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CHARACTERIZATION OF CONDITIONALLY COMPLETE POSETS

Milan R. Tasković

Abstract. A complete lattice L has the property that every increasing function on L to L has a fixed point. Tarski raised the question whether the converse of this result also holds. The affirmative answer was given by Anne Davis, getting by it one characterization of the completelly ordered lattice. The analogous problem for conditionally complete poset has remained largely unexplored. The purpose of this paper is to characterize conditionally complete poset with the fixed point property.

1. Introduction

We shall use the notation as in [5]. Let $(P, \leqslant)$ be partially ordered set. For $x, y \in P$ and $x \prec y$, the set $]\, x, y\, [$ is defined by

$$]\, x, y\, [:= \{t : t \in P \text{ and } x \prec t \prec y\}.$$

An order-preserving (isotone or increasing) map f of a partially ordered set P to itself has a *fixed point* if there exists an element p in P such that $f(p) = p$. P is said to have the *fixed point property* if every isotone map of P into itself has a fixed point. The first of the fixed point theorems for partially ordered sets goe back to Tarski and Knaster (cf. [2]), who proved that the lattice of all subsets of a set has the fixed point property. In the mid — 1950 Tarski's [5] published a generalization: *Every complete lattice has the fixed point property.*

Tarski [5] raised the question whether the converse of this result also holds. Davis [1] proved the converse: *Every lattice, with the fixed point property is complete.*

The analogous problem for *conditionally complete* (that is, every nonempty subset of P with upper bound has its supremum) partially ordered sets has remained largely unexplored. At present there are no known necessary and sufficient conditions on a conditionally complete partially ordered set in order that it have the fixed point property. The purpose of this paper is to characterize conditionally complete partially ordered sets with the fixed point property.

It should be pointed out that the result of Davis [1] cannot be transferred to the completely ordered sets that are not lattices. That is seen from the following example.

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Example 1. Let the set $P=\{a, b, c\}$ be ordered by $\leqslant$ so that $a\leqslant b$, $a\leqslant c$ and assume the elements b, c are uncomparable; as shown on the diagram (Fig. 1.):

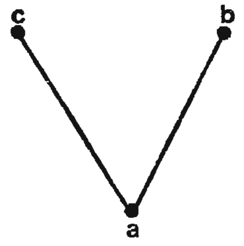

Fig. 1

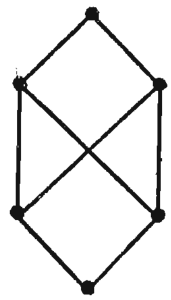

Fig. 2

Every isotone mapping $f: P\to P$ has the fixed point, but yet P is still not completelly ordered set. However, P is conditionally complete.

We begin with a statement for conditionally complete sets.

Theorem 1. (Fixed Point Lemma). *Let* $(P, \leqslant)$ *be a partially ordered set and* f *a mapping from* P *into* P *such that:*

(A) f *is an isotone mapping,*

(B) f *has a fork* i.e. $a\leqslant f(a)\leqslant f(b)\leqslant b$ *for some* $a, b\in P$,

and

(C) *The set* $]\,a, b\,[\subset P$ *is a conditionally complete.*

Then

(1.1) *The set* $I(P,f):=\{x\in P: f(x)=x\}$ *is nonempty,*

and

(1.2) *Neither of the conditions* (A), (B), (C) *can be deleted if* (1.1) *is to be valid.*

Proof. (1.1) Since $a\leqslant f(a)$ for some $a\in P$, then we have $a\leqslant f(a)\leqslant f(f(a))\leqslant \leqslant\ldots\leqslant f(b)\leqslant b$. Hence, the set S of elements $x=f^n(a)\in\,]\,a, b\,[$ for $n=0, 1, 2, \ldots$, such that $x\leqslant f(x)$ is nonempty and bounded from above, and $s=\sup S$ exists, by conditionally, completeness of $]\,a, b\,[$. Since $f: P\to P$ is isotone and $x\leqslant s$ for all $x\in S$, $x\leqslant f(x)\leqslant f(s)$ for all $x\in S$; hence $s=\sup S\leqslant f(s)$. Since f is isotone, it follows that $f(s)\leqslant f(f(s))$, whence $f(s)\in S$. But this implies $f(s)\leqslant s$, since $s=\sup S$ We conclude $s=f(s)$, i.e. $s\in I(P,f)$, and other $I(P,f)$ is a nonempty. This completes the proof of (1.1).

(1.2) Now we prove that the conditions (A), (B) and (C) cannot be removed.

Example 2. Let the poset (interval) $P=[0,2)$ be ordered by the relation order $\leqq$ (totally ordered) and define $f: P\to P$ by $f(x)=x/2+1$ for $x\in[0,2)$. The condition (C) is satisfied (the set P is a conditionally complete set), condition (A) is satisfied (f is a isotone mapping), but condition (B) is not satisfied. Furthemore, f does not have a fixed point.

Example 3. Let the poset $P=:R\setminus\{0\}$ (R denotes the set of all real numbers) and define $f: P\to P$ by $f(x)=x/2$, where P is a totally ordered set by ordinary ordering $\leqq$. The conditions (A) and (B) are fulfilled, but condition (C) is not satisfied (the set $P=\,]a, b\,[=R\setminus\{0\}$ is not conditionally complete), and f has not fixed point.

Example 4. Let the poset $P=R$ and define $f:R\to R$ by $f(x)=-x\ (x<0)$ and $f(x)=x-1\ (x\geqq 0)$. Then, conditions (B) and (C) are fulfilled, but condition (A) is not satisfied. Furthermore, f does not have a fixed point. This completes the proof of (1.2).

The following well known fixed point theorem of Tarski [5], is readily derived from Theorem 1.

Corollary 1. (Tarski, [5]). *An increasing mapping from a nonempty complete lattice into itself has a fixed point.*

Proof. Let a be the minimum of complete lattice L. Obviously, $a\leqslant f(a)$ for every increasing mapping f from L into L i.e. f has a fork. But then the conclusion of the statement follows from Fixed Point Lemma.

On the other hand, an immediate corollary of the preceding statement and of proof for Fixed Point Lemma is:

Corollary 2. *Let X be a nonempty conditionally complete set. Let f be an increasing mapping from X into X with fork. Then f has a fixed point.*

Corollary 3. (Kurepa, [3]). *Let $(P, \leqslant)$ be a conditionally complete set and let $f: P\to P$ be an increasing mapping such that $f(X)\subset X$ for every $X\subset P$. Then every segment $[a, b]$ of P contains a fixed point of f or f permutes two points of this segment.*

Proof. Let us consider the case that $a\leqslant b$. Then, from the conditions of statement, $a\leqslant f(a)\leqslant f(b)\leqslant b$ i.e. f has a fork. Therefore, from the Fixed Point Lemma, f has a fixed point in $[a, b]$. If the elements a, b are uncomparable, then f permutes two points of segment i.e. $\{f(a), f(b)\}=\{a, b\}$, and the function f is a permutation (indentical or non indentical).

2. The main result

With the help of Theorem 1 we now obtain the main result of this paper:

Theorem 2. *Let $(P, \leqslant)$ be a partially ordered set and suppose if x, y are upper bounds of a bounded subsets X of P, then there is an upper bound z for X such that $z\leqslant x$ and $z\leqslant y$. For set P to be conditionally complete it is necessary and sufficient that every increasing function $f: P\to P$ with fork have a fixed point.*

We note, that the partially ordered set P on Fig. 2. is not conditionally complete, however, every isotone mapping $f: P\to P$ has the fixed point. This prove that the condition: if x, y are upper bounds of a bounded subsets X of P, then there is an upper bound z for X such that $z\leqslant x$ and $z\leqslant y$; cannot be removed in preceding statement.

Proof. The necessity follows from Theorem 1. It remains to prove the sufficiency. In other words, we have to show that, *under the assumption that the set P is not conditionally complete, there exists an increasing function f on P to P such that* (B), *without fixed points.*

Suppose that the set P is not conditionally complete. Then there exists a nonempty part U of P which is bounded from above and has not its supremum. Let us denote by V the set of all upper bounds of U. The sets U and V are nonempty, U has no supremum, and V has no infimum. Clearly $\inf V$ does not exists, for if it did, it would coincide with $\sup U$, what contradicts the supposition that U has no supremum. One can show that there exist sequences (generalized) $\{x_\alpha\}$ in U and $\{x_\beta\}$ in V such that:

(1) $\{x_\alpha\}$ is increasing and, for each $t \in U$, there exist $\alpha(t)$ such that $\alpha(t) \prec \alpha$ implies $t \leqslant x_\alpha$, and

(2) $\{x_\beta\}$ is decreasing and, for each $t \in V$, there exists $\beta(t)$ such that $\beta(t) \prec \beta$ implies $x_\beta \leqslant t$.

To define $f: P \to P$ for any element $x \in P$, we distinguish two cases depedent upon whether x is a lower bound of $\{x_\beta\}$ or not. In the first case, by (1) and (2), if x is not an upper bound of $\{x_\alpha\}$ then

$$f(x) = \min \{x_\alpha : x_\alpha \not\leqslant x\}, \tag{3}$$

where $a \not\leqslant b$ will be used to express the fact that $a \leqslant b$ does not hold.

In the second case, we let

$$f(x) = \min \{x_\beta : x \not\leqslant x_\beta\}. \tag{4}$$

Thus we have defined a function f on P to P. From (1) — (4) it follows clearly that either $f(x) \not\leqslant x$ or $x \not\leqslant f(x)$ for every $x \in P$; thus f has no fixpoints, and also then (B) holds.

Let x and y be any elements of P with $x \leqslant y$. If x is a lower bound of $\{x_\beta\}$ but y is not, then, by (1) — (4), $f(x) \leqslant f(y)$. If both x and y are lower bounds of $\{x_\beta\}$ we see from (2) and (4) that $f(x) \leqslant f(y)$. Finally, if x is not a lower bound of $\{x_\beta\}$, then y is not either, and by an argument analogous to that just outlined (using (3) and (4)) we again obtain $f(x) \leqslant f(y)$. Thus the function f is increasing, and the proof of the theorem is complete.

Special cases of theorem 2 have been discussed by Davis [1] and some others.

Corollary 4. (Davis, [1]) *For a lattice $(L, \leqslant)$ to be complete it is necessary and sufficient that every isotone function $f: L \to L$ have a fixed point.*

Corollary 5. *If $(P, \leqslant)$ is a partially ordered set and if every isotone function $f: P \to P$ has a fixed point, then every maximal chain of P is a complete set.*

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KARAKTERIZACIJA USLOVNO KOMPLETNIH SKUPOVA

Milan R. Tasković

Karakterizaciju kompletnih mreža, kao što je poznato dala je Anne Davis, 1955., odgovarajući na jedno pitanje Alfreda Tarskog. U ovom radu data je, u terminima fiksne tačke, karakterizacija jedne šire strukture uređenih skupova, tzv. uslovno kompletnih skupova.

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AN EXTREMAL PROBLEM FOR POLYNOMIALS WITH NONNEGATIVE COEFFICIENTS. II

Gradimir V. Milovanović and Radosav Ž. Đorđević

Abstract. Let W_n be the set of all algebraic polynomials of exact degree n, whose coefficients are all nonnegative. For the norm in $L^2[0, \infty)$ with Freud's weight function, the extremal problem (1. 2) is considered.

1. Introduction

In a previous paper G. V. Milovanović found a complete solution of the following problem of A. K. Varma [2]:

Let W_n be the set of all algebraic polynomials of exact degree n, all coefficients of which nonnegative, i.e.,

$$W_n = \left\{ P_n \,\middle|\, P_n(x) = \sum_{k=0}^{n} a_k x^k,\ a_k \geqq 0 \ (k=0, 1, \ldots, n-1),\ a_n > 0 \right\} \tag{1.1}$$

and let $\|f\|^2 = (f, f)$, where

$$(f, g) = \int_0^\infty w(x) f(x) g(x)\, dx \qquad (f, g \in L^2[0, \infty)),$$

with generalized Laguerre weight function $w(x) = x^\alpha e^{-x}$ $(\alpha > -1)$.

Determine the best constant in the inequality

$$\| P_n' \|^2 \leqq C_n(\alpha) \| P_n \|^2 \qquad (P_n \in W_n),$$

i.e.,

$$C_n(\alpha) = \sup_{P_n \in W_n} \frac{\| P_n' \|^2}{\| P_n \|^2}. \tag{1.2}$$

Namely, Milovanović [1] proved the following result:

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Theorem A. *The best constant* $C_n(\alpha)$ *defined in* (1.2) *is*

$$C_n(\alpha)=\begin{cases}\dfrac{1}{(2+\alpha)(1+\alpha)} & (-1<\alpha\leqq\alpha_n),\\[2ex] \dfrac{n^2}{(2n+\alpha)(2n+\alpha-1)} & (\alpha_n\leqq\alpha<+\infty),\end{cases}$$

where

$$\alpha_n=\frac{1}{2(n+1)}\left((17n^2+2n+1)^{1/2}-3n+1\right).$$

From Theorem A we can see:

(a) $C_n(\alpha_n-0)=C_n(\alpha_n+0)$;

(b) $C_{n+1}(\alpha)\geqq C_n(\alpha)$;

(c) The sequence (α_n) is decreasing.

Remark 1. The statement of Theorem A holds if W_n is the set of all algebraic polynomials $P\,(\not\equiv 0)$ of degree at most n (not only of exact degree n), with nonnegative coefficients.

In this note we consider the extremal problem (1.2) with Freud's weight function

(1.3) $$w(x)=x^\alpha e^{-x^s}\quad(\alpha>-1,\ s>0)$$

on $[0, \infty)$. The corresponding best constant we will denote by $C_n(\alpha;\,s)$.

2. Extremal problem with Freud's weight

Let the set W_n be defined by (1.1) and the weight function $x\mapsto w(x)$ by (1.3) The subset of W_n for which $a_0=0$ (i.e. $P_n(0)=0$) we denote by W_n^0.

By a simple application of integration by parts we can prove:

Lemma 1. *If* $P\in W_n^0$, *then for the inner products*

$$J_n(\alpha;\,s)=(P_n',\,P_n')=\int_0^\infty x^\alpha e^{-x^s}P_n(x)^2\,dx,$$

$$I_{n,i}(\alpha;\,s)=(P_n,\,P_n^{(i)})=\int_0^\infty x^\alpha e^{-x^s}P_n(x)\,P_n^{(i)}(x)\,dx\qquad(i=0,\,1,\,2)$$

the following recurrence relations hold

$$2I_{n,1}(\alpha;\,s)=s\,I_{n,0}(\alpha+s-1;\,s)-\alpha\,I_{n,0}(\alpha-1;\,s)\qquad(\alpha>-2),$$

$$I_{n,2}(\alpha;\,s)=s\,I_{n,1}(\alpha+s-1;\,s)-\alpha\,I_{n,1}(\alpha-1;\,s)-J_n(\alpha;\,s)\qquad(\alpha>-1).$$

In [1] Milovanović proved an interesting inequality for $P_n\in W_n$. Namely, for every $x\geqq 0$ the inequality

$$x\,(P_n'(x)^2-P_n(x)\,P_n''(x))\leqq P_n'(x)\,P_n(x)$$

holds.

From this inequality and Lemma 1 we obtain:

Lemma 2. *If* $P_n \in W_n^0$, *then for* $\alpha > -1$ *and* $s > 0$

$$J_n(\alpha; s) \leqq \frac{1}{4}\{s^2 I_{n,0}(\alpha+2s-2; s) + s(2-2\alpha-s) I_{n,0}(\alpha+s-2; s) + (\alpha-1)^2 I_{n,0}(\alpha-2; s)\}.$$

Since the supremum in (1.2) attained for some $P_n \in W_n^0$ (see [1]), we will consider only such polynomials, i.e., $P_n(x) = \sum_{k=1}^{n} a_k x^k$ ($a_n > 0$ and other $a_k \geqq 0$). Then

$$P_n(x)^2 = \sum_{k=2}^{2n} b_k x^k \quad (b_{2n} > 0 \text{ and other } b_k \geqq 0)$$

and

$$\|P_n\|^2 = I_{n,0}(\alpha; s) = \frac{1}{s} \sum_{k=2}^{2n} b_k \Gamma\left(\frac{\alpha+k+1}{s}\right), \tag{2.1}$$

where Γ is the gamma function.

Using the same method as in the paper [1] we find

$$\|P_n'\|^2 = J_n(\alpha; s) = (P_n', P_n') \leqq \frac{1}{s} \sum_{k=2}^{2n} H_k(\alpha; s) b_k \Gamma\left(\frac{\alpha+k+1}{s}\right), \tag{2.2}$$

where

$$H_k(\alpha; k) = \frac{k^2}{4} \cdot \frac{\Gamma\left(\frac{\alpha+k-1}{s}\right)}{\frac{\alpha+k+1}{s})}. \tag{2.3}$$

According to (2.1) and (2.2) we have

$$\|P_n'\|^2 \leqq (\max_{2 \leqq k \leqq 2n} H_k(\alpha; s)) \|P_n\|^2,$$

and then

$$C_n(\alpha; s) \leqq \max_{2 \leqq k \leqq 2n} H_k(\alpha; s).$$

The case $s=1$ is solved in [1].

For $s=2$ we get a simple result:

Theorem 1. *The best constant* $C_n(\alpha; 2)$ *is given by*

$$C_n(\alpha; 2) = \begin{cases} \dfrac{2}{\alpha+1}, & -1 < \alpha \leqq -\dfrac{n-1}{n+1}, \\[2ex] \dfrac{2n^2}{2n+\alpha-1}, & -\dfrac{n-1}{n+1} \leqq \alpha < +\infty. \end{cases}$$

Proof. In this case, (2.3) reduces to $H_k(\alpha; 2) = \dfrac{k^2}{2(\alpha+k-1)}$. Determining the maximum of $f(x) = \dfrac{x^2}{x+\alpha-1}$ on the interval $[2, 2n]$, we find that

$$\max_{2\leqq k\leqq 2n} H_k(\alpha; 2) = \begin{cases} H_2(\alpha; 2) & \text{if} \quad -1<\alpha\leqq\alpha_n, \\ H_{2n}(\alpha; 2) & \text{if} \quad \alpha_n\leqq\alpha<+\infty, \end{cases}$$

where $\alpha_n = -\dfrac{n-1}{n+1}$.

Similarly as in the paper [1] we show that $C_n(\alpha; 2) = \max\limits_{2\leqq k\leqq 2n} H_k(\alpha; 2)$, The polynomial $x\mapsto x^n$ is an extremal polynomial for $\alpha\geqq\alpha_n$. If $-1<\alpha\leqq\alpha_n$, there exists a sequence of polynomials, for example, $p_{n,k}(x) = x^n + kx$, $k = 1, 2, \ldots$, for which $\lim\limits_{k\to\infty} \dfrac{\| p'_{n,k} \|^2}{\| p_{n,k} \|^2} = C_n(\alpha; 2)$.

Remark 2. The statement of Theorem 1 holds if W_n is a set as in Remark 1. In that case, if $-1<\alpha\leqq\alpha_n$, we can see that $x\mapsto \lambda x\ (\lambda>0)$ is an extremal polynomial.

From Theorem 1 we obtain the following inequality

$$\int_0^\infty e^{-t^2} P_n'(t)^2\, dt \leqq \frac{2n^2}{2n-1} \int_0^\infty e^{-t^2} P(t)^2\, dt$$

for each $P_n\in W_n$.

The case when s is an arbitrary positive number is more complicated. We state the following conjecture:

Conjecture. *Let $s\geqq 1$ and let $\alpha_n\,(>-1)$ be the unique root of the equation*

$$\frac{\Gamma\left(\dfrac{\alpha+1}{s}\right)}{\Gamma\left(\dfrac{\alpha+3}{s}\right)} = n^2 \frac{\Gamma\left(\dfrac{\alpha+2n-1}{s}\right)}{\Gamma\left(\dfrac{\alpha+2n+1}{s}\right)}.$$

The best constant $C_n(\alpha; s)$ is given by

$$C_n(\alpha; s) = \begin{cases} H_2(\alpha; s), & -1<\alpha\leqq\alpha_n, \\ H_{2n}(\alpha; s), & \alpha_n\leqq\alpha<+\infty. \end{cases}$$

At the end we give more one result:

Theorem 2. *If $\alpha>1$ and $s>0$ we have*

$$\| P_n' \|_\alpha \leqq n \| P_n \|_{\alpha-2} \qquad (P_n\in W_n),$$

where $\|f\|_\alpha = \left(\int_0^\infty x^\alpha e^{-x^s} f(x)^2\, dx\right)^{1/2}$.

This result follows immediately from (2.1), (2.2), and (2.3).

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EKSTREMALNI PROBLEM ZA POLINOME SA NENEGATIVNIM KOEFICIJENTIMA. II

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Neka je W_n skup svih algebarskih polinoma egzaktnog stepena n čiji su koeficijenti nenegativni. U radu se razmatra ekstremalni problem (1.2) za normu u $L^2[0, \infty)$ sa Freudovom težinom.

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CONVEXITY CRITERION INVOLVING LINEAR OPERATORS

Lj. M. Kocić and I. B. Lacković

Abstract. The necessary and sufficient conditions for a linear operators family $\{A_\lambda\}_{\lambda\in\Lambda}$ defined on $C[a, b]$ so that $A_\lambda f \geqq 0$ $(\lambda\in\Lambda)$, if and only if f is convex on $[a, b]$. The result is illustrated by several examples. Some of examples are already known criterions of convexity, the other ones are new.

1. Introduction

Let $I\subseteq R$ be a nonempty interval and $C(I)$ be a space of continuous functions $f: I\to R$. We say a function $f\in C(I)$ is convex on I provided that for every $x, y\in I$ holds $f\left(\frac{x+y}{2}\right)\leqq\frac{1}{2}(f(x)+f(y))$. The cone of all convex functions on I will be denoted by $K(I)$ (or by $K[a, b]$ if $I=[a, b]$). Let $S(D)$ be a normed subspace of the space of all real functions defined on an arbitrary interval $D\subseteq R$.

There are a lot of conditions, concerning convex functions which can be expressed in the form $Af\geqq 0$ or, more properly

$$(1)\qquad A_\lambda f\geqq 0,\quad \lambda\in\Lambda,$$

where Λ is an index-set, and A_λ is a linear operator for every λ. Some of this conditions play the role of criterions for convexity of the function f. Such criterions, having the form (1) are:

i) Jensen inequality

$$f(\lambda u+(1-\lambda)v)\leqq\lambda f(u)+(1-\lambda)f(v),$$

where $\lambda\in(0, 1)$, $u, v\in I$, which can be rewriten as

$$A_\lambda f=\lambda f(u)+(1-\lambda)f(v)-f(\lambda u+(1-\lambda)v)\geqq 0,$$

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and which is customary used for definition of convex functions without continuity assumption. In this case A_λ is the functional for every $\lambda \in \Lambda = (0, 1)$.

ii) Another criterion, similar to i) is those concerning the divided difference of the fuction f, i.e.

$$A_\lambda f = [x_1, x_2, x_3] f \geqq 0,$$

where $x_1, x_2, x_3 \in I$. In this case λ is a vector (x_1, x_2, x_3) and $\Lambda = I^3$.

iii) The n-point Jensen inequality

$$A_\lambda f = \sum_{k=1}^{n} a_k f(x_k) - f\Big(\sum_{k=1}^{n} a_k x_k\Big) \geqq 0, \tag{2}$$

where $\Sigma a_k = 1$, $a_k > 0$, $x_k \in I$, $k = 1, \ldots, n$. In this case $\lambda = (x_1, \ldots, x_n, a_1, \ldots, a_n)$.

The following criterions are known as Hermite-Hadamard inequalities (see [8]):

iv) For $\lambda > 0$ and $x - \lambda$, $x + \lambda \in I$,

$$A_\lambda f = f(x - \lambda) - \frac{1}{\lambda} \int_{x-\lambda}^{x+\lambda} f(t)\, dt + f(x + \lambda) \geqq 0. \tag{3}$$

v) For the same assumptions,

$$A_\lambda f = \frac{1}{2\lambda} \int_{x-\lambda}^{x+\lambda} f(t)\, dt - f(x) \geqq 0. \tag{4}$$

Some very interesting criterions are based upon a certain approximation operators. Perheps best known are those concerning the polynomials of S. N. Bernstein:

vi) Let B_n is the n-th Bernstein operator defined by

$$B_n(f, x) = \sum_{k=1}^{n} p_{nk}(x) f\Big(\frac{k}{n}\Big), \quad p_{nk}(x) = \binom{k}{n} x^k (1-x)^{n-k}, \quad x \in [0, 1]. \tag{5}$$

Now, the criterion have the form

$$A_n f = B_n f - f \geqq 0 \qquad (n \in N), \tag{6}$$

or

$$A_n f = B_n f - B_{n+1} f \geqq 0 \qquad (n \in N), \tag{7}$$

where $f \in C[0, 1]$. For this criterions see [5], [6], [9] and [4].

The similar criterions also can set up using the operators of Szász--Mirakjan [3]

$$S_n(f, x) = e^{-nx} \sum_{k=0}^{+\infty} f\left(\frac{k}{n}\right) \frac{(nx)^k}{k!} \qquad (n \in N), \tag{8}$$

and Gauss-Weierstrass [2]

$$W_\lambda(f, x) = \int_{-\infty}^{+\infty} w(\lambda, t) f(x-t)\, dt, \quad w(\lambda, t) = \left(\frac{\lambda}{2\pi}\right)^{1/2} e^{-\lambda t^2/2}. \tag{9}$$

In the next section we deal with establishing the conditions for the linear operators family $\{A_\lambda\}_{\lambda \in \Lambda}$ in order that the set of inequalities $A_\lambda f \geqq 0$, $\lambda \in \Lambda$ is equivalent to $f \in K(I)$.

2. Main result

A common trait for all above criterions are: A_λ is a linear operator for fixed λ. In all cases the index set Λ has a boundless cardinality, or more precise, Λ is at least countable set. So, $\{A_\lambda\}$ is at least countable family of linear operators.

To prove the general theorem we need the concept of one-sided strong local maximum (OSLM) of the real function.

Definition. *The function* $\varphi \in C(I)$ *have* OSLM *in the point* $x_0 \in \operatorname{int} I$ *if exists* $h > 0$ *so that for every* $x \in (x_0 - h, x_0 + h) \subseteq I$, *we have* $\varphi(x) \leqq \varphi(x_0)$ *and* $\varphi(x) < \varphi(x_0)$ *at least in one of the intervals* $(x_0 - h, x_0)$ or $(x_0, x_0 + h)$. *We denote by* $\hat{C}(I)$ *the set of all functions from* $C(I)$, *having* OSLM *in at least one point from* int I.

Now, the following theorem gives the sufficient conditions for convexity of continuous real function.

Theorem 1. *Let* $\{A_\lambda\}$ *be a family of linear operators so that* $A_\lambda : C(I) \to S(D)$ *and* $e_i(t) = t^i\ (i = 0, 1)$. *If*

(a) $A_\lambda e_0 = 0 \quad (\lambda \in \Lambda)$,

(b) $A_\lambda e_1 = 0 \quad (\lambda \in \Lambda)$,

(c) *for every* $\varphi \in \hat{C}(I)$ *there exists at least one* $\lambda_0 \in \Lambda$ *and* $y_0 \in D$ *so that* $A_{\lambda_0}(\varphi, y_0) < 0$,

then the implication

$$A_\lambda f \geqq 0 \ (\textit{for every } \lambda \in \Lambda) \Rightarrow f \in K(I)$$

is valid for every $f \in C(I)$.

Proof. Suppose that $f \in C(I)$ is not convex, but $A_\lambda f \geqq 0$ for every $\lambda \in \Lambda$. Then, we can choose three points $x_1 < x_2 < x_3$ from I, and an affine function $a(x) = Cx + D$ satisfying the conditions $a(x_1) = f(x_1)$, $a(x_3) = f(x_3)$ and

$$f(x_2) > a(x_2). \tag{10}$$

The function g defined by $g(x) = f(x) - a(x)\,(x \in I)$ is continuous on I (and also on $[x_1, x_3] \subset I$) and $g(x_1) = g(x_2) = 0$. So, g achieves its maximum on $[x_1\ x_3]$, say $M = \max g(x)$. From (10) follows that $M > 0$. We introduce the set $P = \{x \mid g(x) = M,\ x \in [x_1, x_3]\}$. It is clear by definition that P is nonempty, so we have $p = \inf P$ and $q = \sup P$, and $x_1 \leq p \leq q \leq x_3$ is valid. Let us prove that the stronger inequality

$$x_1 < p \leqq q < x_3 \tag{11}$$

also takes place. Suppose on the contrary that $p = x_1$. Then, by Weierstrass theorem, one can select from P a convergent sequence (p_n) so that $\lim p_n = x_1 = p$. In virtue of continuity of g on $[x_1, x_3]$ we have $\lim g(p_n) = g(\lim p_n) = g(p) = g(x_1) = 0$, which is in contradiction to the fact that q is continuous, i.e. it must be $x_1 < p$. In the similar way, one can show that $q = x_3$ also leads to the contradiction, so (11) holds.

Now, we get from (11) that $g(x) < M$ for $x \in [x_1, p) \cup (q, x_3]$ and $g(p) = g(q) = M$ because P is a closed set (as a consequence of continuity of g), which means that g has OSLM at the point $p \in (x_1, x_3)$ as well as in $q \in (x_1\ x_3)$. So, at least in one point $x_0 \in \operatorname{int} I$ the function g has OSLM, and according to condition (c) we can find $\lambda_0 \in \Lambda$ and $y_0 \in D$ so that $A_{\lambda_0}(g, y_0) < 0$. On the other hand, in virtue of conditions (a), (b) and linearity of A_λ we have $A_\lambda g = A_\lambda(f - Ce_1 - De_0) = A_\lambda f$, for every $\lambda \in \Lambda$. Since $A_\lambda f \geq 0$ we have $A_\lambda(g, y) \geq 0$ for every $y \in D$ and $\lambda \in \Lambda$. This contradiction proves the assertion of the theorem.

Using the results of [12] we can formulate some stronger conditions for A_λ so that the equivalence

$$A_\lambda f \geqq 0\ (\lambda \in \Lambda) \Leftrightarrow f \in K(I) \tag{12}$$

is valid for every $f \in C(I)$.

Let for fixed $c \in I$ we introduce the functions σ_c and σ_c^+ by $\sigma_c(x) = |x - c|$, $x \in I$, $\sigma_c^+(x) = \frac{1}{2}(x - c + \sigma_c(x))$. Then we have

Theorem 2. (Linear criterion of convexity). *Let $\{A_\lambda\}$ be a family of continuous linear operators $A_\lambda : C(I) \to S(D)$, where I is a finite interval. If the conditions* (a), (b), (c) *and*

(d) $$A_\lambda \sigma_c \geqq 0 \ \textit{for every}\ c \in I,\ \lambda \in \Lambda,$$

are fulfilled, then (12) *holds. The function σ_c in* (d) *can be replaced by σ_c^+.*

Proof. The proof follows from theorem 1 and [12, theorem 3]. The posibility of replace the function σ_c by σ_c^+ is based upon the fact that these functions are disitinguish up to the affine function.

3. Examples

In order to illustrate our theorem 2, we shall prove the criterions iii), vi) and some criterions of more general form than those in iv) and v).

Example 1. Let $I=[a, b]$ and A_λ is a functional given by (2). It is easy to see that $A_\lambda e_0 = A_\lambda e_1 = 0$ for all $\lambda \in \Lambda$, where $\Lambda = I^n \times R_1{}^n$, where $R_1{}^n$ is a set of all n-tuples $(a_1, a_2, \ldots, a_n)$ so that $\Sigma a_i = 1$. Thus, the conditions (a) and (b) are fulfilled. Now, let φ be an arbitrary function from $\hat{C}[a, b]$ with OSLM at the point $x_0 \in (a, b)$. Let $h>0$ be choosen so that for arbitrary $n(\geqq 2)$ the set of points $\{x_1, \ldots, x_n\}$, can be found so that $x_i \in (x_0 - h, x_0 + h)$. Then, by theorem of Caratheodory [11], x_0 can be expressed as a convex combination of n-dimensional (but degenerate) simplex $\{x_i\}_1{}^n$ as follows

$$x_0 = \sum_{i=1}^{n} a_i x_i, \quad \sum_{i=1}^{n} a_i = 1, \quad a_i > 0 \qquad (i = 1, \ldots, n). \tag{13}$$

Thus, we have

$$\varphi(x_i) \leqq \varphi(x_0) \qquad (i = 1, \ldots, n; \quad n \geqq 2), \tag{14}$$

with at least one strong inequality. Multiplying each side of k-th inequality in (14) by $a_k > 0$ $(1 \leqq k \leqq n,\ n \geqq 2)$ and adding all of them we get

$$\varphi(x_0) \sum_{i=1}^{n} a_k > \sum_{i=1}^{n} a_k \varphi(x_k), \quad n \geqq 2, \tag{15}$$

wherefrom, taking (13) into consideration, we have $A_{\lambda_0} \varphi < 0$, where λ_0 is determinated by the points x_i and the weights a_i. The last inequality is the conditon (c).

The condition (d) is also satisfied, as shows the chain

$$|\Sigma a_i x_i - c| = |\Sigma a_i x_i - c \Sigma a_i| = |\Sigma a_i (x_i - c)| \leqq \Sigma a_i |x_i - c|$$

which means that $A_\lambda \sigma_c \geqq 0$, for every $c \in [a, b]$.

Example 2. Jensen integral inequality. The following criterion, concerning integral inequality is valid.

Theorem 3. *Let p and q be integrable on $[u, v]$ and $p(t) > 0$, g is monotone and bounded $(a \leqq g(t) \leqq b)$ on $[u, v]$. Then $f \in C[a, b]$ is convex if and only if for every $t_1 < t_2$ from $[u, v]$ holds the inequality*

$$f\left(\frac{\int_{t_1}^{t_2} p(t)\, g(t)\, \mathrm{d}t}{\int_{t_1}^{t_2} p(t)\, \mathrm{d}t}\right) \leqq \frac{\int_{t_1}^{t_2} p(t)\, f(g(t))\, \mathrm{d}t}{\int_{t_1}^{t_2} p(t)\, \mathrm{d}t}. \tag{16}$$

Proof. Let $l(f)$ and $r(f)$ be the left and right side in (16). Now, we can introdice the family of functionals $A_\lambda : C[a, b] \to S(R)$ as

$$A_\lambda f = r(f) - l(f),\ \lambda \in \Lambda = \{(t_1, t_2) \mid u < t_1 < t_2 < v\}, \tag{17}$$

and one can shows without difficulties that A_λ fulfils the conditions (a), (b) and by similar chain as in example 1, the condition (d).

Now, let $\varphi \in \hat{G}[a, b]$ with OSLM in $x_0 \in (a, b)$. Then, one can find the points $x_1, x_2 \in [a, b]$ so that $x_1 < x_0 < x_2$ and for every $x \in (x_1, x_0) \cup (x_0, x_2)$ holds

$$\varphi(x) \leqq \varphi(x_0) \tag{18}$$

with strong inequality in at least one subinterval (x_1, x_0) or (x_0, x_2). On the other hand, for monotone function g there exists inverse fuction $g^{-1} : [a, b] \to [u, v]$ and the numbers T_1 and T_2 defined by $T_1 = \min\{g^{-1}(x_1), g^{-1}(x_2)\}$, $T_2 = \max\{g^{-1}(x_1), g^{-1}(x_2)\}$ satisfies that $u \leqq T_1 < T_2 \leqq v$, and for every $t \in (T_1, T_2)$, $t \neq g^{-1}(x_0)$ inequality (18) can be rewriten in the form

$$\varphi(g(t)) \leqq \varphi(x_0), \tag{19}$$

with the stong inequality at least in one of two subintervals $(T_1, g^{-1}(x_0))$ or $(g^{-1}(x_0), T_2)$. If we choose t_1 and t_2 so that $T_1 < t_1 < g^{-1}(x_0) < t_2 < T_2$ and

$$x_0 = \frac{\int_{t_1}^{t_2} p(t)\, g(t)\, dt}{\int_{t_1}^{t_2} p(t)\, dt}, \tag{20}$$

by multiplying (19) by $p(t) > 0$ and integrating this inequality over the range (t_1, t_2) we get

$$\int_{t_1}^{t_2} p(t) f(g(t))\, dt < f(x_0) \int_{t_1}^{t_2} p(t)\, dt,$$

which means that $A_{\lambda_0} f < 0$, where $\lambda_0 = (t_0, t_1)$, which completes the proof.

The next theorem establish the criterion of convexity employing an operator of more general form to this in (4).

Theorem 4. *Let $p, q > 0$ and $0 < \theta \leqq 1$. For $a \leqq s < t \leqq b$ we define*

$$M = \frac{ps + qt}{p+q},\quad h = \theta \frac{t-s}{p+q} \min(p, q). \tag{21}$$

Then $f \in C[a, b]$ is convex on $[a, b]$ if and only if the inequlity

$$\frac{pf(s) + qf(t)}{p+q} - \frac{1}{2h} \int_{M-h}^{M+h} f(x)\, dx \geqq 0,$$

holds, for every $s < t$ from $[a, b]$.

The proof can be found in [4].

Example 4. The criterions involving Steklov functions.

For $f \in C(I)$, $h>0$ and $x \in I_1(h)=\{t \mid t-h,\ t+h \in I\}$, the operator S_h defined by

$$(22)\qquad S_h(f, x)=(2h)^{-1} \int_{x-h}^{x+h} f(t)\,\mathrm{d}t,$$

is often called Steklov function, although it is an operator mapping $C(I)$ into $C(I_1)$. For a finite interval $I=[a, b]$, the maximum value of h can be $(b-a)/2$. In this case I_1 degenerates into a point, and S_h becomes a functional. The Hermite-Hadamard inequality (4) now has the form $f(x) \leq S_\lambda(f, x)$, $x \in I_1(\lambda)$, and it is equivalent to convexity of the function f. The itereted Steklov operators (with the step $h>0$) S_h^n $(n \in N)$ are defined by

$$(23)\qquad S_h^0(f, x)=f(x), \quad S_h^n(f, h)=(2h)^{-1} \int_{x-h}^{x+h} S_n^{n-1}(f, x)\,\mathrm{d}t,$$

where $n \in N$, $x \in I_n(h)=\{t \mid t-nh,\ t+nh \in I\}$. We write S_h instead S_h^1.

Now, we have the following theorems:

Theorem 5. *Function $f \in C(I)$ is convex if and only if for every $h>0$ and $x \in I_n(h)$, the inequality*

$$S_h^n(f, x) \geq S_h^{n-1}(f, x) \quad (n \text{ is fixed}),$$

holds.

Theorem 6. *Let $I=[a, b]$. The function $f \in C(I)$ is convex if and only if for every H, $0<H \leq (b-a)/2$ and x such that $x-h,\ x+h \in I$ and every $h \in (0, H)$ the inequality*

$$S_H(f, x) \geq S_h(f, x)$$

holds.

It is easy to see that those theorems generalize the convexity criterion based on the inequality (4). For the proof see [4].

Example 5. Criterions concerning some approximating operators.

Let the Bernstein polyncmials be given by (5) The following holds:

Theorem 7. (Kosmák [6]) *The function $f \in C[0, 1]$ is convex if and only if $B_n(f, x) \geqq f(x)$, $x \in [0, 1]$.*

Proof. If we put $Af=B_n f-f$, it is easy to check that $Ae_0=Ae_1=0$, i.e. the condition (a) and (b) are fulfilled.

Let $\varphi \in \hat{C}[0, 1]$ with OSLM in the point $x_0 \in (0, 1)$ and introduce $g(x)=\varphi(x)-\varphi(x_0)$. So we have $g(x) \leqq 0$ for $x \in (x_0-h, x_0+h)$ and $g(x)<0$ at least in one of intervals (x_0-h, x_0), (x_0, x_0+h). Now we have

$$B_n(g, x)=\sum_{k=0}^{n} g\left(\frac{k}{n}\right) p_{nk}(x), \quad x \in [0, 1],$$

and taking $x=x_0$ we can break up the above sum into two parts

$$B_n(g, x_0)=S_1+S_2$$

where we put

$$S_1=\sum_{\left|\frac{k}{n}-x_0\right|<h} g\left(\frac{k}{n}\right)p_{nk}(x_0), \quad S_2=\sum_{\left|\frac{k}{n}-x_0\right|\geqq h} g\left(\frac{k}{n}\right)p_{nk}(x_0).$$

We see that $S_1<0$ as a consequence of $g\left(\frac{k}{n}\right)<0$ for $\left|\frac{k}{n}-x_0\right|<h$. Now we have

$$B_n(g, x_0)=S_1\left(1+\frac{S_2}{S_1}\right)=S_1\left(1+\frac{M}{n}\cdot\frac{P_2(n)}{P_1(n)}\right), \tag{24}$$

where we denoted $M=\max_{|x-x_0|<h} g(x)$, $m=\min_{|x-x_0|\geqq h} g(x)$, and

$$P_1(n)=\sum_{\left|\frac{k}{n}-x_0\right|<h} p_{nk}(x_0), \quad P_2(n)=\sum_{\left|\frac{k}{n}-x_0\right|\geqq h} p_{nk}(x_0).$$

Now, since $P_1(n)+P_2(n)=\sum_{k=0}^{n} p_{nk}(x_0)=1$, we have $P_1(n)=1-P_2(n)$, and the following estimations takes place [7, p. 6]

$$P_2(n)\leqq\frac{1}{h^2}\sum_{\left|\frac{k}{n}-1\right|\geqq h}\left(\frac{k}{n}-x_0\right)^2 p_{nk}(x)\leqq\frac{x(1-x)}{nh^2}\leqq\frac{1}{4nh^2}$$

and by noting that $\lim_{n\to\infty} P_2(n)=0$, we get from (24) that $B_n(g, x_0)\to S_1$ $(n\to\infty)$, i.e. for n sufficiently large the inequality $B_n(g, x_0)<0$ holds. This means that $B_n(\varphi, x_0)-\varphi(x_0)=A(\varphi, x_0)<0$, or the condition (c) is satisfyed.

The condition (d) follows from positivity of B_n $(n\in N)$

$$B_n(\sigma_c, x)=B_n(|e_1-ce_0|, x)\geqq|B_n(e_1\, x)-cB_n(e_0, x)|=|x-c|$$

i.e., $A\sigma_c\geqq 0$ for every $c\in[0, 1]$. We have established assertion of this theorem.

Theorem 8. (Kosmák [6], Moldovan [9]). *The function $f\in C[0, 1]$ is convex if and only if* (7) *holds.*

Using the theorem 2, we also can prove the result of Horova [3] concerning Szász-Mirakjan operators S_n, defined by (8).

Theorem 9. (Horova). *Let $f\in C$ $[0, a]$, $a>0$ is bounded on $[0, +\infty)$. Then, f is convex on $[0, a]$ if and only if*

$$S_n(f, x)\geqq f(x), \quad x\in[0, a].$$

The similar theorem holds for Gauss-Weierstrass operator (9), see Butzer and Nessel [2, p. 150—153; also remarks, p. 160]. For further examples, concerning exponential type operators see Satô [10, p. 76] where we found very elegant application of parabola technique introduced by Bajšanski and Bojanić. Very interesting results are also obtained by Wang [13]. Those are criterions involving the Fourier coefficients of continuous function f. This is generalization of the criterion iv) employing Hermite-Hadamard inequality (3). For some further generalizations, including Chebyshev systems see Ziegler [14].

4. Some remarks and properties

The class of all operator families $\{A_\lambda\}_{\lambda\in\Lambda}$ satisfying the conditions (a), (b), (c) and (d), we will denote by Φ. If $\{A_\lambda\}\in\Phi$ this means that $A_\lambda f\geqq 0$, $\lambda\in\Lambda$ is a convexity criterion for f on the corresponding interval.

In all criterions cited above, we see that Λ is at least countable set. This fact is implicitely supposed in theorems 1 and 2. We shall clear up this standpoint. The reason hides into the equivalence between the inequality $A_\lambda f\geqq 0$, $\lambda\in\Lambda$ and Jensen inequality

$$\Delta_h(f, x)=f(x+h)-2f(x)+f(x-h)\geqq 0,\quad h\in H, \tag{25}$$

where $H=\{h\,|\,x-h,\ x+h\in I\}$, whenever $f\in C(I)$. Every criterion (i) can, step by step, be transformed into (25), and vice versa. This transformation carrying the index set Λ into H, preserves theirs cardinality. In other words, Λ and H are equipotent sets. The least cardinality of H so that (25) still has the power of criterion is aleph-0, or the countableness. It is easy to see this by taking $h\in H_r$ insteed $h\in H$ in (25), where H_r is a set of all rationals from H. But every irrational $h\in H$ is a limit of certain subsequence h_j from H_r, i.e. $h=\lim h_j$. Thus,

$$\Delta_h f=\Delta_{\lim h_j} f=\lim \Delta_{h_j} f\geqq 0,$$

since $\Delta_{h_j} f\geqq 0$ for every $h_j\in H_r$. So, Λ must be at least countable set. This characterizes mainly the criterions containing approximate operators. Each of them has a form $B_n f\geqq 0$, $n\in N$, as we see in the previous section.

Now, the kernel of linear operator A is a set of all functions satisfying $Af=0$. We have

Theorem 10. *$\{A_\lambda\}\in\Phi$ and $f\in C(I)$, then*

$$\operatorname{kern}\{A_\lambda\}=\alpha\, e_1+\beta\, e_0\quad (\alpha,\ \beta\in R).$$

Proof. Let $f(x)=\alpha x+\beta$, an $\{A_\lambda\}\in\Phi$. Then, A_λ satisfies the conditions (a) and (b) so $A_\lambda f=0\,(\lambda\in\Lambda)$. Thus, $A_\lambda f\geqq 0$, i.e. by theorem 2, f is convex and then satisfies the Jensen inequality

$$f\left(\frac{x+y}{2}\right)\leqq\frac{f(x)+f(y)}{2}. \tag{26}$$

But, on the other hand, $A_\lambda f=0\leqq 0$, i.e. $A_\lambda(-f)\leqq 0$, so $-f$ is convex. This means that

$$f\left(\frac{x+y}{2}\right)\geqq\frac{f(x)+f(x)}{2} \tag{27}$$

is valid. So, we have from (26) and (27)

$$f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}.$$

This is so called Jensen functional equality, which only solution on the set of continuous functions is affine function $f(x)=\alpha x+\beta$, α, $\beta\in R$ (see Aczél [1]), which proves the theorem.

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KRITERIJUMI KONVEKSNOSTI KOJI SADRŽE LINEARNE OPERATORE

Lj. M. Kocić i I. B. Lacković

Dati su potrebni i dovoljni uslovi za familiju linearnih operatora $\{A_\lambda\}_{\lambda\in\Lambda}$ definisanu na $C[a, b]$, tako da je $A_\lambda f\geqq 0\ (\lambda\in\Lambda)$ ekvivalentno konveksnosti funkcije f na $[a, b]$. Rezultat je ilustrovan sa više primera. Neki od tih primera su poznati kriterijumi koveksnosti dok su drugi novi.

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REMARK ON AN INEQUALITY FOR 3-CONVEX FUNCTIONS

Josip E. Pečarić and Radovan R. Janić

Abstract. An inequality for 3-convex functions is considered in this paper. Under conditions (3) and (6) is shown that inequality (4) holds.

1. N. Levinson [1] proved the following result:

If f has a third derivative $f'''\geqq 0$ on $[0, 2a]$, $0\leqslant x_k\leqslant a, p_k>0$ $(1\leqq k\leqq n)$, $P_n=\sum_{k=1}^{n} p_k$, then

$$\text{(1)}\qquad \frac{1}{P_n}\sum_{k=1}^{n} p_k f(x_k)-f\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k x_k\right)\leqq\frac{1}{P_n}\sum_{k=1}^{n} p_k f(2a-x_k)-f\left(2a-\frac{1}{P_n}\sum_{k=1}^{n} p_k x_k\right)$$

If $f'''>0$ then there is equality if and only if $x_1=\cdots=x_n$.

P. M. Vasić and R. R. Janić [2] proved, by taking into account the supplementary assumptions $p_k\geqq 1$ and $\sum_{k=1}^{n} p_k x_k\in[0, a]$, that the inequality

$$\text{(2)}\qquad \sum_{k=1}^{n} p_k f(x_k)-\sum_{k=1}^{n} p_k f(2a-x_k)\geqq f\left(\sum_{k=1}^{n} p_k x_k\right)-f\left(2a-\sum_{k=1}^{n} p_k x_k\right)$$
$$-\left(1-\sum_{k=1}^{n} p_k\right)(f(0)-f(2a))$$

holds.

P. M. Vasić and Lj. Stanković [3] proved the following generalization of (2).

If f has a third derivative $f'''\geqslant 0$ on $[0, 2a]$, $p_k>0$ $(k=0, 1,\ldots, n)$, $x_0, \sum_{k=0}^{n} p_k x_k\in[0, a]$ and if

$$\text{(3)}\qquad (x_i-x_0)\left(\sum_{k=1}^{n} p_k x_k-x_i\right)>0\qquad (i=1,\ldots, n)$$

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hold, then the following inequality is valid:

$$\sum_{i=0}^{n} p_i f(x_i)-\sum_{i=0}^{n} p_i f(2a-x_i)\geqq A\left(f\left(\sum_{i=0}^{n} p_i x_i\right)-f\left(2a-\sum_{i=0}^{n} p_i x_i\right)\right)+B\left(f(x_0)-f(2a-x_0)\right) \tag{4}$$

where

$$A=\frac{\sum_{i=0}^{n} p_i x_i-x_0\sum_{i=0}^{n} p_i}{\sum_{i=0}^{n} p_i x_i-x_0}, \qquad B=\frac{\sum_{i=0}^{n} p_i-1}{\sum_{i=0}^{n} p_i x_i-x_0}\sum_{i=0}^{n} p_i x_i.$$

From (1) it follows that the function $x\mapsto f(2a-x)-f(x)$ is convex on $[0, a]$ (see also [4]). Using this fact, we can obtain inequalities (2) and (4) from the coresponding results for convex functions (i. e. from the generalizations of the well-known Petrović's inequality). However, we shall show that inequalities (2) and (4) are valid with weaken conditions.

2. Let x_0, x_1, x_2, x_3 be distinct points from $[a, b]$ then the third divided difference of f at these points is

$$V_3(f; x_0, x_1, x_2, x_3)=\sum_{k=0}^{3}\frac{f(x_k)}{\omega'(x_k)}$$

where $\omega(x)=\prod_{k=0}^{3}(x-x_k)$.

A real-valued function is said to be 3-convex on $[a, b]$ if and only if for all choices of 4 distinct points in $[a, b]$, $V_3(f; x_0, \ldots, x_3)\geqq 0$. If f has a third derivative on $[a, b]$ then f is 3-convex if and only if $f'''\geqq 0$.

In [5] it is shown that the conditions for the validity inequality (1) can be weakened, i. e. (1) hold for 3-convex function $f:[0, 2a]\to R$ if $p_k>0$ $(1\leqslant k\leqslant n)$ and

$$x_i+x_{n-i+1}\leqq 2a, \quad \frac{p_i x_i+p_{n-i+1}x_{n-i+1}}{p_i+p_{n-i+1}}\leqq a, \tag{5}$$

hold.

Putting $n=2$ in (1) and substituting (see [3]):

$$x_1\to\sum_{k=0}^{n} p_k x_k, \quad x_2\to x_0, \quad p_1\to 1, \quad p_2\to\frac{1}{x_i-x_0}\left(\sum_{k=0}^{n} p_k x_k-x_i\right)$$

for $i=1, \ldots, n$, we find, if (3) is valid,

$$f(x_i)-f(2a-x_i)\geqq\frac{x_i-x_0}{\sum_{k=0}^{n} p_k x_k-x_0}\left(f\left(\sum_{k=0}^{n} p_k x_k\right)-f\left(2a-\sum_{k=0}^{n} p_k x_k\right)\right)$$

$$+\frac{\sum_{k=0}^{n} p_k x_k-x_i}{\sum_{k=0}^{n} p_k x_k-x_0}\left(f(x_0)-f(2a-x_0)\right) \quad (i=1, \ldots, n).$$

Therefore, after multiplication by p_i, summation and addition of $f(x_0)-f(2a-x_0)=f(x_0)-f(2a-x_0)$ to the corresponding sides of the resulting inequality, we obtain (4), and from (5) (for $n=2$) we obtain the condition

(6) $$x_0+\sum_{k=0}^{n} p_k x_k \leqq 2a \text{ and } x_i \leqq a \quad (i=1, \ldots, n).$$

So, the following theorem is valid:

Theorem 1. *Let f be 3-convex function on $[0,2a]$. If x_i, p_i $(1 \leqq i \leqq n)$ are positive numbers and x_0 nonnegative number such that* (3) *and* (6) *are valid, then* (4) *holds.*

For $x_0=0$, from Theorem 1, we obtain:

Corollary 1. *Let f be defined as in Theorem* 1 *and let x_i, p_i $(1 \leqq i \leqq n)$ be positive numbers. If either*

(7) $$x_i \leqq \sum_{k=1}^{n} p_k x_k \leqq a \quad (i=1, \ldots, n)$$

or

(8) $$x_i \leqq a \leqq \sum_{k=1}^{n} p_k x_k \leqq 2a \quad (i=1, \ldots, n)$$

hold, then (2) *is valid.*

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PRIMEDBA O JEDNOJ NEJEDNAKOSTI ZA 3-KONVEKSNE FUNKCIJE

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U radu se razmatra jedna nejednakost za 3-konveksne funkcije. Pod uslovima (3) i (6) pokazuje se da nejednakost (4) važi.

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ON INTERPOLATING OF SOME INEQUALITIES

Dušica D. Đorđević and Petar M. Vasić

Abstract. In this paper, an interpolating of some inequalities is considered. By successive application of this procedure several inequalities are obtained.

1. Introduction

Let I and J be index sets ($I, J \subset N$). We considere inequalities of the form

(1.1) $$F(I) \geqq 0 \qquad \text{or} \qquad F(I) \leqq 0,$$

where F is a corresponding fuction defined on the index set I.

We say the inequalities (1.1) are interpolating if

(1.2) $$F(I) \geqq F(J) \geqq 0 \qquad \text{or} \qquad F(I) \leqq F(J) \leqq 0,$$

where $I \supset J$.

An interesting case is when $I = I_{n+1} = \{1, \ldots, n+1\}$ and $J = I_n = \{1, \ldots, n\}$. Namely, let the function F is superaditive or subaditive, i. e.,

$$F(I \cup J) \geqq F(I) + F(J) \qquad \text{or} \qquad F(I \cup J) \leqq F(I) + F(J),$$

for all $I, J \subset N$. Then for $I = I_{n+1}$ and $J = I_n$, it follows that F is a monotonic function. Thus, the inequalities

$$F(I_{n+1}) \geqq F(I_n) \geqq \ldots \geqq F(I_2) \geqq F(I_1) = 0$$

or

$$F(I_{n+1}) \leqq F(I_n) \leqq \ldots \leqq F(I_2) \leqq F(I_1) = 0$$

hold.

In this paper we will use the notation $N(I)$ for $\sum_I 1$. For finite and disjoint sets I and J, the function N is an aditive function, i.e., $N(I \cup J) = N(I) + N(J)$.

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2. Main results

P. M. Vasić [1] and J. D. Kečkić and I. B. Lacković [2] proved the following result:

Theorem A. *Let f be a twice differentiable function on* $[0, a)$, *and let it on that interval satisfies the following differential inequality*

$$xf''(x)-(k-1)f'(x)\geqq 0, \tag{2.1}$$

where k is a real number not equal to zero. Then

$$f\left(\sqrt[k]{\frac{x_1^k+\cdots+x_n^k}{n}}\right)\leqq\frac{f(x_1)+\cdots+f(x_n)}{n} \tag{2.2}$$

$$\leqq\frac{f(\sqrt[k]{x_1^k+\cdots+x_n^k})+(n-1)f(0)}{n},$$

for $x_i\in[0, a)$ $(i=1,\ldots,n)$ *and* $\sqrt[k]{x_1^k+\cdots+x_n^k}\in[0, a)$.

The inequalities (2.2) are equvalent to following inequalities

$$nf\left(\sqrt[k]{\frac{\sum_{i=1}^{n}x_i^k}{n}}\right)\leqq\sum_{i=1}^{n}f(x_i)\leqq f\left(\sqrt[k]{\sum_{i=1}^{n}x_i^k}\right)+(n-1)f(0). \tag{2.3}$$

We will prove two theorems which are related to the above inequalities.

Theorem 2.1. *Let* $f:[0, a)\to R$ *be a twice differentiable function and such that it satisfies the differential inequality* (2.1). *If* $x=\{x_i\}_{i\in I\cup J}$ *is a nonnegative sequence, such that* x_i $(i\in I\cup J)$ *and* $(\sum_{i\in I\cup J}x_i^k)^{1/k}$ *belong to* $[0, a)$, *where I and J are finite and disjoint sets, then the function F, defined by*

$$F(I)=N(I)f\left(\left(\frac{1}{N(I)}\sum_{i\in I}x_i^k\right)^{1/k}\right)-\sum_{i\in I}f(x_i), \tag{2.4}$$

is a subaditive function, i.e.

$$F(I\cup J)\leqq F(I)+F(J). \tag{2.5}$$

Proof: According to (2.4) we have

$$F(I\cup J)=N(I\cup J)f\left\{\left(\frac{1}{N(I\cup J)}\sum_{i\in I\cup J}x_i^k\right)^{1/k}\right\}-\sum_{i\in I\cup J}f(x_i)$$

$$=N(I\cup J)f\left\{\left(\frac{1}{N(I\cup J)}\left\{\sum_{i\in I}x_i^k+\sum_{i\in J}x_i^k\right\}\right)^{1/k}\right\}-\sum_{i\in I}f(x_i)-\sum_{i\in J}f(x_i).$$

Putting

$$x_i^k\to\frac{\sum_{i\in I}x_i^k}{N(I)}\quad(i\in I)\text{ and }x_i^k\to\frac{\sum_{i\in J}x_i^k}{N(J)}\quad(i\in J),$$

we obtain

$$F(I\cup J)=N(I\cup J)f\left(\left\{\frac{1}{N(I\cup J)}\left(\sum_{i\in I}x_i^k+\sum_{i\in J}x_i^k\right)\right\}^{1/k}\right)$$

$$-N(I)f\left(\left\{\frac{1}{N(I)}\sum_{i\in I}x_i^k\right\}^{1/k}\right)-N(J)f\left(\left\{\frac{1}{N(J)}\sum_{i\in J}x_i^k\right\}^{1/k}\right)$$

$$=N(I\cup J)f\left(\left\{\frac{1}{N(I\cup J)}\sum_{i\in I\cup J}x_i^k\right\}^{1/k}\right)-\sum_{i\in I\cup J}f(x_i)$$

$$-N(I)f\left(\left\{\frac{1}{N(I)}\sum_{i\in I}x_i^k\right\}^{1/k}\right)+\sum_{i\in I}f(x_i)$$

$$-N(J)f\left(\left\{\frac{1}{N(J)}\sum_{i\in J}x_i^k\right\}^{1/k}\right)+\sum_{i\in J}f(x_i).$$

Since $F(I\cup J)\leqq 0$, according to (2.4) we conclude that (2.5) holds.

Corollary 2.1. *Putting* $I=I_n=\{1,\ldots,n\}$ *and* $J=\{n+1\}$, *from the proved subaditivity of function F, it follows*

(2.6) $$F(I_{n+1})=F(I_n\cup J)\leqq F(I_n)+F(J)=F(I_n),$$

because $F(J)=0$ *to* $J=\{n+1\}$.

By succesive application of the inequality (2.6) we obtain

$$F(I_{n+1})\leqq F(I_n)\leqq\ldots\leqq F(I_2)\leqq F(I_1)=0.$$

Thus $F(I_n)\leqq 0$, which is equivalent to the first inequality in (2.3).

Theorem 2.2. *Under conditions of Theorem* 2.1., *the function G, defined by*

(2.7) $$G(I)=f\left\{\left(\sum_{i\in I}x_i^k\right)^{1/k}\right\}-\sum_{i\in I}f(x_i)+(N(I)-1)f(0),$$

is subaditive, i.e.,

(2.8) $$G(I\cup J)\leqq G(I)+G(J).$$

Proof. For $n=2$, the second inequality in (2.3) becomes

(2.9) $$f(x_1)+f(x_2)-f\left(\sqrt[k]{x_1^k+x_2^k}\right)-f(0)\leqq 0.$$

Putting $x_1^k=\sum_{i\in I}x_i^k$ and $x_2^k=\sum_{i\in J}x_i^k$ in (2.9), we obtain

$$f\left(\left(\sum_{i\in I}x_i^k\right)^{1/k}\right)+f\left(\left(\sum_{i\in J}x_i^k\right)^{1/k}\right)-f\left(\left(\sum_{i\in I\cup J}x_k^k\right)^{1/k}\right)-f(0)\leqq 0.$$

This inequality is equivalent to

$$\sum_{i\in I\cup J} f(x_i)-f\Big(\big(\sum_{i\in I\cup J} x_i^k\big)^{1/k}\Big)-(N(I\cup J)-1)f(0)$$

$$-\sum_{i\in I} f(x_i)+f\Big(\big(\sum_{i\in J} x_i^k\big)^{1/k}\Big)+(N(I)-1)f(0)$$

$$-\sum_{i\in J} f(x_i)+f\Big(\big(\sum_{i\in J} x_i^k\big)^{1/k}\Big)+(N(J)-1)f(0)\leqq 0.$$

Then, according to (2.7), we conclude that G is a subaditive function, i.e. the inequality (2.8) holds.

Thus, Theorem 2.2 is proved.

Corollary 2.2. *If* $I=I_n=\{1,\dots,n\}$ *and* $J=\{n+1\}$, *from* (2.8) *it follows*

(2.10) $$G(I_{n+1})=G(I_n\cup J)\leqq G(I_n)+G(J)=G(I_n),$$

because $G(J)=0$ *for* $J=\{n+1\}$.

By succesive application of the inequali.y (2.10), we obtain

$$G(I_{n+1})\leqq G(I_n)\leqq \dots \leqq G(I_2)\leqq G(I_1)=0.$$

The inequality $G(I_n)\leqq 0$ is equivalent to the second inequality in (2.3).

In the case $k=1$, from Theorem 2.2., we obtain:

Corollary 2.3. *Let* $f:[0,a)\to R$ *be a twice differentiable function and* $f''(x)\geqq 0$. *If* $x=\{x_i\}_{i\in I\cup J}$ *is a nonnegative sequence, such that* x_i $(i\in I\cup J)$ *and* $\sum_{i\in I\cup J} x_i$ *belong to* $[0,a)$ $(a>0)$, *where* I *and* J *are finite and disjoint sets, then for function* G, *defined by*

$$G(I)=f\big(\sum_{i\in I} x_i\big)-\sum_{i\in I} f(x_i)+(N(I)-1)f(0).$$

the inequality

(2.11) $$G(I\cup J)\geqq G(I)+G(J)$$

holds.

Corollary 2.4. *If* $I=I_n=\{1,\dots,n\}$ *and* $J=\{n+1\}$, *from the inequality* (2.11) *follows*

(2.12) $$G(I_{n+1})=G(I_n\cup J)\geqq G(I_n)+G(J)=G(I_n),$$

because $G(J)=0$, *for* $J=\{n+1\}$.

By the succesive application of the inequality (2.12) we obtain the following inequalities:

(2.13) $$G(I_{n+1})\geqq G(I_n)\geqq \dots \geqq G(I_2)\geqq G(I_1)=0.$$

Corollary 2.5. *From* (2.13) *immediately follows the inequality of M. Petrović* [3]:

$$f(x_1)+\dots+f(x_n)\leqq f(x_1+\dots+x_n)+(n-1)f(0),$$

where f is a convex function on $[0, a)$ $(a>0)$, $x_1, \ldots, x_n$, $x_1+\ldots x_n\in[0, a)$ *with equality case if and only if* $x_i=0$ $(i=1, \ldots, n)$ *or* $n=1$.

The inequality

$$\prod_{k=1}^{n}(1-a_k)>1-\sum_{k=1}^{n}a_k \qquad (0<a_k<1) \tag{2.14}$$

is a well-know Weierstrass' inequality (see [4]).

We prove the following rezult:

Theorem 2.3. *Let I and J be two finite and disjoint index sets. If $a=\{a_i\}_{i\in I\cup J}$ is real sequence such that $0<a<1$ $(i\in I\cup J)$, then the inequality*

$$H(I\cup J)>H(I)+H(J) \tag{2.15}$$

holds, where

$$H(I)=\prod_{i\in I}(1-a_i)+\sum_{i\in I}a_i-1.$$

Proof. For $n=2$, the inequality (2.14) reduces to

$$(1-a_1)\ (1-a_2)+a_1+a_2-1>0. \tag{2.16}$$

Putting

$$a_1=1-\prod_{i\in I}(1-a_i) \quad\text{and}\quad a_2=1-\prod_{i\in J}(1-a_i),$$

the inequality (2.16) becomes

$$\prod_{i\in I}(1-a_i)\cdot\prod_{i\in J}(1+a_i)+1-\prod_{i\in I}(1-a_i)+1-\prod_{i\in J}(1-a_i)-1>0,$$

i.e.,

$$\prod_{i\in I\cup J}(1-a_i)+\sum_{i\in I\cup J}a_i-1-\{\prod_{i\in J}(1-a_i)+\sum_{i\in I}a_i-1\}$$

$$-\{\prod_{i\in J}(1-a_i)+\sum_{i\in J}a_i-1\}>0,$$

wherefrom, we obtain the statement of Theorem 2.3.

Corollary 2.6. *If $I=I_n=\{1, \ldots, n\}$ and $J=\{n+1\}$, the inequality* (2.15) *becomes*

$$H(I_{n+1})=H(I_n\cup J)\geqq H(I_n)+H(J)=H(I_n), \tag{2.17}$$

because $H(J)=0$ for $J=\{n+1\}$.

Iterating the inequality (2.17), we obtain the following sequences of inequalities

$$H(I_{n+1})\geqq H(I_n)\geqq\ldots\geqq H(I_2)\geqq H(I_1)=0.$$

The inequality $H(I_n)\geqq 0$ is equivalent to Weierstrass' inequality.

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O INTERPOLIRANJU NEKIH NEJEDNAKOSTI

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U radu je interpolirano nekoliko poznatih nejednakosti. Na taj način dobijene su neke nove nejednakosti kao i nejednakosti koje su oštrije od polaznih. Uzastopnom primenom interpoliranja dobijeni su nizovi nejednakosti, iz kojih su, kao specijalni slučajevi, dobijene neke poznate nejednakosti.

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ON THE INCLUSION ISOTONICITY OF TAYLOR'S CIRCULAR CENTERED FORM

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Abstract. The inclusion isotonicity property is important for many applications of interval functions. In this paper it is proved that frequently used the Taylor's cicular centered form, introduced in [6] for approximating the range $\{f(z):z\in Z\}$ of an analytic function f over a disk Z, is inclusion isotone.

1. Introduction

Let f be a closed complex function and Z any disk in the complex plane lying in the domain of f. The range

$$f^*(Z)=\{f(z):z\in Z\}$$

is of importance for numerous considerations. But, in general, $f^*(Z)$ is not a disk and is not computable exactly. In order to exceed these difficuities it is necessary to develope the methods for finding a circular interval $Q\supseteq f^*(Z)$, as small as possible.

Let $K(C)$ be the set of circular intervals (disks) and let $F: D\to H$, $(D, H\subseteq K(C))$ be an *circular interval fnnction* such that

$$F(Z)\supseteq f^*(Z) \quad \text{for all } Z\in D.$$

$F(Z)$ is an outer approximation for the range $f^*(Z)$ and it is called *the inclusion function* for f over Z. Interval analysis gives some effective methods for finding these inclusion functions (cf. [1]–[10]). Among the most important inclusions of the ranges functions are *the centered forms* with the convenient property that the center of inclusion functions $\text{mid}(F(Z))$ is equal to the value of the function f at the center $c=\text{mid}(Z)$ of the domain disk, that is, $\text{mid}(F(Z))=f(c)$. One of these centered forms has been presented in [6] using Taylor development of f about the center c and inclusion isotonocity property:

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Let f be an analytic function on $D \subseteq K(C)$ and let $Z=\{z: |z-c| \leqq r\} \in D$ be a disk with the the center c and the radius r, denoted by $Z=\{c; r\}$. Then the circular interval function F_T, defined by

$$(1) \qquad F_T(Z)=\left\{f(c); \sum_{k=1}^{+\infty} \frac{|f^{(k)}(c)|\, r^k}{k!}\right\},$$

is called *the Taylors's circular centered form.* The centered form (1) has interesting practical and theoretical properties. Since the inclusion $F_T(Z) \supseteq f^*(Z)$ always holds, it can be applyed for approximating the range of a function f. It was proved in [5] that the Taylor's centered form converges to the range of the function with a quadratic convergence in the sense of the DI pseudo-distance.

A fundamental property of interval computations is *inclusion isotonicity* (see [1]). Let $F: D \to H(D, H \subseteq K(C))$ be an circular interval function. F is said to be *inclusion isotone* if

$$Z \subseteq W \text{ implies } F(Z) \subseteq F(W) \text{ for all } Z, W \in D.$$

It is important for many applications that the interval functions have this property. Besides, various properties of centered forms are connected to inclusion isotonicity. In some recent papers the inclusion isotonicity has been considered for some functions having the Taylor's centered form (1). Among these functions, the Taylor's centered form for a polynomial p,

$$(2) \qquad P_T(Z)=\left\{p(c); \sum_{k=1}^{n} \frac{|p^{(k)}(c)|\, r^k}{k!}\right\},$$

is of great importance because this form gives always better results than the other polynomial centered forms: the Horner scheme and the power-sum evaluation (see [9], [10] and, also, [7] for real polynomials). J. Rokne and T. Wu have stated in [10] that the Taylor's form (2) is inclusion isotone, but without a proof. Therefore, an investigation of inclusion isotonicity of (2) is also of interest.

In this paper we prove that, in general, the Taylor's circular centered form (1) is inclusion isotone. The presented proof is very simple and uses a procedure for proving the analytic inequalities.

2. The proof of inclusion isotonicity

Let $Z=\{z; r\}$ and $W=\{w; R\}$ be two disks. Then

$$(3) \qquad Z \subseteq W \Leftrightarrow |w-z| \leqq R-r.$$

Theorem. *The Taylor's circular centered form* (1) *is inclusion isotone.*

Proof. Assume that the disks $Z=\{z; r\}$ and $W=\{w; R\}$ belong to the domain of an analytic function f. The Taylor's centered form (1) will be inclusion isotone if

$$Z \subseteq W \text{ implies } F_T(Z) \subseteq F_T(W),$$

that is, with regard to (3), if the implication

(4)
$$|w-z|\leqq R-r \Rightarrow$$
$$|f(w)-f(z)|\leqq \sum_{k=1}^{+\infty}\frac{|f^{(k)}(w)|R^k}{k!}-\sum_{k=1}^{+\infty}\frac{|f^{(k)}(z)|r^k}{k!}$$

holds.

Let $R-r=h\geqq 0$ and $h\mapsto g(h)$ be the real function defined by

(5)
$$g(h)=\sum_{k=1}^{+\infty}\frac{|f^{(k)}(w)|(r+h)^k}{k!}-\sum_{k=1}^{+\infty}\frac{|f^{(k)}(z)|r^k}{k!}-|f(w)-f(z)|.$$

Now, the implication (4) can be rewritten as

(6)
$$|w-z|\leqq h \Rightarrow g(h)\geqq 0.$$

If $h=0$, then $w=z$ and whence $f(w)=f(z)$. Therefore, from (5) it follows $g(0)=0$. Furthermore,

$$\frac{dg}{dh}=\sum_{k=1}^{+\infty}\frac{|f^{(k)}(w)|(r+h)^{k-1}}{(k-1)!}>0 \text{ all } h\geqq 0,$$

which means that the function g is monotonically increasing on $[0, +\infty)$. According to this and the fact that $g(0)=0$, it follows $g(h)\geqq 0$ for $h\in[0, +\infty)$, which completes the proof.

3. Some remarks

The proved assertion is very useful because a number of the frequently used inclusion functions (for example, the elementary functions e^z, z^n, $z^{1/n}$, $\sin z$, $\cos z$, $\operatorname{arctg} z$, $\ln z$, the polynomials and the others) have often the Taylors's centered form.

Remark 1. As a corollary of Theorem, the conjecture for the inclusion isotonicity of the Taylor's centred form (2) of a polynomial is verified.

Remark 2. The implication (4) can be usefully applied for finding the upper bound of the modulus $|f(z_2)-f(z_1)|$ when the values z_2 and z_1 of the complex argument lie in a disk with the radius $\frac{h}{2}$, that is, if $|z_2-z_1|\leqq h$.

Let us put $R=r+h$ ($r\geqq 0$, arbitrary) in (4) and introduce

$$y(r)=\sum_{k=1}^{+\infty}\frac{|f^{(k)}(z_2)|(r+h)^k}{k!}-\sum_{k=1}^{+\infty}\frac{|f^{(k)}(z_1)|r^k}{k!}.$$

According to (4) we have

(7)
$$|f(z_2)-f(z_1)|\leqq y(r) \quad \text{for all } r\geqq 0.$$

Let $y_{\min}$ be the smallest value of the function $r \mapsto y(r)$ over $[0, +\infty)$. Then, a better upper bound for $|f(z_2)-f(z_1)|$ is given by

$$(8) \qquad |f(z_2)-f(z_1)| \leqq y_{\min}.$$

For example, let $f(z)=e^z$ and $|z_2-z_1|\leqq h$. From (7) it follows

$$(9) \qquad |e^{z_2}-e^{z_1}| \leqq |e^{z_2}|(e^{r+h}-1)-|e^{z_1}|(e^r-1) \qquad (r\geqq 0).$$

Since the function $y(r)=|e^{z_2}|(e^{r+h}-1)-|e^{z_1}|(e^r-1)$ is monotonically increasing on $[0, +\infty)$ (namely, $y'(r)=e^r|e^{z_2}|(e^h-|e^{z_1-z_2}|)\geqq 0$ according to the inequality $e^{|z|}\geqq |e^z|$ for all $z\in C$), one obtains $y_{\min}=y(0)=|e^{z_2}|(e^h-1)$. Therefore, a better estimation for the modulus $|e^z-e^{z_1}|$, compared to (9), is given by

$$|e^{z_2}-e^{z_1}| \leqq |e^{z_2}|(e^h-1) \qquad (|z_2-z_1|\leqq h).$$

Remark 3. Many interesting inequalities can be obtained using (7) and (8). For example, we give the following elementary inequality:

Let w and z be arbitrary complex numbers satisfying $|w-z|\leqq h$, $h\geqq 0$. Then

$$|w^n-z^n|\leqq(|w|+h)^n-|w|^n \qquad (n\in N)$$

(from (8)).

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O INKLUZIVNOJ IZOTONOSTI TAYLOROVE KRUŽNE CENTRALNE FORME

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Osobina inkluzivne izotonosti je važna za mnoge primene intervalnih funkcija. U ovom radu dokazano je da je često korišćena Taylorova kružna centralna forma, uvedena u [6] za aproksimaciju oblasti $\{f(z):z\in Z\}$ analitičke funkcije f na disku Z, inkluzivno izotona.

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THE EQUALITY CASES IN $m^*(f+g)(H) \leqq m^*f(H)+m^*g(H)$

M. J. Pelling

Abstract. Using standard properties of the Banach indicatrix of a function of bounded variation, the equality cases in an inequality considered in the paper [1] are completely characterised.

1. Introduction

The inequality of the title was established in paper [1] in the cases when f, g are continuous real functions and H an interval ([1, theorem 1 (i)]) and when f, g are monotonic and H a bounded subset of R ([1, theorem 2]). Theorem 3 of [1] also established $m^*(f+g)(H)=m^*f(H)+m^*g(H)$ when f, g are both monotonic increasing or both monotonic decreasing, but otherwise the cases of equality were left open. It is easy to see from the proof of [1, theorem 1 (i)] that if f, g are continuous and H an interval then $m(f+g)(H)=mf(H)+mg(H)$ if and only if f and g possess both a simultaneous maximum and a simultaneous minimum in the closure of H. The more interesting case is when f and g are monotonic in different senses, say $f\uparrow, g\downarrow$, and H is arbitrary bounded: by making use of standard properties of the Banach Indicatrix of a function of bounded variation we shall completely characterise the equality cases in this instance.

2. The Banach Indicatrix

If $h:[a, b]\to R$ is of bounded variation and $B\subseteq[a, b]$ is Borel then the Banach Indicatrix $N_B^h: R\to R\cup\{+\infty\}$ is defined by:

$N_B^h(y)=$ the number of solutions $x\in B$ to the equation $f(x)=y$.

The basic properties of N_B^h are that it is an integrable function over R and that if $t(x)$ is the total variation function of $h(x)$ then $mt(B)=\int_{-\infty}^{\infty} N_B^h(y)\,dy$. Since $h(B)=\{y \mid N_B^h(y)\geqslant 1\}$ it follows that $h(B)$ is measurable and that,

$$mh(B)=\int_{-\infty}^{\infty}\min(N_B^h(y), 1)\,dy \leqslant \int_{-\infty}^{\infty} N_B^h(y)\,dy=mt(B).$$

Hence $mh(B)=mt(B)$ if and only if $N_B^h(y)=0$ or 1 almost everywhere in y.

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Theorem. *Suppose* $f\uparrow$, $g\downarrow$ *on* $[a, b]$ *and* $H\subseteq[a, b]$. *Then the necessary and sufficient condition for* $m^*(f+g)(H)=m^*f(H)+m^*g(H)$ *is that, if* $f+g=p-n$ *is the canonical decomposition of* $f+g$ *into the difference of its positive and negative variation functions, there exist a Borel set* $B\supseteq H$ *such that* $m(f-p)(B)=m(g+n)(B)=0$ *and that* $N_B^{(f+g)}(y)=0$ *or* 1 *a.e. in* y — *that is, the graph of* $(f+g)|_B$ *meet each level line* $y=c$ *in at most one point a.e. in* c.

Proof. Suppose the proposed conditions hold. Since $f=(f-p)+p$ and $f\uparrow, (f-p)\uparrow, p\uparrow$ (by the least increase property of the positive variation p) it follows, using [1, theorem 3], that $m^*f(H)=m^*(f-p)(H)+m^*p(H)=m^*p(H)$, and similarly that $m^*g(H)=m^*n(H)$. Writing $h\equiv f+g$ and t for the total variation function of h, then from the remarks above, as $N_B^h(y)=0$ or 1 a.e. in y, and applying [1, theorem 3] to $t=p+n$, we infer $mh(B)=mt(B)=mp(B)+mn(B)$. This is also true for any Borel $B_1\subseteq B$, as $N_{B_1}^h(y)$ satisfies the same condition, and since we may certainly find such a set B_1 with $H\subseteq B_1\subseteq B$ and $m^*h(H)=mh(B_1)$, $m^*p(H)=mp(B_1)$, $m^*n(H)=mn(B_1)$ it follows that,

$$m^*h(H)=mh(B_1)=mp(B_1)+mn(B_1)=m^*p(H)+m^*n(H)=m^*f(H)+m^*g(H).$$

Conversely, if $m^*h(H)=m^*f(H)+m^*g(H)$, then since

$$m^*f(H)=m^*(f-p)(H)+m^*p(H),\quad m^*g(H)=m^*(g+n)(H)+m^*n(H),$$

$$m^*h(H)\leq m^*p(H)+m^*n(H)$$

it follows that $m^*(f-p)(H)=m^*(g+n)(H)=0$ and $m^*h(H)=m^*p(H)+m^*n(H)$. Choosing a Borel $B\supseteq H$ for which $m(f-p)(B)=m^*(f-p)(H)$, $mp(B)=m^*p(H)$ etc. it follows that $mh(B)=mp(B)+mn(B)=mt(B)$ so that by the remarks above on the Banach Indicatrix, $N_B^h(y)=0$ or 1 a.e. in y, or in other words the graph of $h|_B$ meets each line $y=c$ in at most one point a.e. in c. Thus the conditions of the theorem are satisfied with this Borel set B.

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SLUČAJEVI JEDNAKOSTI U $m^*(f+g)(H)\leqslant m^*f(H)+m^*g(H)$.

M. J. Pelling

Korišćenjem standardnih osobina Banachovih indikatora za funkcije ograničene varijacije, kompletno su opisani slučajevi jednakosti razmatrani u [1].

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ON REPRESENTATION OF A LINEAR OPERATOR ON THE SET OF STARSHAPED SEQUENCES

I. Ž. Milovanović, N. M. Stojanović and Lj. M. Kocić

Abstract. The necassary and sufficient conditions for positivity of linear continuous operator, defined on a set of starshaped or generalized starshaped sequences are given. The tool for proving is representation of (generalized) starshaped sequence as a limit (in metric d given by (5)) of some special sequences from the same class.

Let us introduce the notations: $a=(a_0, a_1, \ldots)$, $a_i \in R$,

$$T(a_n)=(a_{n+1}-a_0)/(n+1)-(a_n-a_0)/(n) \qquad (n \in N),$$

$$T_{p,q}(a_n)=(a_{n+1}-\alpha_{n+1}a_0)/W_{n+1}-(a_n-\alpha_n a_0)/W_n,$$

where the sequences α and W are defined by $\alpha_k=(p^k+q^k)/2$ $(p, q \in R)$ and

$$W_k=\begin{cases}(p^k-q^k)/(p-q), & p\neq q,\\ kp^{k-1}, & p=q.\end{cases}$$

The set of all starshaped sequences $S_1 (\subset S)$ is defined by $S_1=\{a \mid a\in S, T(a_n)\geq 0, n\in N\}$ and the set of all p, q-starshaped sequences by

$$S_{p,q}=\{a \mid a\in S,\ T_{p,q}(a_n)\geq 0,\ n\in N\}.$$

The purpose of this work is to determine the necessary and sufficient conditions for a real sequence $p=(p_0, p_1, \ldots)$ such that the inequality

$$\sum_{k=0}^{\infty} p_k a_k \geq 0 \tag{1}$$

holds for all sequences $a=(a_0, a_1 \ldots)\in S_1$, i.e. $a\in S_{p,q}$. Besides, we shall state the necessary and sufficient conditions for linear operators, defined on S_1 or $S_{p,q}$, to be positive.

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Theorem 1. *Let* $p=(p_0, p_1, \ldots)\in S$. *The inequality* (1) *holds for every sequence* a *from* S_1 *if and only if the following conditions*

(2) $$\sum_{k=0}^{n} p_k = 0,$$

(3) $$\sum_{k=1}^{n} kp_k = 0,$$

(4) $$\sum_{k=i}^{n} kp_k \geq 0, \text{ for } i=2, 3, \ldots, n,$$

are fulfilled.

Proof. Suppose that (1) holds. The sequences $a=(c, c, \ldots), -a=(-c, -c, \ldots)$, $c=\text{const}$, belong to S_1, so that necessity of (2) follows. Further, $a=(0, 1, \ldots)$ and $-a$ also belong to S_1 and from (1) we get that (3) is necessary. Finally, the sequences $a=(a_0, a_1, \ldots)$, where $a_0=\cdots=a_i=0$, $a_k=k-1$, $k=i+1, \ldots, n$ and $i=1, \ldots, n$, belong to S_1 so that conditions (4) are necessary too.

The sufficience of the conditions (2), (3), (4) is a consequence of the following identity:

$$\sum_{k=0}^{n} p_k a_k = a_0 \sum_{k=0}^{n} p_k + (a_1 - a_0) \sum_{k=1}^{n} kp_k + \sum_{k=1}^{n-1} \left(\sum_{i=k+1}^{n} ip_i \right) T(a_k).$$

Remark 1. If the sequence $a=(a_0, a_1, \ldots)$ from S_1 satisfies the additional condition $a_1=a_0$, then (3) and (4) can be replaced by

$$\sum_{k=i}^{n} kp_k \geq 0, \quad \text{for } i=1, \ldots, n.$$

Remark 2. In the few papers, e. g. [1–3] the inequality (1) is regarded to other classes of sequences.

Starting from the identity

$$\sum_{k=0}^{n} p_k a_k = a_0 \sum_{k=0}^{n} p_k \alpha_k + \frac{a_1 - \alpha_1 a_0}{W_1} \sum_{k=1}^{n} p_k W_k + \sum_{k=1}^{n-1} \left(\sum_{i=k+1}^{n} p_i W_i \right) T_{p,q}(a_k),$$

by quite similar procedure as in Theorem 1, the following result can be proved:

Theorem 2. *Let* $p=(p_0, p_1, \ldots)$ *be an arbitrary sequence of real numbers. Then, the inequality* (1) *holds for every sequence* $a\in S_{p,q}$ *if and only if the following conditions*

$$\sum_{k=0}^{n} \alpha_k p_k = 0, \quad \sum_{k=}^{n} p_k W_k = 0 \text{ and } \sum_{k=i}^{n} p_k W_k \geq 0 \quad (i=2, \ldots, n)$$

are fulfilled.

Remark 3. For $p=q=1$ from Theorem 2 we get Theorem 1.

In the sequel, we shall use following concepts and notations: Let S be set of all real sequences $a=(a_0, a_1, \ldots)$. The set S of all sequences becomes a vector space if it is supplied by addition and multiplication by scalars in the usual way. Namely, for $\lambda \in R$ and arbitrary $x, y \in S$, we put $\lambda x \in (\lambda x_0, \lambda x_1, \ldots)$, $x+y=(x_0+y_0, x_1+y_1, \ldots)$. The basic sequences in S could be denoted by $e_n=(e_{n0}, e_{n1}, \ldots)$, where

$$e_{nk}=\begin{cases} 0, & n\neq k, \\ 1, & n=k. \end{cases}$$

(For example $e_0=(1, 0, \ldots)$, $e_1=(0, 1, 0, \ldots)$). Metric in S is introduced by

(5) $$d(x, y)=\sum_{k=0}^{+\infty} 2^{-k}\frac{|x_k-y_k|}{1+|x_k-y_k|}.$$

It is easy to see that every sequence $u=(u_0, u_1, \ldots)$ from the metric space (S, d) is the limit of sequences

(6) $$u^{(n)}=\sum_{k=0}^{n} u_k e_k$$

in the sence of metric (5), i.e. $\lim d(u^{(n)}, u)=0$.

Therefore, every $u\in S$ can be represented in the form

(7) $$u=\sum_{k=0}^{+\infty} u_k e_k.$$

Let $E_0=\sum_{k=0}^{+\infty} e_k$ and $E_n=\sum_{k=n}^{+\infty} ke_k$ for $n=2, 3, \ldots$. Let A be linear operator defined on S with values in $F(D)$ which is the set of all real functions $f: D\to R$, where $D\subset R$. We also suppose that A is continuous, i.e. for every $a^{(n)}\to a$ $(n\to+\infty)$, $A(a^{(n)})\to A(a)$ $(n\to+\infty)$ holds.

Theorem 3. a) *Every seqeuence $a^{(n)}=(a_0^{(n)}, a_1^{(n)}, \ldots)$ of the form*

(8) $$a^{(n)}=\alpha^{(n)}E_0+\beta^{(n)}E_1+\sum_{k=2}^{n}\gamma_k^{(n)}E_k,$$

where $\alpha^{(n)}, \beta^{(n)}\in R$, $\gamma^{(n)}\geq 0$, for $k=2, 3, \ldots, n$, and fixed n belongs to S_1.

b) *Every sequence $a\in S_1$ is a limit (in d-metric) of sequences $a^{(n)}$ given by* (8).

c) *Let $A: S\to F(D)$ be a continuous linear operator. Then, for every a $(a\in S)$ the implication*

(9) $$a\in S_1 \Rightarrow Aa\geq 0$$

holds, if and only if

(10) $$AE_0=AE_1=0 \text{ and } AE_k\geq 0 \text{ for } k=2, 3, \ldots .$$

Proof. a) This assertion follows from (8) directly.
b) In virtue of representation

$$a_k = a_0 + k(a_1 - a_0) + k \sum_{j=1}^{k-1} T(a_j),$$

from (7) follows

$$a = a_0 e_0 + a_1 e_1 + \sum_{k=2}^{+\infty} \left(a_0 + k(a_1 - a_0) + k \sum_{j=1}^{k-1} T(a_j) \right) e_k$$

$$= a_0 \sum_{k=0}^{+\infty} e_k + (a_1 - a_0) \sum_{k=1}^{+\infty} k e_k + \sum_{k=2}^{+\infty} \left(\sum_{j=k}^{+\infty} j e_j \right) T(a_{k-1})$$

$$= a_0 E_0 + (a_1 - a_0) E_1 + \sum_{k=2}^{+\infty} E_k T(a_{k-1}).$$

Using this identity we can get that every sequence $a \in S_1$, is a limit of sequence $a^{(n)}$, where

$$a^{(n)} = a_0 E_0 + (a_1 - a_0) E_1 + \sum_{k=2}^{n} E_k T(a_{k-1})$$

in metric space (S, d).

c) Suppose that (9) holds. By the fact that the sequences $\pm E_0$, $\pm E_1$, E_k, for $k = 2, 3, \ldots$, belong to S_1, we get that the conditions (10) are necessary. If we suppose that the conditions (10) are fulfilled, then

$$Aa = A(\lim_{n \to \infty} a^{(n)}) = \lim_{n \to \infty} (Aa^{(n)})$$

$$= \lim_{n \to \infty} \left(\alpha^{(n)} AE_0 + \beta^{(n)} AE_1 + \sum_{k=2}^{n} \gamma_k^{(n)} AE_k \right) = 0,$$

from which we see that the conditions (10) are sufficient.

Remark 4. For representation of starshaped sequences by using the positive ones, see [5].

Let $\hat{E}_0 = \sum_{n=0}^{+\infty} \alpha_n e_n$ and $\hat{E}_n = \sum_{k=n}^{+\infty} w_k e_k$ for $n \geq 1$.

By similar treatment as in the proof of Theorem 3 we can prove the more general theorem:

Theorem 4. a) *Every sequence* $a^{(n)} = (a_0^{(n)}, a_1^{(n)}, \ldots)$ *of the form*

$$a^{(n)} = \alpha^{(n)} \hat{E}_0 + \beta^{(n)} \hat{E}_1 + \sum_{k=2}^{n} \gamma_k^{(n)} \hat{E}_k, \tag{11}$$

where $\alpha^{(n)}$ *and* $\beta^{(n)}$ *are real numbers and* $\gamma_k^{(n)} \geq 0$, *for* $k = 2, 3, \ldots, n$, *belongs to* $S_{p,q}$.

b) *Every sequence* $a \in S_{p,q}$ *is a limit (in d-metric) of sequences* $a^{(n)}$ *given by* (11).

c) *Let* $A: S_{p,q} \to F(D)$ *be a continuons linear operator. Then, for every* $a\,(a \in S_{p,q})$ *the implication*

$$a \in S_{p,q} \Rightarrow Aa \geq 0$$

holds, if and only if

$$A\hat{E}_0 = A\hat{E}_1 = 0 \text{ and } A\hat{E}_k \geq 0 \text{ for } k = 2, 3, \ldots .$$

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O REPREZENTACIJI LINEARNOG OPERATORA NA SKUPU ZVEZDASTIH NIZOVA

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U radu su dati potrebni i dovoljni uslovi za pozitivnost neprekidnog operatora, definisanog na skupu zvezdastih ili generalisanih zvezdastih nizova. U dokazima se koristi reprezentacija zvezdastih nizova kao granice nekih specijalnih nizova iz iste klase.

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ZUR VERALLGEMEINERUNG EINES SATZES VON CENSOR UND DE VORE

Jutta Meier

Zusammenfassung. In der vorliegenden Arbeit beweisen wir die Verallgemeinerung einer Aussage von R. A. DeVore und E. Censor zur punktweisen Appro imationsgüte stetig differenzierbarer Funktionen durch positive lineare Operatoren. Unser besonderes Augenmerk gilt dabei den Termen $L(e_1-x, x)$ und $L(|e_1-x|, x)$, die im Falle der Approximation differenzierbarer Funktionen von besonderer Bedeutung sind. Verschiedene Beispiele und Bemerkungen belegen diese Aussage.

In einer im Jahre 1976 erschienenen Arbeit modifizierte B. Mond [10] die Aussage eines auf O. Shisha und B. Mond [15] zurückgehenden Satzes über die Konvergenzgüte bei der Approximation durch Folgen positiver linearer Operatoren. Im Jahre 1969 erweiterte R. A. DeVore [4,5] dieses Ergebnis in dem Sinne, daß er eine ebenso allgemeingültige Aussage bezüglich der Konvergenzgüte differenzierbarer Funkionen herleitete. Unabhängig von DeVore bewies E. Censor [3] einen ähnlichen Satz.

Es ist das Anliegen der vorliegenden Untersuchung aufzuzeigen, daß sich die Vorgehensweise von Mond auf den Beweis des erwähnten Satzes von Censor und DeVore übertragen läßt und man auf diese Weise eine Abschätzung erhält, die bei Anwendung auf spezielle Operatorenfolgen bessere Ergebnisse liefert als dies mit Hilfe des Satzes von Censor und DeVore möglich ist. Der DeVoreschen Vorgehensweise folgend beweisen wir eine punktweise Aussage, da hierdurch die lokale Qualität eines Approximationsprozesses besser wiedergegeben wird als durch gleichmäßige Abschätzungen. Unsere Aussage zeigt gleichzeitig, daß es möglich ist, auch andere quantitative Testfunktionen heranzuziehen als die üblicherweise benutzten ersten drei Monome e_0, e_1 und e_2 bzw. deren Linearkombination $(e_1-x)^2$. Unsere Darstellung folgt weitgehend der von H. Gonska [7, S. 159], der dort Theorem 7 von Censor bereits in eine punktweise Version umformulierte und auf eine größere Klasse möglicher Testfunktionen ausdehnte. Wir weisen darauf hin, daß zum Beweis der ange-

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kündigten Abschätzung die bei Censor gemachte Voraussetzung bezüglich der gleichmäßigen Beschränktheit der Bilder von e_0 unter einer Folge positiver linearer Operatoren völlig überflüssig ist.

Satz. *Es sei L ein positiver linearer Operator, der den Banachraum $C[a, b]$ in den Raum aller beschränkten Funktionen $B[c, d]$ abbildet, wobei $[c, d] \subset [a, b]$ gelte. Es sei $\{u_0, \ldots, u_p\} \subset C[a, b]$ ein System von Funktionen, das für jedes $x \in [a, b]$ eine Linearkombination*

$$F(\cdot, x) = \sum_{k=0}^{p} a_k(x) \cdot u_k(\cdot)$$

zuläßt, die der folgenden Bedingung genügt:

(B) $F(t, x) \geq C \cdot (t-x)^2$ für alle $t, x \in [a, b]$, mit einer von t und x unabhängigen Konstanten C.

Dann gelten unter Benutzung der Schreibweise $\mu_L(x) := \left(\frac{1}{C} \cdot L(F(\cdot, x), x)\right)^{1/2}$ für jede in $[a, b]$ differenzierbare Funktion f, alle $x \in [c, d]$ und jedes $h > 0$ die Ungleichungen

$$(1) \quad |L(f, x) - f(x)| \leq |L(e_0, x) - 1| \cdot |f(x)| + |L(e_1 - x, x)| \cdot |f'(x)| + \left[L(|e_1 - x|, x) + \frac{1}{h} \cdot L((e_1 - x)^2, x)\right] \cdot \omega(f', h),$$

$$(2) \quad |L(f, x) - f(x)| \leq |L(e_0, x) - 1| \cdot |f(x)| + |L(e_1 - x, x)| \cdot |f'(x)| + \left[L(e_0, x)^{1/2} + \frac{1}{h} \cdot \mu_L(x)\right] \cdot \mu_L(x) \cdot \omega(f', h),$$

$$(3) \quad |L(f, x) - f(x)| \leq |L(e_0, x) - 1| \cdot |f(x)| + L(e_0, x)^{1/2} \cdot \mu_L(x) \cdot |f'(x)| + \left[L(e_0, x)^{1/2} + \frac{1}{h} \cdot \mu_L(x)\right] \cdot \mu_L(x) \cdot \omega(f', h).$$

Beweis. Zunächst gilt für jedes $f \in C[a, b]$ und jedes $x \in [a, b]$ die Ungleichung

$$|L(f, x) - f(x)| \leq |L(f, x) - f(x) \cdot L(e_0, x)| + |f(x)| \cdot |L(e_0, x) - 1|;$$

uns interessiert also nur noch der erste Summand auf der rechten Seite. Dazu schreiben wir für $t, x \in [a, b]$:

$$f(t) = f(x) + f'(x)(t - x) + [f(t) - f(x) - f'(x)(t - x)].$$

Dies führt auf die (Un-) Gleichungskette

$$\begin{aligned} |L(f, x) - f(x) \cdot L(e_0, x)| &= |L(f - f(x), x)| \\ &= |L(f'(x) \cdot (e_1 - x) + (f - f(x) - f'(x)(e_1 - x)), x)| \\ &\leq |f'(x)| \cdot |L(e_1 - x, x)| + L(|f - f(x) - f'(x)(e_1 - x)|, x). \end{aligned}$$

Nun ist

$$\begin{aligned}|f(t)-f(x)-f'(x)(t-x)| &= |f'(\xi_t)(t-x)-f'(x)(t-x)| \\ &= |f'(\xi_t)-f'(x)|\cdot|t-x| \\ &\leq \omega(f', |\xi_t-x|)\cdot|t-x| \\ &\leq \omega(f', |t-x|)\cdot|t-x|,\end{aligned}$$

da ξ_t nach dem Mittelwertsatz zwischen t und x liegt. Bei Vorgabe eines beliebigen $h>0$ erhalten wir ferner, daß der letzte Ausdruck kleiner als der bzw. gleich dem folgenden Term ist:

$$|t-x|\cdot\left(1+\left]\frac{|t-x|}{h}\right[\right)\cdot\omega(f', h).$$

Hierbei bezeichnet für $r>0$ das Symbol $]r[$ die größte ganze Zahl, die kleiner als r ist (vgl. [11, S. 100]).

Damit können wir den letzten Term in der vorletzten Ungleichungskette weiter nach oben abschätzen durch

$$\begin{aligned}&|f'(x)|\cdot|L(e_1-x, x)|+\omega(f', h)\cdot\left[L(|e_1-x|, x)+L\left(|e_1-x|\cdot\left]\frac{|e_1-x|}{h}\right[, x\right)\right] \\ &\leq|f'(x)|\cdot|L(e_1-x, x)|+\omega(f', h)\cdot\left[L(|e_1-x|, x)+\frac{1}{h}\cdot L((e_1-x)^2, x)\right].\end{aligned}$$

Dies liefert die unter (1) behauptete Ungleichung.

Um den Zusammenhang zur allgemeineren Klasse von Testfunktionen $\{u_0, \ldots, u_p\}$ herzustellen und damit zu der unter (2) formulierten Aussage zu gelangen, gehen wir wie folgt vor:

Der in (1) auftretende Ausdruck $L(|e_1-x|, x)+\frac{1}{h}\cdot L((e_1-x)^2, x)$ läßt sich auf Grund der Cauchy-Schwarzschen Ungleichung (siehe z. B. J. Dieudonné [6]) und der im obigen Satz gemachten Voraussetzungen nach oben durch

$$\begin{aligned}&L(e_0, x)^{1/2}\cdot L((e_1-x)^2, x)^{1/2}+\frac{1}{h}\cdot L((e_1-x)^2, x) \\ &\leq L(e_0, x)^{1/2}\cdot L\left(\frac{1}{C}\cdot F(\cdot, x), x\right)^{1/2}+\frac{1}{h}\cdot L\left(\frac{1}{C}\cdot F(\cdot, x), x\right)\end{aligned}$$

abschätzen; dies liefert die unter (2) behauptete Aussage.

Um zu der (manchmal groben) Abschätzung unter (3) zu gelangen, schätzen wir die Terme in (2) weiter ab. Wegen der Positivität von L ergibt sich unter Anwendung der Cauchy-Schwarzschen Ungleichung:

$$\begin{aligned}|L(e_1-x, x)| &\leq L(|e_1-x|, x)\leq L(e_0, x)^{1/2}\cdot L((e_1-x)^2, x)^{1/2} \\ &\leq L(e_0, x)^{1/2}\cdot L\left(\frac{1}{C}\cdot F(\cdot, x), x\right)^{1/2}.\end{aligned}$$

Damit erhalten wir die Abschätzung

$$|L(e_1-x, x)|\cdot|f'(x)|+\left[L(e_0, x)^{1/2}\cdot L\left(\frac{1}{C}\cdot F(\cdot, x), x\right)^{1/2}+\right.$$

$$\left.+\frac{1}{h}\cdot L\left(\frac{1}{C}\cdot F(\cdot, x), x\right)\right]\cdot\omega(f', h)$$

$$\leqslant L(e_0, x)^{1/2}\cdot\mu_L(x)\cdot|f'(x)|+\left[L(e_0, x)^{1/2}\cdot\mu_L(x)+\frac{1}{h}\cdot\mu_L^2(x)\right]\cdot\omega(f', h).$$

Dies führt zu der unter 3) behaupteten Aussage.

Wir betrachten im folgenden zwei Beispiele, die aufzeigen, inwieweit der obige Satz bessere Aussagen liefert als bekannte Ergebnisse zur Approximation durch positive lineare Operatoren.

Beispiel 1. Bei Betrachtung der klassischen Operatoren B_n, die von S. N. Bernstein [2] eingeführt wurden, erhielten F. Schurer und F. W. Steutel [12] die Darstellung

$$B_n(|e_1-x|, x)=\frac{2}{n}(n-r)\binom{n}{r}x^{r+1}(1-x)^{n-r};$$

wobei $r=[nx]$ die größte ganze Zahl bezeichnet, die nicht größer als nx ist. Wegen Gültigkeit der Gleichungen ($n\geqslant 1$)

$$B_n(e_0, x)=1,\quad B_n(e_1, x)=x\quad\text{und}\quad B_n((e_1-x)^2, x)=\frac{x(1-x)}{n},$$

liefert unser Satz zunächst die Abschätzung

$$|B_n(f, x)-f(x)|\leqslant\left[\frac{2}{n}(n-r)\binom{n}{r}x^{r+1}(1-x)^{n-r}+\frac{1}{h}\cdot\frac{x(1-x)}{n}\right]\cdot\omega(f', h)$$

für $h>0$. Die spezielle Wahl $h=n^{-1/2}$ führt auf die Ungleichung

$$|B_n(f, x)-f(x)|\leqslant\left[\frac{2}{n}(n-r)\binom{n}{r}x^{r+1}(1-x)^{n-r}+\frac{x(1-x)}{n^{1/2}}\right]\cdot\omega(f', n^{-1/2}).$$

Wie von Schurer und Steutel gezeigt wurde, ist der erste Summand in der eckigen Klammer stets kleiner bzw. gleich $\frac{1}{2\cdot n^{1/2}}$. Wegen $x(1-x)\leqslant\frac{1}{4}$ für $x\in[0, 1]$ ergibt sich also die gleichmäßige Abschätzung

$$|B_n(f, x)-f(x)|\leqslant\left(\frac{1}{2\cdot n^{1/2}}+\frac{1}{4\cdot n^{1/2}}\right)\cdot\omega(f', n^{-1/2})=\frac{3}{4}\cdot n^{-1/2}\cdot\omega(f', n^{-1/2}),$$

die auch im Buch von G. G. Lorentz [9] zu finden ist. Wir bemerken, daß sich die Konstante $\frac{3}{4}$ weder aus dem Ergebnis von Censor [3] noch aus dem von DeVore [5] ergibt.

Bemerkung. Es ist festzuhalten, daß einige allerdings allein auf die Folge der klassischen Bernstein-Operatoren bezogene Untersuchungen bezüglich der vor $n^{-1/2} \cdot \omega(f', n^{-1/2})$ auftretenden Konstanten etwas bessere Ergebnisse geliefert haben.

So zeigte Li Ven'-Tin [8] die Ungleichung

$$|B_n(f, x)-f(x)| \leqslant \frac{11}{16} \cdot n^{-1/2} \cdot \omega(f', n^{-1/2}).$$

Darüberhinaus besagt ein Ergebnis von F. Schurer und F. W. Steutel [12, 13, 14], daß

$$|B_n(f, x)-f(x)| \leqslant \frac{1}{4} \cdot n^{-1/2} \cdot \omega(f', n^{-1/2})$$

gilt und daß die darin auftretende Konstante $\frac{1}{4}$ nicht verkleinert werden kann. Der Vollständigkeit halber erwähnen wir an dieser Stelle auch ein Ergebnis von I. Badea [1, Théorème 2], welches allerdings den vor $\omega(f', n^{-1/2})$ auftretenden Faktor $n^{-1/2}$ völlig außer Acht läßt Allerdings liefert eine geringfügige Modifikation der Beweisführung von I. Badea ebenfalls eine Abschätzung gegen $\frac{3}{4} \cdot n^{-1/2} \cdot \omega(f', n^{-1/2})$.

Die im obigen Satz unter (2) angegebene Abschätzung läßt sich z. B. dann nutzbringend anwenden, wenn die Berechnung von $L(|e_1-x|, x)$ umgangen werden soll, da dies (vgl. Beispiel 1) selbst bei im allgemeinen gut handhabbaren Operatoren nicht ganz einfach ist. Die Wahl $F(t, x)=(t-x)^2$ führt in diesem Fall auf die Ungleichung

$$\begin{aligned}|L(f, x)-f(x)| \leqslant &|L(e_0, x)-1| \cdot |f(x)|+|L(e_1-x, x)| \cdot |f'(x)| \\ &+\left[L(e_0, x)^{1/2} \cdot L\left((e_1-x)^2, x\right)^{1/2}+\frac{1}{h} \cdot L\left((e_1-x)^2, x\right)\right] \cdot \omega(f', h),\end{aligned}$$

die im Falle $L=B_n$, $n \geqslant 1$, $h=n^{-1/2}$ nun ebenfalls die abschließende Abschätzung in Beispiel 1 impliziert. Es ist allerdings nicht grundsätzlich günstig, von $L(|e_1-x|, x)$ zu $L(e_0, x)^{1/2} \cdot L((e_1-x)^2, x)^{1/2}$ überzugehen, da hierdurch z. B. im Falle Hermite-Fejérscher Interpolationsoperatoren Information über deren Verhalten auf $C^1[-1, 1]$ verlorengeht.

Als Anwendung der unter (2) bewiesenen Ungleichung behandeln wir das folgende Beispiel 2, in dem zugleich deutlich wird, warum wir eine Aussage bewiesen haben, die eher der von DeVore als der von Censor entspricht.

Beispiel 2. Bei P. C. Sikkema [16] wird gezeigt, daß die für $f \in C[0, 2]$ und $x \in [0, 1]$ durch

$$B_{n,1}(f, x)=\sum_{k=0}^{n+1} f\left(\frac{k}{n}\right)\binom{n+1}{k} x^k(1-x)^{n+1-k}$$

definierten modifizierten Bernstein-Operatoren positiv und linear sind und den Bedingungen

$$B_{n,1}\, e_0 = e_0, \quad B_{n,1}\, e_1 = e_1 + \frac{1}{n} \cdot e_1, \text{ sowie}$$

$$B_{n,1}\, ((e_1 - x)^2, x) = \frac{1}{n^2}(x^2 + (n+1)(x - x^2))$$

genügen.

Dabei gilt für jedes $x \in [0, 1]$ die Abschätzung

$$B_{n,1}\, ((e_1 - x)^2, x) \leqslant \frac{1}{n}.$$

Die im obigen Satz unter (2) gemachte Aussage liefert also für jedes $f \in C^1[0, 2]$ mit $h = n^{-1/2}$ die Ungleichung

$$|B_{n,1}(f, x) - f(x)| \leqslant \frac{1}{n} \cdot x \cdot |f'(x)| + \frac{2}{n} \cdot (x^2 + (n+1)(x - x^2))^{1/2} \cdot \omega(f', n^{-1/2}).$$

Ist $f \in C^2[0, 1]$, also $\omega(f', n^{-1/2}) \leqslant \|f''\| \cdot n^{-1/2}$, so gilt

$$|B_{n,1}(f, x) - f(x)| \leqslant \frac{1}{n} \cdot x \cdot |f'(x)| + \frac{2}{n^{3/2}} \cdot (x^2 + (n+1)(x - x^2))^{1/2} \cdot \|f''\| = O\left(\frac{1}{n}\right).$$

Eine so günstige Ordnung läßt sich im vorliegenden Fall nicht mit der im obigen Satz unter (3) angegebenen Ungleichung, dem Analogon der Aussage von E. Censor, erzielen, da in dieser Abschätzung die Größe $|L(e_1 - x, x)|$ zunächst nach oben durch $L(|e_1 - x|, x)$ und dann weiter durch $L(e_0, x)^{1/2} \cdot L((e_1 - x)^2, x)^{1/2}$ abgeschätzt wird. Im soeben betrachteten Beispiel würde dies auch für $f \in C^2[0, 2]$ nur zur Ordnung $O(n^{-1/2})$ führen.

Schlußbemerkungen. 1. Die Verwendung der Abschätzung (2) ist auch dann vorteilhaft, falls mit einem Testfunktionensystem $\{u_0 = e_0, u_1 = e_1, u_2\}$ gearbeitet wird, das die Voraussetzungen des obigen Satzes erfüllt, wo aber $u_2 \neq e_2$ gilt. Ein Beispiel dieser Art wird bei H. Gonska [7, S. 155] behandelt. Wir bemerken, daß in diesem Fall unter der Zusatzvoraussetzung

$$(B') \qquad F(x, x) = \sum_{k=0}^{2} a_k(x) \cdot u_k(x) = 0, \qquad x \in [a, b],$$

die Gleichung

$$L(F(\cdot, x), x) = L\left(\sum_{k=0}^{2} a_k(x) \cdot u_k(\cdot), x\right) - \sum_{k=0}^{2} a_k(x) \cdot u_k(x)$$

$$= \sum_{k=0}^{2} a_k(x) \cdot [L(u_k, x) - u_k(x)]$$

gilt. Reproduziert also L die Funktionen $u_0 = e_0$ und $u_1 = e_1$, so reduziert sich die Ungleichung unter (2) auf die einfachere Form

$$|L(f, x) - f(x)| \leqslant \left[\left(\frac{a_2(x)}{C}\right)^{1/2} \cdot (L(u_2, x) - u_2(x))^{1/2} + \frac{a_2(x)}{C \cdot h} \cdot (L(u_2, x) - u_2(x))\right] \cdot \omega(f', h).$$

2. Die Abschätzung unter (3), welche sich aus der unter (2) durch eine erneute Verwendung der Cauchy-Schwarzschen Ungleichung ergibt, ist für Anwendungen weniger interessant, obwohl eine ganz ähnliche Form bei Censor als Hauptaussage auftritt.

Schreibt man die rechte Seite in (3) nämlich in der Form

$$|L(e_0, x)-1| \cdot |f(x)| + \mu_L(x) \cdot \Big[L(e_0, x)^{1/2} \cdot |f'(x)| + (L(e_0, x)^{1/2} + \frac{1}{h} \cdot \mu_L(x)) \cdot \omega(f', h) \Big],$$

so ergibt sich unter der Annahme, daß $L(e_0, x) = 1$ ist, der Ausdruck

$$\mu_L(x) \cdot \Big[|f'(x)| + \Big(1 + \frac{1}{h} \cdot \mu_L(x)\Big) \cdot \omega(f', h) \Big].$$

Betrachtet man nun eine Folge (L_n) positiver linearer Operatoren, so ergibt sich als Majorante für $|L_n(f, x)-f(x)|$ ein Ausdruck der Form

$$\mu_{L_n}(x) \cdot \Big[|f'(x)| + \Big(1 + \frac{1}{h} \cdot \mu_{L_n}(x)\Big) \cdot \omega(f', h) \Big].$$

Hieran zeigt sich, daß die aus (3) herleitbare Größenordnung des Verschwindens von $|L_n(f, x)-f(x)|$ für $f'(x) \neq 0$ nicht besser als die von $\mu_{L_n}(x)$ sein kann. Betrachtet man zum Beispiel die Bernstein-Operatoren, so ist dort mit $F(t, x) = (t-x)^2$

$$\mu_{L_n}(x) = \left(\frac{x(1-x)}{n}\right)^{1/2},$$

also $|B_n(f, x)-f(x)| \leq c \cdot \left(\frac{x(1-x)}{n}\right)^{1/2}$ für alle $f \in C^1[0, 1]$, ohne daß Differenzierbarkeitseigenschaften von f' diese Ordnung im positiven Sinne beeinflussen. Die Abschätzungen unter (1) und (2) haben diesen Nachteil im genannten Beispiel nicht.

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UOPŠTENJE JEDNOG STAVA CENSORA I DEVOREA

Jutta Meier

U radu je dokazano uopštenje jednog stava Censora i DeVorea o aproksimaciji neprekidno-diferencijabilnih funkcija pomoću pozitivnih linearnih operatora i dato više komentara za slučaj tzv. Bernsteinovih operatora.

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EXAMPLES AND REMARKS TO A FIXED POINT THEOREM

Janusz Matkowski and Jerzy Miś

Abstract. A fixed point theorem for nonlinear contractions is under discussion. Three examples and an observation about metric convexity are given.

The first author of this note proved the following fixed point theorem (cf. [3] and J. Dugundji, A. Granas [2], p. 12).

Theorem 1. *Let (X, d) be a complete metric space. Suppose that $F: X \to X$ is a φ-contraction, i.e. that*

(1) $$d(Fx, Fy) \leqslant \varphi(d(x, y)) \qquad (x, y \in X)$$

where $\varphi: R_+ \to R_+$ satisfies the following two conditions

(i) *φ is non-decreasing not necesserily continuous,*

(ii) $\lim\limits_{n\to\infty} \varphi^n(t) = 0$ *for each* $t > 0$.

Then F has a unique fixed point $a \in X$ and $\lim\limits_{n\to\infty} F^n(x) = a$ for each $x \in X$.

(Here and in the sequel $R_+ = [0, \infty)$, φ^n and F^n denote the n-th iterate of φ and F respectively).

This theorem appeared to be useful and easy to handle with in the theory of functional equations (cf. [3]) and therefore it seems to be of interest to answer the question if some of the assumptions (concering φ) can be weaken. In the present note we discuss in detail this question showing in particular, that the answer is no. Example 1 shows that the condition (i) on monotonicity of φ cannot be omitted. One can easily observe that the both conditions (i) and (ii) imply

(iii) $$\varphi(t) < t \quad \text{for} \quad t > 0.$$

This inequality plays an important role in the proof of Theorem 1 (cf. [3] as well as [2], p. 12), therefore one could expect that in remains true after changing (i) by (iii). Our Example 2 settles in negative this a little more delicate problem. In the Example 3 we show that also conditions (i), (ii) cannot be replaced by (i), (iii).

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On the other hand let us note that if (X, d) is convex in Menger's sense i.e., if for every $x, y \in X$ $(x \neq y)$ there is a $z \in X$ $(z \neq x, z \neq y)$ such that $\mathrm{d}(x, z) + \mathrm{d}(z, y) = \mathrm{d}(x, y)$ (which is usually satisfied in applications), then both the conditions (i) and (ii) can be replaced by (iii) (cf. Boyd-Wong [1]). In this case even much more can be proved. Namely, as a corollary of the main theorem in [4] we have the folloving result:

Theorem 2. *If* (X, d) *is complete and convex,* $F: X \to X$ *satisfies* (1) *and* φ *satisfies* (iii) *then there exists a nondecreasing, concave and continuously differentiable function* $\gamma: R_+ \to R_+$ *such that* $\gamma(t) < t$ *for* $t > 0$ *and*

$$\mathrm{d}(Fx, Fy) \leqslant \gamma(\mathrm{d}(x, y)) \quad (x, y \in X).$$

Example 1. Let $X = [0, 1]$, $\mathrm{d}(x, y) = |x - y|$ and let Q be the set of rational numbers. The relation „$\sim$" defined as follows:

$$x \sim y \Rightarrow x - y \in Q$$

is an equivalence relation in X. Let $\{X_\alpha\}_{\alpha \in A}$ be the family of all equivalence classes. Thus

$$X = \bigcup_{\alpha \in A} X_\alpha, \; X_\alpha \cap X_\beta = \varnothing \; (\alpha \neq \beta), \; X_\alpha \neq \varnothing \; (\alpha \in A).$$

By the axiom of choice there exist two functions

$$A \ni \alpha \mapsto x_\alpha \in X_\alpha, \qquad A \ni \alpha \mapsto y_\alpha \in X - X_\alpha.$$

Now we define the functions $F, \bar{F}: X \to X$ by the formulas

$$F(x) = x_\alpha \text{ iff } x \in X_\alpha; \quad \bar{F}(x) = y_\alpha \text{ iff } x \notin X_\alpha,$$

and the function $\varphi: R_+ \to R_+$ by the formula

$$\varphi(t) = \begin{cases} 1, & t \notin Q, \\ t/2, & t \in Q. \end{cases}$$

To check that (1) holds take $x, y \in X$. If $x, y \in X_\alpha$ for some $\alpha \in A$ then, of course, $\mathrm{d}(Fx, Fy) = 0$. If there is no such an α, then $\mathrm{d}(x, y) \notin Q$ and, consequently, $\varphi(\mathrm{d}(x, y)) = 1$. Thus F and φ satisfy (1) i.e., F is a φ-contraction. In the same way we can verify that $\bar{F}$ is a φ-contraction too. Moreover, it follows immediately from the definition of φ that $\varphi^n(t) \to 0$ for each $t > 0$, i.e. condition (ii) holds. Thus, except of (i), all the conditions of of the Theorem 1 are fulfilled. On the other hand one can easily observe that the set of fixed points in the case of mapping F is infinite (equal to $\{x_\alpha\}_{\alpha \in A}$) and in the case of mapping $\bar{F}$ is empty.

Remark 1. Conditions (1) and (ii) does not even assure the uniqueness of fixed point of F.

Example 2. Let $X=\{x_1, x_2, \ldots\}$ be an arbitrary countable set. Define $d: X\times X\to R$ as follows

$$d(x_m, x_n)=1+\frac{1}{2^m}+\frac{1}{2^n} \text{ for } n\neq m \text{ and } d(x_n, x_n)=0.$$

Clearly (X, d) is a complete metric space. Note that the metric d has the following property

$$(2) \qquad d(x_m, x_n)=d(x_k, x_1)\ (m<n)\wedge k<1) \Leftrightarrow x_m=x_k \wedge x_n=x_1$$

i.e. all the distances $d(x_m, x_n)$ $(m<n)$ are different.

Let $F: X\to X$ be the shifting map $Fx_n=x_{n+1}$. Since the set

$$D=\{d(x_m, x_n): n, m\in N\ (n\neq m)\}$$

is countable and

$$d(Fx_m, Fx_n)=d(x_{m+1}, x_{n+1})<d(x_m, x_n) \quad (n\neq m)$$

we can choose a positive real $\alpha_{m,n}$ such that

$$(3) \qquad \alpha_{m,n}\in(d(x_{m+1}, x_{n+1}),\ d(x_m, x_n))-D,\ m\neq n \quad (m, n\in N).$$

Define now $\varphi: R_+\to R_+$ as follows

$$\varphi(t)=\begin{cases}0, & t\notin D,\\ \alpha_{m,n}, & t=d(x_m, x_n)\in D.\end{cases}$$

It follows from the property (2) that φ is correctly defined. By (3) we have $\varphi(t)<t$ for $t>0$ and, because $\varphi^2(t)=0$ for all $t\in R_+$, we also have $\lim_{n\to\infty}\varphi^n(t)=0$. Thus φ satisfies both the conditions (ii) and (iii). Moreover, by (3) and it easily follows that (1) holds, i.e. F is φ-contraction. But clearly F has no fixed point.

Example 3. Let the complete metric space (X, d), D and the map F be as in Example 2. Let use define $\varphi: R_+\to R_+$ as follows

$$(4) \qquad \varphi(t)=\begin{cases}t/2, & t\in[0, 1],\\ \sup\{d(x_{m+1}, x_{n+1}): d(x_m, x_n)\leq t;\ m, n\in N\}, & t>1.\end{cases}$$

It follows from the definitions of F and φ that F is a φ-contraction. Evidently, φ is nondecreasing, i.e. φ satisfies condition (i). To see that φ satisfies also (iii) note that every accumulation point t of D such that $t>1$ has the form $t=1+\frac{1}{2^k}$; $k\in N$.

To calculate $\varphi(t)$ take an arbitrary m and n such that

$$d(x_m, x_n)=1+\frac{1}{2^m}+\frac{1}{2^n}\leq t=1+\frac{1}{2^k}.$$

Hence $m \geq k+1$, $n \geq k+1$ and, in view of (4), we have

$$\varphi(t) \leq 1 + \frac{1}{2^{k+1}} + \frac{1}{2^{k+2}} = 1 + \frac{1}{2^{k+1}} < t.$$

If $t > 1$ is not an accumulation point of the set D then there is a maximal $\mathrm{d}(x_m, x_n)$ such that $\mathrm{d}(x_m, x_n) \leq t$. Consequently, we have

$$\varphi(t) = \mathrm{d}(x_{m+1}, x_{n+1}) < \mathrm{d}(x_m, x_n) \leq t.$$

Thus $\varphi(t) < t$ for $t > 0$ which shows that φ satisfies (iii). This completes the required construction.

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PRIMERI I PRIMEDBE ZA TEOREMU O NEPOKRETNOJ TAČKI

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U radu se razmatra teorema o nepokretnoj tački za nelinearne kontrakcije. Daju se tri primera i jedno opažanje oko konveksnosti metrike.

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НЕКОТОРЫЕ ОБОБЩЕНИЯ СОВЕРШЕННОЙ НОРМАЛЬНОСТИ

Любиша Кочинац

Резюме. Настоящая работа является естественным продолжением работы [4], в которой введен класс слабо совершенных пространств. Здесь определяются и изучаются классы почти совершенных и почти слабо совершенных пространств.

1. Введение

Как известно, пространство X называется совершенным, если в нем каждое замкнутое (открытое) подмножество является множеством типа G_δ (F_σ). Следующие два определения дают естественные обобщения этого понятия:

Определение 1.1. *Пространство X называется слабо совершенным, если каждое замкнутое в X множество F содержит множество A всюду плотное в F и типа G_δ в X.*

Определение 1.2. *Назовем пространство X почти совершенным, если каждое множество U открыто в X содержит множество всюду плотное в U и типа F_σ в X.*

Если конкретно не будет названа аксиома отделимости, все пространства в настоящей работе надо считать хаусдорфовым; отображения предполагаются непрерывными и сюръективными. Мы пользуемся стандартными обозначениями и терминологией из [1] и [2]. Например, если $f: X \to Y$ отображение, тогда через $f^{\#}(A) = \{y \in Y : f^{-1}(y) \subset A\}$ обозначаем малый образ множества $A \subset X$; $c(X) = \aleph_0$ означает, что каждое семейство дизъюнктных открытых в X множеств счетно; $C_p(X)$ — пространство непрерывных вещественных функций на X в топологии поточечной сходимости. Ряд определений и утверждений цитируются также по [1] и [2], а не по первоисточникам.

2. Почти слабо совершенные пространства

Класс слабо совершенных пространств введен в [4]. Теперь мы докажем следующее

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Предложение 2.1. *Пусть X — слабо совершенное P-пространство и $f: X \to Y$ замкнутое неприводимое отображение. Тогда и пространство Y слабо совершенно.*

Доказательство. Пусть H произвольное замкнутое подмножество пространства Y. Так как X слабо совершенно, то в замкнутом в X множестве $F=f^{-1}(H)$ содержится множество $A=\bigcap\{U_i: i\in N\}$ типа G_δ в X и всюду плотное в F. Множество A открыто в X, ибо X P-пространство. Поскольку f замкнутое и неприводимое, то $H=f(F)=f(\overline{A})=\overline{f^{\#}(A)}$ (см. [2; 110/VI]). Без труда проверяется, что множество $f^{\#}(A)=f^{\#}(\bigcap\{U_i: i\in N\})=\bigcap\{f^{\#}(U_i): i\in N\}$ имеет тип G_δ в Y; так как оно, кроме того, содержится в H и всюду плотное в нем, то заключаем, что пространство Y слабо совершенно.

Понятие слабой совершенности можем обобщить следующим образом:

Определение 2.2. *Пространство X будем называть* п о ч т и с л а б о с о в е р ш е н н ы м, *если каждое замкнутое в X множество F содержит множество A всюду плотное в F и типа G_δ в F.*

Опираясь на один результат И. Намиоки, замечен А. В. Архангельским (см. 4.1.5 в [1]), заключаем, что каждый бикомпакт Эберлейна [1] является почти слабо совершенным пространством. Если пространство полное в смисле Чеха и имеет счетное измельчение [2], то оно почти слабо совершенно; это фактически доказал Б. Шапировский (ДАН СССР 207 : 4, 1972).

Очевидно, почти слаба совершенность наследуется замкнутыми подпространствами.

Ниже мы дадим одно достаточно условие для того чтобы регулярное пространство было почти слабо совершенным, немедленно вытекающее из предложения 5 работы [3].

В регулярном пространстве T рассматрываются следующие игры $A(T)$ и $B(T)$ для двух игроков — Первый и Второй. Первый выбирает непустое открытое множество $\Gamma_0 \subset T$, а Второй открытое множество Γ_1, такое, что $\Gamma_1 \subset \overline{\Gamma}_1 \subset \Gamma_0$; потом Первый опять берет открытое множество Γ_2 для которого $\overline{\Gamma}_2 \subset \Gamma_1$, итд. Второй выигрывает:

а) в игре $A(T)$ если последовательность $\{\Gamma_i: i\in N\}$ обладает свойством: если $x_i \in \Gamma_i$, то последовательность $\{x_i: i\in N\}$ содержит сходящуюся подпоследовательность;

б) в игре $B(T)$ если $\{\Gamma_i: i\in N\}$ является базой окрестностей некоторой точки в T.

Предложение 2.3. *Регулярное пространство X является почти слабо совершенным, если для каждого замкнутого в X множества F в игре $B(F)$ у Второго существует выигрышная стратегия.*

3. Почти совершенные пространства

Вернемся теперь к почти совершенным пространствам. К числу почти совершенных пространств относятся все совершенные пространства (в частности, все M_1 и все кружевные пространства [5; с. 201]) и все пространства с σ-локально конечной, π-базой, как это показал В. И. Пономарёв [6; лемма 7.1]. На замечательную широту этого класса указывает и следующая теорема (которую надо сравнить с хорошо известным фактом, что для совершенно нормального финально компактного пространства X, $c(X)=\aleph_0$. [2; 207/III]).

Теорема 3.2. *Пусть X-регулярное финально компактное пространство (в частности, бикомпакт). Тогда X почти совершенно если и только если $c(X)=\aleph_0$.*

Доказательство: Необходимость. Предположим противное, что $c(X)>\aleph_0$ и пусть $\{U_s: s\in S\}$, $|S|>\aleph_0$, несчетная система попарно непересекающихся непустых открытых подмножеств в X. Положим $U=\bigcup\{U_s: s\in S\}$. Так как X почти совершенно, то U содержит множество $A=\bigcup\{F_i: i\in N\}$ типа F_σ в X и всюду плотное в $U: A\subset U\subset \overline{A}$. Поскольку для каждого $s\in S$ пересечение $U_s\cap A$ непусто, то из всякого $U_s\cap A$ выберем какую-нибудь точку x_s. В результате получаем множество $B=\{x_s: s\in S\}$; нетрудно убедится в том, что это несчетное множество замкнуто и дискретно, что не может быть, так как X финально компактно.

Достаточность. Пусть U-произвольное открытое в X множество. Для каждого $x\in U$ зафиксируем, в силу регулярности X, открытое множество V_x, такое, что $x\in V_x\subset \overline{V}_x\subset U$. В семействе $\{V_x: x\in U\}$ выберем максимальное счетное подсемейство попарно непересекающихся множеств $V_1, V_2, \dots$; это можно сделать, так как $c(U)\leqq c(X)=\aleph_0$. Тогда U содержит F_σ-множество $\bigcup\{\overline{V}_i: i\in N\}$ которое в силу максимальности семейства $V_1, V_2, \dots$, всюду плотное в U. Теорема доказана.

Из предыдущей теоремы следует, что все диадические бикомпакты и все $\mathcal{H}$-метризуемые бикомпакты (в смысле Е. В. Шепина), являются почти совершенными пространствами; надо только заметить, что если X либо диадический бикомпакт, либо $\mathcal{H}$-метризуемый бикомпакт, то тогда $c(X)=\aleph_0$.

Из теоремы 3.1. и того, что произведение X любого семейства сепарабельных бикомпактов, удовлетворяет условию $c(X)=\aleph_0$ ([1; 1.5.2.]), вытекает

Предложение 3.2. *Произведение любого семейства сепарабельных бикомпактов является почти совершенным пространством.*

Замечание 3.3. Согласно теореме 1.5.18 из [1], совместимо с ZFC считать, что произведение любого семейства почти совершенных бикомпактов является почти совершенным.

Замечание 3.4. Вот несколько примеров почти совершенных пространств:

3.4.1. Квадрат известного пространства „две стрелки". Это пример почти совершенного, но не совершенного пространства;

3.4.2. Квадрат пространства Хелли (см. [2; 82/III]);

3.4.3. Σ-произведение и σ-произведение [1] несчетного числа экземпляров обычного отрезка [0, 1] (ср. с 1.5.29. и 1.5.30 в [1]).

Следствие 3.5. *Если X и Y — бикомпакты и пространства $C_p(X)$ и $C_p(Y)$ бомгоморфны, то тогда X и Y одновременно почти совершенные пространства г(или нет).*

Это просто следствие того, что при предположениях утверждения $c(X)=c(Y)$ (А. В. Архангельский).

Следствие 3.6. *Пусть X — бикомпакт Эберлейна* [1]. *Тогда:*

а) *Следующие условия равносильны:*

(i) *X метризуем;*

(ii) *X совершенно нормален;*

(iii) *X почти совершенно нормален.*

б) *$C_p(X)$ почти совершенное пространство.*

Для доказательства а) надо проверить только что из (iii) следует (i). По одной теореме А. В. Архангельского (см. 4.1.8. в [1]), для бикомпакта Эберлейна X всегда выполняется $w(X)=c(X)$; поэтому если X является почти совершенным, то тогда оно имеет счетную базу и, следовательно, метризуемо.

М. Талагран доказал (см. [1; 4.1.12]), что если X бикомпакт Эберлейна, то $C_p(X)$ финально компактное пространство и, поскольку всегда $c(C_p(X))=\aleph_0$, из теоремы 3.1. следует б).

Следствие 3.7. *Почти совершенный бикомпакт, который является либо секвенциальным, либо псевдорадиальным* [1], *имеет мощность не более континуума* (ср. с 2.1.12 и 1.3.15 в [1]).

Заметим, что, в отличие от совершенно нормального (и слабо совершенно нормального) бикомпакта, почти совершенно нормальный бикомпакт может быть сколь угодно большой мощности и наследственной плотности. В качестве примера годится обобщенный канторов дисконтинуум D^k веса k. А если почти совершенно нормальный бикомпакт удовлетворяет первой аксиоме счетности, верно ли, что его наследственная плотность не превосходит $\aleph_1$ (что справедливо в классе совершенно нормальных бикомпактов)?

4. Некоторые свойства почти совершенных пространств

Теперь мы остановимся на некоторых вопросах, из которых можем увидеть насколько почти совершенные пространства близки к совершенным пространствам.

Предложение 4.1. *Если X почти совершенно нормальное пространство и A открытое или плотное в X множество, то и A почти совершенно.*

Доказательство. Для случая открытого множества A это очевидно и, кроме того, не нуждается нормальность X.

Пусть A — всюду плотное в X и $V\subset A$ произвольное открытое в A множество. Возьмем множество U открыто в X, такое, что $V=U\cap A$. Так как X почти совершенно, существует множество $B=\cup\{F_i:i\in N\}$ типа F_σ в X, такое что $B\subset U\subset \overline{B}$. Для каждого $i\in N$ существует множество G_i открыто в X, такое, что $F_i\subset G\subset \overline{G_i}\subset U$, ибо X нормально. Разумеется, $A\cap G_i\neq\varnothing$ для всякого $i\in N$. Множество $\cup\{A\cap \overline{G_i}:i\in N\}$ имеет тип F_σ в A, содержится в V и всюду плотно в V, так как $Cl_X(A\cap\cup\{G_i:i\in N\})=Cl_X(\cup\{G_i:i\in N\})\supset U$.

Предложение 4.2. *Пусть $f:X\to Y$-замкнутое неприводимое отображение. Тогда пространство X почти совершенно если и только если Y таково.*

Доказательство. Предположим, что X почти совершенно и пусть V-произвольное непустое открытое множество в Y. Открытое в X множество $U=f^{-1}(V)$ содержит множество $A=\cup\{F_i:i\in N\}$ типа F_σ,для которого $U\subset\overline{A}$. В силу замкнутости и непрерывности отображения f, множество $f(A)$ имеет тип F_σ в Y, целиком лежит в V и плотно в V, т.е., Y — почти совершенно.

Пусть теперь Y — почти совершенно и U открытое подмножество в X. Так как f замкнуто и неприводимо, то множество $f^{\#}(U)$ непусто и открыто в Y. Поэтому найдется множество $B=\cup\{H_i:i\in N\}$ типа F_σ в Y, такое, что $B\subset f^{\#}(U)\subset\overline{B}$. Рассмотрим множество $f^{-1}(B)$ типа F_σ в X. Прежде всего, оно содержится в U, поскольку $f^{-1}(B)\subset f^{-1}(f^{\#}(U))\subset U$. С другой стороны, это множество плотно в U. Действительно, если $G\subset U$ открыто, то $f^{\#}(G)$ открыто в $f^{\#}(U)$ и, следовательно, пересекается с B. Поэтому, $G\cap f^{-1}(B)\supset f^{-1}(B)\cap$ $\cap f^{-1}(f^{\#}(G))\neq\varnothing$. Значит пространство X почти совершенно.

Замечание 4.3. Если X — почти совершенный бикомпакт, то каждый непрерывный образ Y этого пространства будет почти совершенным бикомпактом.

Это вытекает из теоремы 3.1., учитывая, что $c(Y)\leqq c(X)$.

Теорема 4.4. *Произведение $X\times Y$ почти совершенного пространства X и метризуемого пространства Y, является почти совершенным пространством.*

Доказательство. В пространстве Y фиксируем σ-окально конечную базу $\mathcal{B}=\cup\{\mathcal{B}_i:i\in N\}$, $\mathcal{B}_i=\{B_{is}:s\in S_i\}$ (и положим $S=\cup\{S_i:i\in N\}$. Пусть G-открытое в $X\times Y$ множество. Для каждого $(x, y)\in G$, существуют открытое в X множество U_x и множество B_y из $\mathcal{B}$, такие, что $(x, y)\in U_x\times$ $\times B_y\subset U_x\times\overline{B_y}\subset G$; иначе говоря, существует $T\subset S$, такое, что $G=\cup\{U_s\times$ $\times\overline{B_t}:t\in T\}$. Так как X почти совершенно, то для каждого $t\in T$ най-дется множество типа F_σ в X для которого $\cup\{F_{tj}:j\in N\}\subset U_t\subset\overline{\cup\{F_{tj}:j\in N\}}$ Положим

$$\mathcal{A}_{ij}=\{F_{tj}\times\overline{B_t}:B_t\in\mathcal{B}_i,\ t\in T\},\ A_{ij}=\cup\{A:A\in\mathcal{A}_{ij}\}.$$

Ввиду того, что каждое $\mathcal{A}_{ij}$-локально конечное семейство, каждое множество A_{ij} замкнуто. Легко проверить, что $\cup\{A_{ij}:i, j\in N\}\subset G\subset\overline{\cup\{A_{ij}:i, j\in N\}}$, что и означает, что $X\times Y$ почти совершенное пространство. Теорема доказана.

Б. С. Клебанов называет ψ-пространством замкнутый образ произведения диадического бикомпакта и метрического пространства. Из теоремы 4.4., замечания после теоремы 3.1. и предложения 4.2. получаем

Следствие 4.5. *Каждое ψ-пространство является почти совершенным.*

Автору неизвестно можно ли это утверждение распространить на класс φ-пространств, как называются замкнутые образы произведений любого семейства метрических пространств.

Прежде чем доказывать следующую теорему, приведем несколько легко проверяемых лемм технического характера.

Напомним, что пространство X называется слабо линделёфовым, если из каждого открытого покрытия $\mathcal{U}$ пространства X можно выделить счетную част $\mathcal{V}$, такую, что $\overline{\bigcup\{V : V \in \mathcal{V}\}} = X$.

Лема 4.6. *Если X-нормальное слабо линделёфово пространство, то и каждое его замкнутое подпространство слабо линделёфово.*

Лема 4.7. *Если $X = \bigcup\{X_i : i \in N\}$- объединение счетного семейства слабо линделёфовых пространств, то и X слабо линделёфово.*

Лема 4.8. *Пространство X наследственно слабо линделёфово, тогда и только тогда когда каждое открытое иодмножество в X слабо линделёфово.*

Теорема 4.9. *Для каждого почти совершенно нормального пространства X следующие условия равносильны: (а) X слабо линделёфово; (б) X наследственно слабо линделёфово.*

Доказательство. Ясно, надо доказать только, что из (а) следует (б). Итак, пусть X-слабо линделёфово пространство и пусть U любое открытое множество в X и $\mathcal{P}$ произвольное открытое покрытие множества U. Так как X почти совершенно, то существует множество $A = \bigcup\{F_i : i \in N\}$ типа F_σ в X такое, что $A \subset U \subset \overline{A}$. В силу нормальности пространства X, по лемме 4.6., каждое F_i слабо линделёфово; поэтому и A слабо линделёфово, согласно лемме 4.7. Заключаем, что существует счетное семейство $\{P_i : i \in N\}$ элементов из $\mathcal{P}$, такое, что $\overline{\bigcup\{P_i : i \in N\}} \supset A$, и значит, $\overline{\bigcup\{P_i : i \in N\}} \supset U$. Наконец, надо сослаться на лемму 4.8., чем теорема доказана.

Примечание 4.10. Как стало известно автору В. В. Ткачуком в работе: О кардинальных инвариантах типа числа Суслина, ДАН СССР 270: 4 (1983) получены результаты аналогичные теоремам 3.1. и 4.4.; там же содержится и положительный ответ на вопрос после следствия 4.5.

Примечание 4.11. Эту работу надо было опубликовать в Math. Balkanica 13 (1983), где она поступила 16. 05. 1983; но, статья осталась неопубликованной ибо том 13, по своеобразным причинам, не вышел. Результаты статьи (и статьи [4]) были сообщены на: VII Congress of Balkan Mathematicians, December, 19—23 (1983), Athens.

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NEKA UOPŠTENJA SAVRŠENE NORMALNOSTI

Ljubiša Kočinac

Ovaj rad predstavlja prirodan nastavak rada [4] u kojem je uvedena klasa slabo savršenih prostora. Ovde se definišu i izučavaju klase skoro savršenih i skoro slabo savršenih prostora.

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A REAL ANALYTICAL INTEGRAL FORMULA FOR A SIMPLE ROOT OF A SYSTEM OF TWO NONLINEAR EQUATIONS

N. I. Ioakimidis

Abstract. A method for the closed-form solution of two real nonlinear algebraic or transcendental equations, $f(x, y)=0$, $g(x, y)=0$, possessing one simple root (x_0, y_0) in a finite domain of the Oxy-plane is proposed. This method is based on the Picard method for the calculation of the number of roots of a system of nonlinear equations or, more explicitly, on the classical Gauss (or divergence) theorem in elementary mathematical analysis. The resulting formulae for x_0 and y_0 contain integrals including the functions f and g and their first partial derivatives. The present results generalize earlier relevant results for a single nonlinear equation.

1. Introduction

Nonlinear algebraic or transcendental equations and systems of such equations appear quite frequently in mechanics, physics and engineering. A long series of numerical methods are available in the literature for their approximate solution (see, e. g., [14] for systems of nonlinear equations). Yet, the closed-form solution of these equations is an interesting alternative possibility both from the theoretical point of view (derivation of more or less „elegant" analytical formulae) and from the practical point of view, since numerical methods cannot take into account the variation of the parameters in the nonlinear equations to be solved and, quite frequently, they do not converge (particularly in the case of systems of nonlinear equations) unless sufficiently accurate estimates of the sought roots have been provided.

For these reasons, a moderately long series of methods for the closed-form solution of nonlinear equations and systems of such equations, leading to analytical integral formulae for the sought roots, have been developed. Methods based on the Cauchy theorem and relevant integral formulae in the theory of functions of one complex variable have been extensively used for the determination of zeros of analytic functions of one complex variable (see, e. g., [3 — 5] and the references reported there). Much more comlicated results for systems of nonlinear equations based on integral theorems in the theory of several complex variables are also available (see, e. g., [1, 2, 7]), but not simple at all and not convenient for direct use in practical applications.

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Before two years the old and forgotten Picard's method for the computation of the number of roots of a set of nonlinear equations [14, 16] has been reconsidered by Hoenders and Slump [8]. Picard's method [15, 16] constitutes a modification and extension of earlier relevant and even today more popular results by Kronecker referenced in [15, 16, 8]. Last year, Ioakimidis and Papadakis studied Picard's method and modified it to apply to the closed-form solution of one nonlinear equation $h(x)=0$ [9, 11]. It was also realized that equivalent results can be obtained by the method of integration by parts for ordinary one-dimensional integrals [10, 12, 13]. These results were applied to the classical Kepler's transcendental equation in celestial mechanics [11, 12], as well as to the Lagrange's quintic equation also in celestial mechanics [13].

In this note, we will generalize these results to systems of two nonlinear equations, $f(x, y)=0$, $g(x, y)=0$, possessing one simple root (x_0, y_0) in an open region R of the Oxy-plane. Our method is a slight modification of Picard's method [15, 16], but with the classical Gauss (or divergence) theorem in elementary mathematical analysis explicitly used, as well as the aforementioned method of integration by parts. To the best of our knowledge, no competitive method based on real variables is available.

2. Derivation of the formula

We assume that we have to solve a system of two nonlinear algebraic or transcendental equations $f(x, y)=0$, $g(x, y)=0$ possessing one simple real root (x_0, y_0) in the open region R surrounded by a simple smooth closed contour C on the Oxy-plane. We assume also that f and g possess continuous first and second partial derivatives with respect to both variables x and y in R, as well as on C. Next, we consider the third equation $z=0$, as well as the finite cylinder $V=\{(x, y, z)\colon (x, y)\in R,\ -\varepsilon<z<\varepsilon\}$, where ε is an arbitary positive constant [15, 16] (Picard [15, 16] had considered the equation $zD=0$, where D is the Jaciobian determinant $\partial(f, g)/\partial(x, y)$, instead of $z=0$, to avoid an ambiguity in sign at the roots of f and g; for just one root (x_0, y_0) this ambiguity is insignificant).

Now, following Picard [15, 16], we consider the equations

$$F_1(x, y, z)=f(x, y)=0,\quad F_2(x, y, z)=g(x, y)=0,\quad F_3(x, y, z)=z=0, \tag{1}$$

as well as the functions [15, 16]

$$A=\begin{vmatrix} F_1 & \partial F_1/\partial y & \partial F_1/\partial z \\ F_2 & \partial F_2/\partial y & \partial F_2/\partial z \\ F_3 & \partial F_3/\partial y & \partial F_3/\partial z \end{vmatrix} \Big/ (F_1^2+F_2^2+F_3^2)^{3/2}, \tag{2a}$$

$$B=\begin{vmatrix} F_1 & \partial F_1/\partial z & \partial F_1/\partial x \\ F_2 & \partial F_2/\partial z & \partial F_2/\partial x \\ F_3 & \partial F_3/\partial z & \partial F_3/\partial x \end{vmatrix} \Big/ (F_1^2+F_2^2+F_3^2)^{3/2}, \tag{2b}$$

(2c)
$$C=\begin{vmatrix} F_1 & \partial F_1/\partial x & \partial F_1/\partial y \\ F_2 & \partial F_2/\partial x & \partial F_2/\partial y \\ F_3 & \partial F_3/\partial x & \partial F_3/\partial y \end{vmatrix} / (F_1^2+F_2^2+F_3^2)^{3/2}.$$

Then, because of (1), we find

(3)
$$A=\frac{fg_y-gf_y}{(f^2+g^2+z^2)^{3/2}},\quad B=\frac{gf_x-fg_x}{(f^2+g^2+z^2)^{3/2}},\quad C=\frac{(f_x g_y-f_y g_x)\, z}{(f^2+g^2+z^2)^{3/2}}.$$

Then, on the basis of the developments of Picard [15, 16], it follows directly that

(4)
$$\frac{\partial A}{\partial x}+\frac{\partial B}{\partial y}+\frac{\partial C}{\partial z}=0.$$

Furthermore, we notice that

(5)
$$\frac{\partial (xA)}{\partial x}+\frac{\partial (xB)}{\partial y}+\frac{\partial (xC)}{\partial z}=x\left(\frac{\partial A}{\partial x}+\frac{\partial B}{\partial y}+\frac{\partial C}{\partial z}\right)+A=A.$$

Now we apply the classical Gauss (or divergence) theorem:

(6)
$$\frac{1}{4\pi}\iiint\limits_{V-E} \operatorname{div} \boldsymbol{F}\, \mathrm{d}V=\frac{1}{4\pi}\iint\limits_{S} \boldsymbol{F}\cdot\boldsymbol{n}\, \mathrm{d}S$$

(where V is the aforementioned finite cylinder, E an ellipsoid with centre the point $(x_0, y_0, 0)$ [16], S the surface of $V-E$ and n the unit outer normal vector to S) for the vector field

(7)
$$\boldsymbol{F}=x(A\boldsymbol{i}+B\boldsymbol{j}+C\boldsymbol{k}) \Rightarrow \operatorname{div} \boldsymbol{F}=A$$

because of (5). This vector field is continuously differentiable because of (3) and the assumptions already made for f and g since the point $(x_0, y_0, 0)$ does not belong to $V-E$.

The evaluation of the surface integral in the right-hand side of (6) (as the dimensions of E tend to zero) is analogous to the evaluation of the corresponding integral by Picard [15, 16]. It is found that

(8)
$$\frac{1}{4\pi}\iint\limits_{S} \boldsymbol{F}\cdot\boldsymbol{n}\, \mathrm{d}S=I_1+I_2\pm x_0,$$

where

(9)
$$I_1=\frac{\varepsilon}{2\pi}\iint\limits_{R} x\,\frac{f_x g_y-f_y g_x}{(f^2+g^2+\varepsilon^2)^{3/2}}\,\mathrm{d}x\,\mathrm{d}y,$$

(10)
$$I_2=\int\limits_{C}(P\,\mathrm{d}x+Q\,\mathrm{d}y)$$

with

(11a) $$P=\frac{x}{2\pi}\cdot\frac{fg_x-gf_x}{f^2+g^2}\cdot\frac{\varepsilon}{(f^2+g^2+\varepsilon^2)^{1/2}},$$

(11b) $$Q=\frac{x}{2\pi}\cdot\frac{fg_y-gf_y}{f^2+g^2}\cdot\frac{\varepsilon}{(f^2+g^2+\varepsilon^2)^{1/2}}.$$

As far as the sign before x_0 in (8) is concerned, it depends on the sign of the Jacobian detrminant D. This means that just the absolute value of x_0 can be determined by the present method. If the sign of D is constant in V, then no ambiguity of sign in (8) exists. The same holds true if the region R lies in the halfplane $x>0$ (or in the halfplane $x<0$) enterily. (This can also be accomplished by a change of variable of the form $x^*=x-c$, where c is an appropriate constant.) In any case, after the determination of $\pm x_0$, $\pm y_0$, their signs can be determined directly by inspection (that is, by checking which of the points $(\pm x_0, \pm y_0)$ is really a root of the system of nonlinear equations $f(x, y)=0$, $g(x, y)=0$. In our opinion, it is not worthwhile to modify the above approach by using the function zD instead of z as F_3 in order to avoid completely this sign ambiguity [15, 16] since in such a case the present formulae would become more complicated.

We have finally to evaluate the left-hand side integral in (6). Because of the first of (3) and the second of (7), it is clear that the ellipsoid E]16] can be ignored when evaluating this integral (since its dimensions have been assumed tending to zero and $\operatorname{div} F$ behaves like $1/\rho^2$, with $\rho^2=(x-x_0)^2+$ $+(y-y_0)^2+z^2$, near $(x_0, y_0, 0)$). Hence,

(12) $$\frac{1}{4\pi}\iiint_V \operatorname{div} F\,\mathrm{d}V=I_3,$$

where

(13) $$I_3=\frac{1}{4\pi}\iiint_V A\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=\frac{1}{4\pi}\iint_R\left[\int_{-\varepsilon}^{\varepsilon}A\,\mathrm{d}z\right]\mathrm{d}x\,\mathrm{d}y.$$

But, because of the first of (3),

(14) $$\int_{-\varepsilon}^{\varepsilon}A\,\mathrm{d}z=\int_{-\varepsilon}^{\varepsilon}\frac{fg_y-gf_y}{(f^2+g^2+z^2)^{3/2}}\,\mathrm{d}z=(fg_y-gf_y)\int_{-\varepsilon}^{\varepsilon}\frac{\mathrm{d}z}{(f^2+g^2+z^2)^{3/2}}.$$

Moreover,

(15) $$\int\frac{\mathrm{d}z}{(f^2+g^2+z^2)^{3/2}}=\frac{1}{f^2+g^2}\cdot\frac{z}{(f^2+g^2+z^2)^{1/2}}.$$

Therefore,

(16) $$\int_{-\varepsilon}^{\varepsilon}A\,\mathrm{d}z=\frac{fg_y-gf_y}{f^2+g^2}\cdot\frac{2\varepsilon}{(f^2+g^2+\varepsilon^2)^{1/2}}$$

and, in this way,

$$I_3 = \frac{\varepsilon}{2\pi} \iint_R \frac{fg_y - gf_y}{f^2+g^2} \cdot \frac{dx\,dy}{(f^2+g^2+\varepsilon^2)^{1/2}}. \tag{17}$$

The integrand in this last integral behaves like $1/r$ near (x_0, y_0) (with $r^2 = (x-x_0)^2+(y-y_0)^2$) and, therefore, this integral is a weakly singular integral.

If we assume that the region R is of type I, that is, $R=\{(x, y): \alpha<x<\beta$ and $\gamma(x)<y<\delta(x)\}$ and take into consideration that

$$\frac{\partial}{\partial y} \tan^{-1} \frac{fg_y - gf_y}{f^2+g^2}, \tag{18}$$

where the function $\tan^{-1}(g/f)$ is defined in such a way that it is a continuous function with respect to the variable y (independently of the selected branch or branches) so that integrating (18) is possible, then (17) takes the form

$$I_3 = \frac{\varepsilon}{2\pi} \int_\alpha^\beta \left\{ \int_{\gamma(x)}^{\delta(x)} \left[\frac{\partial}{\partial y} \tan^{-1} \frac{g}{f} \right] \frac{dy}{(f^2+g^2+\varepsilon^2)^{1/2}} \right\} dx \tag{19}$$

or further (after an integration by parts)

$$I_3 = \frac{\varepsilon}{2\pi} \int_\alpha^\beta \left[\frac{\tan^{-1}(g/f)}{(f^2+g^2+\varepsilon^2)^{1/2}} \Bigg|_{\gamma(x)}^{\delta(x)} + \int_{\gamma(x)}^{\delta(x)} \tan^{-1} \frac{g}{f} \, \frac{ff_y+gg_y}{(f^2+g^2+\varepsilon^2)^{3/2}}\, dy \right] dx \tag{20}$$

(with $h(x)\Big|_\gamma^\delta \equiv h(\delta) - h(\gamma)$), since

$$\frac{\partial}{\partial y} \frac{1}{(f^2+g^2+\varepsilon^2)^{1/2}} = -\frac{ff_y+gg_y}{(f^2+g^2+\varepsilon^2)^{3/2}}. \tag{21}$$

The integrands in (20) present just simple jump discontinuities and, therefore, the corresponding integrals can be evaluated as ordinary integrals.

Finally, taking into account (6), (8) and (12), we find

$$x_0 = \mp (I_1 + I_2 - I_3), \tag{22}$$

where I_1, I_2 and I_3 are determined from (9), (10) and (20). We have already commented on the ambiguity of sign in (22). Although (22) gives only x_0, an analogous formula for y_0 can also be directly derived in a similar way, although after the determination of x_0 our equations $f(x, y)=0$, $g(x, y)=0$ have just one unknown, y_0, and the methods of solution of one nonlinear equation [9—13] are now applicable. Another possibility consists in using the formula

$$\frac{\partial}{\partial y} \cot^{-1} \frac{g}{f} = \frac{gf_y - fg_y}{f^2+g^2} = \frac{\partial}{\partial y} \tan^{-1} \frac{f}{g} \tag{23}$$

instead of (18) (where again $\cot^{-1}(g/f)$ should be defined in such a way that this function be a continuous function with respect to the variable y) or equivalently, alternating the roles of f and g, which is completely permissible.

Although the previous results aimed to the closed-form solution of the system of nonlinear equations $f(x, y)=0$, $g(x, y)=0$, numerical results can also be obtained from (9), (10) and (20) by using classical quadrature rules, like the trapezoidal, the Simpson and the Gauss quadrature rules [6], which converge (as the numbers of nodes tend to infinity) even for integrands with jump discontinuities (like the integrands in (20)) [6]. Numerical results were obtained from the previous formulae (by using the trapezoidal quadrature rule) in few special cases and this verified their vailidity. From the numerical point of view, moderately accurate numerical results obtained by the present method can be used as starting approximations to iterative methods (like the Newton-Raphson method) [14] so that the convergence of these methods be assured.

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REALNA ANALITIČKA INTEGRALNA FORMULA ZA PROST KOREN SISTEMA OD DVE NELINEARNE JEDNAČINE

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U radu se predlaže metod za dobijanje analitičke formule za prost koren (x_0, y_0) sistema od dve realne nelinearne algebarske ili transcendentne jednačine, $f(x, y)=0$, $g(x, y)=0$. Metod se bazira na Picardovom metodu za izračunavanje broja korena sistema nelinearnih jednačina. Dobijene formule za x_0 i y_0 sadrže integrale koji uključuju funkcije f i g i njihove prve parcijalne izvode. Izloženi rezultati generališu ranije rezultate za jednu nelinearnu jednačinu.

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LEAST SQUARES APPROXIMATION WITH CONSTRAINT: GENERALIZED GEGENBAUER CASE

Gradimir V. Milovanović and Milan A. Kovačević

Abstract. This paper consideres the least squares approximation of function $f \in L^2[-1, 1]$, $f(-1)=f(1)=0$. Using generalized Gegenbauer weight function $p(x)=|x|^{\mu}(1-x^2)^{\alpha}$ ($\mu, \alpha>-1$), some of the results from [8] are generalized. This approximation is compared with the least square approximation without constraint. The approximation is illustrated on two numerical examples.

1. Introduction

In [11] Wrigge and Fransén considered two families of functions and showed how these functions can be approximated on [0, 1] by the polynomials of the form $\sum_{n=1}^{k} c_{n,k}(x(1-x))^n$ and $(1-2x)\sum_{n=1}^{k} C_{n,k}(x(1-x))^n$. They used the L^2-norm with respect to the weight function $p(x)=(x(1-x))^q$, where $q \in \{0, 1, \ldots\}$. In [8] Milovanović and Wrigge presented a better and more natural way of approximation using Gegenbauer polynomials $C_{k,\lambda}(x)$ orthogonal with respect to the weight function $p(x)=(1-x^2)^{\lambda-1/2}$, $x \in [-1, 1]$, $\lambda>-1/2$. In this way they generalized the results from [11] and also avoided complicated manipulations with matrices.

Further generalization of these results can be obtained by using generalized Gegenbauer monic polynomials $W_k^{(\alpha,\beta)}(x)$, orthogonal on $[-1, 1]$ with respect to the weight function $p(x)=|x|^{\mu}(1-x^2)^{\alpha}$, $\mu, \alpha>-1$, $\beta=(\mu-1)/2$, which was introduced by Lascenov in [7] (see, also, [2, pp. 155—156]). It is interesting to say that these polynomials have been again „discovered" by J. Radecki ([9]).

2. Preliminaries

The relation between generalized Gegenbauer (monic) polynomials $W_k^{(\alpha,\beta)}(x)$ and Jacobi polynomials $P_k^{(\alpha,\beta)}(x)$ is given by

$$W_{2k}^{(\alpha,\beta)}(x)=\frac{k!}{(k+\alpha+\beta+1)_k}P_k^{(\alpha,\beta)}(2x^2-1), \tag{2.1}$$

$$W_{2k+1}^{(\alpha,\beta)}(x)=\frac{k!}{(k+\alpha+\beta+2)_k}xP_k^{(\alpha,\beta+1)}(2x^2-1). \tag{2.2}$$

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Notice that

$$W_{2k+1}^{(\alpha,\beta)}(x)=xW_{2k}^{(\alpha,\beta+1)}(x). \tag{2.3}$$

Using relations (2.1), (2.2) and Jacobi polynomials theory, it is possible to obtain a set of relations for generalized Gegenbauer polynomials. For example, the recurrence relation is

$$\begin{aligned} &W_{k+1}^{(\alpha,\beta)}(x)=xW_k^{(\alpha,\beta)}(x)-\Lambda_k W_{k-1}^{(\alpha,\beta)}(x), \qquad k=0,\,1,\,\ldots\,, \\ &W_{-1}^{(\alpha,\beta)}(x)=0, \quad W_0^{(\alpha,\beta)}(x)=1, \end{aligned} \tag{2.4}$$

where

$$\Lambda_{2k}=\frac{k\,(k+\alpha)}{(2\,k+\alpha+\beta)\,(2\,k+\alpha+\beta+1)}, \quad \Lambda_{2k-1}=\frac{(k+\beta)\,(k+\alpha+\beta)}{(2\,k+\alpha+\beta-1)\,(2\,k+\alpha+\beta)},$$

for $k=1,\,2,\,\ldots$, except when $\alpha+\beta=-1$; then $\Lambda_1=(\beta+1)/(\alpha+\beta+2)$.

Starting from the relation ([1, p. 782])

$$(2\,k+\alpha+\beta+3)\,P_{k+1}^{(\alpha,\beta)}(x)=(k+\alpha+\beta+2)\,P_{k+1}^{(\alpha+1,\beta)}(x)-(k+\beta+1)\,P_k^{(\alpha+1,\beta)}(x)$$

and using (2.1), we obtain

$$W_{2k+2}^{(\alpha+1,\beta)}(x)=W_{2k+2}^{(\alpha,\beta)}(x)+\frac{(k+1)\,(k+\beta+1)}{(2\,k+\alpha+\beta+3)\,(2\,k+\alpha+\beta+2)}\,W_{2k}^{(\alpha+1,\beta)}(x). \tag{2.5}$$

In the sequel, the following formulas will be necessary

$$\| W_{2k}^{(\alpha,\beta)}\|^2=\int_{-1}^{1} W_{2k}^{(\alpha,\beta)}(x)^2\,p\,(x)\,dx=\frac{k!}{(k+\alpha+\beta+1)_k}\,B(k+\alpha+1,\;k+\beta+1),$$

$$\begin{aligned} &\| W_{2k+1}^{(\alpha,\beta)}\|^2=\| W_{2k}^{(\alpha,\beta+1)}\|^2=\frac{k!}{(k+\alpha+\beta+2)_k}\,B(k+\alpha+1,\;k+\beta+2), \\ &W_{2k}^{(\alpha,\beta)}(1)=\frac{(\alpha+1)_k}{(k+\alpha+\beta+1)_k}, \quad W_{2k+1}^{(\alpha,\beta)}(1)=W_{2k}^{(\alpha,\beta+1)}(1)=\frac{(\alpha+1)_k}{(k+\alpha+\beta+2)_k} \end{aligned} \tag{2.6}$$

and, also

$$\begin{aligned} &W_{2k}^{(\alpha,\beta)}(x)=W_{2k}^{(\alpha,\beta)}(1)\,{}_2F_1(-k,\;k+\alpha+\beta+1;\;\alpha+1;\;1-x^2), \\ &W_{2k+1}^{(\alpha,\beta)}(x)=xW_{2k+1}^{(\alpha,\beta)}(1)\,{}_2F_1(-k,\;k+\alpha+\beta+2;\;\alpha+1;\;1-x^2), \end{aligned} \tag{2.7}$$

where ${}_2F_1$ is the hipergeometric function.

Lemma 2.1. *Let* $h_k=\| W_k^{(\alpha,\beta)}\|^2$. *Then the identities*

$$S_0^{(n)}(x)=\sum_{k=0}^{n}\frac{W_{2k}^{(\alpha,\beta)}(x)\,W_{2k}^{(\alpha,\beta)}(1)}{h_{2k}}=A_n^{(\alpha,\beta)}W_{2n}^{(\alpha+1,\beta)}(x) \tag{2.8}$$

and

$$S_1^{(n)}(x)=\sum_{k=0}^{n}\frac{W_{2k+1}^{(\alpha,\beta)}(x)\,W_{2k+1}^{(\alpha,\beta)}(1)}{h_{2k+1}}=A_n^{(\alpha,\beta+1)}W_{2n+1}^{(\alpha+1,\beta)}(x) \tag{2.9}$$

hold, where

$$A_n^{(\alpha,\beta)}=\frac{W_{2n}^{(\alpha,\beta)}(1)}{h_{2n}}=\frac{\Gamma(2n+\alpha+\beta+2)}{n!\,\Gamma(\alpha+1)\,\Gamma(n+\beta+1)}. \tag{2.10}$$

Proof. Our proof is based on induction. Note that (2.8) is correct for $n=0$. Now, let we suppose that (2.8) holds true for some n. Then we get

$$S_0^{(n+1)}(x)=A_n^{(\alpha,\beta)}W_{2n}^{(\alpha+1,\beta)}(x)+\frac{W_{2n+2}^{(\alpha,\beta)}(x)\,W_{2n+2}^{(\alpha,\beta)}(1)}{h_{2n+2}}.$$

On the basis of (2.10), we obtain

$$S_0^{(n+1)}(x)=A_{n+1}^{(\alpha,\beta)}\left(\frac{A_n^{(\alpha,\beta)}}{A_{n+1}^{(\alpha,\beta)}}W_{2n}^{(\alpha+1,\beta)}(x)+W_{2n+2}^{(\alpha,\beta)}(x)\right).$$

$$=A_{n+1}^{(\alpha,\beta)}\left(\frac{(n+1)(n+\beta+1)}{(2n+\alpha+\beta+3)(2n+\alpha+\beta+2)}W_{2n}^{(\alpha+1,\beta)}(x)+W_{2n+2}^{(\alpha,\beta)}(x)\right).$$

Finally, according to (2.5), we find $S_0^{(n+1)}(x)=A_{n+1}^{(\alpha,\beta)}W_{2n+2}^{(\alpha+1,\beta)}(x)$, showing that identity (2.8) holds true also for $n:=n+1$. The identity (2.9) is simply proved if we multiply identity (2.8), for $\beta:=\beta+1$, by x and using relations (2.3) and (2.6).

3. Approximations with constraint

Following Milovanović and Wrigge [8], we introduce two families of real functions, viz.

$$F_e=\{f: f(-x)=f(x),\ f(1)=0,\ f\in L^2[-1,1]\}$$

and

$$F_o=\{f: f(-x)=-f(x),\ f(1)=0,\ f\in L^2[-1,1]\},$$

where $L^2[-1,1]=L_p^2[-1,1]$, $p(x)=|x|^{\mu}(1-x^2)^{\alpha}$, $\alpha,\mu>-1$, and

$$(f,g)=\int_{-1}^{1}f(x)\,g(x)\,p(x)\,\mathrm{d}x\quad (f,g\in L^2[-1,1]). \tag{3.1}$$

Let further $\mathcal{P}_m$ be the set of all real polynomials of degree at most m and such that the polynomials belong to the set F_e if m is even and to the set F_o if m is odd.

In this section, we will give the least squares approximation Φ_{2n} (or Φ_{2n+1}) for the function $f\in F_e$ (or F_o) in the class $\mathcal{P}_{2n}$ (or $\mathcal{P}_{2n+1}$), with respect to the norm $||f||=((f,f))^{1/2}$, where the inner product $(\cdot,\cdot)$ is defined by (3.1). For this approximation we have

$$\min_{\Phi\in\mathcal{P}_{2n}}||f-\Phi||=||f-\Phi_{2n}||\quad \text{when } f\in F_e, \tag{3.2}$$

or

$$\min_{\Phi\in\mathcal{P}_{2n+1}}||f-\Phi||=||f-\Phi_{2n+1}||\quad \text{when } f\in F_o. \tag{3.3}$$

Applying the same method as in the paper [8] and using the relations (2.6) — (2.10), we can prove two following theorems:

Theorem. 3.1. *If $f \in F_e$ then the least squares approximation in the class $\mathcal{P}_{2n}$ is given by*

$$\Phi_{2n}(x) = \sum_{i=1}^{n} d_{n,i} (1-x^2)^i, \tag{3.4}$$

where

$$d_{n,i} = \frac{(-1)^i}{\Gamma(\alpha+i+2)} \sum_{k=0}^{n} (f, W_{2k}^{(\alpha,\beta)}) \frac{\Gamma(2k+\alpha+\beta+2)}{k!\,\Gamma(k+\beta+1)} s_{k,i}^{(\alpha,\beta)}$$

and

$$s_{k,i}^{(\alpha,\beta)} = \begin{cases} -\binom{n}{i}(\alpha+1)\,(n+\alpha+\beta+2)_i, & k<i, \\ \binom{k}{i}(\alpha+i+1)\,(k+\alpha+\beta+1)_i - \binom{n}{i}(\alpha+1)(n+\alpha+\beta+2)_i, & k \geqq i. \end{cases}$$

Theorem 3.2. *If $f \in F_o$, then the least squares approximation in the class $\mathcal{P}_{2n+1}$ is given by*

$$\Phi_{2n+1}(x) = x \sum_{i=1}^{n} b_{n,i} (1-x^2)^i, \tag{3.5}$$

where

$$b_{n,i} = \frac{(-1)^i}{\Gamma(\alpha+i+2)} \sum_{k=0}^{n} (f, W_{2k+1}^{(\alpha,\beta)}) \frac{\Gamma(2k+\alpha+\beta+3)}{k!\,\Gamma(k+\beta+2)} g_{k,i}^{(\alpha,\beta)}$$

and

$$g_{k,i}^{(\alpha,\beta)} = \begin{cases} -\binom{n}{i}(n+\alpha+\beta+3)_i\,(\alpha+1), & k<i, \\ \binom{k}{i}(k+\alpha+\beta+2)_i(\alpha+i+1) - \binom{n}{i}(n+\alpha+\beta+3)_i(\alpha+1), & k \geqq i. \end{cases}$$

Let $\tilde{\Phi}_{2n+q}(x)$ be the least squares approximation without constraint given by (see, e. g., [10, pp. 50—51])

$$\tilde{\Phi}_{2n+q}(x) = \sum_{k=0}^{n} \frac{(f, W_{2k+q}^{(\alpha,\beta)})}{h_{2k+q}} W_{2k+q}^{(\alpha,\beta)}(x),$$

where $q=0$ or $q=1$. It is easy to see that the approximation with constraint $\Phi_{2n+q}(x)$ turns out to be the truncated expansion in generalized Gegenbauer polynomials with a multiple of $S_q^{(n)}(x) \left(= A_n^{(\alpha,\beta+q)} W_{2n+q}^{(\alpha+1,\beta)}(x)\right)$ added to satisfy the constraint at $x=1$, i. e.,

$$\Phi_{2n+q}(x) = \tilde{\Phi}_{2n+q}(x) - \tilde{\Phi}_{2n+q}(1) \frac{S_q^{(n)}(x)}{S_q^{(n)}(1)}, \tag{3.7}$$

where $q=0$ or $q=1$.

4. General case

In the general case, when f is neither an even nor an odd function, but $f \in L^2[-1, 1]$ and $f(-1)=f(1)=0$, then the least squares approximation ψ_m (in the class of real polynomials of degree $\leq m$), which satisfies the conditions $\psi_m(-1)=\psi_m(1)=0$, is given by

(4.1) $$\psi_m(x)=\Phi_{2n}(x)+\Phi_{2n+1}(x) \qquad \text{when } m=2n+1$$

and

$$\psi_m(x)=\Phi_{2n}(x)+\Phi_{2n-1}(x) \qquad \text{when } m=2n,$$

where Φ_{2n} and Φ_{2n+1} are the solutions of (3.2) and (3.3). This can be seen by writing

(4.2) $$f(x)=f^{(e)}(x)+f^{(o)}(x)=\frac{1}{2}(f(x)+f(-x))+\frac{1}{2}(f(x)-f(-x)).$$

It is of some interest to compare approximations without constraint with our approximations with constraint.

In the set of all real polynomials Π_{2n+1} of degree at most $m=2n+1$, the least squares approximation without constraint for the function f, given by (4.2), can be represented in the form

(4.3) $$\tilde{\psi}_{2n+1}(x)=\tilde{\Phi}_{2n}(x)+\tilde{\Phi}_{2n+1}(x),$$

where the even and odd parts, $\tilde{\Phi}_{2n}$ and $\tilde{\Phi}_{2n+1}$, are given by (3.6) for $q=0$ and $q=1$ respectively.

According to (4.1), (3.7) and (4.3) we have that the corresponding least squares approximation with constraint is given by

(4.4) $$\psi_{2n+1}(x)=\tilde{\psi}_{2n+1}(x)-h(x),$$

where

$$h(x)=\tilde{\Phi}_{2n}(1)\frac{S_0^{(n)}(x)}{S_0^{(n)}(1)}+\tilde{\Phi}_{2n+1}(1)\frac{S_1^{(n)}(x)}{S_1^{(n)}(1)}.$$

Assuming $f \in L^2[-1, 1]$ and $f(-1)=f(1)=0$, define

(4.5) $$D^*=\min_{\psi \in \mathcal{P}_{2n} \cup \mathcal{P}_{2n+1}} \|f-\psi\|^2=\|f-\psi_{2n+1}\|^2$$

and

(4.6) $$\tilde{D}^*=\min_{\psi \in \Pi_{2n+1}} \|f-\psi\|^2=\|f-\tilde{\psi}_{2n+1}\|^2.$$

We note that $\mathcal{P}_{2n} \cup \mathcal{P}_{2n+1} \subset \Pi_{2n+1}$.

According to (4.5), (4.4) and (4.6) we have

$$D^*=(f-\psi_{2n+1}, f-\psi_{2n+1})=\tilde{D}^*+2(f-\tilde{\psi}_{2n+1}, h)+(h, h).$$

It is easy to show that $(f-\tilde{\psi}_{2n+1}, h)=0$ and

$$(h, h)=\frac{(\tilde{\Phi}_{2n}(1))^2}{S_0^{(n)}(1)}+\frac{(\tilde{\Phi}_{2n+1}(1))^2}{S_1^{(n)}(1)}.$$

So, we obtain

$$D^*-\tilde{D}^*=\frac{(\tilde{\Phi}_{2n}(1))^2}{A_n^{(\alpha,\beta)}W_{2n}^{(\alpha+1,\beta)}(1)}+\frac{(\tilde{\Phi}_{2n+1}(1))^2}{A_n^{(\alpha,\beta+1)}W_{2n+1}^{(\alpha+1,\beta)}(1)}.$$

Thus we see that the difference $D^*-\tilde{D}^*$ is a linear combination of the squares of errors $\tilde{\Phi}_{2n}(1)-f^{(e)}(1)=\tilde{\Phi}_{2n}(1)$ and $\tilde{\Phi}_{2n+1}(1)-f^{(o)}(1)=\tilde{\Phi}_{2n+1}(1)$, where $f^{(e)}(1)=$ $=f^{(o)}(1)=f(1)=0$.

A similar result can be obtained if we consider our approximations in the case $m=2n$.

5. Examples

As may be seen from Theorems 3.1 and 3.2, a main difficulty when calculating the least squares approximations is to achieve high — precision values of the inner products $(f, W_k^{(\alpha,\beta)})$.

An appropriate numerical method for the determination of these inner products is the application of Gauss — Christoffel quadrature with the generalized Gegenbauer weight. The parameters of these quadratures can be calculated from corresponding Jacobi matrix by using QR — algorithm ([4], [5]). The elements of Jacobi matrix are determinated by three — term recurrence relation (2.4).

A better approach is use of Gauss — Lobatto quadratures with the same weight ([6]). Namely, we can achieve higher accurace than with the above one, using the same number of knots because of the conditions $f(-1)=f(1)=0$.

Example 5.1. $f(x)=\cos(\pi x/2)$, $x\in[-1, 1]$.

In this case the approximating polynomial is given by (3.4), where the coefficients $d_{n,i}$ $(n=1, 2, 3, 4)$ are displayed in Table 5.1. The corresponding absolute errors

$$e_n=\max_{-1\leq x\leq 1}|f(x)-\Phi_{2n}(x)| \qquad (n=1, \dots, 4)$$

are given, too. Numbers in parenthesis indicate decimal exponents.

Table 5.1

		$\mu=0$.	$\alpha=-0.5$	$\mu=-0.5$	$\alpha=-0.5$
n	i	$d_{n,i}$	e_n	$d_{n,i}$	e_n
1	1	0.962270459871	3.84 (—2)	0.979346973677	4.60 (—2)
2	1	0.777230028062	7.47 (—4)	0.776199638179	9.00 (—4)
	2	0.222048518171		0.223462069048	
3	1	0.785557128489	8.05 (—6)	0.785579574340	9.69 (—6)
	2	0.195401796805		0.195322260565	
	3	0.019033372405		0.019094870042	
4	1	0.785396470018	5.46 (—8)	0.785396215336	6.56 (—8)
	2	0.196365747628		0.196367406887	
	3	0.017380885279		0.017377843941	
	4	0.000856845176		0.000858513050	

We can notice that for $\beta=-0.5$ ($\mu=0$) and $\alpha=\lambda-1/2$, the problem is reduced to the Gegenbauer's case which is considered in [8].

Example 5.2. The error function

$$\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$$

is an odd one. In order to approximate this function by means of Theorem 3.2, let us define a new function f by

$$f(x)=\operatorname{erf}(ax)-\operatorname{erf}(a)x \quad (|x|\leqq 1),$$

where a is a positive constant. For the function $f\in F_o$, according to Theorem 3.2, the approximating polynomial is given by (3.5). Taking into account the above, we obtain the approximation of the form

$$\operatorname{erf}(ax)\cong x\left(\operatorname{erf}(a)+\sum_{i=1}^{n} b_{n,i}(1-x^2)^i\right). \tag{5.1}$$

The coefficients $b_{n,i}$ ($i=1,\ldots,n$) and $b_{n,0}=\operatorname{erf}(a)$, for $a=0.5$ and $\alpha=\beta=-0.5$ and 0.5, are given in Table 5.2.

Table 5.2

		$\alpha=\beta=-0.5$	$\alpha=\beta=0.5$
n	i	$d_{n,i}$	$d_{n,i}$
6	0	0.52049987781305	0.52049987781305
	1	0.04055429417069	0.04055429417017
	2	0.00295376508012	0.00295376508471
	3	0.00017297426446	0.00017297425369
	4	0.00000832270120	0.00000832270347
	5	0.00000033642194	0.00000033643767
	6	0.00000001309590	0.00000001308425

Table 5.3 contains the maximum values of the absolute error, i.e.

$$\max_{|x|\leqq 1} |f(x)-\Phi_{2n+1}(x)|,$$

for $a=0.5\,(0.5)\,2$, when $\alpha=\beta=-0.5$ and $\alpha=\beta=0.5$.

It can be seen that the increase of the parameter a produces an increase of the error.

The obtained approximation for the error function given by (5.1) is very efficient fot usage because it requires a small number of arithmetic operations, i.e. n additions, one substraction, and $n+2$ multiplications, assuming that the Horner's sheme is used.

Table 5.3

	$a=0.5$		$a=1.0$	
n	$\alpha=\beta=-0.5$	$\alpha=\beta=0.5$	$\alpha=\beta=-0.5$	$\alpha=\beta=0.5$
1	2.6 (—4)	2.9 (—4)	6.1 (—3)	7.1 (—3)
2	3.6 (—6)	4.7 (—6)	3.4 (—4)	4.5 (—4)
3	4.3 (—8)	6.4 (—8)	1.6 (—5)	2.4 (—5)
4	4.3 (—10)	7.2 (—10)	6.1 (—7)	1.1 (—6)
5	3.7 (—12)	7.0 (—12)	2.1 (—8)	4.2 (—8)
6	2.9 (—14)	5.9 (—14)	6.4 (—10)	1.4 (—9)
	$a=1.5$		$a=2.0$	
n	$\alpha=\beta=-0.5$	$\alpha=\beta=0.5$	$\alpha=\beta=-0.5$	$\alpha=\beta=0.5$
1	3.1 (—2)	3.6 (—2)	7.7 (—2)	8.9 (—2)
2	3.6 (—3)	4.9 (—3)	1.6 (—2)	2.1 (—2)
3	3.7 (—4)	5.7 (—4)	2.7 (—3)	4.1 (—3)
4	3.2 (—5)	5.7 (—5)	4.0 (—4)	6.9 (—4)
5	2.5 (—6)	4.8 (—6)	5.2 (—5)	1.1 (—4)
6	1.7 (—7)	3.6 (—7)	6.1 (—6)	1.4 (—5)

All computations, which include the error function, have been performed using rational approximation to this function given in [3].

All calculations were performed in double precision arithmetic on a HONEYWELL DPS 6/92 computer.

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SREDNJE-KVADRATNA APROKSIMACIJA SA OGRANIČENJEM: GENERALISANI GEGENBAUEROV SLUČAJ

Gradimir V. Milovanović i Milan A. Kovačević

U radu se razmatra srednje-kvadratna aproksimacija funkcije $f \in L^2[-1, 1]$ pod uslovom da je $f(-1)=f(1)=0$. Korisćenjem generalisane Gegenbauerove težinske funkcije dobijena su uopštenja nekih rezultata Milovanovića i Wriggea [8]. Rezultati su upoređeni sa odgovarajućom srednje-kvadratnom aproksimacijom bez ograničenja. U poslednjem odeljku rada data su dva numerička primera.

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