i \neq \varnothing$ for every $i \in I \setminus I_{om}$. This mapping

(5) $$f: D_m \to I_{om}$$

is a single-valued mapping of the set D_m of cardinality $> \aleph_o$ into a *finite* m-point set $I_{om} = \{i_1, i_2, \ldots, i_m\}$. Therefore for some $j_1 \in I_{om}$ there exists a subset D_m^o of D_m in which the mapping (4) equals j_1 and so that $k D_m^o = k D_m$. Now, let us consider the p_{j_1} — projection of the set D_m^o into the space X_{j_1}; this mapping is a single-valued mapping of the set D_m^o of cardinality $> w$ into the basis B_{j1} of w_{j1} members of the space X_{j1}. Since $k D_m^o > w \geqslant w_{j_1}$, one infers that for some member $O_{j'_1} \in B_{j_1}$ and for some subset D_{m_1} of D_m^o one would have

$$pr_{j_1} D_{m_1} = O_{j'_1}, \quad k D_{m_1} > w.$$

5.5.4. Substituing D_{m_1} for D_m and $I_{om} \setminus \{j_1\}$ for I_{om} the argument of 4.4.3. shows that for: some point $j_2 \in I_{om} \setminus \{j_1\}$, some subset D_{m2} of D_{m1} and some neighbourhood $O_{j'_2} \in B_{j_2}$ one has:

$$f \mid D_{m2} = j_2, \quad pr_{j_2} D_{m2} = O_{j'_2}, \quad k D_{m2} > w.$$

The induction procedure would go on: there woud be a subset D_{m3} of D_{m2}, a point $j_3 \in I_{om} \setminus \{j_1, j_2\}$ and a $O_{j'_3} \in B_{j_3}$ such that

$$f \mid D_{m3} = j_3, \quad pr_{j_3} D_{m3} = O_{j'_3}, \quad k D_{m3} > w; \text{ etc.}$$

The m^{th} step of induction procedure would yield: a disjoint subset D_{mm} of $D_{m\,m-1}$, a point $j_m \in I_{om} \setminus \{j_1, j_2 \ldots, j_{m-1}\}$ and a member $O_{j'_m} \in B_{j_m}$ such that

(6) $$f \setminus D_{mm} = j_m, \quad pr_{j_m} D_{mm} = O_{j'_m}, \quad k D_{mm} > w.$$

Now, for every index $i \in I \setminus I_{om}$ we have $pr_1 D_{mm} = X_i$; therefore $x \in D_{mm} \Rightarrow$ $\Rightarrow X_{j\mu}^+ = O_{j'\mu}$ for $\mu = 1, 2, \ldots, m$ and $X_i^+ = X_i$ for $i \in I \setminus I_{om}$; consequently $k D_{mm} = 1$, in contradiction with the last relation in (6). This contradiction proves the theorem.

5.5.5. Remark. Let us consider the cellularity numbers $cF^{\cdot I\alpha}$ (F and α being any system of sets, and any ordinal number) and the numbers $cS^{I\alpha}$ (S being any topological space); in the first case, as α is increasing so is also the corresponding cellularity; on the contrary, in the case of hyper-cubes of any topological space S the cellularity numbers are always less or equal to the weight wS of S.

5.5.6. Corollary. *For any topological T_2-space S satisfying $cS = wS$ one has $cS^I = cS$ for every non void set I. In particular, $cS^\alpha = cS$ for any positive ordinal α, irrespective whether α is finite or transfinite. In particular this holds provided S is a metrical space or if S is a totally ordered space in which the cellularity equals the separability number of the space.*

A consequence of the corollary 5.5.6. for metrical spaces is this one:

5.5.7. *The hypercube M^I of any metrical space M with $kI > \aleph_0$ is a non metrical space; in particular the real cube $[0, 1]^{I(\omega_1)}$ or $\{0, 1\}^{I(\omega_1)}$ are non metrical spaces.*

5.5.8. Corollary. *For every topological space S the set consisting of the cellularity numbers* cS^{α} *running through ordinal numbers is well determined and has at most wS numbers (cf. theorems* 3.7; 3.8; 5.8 *(ii)).*

5.6. *Comparison between cS* and cS^n *for any space S.*

Theorem. (1). *For any topological space S one has* $cS \leqslant cS^2 \leqslant \inf\{2^{cS}, (cS)^{\varepsilon} \cdot sS\}$; *for every ordinal* $n < \omega$.

(II) *For any ordered pair of topological spaces* S_1, S_2 *we have* $s \leqslant c(S_1 \times S_2) \leqslant 2^s$, $Tr(S_1) s_1 S_1$, $Tr(S_2) \cdot s_1 S_2$ *where* $s = \sup\{cS_1, cS_2\}$: $Tr\, S_1 = \inf kT$, T being quasi-cofinal subset of the family of open sets of S_1 (cf. § 4.4 and § 4.5).

(III) *For any ordered pair of totally ordered spaces* S_1, S_2 *one has* $s \leqslant c(S_1 \times S_2) \leqslant s^{\varepsilon}$, *where* $s^{\varepsilon} \in \{s, s^+\}$; *the relation* $s^{\varepsilon} = s$ *is equivalent to the tree hypothesis.*

The proof is like the one of theorem 3.4. (1) in § 3.5.2; cf also § 3.10 and § 4.9; § 4.10.

5.7 In connexion with the results 3.4. (1), 4.4 and 4.6 let us indicate that there are spaces $S_{\aleph_1}$ satisfying $cS < wS$; such a space is the cartesian product $[0,1]^{\aleph_1}$ of $\aleph_1$ real segments [0,1]; the cellularity and the weight of this product are $\aleph_0$, $\aleph_1$ respectivaly.

5.7.1. Theorem. For any ordered pair (a,b) of cardinal infinite numbers there is a topological space S such that $cS \leqslant a < b \leqslant wS$. Such a space is the cartesian product of $k\alpha$ real segments [0,1] where α is any ordinal of cardinality $\geqslant b$.

5.7.2. Here is also a space S satisfying $cS < wS$ and which was given by Inagaki as the solution of a problem in my doctoral thesis. Let R^* be the set of members of a one-to-one ω_1— sequence x_{α} $(\alpha < \omega_1)$ of real numbers x_{α}; the set R^* is topologized by considering for any $\alpha < \omega_1$ as neighbourhoods of x_{α} the sets of the form $V^*(x_{\alpha}) = \{x_{\beta};\ x_{\beta} \in V(x_{\alpha});\ \alpha \leqslant \beta < \omega_1\}$, $V(x_{\alpha})$ being any ordinary neighbourhood of x_{α}. The space (R^*, V^*) so obtained has the cellularity $\aleph_0$ and the weight $\aleph_1$.

5.8. Main theorem (I). *For any topological space S satisfying* $cS \geqslant \aleph_0$ *and every index set I the cellularity of the cube* S^I *is* $\leqslant 2^{cS}$ *i. e.*

$$\text{cel}\, S \leqslant \text{cel}\, S^I \leqslant (\text{cel}\, S)^{\text{cel}\, S}$$

(ii) *The general continum hypothesis implies*

$$\text{cel}\, S^I \in \{\text{cel}\, S,\ (\text{cel}\, S)^+\}.$$

5.8.1. Proof. First of all we proved that the theorem (1) holds provided $kI = 2$, e. g. $I = \{1, 2\}$ and even for $kI < \infty$ (cf theorem 3.11).

Now we shall prove the theorem (1) for every I.

5.8.2. Lemma. *Let m be any positive integer* $1 < m \leqslant kI$ *and* Δ *any disjoint system of open sets of* S^I *satisfying* $ksx = m$ *for every* $x \in \Delta$; *then* $k\Delta \leqslant 2^{cS}$; *here sx denotes the greatest subset* I_0 *of the index set I satisfying* $x(i) \neq S$ $(i \in I_0)$.

We shall prove the lemma by induction argument on m. Suppose that the lemma holds for every natural number $< m$; let us prove that it holds also for m. Assume on the contrary that there exists a disjoint system Δ of cardinality $> 2^{cS}$ and such that $ksx = m$ for every $x \in \Delta$. Since the set Δ is *disjoint*, the set of $sx\,(x \in \Delta)$ is *jointed*; namely, if $x,\ y \in \Delta,\ x \neq y$, then $x \cap y = \varnothing$ what means that for some $i \in I$ one has $x_i \cap y_i = \varnothing$; this means that

$$x_i \neq S \neq y_i \text{ i. e. } i \in sx \cap sy, \text{ hence } sx \cap sy \neq \varnothing.$$

Now, let $e \in \Delta$; for every $a \in se$ let $\Delta(a)$ be the system of all the members x of Δ satisfying $a \in sx$. Then

$$\Delta = \bigcup_a \Delta(a) \quad (a \in se).$$

Since the set se has just m members, one of the sets $\Delta(a)$ has $k\Delta$ points; let a_0 be such an element of se:

$$k\Delta(a_0) = k\Delta > 2^{cS} \quad \text{and} \quad a_0 \in se.$$

Let us consider the *disjoint* set system $\Delta(a_0) = A$. Let us structurize A by defining that for $x,\ y \in A$ the relation xry means $x_i \cap y_i = \varnothing$ for some $i \in sx \cap sy \setminus \{a_0\}$.

Let L be a r-chain in $(A;\ r)$; then L is a disjoint set in S^I; moreover, let L_I be the system of sets x_0 where for every $x \in L$ one denotes by x_0 the set obtained from x by substituing the a_0-factor of x by the space S. The mapping $x \in L \to x_0$ is one-to-one; let $L_0 = \{x_0;\ x \in L\}$. Then L_0 is disjointed system of sets of S^I such that

(1) $\qquad ksx_0 = m - 1 \ (x_0 \in L_0)$. Namely, $sx_0 = sx \setminus \{a_0\}$.

Now, by induction hypothesis the relations (1) imply that $kL_0 \leqslant 2^{cS}$, what jointly with $kL_0 = kL$ implies the requested relation $kL \leqslant 2^{cS}$. In other words every r-chain L in $(A,\ r)$ is $\leqslant 2^{cS}$; therefore also the k_c-number of $(A,\ r)$ is $\leqslant 2^c$. On the other hand every antichain L' in $(A,\ r)$ is $\leqslant cS$ because if $x,\ y \in L'$ and $x \neq y$, then

$$x_i \cap y_i \neq \varnothing \text{ for } i \in sx \cap sy \setminus \{a_0\} \text{ and therefore } x_{a_0} \cap y_{a_0} = \varnothing.$$

In other words to every antichain L' in $(A,\ r)$ corresponds a well determined disjoint system in S, of cardinality kL'.

In virtue of the chain-antichain theorem for graphs one concludes that $k\Delta \leqslant (2^{cS})^{cS} = 2^{cS}$ i. e. $k\Delta \leqslant 2^{cS}$. *Q. E. D.*

5.8.3. Proof of the theorem (I). For any natural number n let

$$\Delta_n = \{x;\ x \in \Delta,\ ksx = n\}.$$

Then the sets Δ_n exhaust Δ; since by hypothesis $k\Delta > 2^{cS}$, then for some integer n one would have necessarily $k\Delta_m > 2^{cS}$, in contradiction with the foregoing lemma, because every member x of Δ_m satisfies $ksx = m$.

5.8.4. The theorem 5.8. (ii) is an obvious consequence of the theorem 5.8. (i) and of the general continuum hypothesis.

6. On the cartesian multiplication of ordered sets and graphs

6.1. Definition. Let (1) $(O_i,\ <_i)\ (i \in I)$ be a family of ordered sets; the *cartesian product* or the *cardinal product* of the sets (1) is the set $(O,\ <)$ where $O = \prod_i O_i\ (i \in I)$ and where for $x,\ y \in O$ one has

$$x \leqslant y \text{ in } O \Leftrightarrow x \leqslant_i y \text{ in } O_i \ (i \in I).$$

Analogously one defines the cartesian product of any non empty family of graphs (G_i, r_i) on substituing in the preceding definition G_i for O_i and r_i for $<_i$; r_i means any binary relation that is either symetrical (for *symetrical graphs*) or antisymetrical (for *oriented graphs*).

One proves readily the following.

6.2. Theorem. The cartesian product of any system of ordered sets is an ordered set; if every factor is ramified (a tree, a chain), the product need not be so.

We are especially interested to know the connexions between the cellularity number of the product and the cellularity numbers of the factors. In this respect the notion of ramified sets and particularly of ramified tables or trees is of special importance.

6.3. Definition of a node. Every *maximal* subset S of a ramified set R such that

$$(1) \qquad x, y \in S \Rightarrow S(., x) = S(., y)$$

is called a *node of S*.

6.4. Theorem. *Let R be any ramified set i.e. any ordered set $(R; <)$ in which $R(., R^{\cdot})$ is a chain; let I be any non empty index set; let $f, g \in R^I$; then $f \leqslant g$ in R^I means $f_i \leqslant g_i$ in R for every $i \in I$; if $f \neq g$ and if the set $fI = \{f_i;\ i \in I\}$ lays in a node of R as well as does gI and if*

$$kfI > 1, \quad kgI > 1$$

then $f \parallel g$ i. e.: neither $f \leqslant g$ nor $f \geqslant g$. The conclusion holds also provided R contains a subset M such that fI lays in a node of M and that gI lays in a node of M.

Proof. Since by hypothesis the set fI is located in a node of R, the chain $R_f = R(., f_i)$ is well determined and does not depend on an particular choice of i in I. Analogously, one has $R_g = R(., g_i)$ for $i \in I$. Let

$$(2) \qquad C = R_f \cap R_g.$$

The set C is a chain in R and is an initial section of the chains R_f, R_g.

Case (i). C is a proper subset of both R_f and R_g. Then there is an $a \in R$ and $a, b \in R$ such that

$$a \| b,\ C . < a < f_i \text{ and } C . < b < g_i.$$

Hence $f_i \| g_i$, for one has not e. g. $f_i < g_i$, because the antichain $\{a, b\}$ would be $< g_i$, contrary to the ramification condition on R. Thus $f_i \| g_i$, and consequently

$$f \| g.$$

Case (ii). $C = R_f$.

(ii). 1. Subcase: $C \neq R_g$. Then $R_f < g_i$ for some $i \in I$ and there exists one (and only one) point $a' \in R$ satisfying

(3) $a' \sim f_i$, $a' < g_i$ ($x \sim y$ means to be in a same node).

Namely, f_i is in the node following C and the chain R_g intersects this node. Hence we have (3). Since $kfI > 1$ we have $f_j \neq a'$ for some $j \in I$; thus $f_j \| a'$ and $f_j \| g_i$ $(i \in I)$; in particular $f_j \| g_j$ i. e. $f \| g$.

(i i) 2. Subcase; $C = R_g$ i. e. $R_f = R_g$. Since $f \neq g, f_i \neq g_i$ for some $i \in I$; thus $f_i \| g_i$, because f_i, g_i are 2 members of the same node of R; the relation $f_i \| g_i$ implies $f \| g$ by definition. The theorem is proved.

6.4.1. Remark. By counterexamples one might prove that both conditions: (i) R is ramified, (i i) f, g are not constant in I are necessary: dropping either of them, one could have $f \leqslant g$ or $f > g$.

6.4.2. Remark. If R is any ramified subset of a ramified set $(R', <)$, then again any two elements f, g of R^I such that f, g satisfy, with respect to R, the conditions of the preceding theorem:

fI is a part of a node of R and $kfI > 1$

gI ,, ,, ,, ,, ,, R ,, $kgI > 1$

then f, g are incomparable both in R^I and in R'^I.

6.5. The applications of the preceding considerations concern particularly *ramified collections of sets* i. e. collections of sets containing no pair of *interlaced sets* (two sets A, B are interlaced, if both sets $A \setminus B$ and $B \setminus A$ are nonempty).

6.6. We have in particular the following theorem as a particular case of the preceding general theorem (consider I to have just 2 points):

Theorem. *Let F be any family of sets; let $F^{\cdot 12}$ be the set of the cartesian products $x_0 \times x_1$ where $x_0, x_1 \in F$. If R is any ramified subfamily of F [i. e. $X, Y \in R \Rightarrow X \subseteq Y \vee X \supset Y \vee (X \cap Y = \varnothing)$] such that to every $X \in R$ corresponds an $X' \in R$ satisfying $X \cap X' = \varnothing$ and that X, X' have the same predecessors i. e. supersets in R, then the sets*

$$X \times X' \; (X \in R)$$

are mutually disjoint.

Direct proof. Suppose (3) $(X \cap X') \cap (Y \cap Y') \neq \varnothing$ for some $X, X' \in R$ and some $Y, Y' \in R$.

Then X, Y are comparable; X', Y' as well.

Suppose the case $X \subseteq Y$, hence $X' \subseteq Y$ because X, X' have in R same predecessors. Since $Y' \cap Y = \varnothing$ then Y' is disjoint from X and X', contrary to the hypothetical comparability of X', Y'. The impossibility of other cases is proved in an analogous way.

Corollary. *Let R be a ramified set and $N_o R$ the set of nodes of R each containig at least 2 points. Then the system*

$$\bigcup_x [X \times X \setminus diag\,(X \times X)] \qquad (X \in N_o R)$$

is an antichain in the cartesian square R^2 of R.

6.7. Theorem. *For any square or hypersquare of any tree or ramified set R the chain $\times$ antichain relation holds:*

$$kI > 1 \Rightarrow kR^I \leqslant k_c R^I \cdot k_{c'} R^I.$$

First of all if R itself satisfies the chain $\times$ antichain relation, then so also R^I. If R does not satisfy this relation then it contains a tree T of the same cardinality (every T which is cofinal with R is such one); T contains (cf. Kurepa) [1] p. 109) a subtree t of the cardinality $kT\,(=kR)$ and such that every isolated node of t contains at least 2 points (cf. the ambiguous tables in Kurepa (1)); according to theorems 6.4 and 6.6 some subtree t_0 of cardinality kt^I of t^I is „normal" i.e. satisfies the chain $\times$ antichain relation; since $t_0 \subseteq R^I$, $kt^I = kR^I$, R^I is normal too.

6.7.1. Remark of course, if for some index set I such that $kI > 1$, *every* subtree of R^I satisfied the chain $\times$ antichain relation (R beinq any ramified set), *the tree hypothesis would hold; and vice versa.*

6.8. For symmetrical graphs we have theorems that read like the ones we formulated and proved for set systems (we define *the cellularity* cG *of a graph* as the supremum of cardinalities of its antichains). E. g. it is legitimate to substitute „*symmetrical graph G*" instead of „*family F of sets*" in the wording of statements: 3.4, 3.8, 3.9, 3.10, 3.11, 4.10 etc.

BIBLIOGRAPHY

[1] P. Erdös — R. Rado: *A partition calculus in set theory* (Bull. Amer. Math. Soc. 62 (1956), 427—489).

[2] F. Hausdorff: *Grundzüge einer Theorie der geordneten Mengen*, Math. Annalen, 65 (1908), 435—505.

[3] T. Inagaki: *Les espaces abstraits et les ensembles ordonnés* Journal of the Faculty of Science HOKKAIDO Univ. Ser 1, Vol. 8 (1940), N^{os} 2—3, pp. 145—162.

[4] Đ. Kurepa: 1. *Ensembles ordonnés et ramifiés*. Thèse, Paris, 1935. (Publ. math. Belgrade, 4 (1935), 1—138).

2. *Sur une propriété caractéristique du continu linéaire et le problème de Suslin* (Publ. Math. Inst. Ac. Sci. Belgrade, 4, 1952, 97—108).

3. *On binary symmetrical relations or graphs*, Dissertationes Slov. Ac. Sci. Ljubljana IV, 4 (1953), 67—80—92.

4. *On the cardinal number of ordered sets and of symmetrical structures in dependance on the cardinal number of its chains and antichains*, Glasnik Mat.-fiz., Zagreb 14 (1959), 183—203.

[5] W. Sierpiński: *Sur un problème de la théorie générale des ensembles* (Fundamenta math. 33 (1945), 301—302).

[6] E. Szpilrajn: (= E. Marczewski): 1. *Заметка о декартовых топологических пространстве* (Доклады Ак. н. СССР 31 (1941), N. 16, 525—527.

2. *Separabilité et multiplication cartésienne des espaces topologiques*, Fundamenta Mathematicae, 34 ((1947), 127—143).

C. R. Acad. Sc., t. 256, p. 2504-2506. Séance du 18 mars 1963. Groupe 1

THÉORIE DES ENSEMBLES. — *Sur certains complétés des ensembles ordonnés munis d'opérations : complétés de Dedekind et de Kurepa des ensembles partiellement ordonnés.* Note (*) de M. **Lambros Dokas**, présentée par M. Paul Montel.

M. G. Kurepa a défini, dans sa thèse, un complété d'un ensemble totalement ordonné E différent de son complété par la méthode de Dedekind (qui consistait à remplir les lacunes de E).

Cette complétion consistait à remplacer chaque $e \in E$ par :

— le triple e^-, e, e^+ si e est à la fois limite à gauche et à droite dans E;

— la paire e^-, e, si e est une limite à gauche dans E, sans l'être à droite;

— la paire e, e^+, si e est une limite à droite dans E, sans l'être à gauche;

— par e, si e est un élément isolé de E,

et toute lacune l de E par une paire l^-, l^+, tous ces ensembles de un, deux, trois éléments étant rangés les uns par rapport aux autres selon un ordre évident, dérivant de celui de E. M. Krasner a donné, dans le cas particulier où E est la droite réelle $\mathfrak{R}$ (ordonné par son ordre habituel) une autre interprétation de ce complété (qu'il a appelé la droite semi-réelle $\mathfrak{S}$), en montrant qu'on peut l'interpréter à la manière de complété des espaces métriques en remplaçant les suites de Cauchy par les suites asymptotiquement monotones et la condition d'équivalence métrique par une autre, provenant de la relation d'ordre. En vue des applications aux espaces ultramétriques et aux corps valués, il a étendu les opérations rationnelles de $\mathfrak{R}$ à certaines parties de $\mathfrak{S}$ (ou de $\mathfrak{S} \times \mathfrak{S}$ dans le cas des opérations binaires) en indiquant un principe général permettant de déterminer, pour toute opération (de dimension quelconque) sur $\mathfrak{R}$ le plus grand sous-ensemble de $\mathfrak{S}^n$, où l'on peut raisonnablement l'étendre, ainsi que la manière dont cette extension doit être faite. Il a d'ailleurs suggéré que ces méthodes s'appliquent, avec les modifications évidentes, au complété de Kurepa de tout ensemble totalement ordonné.

J'ai pu généraliser les méthodes de Kurepa et de Krasner aux ensembles partiellement ordonnés E où ils donnent en général (contrairement au cas de l'ordre complet) deux complétés différents de E, celui de ~~Krasner~~ $\tilde{E}_{Kr}$ étant, en général, plus fin que celui $\tilde{E}_{Ku}$ de Kurepa, $\tilde{E}_{Ku}$ pouvant être considéré comme quotient de $\tilde{E}_{Kr}$ par une relation d'équivalence compatible avec l'ordre. D'autre part, si E est muni d'une opération (ou de plusieurs opérations), j'ai pu déterminer, en me servant du même principe, que dans le cas totalement ordonné, le domaine maximal, où cette opération (ou ces opérations) peuvent être étendues en tant que celle (ou celles) de $\tilde{E}_{Kr}$ ou $\tilde{E}_{K}$, ainsi que la manière de faire cette extension.

La présente Note est consacrée au rappel de la notion bien connue du complété de Dedekind d'un ensemble partiellement ordonné E et définit son complété de Kurepa.

Soit E un ensemble (partiellement) ordonné. On sait qu'on appelle coupure de E un couple (E_1, E_2) de sous-ensembles disjoints de E tels qu'un $x \in E$ est supérieur à tout $y \in E_1$ si et seulement si $x \in E_2$ et qu'un $x \in E$ est inférieur à tout $y \in E_2$ si et seulement si $x \in E_1$. E_1 est dit la classe inférieure de la coupure considérée et E_2 sa classe supérieure. Si $\overline{E}$ est un sous-ensemble de E, il existe toujours deux coupures sup $\overline{E}$ et inf $\overline{E}$ définies comme suit :

La classe supérieure E_2 de sup $\overline{E}$ est définie comme l'ensemble E_2 de tous les $x \in E$ tels que pour tout $y \in \overline{E}$, $x > y$, et sa classe inférieure est l'ensemble E_1 des $x \in E$ tels que pour tout $y \in E_2$, $x < y$. Inf $\overline{E}$ se définit d'une manière analogue, en renversant les rôles de E_1, E_2 par rapport à $\overline{E}$. On vérifie sans peine que Sup $\overline{E}$ et Inf $\overline{E}$ sont bien des coupures et que $\overline{E}$ est contenu dans la classe inférieure de sup $\overline{E}$ et dans la classe supérieure de inf $\overline{E}$. On dit que $e \in E$ est un élément limite à gauche si la classe inférieure de inf $\{e\}$ n'a pas de dernier élément, et qu'il est un élément limite à droite si la classe supérieure de sup $\{e\}$ n'a pas de premier élément. Une coupure $c = (E_1, E_2)$ est dite une lacune si E_1 n'a pas de dernier élément, ni E_2 de premier élément. On dit qu'une classe d'une coupure est définie par un $e \in E$ si elle est la classe inférieure de sup $\{e\}$ (ce qui a lieu si et seulement si e est son dernier élément) ou la classe supérieure de inf $\{e\}$ (ce qui a lieu si et seulement si e est son premier élément). D'ailleurs,

$$\sup\{e\} = (\{x \in E;\ x \leq e\},\quad \{x \in E;\ x > e\})$$

et

$$\inf\{e\} = (\{x \in E;\ x < e\}, \{x \in E;\ x \geq e\}).$$

Complété de Dedekind. — Soit $\mathfrak{C}$ l'ensemble des classes inférieures et supérieures non vides de toutes les coupures de E (où, dans le cas où deux classes de coupures différentes coïncident en tant qu'ensembles, il y a lieu de les distinguer; ce cas se produit pour E dans les coupures $(\emptyset, E)$ et $(E, \emptyset)$.

Soit $\tilde{d}$ la relation d'équivalence telle que E', $E'' \in \mathfrak{C}$ tels que $E' \neq E''$ soient congrues $(\text{mod}\ \tilde{d})$, si et seulement si :

1° ou bien ils sont deux classes d'une même coupure qui n'est pas un *saut*, c'est-à-dire telle que, au moins une de ces classes ne soit engendrée par aucun $e \in E$;

2° ou bien les deux classes sont engendrées par un même élément $e \in E$ (donc sont les classes des coupures différentes inf $\{e\}$ et sup $\{e\}$). Un $e \in E$ sera identifié avec la classe $(\text{mod}\ \tilde{d})$ contenant les classes qu'il engendre, et une lacune l avec l'ensemble de ses classes.

L'ensemble $\tilde{E} = \mathfrak{C}/\tilde{d}$ qui est ainsi identifié à la réunion de E et de l'ensemble L(E) de ses lacunes est ordonné comme suit : chaque classe $(\text{mod}\ \tilde{d})$ dans $\mathfrak{C}$ contient au moins une (quelquefois deux) classes supérieures de coupures, et l'on posera $x < y$ $(x, y \in \mathfrak{C}/\tilde{d},\ x \neq y)$ si une classe supérieure $E_x \in x$ contient forcément au sens strict une classe supérieure $E_y \in y$ on vérifie sans peine que c'est un ordre partiel prolongeant celui de E).

Complété de Kurepa. — Soit $\sim$ la relation d'équivalence dans $\mathfrak{C}$, telle que $E', E'' \in \mathfrak{C}$, $E' \neq E''$, soient équivalentes (mod $\sim$) si et seulement si ils sont les classes (de $\inf\{e\}$ et de $\sup\{e\}$ engendrées par un même e ce qui montre que $\sim$ est plus fine que $\tilde{d}$).

On posera $E' < E''$ si et seulement si :

a. ou bien chacun des E' et E'' est la classe inférieure d'une coupure et $E' \subset E''$;

b. ou bien chacun des E', E'' est la classe supérieure d'une coupure et $E' \supset E''$;

c. ou bien E' est la classe inférieure d'une coupure et E'' la classe supérieure d'une coupure (pouvant être identique à la précédente ou différente d'elle), et $E' \cap E'' = \emptyset$. On vérifie sans peine que cette relation est un ordre partiel préservé par $\sim$, donc induit un ordre dans $\mathfrak{C}/\sim$. Le complété de Kurepa est $\tilde{E}_{Ku} = \mathfrak{C}/\sim$ organisé par l'ordre précédent, et l'on applique E dans $\tilde{E}_{Ku}$ avec préservation d'ordre en appliquant $e \in E$ sur la classe d'équivalence dans $\mathfrak{C}$ (mod $\sim$) formée de la classe supérieure de $\inf\{e\}$ et la classe inférieure de $\sup\{e\}$, cette application sera considérée comme une identification. Soit $e \in E$, si $(E_1, E_2) = \inf\{e\}$, e est le premier élément de E_2 et si E_1 n'a pas de dernier élément, autrement dit si e est un élément limite à gauche, E_1 (qui est à la fois, dans ce cas, élément de $\mathfrak{C}$ et de $\tilde{E}_{Ku} = \mathfrak{C}/\sim$) sera noté e^- [dans le cas contraire, si e' est le plus grand élément de E_1, E_1 est bien un élément de $\mathfrak{C}$ mais non de $\tilde{E}_{Ku}$ et sa classe (mod $\sim$), qui contient encore la classe supérieure de $\inf\{e'\}$ est notée, par identification précédente, e']. De même, si e est un élément limite à droite, la classe supérieure de $\sup\{e\}$ n'a pas de plus petit élément et sera notée e^+. Enfin, si l est une lacune $l = (E_1, E_2)$, E_1 sera notée l^- et E_2 sera notée l^+.

Si $x = e^-$, e, e^+ ou l^-, l^+, e ou l [qui est la classe (mod $\tilde{d}$) contenant la classe x (mod $\sim$)] sera dite la *valeur de Dedekind* de x; et $\xi = -, o, +$ s'appelle l'*espèce de Kurepa* (ou le *signe d'espace*) de e^ξ (où l'on pose $e = e^0$) ou de l^ξ. Visiblement, si E est le complété de Dedekind avec son ordre, $\tilde{E}_{Ku}$ peut s'identifier avec un sous-ensemble de $\tilde{E} \times \{-, o, +\}$ dont la projection sur $\tilde{E}$ est $\tilde{E}$ et dont l'ordre se déduit lexicographiquement de celui de $\tilde{E}$ et de l'ordre $- < o < +$ de $\{-, o, +\}$.

(*) Séance du 11 mars 1963.

163.547. — Imp. GAUTHIER-VILLARS & C^ie, 55, Quai des Grands-Augustins, Paris (6^e).
Imprimé en France.

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Nouvelle série, tome 4 (18), 1964, p. 101—106.

SOME REFLEXIONS ON SETS AND NON-SETS

Djuro Kurepa

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1. Membership relation. 1.0. The relation $\in$ meaning *"to be an element of"* was introduced as late as at the end of the 19[th] century (G. Peano).

1.1. The relation $\in$ is not characteristic for sets.

1.1.1. One could call granular structure G any thing consisting of members i.e. satisfying the identity

$$G = \{x; x \in G\}.$$

The solution x of $x \in G$ may be of a very various character and complexity.

1.1.2. There are sets S such that $x \in S$ has no solution x; such a set is the empty set $\emptyset$; one convenes that $\emptyset$ is unique; but one might consider a theory of sets in which there are many void sets.

1.1.3. There are non-sets X consisting of all the x satisfying $x \in X$; such a thing are e.g.: the class O of all ordinal numbers, the class KO of all cardinal-ordinal numbers i.e. of all ordinals $< \omega$ and of all ordinal numbers of the form ω_α $(\alpha \in O)$, the class K of all the cardinal numbers, the class of all sets, the hypertree $(P, \dashv)$ consisting of all the sequences

$$s \cdot \cdot \cdot s_0, s_1, \ldots, s_{\alpha'}, \ldots \quad (\alpha' < \alpha, \ \alpha \in O)$$

such that $s_{\alpha,}$ is an ordinal number satisfying

$$s_{\alpha'} < \omega_{[\alpha']} ; \ \omega_{[\alpha']}$$

denotes the first ordinal number such that the cardinal numbers which are $< k\omega_{(\alpha')}$ form a well ordered set of type α'; for sequences s, t one denotes $s = \!\mid t$, provided s be an initial section of t; if $s = \!\mid t$ and $s \neq t$ one writes $a \dashv b$. The *hypertree* $(P, \dashv)$ is connected to permutations of sets and of numbers.

1.2. For sets the binary ε-relation is antireflexive, antisymmetric and antitransitive. In this sense the relation ε is complete negation of equivalence relations (which are: reflexive, symmetric and transitive). Consequently, the theory of sets being based on the theory of ε-relation, is in a particular connexion to the theory of equivalence relations in the frame of the theory of sets itself.

1.3. The property of being a member (element), or class is of a relative character.

1.4. It is interesting that for any thing b — no matter whether b is a set or a non set — one might let correspond the set $\{b\}$ consisting of b as the unique member (cf. § 7.3).

1.5. Repetition sets or spectra.

1.5.1. For any set S and any member s of S one does not allow the relation $s \in S \setminus \{s\}$. Therefore e.g. $\{2, 3, 3, 3, 4, 4\} \setminus \{3\} = \{2, 4\}$.

1.5.2. On the other hand, there are structures in which it is relevant whether a member occurs once, twice or several times. E.g. for an algebraic polynomial $a(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, $a_n \neq 0$, one considers not only the set σ_a of zeros of a but also the unordered sequence

$$Sa = a_{(1)}, a_{(2)}, \ldots, a_{(n)}$$

of all the zeros of $a(x)$, each with the corresponding frequency and such that

$$a(x) = (x - a_{(1)})(x - a_{(2)}) \cdots (x - a_{(n)}) a_n,$$

where n is the *degree* of $a(x)$.

1.5.3. The whole theory of "*sets with repetition*" or of *R-sets* could be built; where *R-set or spectrum* is any ordered pair (S, f) of a set S and a mapping $f: S \to K$ such that for every $x \in S$ the symbol fx denotes a cardinal number > 0 indicating the frequency of x in S. In such *R-sets* the relation $x \in S \setminus \{x\}$ is well allowed.

2. On the *i*-operators. 2.1. For every object b let $\{b\}$ or ib denote the set consisting of b as its unique term (cf. 1.4). 2.2. The opposite operator: the *anti i-operator or* $-i$*-operator*, associates to every set S all the members of S; thus $-i\{b\} = b$, $-i\{1, 2\} = 1, 2$ etc. The anti $-i-$ operator is a multi-valued function defined on every system of sets, granular structures etc.

2.2. The *i-operator by iteration yields the* i^2*-operator:*

$$i^2 = i\,i \quad \text{i.e.} \quad i^2 b = i\,i b\,.$$

For any object b one could set

$$i^0 b = b,\ i^1 b = ib = \{b\},\ i^{(\alpha+1)^*} b = i(i^{\alpha^*} b),\ i^{\lambda^*} b = \cdots i\,i \cdots i\,i\,b$$

for every limit ordinal λ.

2.2.1. Example. We have the entities:

$$1,\ i\,1 = \{1\},\ i^2 i = \{\{1\}\}, \ldots, i^{\omega^*} 1 = \cdots \{\{\{1\}\}\} \cdots$$

The first term 1 is not a set; the last one neither. While 1 has no "elements" and no structure — 1 is an atom — the entity $c = i^{\omega^*} 1$ has a structure; in particular c is a kind of infinitely complex unity.

2.3. Obviously, the objects b; $\{b\} = ib$, $i^2 b = \{\{b\}\}$ are mutually related. We shall say that b is in ε^2-relation to $i^2 b$ and write $b \in^2 i^2 b$. More generally, we shall write $b \in^2 S$, provided $ib \in S$; and for any ordinal number α we shall write $b \in^{(\alpha)} S$, provided $i^{(\alpha-1)^*} b \in S$ or $i^{\alpha^*} \in S$, according as α is of the first or of the second kind. We get in this way the ε-relations:

$$\varepsilon^0 \text{ meaning } =,\ \varepsilon^1 = \varepsilon,\ \varepsilon^2, \ldots, \varepsilon^\alpha, \ldots$$

for every ordinal number α.

2.4. Sub-element relation. The "logical sum" of these relations might be called the sub-element relation and denoted by E; xEy is read: *x is a sub-element of y*; in particular case that E means $\in$, x is an element of y. It is to be observed that a set S can contain a thing x as its element or subelement of various degrees, as it is shown by considering the set

$$\{1, i\,1, i^2\,1, i^3\,1, \ldots\}$$

2.5. *For sets S the objects*

$$i^0 S = S,\ iS,\ i^2 S,\ \ldots,\ i^{\alpha^*} S,\ \ldots \qquad \text{(for any ordinal } \alpha)$$

are pairwise different. For non-sets S, it is conceivable that the foregoing objects are not all pairwise different.

2.6. It is interesting to observe that there are sets S such that if $x \in^2 S$ then $x \in S$; such are the sets

$$S = \{\emptyset, i\,\emptyset, i^2\,\emptyset\},$$

$$\{\emptyset, i\,\emptyset, i^2\,\emptyset, \ldots\},$$

where $\emptyset$ is the empty set.

2.7. For every object b we have the hypersequence

$$i^0 b = b,\ i^1 b,\ \ldots\ i^{\alpha^*} b,\ \ldots$$

of sets $i^{\alpha^*} b$ for $\alpha^- < \alpha$ and of non sets $i^{\alpha^*} b$ for $\alpha^- = \alpha$; as to $i^0 b = b$, b might be a set as well as a non set.

3. Granular structures and non granular structures.

3.1. In every granular structure G the elements of G are differentiated; there are granular structures which are nonsets. Such structure is every class which is "too extensive" to be a set, e.g. the class of all ordinal numbers or the class of all sets.

3.2. A new kind of non-set structure is obtained by considering *things with non differentiated elements*, the "elements" having no individuality (in atomic physics, in biology and in the theory of big molecules one is dealing with such non-differentiated non granular structures).

3.3. Another kind of non-set structures is obtained, when the "evolution" of S is too much put forward in such a way that the constituents of S need not be elements of S. Such one is the structure $i^{\omega^*}\,1 = \cdots\{\{\{1\}\}\}\cdots$: too many shells are present and we are not able to reach from outside any constituent. In this example there is a unique constituent; it is ready to form more complicated structures with many constituents, tied and quite nonseparable mutually.

3.4. The *notion of structure* — granular or non granular — *is very general and multivalent*. The study of various structures is the very object of many human activitities. Every science is a structure. Every machine is a structure. Language is a structure. Mental structures are of vital importance; mathematical structures are reflecting some special observed structures. The classification of various structures, the interconnections between them are very important topics. It is very important to examine the transitions and mutations of a structure moving from a domain in another domain. As example let us mention the following structures: relation, group, system, family, operation etc. which generated in biology but are transplanted in other fields, particularly into mathematics.

4. *Vacuous or void set. All-sets.*

4.1. We assume the existence of a set without elements or proper parts and being part of every set; it is called the vacuous or empty set and denoted by $\bigcirc$ or v or $\wedge$ or $\emptyset$. Consequently, v is a set but the relation $x \in v$ does not hold. We assume that v is unique. The relation $v \subseteq S$ for every S is really a *definition* of v.

The consequence of the convention $v \subseteq S$ is $v \in PS$ for every set S (as usually, PS denotes the set of all the parts of S).

The set $\{v\}$ is not empty: $\{v\}$ consists of v as its single element.

The philosophical aspect of the distinction of v from $\{v\}$ is evident. The mathematical implications of this distinction are very far-reaching. Is it really non-contradictory to form $\{v\}$ and to distinguish $\{v\}$ and v?

v consists of nothing on the one hand, and on the other hand *v is a part of every set S* and even an *effective element* of every *P-set PS*. Hence, S being any set the vacuous set v is an element and a part of PS i.e. $v \in PS \cap PPS$.

The void set is connected to a number — with 0[1].

The notion of void set is a useful convention and presents a magistral dialectical unification of two different items: void and non void. It is to be observed that void sets were introduced as late as the beginning of this century. There is a unique void set, although by provenience one could classify the void sets in very various ways. The properties, conventions, terminology concerning vacuous set might be very various and in mutual contradiction. For instance, the set (space) v is considered as dense, non dense, nowhere dense, finite, etc.

The considerations about the number 0 form a chapter of the theory of void sets; dynamic theory of 0 is the very basis of infinitesimal processes.

4.2. The logical counterpart of void set — the "all-set" is not conceivable as a *set* because such an idea would lead to contradictions. The non-vacuous sets are either finite or transfinite. The theory of transfinite sets is of great philosophical importance, and closely tied to logical quantors and hierarchy types. The theory of void set(s) on the one hand and that one of transfinite sets on the other hand are two aspects of human mathematical and philosophical activity.

4.3. *The void set and atoms.* The void set v is to be distinguished from general "*atoms*". Atoms have no elements; e.g. such are the points in the sense of Euclid. Various points are elements of sets. Since the points are distinguishable, we are able to adjoin every individual point p to every set S — the result is again a set, the set $\{p\} \cup S$; in general, this set differs from S; what is to be compared with the identity $S \cup v = S$ for every set S.

5. On the operator $x \cup \{x\} = ux$ for every thing x.

5.1. Definition: *ub is obtained by adjoining to b the thing b as an element* i.e. ub consists of the elements of b and of b as a member:

(1) $$x \in ub \Leftrightarrow x = b \vee x \in b.$$

5.2. Value of *ub* for any set *b*. If b is a set, then ub is a set containing as well b as an element as well as all the elements of b; since $b \notin b$ (b being

[1] The role of the number 0 is tremendous. How the role of 0 might be of a relative character, let us remember that 0 assumes the role of neutral element in a group.

If we are dealing with single-valued fonctions f on a set S, we could realize f as changing every x of S into fx; by idealization, we consider the identity *mapping* too as a "*changing*", although there is no changing at all. Similarly, the *resting* is called a *moving* with the speed 0.

a set), we see that both $ub \supset b$ and $ub \ni b$ and more precisely $ub \setminus \{b\} \ni b$. In particular, $ub \neq b$ and moreover, $b \neq ub \neq \{b\}$ for every non void set b. One has $uv = v \cup \{v\} = \{v\}$.

5.3. Ub for any non set *b*. Let us consider the case when b. is not a set.

5.3.1. Case: *b has no element(s)*: the relation $x \in b$ is not possible. In this case $ub = \{b\}$. In fact, since the relation $x \in b$ has no solution x, then $x \in ub \Rightarrow x = b$ and consequently $ub = \{b\}$.

5.3.2. Case: *b is a non-set containing at least one element:* $x \in b$ is possible for some x (this case occurs e.g. if b contains very many elements—*b is a class, a superset*). In this case again, ub contains as elements all the elements of b as well as b itself. Only, in this case the relation $b \in b$ is not excluded. If $b \in b$, then $ub = b$ and $b \in ub$. If $b \notin b$, then $ub \ni b$ and $ub \setminus b \ni b$; consequently, $ub \neq b$, $\{b\}$. In particular, for every non void set b we have $b \notin b$ and therefore $ub \neq b$, $\{b\}$, as stated already in § 5.2.

5.4. Theorem. *The system*

(1) $$ub = \{b\}, \ b \text{ non} \in b$$

is characteristic for atoms or points i.e. for things containing no element; in particular, for the vacuous set $\emptyset$ *one has* $u\emptyset = \{\emptyset\}$ (the vacuous set is considered also as an atom). In other words, if (1) holds, then b is an atom; and conversely, if b is an atom then (1) holds.

First, if b is an atom, then (1) holds, as was shown in 5.3.1. Let us prove the converse: (1) implies that b is an atom. In virtue of 5.2. b is not a non vacuous set; consequently, b is either vacuous set $\emptyset$ or b is a non-set; in the last case, there is no x satisfying $x \in b$ i.e. b is an atom. Suppose on the contrary that the relation $x \in b$ be possible. Since b non $\in b$, then b and x would be two different elements of ub, in contradiction to the hypothesis (1) stating that ub is a single-pointset $\{b\}$.

The theorem 5.4. may be expressed in the following form.

5.5. Theorem. *The relation* $x \in b$ *holds for at least one* x *if and only if* $ub \neq \{b\}$ *or* $b \in b$.

Let us prove this theorem directly.

1. First, if $x \in b$ is possible, b is either a non empty set or a superset; if b is a non vacuous set, then $b \neq ub \neq \{b\}$; if b is a superset, then ub also is a superset and might not be equal to the set $\{b\}$ consisting of the single member b. Consequently, $x \in b \Rightarrow ub \neq \{b\}$.

2. Conversely, let us prove that $ub \neq \{b\} \Rightarrow x \in b$ for some x. We have to distinguish two cases.

The implication being obvious for the case $b \in b$, let us suppose that b non $\in b$.

First case: *b is a set;* since $u\emptyset = \{\emptyset\}$, one has necessarily $b \neq \emptyset$ and hence $x \in b$ for some x.

Second case: *b is a non set;* b is not an atom because every atom satisfies $ub = \{b\}$. Consequently, b is a superset and consequently, *one ha* $x \in b$ for some x.

5.6. Theorem. If

(1) $b \cup \{b\} = b$, then $b \in b$; and conversely,

(2) $b \in b$ implies (1); *consequently* (3) $b \cup \{b\} = b \Leftrightarrow b \in b$.

First, by definition $b \in ub$ i.e. $b \in b \cup \{b\}$ and hence by (1) one has $b \in b$. This means that $(2) \Leftrightarrow (1)$.

Secondly, if $b \in b$, then $\{b\} \subseteq b$ and $b \cup \{b\} = b$; in other words (2) $\Rightarrow$ (1).

6. Operator u_α for any ordinal α.

6.1. Definition. Let α be any ordinal $\neq 0$; for every thing b we set $u_1 b = i^0 b = b$

$$u_2 b = b \cup \{b\} = i^0 b \cup i^1 b$$
$$u_3 b = b \cup \{b\} \cup \{\{b\}\} = i^0 b \cup i^1 b \cup i^2 b$$
$$\cdots\cdots\cdots$$
$$u_\alpha b = \bigcup i^{\alpha'} b, \ (\alpha' < \alpha) \text{ for every ordinal } \alpha > 0.$$

One could say that

$$u_\alpha = \bigcup_{\alpha'} i^{\alpha'} \qquad (\alpha' < \alpha).$$

In the foregoing sections we considered the operator u_2; the index α in u_α indicates the order type of summands in the definition of u_α.

6.2. Of course, it is a particular problem to study the foregoing functions u_α as well as other ones connected to $i^{\alpha*}$-operators.

6.2.1. It should be particularly interesting to study the function $i^0 \cup i^2$ i.e. to form $b \cup \{\{b\}\}$ for any object b.

6.3. Theorem. *$b \in b \Rightarrow u_\alpha b = b$ as well as $i^\alpha b \in b$ for any positive ordinal $\alpha \leq \omega$.*

As a matter of fact, we have the following chain of implications:

$$b \in b \Rightarrow \{b\} \subseteq b \Rightarrow ib \subseteq i^0 b \Rightarrow i^2 b \subseteq ib \subseteq i^0 b \Rightarrow ib \in b;$$

from here, by iteration one gets $i^2 b \in b$, $i^3 b \in b$, ...

Therefore we have the following

$$u_2 b = u_0 b \cup u_1 b = b \cup \{b\} = b$$
$$u_3 b = b \cup \{b\} \cup \{\{b\}\} = b$$
$$\cdots\cdots\cdots$$
$$u_n b = b \text{ and}$$
$$u_\omega b = b \cup ib \cup i^2 b \cup \cdots = b.$$

7. The foregoing considerations show a particular and very important role of the relation

(1) $$b \in b$$

and of the mapping

(2) $$b \to \{b\}.$$

The simplest standpoint is the following one:

7.1. *No set satisfies $b \in b$;*

7.2. *No non-set satisfies $b \to \{b\}$.*

7.3. It is a special task to consider *axiomatically* also such theories of sets in which one of the propositions 7.1, 7.2 or both are not accepted.

BIBLIOGRAPHY

P. Bernays — A. A. Fraenkel, *Axiomatic Set Theory*, Amsterdam 1958, VIII + 226;

A. A. Fraenkel, *Abstract Set Theory*, Amsterdam 1953, XII + 479;

Đ. Kurepa, *Some philosophical aspects of the set theory* (Atti XII Congres intern. Filosophia, Venezia 1958, Vol. 5, Firenze 1960, 323—326.

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ON TRIANGULAR MATRICES

Đuro Kurepa, Zagreb

0. Introduction[1]

0.0. Since triangular matrices are much simpler than general matrices it is obvious that the study of a matrix which is representable as a product of triangular matrices is easier to be effectuated than if no such factorization is known or possible. This is so in particular if a matrix a is the product of triangular matrices x, y such that the last factor y is not only triangular but also such that its diagonal is the constant 1. One has this situation e. g. in the method of solving systems of linear equations by means of *Banachiewicz's method.*

0.1. Something analogous could be said for determinants.

0.2. Now, the question arises as to whether a triangular factorization of a given (finite-size) square matrix is or is not possible.

We shall indicate a class of matrices which consists exactly of products of ordered pairs of regular matrices (7.1; *G-matrices*); consequently, it will be easy to indicate finite-size matrices with are not products of any two triangular matrices.

0.3. Since every finite-size square matrix a is representable as a product of triangular matrices, it is natural to associate with a the minimal cardinal number $K(a)$ such that a be the product of a series of $K(a)$ *triangular matrices* (K is the field or the ring over which the matrix a is defined). For finite-size matrices a over the field R of real numbers or over the field of complex numbers one has $Ka \leq 3$ (cf. 8).

0.4. The matrices we shall consider have values in a given ring A with unit[2]; we shall suppose that A has no zero-divisors, i. e. that the product of two non zero elements of A is a non-zero ele-

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[1] Some results of this paper were presented at the Colloquium of the Society of Mathematicians and Physicists of Croatia on 27. 03. 1963.

[2] Every ordered triplet $(A, +, \cdot)$ consisting of a set A and of the operations $+, \cdot$ such that $(A, +)$ be a commutative group and that $(A, \cdot)$ be a groupoid such that operation $\cdot$ is left and right distributive on $+$ is called a *ring; ring* will be supposed to be *associative,* i. e. $(ab)c = a(bc)$ holds good. A unit of R is such an element $1 \in R$ that $x \cdot 1 = 1 \cdot x = x$, for every $x \in R$.

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ment of A[3]. There is a great difference between the case when the multiplication in the ring is commutative and the case when it is not commutative; in particular, in a commutative and associative ring with division one has a theory of determinants and one of linear equations the same as for the ring of real or complex numbers.

1. Triangular matrices

1.1 A *(n, n)-triangular lower* or *L-matrix* or *d-matrix* is such a matrix that all the elements *above* the main diagonal are 0. The transpose of a *d*-matrix is an *upper matrix* or a *g-matrix*. Consequently, if $a = xy$ is a triangular 2-factorization of a (two factors!), then theoretically one has the following possibilities of multiplication:

$$d\,d,\ d\,g,\ g\,g,\ g\,d.$$

1.2. *Left (right) triangular matrices.* Analogously, one defines left and right triangular matrices l and r. Instead of the main diagonal d one could consider also the *second diagonal* d', consisting of the values

$$a_{1n},\, a_{2,\,n-1},\, \ldots,\, a_{i,\,n+1-i},\, \ldots,\, a_{n1};$$

if all elements *below* (*above*) of d' are 0 the matrix is said to be a *right* (resp. *left*) triangular matrix.

1.3. *Four forms triangular matrices.* Consequently, we have the following definitions concerning any (n, n)-matrix $a = [a_{ik}]_{i,k}$:

1.3.1. a is a *d-matrix* $\langle = \rangle$ lower-triangular matrix $\langle = \rangle$

$$\langle = \rangle\ i < k \Rightarrow a_{ik} = 0;$$

1.3.2. a is a *g-matrix* $\langle = \rangle$ upper-triangular matrix $\langle = \rangle$

$$\langle = \rangle\ i > k \Rightarrow a_{ik} = 0;$$

1.3.3. a is a *l-matrix* $\langle = \rangle$ left triangular matrix $\langle = \rangle$

$$\langle = \rangle\ i + k > n + 1 \Rightarrow a_{ik} = 0;$$

1.3.4. a is a *r-matrix* $\langle = \rangle$ right triangular matrix $\langle = \rangle$

$$\langle = \rangle\ i + k < n + 1 \Rightarrow a_{ik} = 0.$$

1.3.5. Consequently, one has four forms of triangular matrices: 2 forms d, g of the *first kind* and two forms l, r of the second kind.

1.3.6. The symmetry with respect to the vertical (horizontal) median of the square carries the matrices of the first (second) kind onto the matrices of the second (first) kind.

[3] An element a of R is regular provided that for some $x \in R$, one has $ax = 1 = xa$; x is denoted a^{-1}.

2. Some kinds of transposition of matrices

For an (n, s)-matrix a-regardless whether of size $n = s$ or $n \neq s$ we define the following four transposes:

2.1. $\underset{(n,s)}{a^T}$; $(a^T)_{i\nu} = a_{\nu i}$ (symmetry with respect to the first diagonal);

2.2. a^{T_2} ; $(a^{T_2})_{i\nu} = a_{n+1-\nu,\, s+1-i}$ (symmetry with respect to the second diagonal);

2.3. $\underset{(n,s)}{a^{|T}}$; $(a^{|T})_{i\nu} = a_{i,\, s+1-\nu}$ (symmetry with respect to the vertical median);

2.4. $\underset{(n,s)}{a^{-T}}$; $(a^{-T})_{i\nu} = a_{n+1-i,\, \nu}$ (symmetry with respect to the horizontal median).

In particular, for finite-size (n, n)-matrices a one has

$$a_{ik}^{T_2} = a_{n+1-k,\, n+1-i},$$
$$a_{ik}^{|T} = a_{i,\, n+1-k},$$
$$a^{-T} = a_{n+1-i,\, k}.$$

2.5. Remark. For finite-size non-square (n, s)-matrices one could try to define two transposes more: as symmetric mappings with respect to the diagonal issuing from the lower left corner and from the lower right corner (observe that every finite-size non-square matrix has four diagonals, one corresponding to its own corner). But one sees that the first operation yields a^{T_2} and that the second operation yields a^T.

2.6. *Central transpose of a finite-size matrix* a is defined as the matrix a^c obtained from a by the *central symmetry mapping*:

$$(a^c)_{i\nu} = a_{n+1-i,\, s+1-\nu}.$$

2.7. One verifies readily that

$$a^c = (a^{|T})^{-T} = (a^{-T})^{|T}.$$

2.8. *Every kind of the foregoing transposes is idempotent: its first iteration yields the original matrix from which we started.*

2.9. *For any diagonal square matrix a one has*

$$a^{|T} = a^{-T} = \text{the second diagonal matrix} = \begin{bmatrix} & & & a_{1n} \\ & & a_{2n} & \\ & \cdot^{\cdot^{\cdot}} & & \\ a_{n1} & & & \end{bmatrix}$$

in which all elements outside the second diagonal vanish.

2.10. In particular, we consider the matrix

$$1' = 1'(n) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}^{-T} = \begin{bmatrix} & & & 1 \\ & & \cdot^{\cdot^{\cdot}} & \\ & 1 & & \\ 1 & & & \end{bmatrix}.$$

2.11. *Several kinds of symmetries.* In connexion with every preceding transpose one defines the *corresponding symmetry* and *skew symmetry*, respectively; e. g. a is *centrally symetric* (skew symmetric) provided

$$a^c = a \text{ (resp. } a^c = -a).$$

In a particular case that the values a_{iv} belong to the field of complex numbers or to some other field with conjugation $z \to \bar{z}$ one defines not only the scalar symmetry (skew symmetry) but also the *hermitian symmetry* and *hermitian skew symmetry* by the equalities

$$a^c = \bar{a} \text{ and } a^c = -\bar{a}, \text{ respectively}.$$

E. g. a is *horizontally hermitian skew symmetric* provided

$$a^{-T} = -\bar{a}.$$

3. Square root unit 1′ (n) and various transposes

For any n we defined the vertical transpose $1'(n)$ of the unit (n, n)-matrix $1(n)$:

$$1'(n) = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{bmatrix}, \ n \text{ rows.}$$

3.1. For any positive integer n one has $1'(n)^2 = 1(n)$ or shorter $1'^2 = 1$. Therefore, the matrix $1'(n)$ is called the *square root unit matrix of order* n.

3.2. Theorem. 3.2.1. $\underset{(n,s)}{a} \cdot 1'(s) = a^{|T}$, *i. e.*

$$\underset{(n,s)}{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1s} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \dots & a_{ns} \end{bmatrix}} \underset{(s,s)}{\begin{bmatrix} & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{bmatrix}} = \begin{bmatrix} a_{1s} & a_{1\,s-1} & \dots & a_{11} \\ \cdot & \cdot & \cdot & \cdot \\ a_{ns} & a_{n\,s-1} & \dots & a_{n1} \end{bmatrix}.$$

3.2.2. $1'(n) \cdot \underset{(n,s)}{a} = a^{-T}$, *i. e.*

$$\begin{bmatrix} 0\,0 & \dots & 1 \\ & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix} \begin{bmatrix} a_{11} & \dots & a_{1s} \\ \cdot & \cdot & \cdot \\ a_{n1} & \dots & a_{ns} \end{bmatrix} = \begin{bmatrix} a_{n1} & \dots & a_{ns} \\ \cdot & \cdot & \cdot \\ a_{11} & \dots & a_{1s} \end{bmatrix}$$

3.2.3. $\underset{(n,s)}{a^c} = 1'(n)\, a\, 1'(s)$.

3.2.4. $a^{T_2} = \left((a^{|T})^T\right)^{|T} = a^{|T\,T\,|T} = \left(a\, 1'(s)^T\right) 1'(n) =$
$= 1'(s)\, a^T\, 1'(n)$, i. e. $a^{T_2} = 1'(s)\, a^T\, 1'(n)$.

3.2.5. $a^{T_2} = \left((a^{-T})^T\right)^{-T}$.

3.2.6. $a^T = \left((a^T)^{T_2}\right)^{|T} = \left((a^{-T})^{T_2}\right)^{-T}$.

3.2.7. $(ab)^{T_2} = b^{T_2} a^{T_2}$, *i. e. the second diagonal transposition follows the same rule as the first diagonal transposition.*

3.2.8. $(a\,b)^c = a^{-T}\, b^{|T}$, $(a\,b)^c = a^c\, b^c$; *in particular, if* a, b *are square finite-size matrices, then* $(ab)^c = a^c\, b^c$. *In other words, the central transposition is distributive with respect to the multiplication of matrices.*

3.2.9. Corollary.

$$\begin{aligned} d\,1' &= r, & 1'\,d &= l, \\ g\,1' &= l, & 1'\,g &= r, \\ r\,1' &= d, & 1'\,r &= d, \\ l\,1' &= g, & 1'\,l &= d. \end{aligned}$$

The proof of Theorems 3.2. is easy. Let us prove e. g. Theorem 3.2.4. $(((a^{|T})^T)^{|T})_{ij}$ (going from outside, in the order as the operators $|T, T, |T$ act) equals

$$((a^{|T})^T)_{i,\, n+1-j} = (a^{|T})_{n+1-j,\, i} = a_{n+1-j,\, s+1-i}.$$

Or, in this way:

$$(a\,1'(s)^T\,1'(n) = (1'(s)^T\,a^T)\,1'(n) = 1'(s)\,a^T\,1'(n)\,.$$

Let us prove the formula concerning the second diagonal transposition:

$$\begin{aligned} (a\,b)^{T_2} &= 1'(t)\,(ab)^T\,1'(n) = 1'(t)\,(b^T\,a^T)\,1'(n) = 1'(t)\,b^T\,1(s)\,a^T\,1'(n) = \\ &= (1'(t)\,b^T\,1'(s))\,(1'(s)\,a^T\,1'(n)) = b^{T_2}\,a^{T_2}, \end{aligned}$$

because of $1(s) = 1'(s)\,1'(s)$.

Analogously,

$$(ab)^c = 1'(n)\,(ab)\,1'(t) = (1'(n)\,a\,1'(s))\,(1'(s)\,b\,1'(t)) = a^c\,b^c\,.$$

3.3. Theorem. *$a^c = a$ is equivalent to $1'(n)\,a = a\,1'(s)$; if $n = s$, then for every positive integer m*

$$(a^m)^c = a^m\,;$$

if moreover, a is regular, then for every integer m the preceding equation holds.

For every polynomial f one has $a^c = a \Rightarrow (f(a))^c = f(a)$.

Proof. One has $\underset{(n,s)}{a^c} = 1'(n)\,a\,1'(s)$. From here, multiplying on the right side by $1'(s)$, one obtains the requested equality, because $1'(s)^2 = 1(s)$.

Let a be a square finite-size centrally symmetric matrix; then

$$\begin{aligned} a^2 = a\,a &= (1'(n)\,a\,1'(n))\,(1'(n)\,a\,1'(n)) = 1'(n)\,(a\,(1'(n)\,1'(n)\,a)\,1'(n) = \\ &= 1'(n)\,(a\,1\,(n)\,a)\,1'(n) = 1'(n)\,a^2\,1'(n)\,, \end{aligned}$$

i. e. a^2 is centrally symmetric.

Analogously one proves by an induction argument that $a^3, a^4, \ldots$ are centrally symmetric:

$$a^{m+1} = a^m a = (1'(n) a^m 1'(n)) (1'(n) a 1'(n)) = 1'(n) a^m a 1'(n) = \\ = 1'(n) a^{m+1} 1'(n), \text{ i. e. } 1'(n) a^{m+1} 1'(n) = a^{m+1}.$$

3.4. Theorem. The set of all centrally symmetric matrices over a given ring R of a given size (n, s) forms an additive group; if $n = s$, they form a ring, in which for any member a the set $(a^{I\omega}, \cdot)$ is a multiplicative semi-group; $a^{I\omega} = \{a^n; n \leqslant I\omega = \text{the set of all non negative integers}\}$; if $n = s$ and if a is regular, then $a^D = \{a^x; x \leqslant D, D = \text{set of all rational integers}\}$ is a multiplicative group; in particular a^{-1} is centrally symmetric and one has

$$(a^{-1})^c = (a^c)^{-1}. \tag{1}$$

For every finite-size (n, n)*-matrix* a one has the unique decomposition

$$a = 2^{-1}(a + a^c) + 2^{-1}(a - a^c)$$

into a centrally symmetric part and a skew centrally symmetric part. By definition of a^{-1}, we have

$$a\, a^{1-} = a^{-1} a = 1. \tag{2}$$

The premultiplication by $1'(n)$ and the postmultiplication by $1'(n)$ and the intercallation of $1 = 1' \cdot 1'$ yield

$$(1' a 1')(1' a^{-1} 1') = (1' a^{-1} 1')(1' a 1') = 1,$$

and since by hypothesis $1' a 1' = a$, one has

$$a (a^{-1})^c = (a^{-1})^c a = 1,$$

which together with (2) yields

$$a (a^{-1})^c = a\, a^{-1}.$$

The premultiplication by a^{-1} yields the requested central symmetry $(a^{-1})^c = a^{-1}$. Now, from (2) we have

$$(a\, a^{-1})^c = 1^c = 1,$$

which in virtue of Theorem 3.2. yields

$$a^c (a^{-1})^c = 1, \text{ i. e. } (a^{-1})^c = (a^c)^{-1}.$$

Analogously, one proves that $(a^m)^c = (a^c)^m$, for every rational integer m.

3.5. Remark. If a^m is centrally symmetric, the matrix a need not be centrally symmetric.

3.6. A particular set of matrices is that consisting of *power centrally symmetric matrices* (a matrix x is *power centrally symmetric* if for some positive integer n the power x^n is centrally symmetric); all nilpotent matrices belong to the set (it is not difficult to indicate a square matrix a such that no a^m is centrally symmetric, m being a positive integer). Such a matrix is $a = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, because

$$a^m = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix} \text{ and } 1' a^m - a^m 1' = m \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \neq 0.$$

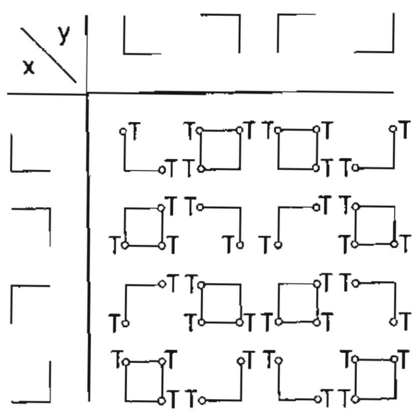
4. Products of two triangular matrices

If x, y are triangular matrices, then the product xy need not be triangular (of course, one supposes that the conformability condition[4] $D_2 x = D_1 y$ for the existence of xy is fulfilled). One proves readilly the following theorem.

4.1. T h e o r e m. *For any finite-size matrix* (n, n) *one has the following table of multiplication of matrices:*

x/y	d	g	l	r
d	d	□	□	r
g	□	g	l	□
l	l	□	□	g
r	□	r	d	□

In particular, *for regular triangular matrices x, y one has the following table:*



If the matrices x and y are regular, then the T-vertex of the product xy is $\neq 0$.

[4] For a matrix a we denote by $D_1 a$ (resp. $D_2 a$), the cardinal number of all the rows (columns) of a.

A *T-vertex* of the product xy is such a one that the corresponding value of the product xy is the product of one hypothenuse value of the factor x and of one hypothenuse value of the factor y.

4.2. Corollary. *No regular square matrix the corners of which are* 0 *is the product of two triangular matrices.*

E. g. the regular (4, 4)-matrix

$$a = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 3 & 4 & 2 \\ 4 & 1 & 1 & 0 \\ 0 & 2 & 3 & 0 \end{bmatrix}$$

is not of the form xy, where x, y are triangular.

The statement holds for matrices over any ring having no zero divisors.

5. Regular triangular matrices

5.1. A square matrix a is termed *regular* provided there exists at least one matrix x such that $a\,x = 1 = x\,a$.

One proves that this solution x is unique; it is denoted by a^{-1}.

5.2. Non-regular square matrices are said to be *singular*.

5.3. Lemma. For any positive integer n the matrices $1(n)$, $1'(n) = 1(n)^{-T}$ are regular.

As a matter of fact, they both satisfy the equation $xx = 1(n)$.

5.4. Theorem. (0). *The product of two regular matrices of size* (n, n) *is regular.*

(00). *The product of a regular (singular) and a singular (regular) matrix, both of size* (n, n), *is singular.*

(000). *For finite-size* (n, n)*-matrices over a field K, the product of two singular matrices is singular.*

(0000). *There are singular not finite-size square matrices* a, b *over the field R of reals such that* ab *be regular.*

(00000). *Schematically one has the following multiplication table:*

	1	0
1	1	0
0	0	$\left\{\begin{matrix}0\\1\end{matrix}\right.$

where 1 *means the property to be a regular matrix and* 0 *means the property to be a singular matrix.*

Let us prove the case (00): if x is regular and if y is singular, then xy is singular.

In the other case, the product xy would be regular and one would have a matrix z such that

$$(xy)z = 1 = z(xy) .$$

From here, the premultiplication by x^{-1} would yield $yz = x^{-1}$, which (post) multiplied by x yields $yzx = 1$ and thus, since $1 = = zxy$,

$$y(zx) = 1 = (zx)y ;$$

in other words, y would be regular and $y^{-1} = zx$, contrary to the hypothesis that y is singular.

The proof of the case (000), for any commutative K, runs as for the case of real-valued or complex-valued matrices, because in K one has a theory of determinants like in the case $K = R$ (field of reals) or $K = R(i)$ (field of complex numbers).

P r o o f o f (000). For any K (regardless of whether commutative or non commutative) the proof runs like this.

Suppose on the contrary that for some integer $n > 0$ and some singular (n, n)-matrices x, y the product $xy = a$ be regular. In any case there is a regular matrix p such that $px = d$ (the proof is like in real and commutative K). Hence, $(px)y = dy$, i. e. $pa = dy$; as a product of regular matrices p and a the product $(pa =)\ dy$ is a regular matrix, say b:

$$dy = b \text{ (regular)} ;$$

hence,

$$d(yb^{-1}) = 1(n) .$$

The product yb^{-1} as product of the singular matrix y and of the regular matrix b^{-1} is singular; for the same reason, the matrix d should be singular, which is in contradiction with the following lemma.

5.4.1. L e m m a. *If* $dx = 1(n)$, *then* d *is regular.*

The lemma is a consequence of Lemma 5.6.1.1 and of Lemma 5.6.2.

In order to prove (0000), let us consider the following infinite shift matrices S_1, S_{-1} (cf. M a c D u f f e e [1], p. 108):

$$S_1 = \delta_{i+1,k} = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \cdot & \vdots \\ & & & & \cdot \;\; \cdot \end{bmatrix} =$$

= characteristic function of the first upper diagonal;

$$S_{-1} = \delta_{i-1,k} = \begin{bmatrix} 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} =$$

= characteristic function of the first lower diagonal.

One sees readily that

$$S_1 S_{-1} = 1(\omega) = \text{infinite unit matrix}, \tag{1}$$

$$[S_{-1} S_1]_{11} = 0, \tag{2}$$

otherwise

$$(S_{-1} S_1)_{ij} = \delta_{ij}.$$

Now, S_1 is singular; otherwise, if S_1 was regular, the relation (1) would imply $S_1^{-1} = S_1^{-1}(S_1 S_{-1}) = S_{-1}$, which jointly with $S_1^{-1} S_1 = 1$ would yield $S_{-1} S_1 = 1$, contradicting (2). In the same way one sees that S_{-1} is singular. Consequently, S_1, S_{-1} are two singular matrices and their product is the regular matrix $1(\omega)$.

5.4.2. Remark. In the foregoing example of an ordered pair of singular matrices with a regular product the matrices are of infinite domain but over a commutative field; this fact is to be compared with the statement 5.4 (000).

5.5. Theorem. (0). *If a triangular matrix is regular, then all of its hypothenuse values are regular; and vice versa; more explicitly:*

$$d \text{ is regular} \Leftrightarrow d_{\nu\nu} \neq 0 \ (\nu \leq n);$$
$$g \text{ is regular} \Leftrightarrow g_{\nu\nu} \neq 0 \ (\nu \leq n);$$
$$r \text{ is regular} \Leftrightarrow r_{\nu,n+1-\nu} \neq 0 \ (\nu \leq n);$$
$$l \text{ is regular} \Leftrightarrow l_{\nu,n+1-\nu} \neq 0 \ (\nu \leq n).$$

(00). *In any regular triangular matrix all corresponding principal corner square matrices are regular.*

5.5.1. Definition. For a finite-size (n, s)-matrix we have the following four *corners of the matrix:*

$$(1, 1), (1, s), (n, 1), (n, s).$$

For a given corner V of a matrix a, any submatrix whose domain is a subrectangle of Dom a containing V as its corner is called a *principal submatrix of a, relative to the corner V.*

5.5.2. Examples. For the matrix

$$a = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \end{bmatrix}$$

the principal matrices corresponding to the top left corner (1,1) are:

$$[1], \quad \begin{bmatrix} 1 & 2 \\ 6 & 7 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 6 & 7 & 8 \\ 11 & 12 & 13 \end{bmatrix};$$

the principal matrices corresponding to the corner (1,5) (top right corner) are:

$$[5], \quad \begin{bmatrix} 4 & 5 \\ 9 & 10 \end{bmatrix} \quad \begin{bmatrix} 3 & 4 & 5 \\ 8 & 9 & 10 \\ 13 & 14 & 15 \end{bmatrix}.$$

5.6. Proof of Theorem 5.5. If one deals with matrices over the field of reals or over the field of complex numbers Theorem 5.5 is obvious: the theorem is an easy consequence of the theory of determinants.

5.6.1. Lemma. *If d is regular, then the inverse d^{-1} of d is of the same form and $(d^{-1})_{\nu\nu} = (d_{\nu\nu})^{-1} \neq 0 \neq d_{\nu\nu}$; i. e. diag $d^{-1} =$ $= (\text{diag } d)^{-1} = \text{diag } [d_{11}^{-1}, d_{22}^{-1}, \ldots, d_{nn}^{-1}]$.*

Let x be the inverse of d which is of size (n, n):

$$d\,x = 1 = x\,d.$$

We have $d_{1.} x_{.\nu} = \delta_{i\nu}$; in particular

$$d_{1.} x_{.1} = 1 \text{ i. e. } d_{11} x_{11} = 1, \text{ hence, } x_{11} = d_{11}^{-1}$$

and the value d_{11} is regular. For $\nu > 1$ we have

$$\delta_{i\nu} = 0 = d_{11}\, x_{1\nu}$$

(because $d_{1j} = 0$, for $j > 0$); hence,

$$x_{1\nu} = 0, \text{ for } \nu > 1.$$

Let $1 < j \leq n$ and suppose that for every $i < j$ one has proved

$$x_{ii} = d_{ii}^{-1}, x_{i\nu} = 0, \text{ for } \nu > i. \tag{3}$$

We have

$$1 = \delta_{jj} = d_{j1} x_{1j} + \ldots + d_{j\,j-1} x_{j-1\,j} + d_{jj} x_{jj},$$

i. e. (because of (3))

$$1 = d_{jj} x_{jj}, x_{jj} = d_{jj}^{-1}$$

and, therefore, $d_{jj} \neq 0$. Thus, if $j < n$ and $j < k < n$, one has

$$\delta_{jk} = 0 = d_{jj} x_{jk}, \text{ for } k > j;$$

and since $d_{jj} \neq 0$,

$$x_{jk} = 0; \text{ for } j < k.$$

This proves the lemma. The foregoing proof yields also the following lemma.

5.6.1.1. Lemma. *If for finite-size (n,n)-matrices d, x over a unit ring without zero-divisors, one has $d\,x = 1\,(n)$, then the diagonal values of d are regular and one has $x_{\nu\nu} = d_{\nu\nu}^{-1} \neq 0$, for $\nu =$ $= 1, 2, \ldots, n$.*

5.6.2. Lemma. *If in a lower triangular matrix d the elements on the hypothenuse are regular, then d is regular.*

As a matter of fact, as in 5.6.1 one proves that the equation $d\,x = 1$ has a solution of the d-form d and that the same solution x satisfies $x\,d = 1$; therefore, the solution x of $d\,x = 1$ is precisely d^{-1}.

5.6.3. Theorem 5.5. (00), for $a = d$, is a consequence of Lemmas 5.6.1 and 5.6.2.

5.6.4. Theorem 5.6. holds for other three forms of triangular matrices.

5.6.5. E. g. if r is regular, then $r\,1'(n) = d$ is regular too; in virtue of Theorem 5.5. (0) applied to d, the elements $d_{\nu\nu}$ are $\neq 0$; now $d_{\nu\nu} = r_{\nu\, n+1-\nu}$, and we have $r_{\nu\, n+1-\nu} \neq 0$; in other words, the regularity of r implies the regularity of every element on the hypothenuse of r. And vice versa:

$r_{\nu\, n+1-\nu} \neq 0 \Rightarrow r_{\nu\nu} \neq 0 \Rightarrow d$ is regular $\Rightarrow d\,1'(n)$ is regular $\Rightarrow r$ is regular.

Hence, Theorem 5.5. (0) holds for r-matrices too.

5.6.6. Considering the relation $1'(n)\,r = g$, one proves, as in 5.6.5 for r, that Theorem 5.5. (0) holds for g-matrices too.

Finally, the substitutions $g \rightarrow l$, $d \rightarrow g$ in the reasonning 5.6.5. yields as a result that Theorem 5.5. (0) can be applied to l-matrices too. Hence, Theorem 5.5. (0) is completely proved.

Theorem 5.5. (00) being a consequence of Theorem 5.5. (0), Theorem 5.6. (00) also holds; consequently, Theorem 5.6 is completely proved.

5.6.7. Theorem. *The inverse of any triangular regular matrix is of the same form as the matrix itself; the elements on the hypothenuse of the inverse matrix are the inverses of the elements on the hypothenuse of the matrix:*

a being triangular and regular, then a^{-1} is triangular and of the same form and

$$(\text{hypothenuse of } a)^{-1} = \text{hypothenuse of } a^{-1}.$$

The proof is the same as the proof for the d-matrices.

6. Product of two regular triangular matrices

Of course, such a product is regular (cf. 5.4. (0)) but the specific fact that both factors are triangular matrices implies much more, as is seen from the following.

6.1. Theorem. (0). *If x, y are triangular regular matrices of finite size (n, n) both, then the product $x\,y$ has at least one complete sequence of regular corner principal square submatrices.*

(00). *Let us agree that:*

NW means any matrix in which all principal square submatrices issuing from the left top or North-West corner (1, 1) are regular;

SE means any matrix in which all principal square submatrices issuing from the right bottom or South-East corner are regular;

NE means any matrix in which all principal square submatrices issuing from the right top or North-East corner are regular;

SW means any matrix in which all principal square submatrices issuing from the left bottom or South-West corner are regular;

(000). *Then for regular triangular matrices one has*

$$d\,d = N\,W,$$
$$d\,g = N\,E,$$
$$d\,l = N\,E \text{ etc.}$$

(0000). *One has the following table*

	d	g	l	r
d	$NW \wedge SE$	NW	NE	$NE \wedge SW$
g	SE	$NW \wedge SE$	SE	SW
l	$NE \wedge SW$	SW	$SE \wedge SW$	$NW \wedge SE$
r	NE	$NE \wedge SW$	$NW \wedge SE$	NW

Here the symbol S denotes the conjunction.

The theorem is a consequence of the multiplication of block matrices and of the results of 5.

6.2. E. g. $d\,g = \left[\begin{array}{c|c} A & 0 \\ \hline C & D \end{array}\right] \left[\begin{array}{c|c} A' & B' \\ \hline 0 & D' \end{array}\right] = \left[\begin{array}{cc} \boxed{AA'} & AB' \\ CA' & CB' + DD' \end{array}\right],$

here A (resp. A') denotes the $(\nu\nu)$-left principal submatrix of d (resp. of y); D (resp. D') denotes the right bottom principal square submatrices of size $(n-\nu,\ n-\nu)$; B (resp. B') denotes $(\nu,\ n-\nu)$-submatrices issuing from the right top corner; B, B' need not be square matrices. In the product $d\,g$ the matrix AA' is a left top block, a principal square block; as product of regular matrices A, A' the block AA' is regular; this holding for every $\nu \in \{1, 2, \ldots, n\}$, the theorem $d\,g = NW$ is proved.

6.3. Analogously, one proves all other 15 theorems of the statement 6.1. (0000).

$$\text{E. g. } g\,d = \left[\begin{array}{c|c} \boxed{A} & B \\ 0 & \boxed{D} \end{array}\right] \cdot \left[\begin{array}{c|c} \boxed{A'} & 0' \\ C' & \boxed{D'} \end{array}\right] = \left[\begin{array}{c|c} AA' + BC' & BD' \\ DC' & \boxed{DD'} \end{array}\right];$$

DD' is a square principal submatrix issuing from the South-East corner (n, n). Therefore, $g\,d = SE$ as is indicated in the table.

6.4. Some matrix might have all four complete series of regular principal square corner submatrices. Such a matrix is $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$; probably so is also the matrix of any consecutive positive integers having the from

$$a(k, n) = \begin{bmatrix} k+1 & \cdots\cdots\cdots\cdots & k+n+1 \\ & k+2 \cdots\cdots k+n+2 & \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ k+2n & & k+n \end{bmatrix};$$

in other words the diagonals of $a(k, n)$ are $k+1, k+2, \ldots, k+n$ and $k+n+1, k+n+2, \ldots, k+2n$, respectively; the values outside the diagonals are 0.

7. G-matrices

7.1. Definition. *Every matrix having at least one complete sequence of regular principal square corner submatrices is called a G-matrix;*[5]

In the particular case, when the corner of all members of this complete sequence is (r, s), one speaks of a $G_{(r,s)}$-matrix. So e. g. if a square G-matrix is infinite, then it is necessarily a G_{11}-matrix. An infinite matrix might have at most two corners and, therefore, also at most two complete series of regular principal square submatrices.

Theorem 6.1 might now be stated also in this form:

7.2. Theorem. *The product of any two triangular G-matrices of the same size is a G-matrix.*

7.2.1. Remark. In the wording 7.2 it is not allowed to delete the word »triangular« as it is easy to show by counterexamples. The product of two G-matrices need not be a G-matrix.

We shall need the following

7.3. Lemma. *The product of a lower regular matrix d and of a G_{11}-matrix a is a G_{11}-matrix.*

The proof is visible from the following diagram:

$$\underset{d}{\begin{array}{c|c} \boxed{A} & 0 \\ \hline C & D \end{array}} \cdot \begin{array}{c|c} \boxed{A'} & B' \\ \hline C' & D' \end{array} = \begin{array}{c|c} \boxed{AA'} & AB' \\ \hline CA' + DC' & CB' + DD' \end{array};$$

the involved principal corner matrices are put in square brackets $\Box$.

7.4. (0). Every transpose of a G-matrix is a G-matrix. In particular,

(00) $NW\,1' = NE$; more explicitly

$$\left[\begin{array}{c|c} \boxed{A} & B \\ \hline C & D \end{array}\right] \cdot \left[\begin{array}{c|c} 0 & \boxed{B'} \\ \hline C' & 0 \end{array}\right] = \left[\begin{array}{c|c} BC' & \boxed{AB'} \\ \hline DC' & CB' \end{array}\right];$$

(000) $1'\,NW = SW$; more explicitly

$$\left[\begin{array}{c|c} 0 & \boxed{B} \\ \hline C & 0' \end{array}\right] \cdot \left[\begin{array}{c|c} \boxed{A'} & B' \\ \hline C' & D' \end{array}\right] = \left[\begin{array}{c|c} BC' & BD' \\ \hline \boxed{CA'} & CB' \end{array}\right];$$

---

[5] The denomination is connected with the fact that to any G-matrix a the classical Gauss algorithm is appliquable yielding a factorization of a into two triangular matrices.

(0000) $1' NW 1' = SE$; more explicitly

$$\left[\begin{array}{c|c} 0 & \boxed{B} \\ \hline \boxed{C} & 0 \end{array}\right] \cdot \left[\begin{array}{c|c} \boxed{A'} & \\ \hline & \end{array}\right] \cdot \left[\begin{array}{c|c} & \boxed{B''} \\ \hline \boxed{C''} & \end{array}\right] = \left[\begin{array}{c|c} & \\ \hline & \boxed{CA'B''} \end{array}\right].$$

The proof of Theorems (00), (000), (0000) is visible from the diagram of block multiplications; in either case a principal regular corner square matrix is put into a square bracket $\square$.

7.5. Theorem. *Every G_{11}-matrix a of finite size (n, n) is the product of triangular matrices d, g.*

One might request that the hypothenuse of one factor be the constant 1 (ordinarilly, one requests that the hypothenuse of the second factor be the constant 1; then the determination of d, g from $a = dg$ is performed by means of the Banachiewicz's method).

7.5.1. Proof. We are going to prove that the common Gauss algorithm may be applied on the G_{11}-matrix a of size (n, n).

The matrix a being G_{11}, one has $a_{11} \neq 0$; therefore, we can consider the lower d-matrix

$$d(1) = \begin{bmatrix} 1 & & & & \\ -a_{21} a_{11}^{-1} & 1 & & & \\ -a_{31} a_{11}^{-1} & & 1 & & \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -a_{n1} a_{11}^{-1} & & & & 1 \end{bmatrix};$$

the hypothenuse of the matrix $d(1)$ being a constant $\neq 0$, the matrix $d(1)$ is regular and even a G_{11}-matrix; let us consider the product

$$a^{(1)} = d(1) \cdot a;$$

one has

$$a_{1\nu}^{(1)} = a_{1\nu},$$
$$a_{i1}^{(1)} = 0, \text{ for every } 1 < i \leq n.$$

In virtue of Lemma 7.3. the matrix $a^{(1)}$ is a G_{11}-matrix; in particular, the second member of the (1, 1)-series is regular, i. e.

$$\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22}^{(1)} \end{bmatrix}$$

is regular; this matrix being triangular, its hypothenuse values are $\neq 0$; in particular, $a_{22}^{(1)} \neq 0$. Therefore, we might consider the (n, n)-matrix $d(2)$, for which $d(2)_{ik} = \delta_{ik}$, $k \neq 2$, and for $i \leq 2 = k$, $d(2)_{i2} = a_{i2}^{(1)} a_{22}^{(1)-1}$, for $i > 2$; let $a^{(2)} = d(2) \cdot a^{(1)}$; then $a_{i.}^{(2)} = d(2)_{i2} = -a_{i2}^{(1)} = a_{i.}^{(1)}$, for $i = 1, 2$; $a_{i\nu}^{(2)} = 0$, for $i > \nu \leqslant \{1, 2\}$. Reasoning as in the case of the matrix $a^{(1)}$, one sees that $a^{(2)}$ is a G_{11}-matrix for which the (3, 3)-top left principal matrix is upper triangular, regular, and therefore, in particular $a_{33}^{(2)} \neq 0$.

The procedure continues. One defines the matrix $d(3)$ obtained from the unit matrix $1(n)$ in such a way that the part of the column

$1(n)_{.3}$ is replaced by the corresponding part of the column-vector

$$a_{.3}^{(2)} a_{33}^{(2)-1}.$$

The matrix

$$a^{(n-1)} = d(n-1) \cdot d(n-2) \ldots d(2) \cdot d(1) \cdot a$$

is an upper d-matrix, say y. Now, the coefficient of a as a product of lower matrices is a lower triangular matrix, say b; since the matrices $d(i)$ were regular, b is regular too, and one has

$$b\,a = y;$$

hence,

$$a = b^{-1} y.$$

Since the inverse of a lower matrix b is a lower matrix, the requested factorization of a is obtained: $a = x\,y$, where $x = b^{-1}$.

7.5.2. Remark. In case that the matrix is over a commutative field (or over a commutative division ring) the foregoing procedure might be directly described by determinants: $a = d\,g$ implies

$$d_{11}\, g_{11} = D_1,\ d_{22} = \frac{D_2}{D_1}, \ldots,\ d_{\nu\nu}\, g_{\nu\nu} = \frac{D_\nu}{D_{\nu-1}},\ (\nu = 1, 2, \ldots, n),$$

where

$$D_\nu = \det \begin{bmatrix} a_{11} \cdots a_{1\nu} \\ \cdots\cdots \\ a_{\nu 1} \cdots a_{\nu\nu} \end{bmatrix}$$

and

$$d_{ij} = d_{jj} \frac{\det a \begin{bmatrix} 1\,2 \cdots j-1\; i \\ 1\,2 \cdots j-1\; j \end{bmatrix}}{D_j},\quad g_{ji} = g_{jj} \frac{\det a \begin{bmatrix} 1\,2 \cdots j-1\; j \\ 1\,2 \cdots j-1\; i \end{bmatrix}}{D_j};$$

one puts

$$a \begin{bmatrix} i_1 \cdots i_\nu \\ \cdots\cdots \\ j_1 \cdots j_\nu \end{bmatrix} = \begin{bmatrix} a_{i_1 j_1} \cdots a_{i_1 j_\nu} \\ \cdots\cdots\cdots \\ a_{i_\nu j_1} \cdots a_{i_\nu j_\nu} \end{bmatrix}$$

(cf. F. P. Gantmaher [1], p. 37, Theorem 1).

7.6. Theorem 7.5. might be written in the following way:

$$NW = d\,g.$$

Now, $1'(n)^2 = 1$; therefore,

$$d\,g = d\,(1'(n)\ 1'(n))\, g = (d\,1'(n))\ (1'(n)\, g) = r\,r,$$

and one obtains the following relation

$$NW = r\,r,$$

saying that every G_{11}-matrix is a product of two *right* triangular matrices.

7.7. Theorem. *Every square G-matrix of finite size is a product of two triangular matrices; in particular one has the following factorizations:*

$$
\begin{aligned}
&(0) && NW = d\,g\,, && NW = r\,r\,,\\
&(00) && NE = r\,d\,, && NE = g\,l\,,\\
&(000) && SW = l\,g\,, && SW = g\,r\,,\\
&(0000) && SE = l\,l\,, && SE = g\,d\,.
\end{aligned}
$$

The two factorizations 7.7. (0) were proved and yield Theorems 7.5, 7.6. The other factorizations (00), (000), (0000) are obtained from (0) by premultiplications and/or postmultiplications with $1'(n)$ and then by intercallation, between the triangular factors, of the term $1(n) = 1'(n) \cdot 1'(n)$ as in the proof of 7.6.

E. g. the premultiplication of (0) by $1'(n)$ yields

$$
\begin{aligned}
1'\,NW &= 1'\,d\,g\,,\\
SW &= l\,g\,;
\end{aligned}
$$

and this is the first equality in (000). The other equality in (000) is proved in the following way:

$$SW = l\,(1'\,1')\,g = (l\,1')\,(1'\,g) = g\,r\,.$$

7.8. Main theorem. *In order that a regular finite-size square matrix be the product of two triangular matrices it is necessary and sufficient that it be a G-matrix. In other words, the set of all G-matrices of a given finite-size (n, n) coincides with the products $x\,y$, where x, y mean triangular regular matrices of size (n, n).*

7.8.1. Corollary. *If a finite-size regular square (n, n)-matrix a contains at least one singular principal corner square submatrix in each of the four sequences of principal corner matrices, then the matrix a is not a product of two triangular matrices; such a situation occurs if all four vertices of a are singular* (cf. 4.2.).

8. Factorization of any square finite-size matrix into two or more triangular matrices

8.1. One knows that any square matrix a of finite domain is transformable into a g-matrix g (or d-matrix) using only:

I. permutations of rows (columns),

II. adding to a row (column) of a any scalar multiple of another row (column) of a,

III. multiplication of a row (column) of a by a scalar $\neq 0$.

8.2. On the other hand, addition of the row $\lambda\, a_{j.}$ (of the column $\lambda\, a_{.i}$) to the row $a_{i.}$ (to the column $a_{.j}$) is equivalent to the passage $a \to 1 + \lambda\, e\,(i\,j)\, a$ (resp. $a \to a \cdot (1 + \lambda\, e\,(i, j))$, where $e\,(i, j)$ means the square matrix equals 1 in (i, j) and is 0 elsewhere.

8.3. If $\lambda \neq 0$ and if a matrix b is such that $b_{\nu .} = a_{\nu .}$ for any $\nu \neq i \leqslant [1, n]$ and $b_{i.} = \lambda a_{i.}$, then

$$b = (1(n) + (\lambda - 1) e(i, i)) a.$$

8.4. Now, the matrices $1(n) + \lambda e(i, j)$ are triangular of the first kind; they are lower for $i \geq j$, and upper for $i \leq j$.

8.5. If $a^{(i,j)}$ resp. $a_{(i,j)}$ means the matrix obtained from the matrix a by permuting the rows (columns) i and j one has this sequence of transformations (of columns):

$$a = [a_{.1} a_{.2} \dots a_{.i} a_{.i+1} \dots a_{.j} \dots] \to [\dots a_{.i} \dots a_{.i} + a_{.j} \dots] \to$$
$$\to [\dots a_{.i} - (a_{.i} + a_{.j}) \dots a_{.i} + a_{.j} \dots] \to [\dots - a_{.j} \dots a_{.i} +$$
$$+ a_{.j} \dots] \to [\dots - a_{.j} \dots - a_{.j} + (a_{.i} + a_{.j}) \dots] \to$$
$$\to [\dots - a_{.j} \dots a_{.i} \dots] \to [\dots a_{.j} a_{.i} \dots] = a_{(i,j)}.$$

To every transition $\to$ corresponds a multiplication on the right. Hence,

$$a(1(n) + e(i,j)) \cdot (1(n) - e(j, i)(1(n) + e(i, j))(1 - 2e(i, i)) = a_{(i,j)}. \qquad (1)$$

Analogously one gets $a^{(i,j)}$.

Since all the matrices (1) are triangular, we conclude that all thre types I, II, III of elementary transformations are equivalent to multiplications with special regular triangular matrices. Consequently, there is a sequence of triangular transformations of the type $1(n) + \lambda_k e(i_k, j_k)$ such that the matrix

$$a \prod_{k=1}^{m} (1(n) + \lambda_k (i_k j_k)) = g;$$

hence

$$a = g \cdot (1(n) + \lambda_m (i_m j_m))^{-1} \dots (1(n) + \lambda_1 (i_1, j_1))^{-1}.$$

8.6. Theorem. *Any matrix is the product of triangular matrices of the first kind.*

8.7. *Operator* $K(a)$. Let K be the set of complex or real numbers and a any square matrix over K of finite size: let[6] $K(a)$ denote the minimal number of triangular matrices of the first kind the

---

[6] The number $K(a)$ depends on the matrix a and on the division ring K; analogous considerations could be done substituting for K any division ring and particularly any field F; of course, the boundary 3 of the theorem might vary with F. So far the present author did not succeed in determining

$$\sup_a G_p(a) \qquad (1)$$

where G_p is the abbreviation for the p-point-field; a is running through the set of all finite-size square matrices over the field G_p; p denotes any prime number.

product of which equals a. We saw (4.2) that for some regular matrix a one has not $K(a) \leq 2$. Is necessarily

$$K(a) \leq 3\,? \tag{2}$$

Yes, as will be shown in another paper!

E. g. the matrix $1'(2)$ is not of the form $d\,g$, but one has

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}; \tag{3}$$

therefore, also (premultiplication of (3) by $\left[\begin{smallmatrix} a_{12} & 0 \\ a_{22} & a_{21} \end{smallmatrix}\right]$):

$$\begin{bmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{12} & 0 \\ a_{22}+a_{21} & a_{21} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix},$$

Since, for $a_{11} \neq 0$,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} - a_{21}\, a_{11}^{-1}\, a_{12} \end{bmatrix} \begin{bmatrix} 1 & a_{11}^{-1}\, a_{12} \\ 0 & 1 \end{bmatrix},$$

the relation holds for any a of size (2,2).

Institute of Mathematics
University of Zagreb

BIBLIOGRAPHY:

T. Anđelić:
[1] Matrice, Beograd, 1962, str. 267,

Ф. Р. Гантмахер:
[1] Теория матриц, Москва, 1954, стр. 492,

Đ. Kurepa:
[1] On a definition and notation of matrices. On a kind of switch matrices, Vesnik Društva Mat. Fiz. N. R. Srbije 4 (1952), 1—7,
[2] Une généralisation des matrices, C. R. Acad. Sci. Paris 239 (1954), 19—20,
[3] Viša algebra, Zagreb, 1965, 32°+765, II 20°°, 764—1432,

C. C. Mac Duffee:
[1] The theory of matrices, Ergebnisse d. Math. 2, Berlin 2, 1933, V—110,

R. Zurmühl:
[1] Matrizen, Berlin—Heidelberg, 1950, XVI—427.

O TROKUTNIM MATRICAMA

Đuro Kurepa, Zagreb

Sadržaj

Trokutne matrice su s jedne strane jednostavnije od općih matrica, a s druge strane služe kao materijal u izgradnji općih matrica.

1.3. Imamo četiri oblika trokutnih matrica: *d-matrice, g-matrice* (prva vrst), te *l-matrice, r-matrice* (druga vrst); definicije su im u 1.3.1 — 1.3.4.

2. *Transpozicije matrice a.* U 2.1.—2.4 dane su 4 vrsti transpozicija zadane matrice a.

2.6. *Centralna transpozicija* $a \to a^c$ definira se kao u 2.6, a vrijedi i obrazac 2.7.

2.10. Važna je matrica $1'(n)$ kojoj je kvadrat $1(n)$.

3.2. Teorem. Razabire se iz napisanih obrazaca.

4. *Produkt trokutnih matrica konačna formata.* Množenje je prikazano tablicom 4.1.

4.2. Korolar. *Regularna kvadratna konačna matrica kojoj su uglovi 0 nije produkt od dvije trokutne matrice. Takva je npr. napisana matrica a.*

5. *Regularne trokutne matrice.* Produkt para matricâ vlada se po shemi (00000); pri čemu 1 (odn. 0) znači svojstvo biti regularan (singularan); posebno produkt singularnih matrica može biti i regularan.

6.1. (0). *Produkt regularnih trokutnih matrica sadrži bar jedan glavni niz regularnih kvadratnih podmatrica;* već prema tome da li taj niz počinje u sjeverozapadnom vrhu *NW*, sjeveroistočnom vrhu *NE*, jugozapadnom vrhu *SW* ili jugoistočnom vrhu *SE* govori se o *NW-matrici*, o *NE-matrici, SW-matrici, SE-matrici.* Tablica (00000) pokazuje pravilnosti koje tu postoje.

7. *G-matrice* jesu one koje posjeduju bar jedan potpun niz regularnih glavnih podmatrica.

7.2. Skup svih trokutnih (n, n)-G-matrica čini množidben grupoid.

7. 7. *Teorem o faktorizaciji G-matrica* u trokutne matrice vidi se iz obrazaca (0)—(0000).

7.8. Glavni teorem. *Ako je matrica G-matrica, onda je ona produkt dvojke regularnih trokutnih matrica; i obrnuto.*

8.7. $K(a)$ naznačuje minimalan broj trokutnih matrica prve vrste kojima je produkt matrica a. Npr. $K\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 3$. Uopće je $K(a) \leq 3$ za svaku kvadratnu realnu ili kompleksnu matricu konačna formata.

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ON ORDER-ISOMORPHISM OF TREES

Đuro Kurepa, Zagreb

1. Theorem. *Let*

$$(T, <),\ (T', <')\ (0)$$

be trees satisfying the following two conditions:

1. *Every maximal chain of one tree is isomorph to every maximal subchain of the other tree (therefore, in particular, all maximal chains in $(T, <)$ as well as in $(T', <')$ are isomorph).*

2. *In $(T, <)$ as well as in $(T', <')$ the cardinality of every node N depends uniquely of the order type $t(\cdot, N)$ of the chain, the points of which preceed the set N; in other words if N is a node in $(T, <)$, and if N' is a node in $(T', <')$, then*

$$t\,T(\cdot, N) \equiv t\,T'(\cdot, N') \Rightarrow k\,N = k\,N'. \tag{1}$$

Then the trees $(T, <), (T', <')$ are isomorph, i. e. there exists a one-to-one mapping i of T onto T' such that

$$\{a, b\}_< \subseteq (T, <) \Leftrightarrow \{i\,a, i\,b\}_{<'} \subseteq (T', <'). \tag{2}$$

If γ is the order-type of every maximal chain of $(T, <)$ as well as of $(T', <')$ and if for every $\gamma' < \gamma$ one denotes by $k_{\gamma'}$ the cardinality of every node N (resp. N') of $(T, <)$ (resp. of $(T', <')$) satisfying $t\,T(\cdot, N) = \gamma' = t\,T'(\cdot, N')$, then the rank γT of $(T, <)$ equals γ:

$$\gamma(T, <) = \gamma = \gamma(T', <') \tag{3}$$

and

$$k\,T = \sum_{\gamma' < \gamma} \prod_{\alpha < \gamma'} k_{\alpha'} = k\,T'. \tag{4}$$

1.1. Remark. Obviously, any isomorphism i from $(T, <)$ onto $(T', <')$ carries the set of all maximal chains (antichains) of $(T, <)$ onto the set of all maximal chains (antichains) of $(T', <')$.

1.2. Proof of the theorem 1. The sets T, T' are both either empty or both non empty. The isomorphism $i: T \to T'$ is defined by induction argument involving the ordinal numbers $< \gamma$.

At first, let $R_0 T$, $R_0 T'$ be the first row of $(T, <)$, and of $(T', <')$ respectively; since $x \in R_0 T$, $x' \in R_0 T'$ imply $T(\cdot, x) = T'(\cdot, x') = \emptyset$, the node $R_0 T$ has the same cardinality k_0 as the node N' in $(T', <')$; according to the condition (1), there exists a one-to-one mapping i_0

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of $R_0 T$ onto $R_0 T'$. Now, suppose that a is an ordinal such that $0 < a < \gamma T$ and that a strongly increasing sequence

$$i_{a'} \, (a' < a)$$

of isomorphisms

$$i_{a'} : T(\cdot, a') \to T'(\cdot, a')$$

be defined such that

$$a_1 < a_2 < a \Rightarrow i_{a_1} \,|\, \mathrm{Dom}\, i_{a_1} = i_{a_2} \,|\, \mathrm{Dom}\, i_{a_1}.$$

We define

$$T(\cdot, a) = \bigcup_{\xi < a} R_\xi, \; T(\cdot, a] = T(\cdot, a) \cup R_a T.$$

We extend $i_{a'}$ into i_a in the following way.

If $a-1$ exists, we consider for every $x \in R_{a-1} T$ the node $N_x = R_0 T(x, \cdot)$ consisting of all the immediate followers of x in $(T, <)$ as well as the corresponding node $N'_{x'} = R_0 T'(i_{a-1}, \cdot)$. The sets $N_x, N'_{x'}$ being of the same cardinality, let $y \in N_x \to f_x y$ be an isomorphism between N_x and $N'_{x'}$; then we extend i_{a-1} into i_a by putting $i_a y = f_x y$ for every $x \in R_{a-1} T$ and for every $y \in N_x$.

If $a^- = a$, we consider the set C_a of all the a-chains C of type a each and such that C contains a single point of every $R_\xi T (\xi < a)$. We consider the node N_C consisting of all the points $x \in T$ satisfying $T(\cdot, x) = C$; the chain C has no last element and for every ordinal $\xi < a$ the isomorphism i_ξ is defined for every ξ-section of C; we denote by $f_a = \sup_{\xi < a} i_\xi$ the mapping, the domain of which is the union of the domains $\mathrm{Dom}\, i_\xi$, i. e. the set $T(\cdot, a)$. Consequently, we have the chain

$$C' = f_a C = \{f_a c \,;\, c \in C\} \text{ and the node}$$

$$N'_{C'} = \{x' ; x' \in T' ; T'(\cdot, x') = C'\}.$$

The mapping f_a is an isomorphism between $T(\cdot, a)$ and $T'(\cdot, a)$. Now we define i_a on $T(\cdot, a]$ as extension of $f_a \,|\, T(\cdot, a)$ such that $i_a \,|\, N_C$ be any one-to-one mapping of N_C onto $N'_{C'}$ for every $C \in C_a$. Obviously, i_a is an isomorphism between $T(\cdot, a]$ and $T'(\cdot, a]$. This being true for every $a < \gamma$, this means in particular that $\sup_{\xi < \gamma} i_\xi \equiv i$ is an isomorphism between $T(\cdot, \gamma) = T$ and $T'(\cdot, \gamma) = T'$, what proves the first part of the theorem.

1.3. Let us prove the equality (4). For this, let us consider the γ-sequence

$$k_0, k_1, \ldots, k_\xi, \ldots \quad (\xi < \gamma) \tag{1}$$

of cardinals and for every $\xi < \gamma$ a set S_ξ of cardinality k_ξ (e. g. S_ξ might be the smallest initial section of cardinality k_ξ of ordinal numbers); we form the set $T = (k_0, k_1, \ldots)$ of all the sequences $s_0, s_1, s_2, \ldots, s_\xi, \ldots$ of length $a + 1$, under conditions that $a < \gamma$ and $s_\xi \in S_\xi$; then we order T by means of the relation $=\!|$ meaning »to *be an initial section of«*; then one verifies readily that $(T, =|)$ is a tree satisfying all the conditions of the theorem 1; consequently, this tree is isomorph to any tree $(T, <)$ occuring in the wording of the theorem 1.

1.3.1. The sequence (1) is called the characteristic sequence of the tree T; therefore the tree T might be denoted

$$T(k_0, k_1, \ldots, k_\xi, \ldots)_{\xi < \gamma}. \tag{2}$$

1.3.2. Now, the cardinality of the tree (2) is not difficult to determine. Namely the set of all the $(\alpha + 1)$-sequences equals $\prod_{\xi \leqslant \alpha} k_\xi$ according to the definition of product of cardinal numbers. Therefore, the formula (4) of the theorem 1 holds true and the theorem is completely proved.

1.4. *Some particular cases.*

1.4.1. Characteristic sequence $k_0, k_1, \ldots$ is a constant b, e. g. 2 or 3 or any finite natural number;

1.4.2. If $\alpha^- = \alpha$, then $k_\alpha = 1$; if $\alpha^- < \alpha$, then k_α is a constant b (the constant number b might be finite or transfinite); such a case is obtained if one considers all the initial sections of the set k^{I_γ} of all the γ-sequences of ordinal numbers $< \omega_{(b)}$, where $\omega_{(b)}$ denotes the smallest ordinal number of cardinality b. The study of subtrees of such a tree for $\gamma = \omega_\delta$ and $k_\xi = k\,\omega_{\delta^-}$ is particularly interesting (more particularly for the case $\delta^- < \delta$).

1.4.3. If $\alpha^- = \alpha \Rightarrow k_\alpha = 1$ and if the sequence $k_{\xi+1}$ for $\xi < \gamma$ is just the increasing sequence of the initial γ-section of positive cardinals, then the T_γ-tree is closely connected to the factorial numbers; in particular, $k\,T_\gamma = !\,n = \sum_k k\,!\ldots, k < n$ provided moreover that every maximal subchain of the tree equals n^- (cf. [6] § 6.1).

2. *Chain-cover preserving isomorphisms.*

2.1. In connexion with the theorem 1. let us remark that the property of a family to be a chain-cover (resp. an antichain-cover) of an ordered set is preserved by order-isomorphisms. In other words, let $(O, <)$, $(O' <')$ be any ordered sets (partially or totally); if F is a family of chains (antichains) exhausting $(O, <)$ and if i is any isomorphism of $(O, <)$ onto $(O', <')$, then $i\,F := \{i\,X; X \in F\}$ is a family of chains (antichains) of $(O', <')$ exhausting this set.

2.2. *A problem.* Now the following question arises: If $(O, <)$, $(O', <')$ are isomorphic and if F, F' is a *minimal* family of maximal chains (antichains) exhausting O, O' respectively, does there exist an isomorphism from $(O, <)$ onto $(O', <')$ transforming the cover F of O onto the cover F' of O'? In general, there is no such isomorphism. But there are cases in which for chain-covers there are such isomorphisms. In particular we have the following.

2.3. Theorem. (0) *Let*

$$k_0, k_1, \ldots, k_n, \ldots (n < \omega_0) \tag{1}$$

be any ω_0*-sequence of positive cardinals* $\leq k\,\omega_0 = \aleph_0$; *let*

$$(T, <),\ \ (T', <') \tag{2}$$

be any trees satisfying the following two conditions:

Condition I. Every maximal chain is of the type ω_0;

Condition II. Every node belonging to the n^{th} *row has* k_n *members.*

Let S, S' respectively be any set of $s(T, <)$, $s(T', <')$[1] *maximal chains of the sets* (2) *respectively satisfying*

$$\bigcup_{X \in S} X = T, \quad \bigcup_{X' \in S'} X' = T'. \tag{3}$$

Then there exists an isomorphism I from $(T, <)$ *onto* $(T', <')$ *carrying the chain-cover S of* $(T, <)$ *onto the chain-cover S' of* $(T', <')$: $IS = S'$ *(of course the mapping* $I: S \to S'$ *is one-to-one and onto).*

(00) *If* ω_σ *is a regular initial number such that*

$$a < k\omega_\sigma \Rightarrow 2^a < \aleph_\sigma, \tag{4}$$

then in the wording of theorem (0) *it is legitime to replace* ω_0, $\aleph_0$ *by* ω_σ, $\aleph_\sigma$ *respectively.*

(000) *The general continuum hypothesis implies the proposition obtained from* (0) *by replacing* ω_0, $\aleph_0$ *by any regular initial numbers* ω_σ, $\aleph_\sigma$ *respectively.*

2.3.1. Remark. The foregoing theorem 2.3.(0) for the case that the sequence (1) is a constant sequence $r, r, \ldots, r$ being some positive integer is due to M. Dolcher (v. [2], theorem 3); the present proof is different.

2.3.2. Proof of the statement (0). Let us normally well-order the set S as well as the set S':

$$S = s_0, s_1, \ldots, s_\xi, \ldots (\xi < \lambda) \tag{5}$$

$$S' = s'_0, s'_1, \ldots, s'_\xi, \ldots (\xi < \lambda). \tag{5'}$$

We well-order normally every node of $(T, <)$ as well as every node of $(T', <')$.

One has necessarily

$$s(T, <) \leq k\,T = \aleph_0 \text{ and } \lambda \leq \omega_0. \tag{6}$$

[1] The star number $s(O, <)$ of any ordered set means the least number of $(O, <)$ exhausting O (cf. [5]).

A requested isomorphism I will be now defined as supremum of an increasing sequence of order-isomorphisms I_ξ:

$$I = \sup_{\xi < \lambda} I_\xi . \tag{7}$$

To start with let I_0 be the isomorphism from s_0 onto s'_0, (let us remark that every $s_\xi \in S$ is a strictly increasing ω_0-sequence of members of T); we put $r_0 = 0$ and thus $I_0 s_0 = s'_{r_0}$. Let $0 < \alpha < \lambda$ and let us assume that the increasing sequence of isomorphisms

$$I_\nu \,|\, (s_0 \cup s_1 \cup \ldots s_\mu \cup \ldots)_{\mu < \gamma} (\gamma < \alpha) \tag{8}$$

as well as the numbers r_μ be defined such that

$$I_\mu s_\mu = s'_{r_\mu} \quad (\mu < \nu). \tag{6}$$

We define now I_α as the extension of the isomorphism $\sup\limits_{\nu<\alpha} I_\nu$ on defining $I_\alpha \,|\, s_\alpha$ in the following way.

If $a = a_0, a_1, \ldots$ and $b = b_0, b_1, \ldots$ are distinct sequences or well-ordered sets we denote by $i(a, b)$ the first index n such that $a_n \neq b_n$.

Now, we consider the numbers $i(s_\nu, s_\alpha)$ $(\nu < \alpha)$ and its supremum

$$\beta = \sup_{\nu < \alpha} i(s_\nu, s_\alpha). \tag{10}$$

Then $\beta < \omega_0$; therefore, for some $\nu_0 < \alpha$ we have $\beta = i(s_{\nu_0}, s_\alpha)$. Consequently we have the point $s_{\nu_0\beta} \in T$ and the point $x' = I_{\alpha-1} s_{\nu_0\beta} \in T'$; in the assumed well-order of the corresponding node $T'(x')$ of $(T' <')$ we consider the first point $y \in T' \setminus \bigcup\limits_{\nu<\alpha} s'_{r_\nu}$; the first member of (5′) which contains the point y is the requested s'_{r_α}. The construction is uniquely determined for every $\alpha < \lambda$ and one proves readily that $\{r_\alpha; \alpha < \lambda\} = [0, \lambda)$, i. e. $\{s'_{r_\alpha}; \alpha < \lambda\} = S' = I\,S$, where $I = \sup\limits_{\xi<\lambda} I_\xi$. In virtue of (3) the theorem (0) is proved.

2.3.3. Proof of the statement 2.3.(00). Under the hypothesis (4) we have again relations like (5), (5′), (6) (7) (replacing ω_0 by ω_σ). The definition of I_α for $0 < \alpha < \lambda$ is slightly modified. By induction, we define two sequences $p_\xi (\xi < \alpha)$ and $r_\xi (\xi < \alpha)$ of ordinals $< \lambda$. Putting $p_0 = 0 = r_0$, let $0 < \alpha < \lambda$ such that for $\xi < \alpha$ the numbers p_ξ, r_ξ are defined. If α is odd we denote by p_α the first ordinal of $[0, \lambda) \setminus \{p_\nu; \nu < \alpha\}$ and consider: the chain s_{p_α}, the number $\sup\limits_{\nu<\alpha} i(s_{p_\nu}, s_{p_\alpha}) = : \beta$ and the chain $\{s_{p_\alpha \gamma}; \gamma < \beta\}$; we have also the chain $\{s'_{r_\gamma}; \gamma < \beta\} = : C'$. Since C' is not maximal (the type of C' is α, thus $< \lambda \leq \omega_\sigma$) the node n' of all the points $y' \in T'$ comming immediately after the chain C' is non empty; we consider:

the first point y'_0 in the well-order $w\,n'$ which is not contained in $\bigcup_{\xi<\alpha} s'_{r_\xi}$ and denote by s'_{r_α} the first member of (5') containing the point y'_0; we define $I_\alpha \mid s_{p_\alpha}$ by putting $I_\alpha s_{p_\alpha} = s'_{r_\alpha}$. If α is even (in particular if $\alpha^- = \alpha$) the numbers p_α, r_α are defined in a similar way reversing the roles of p_α, r_α: we define r_α as the first number of $[0,\lambda) \setminus \{r_\nu; \gamma < \alpha\}$; we consider: the chain s'_{r_α}, the number $\sup_{\nu<\alpha} i(s_{r_\nu}, s'_{r_\alpha}) = \beta'$, the chain $C = \{s_{p_\gamma}; \gamma < \beta'\}$, the node n of $(T, <)$ in which are immediate followers of C, the first point $y_0 \in w\,n \setminus \setminus \bigcup_{\xi<\alpha} s_{p_\xi}$ and the first member s_{p_α} of (5) containing the point y_0; we define $I_\alpha^{-1} \mid s'_{r_\alpha}$ by putting $I_\alpha^{-1} s'_{r_\alpha} = s_{p_\alpha}$. Consequently, for every $\alpha < \lambda$ the isomorphisms I_α, I_α^{-1} are defined; putting $I = \sup_{\alpha<\lambda} I_\alpha$, one obtains a requested isomorphism I of the theorem 2.3. (00).

2.3.4. The statement 2.3. (000) is implied by the statement 2.3. (00).

2.3.5. R e m a r k o n t h e t h e o r e m 2.3. (0). In some cases the theorem 2.3. is implied by the theorem 1.1., in particular, if the sequence $k_0, k_1, \ldots$ is almost the constant sequence $1, 1, \ldots$. If up to m we have $k_\xi = 1$ $(m \leq \xi < \omega_0)$, then

$$k\,R_n(T, <) = k\,R_n(T' <')$$

and necessarily

$$S = \{T[a]\,;\, a \in R_n T\},\ S' = \{T'[a]\,;\, a \in R_n T'\};$$

the sets S, S' coincide with the set of all the maximal chains of $(T, <)$, $(T', <')$ respectively, and consequently the theorem 2.3 is implied by 1.1 and the corollary 1.2.

2.3.6. T h e o r e m. *If the sequence* (1) *in* 2.3. (0) *is constant, it is legitimate to drop, in the wording of the theorem* 2.3. (0), *the conditions*

$$„\leq k\omega_0 = \aleph_0".$$

The proof is like the one in the section 2.3.3 of the theorem 2.3. (00).

3. *On equality of order types.*

3.1. If $(O, <)$, $(O', <')$ are ordered sets, we write

$$t(O, <) \leq t(O', <')$$

provided $(O, <)$ is order-imbeddable into $(O', <')$, i. e. provided $(O, <)$ is similar to a subset of $(O', <')$. One defines

$$t(O, <) = t(O', <') \Longleftrightarrow t(O, <) \leq t(O', <') \wedge t(O', <') \leq t(O, <).$$

3.2. Theorem. *Let ω_σ be any regular initial number and $(T, <)$, $(T', <')$ any trees in which every maximal chain is of the type ω_σ; if for every $x \in T$ and every $x' \in T'$ the sequences*

$$k\, R_\alpha T(x, \cdot) \qquad (\alpha < \omega_\sigma) \tag{1}$$

$$k\, R_\alpha T'(x', \cdot) \qquad (\alpha < \omega_\sigma) \tag{2}$$

are confinal in the sense that for every $\alpha < \omega_\sigma$ there is some $\beta = \beta(\alpha, x) < \omega_\sigma$ such that

$$\alpha \leq \beta(\alpha, x) \text{ and } k\, R_\alpha T(x, \cdot) \leq k\, R_\beta T'(x', \cdot), \tag{3}$$

and for every $\alpha' < \omega_\sigma$ there is some $\beta' = \beta'(\alpha', x')$ such that

$$\alpha' \leq \beta'(\alpha', x') \text{ and } k\, R_{\alpha'} T'(x', \cdot) \leq k\, R_{\beta'} T(x, \cdot); \tag{3'}$$

if

$$k\, R_0 T \leq k\, R_{\alpha'} T' \text{ for some } \alpha' < \omega_\sigma \tag{4}$$

and

$$k\, R_0 T' \leq k\, R_\alpha T \text{ for some } \alpha < \omega_\sigma, \tag{5}$$

then

$$t(T, <) = t(T', \leq'). \tag{6}$$

In general, the trees $(T, <)$, $(T', <')$ are not isomorph but there are decompositions

$$T = T_1 \cup T_2, \quad T_1 \cap T_2 = \emptyset \tag{7}$$

$$T' = T'_1 \cup T'_2, \quad T'_1 \cap T'_2 = \emptyset \tag{8}$$

such that

$$t(T_i <) \equiv t(T'_i, <') \quad (i = 1, 2). \tag{9}$$

Proof. Let us prove that

$$t(T, <) \leq t(T', <'). \tag{10}$$

By induction, we shall imbed $(T, <)$ into $(T', <')$. Now, to start with, we define $f_0 \mid R_0 T$ as any one-to-one mapping of $R_0 T$ into $R_{\alpha'} T'$, where α' is assured by the condition (4). For every point $a \in R_0 T$ we transform biuniquely, by some f_1, the antichain $R_0 T(a, \cdot)$ into a row of $T'(f_0 a, \cdot)$; the cofinality condition of the theorem applied for $x = a$, $x' = f_0 a$ guarantees the existence of $f_1 \mid R_0 T(a, \cdot)$ and hence also of

$$f_1 \mid \bigcup_{a \in R_1 T} R_0 T(a, \cdot);$$

we extend f_1 also on $R_0 T$ by putting $f_1 \mid R_0 T = f_0 \mid R_0 T$. Now, let $0 < \nu < \omega_\sigma$ and let a strictly increasing ν-sequence of isomorphisms $f_\xi \mid \bigcup_{\alpha \leq \xi} R_\alpha T$, $(\xi < \nu)$ be defined with antidomain $\subseteq (T', <')$. We define also $f_\nu \mid \bigcup_{\alpha \leq \nu} R_\alpha T$ as extension of every $f_\xi\, (\xi < \nu)$; on $R_\nu T$ we define f_ν in the following way.

If $\nu^- < \nu$ we consider every $a \in R_{\nu-1} T$, the row $R_0 T(a, \cdot)$, the point $f_{a-1} a \in T'$ and a number $\beta(a) > \gamma f_{a-1}$ such that

$$k\, R_{\beta(a)} T'(f_{a-1}\, a, \cdot) \geq k\, R_0\, T(a, \cdot).$$

The existence of such a number $\beta(a) < \omega_\sigma$ is assured by the cofinality condition. Then for every $a \in R_{a-1} T$ we define $f_\nu \mid R_0 T(a, \cdot)$ as any one-to-one mapping into $R_{\beta(a)} T'(f_{a-1} a, \cdot)$. If $\nu^- = \nu > 0$, i. e. if ν is a limit ordinal, let C be any maximal chain of $T(\cdot, \nu) \equiv \equiv \bigcup_{\xi < a} R_\xi T$; necessarily $t\, C = \nu$ because every maximal chain of $(T, <)$ is of the type $\omega_\sigma > \nu$. Let then $n(C)$ be the set of all points of $R_\nu T$ succeeding to every point of C. If $C = \{f_\xi C_\xi ; \xi < \nu\}$, then we have also the chain $C' = \{f_\xi C_\xi ; \xi < \nu\}$ and the node $n(C')$ consisting of all points of T' comming immediately after the set C'. Let then $a' \in n(C')$; let $\beta(C)$ be an ordinal number such that

$$k\, n(C) \leq k\, R_{\beta(C)}\, T'(a', \cdot).$$

The existence of $\beta(C) < \omega_\sigma$ is a consequence of the cofinality condition because if $c \in C$, then $n(C)$ is in some row $R_a T(a, \cdot)$; by the cofinality condition there is some ordinal $\eta < \omega_\sigma$ such that $k\, R_a\, T(a, \cdot) \leq k\, R_\eta\, T'(a', \cdot)$. The existence of $\beta(C)$ being proved we define f_ν on $n(C)$ as any one-to-one mapping of $n(C)$ into $R_{\beta(C)} T'(a', \cdot)$. Doing so for every chain $C = \{c_\xi ; c_\xi \in R_\xi T, \xi < \nu\}$ one defines f_ν on $R_\nu T$; we define f_ν also on every $R_\xi T$ as the function $f_\xi \mid R_\xi T (\xi < \nu)$.

Putting $f = \sup_{\nu < \omega_\sigma} f_\nu$, we obtain a requested order-isomorphism of $(T, <)$ into $(T', <')$: the relation (10) is proved.

Permuting in the preceding proof $T \langle - \rangle T'$, $< \langle - \rangle <'$ one obtains

$$t(T', <') \leq t(T, <). \tag{10'}$$

The relations (10), (10') mean that (6) holds. The relations (7), (8), (9) are a consequence of a known Banach's theorem (v. [1]).

3.3. C o r o l l a r y. *Let ω_σ be any regular number and $(T, <)$, $(T', <')$ trees in which every maximal chain is of the type ω_σ; if for $\nu < \omega_\sigma$ satisfying $\nu^- = \nu$ every node of the ν^{th} row has a single point, and if for every $x \in T, x' \in T'$ the ω_σ-sequences* (1), (2) *are cofinal, then* $t(T, <) = t(T', <')$.

4. *Sequences of numbers and cofinal increasing subsequences.*

In theorems 1.1, 2.3 occur sequences of (cardinal) numbers; the question arises as whether such a sequence contains a cofinal increasing sequence.

4.1. T h e o r e m. *Let ω_σ be any initial number; let*

$$s_0, s_1, \ldots, s_\xi, \ldots \quad (\xi < \omega_\sigma). \tag{1}$$

be any ω_σ-sequence of ordinal numbers (of cardinal numbers); then *the sequence* (1) *contains a cofinal increasing subsequence; in other words, there exists a set S of ordinals such that sup* $= \omega_\sigma$ *and that*

$$\alpha < \beta \wedge \{\alpha, \beta\} \subseteq S \Rightarrow s_\alpha \leq s_\beta.$$

4.2. Corollary. *Every* ω_0*-sequence of ordinals (cardinals) contains an increasing* ω*-subsequence.*

4.3. Proof. 4.3.1. First case: the sequence (1) is one-to-one, i. e. $\xi < \eta < \omega_\sigma \Rightarrow s_\xi \neq s_\eta$. Let $O = \{s_\xi\,;\, \xi < \omega_\sigma\}$; we put

$$k_m \prec k_n \Longleftrightarrow (k_m < k_n) \wedge (m < n) \text{ for } m, n < \omega_\sigma.$$

We obtain the ranged set $(O, \succ)$ in which every chain is finite. As a matter of fact, if O contained an infinite antichain A, then A determines an infinite subsequence

$$s_{a_1}, s_{a_2}, \ldots \text{ of (1) such that}$$

$a_1 < a_2 < a_3 <$ and $s_{a_1} > s_{a_2} > s_{a_3} > \ldots$ -contradiction, because every regression of ordinal (cardinal)[1] numbers is finite. Since every antichain of O is finite and since $k\,O = \aleph_\sigma \geq \aleph_0$ the ordered set $(O, \prec)$ has the same power as some subchain C of $(O, \prec)$ (cf. Dushnik-Miller [3], theorem 5.25); and Đ. Kurepa [4] § 1.1); the elements of C form a subsequence of (1) of cardinality $\aleph_\sigma$-it is cofinal with the sequence (1) itself.

4.3.2. Analogously, one proves the theorem for every sequence (1) containing $\aleph_\sigma$ distinct members.

4.3.3. The theorem is obvious also provided the sequence (1) is cofinal with a constant subsequence.

4.3.4. There remains the case that the frequency of every member of (1) is $< \aleph_\sigma$ and that there is no constant subsequence of (1) which is cofinal with (1). By induction argument one proves readily that there is then a cofinal one-to-one subsequence S of (1); we might suppose that the order-type of S be regular initial number ω_i. Substituting (1) by S in reasoning of the first case one sees that there exists a strictly increasing subsequence s of (1) which is cofinal to S and consequently to (1) too.

Institute of Mathematics
University of Zagreb

BIBLIOGRAPHY:

[1] *St. Banach*, Un théorème sur les transformations biunivoques, Fund. Math. Warszawa **6** (1924), 236—239,

[2] *M. Dolcher*, Una proprietà delle famiglie numerabili di catene supra un albero, Seminario mat. Univ. Ferrara, Ser. VII, Scienze mat. **8** N. z (1952), 7—16,

[3] *B. Dushnik* and *E. W. Miller*, Partially ordered sets, Amer. J. of Mathematics **43** (1941), 600—610,

[1] This is so at least provided the axiom of choice is used.

[4] *Đ. Kurepa,* On two problems concerning ordered sets, Glasnik Mat.-Fiz. Astr. 13 (1958) 229—234,

[5] *Đ. Kurepa,* Star number and antistar number of ordered sets and graphs, Glasnik Mat.-Fiz. Astr. 18 (1963), 27—37,

[6] *Đ. Kurepa,* Factorials of cardinal numbers and trees, Glasnik Mat.-Fiz. Astr. 19 (1964), 7—21.

O IZOMORFIZMU DRVETA

Đuro Kurepa, Zagreb

Sadržaj

1. Teorem. *Neka drveta* $(T, <)$, $(T', <')$ *iz* (0) *zadovoljavaju ovim uslovima:*

1. *Svaki maksimalni lanac jednog drveta je izomorfan svakom maksimalnom lancu drugog drveta;*

2. *I u jednom i u drugom drvetu kardinalni broj svakog čvora zavisi jedino od skupa svih prethodnika toga čvora; drugim riječima, za svaki čvor N u* $(T, <)$ *i svaki čvor N′ u* $(T', <')$ *vrijedi* (1).

Tada su drveta (0) *izomorfna.*

Za glavni broj kT *vrijedi* (4), *pri čemu je* k_α *broj članova svakog čvora N iz* $R_\alpha T$.

2.3. Teorem (0). *Neka* (1) *označuje proizvoljan* ω_0*-niz pozitivnih glavnih brojeva* $\leq k\omega_0 = \aleph_0$; *neka drveta* (2) *zadovoljavaju uslovima:*

I *Svaki maksimalni lanac je tipa* ω_0;

II *Svaki čvor iz n-tog sloja ima* k_n *članova.*

Neka su S, S' *proizvoljne obitelji po* $s(T, <)$ *odnosno* $s(T', <')$ *maksimalnih lanaca iz* (2) *za koje vrijedi* (3).[1]

Tada postoji izomorfizam I od $(T, <)$ *na* $(T', <')$ *koji prevodi S na S′.*

(00). *Ako je* ω_σ *regularan početan broj sa svojstvom* (4), *tada je dopušteno u iskazu* (0) *zamijeniti* ω_0 *sa* $\aleph_\sigma$ *te* $\aleph_0$ *sa* $\aleph_\sigma$.

(000). *Opća hipoteza o kontinuumu daje sud koji se iz suda* (0) *dobije zamjenjujući* ω_0 *sa* ω_σ *a* $\aleph_0$ *sa* $\aleph_\sigma$.

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[1] Pri tom $s(O, <)$ (za uređen skup $(O, <)$) označuje najmanji broj lanaca koji iscrpljuju O.

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IMBEDDING OF ORDERED SETS IN MINIMAL LATTICES

Đuro Kurepa

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1. Let

$$(O, <) \tag{1}$$

be any ordered set; it is known that the set (1) is imbeddable into a complete lattice $L(O)$ (Mac Neille; cf. Birkhoff, p. 58; Szász p. 73). In general case, the cardinality kO of O is less than the cardinality $kL(O)$ of $L(O)$; now the question arises as whether the imbedding of $(O, <)$ into a lattice $(O_R, <_R)$ is feasible *under the condition that O and O_R be of same cardinality.*

We are going to prove that for any infinite O the answer is by affirmative; for any finite ordered set $(0, <)$ which is not a lattice the answer is by negative.

2. Let us consider the following problem:

P r o b l e m. *Let n be any cardinal number; consider the minimal number $m(n)$ such that every ordered set of a cardinality n be isomorph to a subset of a lattice of cardinality $\leqslant m(n)$.*

The existence of $m(n)$ for any given n is obvious: the question is to determine $m(n)$ as function of n. E. g. $m(0)=0, m(1)=1$ (the empty set as well as every one-point set are considered as lattices); $m(2)=4$.

3. T h e o r e m. *For every cardinal number $n>1$ one has*

$$m(n) \leqslant 2^n. \tag{1}$$

If $n \geqslant \aleph_0$, then $m(n)=n$.

In particular, every infinite ordered set $(0, <)$ is imbeddable into a lattice $(O_R, <_R)$ of the cardinality kO.

4 P r o o f. 4.1. The first step in the transition $(O, <) \rightarrow (O_R, <_R)$ consists to adjoin to O: a first member, 0, provided it is not present in $(O, <)$ and to adjoin a last member, 1, provided it is not present in $(O, <)$.

The relation (1) is an immediate consequence of the forming of $L(O)$ by means of subsets of O, all these subsets forming the partitive set PO of cardinality 2^n.

4.2 As to the existence of $(O_R, <_R)$, at first we define O_a as the family of all the sets of the form

$$0\,1X, (X \subseteq O, kX < \aleph_o)$$

$$\text{where } 1\ X = \{y; y \in O, X < y\}$$

$$0\,X = \{x; x \in O, x < X\}$$

$$0\,1\,X = 0\,(1\,X).$$

Then $(O_a; \subseteq)$ is a lattice.

4.3 If for every $x \in O$ we substitute x for $O\ (., x)$ and $\leqslant_R$ for $\subseteq$, then O_a yields a set, say O_R and $O_R \supseteq O$; the set $(O, <)$ is imbedded in $(O_R, <_R)$.

4.4 *l-extension of* $(O, <)$. Moreover, the set $(O_R, <_R)$ is an *l-extension of* $(O, <)$ in the sense that not only $O_R \supseteq O$ and

$$a \leqslant b \text{ in } O \Rightarrow a \leqslant_R b \text{ in } O_R$$

but also that for $\{a, b\} \subseteq O$ one has

$$\inf_0 \{a, b\} \in O \Rightarrow \inf_0 \{a, b\} = \inf_{O_R} \{a, b\},$$

$$\sup_0 \{a, b\} \in O \Rightarrow \sup_0 \{a, b\} = \sup_{O_R} \{a, b\}.$$

4.5. The cardinality of O_a is such that

$$k\,0 \leqslant k\,0_a \leqslant k\,0 + k\,0^2 + k\,0^3 + \ldots$$

4.6. Consequently, assuming the axiom of choice one has $k\,O^r = k\,O$ for every natural integer r and every infinite O. Therefore $k\,O_a = k\,0$ and also $k\,0_R = k\,0$ because $k\,0_a = k\,0_R$.

4.7. For another proof of the theorem 3 cf. section 8.

5. *Extension of the validity of relations inf* $\{a, b\} \in O$ *and sup* $\{a, b\} \in O$.

5.1 Sometimes it is interesting to imbed $(O, <)$ in a (minimal) l-extension (M, ρ) of $(O, <)$ in such a way that for a given

$$\text{set } E \subseteq \binom{0}{2} = \{X; X \subseteq O, k\,X = 2\}, \text{ one has}$$

$$x \in E \Rightarrow \inf_M x \in M \text{ or}$$

$$x \in E \Rightarrow \sup_M x \in M \text{ or both.}$$

In general, the ordered set (M, ρ) is not a lattice. The simplest case is that E consists of a single 2-point-subset of O.

5.2. We are going to indicate a construction of $M = M(E)$ leaving aside the question whether the construction of $M(E)$ is as economical as possible in the sense to introduce in $M(E)$ as many comparable elements as possible.

5.3. Lemma. *Let* $(O, <)$ *be an ordered set; let* $\{a, b\} \subseteq O$ *and*

$$i = \inf \{a, b\} \in 0;$$

let (M, ρ) *be an ordered set extending* $(0, <)$ *by adjoining to the set O a single member* $x \notin O$.

Then we have the following equivalence $(1) \Leftrightarrow (2)$ *where*

(1) $\quad i \,\|_\rho\, x$,[1] $x \,\rho\, \{a, b\}$

[1]) $i \,\|_\rho\, x$ means that neither $i \,\rho\, x$ nor $x \,\rho\, i$ (i. e. that i, x are ρ incomparable

(2) $\inf_M\{a,b\} \notin M.$

The implication (1) $\Rightarrow$ (2) is obvious, because (1) and the relation $i \in O$ imply that the set of all predecessors of a, b in (M, ρ) consists of x and of O[i,.) and has 2 initial points i, x.

Conversely, (2) $\Rightarrow$ (1). At first, from (2) we infer that

(3) $M(., a] \cap M(., b] = O(., a] \cap O(., b] \cup \{x\}$ and thus $x \rho \{a, b\}$.

One has neither $x \rho i$ nor $i \rho x$ because these relations would imply that $\inf_M \{a,b\}$ equals x and i respectively, in contradiction with (2).

The dual of 5.3 reads as follows.

5.4. *Lemma. Let $(O, <)$ be an ordered set; let (M, ρ) be an order extension of $(0, <)$ obtained from $(0, <)$ by adjoining a single point y; if $\{a\, b\} \subseteq O$ and $s = sup_0 \{a, b\} \in O$, then the following conditions* (4), (5) *are logically equivalent*:

(4) $s \|_\rho y, \ \{a, b\} \rho y$

(5) $\sup_M \{a, b\} \notin M.$

5.5. Lemma. Let $(O, <)$ be any ordered set; if $u, v \in O$ and if $\inf_0 \{u, v\}$ does not exist, let $x = x(u, v)$ be an object which is neither a member nor a part of O: let locate the object x immediately before a and before b and immediately after the set

(1) $O(., u] \cap O(., v]$; in particular $u \bar{=} x = \bar{v}$;

for any other point t of O we consider t, x to be incomparable.

If the set (1) is empty we define x to follow to every point of $(0, <)$. The ordered set $(M = O \cup \{x\}; <')$ so obtained is an extension of the given ordered set $(O, <)$ leaving invariant supremum as well as infimum of any 2-point-subset $\{a, b\}$ of O; i. e. if $\{a, b\} \subseteq O$, then

(2) $i = \inf_O \{a, b\} \in O \Rightarrow \inf_O \{a, b\} = \inf_M \{a, b\}$:

(3) $s = \sup_O \{a, b\} \in O \Rightarrow \sup_M \{a, b\} = \sup_M \{a, b\}$.

Let us prove the implication (2).

First $\inf_M\{a, b\}$ *exists and is a member* t *of* M; in opposite case, the implication (2) $\Rightarrow$ (1) in the lemma 5.3. would yield

(4) $x \rho \{a, b\}$ and $i \|_\rho x$.

In particular, $x \rho \{a, b\}$ implies $u \leqslant \{a, b\}, v \leqslant \{a, b\}$ and from here, by the definition of $i = \inf_O \{a, b\}$, one would have $\{u, v\} < i$, and from here $x \rho i$, contradicting the second relation of (4).

Hence $t \in O$. Therefore $i \rho t$. We say that $t \in O$ i. e. $i = t$ and that (2) holds. In opposite case, one would have $t = x$, thus $i \rho x$, and $i \leqslant \{u, v\}$. Now $x (= t) = \inf_M \{a, b\}$; therefore $x \rho \{a, b\}$ and $u \leqslant \{a, b\}, v \leqslant \{a, b\}$; thus $\{u, v\} \leqslant i$ and $x \rho i$, contradicting $i \rho x, i \neq x$.

Analogously, (3) is holding.

First, $\sup_M\{a,b\}$ exists and is a member z of M. Otherwise one would apply the implication (5) $\Rightarrow$ (4) in the lemma 5.4 and consequently one would have

(5) $\quad s\,\|_\rho x, \qquad \{a,b\}\,\rho\,x.$

The last relation implies $\{a,b\} \leqslant u, \{a,b\} \leqslant v$, hence by the definition of $s=\sup_O\{a,b\}$, $s \leqslant \{u,v\}$ and therefore $s\,\rho\,x$, contradicting (5).

On the other hand, $z \in M \Rightarrow z\,\rho\,s$. If moreover $z \in O$, then necessarily $s=z$ and the requested implication (3) is proved. Now, suppose by contradiction that $z \in M \setminus O$, i. e. $z=x$. Then $x\,\rho\,s$, $x \neq s$ and $\{u,v\} \leqslant s$. Further, $x=z= =\sup_M\{a,b\}$ implies $\{a,b\}\,\rho\,x$; from here by the definition of ρ we have $\{a,b\} \leqslant u, \{a,b\} \leqslant v$ and therefore also $s \leqslant \{u,v\}$, hence $s \leqslant x$ contradicting the assumption $x\,\rho\,s$, $x \neq s$.

Since the last sentence in the lemma is obvious, the proof of the lemma is completed.

The dual of the lemma 5.5. reads as follows.

5.6. Lemma. Let $(O, <)$ be any ordered set; if $\{u,v\} \subseteq O$ and $\sup_O\{u,v\} \notin O$, then adjoining to O an object $y=y(u,v)$ which is neither a member nor a part of 0 and defining in $M=O\cup\{y\}$ the extension ρ of $\leqslant$ in such a way that $u^+=y=v^+$ and that every member of the set $O[u,.)\cap O[v,.)$ precedes immediately to y, while else y is incomparable to every other point of O then for any $\{a,b\}\subseteq O$ one has the implications:

$$\inf_O\{a,b\} \in O \Rightarrow \inf_{(O,<)}\{a,b\} = \inf_{(M\rho)}\{a,b\},$$

$$\sup_O\{a,b\} \in O \Rightarrow \sup_{(O,\leqslant)}\{a,b\} = \sup_{(M,\rho)}\{a,b\}$$

$$y=\sup_{(M,\rho)}\{u,v\}.$$

The proof runs dually to that of the Lemma 5.5.

6. Theorem. *Let* $(O, \leqslant)$ *be any ordered set and* $E \leqslant \binom{O}{2}$ *any set of 2-point-subsets of* O; *there exists an ordered set* $(O(E), \leqslant(E))$ *extending the ordered set* $(O, \leqslant)$ *such that* $\{u,v\} \in E \Rightarrow \inf_{O(E)}\{u,v\} \in O(E)$: *moreover, if* $k\,0 \leqslant \aleph_o$, *then* $k\,0 = k\,0\,(E)$.

And dually for the supremum for every $\{x,y\} \in F$ *where* $F \subseteq \binom{O}{2}$ *is given.*

Proof. Let

$$a_0, a_1, \ldots a_\varphi, \ldots (\varphi<\Psi)$$

be any normal well-order of E; for every $\varphi<\Psi$ we have

$$a_\varphi=\{a_{\varphi 0}, a_{\varphi 1}\}\subseteq O, \qquad a_{\varphi 0}\neq a_{\varphi 1}.$$

6.1. We define

$iE=\{(i,x); x\in E\}$; i is the first character of the word infimum; s is the initial character of supremum; obviously, the sets E, iE, are disjoint.

6.2. Let us consider the following ordered sets

$$(0_{i\varphi}, \leqslant_{\varphi i})\,(\varphi<\Psi)$$

extending $(0, \leqslant)$.

If $\inf a_0 \in (0, \leqslant)$ we put $(0_{10}, \leqslant_{10}) = (0, \leqslant)$; if

$\inf a_0 \notin (0, \leqslant)$, then $0_{10} = 0 \cup \{i a_0\}$ and we define $\leqslant_1$ by

intercalling in $(0, \leqslant)$ the element $(i a_0)$ between a_0 and the set

(4) $\quad 0(., a_{00}] \cap O(., a_{01}]$.

By the lemma 5.5, in the set $(0_{10}, \leqslant_{10})$ the infimum of $a_0 = \{a_{00}, a_{01}\}$ exists: if $\sup \{x, y\}$ exists in $(0, <)$ so also in $(0_{10}, \leqslant_{10})$ and is the same.

6.3. One defines $(0_{11}, \leqslant_{11})$ substituting in the foregoing consideration 5.5, 5.6

$$0 \to 0_{10}, \quad \leqslant \to \leqslant_{10}, \quad a_0 \to a_1; \text{ if}$$

$0 < \gamma < \Psi$ and if the set $(0_{1\beta}, \leqslant_{1\beta})$ is defined for every $\beta < \gamma$, we define also $(0_{1\gamma}, \leqslant_{1\gamma})$ as previously on substituting

$$O \to \bigcup_{\beta<\gamma} O_{1\beta}, \quad < \to \bigcup_{\beta<\gamma} <_{1\gamma}, \quad a_0 \to a_\beta,$$

By definition

$$x \,(\bigcup_{\beta<\gamma} \leqslant_{1\beta})\, y \text{ means } x \leqslant_{1\beta} y \text{ for at least one } \beta < \gamma.$$

6.4. The ordered set $(0_{1\varphi}, \leqslant_{1\varphi})$ being defined for every $\varphi < \Psi$ we define

$$g(O, <) = (0_1, \leqslant_1) \underset{=\!=}{\text{def}} (\bigcup_{\varphi<\psi} O_{1\varphi}, \sup_{\varphi<\psi} \leqslant_{1\varphi}).$$

6.5. The ordered set $(0_1, \leqslant_1)$ is an extension of $(0, \leqslant)$ and one sees readily that for $\{x, y\} \in E$ one has

$$\inf \{x, y\} \in (0_1, \leqslant_1);$$

moreover, if $x, y \in O$ have its infimum in $(0, \leqslant)$ so do they the same in $(0_1, \leqslant_1)$ with the same value.

6.6. One has $k\,0_1 = k\,0$ because $k\,0 \leqslant k\,0_1 \leqslant k\,0 \;\; k\,i\,E \leqslant$

$$k\,0 + k\,0^2 = k\,0.$$

The proof for the dual form runs analogously on considering instead of the set $i\,E$ the set $s\,E = (s\,x), x \in F$; s is the initial of the word supremum.

Of course, the sets 0, $i\,E$, $s\,E$ are pairwise disjoint.

7. Theorem. Let $(0, \leqslant)$ be any ordered set and $E \subseteq \binom{0}{2}$, $F \subseteq \binom{0}{2}$; there exists an l-order extension $(M, \leqslant_R)$ of $(0, \leqslant)$ such that

$$e \in E \Rightarrow \inf_M e \in (M, \leqslant_R) \text{ and } f \in F \Rightarrow \sup_M f \in (M, \leqslant_R).$$

The theorem 7 is an easy consequence of the theorem 6 for $E = F = \binom{0}{2}$.

8. *Another proof of the theorem* 3.

Put in the foregoing proof $E = \binom{0}{2} = F$.

Then putting $g(O, \leqslant) = g_1(O, <)$ and defining

$$g(g_r(O, \leqslant)) = g_{1+r}(O, \leqslant) = (0_{1+r}, \leqslant_{1+r})$$

the requested ordered set $(0_R, \leqslant_R)$ is defined in the following way:

$$0_R = \bigcup_{r<r} O, \quad \leqslant_R = \bigcup_r \leqslant_r, \qquad (0 < r < \omega).$$

One proves readily that the ordered set $(O_R, \leqslant_R)$ is a lattice of cardinality $k\,0$ and that $(0_R, \leqslant_R)$ is an l-extension of $(O, \leqslant)$.

BIBLIOGRAPHY

Birkhoff, G., *Lattice Theory*, Amer. Math. Soc., New York, 1948_2 14 + 284.

Szász, Gábor, *Introduction to Lattice Theory*, Budapest, 1963, 230.

GLASNIK MATEMATIČKI
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DENDRITY OF SPACES AND OF ORDERED SETS

Đuro Kurepa, Beograd

The aim of this paper is to examine the supremum of cardinalities of some sets connected with a topological space. Especially, we want to know the supremum of cardinalities of trees, of increasing F-sets, G-sets, and analogous numbers for decreasing trees. In order to examine the situation for the most general V-spaces, i. e. spaces defined by neighborhoods, or equivalently by closure satisfying the isotonity condition and the condition that the closure of the empty set is empty, we consider systems of sets, well-ordered by inclusion, which are not necessarily closed, but in simple cases are connected with closed sets (cf. the definition of the number F_C in § 4.2.2 and of F_D in § 4.3.1).

The main results are contained in Theorems 1.1 (equality of dr M, dr* M, cel M, w M for metric spaces), 2.2 (the same for pseudometric spaces), 3.6 (equality of sep, w, dr, dr* for *R-spaces*), 4.2.4 (ii) (equality of dr, Sep, $k_c(F)$ for idempotent V-spaces), 4.3.4 (equality dr* V = dp $V = k_d(FV)$.

0. Dendrity of ordered sets and spaces. Definitions

0.1. *Definition of* dr. For any ordered set $(O, \leq)$ we define the dendrity dr $(O, \leq)$ of $(O, \leq)$ as the supremum of cardinalities kT of trees T[1] contained in $(O, \leq)$, i. e. dr $(O, \leq) = \sup kT$, T being any subtree of $(O, \leq)$; instead of dr $(O, \leq)$ we shall write also dr O.

0.1.1. In particular, for every family H of sets we have the ordered sets $(H, \supseteq)$, $(H, \subseteq)$ and the corresponding numbers dr $(H, \supseteq)$, dr $(H, \subseteq)$.

0.2 *Definition of dual dendrity* dr*. We put

$$\mathrm{dr}^*(O, \leq) = \mathrm{dr}(O, \geq).$$

Obviously, for any tree T we have dr$T = kT$ (= cardinality of T). For any ordered set $(O, \leq)$ the number dr$(O, \leq)$ is well-determined.

[1] Recall that a tree is any ordered set $(O, \leq)$ such that for every $a \in O$ the set $O(\cdot, a) = \{x; x \in O, x < a\}$ is a well-ordered subset of $(O, \leq)$.

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0.3. *Dendrity* dr S *of a space* S. For any space S we denote by GS the system of all open sets of the space; the *dendrity of* S is, by definition, the number $\mathrm{dr}\, S \overset{\mathrm{def}}{=} \mathrm{dr}\,(GS, \supseteq)$.

We shall see that dr S = Sep S (we distinguish Sep S from sep S; cf. 2.2, 2.3, 2.4 (ii)).

0.4. *Dual dendrity* dr* S *of a space* is defined by

$$\mathrm{dr}^* S = \mathrm{dr}\,(GS, \subseteq).$$

We shall see that dr* S = dp S (cf. 4.3.2, 4.3.3, 4.3.4 (ii)).

0.5. S being a space, let FS be the system of all closed sets of S; the numbers $\mathrm{dr}\,(FS, \supseteq)$, $\mathrm{dr}\,(FS, \subseteq)$ are well-defined and easily evaluated to be $1 + kS$ for every V-space satisfying $\overline{\{x\}} = \{x\}$.

0.6. Problem. *Is there a space* S *for which (one of) the numbers* dr S, dr* S *(is) are not reached?*

The corresponding problem for ordered sets or general set systems is answered in the affirmative. E. g., for every infinite well-ordered set $(W, \leq)$ we have $\mathrm{dr}^*(W \leq) = \mathrm{dr}\,(W, \geq) = \aleph_0$, although there is no infinite increasing sequence in $(W, \geq)$[2].

For the corresponding ordered space W (interval topology) one has $\mathrm{dr}\, W = \mathrm{dr}^* W$, as it is easily shown.

1. Metric spaces

M shall denote any metric space.

1.1. Theorem. *For every metric space* M *we have*

$$\mathrm{cel}\ M = \mathrm{sep}\, M = \mathrm{w}M = \mathrm{dr}\, M = \mathrm{dr}^* M \text{ (cf. § 1.2).}$$

The equality cel M = wM was proved by K. Haratomi [1]. Therefore, it is sufficient to prove the equality wM = dr M.

1.2. At first, $\mathrm{w}M \leq \mathrm{dr}\, M$. As a matter of fact, let

$$S_0, S_1, \ldots, S_n, \ldots (n < \varphi)$$

be a system of wM disjoint open sets. If for any $n < \varphi$ we put

$$G_n = \bigcup_m S_m \quad (m < n)$$

one obtains a strictly decreasing φ-sequence of G-sets; this sequence being a tree $(T, \supseteq)$ one has $\mathrm{dr}\, M \geq \mathrm{k}n = \mathrm{w}M$, i. e. $\mathrm{dr}\, M \geq \mathrm{w}M$.

[2] Each time a supremum of ordinal (cardinal) numbers is involved, one has to examine whether this supremum is reached under given conditions (cf. the wording of the ramification hypothesis stating that for every tree $(T, \leq)$ the number bT is reached *in* $(T, \leq)$; bT is the supremum of cardinalities of degenerated trees $\subset (T, \leq)$; (an ordered set is degenerated provided the comparability relation is transitive).

Assume the contrary that $\mathrm{dr}\, M > \mathrm{w}M$ and that consequently there exists some tree $(T, \supseteq)$ of open sets of M satisfying

$$\mathrm{k}T > \mathrm{w}M. \tag{1}$$

1.3. Every row $R_n T$ being a system of disjoint open sets the cardinality of $R_n T$ is $\leq \mathrm{cel}\, M$; the disjoint partition of T into rows $R_n T (n < \gamma T)$, γT denoting the ordinal *height or rank* of T, we infer that

$$\mathrm{k}T = \mathrm{k}\,\gamma T \text{ and that consequently} \tag{2}$$

$$\gamma T \geq \omega_{\alpha+1} \text{ provided } \mathrm{k}\omega_\alpha = \mathrm{w}M. \tag{3}$$

1.4. The relation $\gamma T > \omega_{\alpha+1}$ does not hold, because otherwise one would have some $X \in R_{\omega_{\alpha+1}} T$; now, all members Y of T such that $Y \supset X$ would constitute a subchain of $(T, \supseteq)$ of cardinality $\aleph_{\alpha+1}$, in contradiction with the Baire statement that every strictly decreasing sequence of G-sets in M has $\leq \mathrm{w}M$ members.

1.5. *The relation* $\gamma T = \omega_{\alpha+1}$ *does not hold neither.*
As a matter of fact, let B be any set everywhere dense ($\overline{B} = M$) and such that $\mathrm{k}B = \mathrm{sep}\, M = \aleph_\alpha (\equiv \mathrm{w}M)$. For every $b \in B$ there exists some index $n(b) < \omega_{\alpha+1} (= \gamma T)$ such that b belongs to no member of $R_{n(b)}\, T$ (in the opposite case, all members of T containing b would form a chain of cardinality $k\,\gamma T = \aleph_{\alpha+1}$, contrary to Baire's statement). Let

$$n = \sup_{b \in B} n(b).$$

Since, for $b \in B$, we have $n(b) < \omega_{\alpha+1}$ and since $\mathrm{k}B = \aleph_\alpha$, we infer that $n < \omega_{\alpha+1}$. If then $\gamma T = \omega_{\alpha+1}$, the row $R_{n+1} T$ would contain a member X which is an open set of the space M; therefore, there would be some $b_0 \in B \cap X$; consequently $n(b_0) = n + 1$, i. e. $n(b_0)$ would be greater than the supremum of all the numbers $n(b)$, for $b \in B$, which is absurd.

1.6. *The number* $\mathrm{dr}^* M$ *equals* $\mathrm{w}M$.
As a matter of fact, every increasing tree $(T, \subseteq)$ of sets is composed of a family of chains $T[X, \cdot) = \{Y; Y \in T, X \subseteq Y\}$, $(X \in R_0 T)$; these chains are mutually incomparable, i. e. every member of one chain is disjoint with every member of another chain. Since $\mathrm{k}R_0 T \leq \mathrm{cel}\, M \leq \mathrm{w}M$ and since $\mathrm{k}T[X, \cdot) \leq \mathrm{w}M$ (Baire's theorem), we infer that $\mathrm{k}T \leq \mathrm{w}M \cdot \mathrm{w}M = \mathrm{w}M$, i. e. $\mathrm{k}T \leq \mathrm{w}M$; consequently, $\sup \mathrm{k}T \leq \mathrm{w}M$, i. e. $\mathrm{dr}^* M \leq \mathrm{w}M$. To prove the dual relation $\mathrm{dr}^* M \geq$ $\geq \mathrm{w}M$, it is sufficient to consider any isolated set $I = \{i_0, i_1, \ldots\}$ of cardinality $\mathrm{w}M$ and to associate with every $n < \omega_\alpha$ a neighborhood $O_n(x_n)$ having no other point of I but x_n in common and, for every $n < \omega_\alpha$, to put $G_n = \bigcup_{m<n} O_m(x_m)$ in order to obtain a strictly increasing ω_α-sequence of G-sets.

2. The case of pseudo-metric spaces

2.1. Theorem. *If M is any metric or pseudo-metric space, then*

$$\text{sep } M = \text{w}M = \text{cel } M \tag{1}$$

(for pseudo-metric spaces, see Đ. Kurepa [1], [8], [10] *and P. Papić* [3]).

2.1.1. If M is metric, the relation (1) was proved earlier. Therefore, let us assume that $M \in (D_\alpha)$ for some regular initial number ω_α.

2.1.2. Then the space M is definable by means of neighborhoods forming a decreasing tree $(T, \supseteq)$ of sets of rank ω_α each nonisolated point x of M yielding the intersection $\{x\}$ of the ω_α-sequence of decreasing sets $\in T$ (cf. P. Papić [3], théorème 2, p. 218). Consequently, if M is not isolated, there is some non isolated point x of M and the ω_α-sequence of all members $X_n \in T$ such that $x \in X_n$ $(n < \gamma M = \omega_\alpha)$. Now, for $n < \omega_\alpha$ we have $X_n \supset X_{n+1}$; on the other hand, the set $X_n \setminus X_{n+1} = X_n \cap C\, X_{n+1}$ is open, the members of T being closed and open (cf. Papić [1], p. 31). Therefore, one has $\aleph_\alpha$ disjoint open sets

$$X_n \setminus X_{n+1} \quad (n < \omega_\alpha)$$

and consequently, cel $M \geq \text{k}\, \omega_\alpha$ for every $M \in (D_\alpha)$.
On the other hand, T is a basis of neighborhoods and $\text{k}T = \text{w}M$. Since $\text{k}T \leq \text{Sup k } R_\alpha\, T \cdot \gamma\, T \leq \text{cel } M \cdot \aleph_\alpha = \text{cel } M$, we obtain the relation $\text{w}M = \text{k}T \leq \text{cel } M$ and thus

$$\text{w}M = \text{cel } M, \tag{2}$$

the inequality $\text{w}M < \text{cel } M$ being absurd.
Relation (2) is the main part in relation (1) in the theorem, because obviously, $\text{cel } M \leq \text{sep } M \leq \text{w}M$.

2.2. Theorem. *For every regular ω_α and every pseudo-metric space $M \in (D_\alpha)$ the dendrity number* dr M *equals* cel M:

$$\text{cel } M = \text{sep } M = \text{w}M = \text{dr } M = \text{dr}^* M \text{ (cf. } \textit{Theorem } 1.1).$$

At first, let us prove the following generalizations of Baire's theorems concerning well-ordered decreasing (increasing) systems of G-sets of metric spaces.

2.2.1. Theorem. *The number* sep M *equals the supremum of cardinalities of increasing sequences of F-sets:*

(1) $\text{sep } M = \sup \text{k}F$, *$F$ being any increasing sequence of F-sets in the space.*

Proof. We have to prove that in (1) the symbol = means $\leq$ and $\geq$.

At first, let $F_0 \subset F_1 \subset \ldots \subset F_\nu \subset \ldots (\nu < \sigma)$ be any strictly increasing sequence of closed sets in the space M of separability sep M; then $k\,\sigma \leq \operatorname{sep} M$; otherwise, $k\,\sigma > \operatorname{sep} M$; now, the sets $F_{\nu+1} \setminus F_\nu (\nu < \sigma)$ being non empty disjoint sets, one would have a σ-sequence a_ν of pairwise distinct point $a_\nu \in F_{\nu+1} \setminus F_\nu (\nu < \sigma)$. Now, the set F_ν being closed, there would be a neighborhood O_ν of a_ν such that $O_\nu \cap F_\nu = \emptyset$; therefore, the members of the σ-sequence O_ν $(\nu < \sigma)$ would be pairwise distinct sets. Since obviously, it is permitted to assume the sets O_ν to be members of a neighborhood basis B of minimal cardinality wS, and since wS = sep S for every pseudo-metric space S, the cardinality kB of B would be $\geq k\sigma > \operatorname{sep} S$, i. e. w$S$ > sep S, which is absurd.

On the other hand, let $a_0, a_1, \ldots, a_\nu \ldots (\nu < \omega_\beta)$ be a normally well-ordered set of sep M points, which is everywhere dense. We say that $k\,\omega_\beta = (1)_2$. As a matter of fact, we have the increasing sequence

$$F_\nu = \operatorname{cl} \{a_0, a_1, \ldots, a_\xi, \ldots\}_{\xi<\nu}$$

of closed sets containing necessarily sep M distinct members; in other words, there is a strictly increasing sequence of sep M closed sets; this means that in (1) the symbol = is replaceable by $\leq$.

Analogous considerations lead to the following

2.2.2. Theorem. *For every pseudo-metric space* $M_\alpha \in (D_\alpha)$ *the number* sep M_α *equals the supremum of cardinalities of all strictly decreasing sequences of closed sets.*

2.3. The proof of Theorem 2.2.2 runs like the one of the Theorem 2.2.1.

3. *R*-spaces and the corresponding numbers w, sep, cel and dr

3.1. Every *R-space R* is defined by neighborhoods which are comparable or disjoint (as concerns *R*-spaces, see Đ. Kurepa [4], [7], [10] and P. Papić [1]—[5]). According to P. Papić ([1], théorème 3) every R is definable by a decreasing tree of sets.

3.2. The question of relations between the numbers cel R, wR is connected with the ramification hypothesis.

3.3. We shall assume that Fréchet's T_1-separation axiom holds for R; therefore, we shall deal with T_1-R-spaces R, defined by some trees T of sets such that for every $x \in R$ the intersection of all members of T containing x is the one-point set $\{x\}$.

3.4. Since every member of T is *clopen* (closed and open) we infer that for any $A, B \in T$ the set $A \setminus B$ is clopen too. Therefore, if $A_0 \supset A_1 \supset \ldots \supset A_n \supset \ldots (n < \beta)$ is any well-ordered decreasing system of members of T, then the sets $A_n \setminus A_{n+1}$ $(n < \beta)$ are open and disjoint. Therefore, $k_c R \leq \operatorname{cel} R$, $k_d R \leq \operatorname{cel} R$. On the other hand, the sets of rows $R_n T$ being pairwise disjoint, the decomposition of T into rows $R_n T$ $(n < \gamma T)$ yields

$$\mathrm{k}T \leq \mathrm{cel}\, T \cdot \mathrm{k}\,\gamma T, \text{ i. e.}$$
$$\mathrm{w}R \leq \mathrm{cel}\, R \cdot \mathrm{k}\,\gamma T.$$

Now, the number $\mathrm{k}\,\gamma T$ is either cel R or $(\mathrm{cel}\, R)^+$; therefore

$\mathrm{cel}\, R \leq \mathrm{w}R \leq (\mathrm{cel}\, R)^+$, i. e. we have the following

3.5. Theorem. *For every T_1 R-space R we have*

$$\mathrm{w}R \in \{\mathrm{cel}\, R,\ \mathrm{cel}\,{}^+R\}.$$

3.6. Theorem. *For any T_1 R-space R we have* $\mathrm{cel}\, R \leq \mathrm{sep}\, R = \mathrm{w}R = \mathrm{dr}\, R = \mathrm{dr}^* R$.

Proof. The first relation $\mathrm{cel}\, R \leq \mathrm{sep}\, R$ is obvious.

3.6.1. If $\mathrm{w}R = \mathrm{cel}\, R$, then $\mathrm{sep}\, R = \mathrm{cel}\, R = \mathrm{w}R$.

3.6.2. If $\mathrm{w}R = \mathrm{cel}\,{}^+R$, then R is definable by a tree T of neighborhoods, such that γT be an initial ordinal number and $\mathrm{k}R_n T < \mathrm{k}\,\gamma T$ for every $n < \gamma T$. Then $\mathrm{k}T = \mathrm{k}\,\gamma T$.

3.6.3. Assume γT regular. Then $\mathrm{k}T = \mathrm{sep}\, R$ because if B is a set of sep R points, such that $\overline{B} = R$, then to every point $b \in B$ corresponds some index $n(b)$ such that no member of $R_{n(b)} T$ contains the point b. If $\mathrm{k}B < \mathrm{k}\,\gamma T$, then the number $n = \sup\limits_{b \in B} n(b)$ would be $< \gamma T$, and no member X of $R_{n+1} T$ would contain any point of B, which is absurd, because X as an open set should contain some point of B. Consequently, $\mathrm{k}B = \mathrm{k}\,\gamma T$.

3.6.4. If γT is singular, again the relation $\mathrm{k}B < \mathrm{k}\gamma T$ does not hold, because this relation would imply the existence of some ordinal number δ such that

$$\mathrm{k}B < \mathrm{k}\,\delta < \mathrm{k}\,\gamma T,$$

the number γT being initial. Now, let $X \in R_\delta T$; then the chain $T(\cdot, X)$ is of cardinality $\mathrm{k}\delta > \mathrm{k}B$ and would yield a disjoint system of open sets $X_\xi \setminus X_{\xi+1}$ $(\xi < \delta)$, where $X_\xi \supset X$ and $X_\xi \in R_\xi T$. In either case $\mathrm{k}B = \mathrm{k}T$, i. e. $\mathrm{sep}\, R = \mathrm{w}R$.

3.6.5. There remains still to prove $\mathrm{w}R = \mathrm{dr}\, R$.

Now, let $T(G)$ be any decreasing tree of open sets of the space R; we have to prove that

$$\mathrm{k}T(G) \leq \mathrm{w}R, \text{ i. e., that the inequality}$$
$$\mathrm{k}T(G) > \mathrm{w}R$$

does not hold. First of all, every row of $T(G)$ is $\leq$ cel R, the members of $R_\xi\, T(G)$ being disjoint G-sets.

3.6.6. Every decreasing sequence S

$$G_0 \supset G_1 \supset \ldots \supset G_\xi \supset \ldots \quad (\xi < \beta)$$

in any R-space R has $\leq \mathrm{w}R$ members.

As a matter of fact, since every G_ξ is a union of a subsystem of the basis B of neighborhoods of the space R, let us define for every $\xi < \beta$ a member $a_\xi \in G_\xi \setminus G_{\xi+1}$ and a member $V_\xi \in B$ such that $a_\xi \in V_\xi$ and $V_\xi \subseteq G_\xi$. We obtain a mapping $f: S \to B, f G_\xi = V_\xi$; the mapping f being one-to-one, we infer that $\mathrm{k}S \leq \mathrm{w}R\,(= \mathrm{k}B)$.

Similarly one proceeds for every increasing sequence of G-sets.

3.6.7. Let $\mathrm{w}R = \mathrm{k}\omega_\beta$; since every chain from $(T(G), \supseteq)$ is $\leq \mathrm{k}\omega_\beta$ one has (*) $\gamma T(G) \leq \omega_{\beta+1}$.

If the sign $\leq$ in (*) means $<$ all is proved, because $\mathrm{k}R_a T(G) \leq \leq \mathrm{cel}\, R \leq \omega R$, and therefore, $kT(G) \leq \mathrm{w}R\, \mathrm{k}\,\gamma T(G) = \aleph_\beta = \mathrm{w}R$. Therefore, let us assume

$$\gamma T(G) = \omega_{\beta+1}.$$

This equality is not possible neither, as is readily shown by the argument like the one in section 3.6.3.

3.6.8. The proof of the equality $\mathrm{dr}^* R = \mathrm{w}R$ runs like the corresponding proof in § 1.6 for metric spaces M.

3.7. Theorem. *For every T_1 R-space R one has*

$$\mathrm{dr}\, R = \mathrm{dr}^*\, R \text{ and } \mathrm{dr}\, R \in \{\mathrm{cel}\, R,\ \mathrm{cel}^+ R\}.$$

Theorem 3.7 is a consequence of Theorems 3.5 and 3.6. The question whether necessarily $\mathrm{dr}\, R = \mathrm{cel}\, R$ is connected with the ramification hypothesis, or rather with the rectangle hypothesis stating that for every tree T, one has $\mathrm{k}T \leq \mathrm{k}_c T \times \mathrm{k}_a T$ ($\mathrm{k}_a T$ is the supremum of cardinalities of all antichains $\subseteq T$).

3.8. Corollary. *For every T_1 R-space, in particular for every pseudo-metric and for every metric space, one has* $\mathrm{dr} = \mathrm{dr}^*$.

4. *V*-spaces

4.1. According to Fréchet, every V-space V is definable either by neighborhoods or by any isotone closure, such that $\overline{\emptyset} = \emptyset$.

Let FV [resp. GV] be the system of all closed [open] sets in the space. Then we have the ordered sets

$$(FV, \supseteq), \quad (FV, \subseteq)$$
$$(GV, \supseteq), \quad (GV, \subseteq)$$

and the corresponding dendrite numbers.

We shall see how the numbers $\mathrm{dr}\, V = \mathrm{dr}(GV, \supseteq)$, $\mathrm{dr}^* V = \mathrm{dr}(GV, \subseteq)$ depend on density and dispersity properties of subspaces of V (cf. 4.2.4. (ii), 4.3.4 (ii). See also 0.5).

4.2. *Functions* Sep *and* $\mathrm{F_C}$ *for spaces.* The main aim of this section shall be to establish the equality 4.2.4. (ii).

4.2.1. Definition. Sep $V = \inf_n n$, n being such a cardinal that every subspace S of V be n-separable in the sense that for some $X \subseteq S$, one has $\mathrm{k}X \leq n$ and $\overline{X} \supseteq S$; in other words Sep $V =$ $=$ Sup$_s$ sep S, S designating any subspace of V. The spaces satisfying $\aleph_\alpha = \operatorname{sep} S = \operatorname{Sep} S$ are called *hereditarily $\aleph_\alpha$-separable.*

4.2.1.1. Problem. Is the number Sep S reached for every space S?

4.2.2. Definition. $\mathrm{Fc}\, V = \sup \mathrm{k} f W$, $W = \{W_n\}_{n<\alpha}$ being any strictly increasing well-ordered system of sets, such that $n < (\gamma W)^- \Rightarrow \urcorner (W_n \supseteq W_{fn})$ for some ordinal fn satisfying $n < fn \leq (\gamma W)^-$.

4.2.3. Theorem. *For every topological V-space for which*

$$\operatorname{Sep} V + \mathrm{Fc}\, V \geq \aleph_0 \tag{0}$$

we have

$$\operatorname{Sep} V = \mathrm{Fc}\, V. \tag{1}$$

Proof. We shall prove that neither

$$\operatorname{Sep} < \mathrm{Fc} \tag{2}$$

nor

$$\operatorname{Sep} > \mathrm{Fc}. \tag{3}$$

According to the assumption (0) the number $\operatorname{Sep} V + \mathrm{Fc}\, V$ is infinite. Then one of the relations $\operatorname{Sep} V \geq \aleph_0$, $\mathrm{Fc}\, V \geq \aleph_0$ holds.

2. *Case* Sep $V > \mathrm{k}\,\omega_0$; put Sep $V = \aleph_\alpha$.

2.1. Assume that (2) holds, i. e., that $\mathrm{k}\,\omega_\alpha < \mathrm{Fc}\, V$ and that consequently there exists some $\omega_{\alpha+1}$-sequence of increasing sets

$$F_0 \subset F_1 \subset \ldots F_n \subset \ldots \quad (n < \omega_{\alpha+1}) \text{ and}$$

some ordinal function f mapping every $n < \omega_{\alpha+1}$ into some fn between n and $\omega_{\alpha+1}$ such that

$$\urcorner\, (\overline{F}_n \supseteq F_{fn}) \quad (n < \omega_{\alpha+1})\,; \text{ consequently}$$

$$F_{fn} \setminus \overline{F}_n \neq \emptyset, \text{ for every } n < \omega_{\alpha+1}. \tag{4}$$

2.2. We define a set $P = \{p_0, p_1, \ldots, p_n, \ldots\}$ $(n < \omega_{\alpha+1})$ in the following way. Let p_0 be a point in $F_{f0} \setminus \overline{F}_0$. Let $n < \omega_{\alpha+1}$ and suppose that a strictly increasing n-sequence φ_m $(m < n)$ of ordinals $< \omega_{\alpha+1}$ as well as a one-to-one n-sequence of points p_m be defined, such that $p_m \in F_{fm} \setminus \overline{F}_m$; let fn be the first ordinal such that $p_m \in F_{fn}$ $(m < n)$, and $n < fn$, $F_{fn} \setminus \overline{F}_n \neq \emptyset$. We define p_n as any point satisfying

$$p_n \in F_{f\varphi(n)} \setminus \overline{F}_{\varphi(n)}.$$

The number $\omega_{\alpha+1}$ being regular the construction of p_n for every $n < \omega_{\alpha+1}$ is secured in virtue of (4). The points p_n being pairwise distinct, we have

$$\mathrm{k}\, P = \mathrm{k}\, \omega_{\alpha+1}.$$

2.3. The subspace P of V is not $\aleph_\alpha$-separable, because for any subset X of P auch that $\mathrm{k}X < \aleph_{\alpha+1}$ there exists some $r < \omega_{\alpha+1}$ such that

$$X \subseteq F_r,\ p_n \notin \overline{F}_n.$$

These relations would imply

$$\overline{X} \subseteq \overline{F_r}$$

and thus

$$F_{fr} \setminus \overline{X} \supseteq F_{fr} \setminus \overline{F}_r;$$

consequently, since $p \in F_{fr} \setminus \overline{F}_r$, we have $p_r \in F_{fr} \setminus \overline{X}$, and this means exactly that the subset X of P would not be everywhere dense in the subspace P of V; thus, $\operatorname{sep} P > \operatorname{Sep} V$, which is absurd. The relation (3) holds neither.

2.4. On the contrary, we have

$$\mathrm{F_c}\, V \geq \operatorname{Sep} V. \tag{5}$$

As a matter of fact, let X be any subspace of V. We define a most extensive sequence $W = \{X_0 \subset X_1 \ldots\}$ of subsets of X in the following way. Let $p_0 \in X$ and $X_0 = \{p_0\}$; by induction let $X_n = \{p_0, p_1, \ldots, p_m, \ldots\}_{m<n}$, where $p_m \in X \setminus \overline{P_m}$, $P_m = \{p_0, p_1, \ldots, p_e, \ldots\}_{e<m}$. Putting $fn = n+1$, for every $n < (\gamma W)^-$, the *sets* W, fW are of the same cardinality and have $\geq \operatorname{sep} X$ members each. Since for every $n < (\gamma W)^-$ we have

$$p_{n+1} \in X_{n+1} \setminus \overline{X}_n,$$

i. e. the conditions of the definition 4.2.2 are satisfied for $X_n = F_n$, $fn = n+1$ and we infer that

$$\mathrm{F_C}\, X \geq \mathrm{k}\, \gamma W\ (\geq \operatorname{sep} X).$$

The negation of (2) and the relation (5) yield the requested equality (1).

4.2.4. Theorem. *For every idempotent*[3] *V-space V such that the number*

$$\operatorname{Sep} V + \mathrm{F_c}\, V \quad \textit{is infinite, one has}$$

$$\text{(i)}\quad \operatorname{Sep} V = \mathrm{F_c}\, V = \mathrm{k_c}(FV, \subseteq) = \mathrm{k_d}(GV, \supseteq).$$

$$\text{(ii)}\quad \operatorname{sep} V \leq \operatorname{dr} V = \operatorname{Sep} V = \mathrm{k_c}(FV, \subseteq) \leqq \mathrm{w}V.$$

Relations (i) are direct consequences of Theorem 4.2.3 and of the fact that for any idempotent space the members of W in Definition 4.2.2 may be assumed to be closed sets (thus also $fn = n+1$).

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[3] The space is idempotent provided $\overline{\overline{X}} = \overline{X}$, for every subset X of the space.

As to the relation (ii) the question arises on kT for any subtree of $(GV, \supseteq)$; since every chain in T is $\leq$ Sep V (this is the content of Theorem 3), and since obviously every antichain of T is $\leq$ Sep T, the question is to decide that kT = Sep$^+$ S does not hold. The disproof of the last equality is like the argument in § 1.5.

4.3. *Functions* $\mathrm{F_D}$ *and* dp *for spaces.*

4.3.1. Definition. $\mathrm{F_D}\, S = \mathrm{Sup}\ \mathrm{k}fW$; $W = (W_n \supset W_1 \ldots \supset \supset W_n \supset \ldots)_{n<\gamma W}$ denotes any strictly decreasing well-ordered family of sets, such that for any $n < (\gamma W)^-$ one has some fn such that $n < fn < \gamma W$ and

$$\neg\, (W_n \subseteq \overline{W}_{fn}); \tag{1}$$

one puts

$$fW = \{fx;\ x \in W\} \text{ (cf. T. Inagaki [2], p. 198).}$$

For idempotent spaces (condition $\overline{\overline{X}} = \overline{X}$) the conditions on members of W are equivalent to the condition that they are closed (in the notation for $\mathrm{F_D}$, F stands for closed (fermé) and D for decrease).

4.3.2. *Definition of dispersity* dp S *of* S.

We put dp S = sup kX, X being any dispersed set in the space, i. e. such one in which the empty set is the unique dense subset.

4.3.3. Theorem. *For any topological V-space V we have*

$$\mathrm{dp}\, V = \mathrm{F_D}\, V. \tag{2}$$

We shall prove that neither

$$\mathrm{dp}\, V < \mathrm{F_D}\, V \quad \textit{nor} \tag{3}$$

$$\mathrm{dp}\, V > \mathrm{F_D}\, V. \tag{4}$$

4.3.3.1 Let us prove that (3) does not hold. Assume, by contradiction, that (3) holds, and that for some V we have dp $V = \mathrm{k}\,\omega_\alpha$ and that there exists some $\omega_{\alpha+1}$-sequence W of sets W_n $(n < \omega_{\alpha+1})$ and a mapping $n < \omega_{\alpha+1} \to fn \in (n, \omega_{\alpha+1})$ satisfying (1). Since γW is regular, one has kfW = kW $(= \aleph_{\alpha+1})$.

Let us define a set

$$P = \{p_0, p_1, \ldots, p_n, \ldots\}_{n<\omega_{\alpha+1}} \tag{5}$$

in the following way. Let p_0 be any point of the non void set $W_0 \setminus \overline{W}_{f0}$. Let $0 < n < \omega_{\alpha+1}$ and assume that the set $\{p_0, \ldots, p_m, \ldots\}_{m<n}$ of cardinality kn be defined such that

$$p_m \in W_{gm} \setminus \overline{W}_{fgm}$$

for some strictly increasing n-sequence $g_0 < g_1 < \ldots < g_n < \ldots$ of ordinals $< \omega_{\alpha+1}$. Let

$$r_n = \sup_{m<n} fgm.$$

Then $r_n < \omega_{\alpha+1}$ because $fgm < \omega_{\alpha+1}$ for every $m < n$ and because $n < \omega_{\alpha+1}$. We consider the number $fr_n > r_n$, and the non empty set $W_{r_n} \setminus \overline{W_{fr_n}}$; we denote by p_n any point such that $p_n \in W_{r_n} \setminus \overline{W_{fr_n}}$. By induction, the set P is defined. At the same time we see that

$$p_n \notin \overline{P_n}, \tag{6}$$

where $P_n = \{p_s;\ n < s < \omega_{\alpha+1}\}$ because, by construction, one has $P_n \subseteq W_{fn}$ and $p_n \notin \overline{W_{fn}}$.
Therefore, the set P would be dispersed, i. e. if X is any set satisfying

$$X \subseteq P, X \subseteq \text{Der } X, \text{ then } X = \emptyset. \tag{7}$$

In the opposite case, there would be some non void X satisfying (7); let then p_r be the first member of X in the well-ordered set (5); then the relations (7) would imply $p_r \in \text{Der } X$, and consequently $p_r \in \overline{P_r}$, which contradicts (6). Consequently, the assumption (3) would yield a dispersed set X of cardinality $\aleph_{\alpha+1} > \aleph_\alpha = \text{dp } V$, and this is contradiction. Therefore, (3) does not hold.

4.3.3.2. Let us prove that (4) holds neither. Let V be any space; put $\text{F}_\text{D}\, V = \text{k}\, \omega_\alpha$. Assume, by absurdity, that the space V contains a dispersed set X of cardinality $> \omega_\alpha$. Without loss of generality we may suppose $\text{k}X = \aleph_{\alpha+1}$. Let

$$X = \{x_0 < x_1 < \ldots < x_n < \ldots\}_{n < \omega_{\alpha+1}} \tag{8}$$

be a well-ordering of X.

4.3.3.3. There should be a $\text{k}X$-limit point in X; i. e. there exists some $a \in X$ such that for every neighborhood $O\,(a)$ of a one has

$$\text{k}\, O\,(a) \cap X = \text{k}X\, (= \aleph_{\alpha+1}). \tag{9}$$

In the opposite case, to every $x_n \in X$ there would correspond some $O_n\,(x_n)$ such that

$$\text{k}\,(O_n\,(x_n) \cap X) < \text{k}\,X. \tag{10}$$

This would enable us to form a $\omega_{\alpha+1}$-sequence F_n $(n < \omega_{\alpha+1})$ in the following way.

At first, let $F_0 = X$; assume $0 < n < \omega_{\alpha+1}$ and assume F_m $(m < < n)$ be defined; let r_n be the first member in (8) of the set $X \setminus \bigcup_{m<n} O_{r\,m}\,(x_{r\,m})$; putting $F_n = X \setminus \bigcup_{m<n} O_{r(m)}\,(x_{r\,m})$, the set F_n would be defined for every $n < \omega_{\alpha+1}$. One sees that $F_n\,(n < \omega_{\alpha+1})$ should be a strictly decreased sequence of sets such that $F_n \setminus \overline{F_{n+1}} \neq \emptyset$, contradicting the assumed equality $\text{F}_\text{D}\, V = \aleph_\alpha$.

4.3.3.4. Let X_m be the set of all the $\text{k}X$-limit points belonging to X. According to 4.3.3.3 the set X_m is not empty. We say that X_m is dense in itself.

As a matter of fact, let $R = X \setminus X_m$.

4.3.3.5. Then $kR < \aleph_\alpha$; the proof is indirect and runs like the one in 4.3.3.3.

4.3.3.6. Now, let $b \in X_m$; every $O(b)$ contains kX points of $X = R \cup X_m$; i. e.

$$k[O(b) \cap (R \cup X_m)] = \aleph_{\alpha+1}$$

$$k[O(b) \cap R] + k[O(b) \cap X_m] = \aleph_{\alpha+1}$$

$$k[O(b) \cap X_m] = \aleph_{\alpha+1} - k[O(b) \cap R] \text{ and thus}$$

according to 3.3.5. one should infer that

$$k[O(b) \cap X_m] = \aleph_{\alpha+1} = kX \text{ for every } b \in X_m$$

and every neighborhood $O(b)$ of b; in other words, X_m would be non void dense-in-itself subset of the dispersed set X, which is absurd. This finishes the proof of Theorem 4.3.3.

4.3.4. Theorem. *For every idempotent V-space V one has*

(i) $\text{dp}\, V = F_D V = k_d(FV, \supseteq) = k_c(GV, \subseteq)$,

(ii) $\text{dr}^* V = \text{dp}\, V = k_d(FV, \subseteq) = k_c(GV, \subseteq) \leq wV$.

The relations are direct consequences of Theorem 3.3. and of the fact that for any idempotent space the members of W in Definition 3.1 may be assumed to be closed sets (and also to have $fn = n+1$). Concerning the relations (ii), one has to evalute kT for every subtree T of $(GV, \subseteq)$; since every chain in T is $\leq \text{dp}\, V$ (cf Theorem 3.3.), and since every antichain of T and particularly $R_0 T$ is $\leq \text{dp}\, V$ we infer that $kT \leq \text{dp}\, V \cdot \text{dp}\, V = \text{dp}\, V$; therefore, also $\sup_T kT \leq \text{dp}\, V$; the sign of equality holding here in virtue of Theorem 3.3. for the simplest case, that T is a chain, i. e. without non comparable members.

4.3.5. Corollary. *For every idempotent V-space, the equality*

$$\text{dr}\, V = \text{dr}^* V$$

is equivalent to the equality

$$\text{Sep}\, V = \text{dp}\, V \quad (\text{cf. } 4.2.1,\ 4.3.2).$$

The corollary is implied by Theorems 4.2.4. (ii), 4.3.4 (ii).

4.4. *Systems* $(\varphi GV, \subseteq)$, $(\varphi GV, \supseteq)$.

4.4.1. *φ-operator.* For any set E in the V-space V let φE be the family of components of E (every member of φE is a most extensive connected subset of E). For a family H of sets, let

$$\varphi H = \cup_E \varphi E \quad (E \in H).$$

In particular, for the family GV of all the open sets in V we have the system $\varphi GV = H$ and the numbers $\text{dr}(H, \supseteq)$, $\text{dr}(H, \subseteq)$.

Earlier we proved the following theorem and the following corollary:

4.4.2. Theorem. (a) *For every infinite locally connected space V one has*

$$\mathrm{k_c}\,(G\,V,\ \subseteq) = \mathrm{k_c}\,(\varphi\,G\,V,\ \subseteq) + \mathrm{cel}\,\varphi\,G\,V\,.$$

(b) *For every infinite locally connected V-space V we have*

$$\mathrm{k_d}\,(G\,V,\ \supseteq) \geq \mathrm{k_d}\,\varphi\,G\,V + \mathrm{cel}\,\varphi\,G\,V\,.$$

(c) *In order that for every infinite locally conneted V-space V one has* $\mathrm{k_d}\,G\,V = \mathrm{k_d}\,\varphi\,G\,V +\ \mathrm{cel}\ \varphi\,G\,V$, *it is necessary and sufficient that every infinite tree T satisfies* $\mathrm{k}T = \mathrm{b}T$ *(Đ. Kurepa* [5], *Theorem* 2, p. 468).

4.4.2.1. Corollary. *Every infinite locally connected V-space V satisfying* $\mathrm{k_c}\,\varphi\,G\,V \leq \mathrm{cel}\,G$ *satisfies* $\mathrm{k_c}\,G\,V = \mathrm{cel}\,V$ (Ibidem, Corollary 2, p. 469).[4] Now, using these results, we shall prove the following.

4.4.3. Theorem. *Every infinite locally connected V-space satisfying*

$$\mathrm{k_c}\,\varphi\,G\,V \leq \mathrm{cel}\,V \tag{0}$$

satisfies

$$\mathrm{dr}^*\,V = \mathrm{cel}\,V \tag{1}$$

and

$$\mathrm{Sep}\,V = \mathrm{dr}\,V \geq \mathrm{k_d}\,\varphi\,G\,V + \mathrm{cel}\,V\,. \tag{2}$$

As a matter of fact, let T be any increasing tree of G-sets of V; then

$$T = \bigcup_{X \in R_0 T} T\,[X,\,\cdot\,)\,; \tag{1}$$

for every $X \leqslant R_0\,T$ the system $T\,[X,\,\cdot)$ is a chain; and according to the Corollary 5.2.1. we have $\mathrm{k}T\,[X,\cdot\,) \leq \mathrm{cel}\,V$. Since $R_0\,T$ is a disjoint system of G-sets, $\mathrm{k}\,R_0\,T \leq \mathrm{cel}\,V$ and the partition (1) yields $\mathrm{k}\,T \leq \mathrm{k}\,R_0\,T\,\cdot\,\mathrm{cel}\,V = \mathrm{cel}\,V$, i. e. $\mathrm{k}\,T \leq \mathrm{cel}\,V$. Therefore, also

$$\sup\nolimits_T \mathrm{k}\,T \leq \mathrm{cel}\,V\ (T \subseteq (G\,V,\ \subseteq))$$

and consequently

$$\mathrm{dr}\,(G\,V,\ \subseteq) \leq \mathrm{cel}\,V\,.$$

The dual relation of the last one being obvious, the requested equality (1) is established. The relations (2) are implied by the Theorem 4.2.4. and the Theorem 4.4.2. b.

[4] The paper was written before World War II and was printed during the War; because of the War, I had no opportunity to read the paper. Therefore, the paper contains many misprints as well as some material errors. E. g., the wording of Theorem 1, p. 464 is false for increasing trees τ because some member $X \leqslant \varphi\,\tau$ may contain also incomparable members of $\varphi\,\tau$; this error does not affect the implications involved.

4.4.4. *On the condition* $k_c \varphi G V \leq \text{cel } V$ *(Mardešić's example).* This condition arose in the Corollary 5.2.1; then I added »Je ne connais aucun espace localement connexe vérifiant $p_c \varphi G > p_s G$« (i. e. $k_c \varphi G > \text{cel } G$, Đ. Kurepa [5], p. 469, footnote[15]). Now here is an example of such a space exhibited by S. Mardešić in 1962.

Let $I = R[0,1]$ be the real unit interval and

$$X = \Pi_\alpha I_\alpha \, (I_\alpha = I, \alpha < \omega_\xi)$$

for some $\xi > 0$. Then X is a locally connected continuum. One has $\text{cel } X = \aleph_0$ (Marczewski [1], [2]; cf. also Đ. Kurepa [5] 5.5.6). On the other hand, let

$$F_n = (\prod_{0 \leq \mu \leq n} 0_\mu) \times (\prod_{n < \nu < \omega_\xi} I_\nu), \quad 0_\mu = 0 \in I_\mu .$$

Then F_n is closed and obviously $F_n, n < \omega_\xi$, is a strictly decreasing ω_ξ-sequence of closed sets; therefore, the complements $U_n = X \setminus F_n$ yield an ω_ξ-sequence of strictly increasing G-sets, and one sees that U_n is connected. Thus $U_n \in \varphi G$. Consequently

$$k_c \varphi G X \geq \aleph_\xi > \aleph_0 = \text{cel } X .$$

4.5. *The relation between* sep *and* Sep. *Function* Is. Obviously, sep $\leq$ Sep, i. e. for every space S we have

$$\text{sep } S \leq \text{Sep } S \quad \text{and} \tag{1}$$

$$\text{sep } S \leq \text{Is } S \leq \text{Sep } S, \quad \text{where} \tag{2}$$

4.5.1. Definition. Is $= \text{Sup}_I \, kI$, I being any isolated set in the space S.

4.5.2. There are spaces S for which the sign $\leq$ in (1) means $=$ (such are metric spaces, pseudo-metric spaces, R-spaces, etc.); but there are spaces S for which $\text{sep } S < \text{Sep } S$. Such is the space of Kuratowski [1] consisting of real numbers and for which the closure operator K is defined by

$$\text{K} X = \text{Der}(-X) \cup X .$$

This space $(R;\text{K})$ is separable but $\text{Is}(R,\text{K}) = \text{Sep}(R,\text{K}) = c$ because, e. g., the set of all positive numbers is isolated.

4.5.3. Theorem. *For every totally ordered space C we have* $\text{sep } C = \text{Is } C = \text{Sep } C$.

As a matter of fact, let S be any subspace of C. If B is any subset of C, such that $k B = \text{sep } C$ and $\overline{B} = C$, we shall prove that S contains a subset S_0 of cardinality $\leq k B$ and such that S_0 be everywhere dense in S, in the sense that every open non void interval of S contains a point of S_0. Now, let $I(S)$ be the set of all the isolated points of S; then to every $x \in I(S)$ we associate an open interval I_x of S, such that x be no extremal point of I_x and $x \in I_x$, and that the intervals I_x $(x \leqslant I(S))$ be pairwise disjoint (cf. Đ. Kurepa [6], Theorem 1.1). Every corresponding coextensive interval $C(I_x)$ of C contains a point of B; B satisfying $\overline{B} = C$; this can be done in such a way that the intervals $C\,I_x\,(x \leqslant I(S))$ be pairwise disjoint. Therefore,

$$\mathrm{k}\, I(S) \leq \mathrm{k}\,(B \cap \bigcup_{X \in IS} C\, I_x). \tag{3}$$

On the other hand, let us consider the remaining set $S \setminus I(S) = R$; every $y \leqslant R$ is non isolated in S; therefore, it is possible to associate with y an interval $O_B(y)$ with extremal points in B; again, it is possible to associate with $O_B(y)$ some element $s(O_B(y)) \leqslant O_B(y) \cap S$. Let

$$S_0 = \bigcup I(S) \cup \bigcup_{y \in R} \{s(O_B(y))\}; \tag{4}$$

S_0 is a subset of S everywhere dense in S and, as a consequence of (3), (4), one proves readily that $\mathrm{k}\, S_0 \leq \mathrm{k}\, B$.

5. Totally ordered spaces

5.1. Theorem.

(i) *Every totally ordered spaces C satisfies*

$$\mathrm{dr}\, C \leqslant \{\mathrm{cel}\, C,\ \mathrm{cel}^+ C\}, \quad \mathrm{dr}^* C = \mathrm{cel}\, C,\ \mathrm{dr}\, C = \mathrm{sep}\, C; \tag{1}$$

(ii) *The identity* (2) $\mathrm{dr} = \mathrm{dr}^*$ *for totally ordered spaces is equivalent to the tree rectangle hypothesis stating that*

$$\mathrm{k}\, T \leq \mathrm{k}_{c'} T \cdot \mathrm{k}_d T, \ \textit{for every tree } T. \tag{3}$$

1. Let us consider the case that C is connected and consequently also locally connected. Since in this case $\mathrm{k}_0\, \varphi\, \mathrm{G}\, C \leq \mathrm{cel}\, C$, Theorem 4.4.3. implies (4) $\mathrm{dr}^* V = \mathrm{cel}\, C$.

2. On the other hand, $\mathrm{sep}\, C = \mathrm{Sep}\, C$ (cf. Theorem 4.5.3); therefore, by Theorem 4.4.3 we infer that

$$\mathrm{sep}\, C = \mathrm{dr}\, C. \tag{5}$$

3. Now, by successive subpartitions of C one proves that

$$\mathrm{sep}\, C \leqslant \{\mathrm{cel}\, C,\ \mathrm{cel}^+ C\} \tag{6}$$

(cf. Đ. Kurepa, Thèse 2, p. 121, Theorem 2). From (5), (6) we obtain the first relation of (1).

4. In our Thesis we proved that (3) is equivalent to $\mathrm{cel}\, C = \mathrm{sep}\, C$ for chains (cf. $P_2 \Leftrightarrow P_5$ in Théorème fondamental, p. 132). In virtue of (1), this means exactly that the identity (2) for chains and the identity (3) for trees are equivalent.

5. If the chain C is not connected, the filling of holes of C and of gaps of C by single points and sets of type $\lambda \equiv t\,R$ respectively yields a connected chain $\tilde{C}$ for which the numbers occuring in the theorem are the same as for C; the theorem being proved for $\tilde{C}$ one concludes that the proof of Theorem 5.1 is finished.

BIBLIOGRAPHY:

B. Dushnik:
Concerning Fréchet limit spaces, Fund. Math. **23** (1934), 162—165,

K. Haratomi:
Über höherstufige Separabilität und Kompaktheit (Erster Teil), Jap. Journal Math. **8** (1931), 113—141,

T. Inagaki:
[1] Le problème de Souslin et les espaces abstraits, J. Fac. Sci. Hokkaido Imp. Univ. Ser. I **7** (1939), 191—201,
[2] Quelques propriétés des espaces abstraits, Ibidem **9** (1940), 193—208,

K. Kuratowski:
Une remarque sur les classes (L) de M. Fréchet, Fund. Math. **3** (1922), 41—43,

Đ. Kurepa:
[1] Tableaux ramifiés d'ensembles. Espaces pseudo-distanciés, C. R. Acad. Sci. Paris **198** (1934), 1563—1565,
[2] Ensembles ordonnés et ramifiés, Thèse, Paris, 1935, Publ. Math. Belgrade **4** (1935), 1—138,
[3] L'hypothèse de ramification, C. R. Acad. Sci. Paris **202** (1936), 185—187,
[4] Sur les classes (E) et (D), Publ. Math. Belgrade **5** (1936), 124—132,
[5] Le problème de Souslin et les espaces abstraits, Revista de Ciencias Lima **47** (1945), 457—488,
[6] Primjedbe o apstraktnim prostorima, Rad Jugosl. Akad. Znan. Umjetn. **296** (1953), 95—103,
[7] Sur l'écart abstrait, Glasnik Mat.-Fiz. Astr. **11** (1956), 105—134,
[8] Cartesian multiplication and the cellularity number, Publ. Math. Beograd **2** (1962), 121—139,
[9] On the existence of pseudometric non totally ordinable spaces, Glasnik Mat.-Fiz. Astr. **18** (1963), 183—194,
[10] Distanza numerica. Distanza non numerica, Publ. Seminario Matematico Bari **91** (1963), 23 pp.,

E. Marczewski:
(E. Szpilrajn): 1. Zametka o dekartovyh topologičeskih prostranstv, Dokl. Akad. Nauk SSSR **31** (1941), 525—527,
2. Séparabilité et multiplication cartesienne des espaces topologiques, Fund. Math. **34** (1947), 127—143,

E. W. Miller:
On certain properties of Fréchet L-spaces, Fund. Math. **26** (1936), 116—119,

P. Papić:
[1] Sur les espaces admettant une base ramifiée de voisinages, Glasnik Mat.-Fiz. Astr. **8** (1953), 30—40,
[2] Sur une classe d'espaces abstraits, Ibidem **9** (1954), 197—216,
[3] Sur les espaces pseudo-distanciés, Ibidem **9** (1954), 217—228,

[4] Quelques propriétés des espaces totalement ordonnés et des espaces de la classe R, Rad Jugoslav. Akad. Znan. Umjetn. **302** (1957), 171—196,

[5] Sur l'ordinabilité d'espaces de la classe R, Glasnik Mat.-Fiz. Astr. **18** (1963), 75—84.

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Institute of Mathematics
University of Beograd

DRVNI BROJ PROSTORA I UREĐENIH SKUPOVA

Đuro Kurepa, Beograd

Sadržaj

0.1. *Drvni broj* dr. Za uređeni skup $(O, \leq)$ drvni broj dr $(O, \leq)$ definira se kao suprem glavnih brojeva poddrveta toga skupa.

0.2. *Dualni drvni broj* dr* $(O, \leq)$ definira se kao dr $(O, \geq)$.

0.3. *Drvnost prostora* S definira se ovako: dr $S =$ dr $(G S, \supseteq)$, pri čemu $G S$ znači skup svih otvorenih skupova prostora.

0.4. dr* $S \overset{\text{def}}{=}$ dr $(G S, \subseteq)$.

0.6. *Problem.* Ima li koji prostor S za koji dr S, dr* S ne bi bili dostignuti?

1.1. Teorem. Za svaki metrički prostor M stoje jednakosti ispisane u engleskom tekstu.

2.1. Teorem. Za svaki metrički ili pseudometrički prostor stoje jednakosti (1).

3.5. Teorem. Za svaki T_1 R-prostor R važe odnosi ispisani u engleskom tekstu.

4. V-prostori V. Neka je V proizvoljan V-prostor, tj. topološki prostor s izotonim zatvorenjem i da je $\overline{\emptyset} = \emptyset$.

4.2.1. Definicija. Sep V je infimum glavnih brojeva n tako da svaki potprostor od V bude n-separabilan.

4.2.2. $\mathrm{Fc}\, V = \sup_W k f W$; $W = \{W_n\}_{n<\alpha}$ označuje dobro uređen sistem skupova s ispisanim svojstvom za bar neki redni broj fn za koji je $n < fn < (\gamma W)^-$.

4.2.3. Teorem. Uz uslov (0) vredi (1).

4.3.1. Definicija od $F_D\, S$ se razabire iz engleskog teksta.

4.3.2. Raspršenost dp S prostora S je supremum glavnih brojeva raspršenih podskupova (skup je raspršen ako mu je prazan skup jedini u sebi gust podskup).

4.3.3. Teorem. Vredi (2).

4.3.4. Teorem. Ako je V-prostor idempotentan (tj. $\overline{\overline{X}} = \overline{X}$ za svako $X \subseteq V$), tada vredi (*i*) i (*ii*).

4.4.1. φ — operator. Za svaki skup E iz V neka φE znači skup svih svezanih komponenata od E; ako je H obitelj skupova iz V, tada φH znači uniju od φE pri $E \leqslant H$.

Nadovezujući na neke prethodne rezultate (Kurepa [5], Theorem 2, p. 468, str. 2, p. 469) dokazuje se

4.4.3. Teorem. Svaki beskonačan lokalno svezan V-prostor za koji vredi (0) zadovoljava (1) i (2).

4.4.4. Navodi se primer S. Mardešića iz 1962. prostora V koji je lokalno svezan i za koji je $k_c \varphi GV >$ cel V.

4.5. sep, Sep, Is.

4.5.1. Is V je supremum glavnih brojeva izoliranih skupova prostora V.

4.5.2. Teorem. Za svaki lančasto uređen prostor C imamo sep C = Is C = Sep C.

5. Potpuno uređeni prostori.

5.1. (*i*) Svaki lančasto uređen prostor C zadovoljava (1);

(*ii*) Identitet (2) ekvivalentan je s hipotezom (3); pri čemu T označuje drvo.

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ON THE CATEGORY NUMBER OF TOPOLOGICAL SPACES

Đuro Kurepa (Beograd)

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Introduction. René Baire introduced the notion of category of a space S and defined that S is of the first or of the second category, according as the space S is or is not a union of $\leqslant \aleph_0$ nowhere dense sets. In this connexion we consider the *minimal number* of nowhere dense sets exhausting the derivative DS of the space S. Also we were lead to consider the *minimal number* of nowhere dense sets the union of which should be everywhere dense in DS.

We shall examine how these notions are connected with some other properties of space. We consider also the class of all corresponding category numbers for some classes of spaces.

1.1. **Definition of the category number** ctS **of** S. Let be: S a topological space, IS the set of all isolated points and DS the derivative of S (i. e. $DS := S \diagdown IS$). The minimal number n such that there are n nowhere dense sets the union of which coincides with DS is called the category (*number*) *of* S and is denoted by ctS; in other words

$$ctS = \inf_Y kY, \quad \bigcup_{X \in Y} X = DS, \quad X \in Y \Rightarrow S = \overline{C\overline{X}}.$$

In particular we put $ct\varnothing = 0$.

1.2. **Definition of the subcategoricity** $sctS$ **of** S. The infimum of cardinalities of set systems Y of nowhere dense sets, the union of which yields an everywhere dense set on DS is called *the subcategory number of* S and is denoted by $cstS$.

For example, if S is the real line or any complete dense metrical space then $sctS = \aleph_0$ and $ctS > \aleph_0$; if $2^{\aleph_0} = \aleph_1$, then $ctR = \aleph_1$.

If sS denotes the *separability* of S (i. e. sS is the infimum of cardinalities of everywhere dense sets) then obviously

$$sctS \leqslant sS.$$

We shall now exhibit a class of totally ordered spaces showing that $sct\ S$, ctS might be $\aleph_0$, while kS, sS are arbitrarily high.

2. **Transition** $(O, \leqslant) \rightarrow (\sigma O, =\!|)$ **for any ordered set** $(O, \leqslant)$.

2.1. We define σO as the set of all well ordered subsets X of $(O, \leqslant)$ such that $Y \leqslant y$ for some $y \in O$. For $a, b \in \sigma O$ we denote $a =\!| b$ the fact that

a is an initial section of b. If L is any ordered chain we extend the partial order of $(\sigma L, =\!|)$ into a total order $\leqslant_n$ defining $a \leqslant_n b$ to mean $a =\!| b$ or $a_e < b_e$ in $(L, \leqslant)$ for any $a, b \in (\sigma L, =\!|)$; here $a = \{a_0, a_1, \ldots\}$, $b = \{b_0, b_1, \ldots\}$, $a_e < b_e$, $e := e(a, b)$ denoting the first ordinal number ξ such that $a_\xi \neq b_\xi$.

2.2. We proved (Đ. Kurepa [2] that $(\sigma L, \leqslant_n)$ is a linearly ord red set. Consequently, for any totally ordered set $(L, \leqslant)$we have also the ordered space $(\sigma L, \leqslant_n)$ defined by the interval topology on $(\sigma L, \leqslant_n)$.

2.3. Let us consider the particular example of the ordered set $(Q, \leqslant)$ of rational numbers and of its order type η_0. We use a similar terminology for any ordered set and for its order type.

2.4. Lemma. *Let* $x \in \eta_0$; *then the totally ordered subset*

(1) $$(\sigma\eta_0(\cdot, x), \leqslant_n)$$

is nowhere dense in the chain

(2) $$(\sigma\eta_0; \leqslant_n), \text{ i.e.}$$

every non void interval I *of the chain* (2) *contains an interval* I_0 *disjoint from the chain* (1).

Proof. First (cf. Đ. Kurepa [3] p. 91 Lemma 4.2 and [4] L. 4.2. p. 41) there exists some $c \in \sigma\eta_0$ such that

$$(\sigma\eta_0[c, \cdot) =\!|) \subseteq I.$$

Since for every $c \dashv c' \in \sigma\eta_0$ we have also $\sigma\eta_0[c', \cdot) \subset I$ we might suppose that the set c contains at least one point c_ν such that $x < c_\nu$ and $c = \{c_\xi\}_{\xi < \gamma c}$; γc denotes the order type of the well ordered set c.

Now, let $a \in \sigma(\eta_0(\cdot, x))$. Then $e(a, c) < \nu$. According as $a \leqslant_n c$ or $a \geqslant_n c$ we have $a <_n \sigma\eta_0[c, \cdot)$ or $a >_n \sigma\eta_0[c, \cdot)$ and never $a \in \sigma(\eta_0[c, \cdot) \dashv) = A$. Since A is a non empty section of the chain $(\sigma\eta_0, \leqslant_n)$, every interval it contains is disjoint from $(\sigma\eta_0(\cdot, x), \leqslant_n)$ what was to be proved.

.25. Theorem. *Let* $(L, \leqslant)$ *be any dense ordered chain; then the category* $ct(\sigma(L, \leqslant), \leqslant_n)$ *and the cofinality number* $cf(L, \leqslant)$ *satisfy the relation*

$$ct(\sigma(L, \leqslant) \leqslant_n) \leqslant cf(L, \leqslant).$$

As a matter of fact, let W be any well ordered subset of $(L, \leqslant)$ such that $kW = cfL$ and that $(W, \leqslant)$ should be cofinal to $(L, \leqslant)$. Then $(\sigma L, \leqslant_n)$ is the union of kW nowhere dense sets $\sigma L(\cdot, x)$ $(x \in W)$.

2.6. Theorem. (i) *The category number of the chain* (1) $\sigma(\eta_0 \leqslant_n)$ *is* $\aleph_0$. *If* L *is any ordered chain cofinal with* ω_0, *then the category number of the ordered space* $(\sigma\omega_0, \leqslant_n)$ *equals* $\aleph_0$.

(ii). *The chain* (1) *is not a union of* $\aleph_0$ *special nowhere dense sets, each being an antichain in the tree* $(\sigma\eta_0, =\!|)$.

The first part (i) is implied by the theorem 2.5. As to the second part (ii) we refer to the theorem 4.1 p. 37 of the paper Đ. Kurepa [4].

2.7. Corollary. *There are spaces of the categoricity* $\aleph_0$ *of any high cardinality* $\geqslant a$ *and of any separability degree* $s \geqslant \aleph_0$.

As a matter of fact, it is sufficient to consider any ordered chain L such that $cf L = \aleph_0$, $kL \geqslant a$; then $cf(\sigma L, \leqslant) = \aleph_0$, $k \sigma L \geqslant a$. Of course, we might suppose also that sep $L \geqslant s$; this with sep $\sigma L \geqslant$ sep L implies all requested conditions.

3. **By an analogous argument one proves the following.**

3.1. Theorem. *If* $(L, \leqslant)$ *is any dense ordered chain cofinal to a regular initial number* ω_δ, *then the categoricity of the natural order*

$$(\sigma(L, \leqslant), \leqslant_n) \text{ is } \leqslant \aleph_\delta: \tag{1}$$

$$ct(\sigma(L, \leqslant), \leqslant_n) \leqslant \aleph_\delta, \tag{2}$$

in particular if S *is any subset of* L *cofinal to* S *and to* ω_δ, *then the sets*

$$(\sigma(S(\cdot, x)) \leqslant_n) \ (x \in S) \tag{3}$$

are nowhere dense in (3) *and exhaust the chain* (1).

3.2, Corollary. *For every regular ordinal number* ω_δ *one has*

$$ct(\sigma \eta_\delta, \leqslant_n) \leqslant \aleph_\delta.$$

4. **Some propositions concerning the category numbers.**

4.1. Theorem. *For a given infinite number* $ctS = n$ *and any infinite cardinal number* $m > n$ *there exists a space* X *satisfying* $ctX = n$, $kX = m$.

As a matter of fact, it suffices to consider a family F of cardinality m of replicas of S and the union U of all the members of F topologysed in such a way that every member X of F be the subspace in U identical with the given space X. Then $ctU = ctS$, $kU = mkS = m \cdot n = m$.

4.2. Theorem. *For every infinite number* b *there exists a metrical space* M_b *satisfying* $ctM = b$.

As a matter of fact, let $R(b)$ be the set of all the ω-sequences of ordinals $< \omega(b)$, each of which is almost the constant sequence 0, 0 . . ; we metrisize $R(b)$ by the function $\rho(a, b) = \dfrac{1}{1 + \varphi(a, b)}$ for every pair $a = (a_0, a_1, \ldots)$, $b = (b_0, b_1, \ldots)$ of distinct members of $R(b)$; $\varphi(a, b)$ is the first index n at which $a_n \neq b_n$.

Then the space $(R(b), \rho)$ is metrical. dense in itself and satisfies $sep R(b) = b = kR(b) = ctR(b)$.

4.2.1. Corollary. M *running through the class of metrical spaces the class* $\{ctM \ldots\}_M$ *is not a set, because it contains every infinite cardinal number* b.

4.3. Problem. Let S be any topological space; if $ctS > \aleph_0$ and if n is any cardinal number satisfying $\aleph_0 \leqslant n < ctS$, is there a subspace S' of S satisfying $ctS' = n$?. In other words, do the numbers ctX, $(X \subseteq S)$ fill the cardinal interval $[sep S, ct S]$?

4.3.1. For metrical spaces, the general continuum hypothesis implies a positive answer.

4.3.2. The question arises to establish the result 4.3.1. without assuming the continuum hypothesis.

BIBLIOGRAPHY

[1] Baire R., Analli di Matem. (3) 3, (1899) in particular p. 65.

[2] Đ. Kurepa. *Ensembles ordonnés et ramifiés.* Thèse Paris, 1935 (Publ. Math. Beograd 4 (1935) 1—138).

[3] Đ. Kurepa. *O realnim funkcijama u obitelji uređenih skupova racionalnih brojeva.* Rad Jugosl. Ak. Zagreb **296** (1953) 85—93.

[4] Đ. Kurepa. *Sur les fonctions réelles dans la famille des ensembles bien ordonnés des nombres rationnels.* Bull. Ac. Sc. Zagreb **4** (1954) 35—47.

Matematički institut,
Beograd.

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ON A-TREES

Đuro Kurepa (Beograd)

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1. Introduction.

1.1. At several opportunities (cf. the bibliography at the end of the article; in particular see my memory [8] written in 1937) one was lead to consider trees $A=A_\nu$ having the following properties:

1. *The height* γA *of* A *is an initial ordinal number* ω_ν;

2. *Every row* $R_\xi A := \{x;\ x \in A,\ \text{type of } A(\cdot, x) = \xi\}$ *is of a cardinality* $< k\,\omega_\nu$ *and* $\sup_\xi kR_\xi A = k\,\omega_{\nu-}$ *(for an ordinal* ν *we define* ν^- *to be* $\nu-1$ *or* ν, *according as* $\nu-1$ *exists or does not exist; in particular* $0^-=0$).

3. *The cardinality of every chain in* A *is* $<kA$.

4. *For every* $x \in A$ *the height* $\gamma[a]$ *of all the points of* A *each of which is comparable to* a *equals* γA.

5. *For every* $x \in A$ *the node* $|x|_A := \{x;\ x \in A,\ A(\cdot, x) = A(\cdot, a)\}$ *has* 1 *or* $k\gamma x$ *points, according as* $(\gamma x)^- = \gamma x$ *or* $(\gamma x)^- < \gamma x$; *here* γx *is the ordinal satisfying* $x \in R_{\gamma x} A$.

1.2. If ω_ν is cofinal to ω_0, A_ν does not exist; A_1 does exist (N. Aronszajn, v. [5], p. 96) as well as $A_{\alpha+1}$ for every ordinal α; in this paper we shall prove it without continuum hypothesis; the proof is analogous as it was done for the case $\alpha=0$ in Đ. Kurepa [8] and is based purely on order considerations (cf. also [1], [15]).

The problem of the existence of A_ν for inaccessible $\nu > \omega_0$ remains open.

1.3. In the section 2 we define an ordered set $H=R^\nu$; the properties of R^ν and of σR^ν shall yield in § 3 a requested set $A_{\nu+1}$. In § 5 we shall prove that every $A_{\nu+1}$ obtained in § 4 contains an antichain of cardinality $kA_{\nu+1}$ — a property closely connected with the author's ramification hypothesis (cf. [6], [12]).

1. The set D_ν. Let ω_ν be any initial ordinal number; we denote by D_ν the ordered set

$$\{\ldots, -\alpha, \ldots, -2, -1, 0, 1, 2, \ldots, \alpha, \ldots\} \qquad (\alpha < \omega_\nu).$$

The order type of D_ν equals $\overset{*}{\omega}_\nu + \omega_\nu$; i. e. $(D_\nu, \leqslant)$ is coinitial with $\overset{*}{\omega}_\nu$ and cofinal with ω_ν. The members of D_ν might be called ω_ν-integers.

2. The sets R^ν, $R^\nu(\leqslant_n)$.

2.1. Let R^ν be the set of all *finite* sequences of members of D_ν.

2.2. Set $H=(R^\nu, \leqslant_n)$. We order R^ν by putting for members $a=(a_0, a_1 \ldots a_\alpha)$, $b=(b_0, b_1, \ldots, b_\beta)$ of R^ν that $a \leqslant_n b$ means either that a is an initial portion of b (i. e. $\alpha \leqslant \beta$ and $a_i = b_i$ for $i \leqslant \alpha$) or, that $a_i = b_i$ $(i < \varphi)$, $a_\varphi < b_\varphi$, where $\varphi = \varphi(a, b)$ is the first index at which the sequences a, b differ (*natural ordering of complexes*). One sees that

$$H=(R^\nu, \leqslant_n) \tag{2.1.}$$

is a totally ordered set.

2.3. The cardinality of H equals $k\omega_\nu$, i. e. $kR^\nu = \aleph_\nu$.

As a matter of fact, $kR^\nu = \aleph_\nu + \aleph_\nu^2 + \aleph_\nu^3 + \cdots = \aleph_\nu + \aleph_\nu + \cdots = \aleph_\nu$.

2.4. Theorem (i). *The set $H \equiv (R^\nu, \leqslant_\nu)$ is totally ordered and dense.*

(ii) *The ordered set H is order-imbeddable into every of its intervals I, even so that I contains a subinterval similar with H (quasihomogeneity of R^ν).*

(iii) *Every gap $X|Y$ of $(R^\nu, \leqslant_n)$ is of the type (ω_0, ω_0^*) i. e. X is cofinal to ω_0 and $Y = R_\nu \setminus Y$ is coinitial to ω_0^*. Every interval of H contains gaps, i. e. the gaps of H are everywhere dense in* (2.1).

(IV) *Every ordinal number α, $\alpha < \omega_{\nu+1}$ is imbeddable into H; $\omega_{\nu+1}$ is the first ordinal which is not imbeddable into H.*

Proof. (1) For $a=(a_0, a_1, \ldots, a_n) \in (2.1)$, the set $S = R_\nu(\cdot, a)$ of all predecessors of a has no terminating member; in fact, let us consider $a_n \in D_\nu$; if a_n has in D_ν its immediate predecessor a_n^-, i. e. if a_n is isolated ordinal, then the set S is cofinal with the ω_ν-sequence of all sequences of the form $(a_0 a_1 \ldots a_{n-1} a_n^- \omega_\nu')$, ω_ν' running through D_ν; if $a_n^- = a_n$ (i. e. if a_n is an ordinal of the second kind) and equaling $\xi\omega$, then S is cofinal with the υ_n-sequence of members $(a_0 a_1 \ldots a_{n-1} \beta)_{\beta < a_n}$. By dual considerations one proves that a has no immediate follower.

Ad (ii). Let $a = (a_0 a_1 \ldots a_m)$, $b = (b_0, b_1, \ldots, b_n) \in R^\nu$ and $a <_n b$. Then either a is an initial section of b or there is the first index $\varphi < \omega$ such that $a_\varphi < b_\varphi$ (thus $a_\varphi \neq b_\varphi$) and $a_i = b_i$ for every $i < \varphi$. In the first case, $m < n$; let c, d be two members of D_ν such that $c < d < b_{m+1}$; then the mapping $x \in R^\nu \to x' = acx$ is a requested order imbedding of (2.1) into $(R^\nu, \leqslant_n)(a, b)$, because $a <_n x' <_n b$.

In the second case, the mapping $x \in R^\nu \to ax$ furnishes such an imbedding. In both cases, the previous isomorphism carries R^ν onto some subinterval of $R^\nu(a, b)$.

Ad (iii). Let $X|Y$ be a section of (2.1), i. e. X is a non void initial portion of (2.1) and Y is the remainder $(2.1) \setminus X$; suppose $X|Y$ be a gap. Let $b_0 = \sup \xi$ where $(\xi) \in X$; then $b_0 < \omega_\nu$ because the set of all the (ξ) $(\xi < \omega_\nu)$ is cofinal with (2.1); let $b_1 = \sup \xi$ with $(a_0 \xi) \in X$; if $b_1 = \omega_\nu$, then $(a_0 + 1)$ is the first member of Y.

If $b_1 < \omega_\nu$ we consider $b_2 = \sup \xi$ with $(b_0, b_1, \xi) \in X$ etc. If for some $n < \omega_0$ we have $b_n = \omega_\nu$, then $(b_0, b_1 \ldots b_{n-1} + 1)$ is the first member of Y; if $a_n < \omega_\nu$ for every $n < \omega_0$, then X is cofinal with the strictly increasing ω_0-sequence

$$(b_0),\ (b_0 b_1),\ (b_0 b_1 b_2),\ \ldots$$

and consequently has the type of ω_0.

Dual considerations show that Y is coinitial with $\overset{*}{\omega_0}$. As a matter of fact, let $c_0 = \inf \xi$, where $(\xi) \in Y$. Then $-\omega_\nu < c_0$ because the set (2.1) is coinitial with the set of all sequences $(\xi)\,(\xi > -\omega_\nu)$. Let $c_1 = \inf \xi$ where $(c_0\,\xi) \in Y$; if $c_1 = -\omega_\nu$, then $(-1 + c_0)$ is the last member of X. If $c_1 > -\omega_\nu$, we consider $c_2 = \inf \xi$, with $(c_0\, c_1\, \xi) \in Y$, etc.

Since H has gaps, every interval of H has gaps too — consequence of the statement (ii).

Ad (iV). Every ordinal number α, $\alpha < \omega_{\nu+1}$ is imbeddable into (2.1). We prove it by induction argument. For $\alpha \leqslant \omega_\nu$, the fact is obvious: the mapping $\alpha \in I\,\omega_\nu \to (\alpha) \in R^\nu$ is such an isomorphism. Let the statement hold for any α, $\alpha < \beta$, where $\beta < \omega_{\nu+1}$; let us prove it also for $\alpha = \beta$; this being obvious for $\beta = \beta^- + 1$, let us consider the case $\beta^- = \beta$ (β is of second art). Then, there exists a regular number ω_γ such that for some ω_γ-sequence β_ξ of ordinals we have

$$\beta = \Sigma_\xi\, \beta_\xi \qquad (\xi < \omega_\gamma).$$

Now, let $b_\xi\,(\xi < \omega_\gamma)$ be any strictly increasing sequence of points of R^ν; we imbed every ordinal β_ξ into the open interval $R^\nu(b_\xi,\, b_{\xi+1})$ as some well-ordered subset B_ξ; then the union $\bigcup_\xi B_\xi\,(\xi < \omega_\gamma)$ is a well-ordered subset of (2.1) and of type β.

3. The set σ_ν. Let σ_ν or σ be the system of all well-ordered non void subsets $\subseteq$ (2.1), each of which is bounded; consequently, if $a \in \sigma_\nu$, the number γa — the ordinal type of a — is determined as well as the insreasing points

$$a_\xi \qquad (\xi < \gamma\, a)$$

of the set a. The system σ_ν will be ordered by the relation $=\!|$ meaning „*to be an initial segment of*".

Of course, the set $(\sigma_\nu,\ =\!|)$ is (partially) ordered; moreover, it is ramified and even σ_ν is a tree.

3.1. Theorem. *The ordered set* $(\sigma_\nu,\ =\!|)$ *is such that each of its chains is a well-ordered set, the cardinal of which is* $\leqslant \aleph_\nu$; *on the other hand*

(3.1.1) $$\gamma(\sigma_\nu,\ =\!|) = \omega_{\nu+1}.$$

Namely, if C is any chain $\subseteq (\sigma_\nu,\ =\!|)$, then $\bigcup C$ is a well-ordered subset of (2.1) and vice versa: W being any well-ordered bounded subset of (2.1), the system of non void initial intervals of W yields a chain $\subset (\sigma_\nu,\ =\!|)$. Finally, the relation (3.1.1) is an another expression for the statement 2.4. IV

3.2. The set $(R^\nu,\ \leqslant_n)$. Let

(3.2.1) $$\overline{R^\nu} \quad \text{or} \quad \overline{H}$$

denote the totally ordered set obtained from (2.1) by putting a single element in every gap of (2.1). Of course, the set (3.2.1) is continuous in the sense of Dedekind: every section of (3.2.1) is given by a single element of (3.2.1). Consequently, for every bounded chain $C \subseteq$ (3.2.1) the infimum and the supremum of C relatively to (3.2.1) are well determined points of (3.2.1).

3.3. Function f.

In particular, let

(3.3.1) $$f(a) = \sup_{x \in a} x \qquad (a \in \sigma_\nu);$$

3.3.1. Lemma. *The mapping* (3.3.1) *is an increasing mapping of* σ_ν *into* (3.2.1); *every* 3 *points chain of* (σ_ν, $=|$) *is mapped onto at least* 2 *points of* (3.2.1). *The system* $a=|a'$, $f(a)=f(a')$ *is equivalent with the statement that* $\sup a \in$ (3.2.1) *and* $a' \setminus a = \{\sup a\}$.

3.3.2. Lemma. *Let* $e \in \sigma_\nu$, $\beta < \omega_{\nu+1}$; *then the set* $f(R_\beta(e,\cdot)_\sigma)$ *is everywhere dense in the right interval* $(f(e),\cdot)_{\overline{H}}$.

At first, let $\beta=0$; then $R_0(e,\cdot)$ is built up of the sets

$$e \cup \{x\},$$

x running over the set of all the points of H, each of which is $>f(e)$ or $\geqslant f(e)$, according as γe is limit or isolated ordinal; since then

$$f(e) = \sup(e \cup \{x\}) = x,$$

the statement is obvious. Let us suppose now that $0<\beta<\omega_{\nu+1}$ and that the statement holds true for each $\xi<\beta$; to prove it for $\xi=\beta$. If $\beta-1$ exists, the set $fR_{\beta-1}(e,\cdot)_\sigma$ is dense on $(f(e,\cdot)_H$; again, if b is an immediate successor of e in σ, then $f(R_0(e,\cdot)_\sigma)$ is dense on $(f(e),\cdot)_H$; consequently, the join

$$\cup_b f(R_0(b,\cdot)_\sigma = f(\cup_b R_0(b,\cdot)_\sigma = f(R_\beta(e;\cdot)_\sigma (b \in R_{\beta-1}(e;\cdot)_\sigma)$$

is dense on $f(e;)_H$.

If β is a limit number, let β_ξ ($\xi < cf\,\beta = \tau$) be an increasing sequence of ordinals $\to \beta$. Let x be any point of $(f(e),\cdot)_H$ of character c_τ and x_η ($\eta < cf\beta$) any increasing τ-sequence of points of H so that $f(e)<e$, $\sup\limits_{\eta<\tau} x_\eta = x$ and $\sup\limits_{\mu<\eta} x_\mu < x_\eta$ ($\eta<\tau$). The existence of such a chain x_η is obvious. Let $e^\circ \in R_{\beta_0}$ $(e;\cdot)_\sigma$ so that $f(e)<f(e^\circ)<x_0$; inductively, for each $0<\delta<\tau$ let us suppose defined the δ-chain

$$e^\mu \ (\mu<\delta)$$

so that $e^\mu \in R_{\beta_\mu}(e,\cdot)$, $x_\mu < f(e^\mu) < x^{\mu+1}$ ($\mu<\delta$). Let then e^δ be an element of $R_{\beta_\delta}(e,\cdot)_\sigma$ so that $e^\mu =| e^\delta$, ($\mu<\delta$), $x_\delta < f(e^\delta) < x_{\delta+1}$; such one x_δ exists, since $x' = \bigcup\limits_{\mu<\delta} e^\mu$ is a point of σ, inasmuch it is well-ordered and bounded (it is located left to x_δ). Q. E. D.

From the last proof we deduce the following.

3.3.3. Lemma. *If* $\alpha < \omega_{\nu+1}$, *the set* $fR_\alpha \sigma_\nu$ *is everywhere dense on* $\overline{H}$; *if* α *is a limit number, then* $fR_\alpha \sigma_\nu$ *is equal to the set of all the points of* $\overline{H}$ *each of which is of character* $C_{\alpha'}$, $\alpha' = cf\,\alpha$.

4. Construction of the requested sets $A_{\nu+1}$.

We shall construct a requested tree $A = A_{\nu+1}$ as a union of some $k\,\omega_{\nu+1}$ sets D_ξ ($\xi < \omega_{\nu+1}$).

4.1. To start, let G_0 be any subset of cardinality $\aleph_\nu$ of $\overline{H} \setminus H$ so that G_0 be everywhere dense in H; consequently, every member of G_0 is of a countable character (cf. 2.4. (iii)). To every $x \in G_0$ we associate an element $\psi(x) \in R_\omega \sigma_\nu$ such that $f\psi(x) = x$ and that for $x, x' \in G_0$, $x \neq x'$ one has $\psi x \neq \psi x'$. In this way we get the set

$$D_0 := \psi\, G_0 \subseteq R_\omega \sigma_\nu,$$

4.2. Let suppose that $0<\beta<\omega_{\nu+1}$ and that the sets

$$D_0,\ D_1,\ \ldots,\ D_\xi,\ \ldots,\ (\xi<\beta)$$

be constructed so that putting

$$s_\beta = \bigcup_\xi D_\xi \qquad (\xi<\beta)$$

the following conditions $1_\beta - 7_\beta$ hold:

1_β $R_\xi s_\beta = D_\xi \qquad (\xi<\beta)$

2_β $D_\xi \subseteq R_{\omega(1+\xi)}\sigma_\nu \qquad (\xi<\beta)$

3_β $\gamma s_\beta = \beta$

4_β $kD_\xi = \aleph_\nu \qquad (\xi<\beta)$

5_β If $\xi<\beta$, $e\in D_\xi$ and $\xi<\zeta<\beta$, then $fR_\xi[e]_{s_\beta}$ is an everywhere dense set on $\overline{H}(f(e),\cdot)$,

6_β If $\xi<\beta$, the set $fR_\xi s_\beta$ is everywhere dense on $\overline{H}$,

7_β For each $\xi<\beta$ and $x\in D$, fx is a $\omega(1+\xi)$ — point of $\overline{H}$; if $e, e'\in D_\xi$ and $e\neq e'$, then $fe\neq fe'$.

4.3. Let us define D_β and $s_{\beta+1}$.

4.3.1. If $\beta^-<\beta$, let us consider $D_{\beta-1}$; let l_β be a set-mapping of $D_{\beta-1}$ into $\overline{H}$ so that for each $e\in D_{\beta-1}$ the set $l_\beta e$ be a subset of cardinality $\aleph_\nu$ of ω_0-points of $fR_{\omega(1+\xi)}(e,\cdot)$ everywhere dense on it, and that if $e\neq e'$, then the sets $l_\beta e$, $l_{\beta'} e'$ are disjoint. For each $x\in l_\beta e$ let $\varphi(e,\ x)$ be an element of $R_{\omega\beta}[e]_{\sigma_\nu}$ such that $f\varphi(e,\ x)=x$ (the existence of such one ψx is obvious). Then we define

$$D_\beta := \bigcup_e le \qquad (e\in D_{\beta-1}). \tag{4.3.1.1.}$$

Consequently, $fD_\beta = f\bigcup_e l_\beta(e) =$ everywhere dense subset of $\overline{H}(f(e),\cdot)$.

4.3.2. C a s e: β is a limit ordinal. 4.3.2.1. Let β_ζ $(\zeta<\tau,\ \tau=cf\beta)$ be any increasing regular sequence of ordinals $\rightarrow\beta$; let $l_\beta e$ for

$$e\in\bigcup_{\zeta<cf\beta} R_{\omega(1+\zeta)}s_\beta \tag{4.3.2.1}$$

be a disjointed system of sets, so that le be a subset of cardinality $\aleph_\nu$ of $\omega\tau$-points of $fR_{\omega(1+\beta)}[e]_{\sigma_\nu}$ everywhere dense on that set; in particular

$$kle=\aleph_\nu. \tag{4.3.2.2}$$

To each ordered pair $(e,\ x)$ with $x\in l_\beta(e)$ let $\psi(e,\ x)$ be an element of $R_{\beta(1+\zeta)}[e]_{\sigma_\nu}$ so that $f\psi(e,\ x)=x$. The existence of $\psi(e,\ x)$ is clear. As a matter of fact, x being a member in $\overline{H}$, let x_ζ $(\zeta<\tau)$ be a strictly increasing sequence of points of $(f(e),\cdot)_{\overline{H}}$ tending to x. Let e° be a point of D_{β_0} satisfying $e=|e^\circ$, $fe<fe^\circ<x_0$. Inductively, for each $0<\zeta<\tau$ and $0<\xi<\zeta$, let e^ξ be a certain point of D_{β_ξ} succeeding to every e^μ $(\mu<\xi)$ and satisfying $x_\xi<fe^\xi<x_{\xi+1}$; in virtue of conditions 4_β, 5_β the existence of e^ξ is assured. We define, $e^\zeta=\sup_{\xi<\zeta} e^\xi$ and $\psi(e,\ x)=\sup_{\zeta<\tau} e^\zeta$. Thence

$$f\psi(e,\ x)=x.$$

4.3.2.2. The set D_β is defined as consisting of points $\psi(e,\ x)$, x, e running respectively over $\bigcup_e l(e)$ and (4.3.2.1).

In any case, the set D_β is defined.

4.4. Putting

(4.4.1) $$s_{\beta+1} = s_\beta \cup D_\beta$$

one proves that the condititions $1_{\beta+1}$—$7_{\beta+1}$ are satisfied. The condition $1_{\beta+1}$ is satisfied since $R_\xi s_\beta = R_\xi s_{\beta+1}$ $(\xi<\beta)$ and since each $e \in D_\beta$ is preceded by a single element in every D_ξ $(\xi<\beta)$. As to $2_{\beta+1}$, its verification is immediate. As to $3_{\beta+1}$ i. e. that $kD_\beta = \aleph_\nu$, it is a consequence of the formula for D_ξ $(\xi<\beta)$, that means that $ks_\beta = \aleph_\beta$. Now, there is a one-to-$\aleph_0$-mapping of a subset[1] of s_β onto D_β, thus $kD_\beta \leqslant ks_\beta$. $\aleph_\nu = \aleph_\nu \cdot \aleph_\nu = \aleph_\nu$; since for any $\xi<\beta$ each $e \in D_\xi$ is succeeded by $\aleph_\nu$ distinct elements of D_β, the condition $3_{\beta+1}$ is fulfilled.

$4_{\beta+1}$. The case of an isolated β being resolved by (4.4.1), (4.3.1.1), let β be limit number; x running over $l_\beta(e)$; the condition $4_{\beta+1}$ is an immediate cosequence of (4.4.1), (4.3.1.1) and of the assumed density of $l_\beta(e)$. The condition $5_{\beta+1}$ is a consequence of $4_{\beta+1}$ and of the conditions 5_ξ $(\xi<\beta)$. Finally, the condition $7_{\beta+1}$ holds true, first, because the points of D_β are constructed as some $\omega(1+\beta)$ — points of $\overline{H}$ and, secondly, because of the disjointedness of the above sets $l_\beta(e)$ (e variable). Thus the existence of D_β and $s_{\beta+1}$ is proved for every β, $\beta<\omega_{\nu+1}$ and the conditions 1_β—7_β hold true for every $\beta<\omega_{\nu+1}$.

4.5. Putting

$$A_{\nu+1} = \cup_\beta D_\beta \qquad (\beta<\omega_{\nu+1})$$

one sees that conditions $1\,\omega_{\nu+1}$—$6\,\omega_{\nu+1}$ hold true for writing $A_{\nu+1}$ instead of $s_{\omega_{\nu+1}}$. In particular $A_{\nu+1}$ is a tree so that

$$\gamma A_{\nu+1} = \omega_{\nu+1}$$

$$kR_\beta A_{\nu+1} = \aleph_\nu \qquad (\beta<\omega_{\nu+1})$$

$$k_c A_{\nu+1} = \aleph_\nu;$$

moreover, for any $e \in A_{\nu+1}$ the set $(e,\cdot)_A$ satisfies the same last three conditions.

4.6. Total order-extension of $A_{\nu+1}$ to become $I\,\omega_{\nu+1}$ or the partial order destroying in $I\,\omega_{\nu+1}$ to become $A_{\nu+1}$.

4.6.1. The mapping $f(e)$ $(e \in A)$ is a strongly increasing mapping of A into $\overline{H}$. Consequently, f is biunique in every chain of A. According to the previous construction, f is biunique in every set $R_\beta A$ $(\beta<\omega_{\nu+1})$. Thus, if the sets $fR_\beta A$ $(\beta<\omega_{\nu+1})$ are pairwise disjoint, f is a biunique correspondence between A and the subset fA of $\overline{H}$. In such a case, using the mapping f, we can proceed either to destroy partially the order in the chain fA and get an ordered set similar to A, or to transfer the total order of fA onto the set A enlarging so the given partial order of A. Namely, if e, e' are any two incomparable points of A, we can declare incomparable also the corresponding points fe, fe' in fA; fA becomes partially ordered and similar to A. And vice versa, if x, x' are any two points of fA, we can introduce an order $<'$ into A by the procedure that $x<x'$ in fA be equivalent to $f^{-1}x <' f^{-1}x'$ in A.

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[1] Viz. of $D_{\beta-1}$ and $\bigcup_{\varsigma<cf\,\beta} D_{\beta_\varsigma}$ respectively, according as β in isolated or limit number.

4.6.2. *Partial desordonning of $I(\omega_{\nu+1})$ to get a set $A_{\nu+1}$.* Any set $A_{\nu+1}$ has $\aleph_{\nu+1}$ as its cardinality. We can do an extension of order of A to yield a *total order of type* $\omega_{\nu+1}$ of $A_{\nu+1}$. Namely, as $kR_\beta A = \aleph_\nu$, let

$$a_\xi^\beta \,(\xi < \omega_\nu)$$

be an ω_ν-enumeration and at the same time an ordering of the set $R_\beta A$, so that in the new ordering a_ξ^β precedes $a_{\xi'}^\beta$ if and only if $\xi < \xi' < \omega_\nu$. Defining $a_\xi^\beta \leqslant a_{\xi'}^{\beta'}$ if and only if either $\beta \leqslant \beta'$ or $\beta = \beta$, $\xi \leqslant \xi'$ one gets the required total extension of $(A, \leqslant)$.

Putting

$$g\left(a_\xi^\beta\right) = \omega_\nu \beta + \xi \,(\beta < \omega_{\nu+1},\ \xi < \omega_\nu)$$

g is a biunique isomorphic mapping of $A_{\nu+1}$ onto $I(\omega_{\nu+1})$. This isomorphism enables us to destroy partially the total order in $I(\omega_{\nu+1})$ to get in it, as a step of previous ordination of I the partial order of A.[1]

4.6.3. It is not easy to have a simple picture how such an desordonning of $I(\omega_{\nu+1})$ takes place. However, we can realize it in the following manner: let $h(\beta)\,(\beta < \omega_{\nu+1})$ be any uniform mapping of $I(\omega_{\nu+1})$ into $\overline{H}$ so that for each β the mapping h be biunique in $[\omega_\nu \beta,\ \omega_\nu(\beta+1))$ and that the corresponding set

$$h\,[\omega_\nu \beta,\ \omega_\nu(\beta+1)) \tag{4.6.3.1}$$

be everywhere dense in $\overline{H}$ and be composed of very $\omega(1+\beta)$ — points of $\overline{H}$; then the set (4.6.3.1) can be chosen to serve as the set fD_β in the construction in § 4.3: it is sufficient to consider any partition P_β of (4.6.3.1) into $\aleph_\nu$ pairwise disjoint sets, each of which is everywhere dense, establish a biunique correspondence t_β between E_β and P_β and for any $e \in E_\beta$ put $l(e) = t_\beta(e)$. According as β is isolated or limit number, the set E_β means $D_{\beta-1}$ or $\bigcup_{\zeta<\tau} D_{\beta\zeta}$ in previous notations. On the other hand, the existence of such mappings h is easy to establish. Namely, let us consider a bounded ω_ν-sequence a in H; then for each $\zeta \leqslant \nu$ the element $\sup_{\xi<\omega_\zeta} a_\xi$ is a well-determined ω_ζ-point of $\overline{H}$; thus, there are ω_ζ-points in $\overline{H}$ for each $\zeta \leqslant \nu$. In virtue of the quasihomogeneity of H that means that in each interval of $\overline{H}$ there are ω_ζ-points too; thus, for each $\beta < \omega_{\nu+1}$ the $\omega(1+\beta)$ — points are everywhere dense. It is then sufficient to consider a set S_β of power $\aleph_\nu$ of $\omega_\zeta(1+\beta)$ — points everywhere dense, decompose it into a ω_ν-system of disjoint sets $S_\xi^{\beta'}\ \beta' < \omega_\nu)$, each of which is everywhere dense and to consider the sets

$$S_\beta^{\beta'} \,(\beta < \omega_\nu,\ \beta' < \omega_\nu).$$

They are to be used as sets fD_ξ in §§ 4.1.—4.3.

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[1] The precise definition of that idea is the following one [7]: let $(E_1, \leqslant_1)(E_2, \leqslant_2)$ be two ordered sets (in general, they are only partially ordered); of course, one can have $E_1 = E_2$ we say that the order of $(E_1; \leqslant_1)$ is *at least equal to the order* $(E_2; \leqslant_2)$, symbolically $t(E_1; \leqslant_1) \leqslant t(E_2, \leqslant_2)$ if there is a one-to one mapping f of E_1 into E_2 so that *every chain* C *of* $(E_1; \leqslant_1)$ is mapped onto a *similar chain of* $(E_2; \leqslant_2)$, *no matter what happens with subsets of* E_1 *that are no chains*. We say that the *ramification* (*or disorder*) *in* $(E_1, \leqslant_1)$ is at least equal to the ramification (or disorder) of $(E_2; \leqslant_2)$, symbolically $r(E_1, \leqslant_1) \leqslant r(E_2, \leqslant_2)$ if there is a one-to-one mapping of E_1 into E_2 so that if $x, y \in E_1$ and $x \leqslant_1 y$, $x >_1 y$, $x \|_{\leqslant_1} y$ then in $(E_2; \leqslant_2)$ respectively $f(x) \leqslant_2 f(y)$, $f(x) >_2 f(y)$, $f(x) \|_{\leqslant_2} f(y)$ i. e. if $(E_1; \leqslant_1)$ is similar with a subset of $(E_2; \leqslant_2)$.

5. Normality of the preceding set $A_{\nu+1}$.

Theorem. *The set $A_{\nu+1}$ of the foregoing construction contains a set of $kA_{\nu+1}=\aleph_{\nu+1}$ pairwise incomparable points.*

To see it (cf. Kurepa [10]) let $r\in H$ and $a\in A$; let us define $\psi(r, a)$ so that:

(5.1) $$\psi(r, a)=-1 \quad \text{if } a \text{ non } \in a$$

$$\psi(r, a)=\beta \quad \text{if } r\in a \text{ and}$$

just $a_\beta=r$ (let us remind that a is a well-ordered set $\subseteq H$). For any $T\subseteq\sigma H$ let

(5.2) $$\psi(r, T)=\sup_{a\in T}\psi(r, a).$$

There is a point $r_0\in H$ so that

(5.3) $$\psi(r_0, A)=\omega_{\nu+1}.$$

In opposite case, one should have $\psi(r; A)<\omega_{\nu+1}\,(r\in H)$, thus $\delta<\omega_{\nu+1}$, with $\delta=\sup_{r\in H}\psi(r; A)$, because $kH=\aleph_\nu$. Now, since $\gamma A=\omega_{\nu+1}$ and $\delta<\omega_{\nu+1}$, there is an element $a\in R_{\delta+1}A$; the point $a_{\delta+2}$ of H should satisfy $\psi(a_{\delta+2}\,a)>$ $>\delta+2>\delta$, what is a nonsense. Thus the existence of an r_0 satisfying (5.3) is assured.

Now, let us construct a $\omega_{\nu+1}$-sequence

(5.4) $$a^\circ,\ a',\ \ldots,\ a^\xi,\ \ldots,\quad (\xi<\omega_{\nu+1})$$

of points of A, so that $r_0\in a^\xi\,(\xi<\omega_{\nu+1})$ and that for each ξ one has

(5.5) $$\psi(r_0, a^\xi)>\sup_{\zeta<\xi}\psi(r_0, a^\zeta).$$

Because of (5.3) the existence of (5.4) is inductively provable. Since, each a^ξ is a well-ordered subset of H containing r_0 and since (5.5) holds, the elements (5.4) are pairwise incomparable: no one is an initial interval of another one Q. E. D.

BIBLIOGRAPHY

[1] Gaifman, H., Specker, E. P., *Isomorphism of types of trees*, Proc. Amer Math. Soc., 15 (1964), 1—7.

[2] Helson, H., *On a problem of Sikorski*, Colloquium Math., Wroclaw, 2 (1951), 7—8.

[3] Inagaki, T., *Le problème de Souslin et les espaces abstraits*, J. Fac. Sci., Hokkaido Univ., Ser. I (1939), 191—201.

[4] Inagaki, T., *Les espaces abstraits et les ensembles ordonnés*, Ibidem 8 (1940) 146—162.

[5] Kurepa, Đ., *Ensembles ordonnés et ramifiés*, Thèse, Paris, 1935, Publ. math. Univ. Belgrade, 4 (1935) 1—138.

[6] Kurepa, Đ., *L'hypothèse de ramification*, Comptes rendus, Paris, 202 (1936) 185—187.

[7] Kurepa, Đ., *O poredbenim relacijama*, Rad, Zagreb, 1938, knj. 201 (81), 187—219. *Sur les relations d'ordre*, Bull. inter. Ac., Zagreb, 32 (1939) 66—76.

[8] Kurepa, Đ., *Ensembles linéaires et une classe de tableaux ramifiés. Tableaux ramifiés de M. Aronszajn*, Publ. math. Univ. Belgrade, 6 (1937), 129—160.

[9] Kurepa, Đ., *A propos d'une généralisation de la notion d'ensembles bien ordonnés*, Acta Mathematica, 75 (1942). 139—150.

[10] Kurepa, Đ., *Transformations monotones des ensembles partiellement ordonnés*, Revista de Ciencias, №434, 42 (1940), 827—846, № 437, 43 (1941), 483—500.

[11] K u r e p a, Đ., *Ensembles de suites dénombrables d'entiers, Contribution au problème de Suslin*, Bull. Ac. Sci, URSS, 11 (1947), 59—74.

[12] K u r e p a, Đ., *Sur une proposition de la théorie des ensembles*, C. r., Ac. Sci., Paris, 259 (1959) 2698—99.

[13] R i c a b a r r a, R. A. *Conjuntos ordenados y ramificados* (*Contribucion al estudio del problema de Suslin*), Bahia Blanca, 1958, p. 358.

[14] S i k o r s k i, R., *On an ordered algebraic field*, Comptes Rendus Soc. Sci. Lettres, Warszawa, Cl. III, 41 (1948), 69—96.

[15] S p e c k e r, E., *Sur un problème de Sikorski*, Colloquium Math., Wroclaw, 2 (1951) 9—12.

Matematički institut,
Beograd, Yugoslavia.

ĐURO KUREPA (Beograd)

SPEKTRALNI PRINCIPI

1. UVODNO RAZMATRANJE

Ovom prilikom želim precizirati i formulirati neke principe spektra i izneti nekoliko primena.

1.1. Svakako, ideja vodilja spektra sastoji se u preslikavanju, kodiranju, ... podataka tako da se iz slike ili transformata može putem odkidanja, odstranjivanja, prekrivanja, dekodiranja ... doći natrag na podatke.

1.2. Tako npr. svetlu pripada određen spektar sastavljen od crta, pa se od tih crta može postupno doći natrag na vrstu svetla od kojeg svaka crta potiče.

1.3. U skupu ljudi otisci prstiju omogućuju da se pomoću tih otisaka, ljudi međusobno razlikuju i odrede — i to na potpuniji način nego što se to može učiniti ličnim opisom.

1.4. Mihailo Petrović je počev od 1917. godine preneo ideju spektra i na neke matematičke oblasti na način u kojima spektar do tada nije bio upotrebljavan (inače, spektralna metoda u matematici je upotrebljavana i ranije (D. Hilbert), a razvila se naročito u teoriji matrica, odnosno linearnih operatora ... nezavisno od Petrovićevih numeričkih spektara).

2. ŠTO JE MATEMATIČKI SPEKTAR?

2.1. U matematici imamo osnovnu alternativu: skup ili funkcija, pa se na osnovu toga i određuje predikat. Da li je spektar skup ili funkcija?

2.2. *Spektar je funkcija!* Samo treba precizirati što se tiče predstavljanja vrednosti te funkcije.

2.3. Ako je zadan kakav niz podataka

(1) $p_1, p_2, \ldots$

pa ako svakom podatku p_n pridružimo određen redni prirodni broj rp_n, tada se niz dobivenih brojeva rp_n može skupiti tako da rp_n bude odlomak pravog razlomka

(2) $0, rp_1\, rp_2 \ldots$

Samo je pitanje, kako iz toga decimalnog broja (2) doći natrag na podatke (1)? Ako nam je poznat niz

(3) $b_1, b_2, \ldots\ldots$

koji kazuje koliko članovi (2) imaju mesta, onda je lako iz (2) doći na (1) i to naprosto: komadanjem ili rezanjem niza (2) redom: odrezati početak od b_1 članova, pa od ostatka b_2 članova, itd.

2.4. *Što je spektar?*

U opštem slučaju, polazi se od podataka (1) P i svakom podatku p iz P odredi se određeni skup sp tako da imamo skupovnu funkciju

(2) $p \in P \longrightarrow sp$

od P na

$$(3) \quad D = \bigcup_{p \in P} sp \;;$$

zahtevamo, da to preslikavnje bude *tolikovno* tj. obostrano jednoznačno; nadalje, na svakom sp definiramo određeno oblaganje, ispisivanje, kodiranje ili funkciju rp, i najzad sagradimo uniju ili zbir S tih oblaganja:

$$(4) \quad S \mid D = \bigcup_{p \in P} r \mid sp$$

kao oblaganje skupa D koje se na svakom sp podudara sa oblaganjem $r \mid sp$.

2.4.1. Najjednostavnije je da skupovi sp pri $p \varepsilon P$ budu ne samo različni nego i disjunktni. Na taj način, samo oblaganje $S \mid D$ izlazi, kao stavljanje »zakrpe do zakrpe« (pločice do pločice) ili »slova do slova«, između kojih su prazni prostori ili neki posebni materijal kao 0, zarez itd.

2.4.2. Ostvarivanje odlomka (odreska) »pločice« $r \mid p$ može se učiniti bez prethodnog skupovnog preslikavanja $s : P \rightarrow D$, jer se ovo time ostvaruje automatski kao

$$p \varepsilon P \rightarrow sp \equiv D_{om\, r} \mid p.$$

2.4.3. Određivanje tih odlomaka vrši se na razne načine, već prema prilikama.

2.5. *Princip spektra* traži da je veza između podataka P i prirodnog spektra $S \mid D$ pregledna i da se ostvarivanje te uzajamne veze vrši na što prirodniji način, npr., pomoću znakova i pojava koje se lako umnažaju, dobro raspoznavaju i međusobno dobro povezuju.[1]

[1] Tako npr. ekonomičniji je znak ĉ nego č jer je prvi znak celovitiji od drugoga i prvi znak se lakše povezuje nego drugi (kod ispisivanja znaka ˇ imamo prazan hod ruke pre početka i posle svršetka ispisivanja znaka ˇ); nadalje, znak ĉ nije složen od dva slova iste azbuke, a č jest (od c i v); međutim, optički, znak č je uočljiviji i bolje razgovetan od znaka ĉ, osim ako krov ˆ u ĉ ne pišemo previše sitno.

2.5.1. Osnovni i univerzalni znakovi za povezivanje jesu cifre ili znamenke: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Njih prihvataju svi narodi. To je numerička azbuka ili abeceda. Naprotiv, *glasovna azbuka* još nije univerzalna ni jednotna pa i kod jednog te istog naroda.

3. PRIMERI SPEKTARA IZ LINGVISTIKE

Nauka o jeziku pruža vanredno važne brojne primere spektralne metode.

3.1. *Oblaganje* se vrši pismenima ili slovima, koja se raspoređuju u određeni niz sastavljen od redaka ili stubaca (kineski način) koji teku od leva nadesno, odnosno odozgo nadole.

3.2. *Razlaganje* se vrši praznim međuprostorima i raznim posebnim znakovima od kojih su najvažniji: (tačka) (.), zarez (,), tačka-zarez (;), uskličnik (!), upitnik (?) i znak navođenja (»«) i zagrada. Posebno, na taj način iz niza slova u zadanom gradivu dobivaju se nizovi reči, nizovi rečenica, nizovi odlomaka, nizovi paragrafa, odeljaka, poglavlja, delova, svezaka, itd.

3.3. U tome vodećem primeru spektralne metode dobro je držati na umu činjenicu kako je čovečanstvo relativno kasno došlo do slova kao članova i materijala s kojim se oblaže odnosno ispisuje i opet do posebnih znakova za rastavljanje kao što su . , : ; « ? / ().

3.4. Upotreba malih i velikih, kosih, slova također je jedan od načina da se rastavljanje, odnosno čitanje učine preglednijima.

3.5. Premda su znakovi interpunkcije univerzalni, slova nažalost još nisu, i što je najgore: ima *istih slova* sa *raznim značenjima*, pa k tome i u jednom te istom jeziku (kao npr. U, B, C, P, u srpskohrvatskom). Ta se činjenica protivi principu brže i jednoznačne informacije, i principu spektra pa se zato i treba ukloniti.

3.6. Prevođenje sa jednog jezika na drugi jezik je vrlo dobar dalji primer primene principa spektra.

3.7. *Povezivanje pisanih znakova s jedne strane i glasova, odnosno reči s druge strane, dalji je primer principa spektra.*

3.7.1. Tako npr. reč i predmet *igla, I,* je divan primer da bude nosilac glasa i znaka *I,* u svim slavenskim jezicima; zato je zaista šteta što ipak većina Slavena, još ne upotrebljava znak *I* kao znak za početno slovo reči *igla,* pogotovo što je taj znak međunarodan i mnogo jednostavniji od znaka *U* koji treba da preuzme drugu ulogu jer ipak podseća više na *uho* nego znak *Y* koji opet ima drugo značenje (podsetimo se pri tom da *U* i *Y* u grčkom znače jedno te isto slovo i tomalo, odnosno veliko *U* — uho).

3.7.2. Također za znak *V* nalaze se Slaveni u vrlo povoljnom položaju jer svi imamo reč *vile V* kao nosioca glasa i odgovarajućeg znaka; reč *vol* na slavenskim jezicima nosilac je toga glasa *V* a i znaka *V* u vidu rogova. Drugim rečima, *kad znakovi I, V, pa U ne bi već imali svoja značenja kao u latinici, svaki slavenski narod bi imao prirodnih razloga da ga izmisli i uvede upravo u tekućem značenju*, posebno svaki slavenski narod ogrešuje se o spektralni princip ako odbija da u pisanju znakovi I, V, i U imaju upravo ona značenja što danas imaju u nauci, tehnici, odnosno u latinici.

3.8. *Princip spektra zahteva da prenošenjem jednog znaka iz jedne azbuke u drugu bude sačuvan i pripadni glas.*

3.8.1. Zato npr. grčko *P* treba da i u latinici ostane za isti glas npr. PUKA, a znak Π u grčkom i znak R u latinskoj azbuci, koji su i inače dosta slični međusobno, treba da imaju isto značenje i da međusobno jedan drugom odgovara.

3.8.2. Iz istog principa treba znak *𝒯𝓂*, *ϖ* kao komplikaciju od T, t zabaciti i upotrebljavati te znakove kao i u grčkom originalu, odnosno u latinskoj varijanti. Latinsko pisano malo t koje je zašlo i na prvi sprat svakako je preglednije od grčkog τ koje se celinom nalazi u prizemlju; ta ocena stoji pogotovo kad su ti znakovi uključeni u kakvu podulju reč.

3.9. *Tolikovanje između članova azbuke pojedinog pisma i glasova koji se javljaju u dotičnom jeziku dalji je osnovni primer metode spektra.*

U našoj ćirilici taj je princip ostvaren; u našoj latinici on nije ostvaren.

3.9.1. Nakon dugih razmišljanja lično mislim da bi naredna abeceda vrlo dobro odgovarala našem jeziku:

a, A, b, B, c, C, ć, Ć, Ĉ, ĉ, d, D, d̂=d̂, D̂, (d̂=џ u ćirilici), e, E, f, F, g, G, ĝ, Ĝ (=ђ u ćirilici), h, H, i, I, j, J, k, K, l, L, l̂, L̂ (=љ u ćirilici), m, M, n, N, n̂, N̂ (=њ u ćirilici), o, O, r, R (=П), P, p, (=р u ćirilici), s, S, ŝ, Ŝ, (=ш u ćirilici), t, T, u, U, v, V, z, Z, ẑ, Ẑ (=ж u ćirilici).

U tom predlogu, dolazi znak ^ koji se stavlja iznad slova, npr. ĉ=c , L̂ = L̂ , na slovo npr. l̂ , L̂ , a mogao bi se ispisivati i desno gore iznad slova pogotovo ako se piše strojem. Dakle: c^, d^, g^, l^, n^, s^, z^; slično za velika slova: C^, D^, . . . ,Z^. Takva azbuka je jednostavna, tolikovna, lako ostvarljiva i opšta.

3.9.2. (Dodano za vreme korekture). D. Trifunović skrenuo je moju pažnju na članak: M. Petrović: *Husov pravopis (po prof. Dr. J. Gebauer*-u), Prosvetni glasnik — službeni list Ministarstva prosvete i crkvenih poslova, Beograd, 26 (1905) 93—96. Opisanu ulogu znaka ^ (izvrnuto slovo v) vrši pri Janu Husu znak tačke stavljen iznad slova. Zanimljivo je da Petrović u tom članku piše »sveučilište« a ne »univerzitet« (up. str. 95).

3.10. *Sricanje (buhštabiranje ili slabekovanje) dalji je primer spektralnog povezivanja. Zato je važno imati određen naziv i izgovor za svako slovo, npr. ovako:*

A, B (be), C, Ĉ (Ĉa), Ć (Će), D (de), D̂ (D̂e), E, F (ef), G (ge), Ĝ (Ĝu), H (haš), I, J, (je), K (ka), L (el), L̂ (L̂e), M (em), N (en), N̂ (N̂e), O, R (Re), P (ro), S (es), Ŝ (Ŝa), T (te), U, V (ve), Z (Ze), Ẑ (Ẑa).

3.10.1. Primedba o P i R. Posebno, o slovima za glas *p* (pero) i *r* (*r*uka) u raznim azbukama videti na primer knjigu Zvonimir Kujundžić, *Knjiga o knjizi,* I tom: *Historija pisama;* Zagreb, 1957, 868 str.

3.11. *Kodiranje i dekodiranje, odnosno: pisanje i čitanje dalji su vrlo važni slučajevi spektarskih razmatranja* (videti: *J. Wolfowitz, D. A. Novik).*

4. SPEKTRALNA METODA U MUZICI, UMETNOSTI...

U muzici se nižu note i dr. znakovi sa svrhom da jednoznačno odrede kako se nižu odgovarajući glasovi, ritam, melodija; kao jedan od osnovnih znakova odeljivanja služi uspravna taktovna crta, pa se zadani tekst deli na taktove, odlomke, delove, itd.

Slično je u koregrafiji, ornamentici, građevinarstvu, itd. U tim delatnostima ponavlja se pojedini odlomak kao vodeći motiv i doprinosi ugodnijem slušanju, gledanju, razumevanju, itd.

5. PRINCIP JUKSTAPOZICIJE ILI NADOVEZIVANJA

5.1. Najobičniji slučaj toga principa sastoji se u tome *da se na reč nadopisuje reč:* npr. iz reči *nad* i reči *graditi* nastaje reč *nadgraditi.*

5.2. U daljem slučaju radi se o *zbiru procesa ili funkcija* u smislu da uz zadanu uređenu dvojku $f_1 \mid A_1, f_2 \mid A_2$ procesa sa disjunktnim oblastima A_1, A_2 posmatramo i proces $f \mid A$, pri čemu je $A = A_1 \cup A_2$, $f \mid A_i = f_i \mid A_i$ $(i = 1,2)$.

5.3. U opštem slučaju, reč je o skupovnoj funkciji $h \mid P$ pri čemu svakom $p \varepsilon P$ odgovara (neprazan) skup h_p te o funkcijama $f_p \mid h_p$ $(p \varepsilon P)$ tako da različnim članovima $p \varepsilon P$ odgovaraju disjunktni skupovi h_p a onda se definira funkcija — zbir $f \mid A$ tih funkcija sa oblasti Domf$=$ $=A=\bigcup_{p \varepsilon P} h_p$ i s vrednostima f za koje je $f \mid h_p = f_p \mid h_p$.

5.4. U nauci o jeziku dobar primer principa nadovezivanja imamo u *gradnji složenica;* vrlo je korisno i poučno pratiti kako se taj princip pojavljuje u pojedinim jezicima: bilo kao prosto nadopisivanje (npr. u engleskom), bilo prostim pripisivanjem, bilo sa pripisivanjem i određenim promenama na spojnom mestu (npr. u nemačkom).

5.5. U okvir principa nadovezivanja ulaze i rasuđivanja o fonetskom i logičkom (korenskom) pravopisu.

6. PRINCIP GRANANJA ILI RAMIFIKACIJE U OZNAČAVANJU

6.1. Taj se princip sastoji u tom da se jedinke (članovi) zadanog skupa označuju šiframa koje se međusobno razlikuju samo po završetku celokupne oznake. Tako npr. sobe na 5. spratu označuju se sa 51, 52, 53, 54, 55, . . . 5n, ukoliko na tom spratu ima n soba. U trgovini i u ekonomskim naukama dolazi taj princip do znatnog izražaja. U svojoj suštini i mestovni (pozicioni) način pisanja brojeva odraz je principa grananja; tako npr. ako za broj x znamo da je oblika $x=3. \ldots$, onda ispisivanjem naredne niže znamenke npr. 7 dobijamo jedini tačan izbor $x=37, \ldots$ od 10 mogućih izbora

$$x=30, \ldots ; \; x=31, \ldots ; \; x=32, \ldots ; \; x=39, \ldots$$

Ako naredna znamenka treba glasiti 0, a naredna na to 6 i poslednja 9, dobija se tačna tražena vrednost $x=37{,}069$.

6.2. Pisanje i računanje sa nepotpunim ili približnim vrednostima u najužoj je vezi sa spektralnim principom grananja.

6.3. Petrovićeva definicija brojevnog spektra $S=S(n_1 \ldots n_n)$ zadana niza (1) $n_1, \ldots, n_k$ prirodnih brojeva kao decimalan broj $S=o, on_1 \ldots o.an_2o \ldots an_k$ s unapred određenim brojem 0 ispred svakog od znakova niza (1) direktna je primena metode nadovezivanja, odnosno grananja.

6.4. Slično vredi za spektar S ako članovi niza (1) nisu prirodni brojevi nego celi racionalni ili racionalni pa i iracionalni brojevi.

6.5. Valja držati na umu da pri tom nije od bitne važnosti da je spektar S broj nego da u S imamo na neki način smešten i sam zadan niz (1) zadanih podataka u manje više skrivenom i preinačenom ali uočljivom, odgonetljivom obliku.

7. POOPŠTENJA . . .

I baš na tom mestu možemo ukazati na poopštenja.

7.1. S jedne strane, ono što se pripisuje ne moraju biti znamenke 0, 1, . . 9 nego redni brojevi 0, 1, 2, . . , m', . . . koji su manji od m, gde je m bilo koji redni broj; najjednostavnije je uzeti $m=2$. No, može se uzeti ne samo da m bude konačan redan broj nego i beskonačan, npr. $m=\omega$, pa m može biti 0 ili bilo koji prirodni broj. Ako je $m=\omega^2$ ili ω^ω ili ω_1, tada znamenka m' može biti i beskonačan redni broj; posebno se može posmatrati slučaj kad $m-1$ postoji, bez obzira da li je m konačno ili beskonačno.

7.2. Sa druge strane, poopštenje se sastoji u tom da posmatrane *»reči«*, tj. spektri S kao nizovi mogu biti i beskonačni i to proizvoljne *»duljine«* γ; tako npr. pri $\gamma=\omega$, dobiju se obični beskonačni nizovi; ako je $\gamma > \omega$, i $\gamma < \omega_1$, dobiju se beskonačni prebrojivi nizovi-spektri. Ako je $\gamma=\omega_1$ ili $\gamma > \omega_1$ niz je beskonačan i neprebrojiv.

7.2.1. Primer. Posmatrajmo niz 1, 2, 3, . . . prirodnih brojeva n sređenih po veličini; pridružimo svakom n niz

(2) $$|n,2| = \underbrace{011\ldots1}_{1+n}$$

duljine $1+n$ koji počinje sa 0 a ostali su mu članovi 1; tada možemo formirati nov niz

(3) $$|1,2|\ |2;2| \ldots\ |n,2| \ldots = 0101101110111 \ldots$$

On je dijadski i na očigledan način omogućuje da se iz njega rekonstruira zadani niz (1).

7.2.2 Uopšte, neka je X bilo koji skup, a

(4) $$X_1,\ X_2, \ldots X_n, \ldots \quad (n<\xi)$$

bilo koje dobro uređenje od X; tada se umesto niza (4) može posmatrati pripadni dijadski niz (3) pišući dijadski niz (2) umesto X_n. Posebno se može pretpostaviti da je duljina ξ niza (4) minimalna, tako da nijedan pravi početni odsečak od (4) nije istobrojan sa čitavim nizom (4); ako je, k tome, ξ beskonačno, tada će i pripadni dijadski niz (3) biti iste duljine ξ koje je i niz (4).

7.2.3. Neka je α bilo koji redni broj, npr. 0 ili 1; neka je ω_α pripadni početni broj a ω_α glavni broj od ω_α; ako je $2<n<k\omega_\alpha$, tada svakom ω_α -nizu

(5) $$s=s_0,\ s_1, \ldots s_p, \ldots$$

rednih brojeva $\beta=s_p$ za koje je $\beta<W_{(n)}$ pomoću smene

(6) $s_p \to |s_p;2|$ odgovara određen W_α – niz $|s;2|$ od znamenaka 0, 1; to je pridruživanje tolikovno; zato je

(7) $n^{k\alpha} \leq 2^{k\alpha}$ odakle zbog $2^{k\alpha} \leq n^{k\alpha}$ izlazi jednakost brojeva $2^{k\alpha}$, $n^{k\alpha}$ za svako

(8) $2 \leq n \leq kw_\alpha$.

Na osnovu pridruživanja $s \to (s;2)$ odgovara svakom nizu (6) potpuno određen dijadski spektar $|s:2|$ duljine W_α, pri čemu se podela $|s;2|$ vrši u maksimalne intervale sa po jednom jedinom nulom.

7.2.4. Neka je $(S, \leq)$ uređen skup; pri $a \varepsilon S$ neka $S(., a]$ označuje skup svih $x \varepsilon S$ za koje je $x \leq a$; neka $f_a | S$ bude dijadska funkcija na S koja je 1 na $S(., a]$ a 0 na preostalom delu skupa S; tada se lako vidi da pri a, $b \varepsilon S$ imamo

$$a \leq b \rightleftarrows [f_a(s) \leq f_b(s) \text{ za svako } s \varepsilon S].$$

No, $f_a | S$ je određen trivijalni dijadski spektar nad S s podelom na onaj deo $S(., a]$ od S na kojemu je funkcija fa jednaka 1 i na preostali deo od S. Prema tome, svaki uređen skup je sličan s *glavnim uređenjem* neke obitelji dijadskih funkcija-spektara.

7.2.5. Ako je $(S \leq)$ lančasto uređen skup, tada se glavno uređenje iz 7.2.4. može zameniti alfabetskim uređenjem, a dijadske funkcije se mogu pretpostaviti da imaju dobro uređen skup kao svoju oblast. Naravno, umesto dijadskih spektara (funkcija) možemo posmatrati triadske funkcije, funkcije s vrednostima u $(0, 1, \ldots \alpha', \ldots)$ $\alpha' < \alpha$ za bilo koji redni broj α, odnosno s vrednostima u bilo kojem skupu Y. U tom pogledu imamo naredni važni i opšti slučaj.

7.2.6. *Ako je (X, Y) proizvoljan uređen par nepraznih skupova, a $f : X \to Y$ bilo koja funkcija od X u Y, tada je funkcija $f | X$ određen spektar sa slojevima $\{f^{-1} y\}$ $(y \varepsilon fX)$, pri čemu za svako y definiramo*

$$(1) \qquad \{f^{-1} y\} = \{x \mid x \varepsilon X,\ fx = y\},\ y \varepsilon fX.$$

7.2.7. Prethodni primeri pokazuju kako je ideja matematičkog spektra opšta i kako u razmatranjima dolaze ne samo pojedini spektri nego još više skupovi spektara.

LITERATURA

Kujundžić Zvonimir 1. *Knjiga o knjizi* I: *Historija pisama*, Zagreb, 1957_2, str. 865.

Kurepa Đuro 1. *Ensembles ordonnés et ramifiés* (Thèse, Paris 1935), Publ. Math. Beograd 4 (1935) 1—144.

2. *Partitivni skupovi i uređeni lanci — Partitive sets and ordered chains* Rad JAZU, Zagreb 302 (1957) 197—203, 205—235.

D. A. Novik 1. *Effektivnoe kodirovanie*, Izd. Energija, M. L. 1965, 236.

K. Orlov 1. *The fundamentals of practical spectral arithmetic and algebra*, Beograd 1955, 46.

M. Petrović 1. *Les spectres numériques*, Paris 1919, 7+110. — *Leçons sur les spectres mathématiques*, Paris, 1928, 1+90.

J. Wolfowitz 1. *Coding theorems of information theory*, Springer Verlag, 1964_2, r. prevod:
Teoremi kodirovanija teorii insormacii, Moskva 1967, 248.

PRINCIPES SPECTRAUX

ĐURO KUREPA (Beograd)

1. Le but de l'article c'est de formuler quelques principes de spectre et d'indiquer quelques applications.

2. *Qu'est — ce qu'un spectre?*

2.4. L'on part des données (1) et à chaque $p\varepsilon P$ par (2) on associe un ensemble sp en exigeant que (2) soit biunivoque; de plus sur chaque sp l'on définit une fonction sp; enfin, la réunion (4) de ces fonctions est le spectre $S \mid D$.

2.5. On demande que la liaison entre P et $S \mid D$ soit claire, facilement réalisable moyennant des matériaux, signes, ... qui se fabriquent facilement, se distinguent mutuellement et se relient mutuellement.

3. *Exemples de spectres en linguistique.*

3.3. Le principe de spectre demande que l'emploi d'un signe d'un alphabet dans un autre alphabet conserve la prononciacion de ce signe.

3.8.1. En conséquence, le signe P en grec et en latin doit signifier la même chose, à savoir le ro en grec; le R en latin doit signifier π en grec.

3.9. On exige la biunivocité entre les éléments d'un alphabet et des sons pour les prononcer.

3.9.1. On propose la suite des caractères de l'alphabet pour satisfaire au principe 3.9.

3.10. L'action d'epeler est un exemple de spectre et l'on propose une façon comment le faire.

4. Méthode spectrale en musique, en arts... se fait voir partout.

5. *Principe de juxtaposition consiste:*

5.1. Dans le cas le plus simple en juxtaposant un mot, une suite, après l'autre;

5.2. En faisant la réunion d'un couple ordonné de processus $f_i \mid A_i$ $(i=1, 2)$;

5.3. En faisant la réunion de processus $p\varepsilon F \rightarrow h_p$ les h_p étant ensembles disjoints et $h \mid P$ étant une fonction ensembliste quelconque.

6. *Principe de ramification en désignation.*

6.1. Il consiste en différenciant des objects p d'un ensemble de manière que les dénotations correspondantes ne se distinguent que par la fin; par exemple, les symboles 51, 52, .. peuvent désigner des chambres 1, 2, .. du 5-ième étage.

6.3. La définition originel de Petrović du spectre S de la suite (1) de nombres positifs est un exemple faisant voir le principe de ramification.

7. *Généralisations ...*

7.1. Généralisations consistent en ce que, d'une part, les »chiffres« peuvent être quelconques et appartenir à n'importe quel segment 0, 1, .. m', .. de nombres ordinaux (finis ou transfinis) ou appartenir à un ensemble quelconque et, d'autre part, que la longueur de »mots« (ou de suites) peut être infinie.

7.2.3. Par la transformation (2) ou (6) appliquée à (5) on obtient $\mid s; 2 \mid$ et donc (7) pourvu que (8).

7.2.4. On indique une représentation isomorphe de chaque ensemble ordonné $(E, \leqslant)$ par un ordonnement principal de fonctions caractéristiques f_a des demi-cones $S(., a]$.

7.2.6. Chaque fonction $f: X \rightarrow Y$ est un spectre avec des rangées (1).

7.2.6. Exemples précédents montrent qu'on doit considérer non seulement spectres mathématiques isolés mais de systèmes de spectres mathématiques.

КУРЕПА ДЖУРО (Београд, Югославия)

НЕКОТОРЫЕ ФУНКЦИИ НА ТОПОЛОГИЧЕСКИХ СТРУКТУРАХ, ГРАФАХ

Если X пространство, решетка или какая нибудь другая структура, тогда можно определять различные числа в зависимости от X, нпр. главное или кардинальное число kX, сепарабельность или число Fréchet sep X, Sep X, cel $X, wX, \ldots$

1. Целуларность

Если H система множеств пусть

cel $H = \sup k\Phi$, где $\Phi \subseteq H$ пробегает семейство систем попарно непересекающихся множеств. Для любого пространства S пусть cel S = cel GS, где GS множество всех открытых множеств в S (в. Курепа [1], 131, сноска 11).

2. Древовидное число (дендритет)

2.1. Для упорядоченного множества $(O, \leqslant)$ древовидное число dr O определяется следующим образом: dr $O: = \sup kT \mid (T, \leqslant)$ есть дерево в $(O, \leqslant)$ т.е. для всякого $t \in T$ множество $T(\cdot, t) := \{x \mid x < t\}$ есть вполне упорядоченное подмножество в $(O, \leqslant)$ (cf Kurepa [6]).

2.2. Если S пространство, пусть

$$\mathrm{dr}\, S := \mathrm{dr}\,(GS, \supset) \text{ и дуально:}$$

$$\mathrm{dr}^* S := \mathrm{dr}\,(GS, \subset).$$

3. Если (G, ϱ) граф пусть $K_C G$ соотв. $\Gamma_C G$ первое кардинальное соотв. ординальное число которое непредставляемо в (G, ϱ) как цепь.

3.1. Именно для упорядоченного множества $(O, \leqslant)$ имеется $K_C(O, \leqslant)$, $K_C(O, \geqslant)$, $\Gamma_C(O, \leqslant)$, $W_C(O \leqslant) :=$ первое главное число непредставляемо как вполне упорядоченное подмножество в $(O, \leqslant)$. Пусть: $w_C X = (W_C X)^-$, $w_d(O, \leqslant) = W_C(O, \geqslant)^-$.

$K(G, \varrho) :=$ первое главное число которое не кардинальность антицепи в (G, ϱ) некоторого подграфа Φ в (G, ϱ) са свойством $x, y \in \Phi \Rightarrow x = y \vee x$ non ϱy).

3.2. *Теорема. Во всяком пространстве V в котором $\bar{\bar{X}} = \bar{X}$ имеется dr V = = Sep $V = k_C FV$ (FV-множество всех замкнутых множеств в V; Sep V: = = $\underset{X \subset V}{\mathrm{Sup}}$ sep X).*

В метрических пространствах функции cel, sep, Sep, w, dr, dr совпадают.*

3.3. $F_D V := \sup kH$, (H вполне упорядоченная подсистема в $(FV, \supset)$).

3.4. dp $V := \sup kX$ (X рассеяное множество в V).

3.5. *Теорема.* $F_D = dp$; *если* $\overline{\overline{X}} = \overline{X}$, *тогда* $dr^* = dp$ *и* $dr = dr^* \Leftrightarrow Sep = dp$.

4. Числа исчерпывания

4.1. Звездочное число $s(G, \varrho) := \inf_H kH$ (H система цепей в $(G, \varrho), \bigcup H = G$) (в. Курепа [5]).

4.2. Антизвездочное число $s'(G, \varrho) := \inf_H kH$ (H система антицепей в (G, ϱ), $\bigcup H = G$) (в. Курепа [5]).

4.3. Категорическое число ct $X := \inf_X kX$ ($X \subset PS$, X система нигде плотных множеств, $X = \text{Der}\, S$ (в Курепа [7]).

4.31. Подкатегорическое число sct $S := \inf_X kX$ ($X \subset PS$, X система нигде плотных множеств, $\bigcup X$-плотное множество в $S \backslash IS$, IS-множество изолированых точек в S (в. Курепа [7]).

4.4. Изолированное число пространства:

$h_I S := \inf_X kX$ (X система изолированных множеств, $\bigcup X = S$).

4.4.1. Подизолированное число пространства:

$h_{sI} S := \inf_X kX$ (X система изолированных множеств, соединение которых повсюду плотное множество в S) (в. Курепа [2], condition K_0).

4.5. Для множественного свойства P в S соответствующее число исчерпывания $h_P S := \inf_X kX$, X система множеств са свойством P, $\bigcup X = S$); P может значить: быть изолированным, sep $X \leqslant kw_\alpha$, мера 0 итд.

4.5.1. Континуум число $h_C K$ континуума K. Если K континуум ($\equiv$связное компактное множество) пусть $h_C K = \inf_X kX$ (X система попарно непересекающихся подконтинуумов континуума K).

4.5.2. *Проблема.* Можно ли без хипотезы континуума доказать что $h_C I = \aleph_1$? (Sierpiński доказал что $k_C I \geqslant \aleph_1$); $I = R[0, 1] =$ множество всех действительных чисел x для которых $0 \leqslant x \leqslant 1$.

Или же можно предполагать что $h_C I$ любое главное число y для которого $\aleph_1 \leqslant y \leqslant 2^{\aleph_0}$?

5. Число перемени (мутации).

Если P множественное свойство пусть $m_P(S, \bigcup) := \inf_X kX$ (X система P-множеств в S, $\bigcup X$ non$\in P$).

Аналогичным образом определяется $m_P(S, \bigcap)$.

5.1. Для пространства $(R, \leqslant)$ вещественных чисел P может значить: мера 0, категория 1; соответствующие числа $m_0 R$, $m_{ct} R$ таковы что они $> k\,\omega_0$ и $\leqslant 2^{k\omega_0}$ и без хипотезы континуума еще не доказано что эти числа $= k\,\omega_1$ (*проблема*).

6. Задача о sup $(O, \leqslant)$.

Для упорядоченного множества $(O, \leqslant)$ надо испытать существует ли $\operatorname{Sup}(O, \leqslant) := S$ и если $S \in O$. Дуально: существует ли инфимум

$I := \inf(O, <)$ и если $I \in O$? Если $S \in O$, сколько раз число S можно реализовать в связи с $(O, \leqslant)$?

7. Число переменных > 1.

Например: Если (S, S_1) упорядоченная пара пространств, пусть

$C(S, S_1) := \{f : S \to S_1,\ f$ непрерывно$\}$.

$K(S, S_1) := \sup kfS,\ f \in C(S, S_1)$.

Можно доказать что $K(R, I\omega_1) = k\omega_0 = K(I\omega_1, R)$; $I\omega_1$ — пространство упорядоченных чисел $< \omega_1$.

Литература

Appert, A: *Propriétés des espaces abstraits les plus généraux*, Thèse, Paris 1934, XII + 112.
Erdös P., Hajnal A., Rado R.: *Partition relations for cardinal numbers*, Acta Math. Ac. Sc. Hungar. 16 (1965) 93-196.
Kurepa, Đ.:

1. *Ensembles ordonnés et ramifiés*. Thèse, Paris, 1935. Publ. math. Belgrade. 4 (1935) 1-138.
2. *Le problème de Suslin et les espaces abstraits*, Comptes rendus, 203 (1936) 1049-1052.
3. *Le problème de Suslin et les espaces abstraits*. Comptes rendus, 204 (1937) 325-327.
4. *On the cardinal number of ordered sets and of symmetrical structures in dependence on the cardinal numbers of its chains and antichains*, Glasnik Mat.-fiz., Zagreb, 14 (1959) 183-203.
5. *Star number of ordered sets and graphs*. Glasnik Mat. fiz. astr. Zagreb, 18 (1963) 27-37.
6. *Dendrity of spaces and of ordered sets-drvni broj prostora i uređenih skupova*. Glasnik matematički, Zagreb, 2 (22) 1967, 145-161-162.
7. *On the category number of topological spaces*. Publ. Mathem. Beograd 8 (22) (1969) 149-152.
8. *Some functions on topological spaces, graphs.*. Rendiconti Seminario matematico e fisico, Milano 38 (1968 41-51.

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Đuro Kurepa

A REMARK ON EQUIVALENCE RELATION

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In connexion to the article [1], in particular § 3, we formulate here a general theorem on composition of equivalence relations.

1. Theorem (i). *Let S be a non empty set and (R, R_1) any ordered pair of binary equivalence relations on S; in order that*

(1) $$S/_R \circ S/_{R_1} = S^2,$$

it is necessary and sufficient that every $X \in S/_R$ intersects every $X_1 \in S/_{R_1}$, i. e.

(2) $$X \in S/_R \wedge X_1 \in S/_{R_1} \Rightarrow k(X \cap X_1) \geqslant 1.$$

(ii) If

(3) $$X \in S/_R \wedge X_1 \in S/_{R_1} \Rightarrow k(X \cap X_1) = 1, \text{ then}$$

(4) $$X \in S/_R \Rightarrow kX = k\,S/_{R_1} \text{ and}$$

(5) $$X_1 \in S/_{R_1} \Rightarrow kX = k\,S/_{R_1}:$$

all members of $S/_R$ are of a same cardinality, as well are those of $S/_{R_1}$; then the set S is representable in form of a matrix of order $(S/_R, S/_{R_1})$ i. e. there is a one-to-one mapping $f(x, x_1)$ of $S/_R \times S/_{R_1}$ onto S.

2. Proof of (i). 1.1. Necessity: (1) $\Rightarrow$ (2). The set $(1)_1$ is the union of all the sets of the form $X \circ X_1$, where $X \in S/_R$, $X_1 \in S/_{R_1}$, the set $X \circ X_1$ is constituted of all the pairs $(x, x_1) \in S^2$ such that for some $s \in S$ we have $(x, s) \in X^2$, $(s, x_1) \in X_1^2$. The sets X, X_1 being non empty, there is some member (x, x_1) of $X \times X_1$; now, $(x, x_1) \in S^2$ and according to the inclusion of $(1)_2 \subseteq (1)_1$, there exists a $z \in S$ satisfying $xRz \wedge zR_1 x_1$, hence $z \in X \wedge z \in X_1$, i. e. $z \in X_1$. The implication (2) is proved.

1.2. Sufficiency: (2) $\Rightarrow$ (1).

Obviously, $(1)_1 \subseteq (1)_2$; let us prove that dually, also $(1)_2 \Rightarrow (1)_1$, i. e. that $(x, x_1) \in S^2 \Rightarrow (x, x_1) \in (1)_1$. Now, $(x, x_1) \in S^2$ implies $x \in X \in S/_R$ for some X and $x_1 \in X_1 \in S/_{R_1}$ for some X_1; by hypothesis (3) there exists some $z \in X \cap X_1$; therefore, $(x, z) \in X^2$, $(z, x_1) \in X_1^2$ and $(x, y) \in X \circ X_1 \subseteq S/_R \circ S/_{R_1}$.

3. Proof of (ii).

The relation (3) implies that for every $X \in S/_R$

$$X_1 \in S/_{R_1} \rightarrow X \cap X_1$$

is a biunique mapping of $S/_{R_1}$ onto the system $\binom{X}{1} = \{\{x\} \mid x \in X\}$; therefore, (4) is holding; analogously, one proves (5). For the same reason, the mapping

$$(X, X_1) \in S/_R \times S/_{R_1} \Rightarrow X \cap X_1$$

is a one-to-one mapping of $S/_R \times S/_{R_1}$ onto $\binom{S}{1}$, and consequently, the atomization function

$$\iota(X \cap X_1) := \text{the member of } X \cap X_1 \text{ is a one-to-one}$$

mapping of $S/_R \times S_{R_1}$ onto S.

1.1. Corollary. *If E is a binary equivalence relation in a set S, then*

(2) $$E \circ E = S^2 \Leftrightarrow E = S^2.$$

Obviously, it is sufficient to prove that $(2)_1 \Rightarrow (2)_2$. Now, the relation E has no two distinct cosets $E_1, E_2 \in S/_E$. In opposite case, on the one hand we would have $E_1 \cap E_2 = \emptyset$ and on the other hand by the theorem 1 (i) the relation (2) would imply $E_1 \cap E_2 \neq \emptyset$ — contradiction. Consequently, $k\,S/_E = 1$ and therefore $S/_E = \{S\}$. *Q. E. D.*

1.2. Corollary. *If R, R_1 are binary equivalence relations in a set S, then $R \circ R_1 = S^2 \Rightarrow R_1 \circ R = S^2$ and $R \circ R_1 = R_1 \circ R$.*

The corollary is an immediate consequence of the theorem 1 (i).

BIBLIOGRAPHY

D. Jović, J. Ušan, V. Đonin, *Algebarska struktura jedne klasifikacije u vokalnom sistemu srpskohrvatskog književnog jezika*, Mat. vesnik, 6 (21). sv. 1, 1969, str. 17—26

Matematički institut,
Beograd

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Nouvelle série, tome 9 (23), 1969, pp. 217—224

SOME GENERATINGS AND PROPERTIES OF ORDERED SETS

Đ. Kurepa

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1. There are various ways of generating of ordered sets. The simplest way is certainly to consider any system S of sets ordered by the relation $\supset$ or by $\subset$. In particular, the power-sets PX (X being a set) yield the ordered sets $(PX, \supset)$ as the most general ordered sets, every ordered set being similar to a part of some $(PX, \supset)$. The cardinal ordering of any set $\{0,1\}^X$ furnishes the same possibility.

The procedure of intercalation, of inoculation, of hugging yields new ordered sets; it is then interesting how some properties of the obtained output set depend on similar properties of the given input sets.

In the paper we examine the foregoing procedures and examine the property of normality of ramified sets and of trees.

The ω_ν—ordinal dimension of A_ν—sets in the sense of Komm [5] is determined to be $k\,\omega_\nu$ and the problem is formulated as to whether there should be

$$d_0 A_\nu < k\,\omega_\nu{}^{1)}$$

2. *A sufficient condition for the normality of trees.*

2.1 *Degenerated ordered sets.* An ordered set $(0, \leqslant)$ is called degenerated, provided the comparability relation is transitive in the set $(0, \leqslant)$.

2.2. *Normal ordered sets.* An ordered set is called normal if it has the same cardinality as some of its degenerated subsets.

2.3. Theorem. *Let ω_ν be any regular non countable ordinal number. Every tree T of cardinality $\geqslant k\,\omega$ in which, for some $\omega'_\nu < \omega_\nu$, there is a strictly increasing function $f|T$ into a $k\,\omega'_\nu$-separable chain C contains an antichain of the cardinality $k\,\omega_\nu$.*

Proof. Let r_0, r_1, r_n, $(n < \omega'_\nu)$ be any simply ordered everywhere dense subset of C of cardinality $k\,\omega'_\nu$. For every $n < \omega'_\nu$ let T^n be the set of all the points of C satisfying

$$fx \leqslant r_n < f(b(a)),$$

---

1) kX = cardinality of X; $d_0 X$ is the minimal cardinal number n such that there is a system of n linear orderings of X, the superprosition of which produces the given ordering $(X, \leqslant)$.

where for any $a \in T$ we denote by $b(a)$ any successor of a (there is no restriction to assume that every $a \in T$ has infinitely many (even $k\,\omega_\nu$) successors). Obviously

$$T = \cup_n T^n \,(n < \omega_\nu') \text{ and}$$

$$kT = \Sigma_n \, kT^n \,(n < \omega_\nu').$$

The number kT being regular supposedly, there exists some index $m < \omega_\nu'$ such that $kT = kT^m$. Let us prove that T^m contains an antichain of cardinality $k\,\omega_\nu$. We can assume that the tree is of cardinality $k\,\omega_\nu$, that its every row as well as every chain are $< k\,\omega_\nu$, and consequently $\gamma T^m = \omega_\nu$. Moreover, one can suppose that every member a of T^m has $k\,\omega_\nu$ successors. By induction argument, (cf. [5] p. 486) one proves the existence of an ω_ν -sequence

$$a_\xi \in T^m \qquad (\xi < \tau = \omega_\nu)$$

such that the numbers ζ_ξ, defined by $a_\xi \in R_{\zeta_\xi} T$ form a strictly increasing ω_ν -sequence and that the numbers η_ξ defined by $b(a_\xi) \in R_{\eta_\xi} T$ satisfy

$$\zeta_\xi < \eta_\xi < \zeta_{\xi+1} \quad (\xi < \omega_\nu).$$

One proves then that the points $b(a_\xi)\,(\xi < \omega_\nu)$ form a requested antichain.

3. *A theorem on ramified sets.*

3.1. Definition. Any ordered set $(R, \leqslant)$ such that for every $x \in R$ the set $R(\cdot, x] \overset{\text{def}}{=} \{y \mid y \in R, y \leqslant x\}$ is *simply ordered*, is called a *ramified set*. Ramified sets generalize the trees. As an exercise one proves the following.

3.1.1. If D is a degenerated subset of a ramified set $(R, \leqslant)$, then the set

$${}_0D(R) := \cup_x R(.\,, x] \quad (x \in D)$$

is a degenerated subset of $(R, \leqslant)$.

3.2. Theorem. *If a ramified set R is cofinal to a normal tree T, such that for every subset $S \subset T$ of cardinality $cf\ kR$ one has*

(1) $$k \cup R[s] = kR, \ (s \in S),$$

then the set $(R \leqslant)$ is normal.

Proof. Since for every $x \in R$ the set $R(.\,, x]$ is simply ordered we might assume that

(2) $$kR(.\,, x] < kR \qquad (x \in R).$$

Now, by the definition of cofinality of R to T we have

(3) $$R = \cup_x R(.\,, x], \quad (x \in T).$$

3.2.1 First case: kR is regular. Then the relations (1), (2) jointly with the regularity of kR imply $kR = kT$. Now, let us consider the degenerated subset T_0 of T of the cardinality kT. By hypothesis, such a set exists. The number kT being regular, every set $T_0[a, \cdot)$ with $a \in R_0 T_0$ being a chain thus

$<kT$, we conclude that $kR_0T_0 = kT_0 = kR$. Now, for every $a \in R_0T_0$ let a' be any point of R such that $a \leqslant a'$; then the set $R'_0 := \{a' | a' \in R_0T_0\}$ is a requested antichain of R and obviously $kR'_0 = kR_0T_0 = kT_0 = kR$.

3.2.2. Second case: *kR is singular*. Of course, there is no restriction to assume that the cardinality of kT be a regular number $k\omega_\gamma$. The set T being normal, let then T_0 be a degenerated subset of T such that $kT_0 = kT$. Then we have two cases:

3.2.2.1. First case: The first row R_0T_0 of T_0 has kT_0 members. Since (1) and (2) hold for every $s \in R_0T_0$, we conclude easily that in every $R[s, \cdot)$, $(s \in R_0T_0)$ there exists a degenerated set of any cardinality $<kR$. Now, let us consider a well-ordering

$$t_0, t_1, \ldots, t_\alpha, \ldots \qquad (\alpha < \omega_\beta)$$

of the set R_0T_0 and any ω_β-sequence of cardinals k_α such that

$$k_\alpha < kR \text{ and } \Sigma_\alpha k_\alpha = kR, \ (\alpha < \omega_\beta). \tag{4}$$

In every $R[t_\alpha, \cdot), (\alpha < \omega_\beta)$ there exists some *degenerated* subset D_α of cardinality $\geqslant k_\alpha$; then the set $D = \bigcup_\alpha D_\alpha (\alpha < \omega_\beta)$ is a requested degenerated subset of R of cardinality $kR = \sum_\alpha k_\alpha (\alpha < \omega_\beta)$.

3.2.2.2. Second case: $kR_0T_0 < kT_0$. The number kT being regular, one concludes that for some $a \in R_0T_0$ we have

$$kT_0[a, \cdot) = kT_0. \tag{5}$$

Now, the set $T_0[a, \cdot)$ is well-ordered; T_0 being degenerated. Let us consider the chains

$$R(\cdot, x], \ (x \in T_0[a, \cdot)) \text{ and their union}$$

$$A \bigcup_x R(., x], \ (x \in T_0[a, \cdot)).$$

The set A is a simply ordered subset of $(R; \leqslant)$; if $kA = kR$, A is a requested subset of $(R \leqslant)$. Therefore, we have still to consider the case that $kA < kR$. By hypothesis (1) the relation (1) holds for $S = T_0[a, \cdot)$, i.e.

$$k \bigcup_S R(s) = kR \qquad (s \in T_0[a, \cdot)).$$

Consequently, there is a strictly increasing ω_β-sequence of points $a_\alpha \in T_0[a, \cdot)$ and a strictly increasing ω_β-sequence of cardinals k_α with $kA < k_\alpha < kR$ such that

$$kR[a_\alpha, \cdot) \geqslant k_{\alpha+1} \text{ and } \sum_\infty k_\alpha = kR, (\alpha < \omega_\beta).$$

We consider the sets

$$B_\alpha \overset{\text{def}}{=} R[a_\alpha,) \diagdown R[a_{\alpha+1}, \cdot).$$

One has

$$\sup_\alpha kB_\alpha = kR.$$

Therefore, it is possible to choose degenerated sets

$$D_\alpha \subset B_\alpha$$

such that

$$\sup\ kD_\alpha = kR.$$

Now, the set

$$B = \bigcup_\alpha D_\alpha \qquad (\alpha < \omega_\beta)$$

is degenerated and has the cardinality kR, what completes the proof of the theorem.

3.3. *Transition: trees — ramified sets.* By intercalation of chains between consecutive members of a tree T one gets a ramified set; in other words, for any ordered pair (x, y) of consecutive members of a tree T let $c(x, y)$ be an ordered set — empty or non empty; if the sets $c(x, y)$ are pairwise disjoint and in no order relation, we intercalate $c(x, y)$ between the points x, y of T; if $x < y$, the set succeeds to every member of $T(., x]$ and preceeds to every member of $T(y, .)$. Let (T, c) be the ordered set so obtained.

Theorem. *The set* (T, c) *is ramified (a tree) if and only if for every* $\{x, y\}_< \subset (T, \leqslant)$ *the set* $c(x, y)$ *is a totally (well) ordered set.*

4. *Isomorph τ-dimension of ordered sets.*

4.1. *Definition of $i\tau$-dim.* Let Σ be the type of order of some linearly ordered set. If for some ordered set $(0, \leqslant)$ there exists a family F of linear extensions $(0, \leqslant_r)$ of $(0, \leqslant)$, each of order type τ, and such that for any $(a, b) \in 0^2$ one has $a \leqslant b$ if and only if $a \leqslant_r b$ for every $r \in F$, then the minimal cardinality kF of all such families F is called the isomorph τ-dimension of $(0, \leqslant)$ and is denoted $i\tau\text{-dim}\,(0, \leqslant)$.

4.1.1. E.g. for any finite ordered set $(0, \leqslant)$, if $k\,0 \equiv n$, then n-dim $(0, \leqslant)$ exists.

4.2. Theorem. *If*

(1) $(I\omega_0, \leqslant')$ *is any suborder of the linearly ordered set*

(2) $(I\omega_0, \leqslant)$,

then $i\omega_0$*-dim* $(I\omega_0; \leqslant')$ *exists.*

More generally we have the following.

4.3. Theorem. *In order that for some ordered set*

(1) $(0, \leqslant')$ *the* $i\omega_0$*-dim* $(0, \leqslant')$ *exists, it is necessary and sufficient that there is some one-to-one increasing mapping* i *of* (1) *into the chain*

(2) $(I\omega_0, \leqslant)$.

Obviously, the condition of the theorem is necessary. Let us prove that the condition of the theorem is also sufficient: if there exists some (1.1)-mapping i of (1) into (2), then there exists a family of ω_0-extensions of (1), the superposition of which yields the order (1).

For this purpose it is sufficient to prove that every antichain $\{a, b\}$ consisting of 2 incomparable points of (1) is obtainable by such ω_0-extensions of (1). Let $ia = o_1$, $ib = o_2$ and suppose $o_1 < o_2$. The antidomain $O' = iO$ of the set O is the union of the 2-point-set $\{o_1, o_2\}$, of the interval $O'(o_1, o_2)$, of the set $D' = O'(.\,, o_1)$ and of the set $E' = O'(o_2,.)$. The set $I = O'(o_1, o_2)$ is the union of the following 3 subsets:

(3) $A' = i\,0\,(a, \cdot)\overset{\sim}{\leqslant}' \cap I, B' = iO\,(.\,,b)_{\leqslant'} \cap I$ and $C' = i\,[CO\,[a]_{\leqslant'} \cap CO\,[b]_{\leqslant'} \cap I.$

The domains of the corresponding subfunctions are well determined. Put $X = i^{-1}X'$, i.e.

(4) $$A = i^{-1}A', B = i^{-1}B', C = i^{-1}C', D = i^{-1}D', E = i^{-1}E'.$$

There is no restriction to suppose

(5) $$fa > = -fa + fb;$$

as a matter of fact, if this condition were not satisfied, we would consider the function $x \to f'x = fx + (-fa + fb) + 1$, and this function f' would satisfy the condition $f'a > -f'a + f'b$. This being so, let n be a member of $I\omega_0 := I$ such that

(6) $$o_1 > n > -o_1 + o_2;$$

we define a function $g\,|\,I$ in the following way:

(7) $$ga = fb = o_2, gb = o_1 = fa$$

$$g\,|\,A = -n + f\,|\,A,\ g\,|\,B = -n + f\,|\,B,\ g\,|\,C = f\,|\,C,\ g\,|\,D = -n + f\,|\,D,$$

$$g\,|\,E - n + g\,|\,E.$$

Let us define the ordering $(O; \leqslant_g)$ in such a way that for (x, y) we put

(8) $$x \leqslant_g y \Leftrightarrow gx \leqslant' gy.$$

The relation $\leqslant_g$ extends the relation $\leqslant'$ in $(O, \leqslant')$, i.e. for $(x, y) \in O^2$

(9) $$x <' y \Rightarrow gx <_g g_y, \text{ i.e. } g_x < gy.$$

The implication (9) is obvious if $\{x, y\}$ belongs to any of the sets

(10) $$A, B, C, D, E.$$

Therefore, we have to prove (9) if only one of the members of $\{x, y\}$ belongs to one of the sets (10), the another being in some another member of (10) or in $\{o_1, o_2\}$. E.g. assume $x \in A, y \in B$; then $gx = -n + fx$, $gy = -n + {}$ ${}+fy$. Since by hypothesis $x <' y$ so is $fx < fy$ and consequently

$$-n + fx < -n + fy, \text{ i.e. } gx < gy, \text{ i.e. } x <_g y.$$

In all other cases one proves (9) and also that the mapping $g\,|\,O$ is one-to-one. Therefore, $\leqslant_g$ is an order relation in S; in particular, we have $ga > gb$, and this jointly with $a < b$ proves that the superposition of the orderings $(O, \leqslant)$ and $(O, \leqslant_g)$ yields the incomparability of a, b in $(O; \leqslant')$.

4.4. Problem. Probably, in 4.2 and 4.3 it is legitimate to replace everywhere ω_0 by ω_α, for any ordinal α.

5. *On permutations of ordered sets.*

5.1. Let $(O, \leqslant)$ be any ordered set and O_1 the set of all the one-to-one mappings of O into itself. For any subset $F \subset O_1$ we define the order $\leqslant_F$ of O in the following way:

$$x \leqslant_F y \Leftrightarrow fx \leqslant fy \text{ for every } f \in F.$$

Obviously, $\leqslant_F$ is the total unorder for $F = 0_1$; for the identity transformation I of O the relation $\leqslant_{\{I\}}$ equals $\leqslant$.

5.2. Problem. *Is every suborder* $(O, \leqslant')$ of $(O, \leqslant)$ *obtainable as* $(O; \leqslant_F)$ *for some* $F \subset O_1$?

The answer is — yes! at least for $(I\omega_0, \leqslant)$ and probably for every $(I\omega_\alpha, \leqslant)$.

5.3. Theorem (*Superposition of* $< k\omega_\nu$ *orderings of* $I\omega_\nu$). *Let* ω_ν *be regular; any system* F *of cardinality* $< k\omega_\nu$ *of total* ω_ν*-orderings of the set* $I_\nu := I\omega_\nu = \{0, 1, 2, \ldots \alpha, \ldots\}_{\alpha < \omega_\nu}$ *yields by superposition an order* $(O, \leqslant)$ *of* I_ν *possessing an* ω_ν*-sequence in natural order.*

Proof. Let $F = \{f_\xi\}_\xi$ be a normal well order of F; let us define the ω_ν -sequence (1) a_ξ of numbers $< \omega_\nu$ in the following way: let $a_0 = O$; let a_1 be the first member of I_ν coming after a_0 in every member of F. Let $o < \beta < \omega_\nu$ and let suppose that the strictly increasing β-sequence $a_\alpha (\alpha < \beta)$ be defined; we define a_β as the first member of I_ν coming after $\{a_\alpha\}_\alpha$ in every member of F; since $k\beta < k\omega_\nu$ and since ω_ν is regular, the existence of a_β is guaranted. By induction arguments the sequence (1) of $k\ \omega_\nu$ points in strictly increasing order is defined. *Q. E. D.*

5.4. Theorem. *Let* ω_ν *be regular. The ordinal* ω_ν*-dimension of every* A_ν*-set exists and equals* $k\omega_\nu$ (*cf. Problem in 4.4.*).

As to the definition of A_ν-sets s. [5]. The existence of the ω_ν-dimension of A_ν follows from the fact that $(A_\nu, \leqslant_\nu)$ is obtainable by a family of permutations of $(I\omega_\nu, \leqslant)$ on the one hand and on the other hand of the fact that for any permutation p of the chain $I\omega_\nu$ the chain $(I\omega_\nu, \leqslant_p)$ is of the order type ω_ν. Finally, by 4.3. the ω_ν-dimension is not $< k\omega_\nu$ (cf. problem in 4.4).

5.6. Problem. *If a tree* $(T, \leqslant)$ *is the union of* $\leqslant k\omega_0$ *antichains, is then the ordinal dimension* $d_0 T$ *of the tree* $\leqslant k\omega_0$?

5.7. Problem. More generally, if an ordered set $(O, \leqslant)$ is the union of a family of $\leqslant b$ of its antichains, is then $d_0(O, \leqslant) \leqslant b$, i.e. is there a system F of cardinality $\leqslant b$ of total orderings of the set O such that $x \leqslant y$ in $(O, \leqslant)$ if and only if $x \leqslant_f y$ for every ordering $\leqslant_f \in F$.

6. *Hugging and inoculation of ordered sets.*

6.1. Let. $(O, \leqslant)$ be an ordered set. Let $f | O$ be any mapping such that to every point x of O one is associated a single ordered pair $(o_x, 1_x)$ of ordered sets. We denote by $O \otimes f$ the ordered set obtained from $(O, \leqslant)$ in such a way that O be extended by sets $o_x, 1_x$ and ordered in such a way that o_x precedes x, x precedes 1_x for every $x \in O$ and that o_x, 1_x be incomparable to $O(\cdot, x)$ and to $O(x, \cdot)$ respectively. In particular, o_x precedes 1_x in $O \otimes f$ for every $x \in O$.

6.1.1. The set $O \otimes f$ might be defined to consist of points of O and of ordered pairs (a, b) such that either $a \in O$ and $b \in 1_a$ or $b \in O$ and $a \in o_b$.

The ordering of $O\otimes f$ is performed in the following way: If $A, B\in O\otimes f$ then $A\leqslant B$ means exactly the following:

if $A, B\in O$, then $A\leqslant B$ in $(O, \leqslant)$;

if $A\in O$, and $B=(a, b)$, then $a\in O$, $A\leqslant a$ in $(O, \leqslant)$ and $b\in 1_a$;

if $B\in O$, and $A=(a, b)$, then $b\in O, b\leqslant B$ in $(O, \leqslant)$ and $a\in O_a$.

One verifies easily the following:

6.1.2. *Lemma*. For any ordered set $(O, \leqslant)$ and any mapping f defined in 6.1. the set $(O\otimes f, \leqslant)$ is an extension of $(O, \leqslant)$; in particular, for distinct points $x, y\in O$, the sets $0_x, 0_y$ are mutually incomparable as well as are the sets $1_x, 1_y$; one has $0_x\leqslant 1_y$ if and only if $x\leqslant y$; the sets $1_x, 0_y$ are incomparable mutually.

6.1.3. The processus $(O, \leqslant)f\to(O\otimes f; \leqslant)$ is called the hugging or hymerization (*cf* [2] p. 15) or the double inoculation of the sets fx to the set $(O, \leqslant)$.

6.2. *Inoculation*. If o_x equals $\varnothing$ identically, the hugging is called the *inoculation* or *grafting* in $(O, \leqslant)$. If $1_x=\varnothing$, the hugging of fx to x is called the *inverse inoculation* of the set 0_x at the point x of 0. If $f|O=X, Y)$ (constant), the f-hugging is denoted by $(X\leftarrow O\to Y)$; instead of $(\varnothing\leftarrow O\to Y)$ and $(X\leftarrow O\to\varnothing)$ we write also $(O\to Y)$, and $(X\leftarrow O)$ respectively.

6.2.1. Instead of $X\leftarrow(X\leftarrow O)$ we shall write $2X\leftarrow O$ in general, we define

$$(\alpha+1)X\leftarrow O \text{ to be } X\leftarrow(\alpha X\leftarrow O) \text{ and}$$

$$\lambda X\leftarrow O := \cup_\beta \beta X\leftarrow O \quad (\beta<\lambda)$$

for any ordinal α and any limit ordinal λ.

Analogously, we put

$$(O\to Y)\to Y)=O\times 2Y$$

$$(O\to\alpha Y)=O\to(\alpha+1)Y$$

$$O\to\lambda Y=\cup_\beta O\to\beta Y) \qquad (\beta<\lambda).$$

6.2.2. Convention. If (a, b, c) is any ordered triplet of ordered types, we define $a\leftarrow b\to c$ to mean $A\leftarrow B\to C$, where (A, B, C) is an ordered triplet of ordered sets of the ordered types a, b, c respectively.

6.2.3. Example. If (A, B) is an ordered pair of antichains and β any ordinal, then $T:=A\to\beta B$ is a tree; the first row of T is A; the rank $\gamma=\gamma T$ of T is inf. $\{\beta+1, \omega_0\}$; for every positive integer $n<\gamma T$ one has

$$kR_n T=(k\beta-n+1)kA(kB)^n.$$

The proof of the last equality is performed by induction argument on n.

6.2.4. If $(O, \leqslant)$ as well as fx for every $x\in S$ are ramified (ranked) so is also the set $(O\otimes f; \leqslant)$; in particular, if $(O, \leqslant)$ and fx is a tree for every $x\in O$, then so is also the set $O\otimes f$.

6.3. Theorem. *Let $(O, \leqslant)$ be ordered and $x\in O\to fx=(0_x, 1_x)$ like in* 6.1; *then every regular ordinal number r which is representable in $O\otimes f$ is also representable in $(O, \leqslant)$ or in $0_x<x<1_x$ for some $x\in 0$.*

BIBLIOGRAPHY

[1] Adnađević, D., *Dimenzije nekih razvrstanih skupova sa primenama — Dimensions of some partially ordered sets with applications*, VESNIK Društva matematičara, fizičara i astronoma, Beograd, XIII (1961) 49—106, 225—262.

[2] Z. Damjanović — M. Marić — N. Parezanović — P. Pejović, *Investigation of nets composed of dad — neuromimes, I Concept, model and behaviour of dad — — neuromimes* (5 th International Congress of AICA, Lausanne 1967), pp 1—16

[3] Dushnik, B., Miller, E.W., *Partially ordered sets*, Amer. J. Math., 63 (1941) 600—610.

[4] Ginsburg, S., *On the λ-dimension and the A-dimension of partially ordered sets*, Amer. J. Math., 76, 590—598 (1954).

[5] Komm, H., *On the dimension of partially ordered sets*, Amer. J. Math., 70 (1948) 507—520.

[6] Kurepa, Đ., *Transformations monotones des ensembles partiellement ordonnés*, Revista de Ciencias, Lima, No. 434, 42 (1940), 827—846, No.437, 43 (1941) 483—500.

[7] Kurepa, Đ, *On A-trees*, Publ. Inst. Math., Beograd, 8 (22) (1968) 153—161

[8] Sedmak, V., *Quelques applications des ensembles ordonnés*, VESNIK Društva matematičara, fizičara i astronoma, VI, (1954) 12—39, 131—153.

Matematički Institut
Beograd

O ČETIRI FUNKCIJE*

Kurepa R. Djuro, Beograd

1. Neka je n prirodan broj; tada se $n!$ ($Def = 1.2.3...n$) može uvesti kao: 1) broj permutacija skuposti $I(n) = \{0, 1, \ldots, n-1\}$ od n članova; kao i 2) broj maksimalnih lanaca u uređenom skupu ($PI(n), \supset$) svih delova skuposti $I(n)$. Poslednja značenja od $n!$ prenose se neposredno i za svaki beskonačni broj n.

2. Dualni n faktorijal $n¡$ uvodi se kao broj maksimalnih antilanaca u ($PI(n), \supset$). Za razliku od $n!$ koji zadovoljava vrlo jednostavan uslov svođenja $n! = (n-1)!n$ i ima važno proširenje $n! = \Gamma(n+1)$ ne zna se kako javno broj $n¡$ zavisi od $(n-1)¡$ ili od manjih $k¡$ za $k < n$.

3. Uz n faktorijal ($n!$) može se uvesti i levi faktorijal n ($!n$) kao $\Sigma k!$ pri $k < n$, tj. $!n = o! + 1! .. + (n-1)!$

4. Stavimo $M_n = M(!n, n!) =$ najveća zajednička mera od $!n, n!$. Tako imamo posebno funkcije $n \in N \rightarrow !n, M_n$.

5. Od pomenute četiri funkcije posebno je izučena prva u obliku svojeg proširenja kao gama — funkcija. Jasno je da sve tri funkcije $n!$, $n¡$, $!n$ rastu u ∞ pri $n \rightarrow \infty$.

6. Naprotiv izgleda nam da je funkcija $M_n N$ pri $n > 1$ konstanta 2. Ta je pretpostavka ravnovaljana s pretpostavkom da ni za koji prost broj $p > 2$ broj. $¡p$ nije deljiv sa p. Nastaje problem da se odredi po apsolutnoj vrednosti najmanji ostatak broja $!n$ u odnosu na n.

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МАТЕМАТИЧЕСКИЕ СТРУКТУРЫ — ВЫЧИСЛИТЕЛЬНАЯ МАТЕМАТИКА — МАТЕМАТИЧЕСКОЕ МОДЕЛИРОВАНИЕ
Труды, посвященные шестидесятилетию академика Л. Илиева
София, 1975, с. 109—111

ОДНА ПОСЛЕДОВАТЕЛЬНОСТЬ ДЕРЕВЬЕВ И ОДНА ПОСЛЕДОВАТЕЛЬНОСТЬ ЧИСЕЛ *

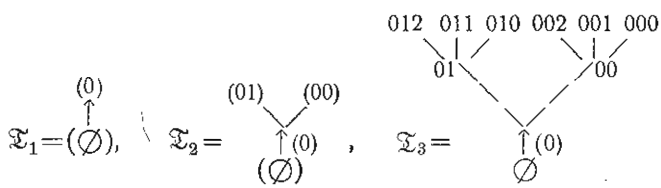
Д. Курепа

Дорогому коллеге и приятелю Любомиру Илиеву в связи с его 60-летием — с лучшими пожеланиями посвящает автор.

0. Обозначения и содержание. kX = мощность X; $[x, .)_S=\{y \mid y \in S, x \leqq y\}$ Определяются дерево T_n и числа K_n для $n \in N$.

1. Совокупность T_n и дерево $\mathfrak{T}_n$. Пусть для каждого $n \in N$ T_n — множество, состоящее из пустых последовательностей $(\emptyset)$ и из всех последовательностей вида $a=(a_1, a_2, \ldots, a_j)$, где $1 \leqq i \leqq n$ и $a_i \in I_i \equiv \{0, 1, 2, \ldots, i-1\}$. Совокупность T_n упорядочивается в отношении $\dashv$, где $a \dashv b$, следовательно, a — начальный отрезок b.

1.1. Таким образом, имеется дерево $\mathfrak{T}=(T_n \dashv)$. Например,



1.2. Начальный слой $R_0\mathfrak{T}_n$ дерева $\mathfrak{T}_n$ есть $\{(\emptyset)\}$. Последний слой дерева $\mathfrak{T}_n$ есть $R_n\mathfrak{T}_n=\{(a_1, a_2, \ldots, a_n)\}$, где $a_1=0$, $a_2 \in \{0, 1\}, \ldots, a_n \in I_n$. Таким образом, ранг или высота $\gamma\mathfrak{T}_n$ дерева $\mathfrak{T}_n$ есть число $1+n$, т. е. $\gamma\mathfrak{T}_n=1+n$.

2. Каждый слой $R_\nu\mathfrak{T}_n$ имеет $\nu!$ членов, т. е.

$$kR_\nu\mathfrak{T}_n=\nu! \quad (\nu=0, 1, 2, \ldots, n).$$

3. Кардинальное или главное число kT_n называется левым факториалом числа n и обозначается через $!(n+1)$ или L_{n+1}:

$$!(n+1) \equiv L_{n+1}=0!+1!+2!+\cdots+n!.$$

4. Правый или обыкновенный факториал $n!=\Gamma(n+1)$ оказывается числом элементов в слое R_nT_m (для $n \leqq m$), а также числом максимальных цепей дерева $\mathfrak{T}_n$.

* София, 20. 4. 1973 г.

5. Соответствие $S \to aS$. Пусть aS для любого упорядоченного множества S обозначает семью всех антицепей $\subset S$, включая и пустую антицепь по определению $\subset S$.

6. Теорема $kaT_1 = 3$.

$$n > 1 \Rightarrow kaT_n = \underbrace{(\ldots(}_{n-1}(2^n+1)^{n-1}+1)^{n-2}+\cdots+1)^2+1)^1+1.$$

Доказательство. $aT_1 = \{\varnothing, \{(\varnothing)\}, \{(0)\}\}$ значит $kaT_1 = 3$.

Пусть $n > 1$. Тогда

(1) $$aT_n = aT_1 \cup \{x \cup y | x \in a[(0, 0), .)_{T_n}, \ y \in a[(0, 1), .)_{T_n}\}.$$

При этом, если $(x, y) \neq (x', y')$, то $x \cup y \neq x' \cup y'$. В частности, если $x = \varnothing = y$, тогда $x \cup y = \varnothing$; $\varnothing$ и только $\varnothing$ — общий член в aT_1 — во втором слагаемом в $(1)_2$.

Для этого левое умножение (1) в k дает

(1′) $$kaT_n = kaT_1 + (ka[(0, 0), .)_{T_n} \,.\, ka[(0, 1), .)_{T_n} - 1).$$

Так как $ka[(00), .)_{T_n} = ka[(0\ 1), .)_{T_n}$ из (1′) следует

$$kaT_n = kaT_1 + [ka[(0)_2, .)_{T_n}]^2 - 1.$$

Аналогичным образом доказывается, что

(2) $$ka[(0)_2, .)_{T_n} = 1 + (ka[(0)_3, .)_{T_n})^3 \text{ и}$$

(i) $$ka[(0)_{i-1}, .)_{T_n} = 1 + (ka[(0)_i, .)_{T_n})^i, \ 1 \leq i \leq n;$$

в частности,

($n-1$) $$ka[(0)_{n-1}, .)_{T_n} = 1 + 2^n \text{ для каждого } n \in \{2, 3, \ldots\}; \ \underbrace{(0,0\ldots0)}_{i} := (0)_i.$$

В самом деле, пусть $3 \leq i \leq n$; тогда

$$a[(0)_{i-1}, .)_{T_n} = \{(0)_{i-1}\} \cup \{x_1 \cup x_2 \cup \cdots \cup x_i | x_j \in a[\underbrace{(0\ 0\ldots0\ j)}_{i}, .)_{T_n}, \quad j \in I_j\}.$$

Пустое множество $\varnothing$ является членом в последнем множестве при $x_1 = x_2 = \cdots x_i = \varnothing$; отсюда

$$ka[(0)_{i-1}, .)_{T_n} = 1 + [ka[(0)_i, .)_{T_n}]^i, \text{ потому что}$$

$$ka[(0)_i, .)_{T_n} = ka[(0)_{i-1}\, j, .)_{T_n} \text{ для } j = 0, 1, 2, \ldots, i-1$$

именно, для $i = n-1$

$$ka[(0)_{n-1}, .)_{T_n} = 1 + [ka[(0)_n, .)_{T_n}]^n = 1 + 2^n,$$

потому что совокупность $a[(0)_n, .)_{T_n} = \{\varnothing, \{(0)_n\}\}$ имеет точно два члена.

Таким образом, соотношения (2), (i), (n) доказаны. Из них немедленно следует теорема 6.

7. Функция $K_n(s)$. В связи с теоремой 6 естественно следующее

7.1. Определение. Если (n, s) — любая упорядоченная пара натурального числа $n > 1$ и произвольной величины s, то

$$K_n(s) := \underbrace{(\ldots(}_{n-1}(s^n+1)^{n-1}+1)^{n-2}+\cdots+1)^2+1)^1+1.$$

7.2. Числа K_n. Пусть

$$K_n := K_n(2).$$

Тогда теорему 6 можно представить в следующей форме:

7.3. Теорема $K_n = K_n(2) = kaT_n$ для $n \in \{2, 3, 4, \ldots\}$.

7.4. В связи с $K_n(s)$ имеются разные разностные уравнения, а именно,

$$K_n(s) = K_{n-1}(s^n + 1) = K_{n-2}((s^n + 1)^{n-1} + 1) = \cdots = K_1(K_n(s) + 1).$$

Для этого естественно определить

$$K_1(y) := y + 1, \; K_0(y) := y \qquad \text{для любого } y,$$

именно $K_0 = K_0(2) = 2$, $K_1 = K_1(2) = 3$.

Функция $K_n(s)$ двух переменных n, s, а именно числа $K_n (n = 0, 1, 2, \ldots)$ будет предметом изучения отдельной статьи.

7.5. Числа K_n возрастают очень быстро:

n	0	1	2	3	4	5
K_n	2	3	6	83	24147398	27818695085249018806026379614017 07603

7.6. Теорема

(1) $$K_{2n-1} \equiv 3 \pmod{10},$$

(2) $$K_{2n} \equiv 8 \pmod{10} \text{ для } n \in \{2, 3, \ldots\},$$

формулу (2) доказал М. Ячимович, участник моего курса „О факториалах".

7.7. Следствие. Числа K_2, $K_3, \ldots$ конгруентны mod. 5.

7.8. Проблема. Функцию K_n/N экстраполировать в функцию K_x/R или даже в $K_x/R(i)$, где R(соотв. $R(i)$) — множество всех действительных (соотв. комплексных) чисел при условии, что эти функции удовлетворяют тем же самым разностным уравнениям, которым удовлетворяют функции $K_n(2)$, $K_n(s)$ при $n \in N$.

8. Последовательность $t_1, t_2, \ldots$

В связи с числом kaT_n для $n \in N$ можно определить также число

$$t_n := kcT_n,$$

где cT_n обозначает множество всех цепей $\subset T_n$, включая пустую цепь $\emptyset \subset T_n$.

Если определить $t_0 = 2$, тогда имеет место следующая

8.1. Теорема. Для $n = 0, 1, 2, \ldots$

$$t_n = 1 + \sum_{k=0}^{n} 2^k k!$$

именно

$$t_0 = 2, \; t_1 = 4, \; t_2 = 12, \; t_3 = 60, \; t_4 = 100, \; t_5 = 3940, \ldots$$

Белград *Югославия* *Поступила 28. 8. 1973 г.*

ĐURO R. KUREPA

PREDMETNA I POJAVNA METODOLOGIJA. POSEBNOST KAO NOSILAC ZAKONITOSTI

1. Među raznovrsnim načinima nastave, učenja, istraživanja i traženja zakonitosti vrlo se ističe način da se pri pojedinom predmetu, pojavi, ... uoči odnosno otkrije i izuči pojedina zakonitost, i ispita gde se sve ona pojavljuje i kako je vezana s raznim drugim pojavama, zakonitostima, i sl. (Na tom načinu razmatranja zasniva se i induktivni način). U tome se i sastoji predmetno-pojavna metodologija ili predmetno-pojavni način. Taj način je vrlo plodan i privlačan. On na svakom koraku korisnika razveseljava jer mu neposredno na delotvoran način proširuje znanje, razvija sposobnosti, neguje volju. Primeri i primene takvog pristupa su bezgranični, sveopšti, svagdašnji i posvudašnji; mogu se primeniti i na svim stupnjevima: i najnižim i najvišim.

2. Posebno, imamo i ovakav model ophođenja i rada:

2.1. Primeti se neki pojav, fenomen, pravilo, ... F u nekom posebnom uočenom slučaju, predmetu, obliku, ... O;

2.2. Onda se to posebno O uzme kao nosilac i predstavnik od F ne samo u vezi s pojavljivanjem od F u O nego i inače.

2.3. Svojstva i zakonitosti o F u posebnom slučaju O su prenosiva i prevodiva u druge slučajeve.

2.4. Pojavljivanje sličnog ili istog F u raznim položajima i prilikama treba iskoristiti da se dogovorno dođe do usaglašavanja, do homomorfizma (ako i ne do izomorfizma).

2.4.1. To posebno vredi za slučaj kad se radi o propisima i dogovorima koji se moraju izvršavati vrlo brzo, automatski i bez razmišljanja.

2.5. Dvojac (O, F). Na taj način imamo važan dvojac (O, F) sastavljen od predmeta O i od fenomena F koji je u O otkriven. Uz dvojac (O, F) ispituju se i razni drugi dvojci (X, F) s istim drugim članom F; u razredu F svih takvi X-ova nalazi se i ono posebno početno O; svi članovi iz F su međusobno vezani bar preko F.

Dvojac (O, F) može biti vrlo raznorodno građen; posebno, nosilac O može biti vrlo raznovrstan kao npr. fizički ili pravni predmet, pojava, pojedina nauka, pojedini model, ... Može se reći da je O manje više veran model ili primerak od F ...

3. Ima mnogo slučajeva u kojima, iz historijskih ili običajnih razloga, vidimo da se stvarno radi protivno prethodnim utanačenjima 2.1 — 2.4. Evo nekoliko primera:

3.1. Kad se već došlo do znakova pismena, I, i za glas I (igla), vrlo je prirodno da predmet *igla* po svojem obliku bude nosilac i znaka i naziva za I; to pogotovo vredi za pisanje u slovenskim jezicima jer je reč *igla* zaista sveslovenska. Slično vredi za oblik i ime slova: U (uho), S (srp), V (vile, viljuška). Posebno bi se izbegao raskorak da npr. Y bude čitano kao ipsilon (dugo i), igrek (grčko i) a da u ćirilici služi za glas u (uho). (Podsetimo se da se danas u grčkom glas i ispisuje na jedan od ovih 7 načina: ι, η, ῃ, υ, υι, ει, οι; zaista neverovatna zamršenost!) Kako čovek na Balkanu može da zna npr. šta je H, kad taj isti znak služi kao haš ili ha (to je najbolje) i kao ita, eta (u grčkom alfabetu) i već nekoliko vekova kao N (Nikola) u ćirilskom alfabetu. Oblik i naziv slova u regionalnim i nacionalnim abecedama trebalo bi usaglasiti s međunarodnim zasadama i s onim kako se radi u nauci, tehnici, umetnosti, a pogotovo u matematici i informatici kao izrazito međunarodnim i sveprimenljivim naukama.

3.2. Nazivi u vezi s brojanjem. Brojevi se svuda upotrebljavaju. Zato bi bilo korisno usaglasiti nazive sa brojevima i stvarnim značenjem. Tako npr. imamo nazive za mesece: 1. mesec, 2. mesec, . . . 6. mesec (jun, lipanj), 7. mesec, 8. mesec, . . . 12. mesec. Nazivi septembar, oktobar, novembar, decembar trebalo bi da označuju redom: sedmi, osmi, deveti i deseti mesec u saglasnosti sa rečima: septem (lat.; sedam), octo (grčki i lat.; osam), novem (l.), ennea (g.: devet), decem (lat.), deka (g.: deset). Učenici i ljudi služe se već vekovima npr. rečima dekadski i decimalni, pa je nezgodno da se s tom opštom upotrebom ne saglašava važan naziv kao što je »decembar«. Za 11. i 12. mesec trebalo bi uvesti nove nazive. Predlažem da se jedanaesti mesec zove Lensk (po Lejninu jer je taj mesec vezan sa važnim svetskim događajem: Velika Socijalistička Revolucija, a predvodio ju je Lenjin); dvanaesti mesec mogao bi se zvati: Dvanaestik, Duodecembar ili slično.

3.3. Uplitanje semantike. Vanredno je korisno i upečatljivo ako uz pojedini pojam dolazi upečatljiv naziv s odgovarajućim značenjem. Zato je neophodno da učeničko štivo i udžbenici ne budu nagomilani stranim nerazumljivim rečima i izrazima. Pojedina reč ili događaj, pojam može upečatljivo da u svesti čoveka zvuči kroz čitav život otkad ju je čuo ili upoznao.

3.4. Koliko bi bilo jednostavnije da se na časovnicima očitavaju časovi od 1—24, a ne 1—12, i da se kretanje kazaljke na časovniku smatra da se izvodi u pozitivnom smislu, a ne u negativnom smislu kao što je danas običaj u matematici.

4. Pojedini oblici matematizacije. Matematizacija odnosno primena matematike u dnevnom životu, nauci, tehnici, umetnosti, . . . , ne sastoji se samo u tome da se primenjuju matematička rasuđivanja, postupci, modeli, formule, pravila, . . . nego i u tome da se to sve primeni što brže, što neposrednije, . . . a pogotovo da ne bude nesuglasja. Jedan od vidova toga usaglašavanja sastoji se u onome što je rečeno u t. 3.1. Stepen matematizacije očituje se i u iznalaženju raznih veza među pojavama i

u pronalaženju dosega pojedine zakonitosti u raznim svojim nijansama (v. t. 2.5).

5. Princip ustaljenosti. Uzme li se neki znak, reč... X iz jedne abecede, jezika u drugu abecedu ili jezik tada značenje, uloga, redosled treba da ostanu nepromenjeni ili slični pogotovu ako su prostorno-vremenski nablizu.

Koliko mi u Jugoslaviji godišnje gubimo što npr. oblici B, C, g, H, P, U, Y, H, u nas nemaju jedan jedini izgovor i jedan jedini redosled. A jednostavnim propisom da se u tom pogledu primene međunarodne norme došli bismo odmah do traženog rešenja. Koliko bismo time pomogli učenicima, tehničarima, slovoslagačima, kompjuterima, prevodiocima,... i — državnoj blagajni. I koliko li bismo se svi time međusobno zbližili.

6. Načelo (princip) provere. Nastavni rad i uopšte rad, posebno sam izvodilac treba da ima pravo i dužnost proveravanja svoga dela. Tako npr. učeniku treba pružiti mogućnost da svaki svoj sastav sam proveri kao i pravo da odmah, na licu mesta, javno sazna, kako je njegov rad, odgovor,... nastavnik ocenio.

7. Tehnička ujednačavanja i usaglašavanja zagovaramo najviše zato da se misaona delatnost ne opterećuje nepotrebno, pogotovo što smo uvereni da ista osoba automatski, bez razmišljanja i gubljenja vremena, ne može u isti mah raditi jednako spretno po dva neusaglašena propisa (npr. brzo naći reč u rečniku, u raznim abecedama s neusaglašenim redosledom ili složiti štamparski slog u neusaglašenim pismenima).

Praktični zahtev: Svaki rečnik, na prvoj ili poslednjoj stranici treba da ima otisnutu i abecedu (oblik i redosled slova) na kojoj je napisan. Ta dva, tri redića neverovatno će pomoći korisniku.

Drugom kolegi Žarku Mijajloviću uz iskrene najbolje želje.

136, 1977.01.14.

Đ Kurepa

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ON THE SYSTEM OF ALL MAXIMAL CHAINS (ANTICHAINS) OF A GRAPH.* [1)]

Kurepa Đuro

To my dear colleague and friend Erdös, Paul

0. Ordered sets constitute an important kind of graphs; it is a very natural way to transfer notions and results concerning ordered sets to general graphs.

0 – 1. From the very beginning of our study of ordered sets $(E, \leq)$ we stressed the importance of the number

(0 – 1 – 1) $\quad p_s(E, \leq) := \sup_A |A|,\ A \in a(E, \leq)$ where

(0 – 1 – 2) $\quad a(E, \leq) := \{X \mid X \subset E;\ X$ is an antichain in $(E, \leq)\}$ [cf Kurepa 1935] p. 1196 and p. 1197, la relation fondamentale $(1)\ |E| \leq (2\,p_s E)^{p_0 E}$]. The number $p_s(E, \leq)$ is called the *liberty degree of* $(E \leq)$ (s is initial of slav words *sloboda* or *svoboda* meaning liberty, freedom).

0 – 1 – 3. The question was whether the number $p_s E$ called also bridth of $(E, \leq)$ is *reached* i.e. whether the family

(0 – 1 – 4) $\quad a_M(E, \leq) := \{X \mid X$ is a maximal antichain of $(E, \leq)$

has a maximum member-one of the greatest cardinality, i.e. the cardinality $p_s(E, \leq)$.

0 – 1 – 5. Obviously, for any graph (G, ρ) the corresponding notions $p_s(G, \rho)$, $a(G, \rho)$ $[a_M(G, \rho)]$ are defined in the same way and are called the independence number, the system of all [maximal] independent subsets.

0 – 2. The systems

(0 – 2 – 1) $\quad L(E, \rho)$, $L_M(E, \rho)$ of all chains resp. of all maximal chains in (E, ρ) are well defined; for the case of graphs one speaks often of *complete subgraphs* instead of subchains in the graph.

0 – 2 – 2. Remark. For every ordered set $(E, \leq)$ the empty set $\varnothing$ is considered as a subchain as well as an subantichain; in other words, $\varnothing$ is member of $L(E, \leq)$ as well as of $a(E, \leq)$.

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0 – 3. Paths.

0 – 3 – 1. Definition A *path* or *way* in graph is every well-ordered subset $W=(W_0, W_1, ..)$ such that for every W_β, $\beta>0$ the set of indices α such $\alpha<\beta$ and $W_\alpha \rho W_\beta$ is cofinal to the set of ordinals $<\beta$; thus in particular if $\beta^-<\beta$, then $W_{\beta^-} \rho W_\beta$.

0 – 3 – 2. Let $\pi(G, \rho)$ [resp. $\pi_M(G, \rho)$] be the system of all paths [resp. of all *maximal* paths] in (G, ρ).

0 – 3 – 3. Let $l(G, \rho):=\sup|X|$; the number $l(G, \rho)$ is called *path-length* of the graph.

0 – 3 – 4. Analogously the number $L_l(G, \rho):=\sup|X|$, $X\in L(G, \rho)$ may be called the chain-length of the graph.

0 – 4. Cellularity. For any system S of sets we define the cellularity cS or $\operatorname{cel} S$ as $cS:=\sup\limits_H |H|$, H running through the system of all disjoint subsystems of S; in particular, for any topological space E we define $cS:=c(GS)$, GS denoting the system of all open sets of the space E [cf. Kurepa [1935] p. 131 where cS was denoted by $p_2 S$.

0 – 5. As always when a supremum is concerned one has to examine whether the number L_l [resp. l] is reached if the graph is transfinite. The same question applies for the cellularity.

1. Cellularity of ordered chains

1 – 1. If $(C, \leq)$ is an ordered chain, then one could consider a *complete subdivision* of $(C, \leq)$ and get a corresponding tree $(T, \supset)$ of subintervals; one proves readily that

1 – 1 – 1. $\operatorname{cel}(C, \leq)=\operatorname{cel}(T, \supset)$ (cf. *Thèse* § 12; for instance Lemme 4 p. 121).

1 – 1 – 2. On the other hand, for every tree $(T, \leq)$ we defined a number $b'T$ as the supremum of cardinals $|F|$, F running through the system of all families of non radial elementary directions in $(T, \leq)$ (cf *Thèse* p. 109 § 4].

1 – 1 – 3. Now, if the rank or height $\gamma(T, \leq)$ is not cofinal to an inaccesible ordinal, then the number $b'(T, \leq)$ is reached (v. *Thèse* p. 110 Théorème 3). As an obvious corollary of this théorème 3 we have the following result.

1 – 1 – 4. Theorem. *If $(C, \leq)$ is any ordered chain such that the cellurarity $c(C, \leq)$ is not cofinal to an inaccessible number, then the cellularity $c(C, \leq)$ is reached, i.e. the chain $(C, \leq)$ contains a disjoint system of cardinality* $\operatorname{cel}(C, \leq)$ *of intervals of $(C, \leq)$. i.e. in the graph $(GC, X\cap Y=\varnothing)$ there is a maximum chain.*

The theorem 1 – 1 – 4 should be compared to the following.

1 – 1 – 5 Theorem. *The cellularity of squares of ordered chains is reached and is equal to the separability number of the chain* (*this result is implicitly contained in our papers* [1950], [1952].

As a matter of fact, it is sufficient to consider any complete bipartition D of $(C, \leq)$ (*Thèse* p. 114); if ψD denotes all elements of D of cardinality >1 each, then $(\psi D, \supset)$ is a tree of intervals in which every $X \in \psi D$ has two immediate followers X_0, X_1 such that $X_0 \leq X_1$; one has the corresponding rectangle $X_1 \times X_0$ in $(C^2, \leq)$; X running through ψD, the corresponding interiors int $X_1 \times X_0$ are open $\neq \varnothing$ and constitute a disjoint family H of cardinality $|\psi D|$; since $|\psi D| = \mathrm{sep}(C, \leq)$ one infers that

$(1-1-5-1)$ $\quad |H| = \mathrm{sep}(C, \leq)$.

On the other hand, obviously

$(1-1-5-2)$ $\quad |H| \leq \mathrm{cel}(C^2, \leq) \leq \mathrm{sep}(C, \leq)$; therefore we conclude that H is a maximum antichain in the graph $(G(C, \leq), \neq \varnothing)$ and this completes the proof ol the theorem $1-1-5$.

1 – 1 – 6. As it was pointed out in *Thèse* (cf. *Principe de réduction* P_2, p. 130) the proposition $|T| = s(T, \leq) \cdot L_l(T, \leq)$ for infinite trees is a postulate; therefore we conclude that the proposition

1 – 1 – 7. *Every infinite ordered chain* $(C, \leq)$ *satisfies*

$(1-1-8)$ $\quad \mathrm{cel}(C^2, \leq) = \mathrm{cel}(C, \leq)$

is a postulate independant of other axioms in the ZF – set theory (cf. Kurepa [1974] for references).

2. Trees $\mathcal{T}_n\ (n = 0, 1, 2, \ldots)$.

$\mathcal{T}_0$ is the empty sequence; if $n \in N$, let T_n be composed of the empty sequence and of all elements of the set

$\{a := (a_1, a_2, \ldots, a_j) \mid 1 \leq i \leq j \leq n,\ a_i \in \{0, 1, \ldots, i-1\}\}$; let $a \dashv b$ mean that a is an initial segment of b; then we have the tree $\mathcal{T}_n := (T_n, \dashv)$ with quite interesting properties. At first we have the following

2 – 1. Theorem. *If* $n \in \{1, 2, \ldots\}$ *then*

$$(2-2) \qquad |a\mathcal{T}_n| = \underbrace{(\ldots((}_{n-1}2^n + 1)^{n-1} + 1)^{n-2} + \cdots + 1)^2 + 1)^1 + 1$$

$$(2-2)_M \qquad |a_M\mathcal{T}_n| = \underbrace{(\ldots((}_{n-2}2^{n-1} + 1)^{n-2} + \cdots + 1)^2 + 1)^1 + 1.$$

Proof. $a\mathcal{T}_0 = \{\varnothing, \{\varnothing\}\}$; thus $|a\mathcal{T}_0| = 2$

$a\mathcal{T}_1 = \{\varnothing, \{\varnothing\}, \{(0)\}\}$; thus $|a\mathcal{T}_1| = 3 = |a_M\mathcal{T}_1|$.

Let $1 < n \in N$, then

$(2-3)$ $\quad a\mathcal{T}_n = a\mathcal{T}_1 \cup \{x \cup y \mid x \in [(0)_2, \cdot)_{\mathcal{T}_n},\ y \in a[(0, 1), \cdot)_{\mathcal{T}_n}\}$ (cf. § 4).

Now, if moreover $(x, y) \neq (x', y')$, then $x \cup y \neq x' \cup y'$. If $(x, y) = (\varnothing, \varnothing)$, then $x \cup y = \varnothing$. Thus $\varnothing$ is a common term – and unique one – of the two summands in $(2-3)$. Therefore considering the cardinal numbers, the formula $(2-3)$ yields

(2 – 4) $|a\mathcal{T}_n| = |a\mathcal{T}_1| + |a[(0, 0, .)_{\mathcal{T}_n}| \cdot |a[(0, 1), .)_{\mathcal{T}_n}| - 1.$ Thus

(2 – 5) $|a\mathcal{T}_n| = 2 + |a[(0, 0), .)_{\mathcal{T}_n}|^2$ because

$$|a[(0, 0), .)|_{\mathcal{T}_n} = |a[(0, 1), .)_{\mathcal{T}_n}|.$$

By a similar argument one proves

(2 – 6) $|a[(0, 0), .)_{\mathcal{T}_n}| = 1 + a[(0, 0, 0), .)_{\mathcal{T}_n}|^3,$

(2 – 7) $|a[0, 0, \ldots), .)_{\mathcal{T}_n}| = 1 + |a[(0)_i, .)_{\mathcal{T}_n}|^i$ for $2 \leq i \leq n$.[1)]

In particular, for $i = n$ we have

(2 – 8) $|a[(0)_{n-1}, .)_{\mathcal{T}_n}| = 1 + |a[(0)_n, .)|^n.$

i.e.

(2 – 9) $|a[(0)_{n-1}, .)_{\mathcal{T}_n}| = 1 + 2^n$ because $a[(0)_n, .)_{\mathcal{T}_n} = \{\emptyset, \{(0)_n\}\}.$

The elimination of the intermediary terms yields

(2 – 10) $|a\mathcal{T}_n| = 2 + (1 + (1 + \cdots + (1 + 2^n)^{n-1})^{n-2} + \cdots)^3)^2,$ i.e.

we get the formula (2 – 2).

If we try to replace the symbol a by a_M in preceding formulas, then we see that we could do it in formulas (2 – 3) – (2 – 8). Only, instead of (2 – 9) we have

(2 – 11) $|a_M[(0)_{n-1}, .)_{\mathcal{T}_n}| = 1 + 1^n = 2,$

because $a_M[(0)_{n-1}, .)_{\mathcal{T}_n}$ consists of the singleton $\{(0)_n\}$. Finally, the formulas (2 – 11) and $(2-3)-(2-8)_M$ yield the requested equality $(2-2)_M$. This completes the proof of the theorem 2 – 1.

2 – 12. Remark. It is remarkable how the formulas (2 – 2), $(2-2)_M$ are tied: subindexing with M in $(2-2)_1$ (cf. § 4) implies replacing of 2^n by 1 in $(2-2)_2$; the removing of the index M in $((2-2)_M)_1$ implies the replacing of the basis 2 in the right part of $(2-2)_M$ by the expression $2^n + 1$.

3. The systems LT_n, $L_M T_n$. Let us prove the following

3 – 0. Theorem. *If* $n \in \{0, 1, 2, \ldots\}$, *then* (3 – 1) $|L_M T_n| = n!$

(3 – 2) $|LT_n| = 1 + \sum_{r=0}^{n} 2^r r!$

(3 – 3) $|LT_n| = |LT_{n-1}| + 2^n n!$;

in particular, we have the following table:

---

1) $(0)_i := (\underbrace{0, 0, \ldots, 0}_{i})$

n	0	1	2	3	4	5	6
2^n	1	2	4	8	16	32	64
$n!$	1	1	2	6	24	120	720
$2^n n!$	1	2	8	48	384	3840	46080
$\sum_{r=0}^{n} 2^r r!$	2	4	12	60	444	4284	50364

Proof of the theorem. The proof of $(3-1)$ is immediate. Therefore let us prove $(3-2)$, $(3-3)$. At first since T_0 is the empty sequence $\varnothing$, we have

$(3-4)$ $|L\mathcal{T}_0|=|\{\varnothing, \{\varnothing\}\}=2.$

$3-4.$ Analogously

$(3-5)$ $L\mathcal{T}_1=\{\varnothing, \{\varnothing\}, \{(0)\}, \{\varnothing, (0)\}, |L\mathcal{T}_1|=4.$

In other words, the formula $(3-2)$ holds for $n=0, 1$. Since $(3-3)$, $(3-4)$ imply $(3-2)$, let us prove still $(3-3)$ for $n\in N$. Now,

$(3-6)$ $L\mathcal{T}_n \backslash L\mathcal{T}_{n-1}=\{\{x\}\cup x' \mid x\in R_n\mathcal{T}_n, x'\subset \mathcal{T}_n(.,x)\}.$

Since x is of the form $x=(x_1, x_2, \ldots, x_n)$ where $x_i\in\{0, 1, \ldots, i-1\}$, x assumes $n!$ values i.e. $|R_n\mathcal{T}_n|=n!$. On the other hand, for every $x\in R_n\mathcal{G}_n$ the set

$$\mathcal{T}_n(.,x):=\{\varnothing, (x_1), (x_1, x_2), \ldots, (x_1, x_2, \ldots, x_{n-1}\}$$

has n elements; therefore x' in $(3-6)$ assumes 2^n values; so does $\{x\}\cup x'$ as well: the system $(3-6)_2$ has just $2^n n!$ members; considering the cardinal numbers of $(3-6)_1$, $(3-6)_2$ we get precisely the requested formula $(3-3)$. Simple evaluations of $(3-2)$ for $n=0, 1, 2, 3, 4, 5, 6$ yielding the values indicated in the table, the theorem $(3-0)$ is completely proved.

$3-7.$ Second proof of the theorem $(3-0)$. At first, $(3-4)$ holds; further, if $n\in N$, $L\mathcal{T}_n$ is formed of the chains obtained by adjoining to every member of $L\mathcal{T}_{n-1}$ a single member of $R_n\mathcal{T}_n$ (remark that $R_n\mathcal{T}_n$ has $n!$ members). Now to each member $x_1\in L\mathcal{T}_1$ we adjoin each of the $n!$ members of $R_n\mathcal{T}_n$; to each member $x_2\in L\mathcal{T}_2\backslash L\mathcal{T}_1$ we can adjoin each of the $\frac{n!}{2!}$ members of $R_n\mathcal{T}_n$ following x_2; to each $0<i<n$ and to every member $x_i\in L\mathcal{T}_i\backslash L\mathcal{T}_{i-1}$ we can adjoin each of the $\frac{n!}{i!}$ members of $R_n\mathcal{T}_n$ following x_i (as a matter of fact, $|R_i\mathcal{T}_n|=i!$ and each member of $R_i\mathcal{T}_i$ is followed by $\frac{n!}{i!}$ members of $R_n\mathcal{T}_n$). In other words

$$(3-8)\quad L\mathcal{T}_n = L\mathcal{T}_{n-1} \cup \{\{x, y\} \mid x \in L\mathcal{T}_0,\, y \in R_n\mathcal{T}_n[x]\} \cup \bigcup_{i=1}^{n-1} (\cup_{x_i, y_i}\{x_i \cup \{y_i\}\},\, x_i \in L\mathcal{T}_i \setminus L\mathcal{T}_{i-1},\, y_i \in R_n\mathcal{T}_n[x_i]).$$

Now, the constituting parts in $(3-8)$ are pairwise disjoint; therefore passing to cardinalities and putting $(3-9)$ $|L\mathcal{T}| = l_i (i = 0, 1, .)$ the formula $3-8$ yields

$$l_n = l_{n-1} + l_0 n! + \sum_{i=1}^{n-1} \frac{(l_i - l_{i-1})}{i!} n!$$

and thus

$$(3-10)\quad \frac{l_n - l_{n-1}}{n!} = l_0 + \sum_{i=1}^{n-1} \frac{l_i - l_{i-1}}{i!}, \qquad (i = 2, 3, \ldots).$$

Put

$$(3-11)\quad q_i = \frac{l_i - l_{i-1}}{i!} \qquad (i = 1, 2, \ldots,);$$

thus in particular (cf. $(3-4)$, $(3-5)$):

$$(3-12)\quad q_1 = l_1 - l_0 = 2,\ q_2 = 2^2.$$

In virtue of $(3-11)$ the relation $(3-10)$ becomes

$$(3-13)\quad q_n = l_0 + \sum_{i=1}^{n-1} q_i \ \ (n = 2, 3, \ldots).$$

Therefore

$$q_{n+1} - q_n = q_n, \text{ i.e.}$$

$$(3-14)\quad q_{n+1} = 2\,q_n \qquad (n = 2, 3, \ldots) \text{ and consequently}$$

$$q_{n+1} = 2^2 q_{n-1} = 2^3 q_{n-2} = \cdots = 2^n q_{n-(n-1)} = 2^{n+1}, \text{ i.e.}$$

$$(3-15)\quad q_{n+1} = 2^{n+1} (n = 2, 3, \ldots).$$

The formula $(3-15)$ joint to $(3-12)$ yields

$$(3-16)\quad q_i = 2^i (i = 1, 2, \ldots),$$

From $(3-16)$ and $(3-11)$ we infer

$$(3-17)\quad l_i = 2^i i! + l_{i-1} \qquad (i = 1, 2, \ldots);$$

therefore

$$l_i = 2^i\, i! + 2^{i-1}(i-1)! + l_{i-2}$$

$$l_i = 2^i i! + 2^{i-1}(i-1)! + l^{i-2}(i-2)! + \cdots + 2^2\, 2 + 2^1\, 1! + l_0$$

$$l_i = 1 + \sum_{r=0}^{n} 2^r r! \quad (n = 0, 1, 2, \ldots). \quad \text{Q.E,D.}$$

4. Notations. If T is a tree, then $R_0T, R_1T, \ldots$ are its rows or levels.

If $x \in T$, then $T(., x)$ or $(\cdot, x)_T$ denotes the set of all members of T preceding x each. Dually, one defines $T(x, .)$ or $(x, .)_T$.

If r is a relation, then r_1 is its first (or left) part; r_2 is the second part of r.

1973. 06. 25.1 means Monday, June 25th, 1973.

BIBLIOGRAPHY

Erdös, P. and Tarski, A. 1943 — *On families of mutually exclusive sets*, Ann. of Math (2) **44**, 315 — 329, M. R. **4** (1943) 269.

Kurepa, Đuro, *Ensembles ordonnés et ramifiés*. Thèse, Paris, 1935, 6 + 138 + 2, idem, Publ. math. Belgrade, **4** (1935), 1—138.

— *L'hypothèse de ramification*, Comptes rendus, Paris, **202**, 1936, 185—187.

— *L'hypothèse du continu et les ensembles partiellement ordonnés*, Comptes rendus, Paris, **202**, 1937, 1196—1198.

— *Sur la puissance des ensembles partiellement ordonnés*, Comptes rendus Soc. Sci. Varsovie, classe Math. **32**. 1939, 61—67, Re-impressed in. Glasnik Math. fiz. Zagreb, **14**, 1959. 205—211.

— *La condition de Suslin et une propriété caractéristique des nombres réels*, Comptes rendus, Paris, **231**, 1950, 1113—1114.

— *Sur une propriété caractéristique du continu linéaire et le problème de Suslin*, Publ. Math. Inst. Ac. Sci. Belgrade, **4**, 1952, 97—108.

— *Sull ipotesi del continuo*, Rendiconti del Seminario Matematico, Torino, **18**, 1958—59, 11—20.

— *On the cardinal number of ordered sets and of symmetrical structures in dependence on the cardinal numbers of its chains and antichains*, Glasnik Math. fiz., Zagreb, **14**, 1959, 183—203.

— *On several continuum hypotheses*, Proc. International summer school, Varenna 1968, 57—64.

— *Around the general Suslin problem*, Proc. Intern. Symp. on Topology in Herceg-Novi 1968, Beograd 1969, 239—245.

— *Nekotorie funkcii na strukturah*. Proc. Intern. Symp. on Topology and itsapp lications, Herceg-Novi 1968, Bgd 1969, 246—248.

— *Odna posledovatel'nost' derev'ev i odna posledovatel'nost' čisel* (posvećeno L. Ilijevu za 60. rođendan), Mat. strukt. Vičislitel. mat. Mat. model. Sofija, 1975, 109—111.

— *Sur une suite de nombres naturels*, Comptes Rendus, Ac. Sci. Paris, Séries A T. **278** 1974, 467—469 Présentée 26. 11. 1973.

— *On trees or ramifications*, Math. Balk. Beograd **4**, 1974, 371—382.

Đ. Kurepa
Matematički institut
Beograd

ĐURO KUREPA

MATEMATIČKI MODELI U PRIRODNIM I DRUŠTVENIM NAUKAMA*

1. POJAVA MODELA. MODELOVANJE

Čim čovek ugleda da se neka pojava odvija, nastojeći upoznati kako se ona odvija, čovek već time stvara neki model (modelovanje!) ili sliku odgovarajuće pojave. Čovek želi da predvidi kako će se događaj u datim uslovima odvijati pa u tu svrhu i modeluje, tj. izgrađuje »modele«, »sheme«, mehanizme pojava koje ispituje .

1. 1. U naše doba, na početku 1966. godine, možemo reći da se modelovanje uopšte, a matematičko posebno, stvarno odnosno misaono sastoji u odražavanju stvarnosti na predmetima i zbivanjima, koja je čovek na zemlji prigotovio opirući se na dostignute duhovne i materijalne tekovine uopšte, odnosno u matematici posebno.

Razvojnost zbivanja, saznanja i mogućnostî imaju svoj odraz i na razvojnost modelovanja i primena . . .

1. 2. Modelovanje i globalizacija. Modelovanje se može vršiti i sa ciljem da se predmet, pojava, . . . bolje shvate kao jedno celo *(globalizacija)*. Takva globalizacija može se ostvariti bilo fizičkim predmetom, bilo shemom, bilo mrežom itd. (isp. i 8. 2.).

Pri tom se možemo služiti i takvim ostvarenjima koja su nekada međusobno dualna pa i protivurečna: tako npr. u modelima atomskih zbivanja služimo se i valovitim pojavama i dualnim korpuskularnim pojavama.

1. 3. Svaku stvar, pojavu, . . . S možemo ispitivati sa raznih gledišta i pomoću raznih pomagala: zato za S i možemo imati raznovrsnih modela: matematičkih, fizičkih, bioloških itd.

*) Članci Đ. Kurepe, E. Stipanića, J. Pop-Jordanova, S. Stankovića i A. Stojkovića su redigovani tekstovi njihovih referata sa naučnog skupa »Marks i savremenost« na temu »Marksizam i savremene matematičke, prirodne i tehničke nauke« održanog decembra 1965. u Opatiji u organizaciji Instituta za radnički pokret i Instituta društvenih nauka.

Model može biti raznovrstan: za svaki predmet, svaku strukturu, svako zbivanje i svaku pojavu može se pomišljati na manje više veran model (homomorfan model, izomorfan model ili uzor). Tako imamo modele kretanja planeta, modele kretanja elektrona u atomu, modele o strukturi materije, modele o ekonomskoj utakmici između pojedinih zajednica itd.

2. 1. Po prilici, možemo reći da se pod matematičkim modelom pojave P razumeva izgrađivanje nekog skupa M relacija čije matematičko izučavanje i obrađivanje dovodi do tačnijeg i preglednijeg upoznavanja pojave P i njenog odvijanja i odvijanja drugih pojava koje su građene poput pojave P. Naravno da model M zavisi od pojave P; to se formalno naznačuje M=M(P). Prelaz od M(P) na M(P') gde je P' pojava slična sa P može se formalno izvesti pomoću pojedinih parametara, kao što su veličina predmeta, težina predmeta, vreme itd. (isp. § 18). Sam model M(P) nije u opštem slučaju definiran jednoznačno: zato pojavi P mogu biti pridruženi razni modeli $M_1(P)$, $M_2(P)$ . . . (tipičan primer u tom pogledu je Ptolomejev model kretanja planeta i Kopernikov model, oba modela odnose se na istu pojavu — kretanje planeta sunčeva sistema).

2. 2. Model M(P) može biti sastavljen od matematičkih relacija kao što su jednačine, funkcije itd; npr. ako je napeta struna (žica) dužine l pričvršćena na svojim krajevima a onda maknuta iz položaja ravnoteže, normalni otklon z u daljini x od prvog kraja je funkcija od x i t, a dokazuje se da zadovoljava diferencijalnu jednadžbu oblika $\frac{\partial^2 z}{\partial t^2} = \omega^2 \frac{\partial^2 z}{\partial x^2}$. Pretpostavlja se da u početnom trenutku t = 0 znamo položaj žice, tj. dato je $z(x, o) = f(x)$ kao i $\left(\frac{\partial z}{\partial t}\right)_{t=o} = g(x)$. $z(0, 0) = 0 = z(l, 0)$: tada se matematičkim metodama nalazi z(x, t).

2.3. Model M(P) može biti umanjena ili uvećana slika od P, poput makete grada, kraja, pojedinog predmeta, (npr. broda) ili rasporeda elektro-energetskih izvora ili odvijanja saobraćaja manjeg ili većeg opsega itd.

2.4. M(P) može biti pribbližna matematička slika odvijanja od P; tako npr. ako P znači međusobnu isprepletenost pojedinih grana privrede u nekoj državi, onda se poslednjih godina za model M(P) uspešno uzima neki skup linearnih relacija među najglavnijim veličinama koje su u privredi P od odlučne važnosti; k tome se traži da neka posebna veličina C koja zavisi od tih osnovnih veličina bude što veća, odnosno što manja (profit i proizvodnost što veći, cena što manja, utrošak materijala što manji i sl.)

Tipičan primer takvog modela jesu tzv. linearni programi, odnosno linearna programiranja i dinamička programiranja. (isp. §3 i § 4).

2.5. Izgradnja modela pomoću aksioma. Jedna od najbitnijih matematičkih metoda je aksiomatska. Pri toj metodi uočavaju se pojedina svojstva i pojedine veze i zapisuju kao da vrede bez izuzetka u odnosu na izučavanu pojavu. Uočene veze nazivaju se aksiome ili postulati. Pomoću njih se manje više dobija uvid u odvijanje pojave ili procesa. Nekad je proces time opisan jednoznačno a nekad mnogoznačno. Tako, na primer, svojstvo grupoidnosti iskazuje se izrekom da je rezultat xoy uočene dvočlane operacije o sa članovima x, y nekog skupa G opet član istog skupa G (kaže se da je G grupoid u odnosu na 0). Kako je ta pojava vrlo česta, jasno je da model (struktura) grupe nije jedan jedini nego da se pojavljuje u vrlo raznolikim prilikama: a posebna je zadaća da se pri tom iskoriste najčešći i najbolje obrađeni grupoidi. To spada u problematiku reprezentacije ili predstavljanja modela, struktura itd. Posebno to vredi za tzv. grupe (o predstavljanju struktura, isp. moju Viša algebra, poglavlje 33 str. 1234—1261).

Realizacija pojedinih aksiomatika pomoću konkretno definisanih modela spada u užu matematičku teoriju modela.

2.6. Algebarska definicija modela. Pod modelom razumevamo uređenu dvojku proizvoljna skupa E i proizvoljna niza $\Delta = (\Delta_1, \Delta_2, \ldots)$ relacija nad E; pri tom Δ_ν označuje podskup od E^{α_ν} gde je α_ν redni broj. Tu definiciju modela možemo uporediti s definicijom *algebre*, koja iz gornje definicije izlazi tako da umesto »relacija« govorimo »operacija«. Glasovoti su modeli geometrije Lobačevskoga pomoću sredstava Euklidove geometrije (Beltramijev model, Poincaré-ov model). U današnjoj fazi matematike teorija modela je usko povezana sa aksiomatskim sistemima (isp. 2.5).

2.7. Tehnička i ekonomska definicija matematičkog modela je drukčija i taj pojam stoji umesto: matematička teorija, matematička relacija, matematička formula, matematički sistem, matematičko opisivanje i sl.

Matematički model od P znači na matematički način povezati veličine (parametre) koji se u P pojavljuju sa svrhom da se dobije pregled pojave P i bolji pogled nego što se to dobije opisivanjem rečima te da se vide međuveze pojedinih faktora koji u P ulaze, njihovu veličinu, povezanost i efektivnost (težinu), te kako se P menja pri delimičnom ili totalnom menjanju parametara koji ulaze u P; nastoje se odabrati takvi modeli koji se mogu potpunije obrađivati *numerički* koristeći efikasnije numeričke strojeve.

3. LINEARNO PROGRAMIRANJE

Linearno programiranje je nastalo 40.-tih godina našega stoleća. Danas, tj, u 1965. godini to je izgrađena matematička disciplina s vrlo brojnim primenama u industriji, proizvodnji, tehnici, vojnoj nauci itd. Linearno programiranje je ekvivalentno s teorijom tzv. *matrične igre* ili *igre sa dva igrača sume 0* od kojih svaki raspolaže *konačnim brojem* strategija. Tipičan primer linearnog programiranja je problem transporta, problem raspodele robe iz m stovarišta na n odredišta (uz minimalne troškove prevoza), raspodela n osoba na n radnih mesta uz maksimalnu proizvodnost, itd.

4. DINAMIČKO PROGRAMIRANJE

4.1. Dinamičko ili višestepeno programiranje je ono koje zavisi od vremena i redosleda učinjenih rešenja i odluka. Pri dinamičkom programiranju problem se rešava postupno, tako da svako delimično rešenje zavisi od prethodnih rešenja. Nizanje odluka može biti strogo funkcionalno, determinističko ili stohastičko, tj. određeno pomoću neke razdeobe verovatnosti.

4.2. Tipičan primer dinamičkog programiranja je uprava i izučavanje zaliha. Koliko preduzeće treba da kupi robe kako bi postiglo što veću zaradu? Pri tom, treba imati u vidu razne činjenice i običaje kao: već prisutnu zalihu, propise, pokvarljivost robe, osetljivost kupaca (ako robu ne mogu odmah dobiti), sistem naručivanja robe, trajanje isporuke itd. Taj se primer može upotrebiti da se objasne pojedini pojmovi i uoče načini rešavanja problema. Posebno se u tu svrhu može upotrebiti vrlo jednostavan slučaj problema, poznat kao problem prodavača novina ili problem inventara (Morse, Kimball; isp. G. E. Forsythe — R. A. Leibner, Matrix inversion by a Monte Carlo Method, Math. Tables Aids Computation 4(1950) 127—129).

4.3. Problem prodavača novina (Morse-Kimball, 1950; *primer za shvatanje stohastičke varijable).* Prodavalac novina proda prosečno $\lambda = 10$ komada nekih novina dnevno; kupuje ih po $b_1 = 20$ din. a prodaje po $b_2 = 30$ din; neprodani primerci su čist gubitak. Koliko primeraka, a, tog dnevnog lista treba dnevno kupovati pa da zarada bude što veća? Naravno, na prečac bi se reklo da treba kupovati po 10 primeraka! Međutim, iskustvo treba da nas pouči po prilici koliko se kupaca pojavljuje pa će stvarni broj, n, kupaca za prosek λ od mnogo posmatranja biti određen tzv. Poissonovom podelom, prema kojoj je verovatnost da će se prodati ukupno n primeraka jednaka

$$p(n,\lambda) = \frac{\lambda^n e^{-\lambda}}{n!} \text{ (isp. [1] str. 227)}$$

4.4. Analogna problematika sreće se u vrlo raznovrsnim oblastima, kao npr. pri zameni, odnosno obnavljanju strojeva u upotrebi, pri potrošnji električne energije, pri sastavljanju privrednih planova, pri izučavanju nastavnih procesa, strukturalne analize, u sociologiji, fiziologiji itd.

5. UPRAVLJANJE PROIZVODNJOM (OPERATIONS RESEARCH)

Upravljanje proizvodnjom je nova ekonomska matematička grana koja istražuje pomoću matematičkih aparata kako da se organizuju rad i proizvodnja da bi se povećala proizvodnost rada, maksimalizirao profit, minimalizirali izdaci i gubitak i postigla sigurnost u obrtu materijalnih dobara. Jedan od takvih problema je organizacija tempa proizvodnje. Linearno programiranje sa matričnim računom i dinamičko programiranje su jedna od radnih metoda u toj oblasti.

6. TEORIJA UPRAVLJANJA I OPHOĐENJA JE U SVOJIM POVOJIMA

Tako npr. *teorija hodanja* čoveka i visokorazvijenih životinja a specijalno izučavanje *uspravna hodanja čoveka* nije još potpuno objašnjeno. Postoje zanimljivi modeli i zanimljiva rasuđivanja s tim u vezi, povezana s teorijom ekstremâ funkcijâ, a posebno s teorijom *ocenskih funkcija*. Stvar predstavlja zajedničko polje rada fiziologa, inženjera, fizičara, kemičara i matematičara (isp. I. M. Gel'fand — M. L. Cetlin, 0 nekotorih sposobah upravlénija složnimi sistemami (Uspehi matem. nauk 17 (1962) 1 (103), 1—26).

7. VEKTOR PRIMER IZ TEORIJE IGARA

Vektor je prestao biti samo predstavnik usmerena puta, duži, odnosno kretanja, sile i sl. Danas vektor i matrice dolaze također posebno u ekonomskim naukama, biološkim naukama itd. pa se govori o vektoru cena, o vektoru raspoređaja tereta, o vektoru komponenata pojedinog leka, jela, o vektoru učinka u pojedinim fiziološkim radnjama, o vektoru strategije itd.

Npr. osnovna teorema o igri sa dva igrača (E. Zermelo 1912. za slučaj šaha, a opšti slučaj J. Neumann 1928) može se izreći ovako:

Neka je a realna matrica konačna formata (k, n); kada $X = [X_1, \cdots X_k]^T$ prolazi skupom svih strategija igrača I, a $x = [x_1 \cdots x_n]$ skupom svih strategija igrača II, tada je vrednost igre $v_1 = \sup_X \inf_{\nu=1,2\cdots n} \sum_i a_{i\nu} X_i$ u odnosu na igrača I jednaka vrednosit igre $v_2 = \inf_x \sup_{i=1\ldots k} \sum a_{i\nu} x_\nu$ u odnosu na igrača II; postoji »optimalna strategija $\overset{\circ}{X}$ od I i optimalna strategija $\overset{\circ}{x}$ od II« tako da bude

$$v_1 = \inf_\nu \sum_{i=1}^{k} a_{i\nu} \overset{\circ}{X}_i = \sup_{i=1..k} \sum_{\nu=1}^{n} a_{i\nu} \overset{\circ}{x}_\nu = v_2 \text{ (isp. [11] 30§ 5.8)}$$

8.1. ANALITIČKI MODELI

Analitički model zbivanja se sastoji od skupa analitičkih relacija u obliku jednakosti ili nejednakosti kojima se određuje tražena veličina. Posebno su važne diferencijalne jednačine, funkcionalne jednačine itd. Analitički modeli su bili uzorom u astronomiji i fizici makrosveta. Izučavanje pojedinih čestica dovelo je do napuštanja analitičkih modela i analitičkih metoda pa su se uveli verovatnosni i statistički modeli kojima obiluju današnja fizika, mehanika, ekonomika, biologija, društvene nauke, tehničke nauke te nauke o igrama.

8.2. T o p o l o š k i   m o d e l i,   m r e ž n i   m o d e l i. Osim analitičkih modela danas se mnogo služimo topološkim modelima, mrežnim modelima, drvetima i sl. Tu se radi o tome da se napravi shema logičkog odvijanja procesa, nizanja i međusobnog vezivanja pojedinih faza procesa. U većim industrijskim, vojnim i sl. pogonima traže se prethodno mrežni programi i planovi pre no što se pristupi arhitektonskom planu i programu izvođenja same gradnje. (isp. knjigu: Ing. J. Brandenberger Ing. R. Konrad, Netzplantechnik, Eine Einführung, Bearbeitung: Zürich 1965, Verlag Industrielle Organisation, pp.222).

9. MATEMATIČKI MODEL PREMA ORIGINALU

Matematički model u vezi sa stvarnosti treba zadovoljavati nekim uslovima; npr. da su na snazi ovi uslovi:

1) M o d e l j e o s t v a r l j i v, tj. ima bar jedno rešenje; također se kaže da model treba »da radi«.

2) postoji jedno *jedino* rešenje.

3) Postoji jedno jedino *stabilno* rešenje. Stabilnost se podrazumeva u smislu da male promene parametara, podataka, ... izazivaju male promene u rešenju.

Uslov stabilnosti je u vezi sa činjenicom da se rešenja traže približnim metodama pa zato nađeno »približno« rešenje mora biti »blizu« pravog rešenja; osim toga, podaci (parametri) dobivaju se eksperimentalno, dakle približno, pa zato ne smeju davati rešenja koja bi previše odudarala od pravih, istinskih rešenja u vezi s prirodnim događajima.

10. PUT OD POJAVE DO MODELA

Kad se dana vrsta predmeta, struktura, pojava želi izučavati pomoću matematičkog modela, onda se može postupiti ovako:

10.1. Intuitivne slike i predožbe o zbivanju pojave nastoje se precizirati ističući najvažnije komponente, momente i veze koje se pri tom uočavaju; pri tom se, bar za prvi početak, zanemaruju druge veličine koje su od nebitnog ili vrlo retkog karaktera; tako se dolazi do pojedinih matematičkih veza među najvažnijim uočenim veličinama.

Tako npr. pri stohastičkim procesima zanemarujemo događaje male verovatnosti i praktički radimo kao da se takvi događaji ne događaju, odnosno da se događaju vrlo retko (Buffon — Cournotov princip). Uočene, odnosno dobivene veze mogu biti već poznatog oblika i tipa, ali mogu biti i novog oblika, pa čak i nosioci novih načina obrađivanja, računanja i novih zakonitosti.

10.2. Učinjene korake treba obrazložiti i povezati te tako doći do željenog, odnosno traženog (nepoznatog) modela, relacije, formule, obrasca, ...

10.3. Izraditi metodiku kako da se model reši (numerički, grafički, mašinski) i

10.4. Izgraditi pomoćna sredstva, mašine, ... za stvarno rešavanje problema.

Stvarno rešavanje problema i način rešavanja zavise također od pomoćnih pribora i tehničkog alata.

Poslednji korak 10.4. je osobito dobio u važnosti od kako su u upotrebi elektronska računala. Time se automatski vanredno raširila oblast matematičkih primena i povećao broj ljudi koji se služe matematikom, odnosno dolaze u vezu s matematičarima.

Pri samoj izgradnji matematičkog modela vrlo je korisna saradnja između matematičara i stručnjaka iz oblasti u koju spada izučavani predmet koji modelujemo. Ta je saradnja korisna da matematičke veze na modelu adekvatno što bolje odgovaraju zbivanjima iz stvarnosti na ispitivanom originalu. Jedan od načina današnjeg naučnog rada sastoji se u ekipnom radu stručnjaka iz raznih i raznorodnih oblasti! Setimo se pri tom kako je nastala npr. kibernetika kao zajedničko delo matematičara, biologa i inženjera.

11. POJAM REŠENJA, EGZISTENCIJA REŠENJA

Funkcionalne jednačine vrlo raznolikih tipova (diferencijske, diferencijalne, integralne, integralno-diferencijalne itd.) jesu primeri važnijih matematičkih modela prirodnih i društvenih pojava. Vrlo je važno ispravno definirati što se podrazumeva pod rešenjem danog modela. Tako npr. kod linearnog programiranja rešenja su neodrečni vektori, tj. vektori sa komponentama koje nisu negativne.

Dalji je problem da se osigura postojanje pa i jedinost rešenja pod danim uslovima.

Kao što smo naveli, obično se zahteva da nađeno rešenje bude stabilno (isp. § 9.3.).

12. EKSTREMALNA REŠENJA. OPTIMALIZACIJA

Traženje ekstrema je vrlo stara i vrlo plodna matematička problematika koja je metodički razrađena i teoretski prostudirana. Vrlo velik broj prirodnih i društvenih pojava obrađuje se matematički tako da se traže ekstremalne vrednosti pojedinih funkcija (uslov stacioniranosti).

Posebno se danas, mašinski, traženje ekstrema smatra kao jedna od osnovnih *prvotnih* zadaća na koju se mnoge druge svode. Tako npr. ne samo rešavanje linearnih jednačina nego i npr. teorija automatičnog upravljanja — automatična optimalizacija — rešavaju se rešavanjem ekstremalnih zadaća. Evo kako to izgleda za linearne algebarske sisteme.

13. POMOĆNA SREDSTVA I MODELI. PRIMERI

13.1. Linearni sistem $a\,x = b$ rešava se na analog-strojevima tako da se traži minimum funkcije

$$f(t) = f(x_1, .., x_n) = \sum_{\nu=1}^{n} \varepsilon_\nu^2 = \sum_{\nu=1}^{n} (a_{\nu k}\, x_k - b)^2; \; x_k \text{ zavisi od } t$$

Pri tome je t nezavisna promenljiva. Možemo pretpostaviti da f(t) opada: $\frac{df}{dt} < 0$.

Taj uslov zbog $\frac{df}{dt} = \sum_i \frac{\partial f}{\partial x_i} \frac{dx_i}{dt}$ i zbog $\frac{\partial f}{\partial x_i} = 2 \sum_j \varepsilon_i a_{ij}$

znači da možemo uzeti $\frac{dx_j}{dt} = - \varepsilon_j$.

Time gornji uslov postaje $-2 \sum_{\nu=1}^{n} \sum_{k=1}^{n} a_{\nu k} \varepsilon_\nu \varepsilon_k < 0$.

Takvi se uslovi lako ostvare na analog-mašini; zato će f težiti prema 0 a x će težiti prema traženu rešenju.

Drugim rečima, linearni sistemi algebarskih jednačina svode se na kvadratne forme i na linearne diferencijalne jednačine. Dakle, obratno nego što se radilo pri računanju olovkom i papirom.

Tako vidimo da prvotnost i jednostavnost matematičkih problema i radnji zavisi od upotrebljenih tehničkih pomoćnih sredstava i metoda.

Evo dalje primera u tom pogledu koji nam kazuje da integracija može biti prvotnija i od množenja.

13.2. Osnovne operacije u Bushovu diferencijalnom analizatoru (1920) su sabiranje (+, izvršeno pomoću diferencijalnih kotača) i integracija (pomoću kola na vretenu).

Množenje konstantom se vrši pomoću nizova diferencijalnih kotača; množenje varijabilnih veličina vrši se po obrascu

$u\,v = \int u\,dv + \int v\,du$ ili

$u\,v = 1/2\,(u + v)^2 - 1/2\,(u - v)^2$, pri čemu je $x^2 = \int 2x dx$.

Funkcije se zadavaju pomoću diferencijalnih jednačina ili pomoću grafova; grafovi se očitavaju bilo veštim obilaskom krivulje ili (u novije vreme) pomoću fotoelektričnih hodala po krivulji (curve follower). U analognim strojevima, varijabla j eistinsko fizičko vreme (naprotiv, u brojčanim strojevima, varijable su raznih drugih tipova).

13.3. Slučajnost u službi preciznosti i zakonitosti. Može izgledati paradoksalno da nizanje slučajnih veličina može dati pravu zakonitost pojava, vrednosti itd. Na tome se i bazira naziv Monte Carlo-metoda za numerička određivanja veličina pomoću tablica slučajnih veličina.

Tako se mogu tražiti približna rešenja sistema algebarskih jednačina, inverzne matrice zadanih matrica itd. Na taj način prividna nezakonitost, proizvoljnost i neposlušnost izgraditelj su reda, zakonitosti i pravilnosti.

Tako matematika doprinosi da se i slučajnost, proizvoljnost i prividni nered u individualnom ili malom iskorištava za izgradnju zakonitosti u velikom. Po stepenu i broju korištenja i takvih situacija kao i po korišćenju što sitnijih no vrlo brojnih aktivnih učesnika — agenasa — (ljudi, životinje, pčele, atomi, molekule itd.) ogleda se visina napretka i prodora čoveka u otkrivanju prirodnih pojava i zakona.

14. MAŠINSKA MATEMATIKA I MATEMATIČKI MODELI

Ona je zajedničko delo matematičara, fizičara i inženjera.

Njena osnova je matematička logika, matematička teorija strojeva i informacija; ona prerasta u kibernetiku koja je zajednička oblast matematičara, biologa, inženjera i dr.

Mašinska matematika omogućuje razvijanje matematike i na eksperimentalnoj bazi; s druge strane, mašinska matematika raširuje zahvat matematike na nove oblasti i probleme u astronomiji, biologiji, ekonomici, inženjerstvu, vojnoj nauci, administraciji, . . . u kojima matematika pre nije mogla efektivno delovati.

Metode rešavanje zadataka u mašinskoj matematici drukčije su nego u matematici u kojoj se radilo samo olovkom i papirom. Tako npr. traženje ekstremalnih vrednosti funkcija nekada se svodilo na rešavanje raznih sistema jednačina; u mašinskoj matematici put je upravo obrnut! (isp. 13. 1).

15. ALGORITAM

Pod algoritmom se razumeva postupak kako se od datih podataka dolazi do tražena rezultata u konačno mnogo određenih koraka. Postoje vrlo raznorodni algoritmi kao npr. Euklidov algoritam za traženje najvećeg zajedničkog faktora dane dvojke prirodnih brojeva ili algoritam za rešavanje datog skupa linearnih jednačina itd.

U današnjoj matematici važno je da se svaki algoritam može ostvariti strojem za računanje, odnosno pomoću rekurzivnih funkcija.

16. DETERMINISTIČKI MODELI. STOHASTIČKI (VEROVATNOSNI) MODELI

Deterministički modeli jesu oni kod kojih su sve veze čisto funkcionalne u klasičnom smislu. Pri stohastičkim, statističkim ili verovatnosnim modelima postoje i veze koje nisu strogo određene u klasičnom smislu nego zavise i od slučajnih ili stohastičkih veličinâ i razmatranjâ. Deterministički modeli su odigrali znatnu ulogu u matematičkom izučavanju makrosveta. Izučavanje mikrosveta, bioloških, društvenih i slučajnih pojava vezano je za stohastičke modele! Tako npr. izbor uzorka i kontrola kvaliteta proizvoda u proizvodnji pomoću uzorka je primer stohastičkih modela i razmatranja. Shvatanje učenja kao stohastički procesi dalji su primeri nedeterminističkih modela. Teorija obuke može se uklopiti u teoriju automata ili robota za koju je vezana problematika o prenošenju informacija, teorija relacija i kibernetika (s tim u vezi spomenimo posebno da je mnogo izučavan i izučava se model mozga, odnosno razni modeli mozga).

Takođe su izučavani modeli rada srca; tako npr. I. M. Gel'fand i M. L. Cetlin su 1961. promatrali neprekidne mreže automata i pomoću njih ostvarivali modele srca.

17. MODEL KAO SREDSTVO (PRIMENE MODELA). MODEL KAO PREDMET

Modeli su nastali u prvom redu kao pomoćna sredstva za objašnjavanje, predskazivanje, primene itd.

Međutim, modeli se izučavaju i kao predmet i to kao samostalan predmet ali i sa gledišta veza sa drugim predmetima, posebno sa drugim modelima. Oba ta aspekta modela posebno su važna u matematici (teorijska matematika-primenjena matematika). S modelom se mogu vršiti razni eksperimenti i izučavanja pa je pogotovo važno izučavati ona svojstva modela koja se bez samog nosioca toga svojstva ne mogu u stvarnom svetu ostvariti fizički odvojeno, samo zasebe.

18. ČETIRI FAKTORA PRI MODELIRANJU (ISP. § 1.)

18.1. Model zavisi od više faktora kao što su npr.

M_1. P a s i v n i f a k t o r i. Stvar, pojava, proces, ... koje hoćemo prikazati.

M_2. A k t i v n i f a k t o r i: jedinke koje prave modele; jedinke su živa bića, strojevi, ...

M_3 Sredstva pomoću kojih se model gradi.

Uz M_1 posebno su važni i aktivni faktori M_2, jer od stepena razvitka aktivnih faktora vrlo mnogo zavisi šta, kako, gde i zašto se modelira, u izgradnji modela raznih pojava ogleda se stvaralačka i duhovna snaga izgraditelja modela.

18.2. Naši dosadašnji modeli o prirodnim i socijalnim zbivanjima bili su vezani za naše dosadašnje zemaljsko iskustvo; posve je sigurno da će naše modelovanje doživeti velikih obogaćenja kad ga budemo sprovodili u vreme kad budemo normalno i fizički saobraćali sa Mesecom, planetama sunčanog sistema, zvezdama itd. Na taj način specijalno ćemo imati npr. modelovanje s aspekta Zemlje, Meseca, Venere, . . . Zvezde itd. pa gornjim faktorima M_1, M_2 M_3 dolazi još i faktor

M_4. Položaj u svemiru gde se model izgrađuje.

19. SPOZNAJNO ZNAČENJE MATEMATIČKIH MODELA

Činjenica da se pojedine pojave javljaju u raznorodnim slučajevima jest izvor matematike i matematičkih modela. Zapravo, postojanje modela je povezano s postojanjem matematike a jedno i drugo odražavaju jedan vid stvarnosti: jedinstvo stvarnosti u smislu međusobne zavisnosti i međusobnog prožimanja. Model je poseban odraz veze deo-celina pa se na to nadovezuje problematika posebno-opšte. Kako su modeli uvek približna slika stvarnosti, ne može se stvarnost svesti na izučavanje određene vrste modela koji su izgrađeni u datom periodu. Matematički model je poput skulpture: kao što kip neke osobe O predstavlja donekle tu osobu, tako i model u nauci predstavlja predmet, strukturu, zakonitost zbivanja. Kao što kip osobe O nije isto što i sama ta osoba i nikada ne može biti isto, tako i model zbivanja, realnosti, . . . nije isto i ne može biti isto što i samo zbivanje, odnosno sama realnost u ime koje je model izgrađen. Međutim, sve nova i nova dostignuća u izučavanju stvarnog sveta i pojava prožeta su jednovremenom izgradnjom i izučavanjem sve novih i novih matematičkih modela. Zato se može govoriti o uspoređivanju modela međusobnom kao što se govori o pridruživanju modela i stvarnih i misaonih pojava. Govori se o raširivanju, odnosno o sužavanju modela, teorije, o modelu koji će obuhvatiti date modele kao posebne slučajeve itd. Govori se o strukturi modela kao što se govori i o strukturi zbivanja; govori se o modelima te i te strukture; govori se o klasifikaciji, hijerarhiji i međuvezama modela kao što se govori o klasifikaciji, hijerarhiji i vezama među strukturama.

Zanimljivo je da se na modele i strukture sve više prenose terminologija i shvatanja u vezi s biotičkim rasuđivanjima a ne u vezi s abiotičkim anorganskim rasuđivanjima (npr. »model radi« — umesto »model ima rešenje«; porodica, operacija, grupa, telo, struktura itd. naslednost ... uobičajeni su nazivi u teoriji matematičkih modela i u matematici).

Izučavanje modela, ostvarivanje modela i veze među modelima te veze model-stvarnost, primene ... jedan je od putova — možda i jedini put-u spoznavanju zakona prirodnih i društvenih pojava; izučavanje modela je primer odnosa pojedinačno (individualno) — svojstveno (egzistencijalno) — generalno (univerzalno). S tim u vezi stoje kvantori (kvantifikatori) s jedne strane a primene sa druge strane.

Dijalektičko jedinstvo i međuprožimanje model-stvarnost vrlo je plodan put u sve obuhvatnijem i sve svestranijem i dubljem otkrivanju zakonitosti u prirodnim i društvenim pojavama. Zakonitosti o vezama: kvalitet-kvantitet, forma-sadržina, usmerenost-evolucija, raštenje-entropija ... primeri su rasuđivanja o modelima, pojavama i spoznanju.

U vezi sa spoznajnim značenjem (matematičkih modela) citiramo ove Lenjinove rečenice:
„Чтобы действительно знать предмет, надо охватить, изучить све его стороны, све связи и „опосредствования". Ми никогда не достигнем этого полностью, но требование всестороности предостережет нас от ошибок и от омертвения. Это во -1-х. Во -2-х, деалектическая логика требует, чтобы брать предмет в его развитии, „самодвижении" (как говорит ингода Гегель) изменении. (Ленин, *Сочинения*, Т. 32, стр. 72.)

LITERATURA

Beckenbach E. F. Modern mathematics for the engineer. New York—Toronto——London, 1956, 20 + 514.

Bellman R. Dynamic Programming. Princeton, 1957, 26 + 340.

А. И. Берг, М. Б. Новик, Б. В. Бирюков, А. Г. Спиртин, И. В. Кузнецов. Кибернетика, Мышлены и жизнь, 1964, стр. 512.

Boulding K. E. — Spivey W. A. Linear of firm. New York, 1960, 10 + 228.

Bush R. — Mosteller Fr. Стохастические модели обучаемости. Москва, 1962, 484 (prevod sa engl.).

Cundy E. — Rollett A. P. Mathematical models. Oxford, 1961, 2 + 286.

А. А. Фелдбаум. Основы оптимальных автоматических систем, Ф. П., М., 1963, 552.

Fréchet M. Les mathématiques et le concret, Paris, 1955, 8 + 438.

Карлин С. Математические методы в теории игр, програмирований и экономике. Москва, 1964, 840, (prevod sa engl.).

Koopmans T. C. Statistical inference in Dynamic economic models. New York — London. J. Wiley, 1960, 16 + 438.

Kurepa Đ. Viša algebra. Zagreb, 1965, T. I, 32 + 764. T. II, 20 + 765—1380.

Petrović M. Elementi matematičke fenomenologije. Beograd, 1911, 11 + 774.

Л. С. Поптрячин, В. Г. Болтянский Р. Б. Гамкрелидзе, Е. Ф. Мищенко. Математическая теория оптимальних процесов, Ф. М., М., 1961, 392.

Применение математических методов и ЭВМ в экономических иследованиях. Ташкент, 1965, 280 .

А. Васони. Научное програмирование в промишленостьи и торговле. Москва, 1963, 388.

ĐURO KUREPA

MATHEMATICAL MODELS IN NATURAL AND SOCIAL SCIENCES RESPECTIVELY

Summary

ARISING OF MODELS. MATHEMATICAL MODELLING.

As soon as we observe a thing, a phenomenon and try to know *how the thing is happening,* we make a model, a scheme of the thing . . . , *we are modelling* . . .

1.1. Nowadays, at 1966, we might say that modelling in general and mathematical modelling in particular consists in reflecting of the reality on objects and phenomena reached by menkind on Earth and based on spiritual and matterial acquisitions in general and in mathematics particularly. Evolution of events, of knowledge and of possibilities is reflected on the evolutionness of modelling and of applications.

1.2. Modelling and globalization. Sometimes, an aim of modelling P consists to consider P as a whole; this might be performed by physical tools, by schemes, networks etc.

2.1. Roughly speaking, a mathematical model of a phenomenon P is any set M of relations, the mathematical study of which yields a better understanding of P and of similar phenomena;M depends of P but need not be uniquely determined by P.

2.2. Modelling as setting up of equations, inequalities, functions . . .

2.3. Modelling of P as similar image of P.

2.4. M might be a math. approximation of a process P . . .

2.5. Modelling by means of axioms . . . consists in observation of connexions between parts, or connexions of phases in sequential processing . . . and in the mathematical and logical examinations of such ties . . . E. g. a groupoid is a modelling of very various situations and phenomena.

2.6. Algebraic definition of a model reads as any ordered pair (E, Δ) of any set E and of any sequence $\Delta = \Delta_0, \Delta_1, . . \Delta_\nu)$ of relations on E; here Δ_ν donotes any subset of the cube E^{α_ν} (α_ν being any ordinal number, finitre or transifinite).

2.7. Technical or economical definition of a math. model means simply: any mathematical theory, mathematical relation, formula, mathematical system, mathematical description, etc.

3. *Linear programming.*

4. *Dynamic programming.*

5. *Operations research.*

6. *Theory of managing, upright walking* of men and of high-developped animals as examples of modelling.

7. *Vector. Example of game theory.*

8.1. *Analytical models* (by means of functional equations ...) muster-models for macro-world.

8.2. *Topological models. Network models:* scheme of logical sequencing and connections between various phases, etc.

9.*Mathematical model and its original.* Any math. model should satisfy some conditions: 9.1 to be *feasible,* to work, i.e. to admit of one solution;

9.2. To have solely one solution;

9.3. To have a unique *stable* solution.

10. *Way from phenomenon to model;* steps: 10.1. Intuitive step and rouph relations and approximations (example: Buffon-Cournot principle); 10.2. Deeper study of relations, data ... of the item 10.1; 10.3. Define the methods how the model should de resolved numerically; 10.4. Set up of real tools how to find solutions (electronical calculating devices are to be stressed particularly). Cooperation between various branches and people is particularly welcome and needed (example: arising of Cybernetics).

11. *Notion of solution. Existence of solution.*

12. *Extremal solutions. Optimalisation.*

13. *Working tool, devices, methodes.* Examples. Methods depend on tools (and vice versa). 13.3. Randomness in service of precision and of law.

14. *Machine mathematics and mathematical models. Experimental approach of mathematics.*

15. *Algorythm.*

16. *Deterministic models. Stochastic (probabilistical) models.*

17. Model as a mean (applications of models). Model as a studied independent item (An aspect of theoretical and applied mathematics).

18. *Four components in modelling.* Any model depends on various factors:

M_1 *Passive factors* (thing, phenomenon, process which are to be modeled);

M_2 *Active factors* (factors which perform models like beings, automata, etc);

M_3 *Means and material serving to produce a model;*

M_4 *Location in the Universe* where a model is to be produced. The item M_4 should de considered particularly because of great recent successes in cosmonautics.

19. *Theoretical and cognition importance of mathematical models.* The fact that various phenomena and laws occur in very different situations is the very source of mathematics and of mathematical modelling. The very existence of models is tied with the existence of mathematics and either of them reflect an aspect of reality: the unity of real world in the sense of mutual interdependence and of mutual interactions. A model is a specific reflecting of ties *Part-Whole* and therefore is connected with the problematics *Particular-General.* Since the models are always only an approximative image of real situations, the study of reality is not exhausted by studying of any kind of models that are obtained at any given timeinterval. A mathematical model is like a sculpture: like a statue of any person O is not the same as O and never shall be so, so any model M of a proces or, of a situation X is not the same as the real original X, in name of which M originated.

Now, ever new results in the study of real world and of phenomena are penetrated by simultaneous constructing and studying of ever new mathematical modellings. And one is allowed to speak of *comparison and ordering of models* as well as of *correspondence between models* and real or thoughtful objects. One speaks of *extensions* as well as of *restrictions of models,* of theories; one considers models to comprehend some given models etc. One speaks of a structure of models like of structure of some phenomena; one studies models of such and such structures; one speaks of classification, hierarchy and interconnections of models and of those of structures, respectively.

It is to be observed that for models and structures one uses the more and more terminology and considerations on *biotic* items and not on abiotic, inorganic statements (e. g. *»model is working«* instead of *»model has a solution«;* family, operation, group, field, structure, heredity . . . are commun terms in the theory of mathematical models and in mathematics. Studying of models, constructing of models, connections between models, links: *model-reality-applications,* . . . is one of the ways maybe the single one-in cognition of laws in natural and social phenomena; the studyng of models is an example of relations *Particular-General Single*. Dialectic unity and interpenetrations *Model-Reality* are a very fruitful method in a the more and the more extensive and the more and the more universal and deeper discovering of lawfulnesses in natural and social processes. Lawfulnesses concerning the relations: quality-quantity, form-content, orientability-evolution, growing-entropy, ... are examples of considerations concerning: models, situations and cognition.

In connection with cognital importance of mathematical models let us quote the following Lenin's sentences:

»At first, in order to know really an item it is needed to comprehend and to study: all its aspects, all connections and »interferences«. We shall never reach it completely, but the request for all-sideness and for universality warns us of errors and of numbness. Secondely, dialectical logic requests to consider any item in its evolution, »self-movement« (as spoke Hegel), in its changing (Lenin, Works, T. 32, 71 — 73).