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NECESSARY AND SUFFICIENT CONDITIONS FOR COMPLETENESS OF LATTICES

Milan R. Tasković

Abstract. In this paper we present a new characterization of complete lattices in terms of fixed points. In this paper we continue the study of the inductiveness of posets in terms of fixed points.

1. Introduction

An order-preserving (isotone or increasing) map f of a partially ordered set P to itself has a *fixed point* if there is an element ξ in P such that $f(\xi)=\xi$. P is said to have *fixed point property* if every isotone map of P to itself has a fixed point. The first of the fixed point theorems for partially ordered sets goes back almost 50 years to Tarski and Knaster (c.f. [10]), who proved that the lattice of all subsets of a set has the fixed point property. In the mid-1950's Tarski [10] published a far-reaching generalization: *Every complete lattice has the fixed point property.*

Tarski [10] raised the question whether the converse of this result also holds. Davis [5] in a companion paper proved the converse: *Every lattice with the fixed point property is complete.*

In paper [12] we have been presents new characterizations of inductiveness of posets in terms of fixed points.

In this paper we present a new characterization completeness of lattices in terms of the fixed points.

Let P a partially ordered set and f a mapping from P into P. For any $f\colon P\to P$ it is natural to consider the following set

$$\overline{f(P)}:=f(P)\cup\{a\in P\mid a=\sup C,\ \text{for some chain } C \text{ in } f(P)\},$$

and dually

$$\underline{f(P)}:=f(P)\cup\{a\in P\mid a=\inf C,\ \text{for some chain } C \text{ in } f(P)\},$$

where $\sup C$ ($\inf C$) is a least upper bound of C (a greatest lower bound of C).

We begin with the following statement which is essential.

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Lemma 1. (Fixed Point Lemma). *Let L be a complete lattice and f a mapping from L into itself such that*

(M) $$x \leqslant f(x), \quad \text{for all} \quad x \in \overline{f(L)}.$$

Then there exists a fixed point of f.

On the other hand, we are now in a position to formulate our the following statement.

Lemma 2. (Dually of (M)). *Let L be a complete lattice and f a mapping from L into itself such that*

(N) $$f(x) \leqslant x, \quad \text{for all} \quad x \in \overline{f(L)}.$$

Then there exists a fixed point of f.

Proof is analogous to the proof of preceding Lemma 1.

Proof of Lemma 1. Since L is a complete lattice, then L is chain complete, i. e., every non-empty chain in L has a supremum in L. By the Zorn's lemma (or Bourbaki's lemma) there exists a maximal element $z := f(x_0)$, $x_0 \in L$, i. e., $\overline{f(L)}$ has a maximal element $z \in \overline{f(L)}$. From the condition (M) we have $z \leqslant f(z)$ and because z is a maximal element of set $\overline{f(L)}$ it will be also $f(z) \leqslant z$. Hence, we obtain the relation $f(z) = z$, i. e., the equation $x = f(x)$ has a solution $z \in \overline{f(L)}$. Hence, from the preceding remark, f has a fixed point. This completes the proof of this statement.

2. Main results

With the help of Lemmas 1 and 2 we now obtain the main result of this paper.

Theorem 1. *For a lattice L to be complete it is necessary and sufficient that every mapping f on L to L with the condition* (M) *or* (N) *has a fixed point.*

First conctruction of proof. Since the condition of the theorem is known to be necessary for the completeness of a lattice, from Lemma 1, and Lemma 2 we have only to show that it is sufficient. In other words, we have to show that, under the assumption that the lattice L is not complete, there exists mapping f on L to L without fixed point and with the condition (M) or (N).

Suppose that the lattice L is not complete. We first notice that there exists at least one subset of L without a least upper bound (for otherwise the lattice would be complete). Hence, we can find a chain A of L with the following properties: least upper bound, i. e., supremum of A, does not exists.

Let U be a chain cofinal with A such that

$$U := \{x \in A \mid x_0 \leqslant x\}, \ x_0 := \text{a fixed element of } A = \min U.$$

Thus all the elements of U can be arranged in a sequence, i. e., one can show that there exists increasing sequence $\{x_\alpha\}$ in U such that: $\{x_\alpha\}$ is strictly increasing and, for each $t \in U$, there exists $\alpha(t)$ such that $\alpha(t) \prec \alpha$ implies $t \preccurlyeq x_\alpha$, and least upper bound of $\{x_\alpha\}$ does not exist.

We define a mapping f from L into itself according to the following prescription

$$(1) \qquad f(x) = \begin{cases} x_\beta, & \text{if } x = x_\alpha \in U \\ x_0 := \min U, & \text{if } x \notin U \end{cases}$$

where $x_\alpha \preccurlyeq x_\beta$ $(x_\alpha \neq x_\beta)$ for any $\alpha \prec \beta \prec w$, and where w any (finite or transfinite) ordinal. Thus we have defined a function f on L to L. Now, for any $x \in U\,(\supset \overline{f(L)})$ we have $x \preccurlyeq f(x)$, i. e., $x = x_\alpha \preccurlyeq x_\beta = f(x_\alpha) = f(x)$, for $\alpha \prec \beta \prec w$; so f satisfied the condition (M), which does not have a fixed point.

In the second case, we first notice that there exists at least one subset of L without a greates lower bound (for otherwise the lattice would be complete). Hence we can find a chain B of L with the following properties: greatest lower bound, i. e., infimum of B does not exist.

Let V be a chain cofinal with B such that

$$V := \{x \in B \mid x \preccurlyeq x_0\},\ x_0 := \text{a fixed element of } B = \max V.$$

Thus all the elements of V can be arranged in a seguence, i. e., one can show that there exists decreasing sequence $\{x_\beta\}$ in V such that: $\{x_\beta\}$ is strictly decreasing and, for each $t \in V$, there exists $\beta(t)$ such that $\beta(t) \prec \beta$ implies $x_\beta \preccurlyeq t$, and greatest lower bound of $\{x_\beta\}$ does not exist.

We define a mapping f from L into itself according to the following prescription

$$(2) \qquad f(x) = \begin{cases} x_\beta & \text{if } x = x_\alpha \in V \\ x_0 := \max V, & \text{if } x \notin V \end{cases}$$

where $x_\beta \preccurlyeq x_\alpha$ $(x_\alpha \neq x_\beta)$ for any $\alpha \prec \beta \prec w$, and where w any (finite or transfinite) ordinal. Thus we have defined a function f on L to L. Now, for any $x \in V(\supset \overline{f(L)})$ we have $f(x) \preccurlyeq x$, i. e., $x = x_\alpha \succcurlyeq x_\beta = f(x_\alpha) = f(x)$ for $\alpha \prec \beta \prec w$; so f satisfied the condition (N), which does not have a fixed point.

Thus the $f: L \to L$, i. e., (1) or (2) is with the condition (M) or (N) and does not have fixed points. This completes the proof of this main statement.

Second conctruction of proof. From Lemmas 1 and 2 we have necessary for the completeness of a lattice. Hence, we have only to show that it is sufficient.

Suppose that the lattice L is not complete. We first notice that there exists at least one subset of L without a least upper bound (for otherwise the lattice would be complete). Hence we can find a subset A of L with the following properties: $\sup A$ does not exists and if X is any subset of L with smaller power that A, then $\sup A$ exists. Thus all the elements of A can be arranged in a sequence, i. e., one can show that there exists increasing sequence $\{x_\alpha\}$ in A such that:

(3) $\{x_\alpha\}$ is strictly increasing and, for each $t \in A$, there exists $\alpha(t)$ such that $\alpha(t) < \alpha$ implies $t \leqq x_\alpha$, and $\sup\{x_\alpha\}$ does not exist.

Let us denote by B the set of all upper bounds of $\{x_\alpha\}$. Clearly $\inf B$ does not exist, for it did, it would coincide with $\sup\{x_\alpha\}$; this result would contradict (3). Now, B, like A, is either empty or infinite. Since B is partially ordered by the relation $\leqq$, there is a strictly decreasing sequence $\{x_\beta\}$ such that $\{x_\beta\}$ is a subset of B with which:

(4) $\{x_\beta)$ is strictly decreasing and, for each $t \in B$, there exists $\beta(t)$ such that $\beta(t) < t$ implies $x_\beta \leqq t$, and $\inf\{x_\beta\}$ does not exist.

To define $f: L \to L$ for any elemet $x \in L$, we distingush two cases depedent upon whether x is a lower bound of $\{x_\beta\}$ or not. In the first case, by (3)–(4), if x is not an upper bound of $\{x_\alpha\}$ then

$$f(x) = \min\{x_\alpha : x_\alpha \nleqq x\}. \tag{5}$$

where, $x_\alpha \nleqq x$ will be used to express the fact that $x_\alpha \leqq x$ does not hold.

In the second case, we let

$$f(x) = \max\{x_\beta : x \nleqq x_\beta\}. \tag{6}$$

Thus we have defined a function f on L to L. From (3)–(6) it follows clearly that either $f(x) \nleqq x$ or $x \nleqq f(x)$ for every $x \in L$, thus f has no fixpoints.

Thus the function $f: L \to L$ is with the condition (M) or (N) and does not have fixed points. This completes the proof of this main preceding statement.

Third construction of proof. Suppose that (3) and (4) holds. Then, we define a mapping f prom L into itself according to the following prescription

$$f(x) = \begin{cases} \min\{x_\alpha : x_\alpha \nleqq x\}, & \text{if } x \text{ is not an upper bound of } \{x_\alpha\} \\ x_{\alpha_0} := \text{a fixed element of } \{x_\alpha\}, & \text{otherwise;} \end{cases}$$

or, on the other hand, in the second case, we define a mapping f from L into itself by

$$f(x) = \begin{cases} \max\{x_\beta : x \nleqq x_\beta\}, & \text{if } x \text{ is not a lower bound of } \{x_\beta\} \\ x_{\beta_0} := \text{a fixed element of } \{x_\beta\}, & \text{otherwise.} \end{cases}$$

Thus, the function $f: L \to L$ is with the condition (M) or (N) and does not have fixed points. Since necessary follows from Lemmas 1 and 2, from preceding, this completetes the proof.

We are now in a position to formulate, L is said to have the *general fixed point property* if every map f of L into itself with the condition (M) or (N) has a fixed point.

Otherwise, many authors have investigated properties of posets satisfying some sort of chain-completess condition and used them in a variety of applications. Tarski's fixpoint theorem generalizes to chain-complete posets, i.e., if $f: P \to P$ is an isotone map and P is a chain-complete poset, then the set of fixpoints is a chain-complete poset under the iuduced order. This sharpens the results of Abian and Brown [1], that every isotone, self-map of chain-complete poset has a fixpoint. Conversely, Markowski [8] show that if every inf-preserving map $f: P \to P$ has a least fixpoint, P is chain-complete.

Also, Klimeš [6] characterized chain complete posets in terms of the fixed points of relatively isotone selfmappings.

Let P and Q be posets and $f: P \to Q$ a map (posets are nonempty by definition). Map f is *inf-preserving* if for all $X \subset P$ such that $\inf X$ exists in P and $f(\inf_P X) = \inf_Q f(X)$.

A mapping f of a poset P into itself is called *comparable* if for each x in P, x comparable with $f(x)$. A mapping f of a poset P into itself is called *relatively isotone* if for $x \leqslant y$, $x \leqslant f(y)$, $f(x) \leqslant y$ implies $f(x) \leqslant f(y)$, for every $x, y \in P$.

We are now in a position to formulate the following statement, from the preceding statements.

Theorem 2. *Let L be a lattice. Then the following statements are equivalent:*

(*a*) *L is a complete lattice,*

(*b*) *L has the general fixed point property,*

(*c*) (Tarski [10], Davis [5]). *L has the fixed point property,*

(*d*) (Klimeš [6]). *Every relatively isotone mapping of L into itself which is comparable has a fixed point.*

Remark. We notice, that first construction of the proof of Theorem 1, is also, and a new construction, i. e., a new proof for statemenmt of Anne Davis [5].

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POTREBNI I DOVOLJNI USLOVI ZA KOMPLETNOST MREŽA

Milan R. Tasković

Nastavljajući studiju o induktivnim uređenim skupovima, u ovom radu, mi na jedan posve novi način karakterišemo kompletne mreže. Naše konstrukcije bitno se razlikuju od konstrukcije Alfreda Tarskog i Anne Davis. Takođe, mi dajemo tri razne konstrukcije za karakterizaciju kompletnih mreža, koje su svaka za sebe od posebnog interesa. Specijalno, prva konstrukcija predstavlja i jednu sasvim novu konstrukciju za dokaz teoreme Tarskog i Davis.

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VARIOUS EXTREMAL PROBLEMS OF MARKOV'S TYPE FOR ALGEBRAIC POLYNOMIALS

Gradimir V. Milovanović

Abstract. Extremal problems of Markov's type for algebraic polynomials in various norms and classes of polynomials are considered. Especially, the problems in L^2-norm on the set of all algebric polynomials of degree at most n or on some its subsets are investigated.

1. Introduction and preliminaries

We begin our investigation by considering the following extremal problem:

Let $\mathcal{P}_n$ be the set of all algebraic polynomials $P\,(\neq 0)$ of degree at most n on an interval (a, b) with a given norm $\|\,.\,\|$.

Determine the best constant in the inequality

$$\|P'\| \leqq A_n \|P\| \qquad (P\in\mathcal{P}_n), \tag{1.1}$$

i. e.,

$$A_n = \sup_{P\in\mathcal{P}_n} \frac{\|P'\|}{\|P\|}. \tag{1.2}$$

The first result at this area is well-known classical inequality of A. A. Markov [19].

Theorem 1.1. *Let* $(a, b) = (-1, 1)$ *and* $\|f\| = \|f\|_\infty = \max_{-1\leqq t\leqq 1} |f(t)|$. *Then*

$$\|P'\|_\infty \leqq n^2 \|P\|_\infty \qquad (P\in\mathcal{P}_n), \tag{1.3}$$

with an equality case for $P(t) = T_n(t)$, *where* T_n *is Chebyshev polynomial of the first kind of degree* n.

An other type of these inequalities is Bernstein's inequality

$$\|P'\|_\infty \leqq n\,(1-t^2)^{-1/2} \|P\|_\infty \qquad (P\in\mathcal{P}_n). \tag{1.4}$$

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Markov's and Bernstein's inequalities are fundamental to the proofs of many inverse theorems in polynomial approximation theory [15], [21], [10].

Recently, these inequalities have been considered on disjoint intervals by P. S. Borwein [2].

In this paper, we will consider only inequalities of Markov's type.

A generalization of the inequality (1.2) for higher derivatives was given by V. A. Markov [20].

Theorem 1.2. *For each* $k=1,\ldots,n$, *the inequality*

$$\|P^{(k)}\|_\infty \leqq \frac{1}{(2k-1)!!}\prod_{i=0}^{k-1}(n^2-i^2)\|P\|_\infty \qquad (P\in\mathcal{P}_n) \tag{1.5}$$

holds. The extremal polynomial is T_n.

We note that the best constant in (1.5) is equal $\|T_n^{(k)}\|_\infty = T_n^{(k)}(1)$. So the inequality (1.5) can be written in the form

$$\|P^{(k)}\|_\infty \leqq T_n^{(k)}(1)\|P\|_\infty \qquad (P\in\mathcal{P}_n).$$

In 1964 G. Szegö [34] studied an extremal problem for the norm $\|f\| = \sup_{t\geqq 0}|f(t)e^{-t}|$ on $(0,+\infty)$. he proved the following:

Theorem 1.3. *Let* $(a,b)=(0,+\infty)$ *and* $\|f\| = \sup_{t\geqq 0}|f(t)e^{-t}|$. *There exists a positive constant* C *such that*

$$\|P'\| \leqq Cn\|P\|$$

for each $P\in\mathcal{P}_n$ $(n=2,3,\ldots)$.

If we put

$$\|f\|_{p,\mu} = \Big(\int_{-1}^{1}|f(t)(1-t^2)^{\mu}|^p\,dt\Big)^{1/p}, \qquad 1\leqq p<\infty,$$

$$= \sup_{-1\leqq t\leqq 1}|f(t)|(1-t^2)^{\mu}, \qquad p=+\infty,$$

where $p\mu>1$ ($\mu\leqq 0$ if $p=+\infty$), we can consider the following general extremal problem (see [11])

$$A_{n,k}(p,\mu;\,q,\nu) = \sup_{P\in\mathcal{P}_n}\frac{\|P^{(k)}\|_{q,\nu}}{\|P\|_{p,\mu}}. \tag{1.6}$$

So the best constant in (1.5) is $A_{n,k}(+\infty,0;\,+\infty,0)$. We note that Bernstein's inequality (1.4) can be represented in the form

$$\|P'\|_{\infty,1/2} \leqq n\|P\|_{\infty,0} \qquad (P\in\mathcal{P}_n).$$

The case $k=n$ is especially interesting. Namely, then we have the following problem: Among all polynomials of degree n, with leading coefficient unity, find the polynomial which deviates least from zero in the norm $\|\cdot\|_{p,\mu}$.

Some more general results in the integral norms are given in [13], [27], [11]. When $p = q$ and $\mu = \nu$, there are several results.

E. Hille, G. Szegö, and J. D. Tamarkin [9] extended Markov's theorem in L^p norm ($p \geqq 1$) on $(-1,1)$ by proving the following result:

Theorem 1.4. *Let* $(a, b) = (-1, 1)$ *and* $\|f\| = \|f\|_{p,0}$ $(p \geqq 1)$. *Then*

$$\|P'\| \leq Cn^2 \|P\| \qquad (P \in \mathcal{P}_n), \tag{1.7}$$

where C is a positive constant which dependens only on p, but not on P or on n.

A. Markov's theorem (with a less precise value of the constant C) is obtained from (1.7) by allowing $p \to +\infty$. Another important case, namely, $p = 1$, was treated by N. K. Bari [1] and recently by S. V. Konyagin [12], who considered the extremal problem (1.6) for $p = q = 1$ and $\mu = \nu = 0$. He found an estimate for $A_{n,k} = A(1, 0;\ 1, 0)$.

Theorem 1.5. *There exist two constants* c_1 and c_2 $(0 < c_1 < c_2 < +\infty)$ *such that*

$$c_1 \frac{nT_n^{(k)}(1)}{(k+1)(n-k+1)} \leq A_{n,k} \leq c_2 \frac{nT_n^{(k)}(1)}{(k+1)(n-k+1)}$$

for each $n \in N$ *and* $k = 1, \ldots, n$.

Especially important cases are $p = q = 2$. In the following section we consider such cases. In Section 3 we give some classical results for the extremal problems on some restricted polynomial classes. In Section 4 we discuss the Varma's extremal problems in L^2-metric. A complete solution of Varma's one and related problems we give in Section 5. Finally, in Section 6 we consider some extremal problems in L^2-metric with Jacobi weight on $(-1, 1)$.

2. Extremal problems in L^2-norm

In the L^2-metric we give first the following result of E. Schmidt [30] and P. Turán [36]:

Theorem 2.1. (*a*) *Let* $(a, b) = (-\infty, +\infty)$ *and* $\|f\|^2 = \int_{-\infty}^{\infty} e^{-t^2} f(t)^2 \, dt$. *Then the best constant in* (1.2) *is* $A_n = \sqrt{2n}$. *An extremal polynomial is Hermite's polynomial* H_n;

(*b*) *Let* $(a, b) = (0, +\infty)$ *and* $\|f\|^2 = \int_0^{\infty} e^{-t} f(t)^2 \, dt$. *Then*

$$A_n = \left(2 \sin \frac{\pi}{4n+2}\right)^{-1}.$$

The extremal polynomial is

$$P(t) = \sum_{\nu=1}^{n} \sin \frac{\nu\pi}{2n+1} L_\nu(t),$$

where L_ν *is Laguerre polynomial.*

The Theorem 2.1, in this form, was formulated by P. Turán. E. Schmidt proved only

$$A_n = \frac{2n+1}{\pi}\left(\frac{\pi^2}{24(2n+1)^2} + \frac{R}{(2n+1)^4}\right)^{-1},$$

where $-8/3 < R < 4/3$.

Recently, L. Mirsky [26] considered the case of L^2-metric with an arbitrary weight function $w:(a,b) \to R_+$ $(-\infty \leqq a < b \leqq +\infty)$ for which all moments are finite.

Theorem 2.2. *There exists a number* $A_n = A_n(a, b; w)$ *such that, for every polynomial P with complex coefficients and of degree not exceeding n, the inequality* (1.1) *holds. Furthermore, we have*

$$A_n \leqq \left(\sum_{k=1}^{n} k \|\pi_k'\|^2\right)^{1/2}, \tag{2.1}$$

where (π_k) *is a system of polynomials orthonormal with respect to the weight function w.*

The main interest of this result is, however, qualitative, for the bound specified by (2.1) can be very crude. For example, when $w(t) = e^{-t^2}$ on $(-\infty, +\infty)$, the estimate (2.1) becomes

$$A_n \leqq \left(\sum_{k=1}^{n} 2k^2\right)^{1/2} \quad O(n^{3/2}).$$

The contrast between this estimate and $A_n = \sqrt{2n}$ (see (*a*) in Theorem 2.1) is evident.

In [6] P. Dörfler considered the analogous inequality for derivatives of higher order and compute the best possible constant:

Theorem 2.2. *Let P be any polynomial with complex coefficients of degree at most n. Then the best possible constant* $A_{n,m}$ *such that*

$$\|P^{(m)}\| \leqq A_{n,m} \|P\|,$$

is the largest singular value of the matrix $A_n^{(m)}$, *where*

$$A_n^{(m)} = \begin{bmatrix} e_{0,0}^{(m)} & \cdots & e_{n,0}^{(m)} \\ \vdots & & \\ e_{0,n-m}^{(m)} & & e_{n,n-m}^{(m)} \end{bmatrix}, \qquad e_{k,j}^{(m)} = \int_a^b \pi_k^{(m)}(t)\,\pi_j(t)\,w(t)\,dt.$$

Moreover, the estimation

$$\max_{0 \leqq k \leqq n} \|\pi_k^{(m)}\| \leqq A_{n,m} \leqq \left(\sum_{k=0}^{n} \|\pi_k^{(m)}\|^2\right)^{1/2}$$

holds.

The exact constant in (1.1) can be found as a maximal eigenvalue of a matrix of Gram's type. Now, we consider a more general case with a given nonnegative measure $d\lambda(t)$ on the real line R, with compact or infinite support, for which all moments

$$\mu_k = \int_R t^k \, d\lambda(t), \qquad k = 0, 1, \ldots,$$

exist and are finite, and $\mu_0 > 0$. There exists, then, a unique set of orthonormal polynomials $\pi_k(\,.\,) = \pi_k(\,.\,; d\lambda)$, $k = 0, 1, \ldots,$ defined by

$$\pi_k(t) = a_k t^k + \text{lower degree terms}, \quad a_k > 0,$$

(2.2) $$\int_R \pi_k(t)\,\pi_m(t)\,d\lambda(t) = \delta_{km}, \qquad k, m \geq 0,$$

(For any polynomial $P \in \mathcal{P}_n$, with complex coefficients, we take

$$\|P\| = \Big(\int_R |P(t)|^2 \, d\lambda(t)\Big)^{1/2}$$

and consider the extremal problem

(2.3) $$A_{n,m} = A_{n,m}(d\lambda) = \sup_{P \in \mathcal{P}_n} \frac{\|P^{(m)}\|}{\|P\|} \qquad (1 \leq m \leq n).$$

Theorem 2.4. *The best constant* $A_{n,m}$ *defined in* (2.3) *is given by*

2.4) $$A_{n,m} = (\lambda_{\max}(B_{n,m}))^{1/2},$$

where $\lambda_{\max}(B_{n,m})$ *is the maximal eigenvalue of the matrix* $B_{n,m} = [b_{ij}^{(m)}]_{m \leq i, j \leq n}$, *which the elements are given by*

(2.5) $$b_{i,j}^{(m)} = \int_R \pi_i^{(m)}(t)\,\pi_j^{(m)}(t)\,d\lambda(t), \qquad m \leq i, j \leq n.$$

An extremal polynomial is

$$P^*(t) = \sum_{k=m}^{n} c_k \pi_k(t),$$

where $[c_m, c_{m+1}, \ldots, c_n]^T$ *is an eigenvector of the matrix* $B_{n,m}$ *corresponding to the eigenvalue* $\lambda_{\max}(B_{n,m})$.

Proof. Let $P \in \mathcal{P}_n$. Then we can write $P(t) = \sum_{k=0}^{n} c_k \pi_k(t)$ and $P^{(m)}(t) = \sum_{k=m}^{n} c_k \pi_k^{(m)}(t)$, $m \leq n$, where the coefficients c_k are uniquely determined. Hence, by (2.2) and (2.5),

$$\|P\|^2 = \sum_{k=0}^{n} |c_k|^2 \quad \text{and} \quad \|P^{(m)}\|^2 = \sum_{i,j=m}^{n} c_i \bar{c}_j b_{ij}^{(m)}.$$

Now we have

$$\frac{\|P^{(m)}\|^2}{\|P\|^2} \leq \frac{\sum_{i,j=m}^{n} c_i \bar{c}_j b_{ij}^{(m)}}{\sum_{i=m}^{n} |c_i|^2} = \frac{\langle B_{n,m}\mathbf{c}, \mathbf{c}\rangle}{\langle \mathbf{c}, \mathbf{c}\rangle}. \tag{2.6}$$

with equality case $c_0 = \ldots = c_{m-1} = 0$, where $\langle .,.\rangle$ is the standard inner product in an $(n-m+1)$-dimensional space.

The matrix $B_{n,m}$ is evidently positive definite. Since the right side in (2.6) is not greater than the maximal eigenvalue of this matrix we obtain

$$\|P^{(m)}\|^2 \leq \lambda_{\max}(B_{n,m})\|P\|^2. \tag{2.7}$$

In order to show that $A_{n,m}$, given by (2.4), is best possible, we note that (2.7) reduces to an equality if we put $P(t) = P^*(t) = \sum_{k=m}^{n} c_k^* \pi_k(t)$, where $[c_m^*, c_{m+1}^*, \ldots, c_n^*]^T$ is an eigenvector of the matrix $B_{n,m}$ corresponding to $\lambda_{\max}(B_{n,m})$.

An alternative result like Theorem 2.3 is the following theorem:

Theorem 2.5. *The best constant* $A_{n,m}$ *defined in* (2.3), *is equal to the spectral norm of one triangular matrix* $Q_{n,m}^T$, $Q_{n,m} = [q_{ij}^{m}]_{m\leq i,j\leq n}$ $(q_{ij}^{(m)} = 0 \Leftarrow i > j)$, *i.e.*

$$A_{n,m} = \sigma(Q_{n,m}^T) = (\lambda_{\max}(Q_{n,m}Q_{n,m}^T))^{1/2}, \tag{2.8}$$

where the elements $q_{ij}^{(m)}$ *are given by the following inner products*

$$q_{ij}^{(m)} = (\pi_j^{(m)}, \pi_{i-m}) \qquad (m \leq i, j \leq n).$$

Alternatively, (2.8) *can be expresed in the form*

$$A_{n,m} = (\lambda_{\min}(C_{n,m}))^{-1/2}, \tag{2.9}$$

where $C_{n,m} = (Q_{n,m}Q_{n,m}^T)^{-1}$.

Proof. It is enough to consider only real polynomial set $\mathcal{P}_n$. Let $P \in \mathcal{P}_n$ and $\pi_j^{(m)}(t) = \sum_{i=m}^{n} q_{ij}^{(m)} \pi_{i-m}(t)$, where $q_{ij}^{(m)} = (\pi_j^{(m)}, \pi_{i-m})$. Then we have

$$P^{(m)}(t) = \sum_{j=m}^{n} c_j \sum_{i=m}^{j} q_{ij}^{(m)} \pi_{i-m}(t) = \sum_{i=m}^{n} \left(\sum_{j=m}^{n} c_j q_{ij}^{(m)}\right) \pi_{i-m}(t)$$

and

$$\|P^{(m)}\|^2 = \sum_{i=m}^{n} \left(\sum_{j=i}^{n} c_j q_{ij}^{(m)}\right)^2 = \sum_{i=m}^{n} Y_i^2,$$

where we put

$$Y_i = \sum_{j=i}^{n} c_j q_{ij}^{(m)}, \qquad i = m, \ldots, n. \tag{2.10}$$

Let $c=[c_m, \ldots, c_n]^T$, $Y=[Y_m, \ldots, Y_n]^T$, and $Q_{n,m}=[q_{ij}^{(m)}]_{m\leq i, j\leq n}$. Since $Y=Q_{n,m}c$ we have

$$\frac{\|P^{(m)}\|^2}{\|P\|}\leq\frac{\langle Y, Y\rangle}{\langle c, c\rangle}=\frac{\langle Y, Y\rangle}{\langle (Q_{n,m}Q_{n,m}^T)^{-1}Y, Y\rangle},$$

wherefrom we conclude that (2.8) and (2.9) hold.

Example 2.1. $d\lambda(t)=\exp(-t^2)\,dt$, $-\infty<t<+\infty$. Here we have $\pi_k(t)=$ $=\hat{H}_k(t)=(\sqrt{\pi}\,2^k k!)^{-1/2}H_k(t)$, where H_k is Hermite polynomial of degree k. Since $H_k'(t)=2kH_{k-1}(t)$, i.e., $\hat{H}_k'(t)=\sqrt{2k}\,\hat{H}_{k-1}(t)$, we have

$$\hat{H}_k^{(m)}(t)=\sqrt{2k}\sqrt{2(k-1)}\cdots\sqrt{2(k-m+1)}\,\hat{H}_{k-m}(t)=\sqrt{2^m m!\binom{k}{m}}\,\hat{H}_{k-m}(t)$$

and

$$b_{i,j}^{(m)}=2^m m!\binom{i}{m}\delta_{i,j},\quad m\leq i, j\leq n,$$

So, we find $\lambda_{\max}(B_{n,m})=2^m m!\binom{n}{m}$ and $A_{n,m}=2^{m/2}\sqrt{n!/(n-m)!}$.

Also, this result can be found in unpublished Ph. D. Thesis of L. F. Shampine [31] and [6].

Example 2.2. $d\lambda(t)=t^\alpha e^{-t}\,dt$, $0<t<\infty$. Here we have the generalized Laguerre case with $\pi_k(t)=\hat{L}_k^\alpha(t)=\sqrt{k!/\Gamma(k+\alpha+1)}\sum_{i=0}^{k}(-1)^{k-i}\binom{k+\alpha}{k-i}\frac{x^i}{i!}$, where Γ is the gamma function.

First, we consider the case $m=1$. Since

$$\frac{d}{dt}\hat{L}_j^\alpha(t)=\sum_{i=1}^{j}q_{ij}^{(1)}\hat{L}_{i-1}^\alpha(t),\quad q_{ij}^{(1)}=-\sqrt{\frac{j!}{\Gamma(j+\alpha+1)}}\cdot\sqrt{\frac{\Gamma(i+\alpha)}{(i-1)!}},$$

from the equalities (2.10) it follows that

$$c_i=Y_{i+1}-\sqrt{\frac{i+\alpha}{i}}\,Y_i,\quad i=1, \ldots, n,$$

where we put $Y_{n+1}=0$. The elements $p_{ij}^{(1)}$ of the matrix $P_{n,1}=Q_{n,1}^{-1}$ are

$$p_{ii}^{(1)}=-\sqrt{1+\frac{\alpha}{i}},\quad i=1, \ldots, n;\quad p_{i,i+1}^{(1)}=1,\quad i=1, \ldots, n-1;$$

$$p_{ij}^{(1)}=0,\ \text{otherwise},$$

so that

$$C_{n,1}=P_{n,1}^T P_{n,1}=-\begin{bmatrix}\alpha_0 & \sqrt{\beta_1} & & & \mathbf{0}\\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot\\ & & & \cdot & \sqrt{\beta_{n-1}}\\ \mathbf{0} & & & \sqrt{\beta_{n-1}} & \alpha_{n-1}\end{bmatrix}=-J_n,$$

where

$$\alpha_0 = -(1+\alpha), \quad \alpha_k = -\left(2+\frac{\alpha}{k+1}\right), \quad \beta_k = 1+\frac{\alpha}{k}, \quad k=1, \ldots, n-1.$$

We see that J_n is the Jacobi matrix for monic polynomials (Q_k), which satisfy the following three-term recurrence relation

$$Q_{k+1}(t) = (t-\alpha_k)\, Q_k(t) - \beta_k Q_{k-1}(t), \qquad k=0, 1, \ldots,$$

$$Q_{-1}(t) = 0, \quad Q_0(t) = 1.$$

The eigenvalues of $C_{n,1}$ are $\lambda_\nu = -t_\nu$, where $Q_n(t_\nu) = 0$, $\nu = 1, \ldots, n$.

The standard Laguerre case ($\alpha = 0$) can be exactly solved. Namely, then for $t = 2(z-1)$ and $-1 \leqq z \leqq 1$, we have

$$Q_k(t) = \cos(2k+1)\frac{\theta}{2} \Big/ \cos\frac{\theta}{2}, \quad z = \cos\theta.$$

The eigenvalues of the matrix $C_{n,1}$ are

$$\lambda_\nu = -t_\nu = 4\sin^2\frac{(2\nu-1)\pi}{2(2n+1)}, \quad \nu = 1, \ldots, n.$$

Since $\lambda_{\min}(C_{n,1}) = \lambda_1$, we obtain $A_{n,1} = \left(2\sin\frac{\pi}{2(2n+1)}\right)^{-1}$. This is Turán's result (Theorem 2.1 (*b*)).

Now, we consider the case when $m = 2$ and $\alpha = 0$. First, we note that

$$\frac{d^m}{dt^m}\hat{L}_j(t) = (-1)^m \sum_{i=m}^{j} \binom{j-i+m-1}{m-1} \hat{L}_{i-m}(t).$$

The formulas (2.10), for $m = 2$, become

$$Y_i = \sum_{j=i}^{n} (j-i+1)\, c_j, \quad i = 2, \ldots, n.$$

Since $\Delta^2 Y_i = c_i$ $(Y_{n+1} = Y_{n+2} = 0)$, we find a five-diagonal symmetric matrix of the order $n-1$

$$C_{n,2} = \begin{bmatrix} 1 & -2 & 1 & & & & & \mathbf{0} \\ -2 & 5 & -4 & 1 & & & & \\ 1 & -4 & 6 & -4 & 1 & & & \\ & 1 & -4 & 6 & -4 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & -4 & 6 & -4 & 1 \\ & & & & 1 & -4 & 6 & -4 \\ \mathbf{0} & & & & & 1 & -4 & 6 \end{bmatrix}.$$

So, using the minimal eigenvalue of this matrix we obtain the best constant $A_{n,2}=(\lambda_{\min}(C_{n,2}))^{-1/2}$. These constants, for $n = 4\,(1)10$ are presented in Table 2.1 (with seven decimal digits). Numbers in parentheses indicate decimal exponents. For $n=2$ and $n=3$ we have $A_{2,2}=1$ and $A_{3,2}=(3+2\sqrt{2})^{1/2}$ respectively.

Table 2.1.

n	$\lambda_{\min}(C_{n,2})$	$A_{n,2}$
4	5.1590055 (—2)	4.4026788
5	2.0635581 (—2)	6.9613208
6	9.8237813 (—3)	10.0892912
7	5.2614253 (—3)	13.7863181
8	3.0685649 (—3)	18.0522919
9	1.9090449 (—3)	22.8871610
10	1.2494144 (—3)	28.2908989

Remark 2.1. The last problem could be interpreted as an extremal problem of Wirtinger's type

$$\sum_{i=2}^{n} Y_i^2 \leq A_{n,2}^2 \sum_{i=2}^{n} (\Delta^2 Y_i)^2, \qquad Y_{n+1}=Y_{n+2}=0.$$

Similar problems were given in [8] by K. Fan, O. Taussky, and J. Todd.

Remark 2.2. In 1965 L. F. Shampine [31] proved that

$$\frac{1}{n^4} A_{n,2}^2 = \frac{1}{k_0^4} - R, \qquad 0<R\leq\frac{1}{2n}-\frac{1}{6n^2},$$

where $k_0=1.8751041\ldots$ (k_0 is the smallest root of the equation $1+\cos k \cosh k=0$).

On the end of this section we consider a case with a special even weight function. Namely, let $d\lambda(t) = w(t)\,dt$ on $(-a,a)$, $0<a<\infty$, where $w(-t) = w(t)$. Then we have

$$\pi_i'(t) = \frac{1}{r_i} \sum_{j=1}^{\left[\frac{i+1}{2}\right]} q_{i,j}\, \pi_{i-2j+1}(t), \qquad r_i \neq 0.$$

Now, we consider a class of weight functions for which $q_{i,j}=q_{i+2,j+1}$ (for example, this property holds for Gegenbauer weight). In this case, for $P\in\mathcal{P}_n$, we have

$$P'(t) = \sum_{i=1}^{n} c_i \pi_i'(t) = \sum_{i=1}^{n} q_{i,1} \Big(\sum_{j\geq 0} c_{i+2j}\, r_{i+2j}^{-1}\Big) \pi_{i-1}(t)$$

and

$$\|P'\|^2 = \sum_{i=1}^{n} Y_i^2,$$

where

(2.11) $$Y_i = q_{i,1} \sum_{j\geq 0} c_{i+2j}\, r_{i+2j}^{-1}, \qquad i=1,\ldots,n.$$

If we put $q_{i,1}=p_i$ and $Y_{n+1}=Y_{n+2}=0$, from (2.11) follows

$$c_i=r_i\left(\frac{Y_i}{p_i}-\frac{Y_{i+2}}{p_{i+2}}\right), \qquad i=1, \ldots, n.$$

Then

$$\| P \|^2=\sum_{i=1}^{n} c_i^2=\frac{r_1^2}{p_1^2} Y_1^2+\frac{r_2^2}{p_2^2} Y_2^2+\sum_{i=3}^{n} \frac{r_i^2+r_{i-2}^2}{p_i^2} Y_i^2-2 \sum_{i=1}^{n-2} \frac{r_i^2}{p_i p_{i+2}} Y_i Y_{i+2}.$$

The corresponding matrix $C_{n,1}$ (see Theorem 2.5) is given by

$$\text{(2.12)} \qquad C_{n,1}=\begin{bmatrix} \alpha_1 & 0 & \beta_1 & & & & & & \\ 0 & \alpha_2 & 0 & \beta_2 & & & & \mathbf{0} & \\ \beta_1 & 0 & \alpha_3 & 0 & \beta_3 & & & & \\ & \beta_2 & 0 & \alpha_4 & 0 & \beta_4 & & & \\ & & \cdot & & \cdot & & \cdot & & \\ & & & \beta_{n-4} & 0 & \alpha_{n-2} & 0 & \beta_{n-2} \\ & \mathbf{0} & & & \beta_{n-3} & 0 & \alpha_{n-1} & 0 \\ & & & & & \beta_{n-2} & 0 & \alpha_n \end{bmatrix},$$

where

$$\alpha_i=\frac{r_i^2+r_{i-2}^2}{p_i^2}, \qquad \beta_i=-\frac{r_i^2}{p_i p_{i+2}} \quad (r_{-1}=r_0=0).$$

Now, we define two sequences of polynomials (R_i) and (S_i) by the following three-term recurrence relations

$$\text{(2.13)} \quad tR_{i-1}(t)=\beta_{2i-1} R_i(t)+\alpha_{2i-1} R_{i-1}(t)+\beta_{2i-3} R_{i-2}(t), \quad i=1, \ldots, \left[\frac{n+1}{2}\right],$$
$$R_{-1}(t)=0, \quad R_0(t)=R_0=\text{const}$$

and

$$\text{(2.14)} \quad tS_{i-1}(t)=\beta_{2i} S_i(t)+\alpha_{2i} S_{i-1}(t)+\beta_{2i-2} S_{i-2}(t), \quad i=1, \ldots, \left[\frac{n}{2}\right],$$
$$S_{-1}(t)=0, \quad S_0(t)=S_0=\text{const}.$$

Theorem 2.6. *The eigenvalues of the matrix $C_{n,1}$, given by* (2.12), *are the zeros of polynomials*

(a) S_{k-1} *and* R_k, *when* $n=2k-1$,

(b) S_k *and* R_k, *when* $n=2k$,

so that

(c) $A_{2k-1,1}=\left(\min\left(s_1^{(k-1)}, r_1^{(k)}\right)\right)^{-1/2}$

and

(d) $A_{2k,1}=\left(\min\left(s_1^{(k)}, r_1^{(k)}\right)\right)^{-1/2}$,

where $s_1^{(m)}$ and $r_1^{(m)}$ are the minimal zeros of the polynomials S_m and R_m respectively.

Proof. Firstly, we put $\nu = \nu(t) = [R_0(t), S_0(t), R_1(t), S_1(t), \ldots]^T$, where the last coordinate of this $(n-1)$-dimensional vector is $R_{k-1}(t)$ (if $n = 2k-1$) or $S_{k-1}(t)$ (if $n = 2k$). Using the matrix notation, the relations (2.13) and (2.14) can be interpreted in the form

$$t\nu = C_{n,1}\nu + w_n, \tag{2.15}$$

where w_n, in depending on n, is given by

$$w_n = \begin{cases} \beta_{n-1} S_{k-1}(t)\, e_{n-2} + \beta_n R_k(t)\, e_{n-1}, & \text{if } n = 2k-1, \\ \beta_{n-1} R_k(t)\, e_{n-2} + \beta_n S_k(t)\, e_{n-1}, & \text{if } n = 2k, \end{cases}$$

and e_s is an $(n-1)$-dimensional vector which s-th coordinate is equal one, and others are zero.

Putting firstly $R_0 = 0$ and $S_0 \neq 0$, and then $R_0 \neq 0$ and $S_0 = 0$, we conclude that w_n is a zero-vector if $S_{k-1}(t) = 0$ and $R_k(t) = 0$, when $n = 2k-1$. In the case $n = 2k$, we have the same situation if $S_k(t) = 0$ and $R_k(t) = 0$. Now, according to (2.14) we can conclude that (a) and (b) in Theorem 2.6 are valid. Finally, (c) and (d) follow from (2.9).

Example 2.3. The conditions $q_{i,j} = q_{i+2,j+1}$ are satisfied for Gegenbauer measure $d\lambda(t) = (1-t^2)^{\lambda-1/2}\,dt$, $-1 < t < 1$. Namely, we have

$$\frac{d}{dt}\hat{C}_i^\lambda(t) = \frac{2}{h_i^{1/2}} \sum_{j=1}^{\left[\frac{i+1}{2}\right]} (i+\lambda-2j+1)\, h_{i-2j+1}^{1/2}\, \hat{C}_{i-2j+1}^\lambda(t),$$

where $\hat{C}_k^\lambda$ is the normalized Gegenbauer polynomial of the order k, $h_i = \|C_i^\lambda\|^2 = \sqrt{\pi}\,\dfrac{(2\lambda)_i\,\Gamma\left(\lambda+\frac{1}{2}\right)}{(i+\lambda)\,i!\,\Gamma(\lambda)}$, and $(p)_i = p(p+1)\cdots(p+i-1)$. This formula follows immediately from [25, Lemma on the p. 552].

Thus,

$$r_i = \frac{1}{2}\sqrt{h_i}, \qquad p_i = q_{i,1} = (i+\lambda-1)\sqrt{h_{i-1}},$$

so, for $n = 1$ and $n = 2$, we have $A_{1,1} = \sqrt{2(\lambda+1)}$ and $A_{2,1} = \sqrt{\dfrac{8(\lambda+1)(\lambda+2)}{2\lambda+1}}$.

In a special case, when $\lambda = 1/2$ (Legendre case), we obtain

$$\alpha_1 = \frac{1}{3}, \quad \alpha_2 = \frac{1}{15}, \quad \alpha_i = \frac{2}{(2i+1)(2i-3)}, \quad i = 3, \ldots, n;$$

$$\beta_i = -\frac{1}{(2i+1)\sqrt{(2i-1)(3i+3)}}, \quad i = 1, \ldots, n-2.$$

Similarly, in Chebyshev case ($\lambda = 0$) we have

$$\alpha_1 = \frac{1}{2}, \quad \alpha_2 = \frac{1}{16}, \quad \alpha_i = \frac{1}{4}\left(\frac{1}{i^2} + \frac{1}{(i-2)^2}\right), \quad i = 3, \ldots, n;$$

$$\beta_1 = -\frac{\sqrt{2}}{4}, \quad \beta_i = -\frac{1}{4i^2}, \quad i = 2, \ldots, n-2.$$

Some numerical results are presented in Table 2.2.

Table 2.2.

n	$A_{n,1}$ ($\lambda = 1/2$)	$A_{n,1}$ ($\lambda = 0$)
1	1.7320508	1.4142136
2	3.8729833	4.0000000
3	6.5215962	7.3948177
4	9.7498094	11.6832385
5	13.5914030	16.8974115
6	18.0596447	23.0482821
7	23.1597543	30.1399752
8	28.8939700	38.1742735
9	35.2633549	47.1520448
10	42.2684628	57.0737521

3. Restricted polynomials

In this section and further we will consider extremal problems on some restricted polynomial classes, i. e.,

$$A_n = \sup_{P \in W_n} \frac{\| P' \|}{\| P \|}, \tag{3.1}$$

where W_n is some subset of $\mathcal{P}_n$. We can restrict (see [17]): (a) zeros of P_n; (b) coefficients of the polynomials. In this way, the inequality (1.1) can be improved.

So for the uniform norm on $[-1, 1]$ P. Erdös [7] proved the following result:

Theorem 3.1. *If $P \in \mathcal{P}_n$ has only real roots, none of which are in $(-1,1)$, then $A_n = \frac{1}{2} en$.*

For the same norm, Q. I. Rahman and G. Schmeisser [28] proved:

Theorem 3.2. *If $P \in \mathcal{P}_n$ has at most $n-1$ distanct zeros in $(-1, 1)$, then $A_n = \left(n \cos \frac{\pi}{4n}\right)^2$. The extremal polynomial is*

$$T_n\left(\pm\left(\cos \frac{\pi}{4n}\right)^2 t + \left(\sin \frac{\pi}{4n}\right)^2\right).$$

There have been several related results (e. g. [4], [32], [33], [3]).

In 1963 G. G. Lorentz [14] introduced polynomials with positive coefficients in t, $1-t$ on $(0,1)$, i. e., the polynomials of the form

$$P(t) = \sum_{k=0}^{n} b_k t^k (1-t)^{n-k}, \quad b_k \geqq 0. \tag{3.2}$$

Also, these polynomials were studied extensively by J. T. Scheick [29].

The Lorentz theorem can be stated in the following form:

Theorem 3.3. *There exists a constant $C>0$ such that for each polynomial P of the form* (3.2),

$$\|P'\|_\infty \leqq Cn\|P\|_\infty \quad (n=1, 2, \ldots), \tag{3.3}$$

for the uniform norm on [0,1].

The inequality (3.3) is much better than basic Markov's inequality (1.3). Namely, the exponent 2 is replaced by 1.

In 1968 G. G. Lorentz [16] considered the problem of G. Szegö (see Theorem 1.3) for the special polynomials with nonnegative coefficients in t,

$$P(t)=\sum_{k=0}^{n} a_k t^k, \quad a_k \geqq 0, \tag{3.4}$$

and the norm of a function on $(0,+\infty)$ is given by $\|f\|=\sup_{t\geqq 0}|f(t)e^{-\omega(t)}|$. Here ω is a positive differentiable function which, together with $t \mapsto t\omega'(t)$, is strictly increasing to $+\infty$.

Theorem 3.4. *Let* ω *satisfy the inequalities*

$$\omega(t)-\omega(0) \leqq At\omega'(t), \quad t \geqq 0,$$

and

$$\omega'(\tau) \leqq A\omega'(t), \quad \tau \leq t,$$

for some positive constant A. Then for some constant $C>0$, the inequality

$$\|P'\| \leqq C\frac{\|p_n'\|}{\|p_n\|}\|P\|, \quad p_n(t)=t^n,$$

is valid, for each polynomial P of the form (3.4).

M. A. Malik [18] studied an extremal problem in the L^p-norm $(p>1)$ on $(-1,1)$. Namely, he found the following improvement of Theorem 1.4 under only a little restriction on the location of the zeros of P:

Theorem 3.5. *Let $p>1$ and $P \in \mathcal{P}_n$ have no zeros in the two cicular regions $|z \pm a|<1-a$ $(0 \leqq a<1)$. Then $\|P'\| \leqq Bn^{1+1/p}\|P\|$, where B is a constant which depends only on p and a, but not on P or n.*

Note that a can be taken as close to 1 as we like, except that $1-a$ has to be positive. Thus, we have the interesting conclusion that

$$\frac{\|P'\|}{\|P\|}=O(n^{1+1/p})$$

howsoever small the two exceptional circles of the theorem may be.

Similarly, S. Zhou [43] showed in $L^p(-1, 1)$, $1 \leqq p \leqq +\infty$:

Theorem 3.6. *If $P \in \mathcal{P}_n$ has at most k roots in $(-1, 1)$ then*

$$\|P'\| \leqq C(k)n\|P\|,$$

where $C(k)$ is a positive constant depending only on k.

The following result was given by V. I. Buslaev [5].

Theorem 3.7. *Let the polynomial P be represented in the form* $P(t)=Q(t)\,R(t)$, *where*

$$Q(t)=\prod_{i=1}^{m}(t-\tau_i), \qquad |\tau_i|\geq 1 \ (i=1, \ldots, m)$$

and R is an arbitrary polynomial of degree r.

Then for every segment $[a, b]$ *lying strictly in the interior if the interval* $[-1,1]$

$$\|P'\|_{L^p(a,b)}\leq C(a, b)\,\mu\,\|P\|_{L^p(-1,1)},$$

where

$$\mu=r+1+\sum_{i=1}^{m}|\tau_i|^{-2},$$

and $C(a, b)$ *depends only on a and b.*

The extremal problems in L^2-norm on the restricted polynomial classes are especially interesting. In the next sections we will investigate such problems. Several results at this area were given by A. K. Varma [37], [38], [39], [40], [41].

4. Extremal problems of A. K. Varma

In several papers A. K. Varma studied the extremal problems of the form (3.1) in L^2-norm on $(-1,1)$ and $(0,+\infty)$ for some restricted classes of polynomials. So he got several inequalities of the form

$$\|P'\|^2\leq C_n\|P\|^2 \quad (P\in W_n).$$

Beside that, he considered some oposite inequalities.

Let W_n be the set of all algebraic polynomials whose degree is n and whose zeros are all real and lie inside $[-1,1]$.

Theorem 4.1. *Let* $\|f\|^2=\int_{-1}^{1}f(t)^2\,dt$. *If* $P\in W_n$ *and* $n=2m$; *then*

$$\|P'\|^2\geq\left(\frac{n}{2}+\frac{3}{4}+\frac{3}{4(n-1)}\right)\|P\|^2,$$

where equality holds iff $P(t)=(1-t^2)^m$. *Moreover, if* $n=2m-1$, *then*

$$\|P'\|^2\geq\left(\frac{n}{2}+\frac{3}{4}+\frac{5}{4(n-2)}\right)\|P\|^2, \qquad n\geq 3,$$

where equality holds iff $P(t)=(1-t)^{m-1}(1+t)^m$ *or* $P(t)=(1-t)^m(1+t)^{m-1}$.

This theorem is proved by Varma in his paper [41] and it is an improvement of an earlier his result [38]. Similar results in uniform norm and L^p-norm $(p\geq 1)$ on $(-1,1)$ were given by P. Turán [35] and S. Zhou [44] respectively.

In 1979 A. K. Varma [39] proved the three following results:

Theorem 4.2. *Let* $\|f\|^2=\int_{-1}^{1}(1-t^2)f(t)^2\,dt$. *If* $P\in W_n$ *then*

$$\|P'\|^2\geqq\left(\frac{n}{2}+\frac{1}{4}-\frac{1}{4(n+1)}\right)\|P\|^2$$

with equality for $P(t)=(1-t^2)^m$, $n=2m$.

Theorem 4.3. *Let P be an algebraic polynomial of degree* $\leqq n$ *having all real roots and no root inside the interval* $[-1,1]$, *then we have*

$$\|P'\|^2\leqq\frac{n(n+1)(2n+3)}{4(2n+1)}\|P\|^2.$$

with equality for $P(t)=(1+t)^n$ *or* $P(t)=(1-t)^n$. *The norm is the same as in the above theorem.*

Theorem 4.4. *Let P be an algebraic polynomial of degree n having all zeros* τ_k $(k=1,\dots,n)$ *inside* $[0,+\infty)$. *Let* $P(0)=0$ *or*

$$\sum_{k=1}^{n}\tau_k^{-1}\geqq\frac{1}{2};$$

then

$$\|P'\|^2\geqq\frac{n}{2(2n-1)}\|P\|^2$$

with equality for $P(t)=t^n$. *Here* $\|f\|^2=\int_{0}^{\infty}e^{-t}f(t)^2\,dt$.

In 1981 Varma has investigated the problem of determinating the best constant in the inequality

$$\|P'\|^2\leqq C_n(\alpha)\|P\|^2, \tag{4.1}$$

for polynomials with nonnegative coefficients, with respect to the generalized Laguerre weight function $t\to t^{\alpha}e^{-t}$ $(\alpha>-1)$.

Theorem 4.5. *Let P be an algebraic polynomial of exact degree n with nonnegative coefficients. Then for* $\alpha\geqq(\sqrt{5}-1)/2$,

$$\int_{0}^{\infty}(P'(t))^2t^{\alpha}e^{-t}\,dt\leqq\frac{n^2}{(2n+\alpha)(2n+\alpha-1)}\int_{0}^{\infty}P^2(t)\,t^{\alpha}e^{-t}\,dt,$$

equality holding for $P(t)=t^n$. *For* $0\leqq\alpha\leqq 1/2$ *we have*

$$\int_{0}^{\infty}(P'(t))^2t^{\alpha}e^{-t}\,dt\leqq\frac{1}{(2+\alpha)(1+\alpha)}\int_{0}^{\infty}P^2(t)\,t^{\alpha}e^{-t}\,dt. \tag{4.2}$$

Moreover, (4.2) is also best possible in the sence that for $P(t)=t^n+\lambda t$ the expression on the left can be made arbitrarily close to the right by choosing λ positive and sufficiently large.

The case $\alpha = 1$ was considered in [39]. The cases $\alpha\in(-1,0)$ and $\alpha\in(1/2, (\sqrt{5}-1)/2)$ were not solved. D. Xie [42] tried to solve this problem in $(1/2, (\sqrt{5}-1)/2$. Namely, he proved the following complicated and crude result:

Theorem 4.6. *Let*

$$b_n=\frac{n^2}{(2n+\alpha)(2n-1+\alpha)},\quad n=1,2,\ldots,$$

and

$$\alpha_n=\frac{1-2n-4n^2+\sqrt{16n^4+32n^3+20n^2+4n+1}}{2(2n+1)},\quad n=1,2,\ldots.$$

Then, for each $P\in W_n$,

$$\|P'\|^2\le b_n(\alpha)\,\|P\|^2,\quad \text{for } \alpha\geqq\alpha_1;$$

$$\|P'\|^2\le\begin{cases} b_1(\alpha)\,\|P\|^2, & \text{for } \alpha_k\leqq\alpha<\alpha_{k-1} \text{ and } n\leqq k,\\ [b_1(\alpha)+b_n(\alpha)-b_k(\alpha)]\,\|P\|^2, & \text{for } \alpha_k\leqq\alpha<\alpha_{k-1} \text{ and } n>k,\end{cases}$$

where $k=2,3,\ldots$.

In the next section we give a complete solution of the extremal problem (4.1.)

5. Extremal problems for polynomial with nonnegative coefficients in $L^2(0,\infty)$ norm

First, we consider the extremal problem (4.1).

Let W_n be the set of all algebraic polynomials of exact degree n, all coefficients of which are nonnegative, i. e.,

$$W_n=\left\{P\mid P(t)=\sum_{k=0}^{n}a_k t^k,\ a_k\geqq 0\ (k=0,1,\ldots,n-1),\quad a_n>0\right\}.$$

We denote by W_n^0 the subset of W_n for which $a_0=0$ (i. e., $P(0)=0$).

Let $w(t)=t^\alpha e^{-t}\,(\alpha>-1)$ be a weight function on $[0,+\infty)$, and let $\|f\|^2=$ $=(f,f)$, where

$$(f,g)=\int_0^\infty w(t)f(t)g(t)\,dt\qquad (f,g\in L^2[0,+\infty)).$$

In the paper [22] we gave a complete solution of Varma's problem (4.1), i. e. we determined

$$C_n(\alpha)=\sup_{P\in W_n}\frac{\|P'\|^2}{\|P\|^2} \tag{5.1}$$

for all $\alpha\in(-1,+\infty)$.

Theorem 5.1. *The best constant* $C_n(\alpha)$ *defined in* (5.1) *is*

$$C_n(\alpha)=\begin{cases}1/(2+\alpha)(1+\alpha) & (-1<\alpha\leqq\alpha_n),\\ n^2/(2n+\alpha)(2n+\alpha-1) & (\alpha_n\leqq\alpha<+\infty),\end{cases} \tag{5.2}$$

where

$$\alpha_n=\frac{1}{2}(n+1)^{-1}\left((17n^2+2n+1)^{1/2}-3n+1\right). \tag{5.3}$$

Note that the supremum in (5.1) is attained for some $P\in W_n^0$. Indeed

$$\sup_{P\in W_n}\frac{\|P'\|}{\|P\|}=\sup_{\substack{P\in W^0\\ a_0\geqq 0}}\frac{\|P'\|}{\|P+a_0\|}=\sup_{P\in W_n^0}\frac{\|P'\|}{\|P\|}.$$

We can see that $P(t)=t^n$ is an extremal polynomial for $\alpha\geqq\alpha_n$. Furthermore, if $-1<\alpha\leqq\alpha_n$, there exists a sequence of polynomials, for example, $p_k(t)=t^n+kt$, $k=1,2,\ldots$, for which

$$\lim_{k\to\infty}\frac{\|p_k'\|^2}{\|p_k\|^2}=C_n(\alpha).$$

From Theorem 5.1 we can see:

(a) $C_n(\alpha_n-0)=C_n(\alpha_n+0)$;

(b) $C_{n+1}(\alpha)\leqq C_n(\alpha)$;

(c) The sequence (α_k) is decreasing, i. e., $\alpha_1>\alpha_2>\alpha_2>\ldots>\alpha_\infty$, where

$$\alpha_1=(\sqrt{5}-1)/2,\quad \alpha_2=(\sqrt{73}-5)/6,\quad \alpha_3=(\sqrt{10}-2)/2,\ \text{etc.},$$

and

$$\alpha_\infty=\lim_{n\to\infty}\alpha_n=(\sqrt{17}-3)/2=0.561552812\ldots.$$

Remark. The statement of Theorem 5.1 holds if W_n is the set of all algebraic polynomials $P(\not\equiv 0)$ of degree at most n (not only of exact degree n), with nonnegative coefficients. In this case, if $-1<\alpha\leqq\alpha_n$, we can see that $\tilde{P}(t)=\lambda t$ $(\lambda>0)$ is an extremal polynomial.

Using the same method as in [22], we can solve the following general extremal problem for higher derivatives

$$C_{n,m}(\alpha)=\sup_{P\in W_n}\frac{\|P^{(m)}\|^2}{\|P\|^2}\qquad(1\leqq m\leqq n).$$

Theorem 5.2. *The best constant* $C_{n,m}(\alpha)$ *is given by*

$$C_{n,m}(\alpha)=\begin{cases}\dfrac{(m!)^2}{(\alpha+1)_{2m}}, & -1<\alpha\leqq\alpha_{n,m},\\[2mm] \dfrac{n^2(n-1)^2\cdots(n-m+1)^2}{(2n+\alpha)^{(2m)}}, & \alpha\geqq\alpha_{n,m},\end{cases}$$

where $\alpha_{n,m}$ *is the unique positive root of the equation*

$$\frac{(2n+\alpha)^{(2m)}}{(2m+\alpha)^{(2m)}}=\binom{n}{m}^2.$$

Here $(p)_k=p(p+1)\ldots(p+k-1)$ *and* $p^{(k)}=p(p-1)\ldots(p-k+1)$.

In the special case, when $n\to+\infty$, we have

$$\lim_{n\to\infty}C_{n,m}(\alpha)=\begin{cases}\dfrac{(m!)^2}{(\alpha+1)_{2m}}, & -1<\alpha\leqq\alpha_m^*,\\[2mm] \dfrac{1}{4^m} & \alpha_m^*\leqq\alpha<+\infty,\end{cases}$$

where α_m^* is the unique positive root of the equation

$$(2m+\alpha)^{(2m)}=2^{2m}(m!)^2.$$

We note that $\alpha_1^*=\alpha_\infty=(\sqrt{17}-3)/2$. These roots α_m^* for $m=2,\ldots,6$ are presented in Table 5.1 (with seven decimal digits).

Table 5.1

m	2	3	4	5	6
α_m^*	0.5515992	0.5461112	0.5425236	0.5399438	0.5379725

The extremal problem for the polynomials with nonnegative coefficients can be investigated with other weight functions on $(0,+\infty)$, for example, $w(t)=t^\alpha\exp(-t^s)$, $\alpha>-1$, $s>0$. The corresponding best constant we will denote by $C_n(\alpha;s)$.

In this case, using the same method we can prove that for $P\in W_n^0$

$$\|P\|^2=(P,P)=\frac{1}{s}\sum_{k=2}^{2n}b_k\Gamma\left(\frac{\alpha+k+1}{s}\right)$$

and

$$\|P'\|^2=(P',P')\leqq\frac{1}{s}\sum_{k=2}^{2n}H_k(\alpha;s)\,b_k\Gamma\left(\frac{\alpha+k+1}{s}\right),$$

where $(f, g) = \int_0^\infty w(t) f(t) g(t) \, dt$ and

$$H_k(\alpha; s) = \frac{k^2}{2} \cdot \frac{\Gamma\left(\frac{\alpha+k+1}{s}\right)}{\Gamma\left(\frac{\alpha+k+1}{s}\right)}.$$

For $s = 2$ we get a simple result:

Theorem 5.3. *The best constant* $C_n(\alpha; 2)$ *is given by*

$$C_n(\alpha; 2) = \begin{cases} \dfrac{2}{\alpha+1}, & -1 < \alpha \leqq -\dfrac{n-1}{n+1}, \\[2ex] \dfrac{2n^2}{2n+\alpha-1}, & -\dfrac{n-1}{n+1} \leqq \alpha < +\infty. \end{cases}$$

If we take, e. g. $\alpha = 0$, we have the following inequality

$$\int_0^\infty e^{-t^2} P'(t)^2 \, dt \leqq \frac{2n^2}{2n-1} \int_0^\infty e^{-t^2} P(t)^2 \, dt$$

for each $P \in W_n$.

In connection with these results see the paper [23].

6. Extremal problems for Lorentz classes of polynomials

Let L_n be the set of algebraic polynomials of the form

$$P(t) = \sum_{k=0}^{n} b_k (1-t)^k (1+t)^{n-k}, \quad b_k \geqq 0 \ (k = 0, 1, \ldots, n). \tag{6.1}$$

These polynomials (transformed to [0, 1]) were introduced by G. G. Lorentz [14] (see Section 3). A subset of Lorentz's class L_n for which $P^{(i-1)}(\pm 1) = 0$ $(i=1, \ldots, m)$ we denote by $L_n^{(m)}$. Notice that $L_n^{(0)} \supset L_n^{(1)} \supset \cdots$, where $L_n^{(0)} \equiv L_n$. The corresponding representation of a polynomial P from $L_n^{(m)}$ is

$$P(t) = \sum_{k=m}^{n-m} b_k (1-t)^k (1+t)^{n-k}, \quad b_k \geqq 0 \ (k = m, \ldots, n-m). \tag{6.2}$$

If $\|f\|^2 = \int_{-1}^{1} (1-t)^\alpha (1+t)^\beta f(t)^2 \, dt$ $(\alpha, \beta > -1)$, we can consider the following extremal problem

$$C_n^{(m)}(\alpha, \beta) = \sup_{P \in L_n^{(m)} \setminus \{0\}} \frac{\|P'\|^2}{\|P\|^2}, \tag{6.3}$$

where $m=0,1,\ldots,\left[\frac{n}{2}\right]$. The corresponding problem in the class L_n for the uniform norm was considered by G. G. Lorentz (see Theorem 3.3).

Here, we mention only some special cases of general results obtained by G. V. Milovanović and M. S. Petković [24].

Theorem 6.1. *If* $P\in L_n$ *and* $\alpha, \beta \geqq 1$, *then the best constant* $C_n^{(0)}(\alpha, \beta)$ *defined in* (6.3) *is*

$$C_n^{(0)}(\alpha, \beta)=\frac{n^2(2n+\alpha+\beta)(2n+\alpha+\beta+1)}{4(2n+\lambda)(2n+\lambda-1)},$$

where $\lambda=\min(\alpha, \beta)$.

Theorem 6.2. *If* $P\in L_n^{(m)}$, $m\geqq 1$, $\alpha=\beta>-1$, *then*

$$C_n^{(m)}(\alpha, \alpha)=\frac{(n+\alpha)(2n+2\alpha+1)\left(\alpha(\alpha-1)n^2+2m(n-m)(n-1+3\alpha-2\alpha^2)\right)}{2(2m+\alpha-1)(2m+\alpha)(2n-2m+\alpha-1)(2n-2m+\alpha)}.$$

In the special case when $\alpha=1$ we obtain:

Corollary 6.3. *If* $P\in L_n^{(m)}$, $m\geqq 1$, *we have*

$$C_n^{(m)}(1,1)=\frac{n(n+1)(2n+3)}{4(2m+1)(2n-2m+1)}. \tag{6.4}$$

From Theorem 6.1 we see that (6.4) holds for $m=0$ (see also Theorem 4.3).

In the proofs of these theorems we use the representations of Lorenz polynomials (6.1) and (6.2) and an analogue of the Lemma 1 from [22]. Another interesting results on this topic can be found in the mentioned paper [27].

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RAZLIČITI EKSTREMALNI PROBLEMI MARKOVLJEVOG TIPA ZA ALGEBARSKE POLINOME

Gradimir V. Milovanović

U radu se razmatraju ekstremalni problemi Markovljevog tipa za algebarske polinome korišćenjem različitih normi i više polinomialnih klasa. Posebna pažnja je posvećena ekstremalnim problemima u L^2-normi na skupu svih algebarskih polinoma ne višeg stepena od n ili na nekim njegovim podskupovima.

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AN INEQUALITY FOR CONCAVE FUNCTIONS WITH APPLICATIONS TO BESSEL AND TRIGONOMETRIC FUNCTIONS

Lee Lorch and Martin E. Muldoon

Abstract. We use an inequality for concave functions to improve the range of validity of an inequality for Bessel functions due to A. Mahajan [Univ. Beograd. Publ. Electrotehn. Fak. Ser. Mat. Fiz. No. 634—677 (1979), 70—71.] We use the same method to derive inequalities for the zeros of the Bessel function $J_\nu(x)$ and its derivative.

1. Introduction

A. Mahajan [6] generalized a result of D. S. Mitrinović [7, pp. 240—241] by showing that

$$(x+1)^{\alpha+1} J_\alpha\left(\frac{\pi}{x+1}\right) - x^{\alpha+1} J_\alpha\left(\frac{\pi}{x}\right) > \left(\frac{\pi}{2}\right)^\alpha \frac{1}{\Gamma(\alpha+1)}, \tag{1}$$

provided

$$x > \pi\,[16(\alpha+2)]^{-1}\,[\pi + \sqrt{\pi^2 + 32(\alpha+2)}] \tag{2}$$

where $a > -1$ and $J_\alpha(t)$ is the Bessel function of the first kind.

Mitrinović had established the case $\alpha = -\frac{1}{2}$ of (1), namely,

$$(x+1)\cos\frac{\pi}{x+1} - x\cos\frac{\pi}{x} > 1, \tag{3}$$

but only for $x \geqq \sqrt{3} = 1.732\ldots$, whereas Mahajan's generalization (1) established a wider interval of validity since, for $\alpha = -\frac{1}{2}$, (2) becomes

$$x > \pi\left[\pi + \sqrt{\pi^2 + 48}\right]/24 = 1.407\ldots. \tag{4}$$

The largest interval of validity for (3) is the interval $x > 1$, as will be shown below;(3) becomes an equality when $x = 1$ and reverses for $0 < x < 1$. We discuss also the largest interval for which the more general inequality (1) is valid and establish some related inequalities. Some of these (Sec. 4) refer to the zeros of $J_\nu(t)$ and of its derivative.

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The proofs here differ from those in [6] and [7]. They stem fundamentally from noticing that (1) and (3) are inequalities for functions that are concave for most of the interval of validity. Inasmuch as this observation may be useful for other functions, we formulate it explicitly.

Remark 1. (*i*) *If $f(t)$ is (strongly) concave for $0\leqslant t\leqslant b$, then*

$$\frac{f(r)}{r}-\frac{f(s)}{s}>\left(\frac{1}{r}-\frac{1}{s}\right)f(0), \quad \text{for} \quad 0<r<s\leqslant b. \tag{5}$$

(ii) *If $f(t)$ is continuous, then this inequality remains valid for some r and s, one or both possibly greater than b, provided that for each $s>b$ we restrict attention to those values of r less than the smallest value of r for which* (5) *becomes an equality.*

Proof. (*i*) The slope of the line joining $(0,f(0))$ to $(r,f(r))$ is (by definition of concavity) algebraically larger that of the line joining $(0,f(0))$ to $(s,f(s))$, i.e.,

$$\frac{f(r)-f(0)}{r}>\frac{f(s)-f(0)}{s},$$

a statement equivalent to (5). (*ii*) is trivial (stated only in view of pending applications below).

Remark 2. To illustrate (5) we may take $y=f(t)$ to be a positive solution of the differential equation $y''+\varphi(t)y=0$ where $\varphi(t)>0$. $0<t\leqslant b$. Then (5) implies that $y(t)/t$ is a decreasing function of t, $0\leqslant t\leqslant b$. This implies, in particular, the well-known result that $(\sin t)/t$ decreases in $0\leqslant t\leqslant\pi$, and consequently also the familiar inequality

$$1\geqslant\frac{\sin t}{t}\geqslant\frac{2}{\pi}, \quad 0\leqslant t\leqslant\pi/2.$$

These observations correspond to the case $\varphi(t)=2$.

More generally, taking

$$\varphi(t)=\beta^2 t^{2\beta-2}, \quad \beta>0,$$

the differential equation $y''+\varphi(t)y=0$ is satisfied by [9, p. 97 (9)] $y=t^{1/2}J_{1/(2\beta)}(t^{\beta})$ so that $t^{-1/2}J_{1/(2\beta)}(t^{\beta})$ decreases for $0\leqslant t^{\beta}\leqslant j_{1/(2\beta),1}$, where $j_{\nu,1}$ is the first positive zero of $J_\nu(x)$.

Putting $\nu=1/(2\beta)$ and $x=t^{\beta}$, we see that $J_\nu(x)/x^\nu$ decreases (from $[2^\nu\Gamma(\nu+1)]^{-1}$ to 0) in $(0, j_{\nu 1})$.

When $\nu=\frac{1}{2}$ we recover the foregoing information about the sine function.

The information that $x^{-\nu}J_\nu(x)$ decreases follows also from the standard differentiation formula [9, p. 45]

$$\frac{d}{dx}\left\{\frac{J_\nu(x)}{x^\nu}\right\}=-\frac{J_{\nu+1}(x)}{x^\nu} \tag{6}$$

and, indeed, for the larger range $0<x<j_{\nu+1,1}$

Other applications of Remark 1 can be made, e. g., to solutions of the Hill, Lamé and Mathieu equations for appropriate values of the parameters [4, Chapters 15, 16].

2. On the Mahajan and Mitrinović inequalities

In the usual notation, let

$$\Lambda_\alpha(t) = 2^\alpha \Gamma(\alpha+1) t^{-\alpha} J_\alpha(t), \quad t>0, \quad \alpha>-1,$$

$$\Lambda_\alpha(0) = \Lambda_\alpha(0+) = 1.$$

From (6), it follows that

$$\Lambda_\alpha''(t) = -2^\alpha \Gamma(\alpha+1) t^{-\alpha-1} [J_{\alpha+1}(t) - tJ_{\alpha+2}(t)],$$

so that $\Lambda_\alpha(t)$ is a (decreasing and) concave function for all sufficiently small positive t, say $0<t<\delta$. Thus, from (5),

$$\frac{\Lambda_\alpha(r)}{r} - \frac{\Lambda_\alpha(s)}{s} > \frac{1}{r} - \frac{1}{s}, \quad 0<r<s<\delta, \tag{7}$$

Putting $r=\pi/(x+1)$ and $s=\pi/x$, we have the following formulation of Mahajan's result:

For $\alpha>-1$ *and* $x>x_0$, *we have*

$$(x+1)\Lambda_\alpha\left(\frac{\pi}{x+1}\right) - x\Lambda_\alpha\left(\frac{\pi}{x}\right) > 1, \tag{8}$$

where x_0 *is the largest root of* $\varphi(x+1) = \varphi(x)$ *and*

$$\varphi(x) = x\{\Lambda_\alpha(\pi/x) - 1\}.$$

The interval (x_0, ∞) is the largest possible domain of validity for (8), but is not indicated in an explicit form. A smaller interval, but one given in a more explicit fashion, follows from Remark 1(i), namely that (8) holds for $x \geqslant \pi/x_1$, where x_1 is the abscissa of the first point of inflection of $\Lambda_\alpha(t)$. In general, however, the interval $[\pi/x_1, \infty)$ would be shorter than Mahajan's interval (2).

The special case $\alpha = -\frac{1}{2}$ of (8) establishes that (3) *holds for all* $x>1$ *and not for* $x=1$. Thus we have obtained the *largest* interval for which (3) is valid.

It is clear (3) becomes an equality for $x=1$ so that we need to prove that $g(x+1) = g(x)$ has no solution on $(1, \infty)$, where

$$g(x) = x\{\cos(\pi/x) - 1\}.$$

To see this, we note that

$$g'(x) = \sin(\pi/x)\{\pi/x - \tan[\pi/(2x)]\}$$

so that $g(x)$ decreases in the right-hand neighbourhood of 1, has precisely one local minimum between $x = 1$ and $x = 2$ and subsequently increases. Thus,

$$g(x+1) \neq g(x), \quad x \geqslant 2.$$

Moreover, for $1<x<2$, $g(x)<g(1) = g(2) = -2$, while $g(x+1)>g(2)$ so that $g(x+1) \neq g(x)$ also when $1<x<2$, completing the proof.

Another consequence of (8), obtained by putting $\alpha = \frac{1}{2}$, is

$$(9) \qquad (x+1)^2 \sin[\pi/(x+1)] - x^2 \sin[\pi/x] > \pi,$$

provided x *exceeds the largest root of* $h(x+1) = h(x)$, *where*

$$h(x) = x^2 [\sin(\pi/x) - \pi/x].$$

This root is about 0.68 so that the interval of validity for (9) as determined by this method is about $(0.68, \infty)$. Putting $\alpha = \frac{1}{2}$ in (2) would establish the truth of (9) only for $x > 0.991$.

Remark 3. The argumentation regarding (3) above can be phrased more transparently by writing (3) in the form

$$(3') \qquad \frac{\cos[\pi/(x+1)]}{\pi/(x+1)} - \frac{\cos(\pi/x)}{\pi/x} > 1/\pi, \quad x > 1,$$

i.e.,

$$(3'') \qquad f(r)/r - f(s)/s > 1/\pi,$$

where $f(z) = \cos z$, $r = \pi/(x+1)$, $s = \pi/x$. Here $0 < r < s < \pi/2$. The inequality (3'') holds for arbitrary r, s, $0 < r < s \leqslant \pi/2$ as is clear from Remark 2(i) upon sketching $f(z) = \cos z$, $0 \leqslant z \leqslant \pi/2$. This re-establishes (3') and (3) for $x \geqslant 2$ (see Sec. 3 (a) for further extensions).

For the remaining portion, namely $1<x<2$, of the interval of validity the same sketch for $f(z)$, now for $0 \leqslant z \leqslant \pi$, makes it clear that the chord connecting $(0, 1)$ to a point $(r, f(r))$ lies above the chord connecting $(0, 1)$ to any other point $(x, f(s))$ provided $0<r\leqslant \pi/2<s\leqslant\pi$. This establishes (3') also for $1<x<2$.

3. Further inequalities for the cosine function

Clearly, (8) can be generalized by choosing $r = \pi/(x+\beta)$, $\beta > 0$, $s = \beta/x$ in **(7)**, instead of restricting β to be 1. In the case $\alpha = -\frac{1}{2}$, this would lead to the following extension of (3):

$$(10) \qquad (x+\beta)\cos\frac{\pi}{x+\beta} - x\cos\frac{\pi}{x} > \beta,$$

if $0<\beta\leqslant 1$, *provided* x *exceeds the largest root of* $g(x+\beta)=g(x)$ *where, as before,*

$$g(x)=x\{\cos(\pi/x)-1\}.$$

A weaker but clearer result reads as follows:

(10) *is valid for* $0<\beta\leqslant 1$ *and* $x>2-\beta$.

The proof requires us to show that $g(x+\beta)-g(x)$ has no zeros for $2-\beta<x<\infty$. This can be done as has already been explained for the special case in which $\beta=1$ and so the details are omitted.

4. Zeros of Bessel functions and their derivatives

Remark 1. can be applied also to the kth positive zero, $j_{\nu k}$, of $J_\nu(t)$, $\nu>0.$, since A. Elbert [1] has shown that $j_{\nu k}$ is a concave function of ν for $\nu\geqslant 0$. (Elbert's result extends to the interval $-k<\nu<\infty$ when $j_{\nu k}$ is interpreted appropriately for negative values of ν, but we shall not consider this aspect here.)

Thus

$$(11)\qquad \frac{j_{\nu k}}{\nu}-\frac{j_{\mu k}}{\mu}>j_{0k}\left[\frac{1}{\nu}-\frac{1}{\mu}\right] \quad \text{for } 0<\nu<\mu, \quad k=1, 2, \ldots .$$

This inequality appears not to have been stated previously except for its limiting case ($\mu\to\infty$)

$$(12)\qquad j_{\nu k}>j_{0k}+\nu, \quad \nu>0, \quad k=1, 2, \ldots,$$

got by using $j_{\nu k}/\mu\to 1$ as $\mu\to\infty$ [8]; (12) has been established in [5] in a slightly different way, but also based on Elbert's concavity theorem.

Remark 4. Still another proof of (12)follows from a different use of Elbert's theorem. From that theorem we see that the positive function $dj_{\nu k}/d_\nu$ decreases in $0<\nu<\infty$ so that $dj_{\nu k}/d\nu-c>0$ as $\nu\to\infty$. Thus, given any $\varepsilon>0$, we have, for sufficiently large ν,

$$-\varepsilon<\mathrm{d}j_{\nu k}/\mathrm{d}\nu-c<\varepsilon.$$

Integration from λ to ν and division by $\nu-\lambda$ implies

$$-\varepsilon<\frac{j_{\nu k}-j_{\lambda k}}{\nu-\lambda}-c<\varepsilon.$$

Letting $\nu\to\infty$, λ fixed, we get

$$-\varepsilon\leqslant 1-c\leqslant\varepsilon.$$

Hence $c=1$. Thus $dj_{\nu k}/d_\nu\downarrow 1$, as $0<\nu\to\infty$, and so

$$(13)\qquad \mathrm{d}j_{\nu k}/\mathrm{d}\nu>1, \quad \nu\geqslant 0.$$

Integration from 0 to ν gives (12). Inequality (13) is also contained in work of A. Elbert and A. Laforgia [2]; see, especially the Lemma in [2, p. 207].

The same argument can be applied to get the inequalities (11), (12) and (13) when $j_{\nu k}$, etc., are replaced by $j'_{\nu k}$, etc., where $j'_{\nu k}$ is the kth positive zero of $J_\nu'(x)$, because this function too has been shown to be concave, for $0<\nu<\infty$, by Elbert and Laforgia [3]. In fact, Elbert and Laforgia [3, Cor. 4.1] have a more general inequality than

$$(12') \qquad j'_{\nu k} > j'_{0 k} + \nu, \quad \nu > 0, \quad k = 1, 2, \ldots .$$

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NEJEDNAKOST ZA KONKAVNE FUNKCIJE SA PRIMENOM ZA BESSELOVE I TRIGONOMETRIJSKE FUNKCIJE

Lee Lorch i Martin E. Muldoon

Korišćenjem jedne nejednakosti za konkavne funkcije proširuje se interval u kome važi A. Mahajanova nejednakost za Besselove funkcije ([6]). Isti metod se koristi za dobijanje nejednakosti za nule Besselove funkcije $J_\nu(x)$ i njenih izvoda.

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QUELQUES REMARQUES RELATIVES AUX FONCTIONES PRIMITIVES DES FONCTIONS RÉELLES

Dušan Adamović

Sommaire. Dans le cadre de considérations plus complexes, on traite ici, en particulier, la question suivante: la fonction f possédant une fonction primitive dans l'intervalle I, quelle propriété (quel degré de régularité) de la fonction g, définie dans I, assure-t-elle l'existence de la fonction primitive dans I du produit $h=f\cdot g$? Les résultats principaux du travail sont contenus dans les théorèmes **P** et **P'**.

0. Dans ce qui suit, l'ensemble des nombres naturels sera désigné par **N**, celui des nombres rationnels par **Q** et celui des nombres réels par **R**.

Comme d'habitude, nous appelons *fonction primitive* de la fonction réelle f dans l'intervalle ouvert I toute fonction réelle F definie dans I et telle que $F'(x)=f(x)$, $x\in I$. Pour une fonction possédant une fonction primitive dans I nous disons qu'elle est une P_I-*fonctions*, ou bien qu'elle *a la propriété* P_I; s'il n'est pas nécessaire de mentioner expressément l'intervalle I, nous disons qu'elle est *P-fonction* ou qu'elle *a la propriété P*.

Il est clair que la somme de deux P_I-fonctions et aussi une P_I-fonction, c'est-à-dire que la propriété P_I est *additive*. Beacoup d'autres propriétés des fonctions réelles sont aussi additives, par exemple: intégrabilité (dans n'importe quel sens), *continuité*, continuité uniforme, *propriété d'être bornée*, *variation bornée*, *dérivabilité* (propriété d'avoir une dérivié finie en tout point), *dérivabilité continue* (propriété d'avoir la dérivée continue) dans un intervalle (ouvert ou fermé selon le cas). De ces propriétés-là, celles soulignées, de même que l'intégrabilité dans le sens de Riemann dans un intervalle borné, sont aussi *multiplicatives*, ce qui veut dire que le produit de deux fonctions jouissant de l'une de ces propriétés — en jouit aussi. C'est par le théorème qui suit que nous allons établier, entre autre, que *la propriété* P_I *n'est pas multiplicative*. Dans la litterature assez vaste, comprenant monographies, manuels et autres publications, dans laquelle on traite (largement ou en passant) les fonctions primitives — par exemple dans la monographie bien connue [2] de H. Lebesgue, de même que dans les livres aussi connus [1] et [3] — nous n'avons nulle part trouvé la constation de ce fait assez élémentaire.

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On peut, bien entendu, poser la question plus complexe suivante: la propriété $\mathrm{P_I}$ de la fonction f étant supposée, quelle propriété (plus forte que $\mathrm{P_I}$) de la fonction g définie dans I assure-t-elle la propriété $\mathrm{P_I}$ du produit des fonctions f et g? Une réponse assez complète à cette question-là est donnée par notre théorème **P** et par le théorème **P'** qui lui est ajouté.

1. Dans la démonstation de ces résultats nous allons utiliser les deux énoncés auxiliaires suivants.

Lemme 1. *Soit $I=(a, b)$, $x_0 \in I$ et f une fonction définie dans I et juissant à la fois des propriétés $P_{(a,\, x_0)}$ et $P_{(x_0,\, a)}$. Alors f est une P_I-fonction si et seulement si une fonction primitive F_1 de f dans (a, x_0) (arbitrairrment choisie) a dans le point x_0 la limite à gauche finie et une fonction primitive F_2 de f dans (x_0, b) a dans le point x_0 la limite à droite finite, et l'égalite suivante*

$$\lim_{x\to x_0-0} \frac{F_1(x)-F_1(x_0-0)}{x-x_0} = \lim_{x\to x_0+0} \frac{F_2(x)-F_2(x_0+0)}{x-x_0} = f(x_0)$$

est valable.

Démonstration. Il est évident que toutes ces conditions sont nécessaires. D'autre part, il est clair que, lorsqu'elles sont toutes remplies et les fonctions F_1 i F_2 sont choises de manière que l'on ait $F_1(x_0-0)=F_2(x_0+0)=A$, alors la fonction F définie par

$$F(x)=\begin{cases} F_1(x), & a<x<x_0 \\ A, & x=x_0 \\ F_2(x), & x_0<x<b \end{cases}$$

est une fonction primitive de f dans I. La fonction f, donc, possède la propriété P_I.

Corollaire. *Sous la condition supplémentaire de l'integrabilité dans le sens de Riemann de f dans tout intervalle fermé et borné contenu dans $(a, x_0)\cup(x_0, b)$ (en particulier, de la continuité de f dans $(a, x_0)\cup(x_0, b)$), cette fonction jouit de la propriété P_I si et seulement si, aves les nombres $x_1 \in (a, x_0)$ et $x_2 \in (x_0, b)$ arbitrairement choisis, les valeurs $\int_{x_1}^{x_0-0} f(t)\,dt$* et *$\int_{x_0+0}^{x_2} f(t)\,dt$ sont finies et*

$$\lim_{x\to x_0-0} \frac{1}{x-x_0}\int_{x_0-0}^{x} f(t)\,dt = \lim_{x\to x_0+0} \frac{1}{x-x_0}\int_{x_0+0}^{x} f(t)\,dt = f(x_0).$$

Si, par surcroît, le graphe de f est symétrique par rapport à la droite $x=x_0=\frac{1}{2}(a+b)$, ou bien si l'intervalle I est remplacé par l'intervalle $[x_0, b)$, alors pour la propriété P_I de f il faut et il suffit que $\int_{x_0+0}^{x_2} f(t)\,dt$ soit fini et que l'on ait

$$\lim_{x\to x_0+0} \frac{1}{x-x_0}\int_{x_0+0}^{x} f(t)\,dt = f(x_0).$$

Lemme 2. *La fonction*

$$f_\alpha(x)=\begin{cases} |x|^\alpha \sin\frac{1}{x^2}, & x\neq 0\\ 0, & x=0\end{cases}$$

dans l'intervalle $I=\mathbf{R}$:

1° *pour* $\alpha>3$ *est continûment dérivable;*

2° *pour* $1<\alpha\leqslant 3$ *est dérivable et n'est pas continûment dérivable;*

3° *pour* $0<\alpha\leqslant 1$ *est continue et n'est pas dérivable;*

4° *pour* $-2<\alpha\leqslant 0$ *a la propriété P est n'est pas continue;*

5° *pour* $\alpha\leqslant -2$ *n'a pas la propriété P.*

Démonstration. On démontre aisément et d'une manière habituelle les faits 1°, 2° et 3°. Évidemment, pour $\alpha\leqslant 0$ la fonction f n'est pas continue dans $\mathbf{R}$. Pour tout $\alpha\in\mathbf{R}$ elle a dans $(0, +\infty)$ la fonction primitive

$$\text{(1)}\qquad G_\alpha(x)=\int_1^x f_\alpha(t)\,dt=\int_1^x t^\alpha \sin\frac{1}{t^2}\,dt=\int_{\frac{1}{x}}^1 u^{-(\alpha+2)}\sin u^2\,du.$$

Il en résulte que, d'après le lemme 1, son corollaire et le fait que f_a est une fonction paire, pour la propriété $\mathbf{P_R}$ de cette fonction il suffit que la limite

$$\text{(2)}\qquad G_\alpha(+0)=\lim_{x\to+0}\int_{\frac{1}{x}}^1 u^{-(\alpha+2)}\sin u^2\,du=\int_{+\infty}^1 u^{-(\alpha+2)}\sin u^2\,du$$

soit finie et que l'on ait en plus

$$\lim_{x\to+0}\frac{G_\alpha(x)-G_\alpha(x+0)}{x}=0=f_\alpha(0).$$

Soit $-2<\alpha\leqslant 0$. On a alors $\alpha+2>0$, et par conséquent l'integrale $\int_1^{+\infty} u^{-(\alpha+2)}\sin u^2\,du$ converge, ce qui signifie, d'après (2), que la limite $G_\alpha(+0)$ existe et que sa valeur est finie. Dans la même cas, en appliquant l'intégration par parties et le théorème de L'Hospital, on obtient

$$\lim_{x\to+0}\frac{G_\alpha(x)-G_\alpha(+0)}{x}=\lim_{x\to+0}\frac{1}{x}\int_{\frac{1}{x}}^{+\infty} u^{-(\alpha+2)}\sin u^2\,du$$

$$=\frac{1}{2}\lim_{y\to+\infty} y\int_y^{+\infty} u^{-(\alpha+3)}\cdot 2u\sin u^2\,du$$

$$=\frac{1}{2}\lim_{y\to+\infty}\left[y\cdot y^{-(\alpha+3)}\cos y^2-(\alpha+3)\,y\int_y^{+\infty} u^{-(\alpha+4)}\cos u^2\,du\right]$$

$$= \frac{1}{2}\left[\lim_{y\to+\infty} y^{-(\alpha+2)}\cos y^2 - (\alpha+3)\lim_{y\to+\infty}\frac{\int_y^{+\infty} u^{-(\alpha+4)}\cos u^2\, du}{y^{-1}}\right]$$

$$= -\frac{\alpha+3}{2}\lim_{y\to+\infty}\frac{-y^{-(\alpha+4)}\cos y^2}{-y^{-2}} = -\frac{\alpha+3}{2}\lim_{y\to+\infty} y^{-(\alpha+2)}\cos y^2 = 0.$$

Il en résulte, d'après ce qui précède, que pour $-2<\alpha\leqslant 0$ la fonction f_α a la propriété $P_{\mathbf{R}}$ et n'est pas continue dans $\mathbf{R}$.

Si $\alpha\leqslant -3$, l'intégrale $\int_1^{+\infty} u^{-(\alpha+2)}\sin u^2\, du$, c-est-à-dire l'intégrale $\int_1^{+\infty} v^{-\frac{1}{2}(\alpha+3)}$ $\sin v\, dv$, diverge, da manière que f_α n'est pas $P_{\mathbf{R}}$ — fonction dans ce cas-là. Enfin, lorsque $-3<\alpha\leqslant 2$, l'intégrale $\int_1^{+\infty} u^{-(\alpha+2)}\sin u^2\, du = \frac{1}{2}\int_1^{+\infty} v^{-\frac{1}{2}(\alpha+3)}\sin v\, dv$ converge, ce qui signifie que $G_\alpha(+0)$ a une valeur finie, mais dans ce cas nous obtenons, avec $y=\frac{1}{x}$, $x>0$,

$$\frac{G_\alpha(x)-G_\alpha(+0)}{x} = y\int_y^{+\infty} u^{-(\alpha+2)}\sin u^2\, du = \frac{1}{2}y\int_y^{+\infty} u^{-(\alpha+3)}\cdot 2u\sin u^2\, du$$

$$= \frac{1}{2}y\cdot y^{-(\alpha+3)}\cos y^2 - \frac{\alpha+3}{2}\cdot\frac{1}{2}y\int_y^{+\infty} u^{-(\alpha+4)}\cdot u^{-1}\cdot 2u\cos u^2\, du$$

$$= \frac{1}{2}y^{-(\alpha+2)}\cos y^2 + \frac{\alpha+3}{4}y^{-(\alpha+4)}\sin y^2 - \frac{(\alpha+3)(\alpha+5)}{4}y\int_y^{+\infty} u^{-(\alpha+6)}\sin u^2\, du$$

$$(3)\qquad = \frac{1}{2}y^{-(\alpha+2)}\cos y^2 + o(1),\ \text{lorsque}\ y\to+\infty,\ \text{c'-est-à-dire}\ x\to+0,$$

puisque $\lim_{y\to+\infty} y^{-(\alpha+4)}\cdot\sin y^2 = 0$ et

$$\lim_{y\to+\infty} y\int_y^{+\infty} u^{-(\alpha+6)}\sin u^2\, du = \lim_{y\to+\infty}\frac{\int_y^{+\infty} u^{-(\alpha+6)}\sin u^2\, du}{y^{-1}}$$

$$= \lim_{y\to+\infty}\frac{-y^{-(\alpha+6)}\sin y^2}{-y^{-2}} = \lim_{y\to+\infty} y^{-(\alpha+4)}\sin y^2 = 0.$$

Il résulte de (3) que dans ce cas $\frac{G_\alpha(x)-G_\alpha(+0)}{x}$ n'a pas de limite lorsque $x\to+0$. Donc, f_α n'est pas $P_{\mathbf{R}}$ — fonction pour $-3<\alpha\leqslant -2$ non plus.

2. Nos résultats principaux sont contenus dans l'énoncé suivant.

Théorème P. *On suppose les fonctions réelles f et g définies dans l'intervalle ouvert I et f possédant la propriété P_I, et soit $h(x)=f(x)\cdot g(x)$, $x\in I$. Alors:*

1° La dérivabilité continue de g dans I entraîne la propriété P_I de h.

2° La dérivabilité de g dans I n'entraîne pas la propriété P_I de h.

3° Si la fonction g possède la propriété P_I et aucune des fonctions f et g n'est continue dans I, la fonction h peut avoir ou ne pas avoir la propriété P_I.

4° Si la fonction f n'est pas continue dans I (possédant toujours la propriété P_I, selon l'hypothèse préalable du théorème), alors pour aucun nombre naturel $n\geqslant 2$ la n-ième puissance de f ne peut être fonction continue dans I, et pour tout tel nombre n cette puissance peut avoir ou ne pas avoir la propriété P_I.

Démonstration. 1° Supposons que la dérivée de g soit continue dans I. D'après la supposition préalable, la fonction f a une fonction primitive F dans I. Alors le produit $F(x)\cdot g'(x)$ est une fonction continue dans I, et par conséquent y possède une fonction primitive G. Il s'ensuit

$$\begin{aligned}(F(x)\cdot g(x)-G(x))' &= f(x)\cdot g(x)+F(x)\cdot g'(x)-G'(x)\\ &= f(x)\cdot g(x)=h(x),\quad x\in I.\end{aligned}$$

Donc, h est une P_I-fonction.

Il est clair que tous les exemples qui suivent (ici ou plus loin dans cet article) et où l'on a $I=\mathbf{R}$ peuvent être simplement appropriés à n'importe quel autre intervalle I.

2° Cette assertion sera prouvée par le cas où $I=\mathbf{R}$, $f=f_{-\frac{3}{2}}$ et $g=f_{\frac{5}{4}}$ (voir le lemme 2), c'est-à-dire où

$$f(x)=\begin{cases}|x|^{-\frac{3}{2}}\sin\dfrac{1}{x^2}, & x\neq 0,\\ 0, & x=0,\end{cases}\qquad g(x)=\begin{cases}|x|^{\frac{5}{4}}\sin\dfrac{1}{x^2}, & x\neq 0\\ 0, & x=0.\end{cases}$$

En effet, dans ce cas, d'après le lemme 2, f est une $P_{\mathbf{R}}$-fonction et la fonction g est dérivable dans $\mathbf{R}$. Puis on a

$$h(x)=\begin{cases}|x|^{-\frac{1}{4}}\sin^2\dfrac{1}{x^2}=\dfrac{1}{2}|x|^{-\frac{1}{4}}\left(1-\cos\dfrac{2}{x^2}\right), & x\neq 0\\ 0, & x=0,\end{cases}$$

de sorte qu'on a

$$h(x)=h_1(x)-h_2(x),\qquad x\in\mathbf{R},\tag{4}$$

avec

$$h_1(x)=\begin{cases}\dfrac{1}{2}|x|^{-\frac{1}{4}}, & x\neq 0,\\ 0, & x=0,\end{cases}\qquad h_2(x)=\begin{cases}\dfrac{1}{2}|x|^{-\frac{1}{4}}\cos\dfrac{2}{x^2}, & x\neq 0\\ 0, & x=0.\end{cases}$$

La fonction h_1, d'après le théorème de Darboux, n'a pas la propriété $P_{\mathbf{R}}$. Étant donné encore que la fonction h_2, d'après une modification légère de l'assertion correspondante du lemme 2 $\left(\text{on a } -\frac{1}{4}\in(-2, 0]\right.$ et le remplacement du sinus par le cosinus et le multiplicateur 2 sous le symbole de cosinus ne changent pas évidemment la validité du lemme), jouit de la propriété $P_{\mathbf{R}}$, on déduit de (4) la conclusion que dans ce cas h n'est pas une $P_{\mathbf{R}}$-fonction.

3° Si

$$f(x)=\begin{cases} 0, & x<0 \\ f_0(x), & x\geqslant 0, \end{cases} \qquad g(x)=\begin{cases} f_0(x), & x\leqslant 0 \\ 0, & x<0, \end{cases}$$

les fonctions f et g ne sont pas continues dans $\mathbf{R}$ et elles sont toutes les deux $P_{\mathbf{R}}$-fonctions, d'après les lemmes 1 et 2. Leur produit — la fonction identiquement nulle — a la propriété $P_{\mathbf{R}}$. D'autre part, si $f=g=f_0$, les fonctions f et g, selon le lemme 2, ont la propriété $P_{\mathbf{R}}$ et ne sont pas continues dans $\mathbf{R}$. Le produit h de ces deux fonctions est donné par

$$h(x)=\begin{cases} \sin^2\frac{1}{x^2}=\frac{1}{2}\left(1-\cos\frac{2}{x^2}\right), & x\neq 0 \\ 0, & x=0, \end{cases}$$

c'est-à-dire par $h(x)=h_1(x)-h_2(x)$, $x\in\mathbf{R}$, où

$$h_1(x)=\begin{cases} \frac{1}{2}, & x\neq 0 \\ 0, & x=0, \end{cases} \qquad h_2(x)=\begin{cases} \frac{1}{2}\cos\frac{2}{x^2}, & x\neq 0 \\ 0, & x=0 \end{cases}$$

La fonction h_1 évidemment n'a pas le propriété $P_{\mathbf{R}}$ et la fonction h_2, d'après le lemme 2 et la remarque correspondante dans notre considération sous 2°, jouit de cette propriété. Par conséquent, h n'est pas une $P_{\mathbf{R}}$-fonction. Ainsi on a prouvé les deux possibilités en question.

4° Supposons que la fonction f ne soit pas continue dans I (tout en jouissant, d'après la supposition générale du théoremè, de la propriété P_I) et posons $\varphi(x)=f^n(x)$, $x\in I$, le nombre $n\in\{2, 3, \ldots\}$ étant fixé. Il existent alors $x_0\in I$ et deux suites (a_k) et (b_k) de points de l'intervalle I telles que $\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = x_0$, $\lim_{k\to\infty} f(a_k)=a$, $\lim_{k\to\infty} f(b_k)=b$ et $a\neq b$. Si n est impair, ou bien si n est pair et $|a|\neq|b|$, on a $\lim_{k\to\infty}\varphi(a_k)=a^n\neq b^n=\lim_{k\to\infty}\varphi(b_k)$, de manierè que la fonction φ n'est pas continue dans le point x_0. Si n est pair et $|a|=|b|\,(>0)$, c'est-à-dire les nombres a et b sont de signés opposés, alors les nombres $f(a_k)$ et $f(b_k)$ sont aussi, pour k suffisamment grand, de signes opposés, et par suite, d'après le théorème de Darboux, il existe une suite (c_k) de points de I, convergeant vers x_0 et telle que l'on a $f(c_k)=0$ pour k suffisamment grand. On a alors $\lim_{k\to\infty}\varphi(a_k)=a^n\neq 0=\lim_{k\to\infty}\varphi(c_k)$, ce qui signifie que la fonction φ n'est pas continue dans point x_0.

Soit ensuite f la fonction dont le graphe est représénté par la figure 1, où l'on a particulièrement mis en relief la partie du graphe correspondant au k-ième ($k\in\mathbf{N}$) des segments consécutifs formant un ensemble dénombrable d'intervalles dont l'union est l'intervalle $(0, \alpha]$, avec $\alpha = \sum_{k=1}^{\infty} k^{-2}$. (Le rapport des grandeurs horizontales à celles verticales n'est pas présenté fidèlement sur la figure). Étant donné que

$$\int_{+0}^{\alpha} f(x)\,\mathrm{d}x = \frac{1}{2} \sum_{k=1}^{\infty} k^{-3} < +\infty$$

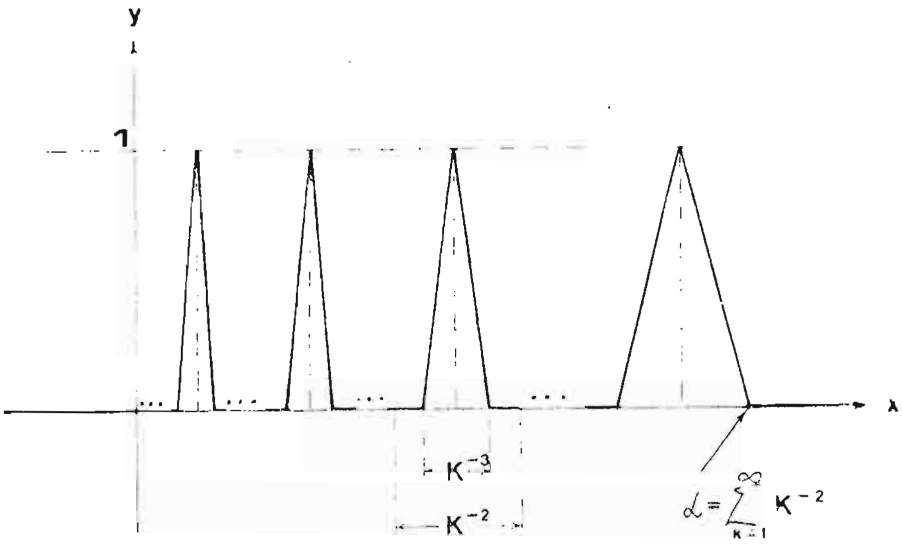

Fig. 1

et que, pour $\sum_{k=n_x+1}^{\infty} k^{-2} < x \leqslant \sum_{k=n_x}^{\infty} k^{-2}$, on a

$$0 < \frac{\int_{+0}^{x} f(t)\,\mathrm{d}t}{x} < \frac{\frac{1}{2} \sum_{k=n_x}^{\infty} k^{-3}}{\sum_{k=n_x+1}^{\infty} k^{-2}} \to 0, \quad x \to +0,$$

(puisque $\lim_{x \to +0} n_x = +\infty$), cette fonction f, discontinue dans le point $x = 0$, possède la propriété $P_{\mathbf{R}}$, d'après le lemme 1. Comme on a, évidemment, $0 \leqslant f^n(x) \leqslant f(x)$, $x \in \mathbf{R}$, chacune des fonctions f^n, $n \in \mathbf{N}$, a aussi la propriété $P_{\mathbf{R}}$. Si, d'autre part, le graphe de la fonction f est représenté par la figure 2, alors cette fonction, discontinue dans le point $x = 0$, a la propriété $P_{\mathbf{R}}$, ce qu'on peut établir de manière semblable que dans le cas précédent, tandis que la fonction f^n, avec $2 \leqslant n \in \mathbf{N}$, n'a pas la propriété $P_{\mathbf{R}}$, puisque

$$\int_{0}^{\alpha} f^n(x)\,\mathrm{d}x \geqslant \sum_{k=1}^{\infty} k^n k^{-3} = \sum_{k=1}^{\infty} k^{n-3} \geqslant \sum_{k=1}^{\infty} k^{-1} = +\infty.$$

3. *Désignons, pour un intervalle fixe I, par les symboles*

(5) $$C_k \ (k = 1, 2, \ldots, 5)$$

respectivement les classes de toutes les fonctions réelles définies dans I et qui: 1) *n'ont pas la propriété* P_I, 2) *ont la propriété* P_I *et ne sont pas continues dans* I, 3) *sont continues et ne sont pas dérivables dans* I, 4) *sont dérivables et*

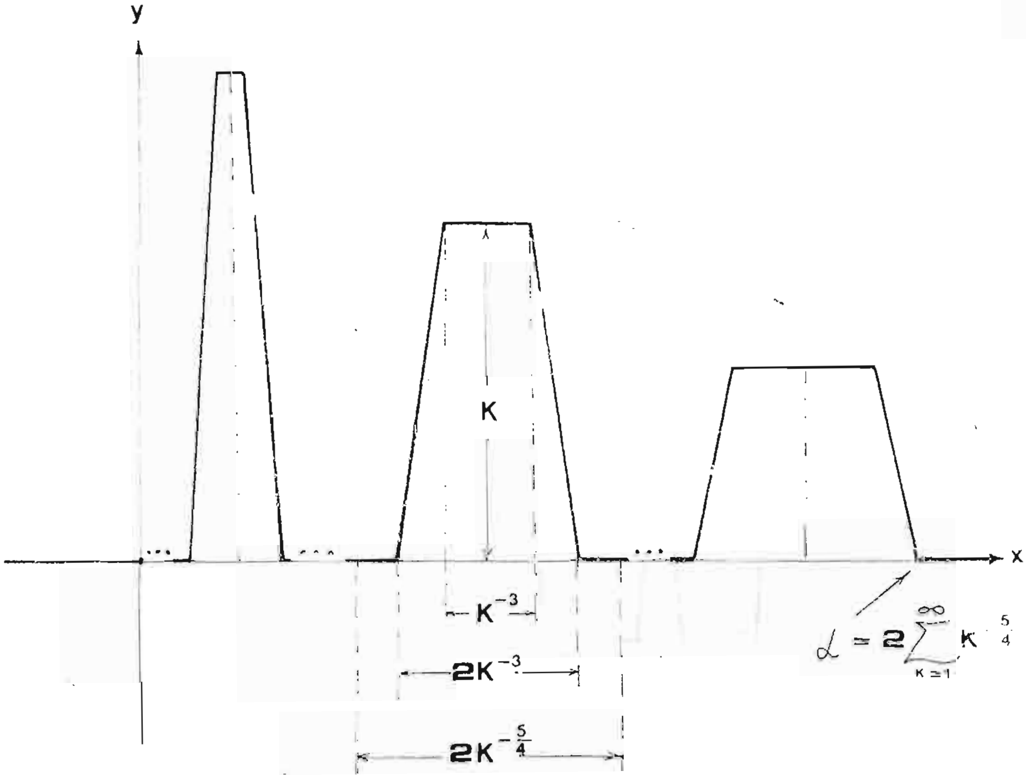

Fig. 2

ne sont pas continûment dérivables dans I *et* 5) *sont continûment dérivables dans* I. C'est en s'appuyant sur le théorème **P** et sur quelques faits élémentaires, et en effectuant quelques considérations supplémentaires, que l'on peut, pour chacun des cas où $f \in C_i$, $g \in C_j$, avec les nombres $i, j \in \{1, 2, \ldots, 5\}$ satisfaisant à $i \leqslant j$ (cette condition sert à éviter la répétition des cas identiques) déterminés, établir exactement auxquelles des classes (5) peut appartenir la produit $f \cdot g$, et aussi auxquelles des classes peut appartenir la ***n***-ième puissance f^n de f, avec $2 \leqslant n \in \mathbf{N}$. Ici il faut considérer que les mots „*peut appartenir*“ signifient que *cette possibilité se réalise effectivement pour une paire concrète de fonctions au moins, ou bien pour une n-ième puissance concrète d'une fonction,* — l'un et l'autre dans le cadre du cas en question.

Pour une partie non vide A de l'ensemble $\{1, 2, \ldots, 5\}$, *soit désigné par le symbole* $(i, j) \to A$ *le fait que, lorsque* $f \in C_i$ *et* $g \in C_j$, *alors A représente l'ensemble des indices de toutes les classes* (5) *auxquelles peut appartenir* (dans le sens précisé ci-dessus) *le produit* $f \cdot g$; *aussi, soit désigné par* $(i)_m \to A$ *le fait que, si* $f \in C_i$, *alors A représente l'ensemble des indices de toutes les classes* (5) *auxquelles peut appartenir la puissance* f^m. Avec ces désignations, le résultat plus complexe que nous avons annoncé sera formulé de la manière suivante:

Théoremè P'. *Nous avons*:

1° $(1, j) \to \{1, 2, 3, 4, 5\}$, $1 \leqslant j \leqslant 5$;

2° $(2, j) \to \{1, 2, 3, 4, 5\}$, $2 \leqslant j \leqslant 4$;

3° $(2, 5) \to \{2, 3, 4, 5\}$;

4° $(i, j) \to \{k : i \leqslant k \leqslant 5\}$, $3 \leqslant i \leqslant j \leqslant 5$;

5° $(\forall n \in \mathbf{N})$ $(1)_{2n} \to \{1, 2, 3, 4, 5\}$;

6° $(\forall n \in \mathbf{N})$ $(1)_{2n+1} \to \{1, 2\}$,

7° $(\forall n \in \mathbf{N})$ $(2)_{n+1} \to \{1, 2\}$;

8° $(\forall n \in \mathbf{N})$ $(i)_{n+1} \to \{k : i \leqslant k \leqslant 5\}$, $3 \leqslant i \leqslant 5$.

Remarque. Le théorème **P**′ contient toutes les assertions du théorème **P** (ainsi, par exemple, 7° dans **P**′ coïncide avec 4° dans **P**, en les complétant et précisant par plusieurs novelles assertions. Nous avons quand-même énoncé le théorème **P** à part — en raison de l'importance des faits qu'il contient (à ce qu'il nous semble) — et nous avons donné au théorème **P**′ la forme précédente afin d'obtenir un aperçu complet et uniforme d'un ensemble de faits.

Démonstration. Comme nous l'avons déjà dit, l'assertion 7° du théorème **P**′ est réellement identique à l'assertion 4° du théorème **P.** Nous n'allons pas exposer complètement, avec tous les détails, les démonstrations des assertions 1°—4° du théorème ***P***, puisque cela exigerait l'élaboration de beaucoup de cas à distinguer, parmi lequels plusieurs sont semblables et quelqu'uns assez simples; quant à ces assertions-là nous allons donner des informations sommaires relatives aux points principaux de leurs démonstrations, en représentant par un nombre d'exemples sélectionnés l'élaboration des cas particuliers. Nous allons exposer, cependant, tous les cas cous 5°, 6° et 8°, prêtant le plus d'attention au dernier des cas sous 6°.

Pour abréger notre exposition, dans chaque cas considéré nous désignerons, sans le dire explicitement, par A l'ensemble figurant dans la relation correspondante de la forme $(i, j) \to A$ ou de la forme $(i)_m \to A$, et le fait que $k \in A$ sera designé par

(6) $$(i, j) \to k,$$

ou bien par

(7) $$(i)_m \to k.$$

Il résulte de l'assertion 1° du théorème **P** que l'on a $A \subseteq \{2, 3, 4, 5\}$ dans le cas 3°. Le fait que les propriétés de continuité, de dérivabilité et de dérivabilité continue sont multiplicatives, dans le sens précisé dans ***O***, entraîne pour tous les cas sous 4° l'inclusion $A \subseteq \{k : i \leqslant k \leqslant 5\}$, $3 \leqslant i \leqslant j \leqslant 5$ et pour tous les cas sous 8° l'inclusion $A \subseteq \{k : i \leqslant k \leqslant 5\}$, $3 \leqslant i \leqslant 5$. Étant donné que, évidemment, la puissance impaire d'une fonction discontinue est aussi fonction discontinue, on a $A \subseteq \{1, 2\}$ dans 6°. Il reste donc à établir que toutes ces inclusions-là se remplacent effectivement par les égalités correspondantes et que l'on a $A = \{1, 2, 3, 4, 5\}$ dans les cas 1°, 2° et 5°. On peut le faire en démontrant chacun des faits valables (6) et (7) (sauf, bien entendu, le fait $(5, 5) \to 5$) par un exemple corrésponant de fonctions f et g concrètes, ou bien, pour quelques cas, par une considération plus ou moins générale.

Ainsi, le fait sous 1°

$$(1, 1) \to 1$$

se prouve par l'exemple suivant:

$$f(x)=g(x)=\begin{cases}0, & x\leqslant 0\\ 1, & x>0.\end{cases}$$

Ensuite, les faits sous 1°:

$$(1,\ 1)\to 2,\quad (1,\ 1)\to 3 \text{ et } (1,\ 1)\to 4$$

sont prouvés par les exemples:

$$f(x)=\begin{cases}0, & x\leqslant 0\\ 1, & x>0,\end{cases}\qquad g(x)=\begin{cases}1, & x<-1\\ 0, & -1\leqslant x\leqslant 0\\ f_\alpha(x), & x>0,\end{cases}$$

où l'on a $\alpha\in(-2,\ 0]$, $\alpha\in(0,\ 1]$ et $\alpha\in(1,\ 3]$, respectivement, et le fait

$$(1,\ 1)\to 5$$

se démontre par l'exemple

$$f(x)=\begin{cases}0, & x\leqslant 0\\ 1, & x>0,\end{cases}\qquad g(x)=\begin{cases}1, & x\leqslant 0\\ 0, & x>0.\end{cases}$$

Le plus grand nombre des exemples servant à démontrer les autres faits particuliers sous 1°—4°, ainsi que les exemples précédents, sont consitués par les fonctions f et g définies dans **R** et telles que leurs restrictions à quelques sous-intervalles de **R** appartiennet à des classes (5) déterminées, la conservation d'une propriété s'effectuant par le fait que l'autre de ces deux fonotions prend la valeur 1 dans l'intervalle en question, et son annulation par la valeur 0 prise par cette autre fonction dans cette intervalle. Les constructions de telles fonctions f et g s'appuyent sur les lemmes 1 et 2 et utilisent des restrictions des fonctions f_α à $(-\infty,\ 0)$ ou à $(0,\ +\infty)$, de même que les translations de ces restrictions ou des restrictions à des intervalles bornés, et la continuité ou la dérivabilité continue dans un intervalle plus large d'une fonction prenant la valeur 1 dans un sous-intervalle et la valeur 0 dans un autre sous-intervalle peut évidemment se réaliser d'une manière simple.

Des exemples ou considérations prouvant le fait du type (6) sous 2°—4°, nous allons citer cependant tous ceux qui diffèrent des exemples que nous venons de décrire.

$(2,\ 2)\to 1,\ 2$: ce fait résulte de l'assertion 4° de **P**).

$(2,\ 2)\to 5$: ce fait est prouvé par l'exemple employé dans la démonstration da la partie positive de l'assertion 3° du théorème **P.**

$(2,\ 3)\to 4$:

$$f(x)=\begin{cases}f_0(x), & x\leqslant 0\\ x, & x>0,\end{cases}\qquad g(x)=\begin{cases}0, & x\leqslant 0\\ f_1(x), & x>0.\end{cases}$$

$(2,\ 4)\to 1$: cela résulte immédiatement des assertion 1° et 2° de **P.**

$(2,\ 4)\to 2$: soit $f\in C_2$ avec le point de desconinuité x_0 et soit g la fonction primitive de f dans I telle que $g(x_0)\neq 0$; on a alors $g\in C_4$ et la fonction $h=f\cdot g$ a la fonction primitive $\frac{1}{2}g^2$ dans I et n'est pas continue dans le point x_0, de manière que $h\in C_2$.

$(2,\ 4)\to 3$: $\quad f=f_{-1},\quad g=f_2.$

$(2,\ 4)\to 4$: $\quad f=f_{-1},\quad g=f_3.$

$(2,\ 5)\to 3$: $\quad f=f_0,\quad g(x)=x,\ x\in\mathbf{R}.$

$(2,\ 5)\to 4$: $\quad f=f_0,\quad g(x)=x^2,\quad x\in\mathbf{R}.$

$(3,\ 3)\to 3$: $\quad f(x)=g(x)=|x|^{\frac{1}{2}},\quad x\in\mathbf{R}.$

$(3,\ 3)\to 4$: $\quad f=f_1,\quad g(x)=|x|,\quad x\in\mathbf{R}.$

$(3,\ 3)\to 5$: $\quad f(x)=g(x)=|x|,\quad x\in\mathbf{R}.$

$(4,\ 4)\to 4$: $\quad f=g=f_{\frac{3}{2}}.$

$(4,\ 4)\to 5$: $\quad f=g=f_3.$

Nous passons aux exemples concernant les assertions 5°, 6° et 8°.

5°

$(1)_{2n}\to 1$:

$$f(x)=\begin{cases}0, & x\leqslant 0\\ 1, & x>0.\end{cases}$$

$(1)_{2n}\to 2$: soit g la fonction dont le graphe est représenté sur la figure 1 et soit

$$f(x)=\begin{cases}(g(x))^{\frac{1}{2n}}, & x\in\mathbf{Q}\\ -(g(x))^{\frac{1}{2n}}, & x\in\mathbf{R}\setminus\mathbf{Q};\end{cases}$$

alors $f\in C_1$, $f^{2n}=g\in C_2$.

$(1)_{2n}\to 3$:

$$f(x)=\begin{cases}0, & x\leqslant 0\\ x^{\frac{1}{2n}}, & 0<x\in\mathbf{Q}\\ -x^{\frac{1}{2n}}, & 0<x\notin\mathbf{Q}.\end{cases}$$

$(1)_{2n}\to 4$:

$$f(x)=\begin{cases}0, & x\leqslant 0\\ f_{\frac{1}{n}}(x), & 0<x\in\mathbf{Q}\\ -f_{\frac{1}{n}}(x), & 0<x\notin\mathbf{Q}.\end{cases}$$

$(1)_{2n} \to 5$:

$$f(x)=\begin{cases} -1, & x\leqslant 0 \\ 1, & x>0. \end{cases}$$

6°

$(1)_{2n+1} \to 1$: le même exemple que pour $(1)_{2n} \to 1$.

$(1)_{2n+1} \to 2$: soit, pour un nombre m impair supérieur à 1 arbitraire, $\sum_{k=1}^{\infty} k^{-\frac{m+2}{m+1}} = \alpha$, et soit f la fonction dont le graphe dans l'intervalle $(0,\ 3\alpha]$ est représenté par la figure 3 et qui s'annule pour les autres valeurs de $x \in \mathbf{R}$. Si l'on désigne par P_k et par $\overline{P}_k$ les aires des triangles isocèles sur la figure 3, et par $P_k^{(m)}$ et $\overline{P}_k^{(m)}$ les aires des figures correspondantes obtenues en remplaçant la ligne $y=f(x)$ par la ligne $y=f^m(x)$, nous avons d'abord

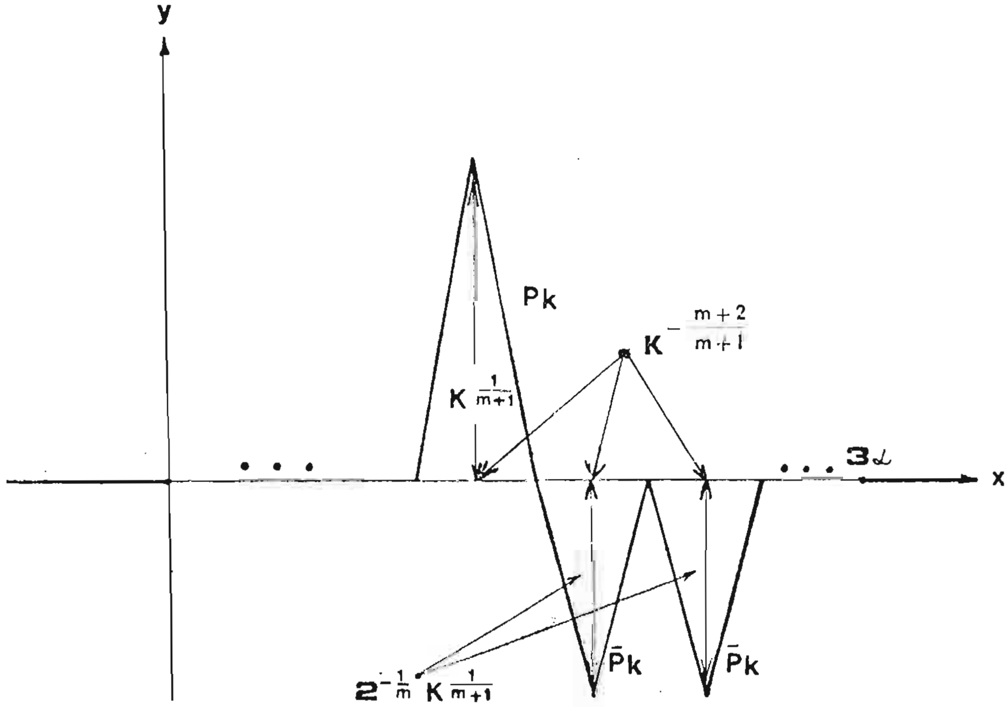

Fig. 3

$$\int_0^{3\alpha} f(x)\,\mathrm{d}x = \sum_{k=1}^{\infty} (P_k - 2\,\overline{P}_k) = \frac{1}{2}\,(1 - 2^{1-\frac{1}{m}}) \sum_{k=1}^{\infty} k^{-1} = +\infty,$$

d'où la conclusion que $f \in C_1$. Étant donné que

$$P_k^{(m)} = \frac{1}{m+1}\, k^{-\frac{2}{m+1}},$$

nous obtenons

$$\int_0^{3\alpha} f^m(x)\,dx = \sum_{k=1}^{\infty} \left(P_k^{(m)} - 2\bar{P}_k^{(m)}\right) = \frac{1}{m+1} \sum_{k=1}^{\infty} \left(k^{-\frac{2}{m+1}} - 2\cdot 2^{-1} k^{-\frac{2}{m+1}}\right) = 0.$$

Avec

$$3 \sum_{k=n_x+1}^{\infty} k^{-\frac{m+2}{n+1}} < x \leqslant 3 \sum_{k=n_x}^{\infty} k^{-\frac{m+2}{n+1}},$$

on aura ensuite

$$0 < \frac{\int_{+0}^{x} f^m(t)\,dt}{x} < \frac{\sum_{k=n_x}^{\infty} \left(P_k^{(m)} - 2\bar{P}_k^{(m)}\right) + P_{n_x}^{(m)}}{3 \sum_{k=n_x+1}^{\infty} k^{-\frac{m+2}{n+1}}}$$

$$= \frac{\frac{1}{m+1} \cdot n_x^{-\frac{2}{m+1}}}{3 \sum_{k=n_x+1}^{\infty} k^{-\frac{m+2}{n+1}}} \sim \frac{1}{3} \cdot \frac{n_x^{-\frac{2}{m+1}}}{n_x^{-\frac{1}{m+1}}} \to 0, \quad x \to +0.$$

Par conséquent, $f^m \in C_2$.

8°

$(3)_{n+1} \to 3$: $f(x) = |x|^{\frac{1}{n+1}}, \quad x \in \mathbf{R}.$

$(3)_{n+1} \to 4$: $f = f_{\frac{2}{n+1}}.$

$(3)_{n+1} \to 5$: $f(x) = |x|^{\frac{2}{n+1}}, \quad x \in \mathbf{R}.$

$(4)_{n+1} \to 4$: soit $g \in C_2$ avec le point da discontinuité x_0 et soit f la fonction primitive dans I de la fonction g, telle que $f(x_0) \neq 0$; alors $f \in C_4$ et $(f^{n+1}(x))' = (n+1)\, f^n(x) \cdot g(x)$, $x \in I$; cette fonction-là n'est pas continue dans le point x_0, de sorte que $f^{n+1} \in C_4$.

$(4)_{n+1} \to 5$: $f = f_2$.

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NEKOLIKO PRIMEDBI O PRIMITIVNIM FUNKCIJAMA REALNIH FUNKCIJA

Dušan D. Adamović

U sklopu širih razmatranja, ovde se na prvom me tu tretira sledeće pitanje: pod pretpostavkom da funkcija f ima na intervalu I primitivnu funkciju, koje svojstvo (koji stepen pravilnosti) funkcije g, definisane na I, obezbeđuje postojanje primitivne fnnkcije na I proizvoda $h=f\cdot g$? Glavni rezultati rada sadržani su u teoremama P i P'.

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SOME REMARKS ON ISOMETRIC MAPPINGS

Themistocles M. Rassias

Abstract. In this paper we present some properties and research problems on isometric mappings between Euclidean (with certain remarks for non-Euclidean) spaces with some emphasis for distance one preserving mappings.

Let X, Y be two metric spaces, d_1, d_2 the distances on X and Y. A bijection mapping $f: X \to Y$, of X onto Y, is defined to be an isometry if $d_2(f(x), f(y)) = d_1(x, y)$ for all elements $x, y)$ of X. If $f: X \to Y$ is an isometry, then the inverse mapping $f^{-1}: Y \to X$ is an isometry of Y onto X. Two metric spaces X and y are defined to be isometric if there exists an isometry of X onto Y. It thus follows that an isometry is an isomorphism for the metric space structures. We now state in which sense an incomplete space can be fattened out to be complete: If (X, d_1) is an incomplete metric space, then there exists a complete metric space X so that $\tilde{X}$ is isometric to a dense subset of $\tilde{X}$ (cf. [2]). Mazur and Ulam [6] have proved that every isometry of a normed real vector space onto a norme real vector space is a linear mapping up to translation. Consider then the following condition (distance one preserving property), for $f: X \to Y$.

(DOPP) *Given* $x, y \in X$ *with* $d_1(x, y) = 1$. *Then* $d_2(f(x), f(y)) = 1$. This condition was considered in [7], [8], [9].

A. D. Aleksandrov had posed the problem:

Under what conditions is a mapping of a metric space into itself preserving unit distance an isometry?

Unless the contrary is stated, it is not assumed in the following that the mapping is one-to-one, onto, or continuous. Furthermore, it is not even assumed is singlevalued. By E^n, L^n will be denoted respectively Euclidean and Lobačevskiĭ spaces of dimension n. A. Guc [4] had proved that a bijective single-valued mapping $f: L^n \to L^n$ $(n > 2)$ such that for some number $r > 0$ and each point $x \in L^n$ satisfies

$$f(S^{n-1}(x, r)) = S^{n-1}(f(x), r)$$

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is a motion, where $S^{n-1}(x, r)$ is the $(n-1)$-dimensional sphere with center x and radius r. E. Beckman and D. Quarles, A. Kuz'minyh [5] had proved that if a is a fixed positive real number then a mapping $f: L^n \to L^n\ (n \geqq 2)$ is anisometry if there exists a positive real number b such that from $\mathrm{d}(x, y) = a$ (where for $x, y \in L^n$, $\mathrm{d}(x, y)$ is the distance between x and y) it follows that for each pair for points $x', y'\ (x' \in f(x),\ y' \in f(y))$, $\mathrm{d}(x', y') = b$. This in particular implies that f is single-valued and $a = b$. E. Beckman and D. Quarles [2] proved that if $f: E^n \to E^n$ for $2 \leqq n < \infty$ satisfies condition (DOPP), then f is anisometry. It can be shown that this result holds in any n-dimensional hyperbolic space for $2 \leqq n < \infty$; it is true in any n-dimensional elliptic or spherical space if the preserved distance is small enough (cf. [1], [9]).

This property does not hold for E^1, the Euclidean line.

Counterexample: Let $f: E^1 \to E^1$ be defined by

$$f(x) = \begin{cases} x+1 & \text{if } x \text{ is an integer point} \\ x & \text{otherwise.} \end{cases}$$

Also this property does not hold for E^∞, a Hilbert space.

Counterexample: Let $\{y_i\}$ be a countable everywhere dense set of points. Define $g: E^\infty \to \{y_i\}$ such that $\mathrm{d}(x, g(x)) < \frac{1}{2}$. Define $h: \{y_i\} \to \{a_i\}$ such that $h(y_i) = a_i$ with $a_i = \left(\frac{\delta_{i1}}{\sqrt{2}}, \frac{\delta_{i2}}{\sqrt{2}}, \ldots, \frac{\delta_{ij}}{\sqrt{2}}, \ldots\right) \in E^\infty$, where δ_{ij} is the Kronecker delta. Then $f = gh: E^\infty \to E^\infty$ satisfies condition (DOPP). However f is not an isometry.

It will be of interest to examine what does it happen when the mapping is required to be continuous.

In E^1 the transformation

$$f: x \to [x] + \{x\}^2$$

(where $[x]$ denotes the integer part of x and $\{x\} = x - [x]$) is continuous and satisfies condition (DOPP) but is not an isometry.

Problem: It is not yet known what does it happen in E^∞ even with the additional condition of continuity on the mapping. My conjecture is that such a mapping, satisfying condition (DOPP), must be an isometry.

It is an open problem whether or not the distance $\frac{\pi}{2}$ can be preserved by a continuous mapping which is not an isometry.

Combining continuity and distance preserving properties for the mapping we can formulate the following conjecture, which according to the evidence I have now seems to be true (see also [1]).

Conjecture. *If M is a locally Euclidean manifold of finite dimension greater or equal to two, then there is a distance a such that for any $b < a$, and any mapping $f: M \to M$, where f preserves distance b implies that f is an isometry.*

If $f: E^n \to E^m$ preserves some distance, it follows that $n \leqq m$. It remains to examine the case when $1 < n < m < \infty$.

In the following we outline a method to show how to construct examples to prove that for each n there exists an m and a unitdistance preserving mapping mapping $f: E^n \to E^m$ that is not an isometry. The following example illustrates the case of a mapping $f: E^2 \to E^8$.

For this consider partitioning the plane into squares of unit diagonal as follows:

	7	8	9	
3	1	2	3	1
6	4	5	6	4
9	7	8	9	7
	1	2	3	

where each square contains the bottom edge, the left edge and the bottom left corner but none of the other corners. Now label the nine vertices of the unit 8-simplex in E^8 and map each square labelled i to the i-th vertex. This mapping satisfies condition (DOPP) but is not an isometry.

Using hexagons instead of squares one can construct such mapping from $E^2 \to E^6$.

This idea extends easily to higher dimensions.

Theorem. *For any integer $n \geqq 1$, there exists an integer n_m such that $N \geqq n_m$ implies that there exists a map $f: E^n \to E^N$ which satisfies condition (DOPP) but is not an isometry.*

Proof. We partition E^n into the regions $D_1, D_2, D_3, \ldots$ such that each region D_i has diameter less than one and also any closed n-sphere of radius one intersects at most k of these regions. We can find an integer n_m so that the regions $D_1, D_2, D_3, \ldots$ can be partitioned into $1+n_m$ sets $U_1, U_2, \ldots, U_{1+n_m}$ such that if $x \in D_i$, $y \in D_j$ and D_i and D_j belong to the same U_k where $k \in \{1, 2, 3, \ldots, 1+n_m\}$, then $d(x, y) \neq 1$.

Define $f: E^n \to E^N$ for $N \geqq n_m$ in such a way that each set

$$S_k = \cup \{D_i : D_i \in U_k\}$$

corresponds to a different vertex of a unit equilateral n_m-simplex in E^N. It follows that $d(x, y) = 1$ implies that both x, y are not in the same set S_k. Thus $d(f(x), f(y)) = 1$. Hence $f: E^n \to E^N$ satisfies condition (DOPP) but is not an isometry.

Q.E.D.

It is not known whether or not there is a distance preserving mapping $f: E^2 \to E^3$ which is not an isometric mapping (cf. [8]). Also, it is still an open problem whether or not there is a continuous mapping $f: E^n \to E^m$ for $m>n$ which satisfies condition (DOPP) but is not an isometry.

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NEKE PRIMEDBE O IZOMETRČKIM PRESLIKAVANJIMA

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U radu se iznose neka svojstva i problemi u istraživanju izometričkih preslikavanja Euklidovih (pod izvesnim uslovima i ne-Eklidovih) prostora sa osvrtom na preslikavanja koja održavaju rastojanje.

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ON THREE TYPES OF MULTIVALUED MAPPINGS

Momir S. Stanojević

Abstract. The purpose of this paper is to investigate some properties of multivalued mapping $F:X\to Y$ whose projection $p_Y:grF\to Y$ is a *bi*-quotient, countaably *bi*-quotient or pseudo-open (hereditarily quotient) function.

1. Introduction

For any sets X and Y, $F:X\to Y$ is a multivalued mapping provided that, for each $x\in X$, $F(x)$ is a nonempty subset of Y. For any $A\subset X$, $F(A)= = U(F(x):\ x\in A\}$ and for any $B\subset Y$, $F'(B)=\{x\in X:\ F(x)\cap B\neq\varnothing\}$ $(\overset{\circ}{F}(A)= = y\in Y:\ F'(y)\subset A\}$ and $\overset{\circ}{F'}(B)=\{x\in X:\ F(x)\subset B\})$.

The graf of multivalued mapping $F:X\to Y$ is $grF=\{(x,y)\in X\times Y:\ y\in F(x)\}$. The functions $p_X:grF\to X$ and $p_Y:grF\to Y$ are defined by $p_X(x,y)=x$ and $p_Y(x,y)=y$. There are very useful results which determine the relationship of a multivalued mapping F and the functions p_X and p_Y (see [2], [7], [8]),

If $A\subset X$, then clA and $intA$ denotes the closure and interior of A, respectivelly.

A point $x\in X$ adheres to a filter base $\mathscr{B}$ in X if $x\in\cap\{clB:\ B\in\mathscr{B}\}$.

For the multivalued mapping $F:X\to Y$ the following terminology is used.

(1) F is upper *semi*-continuous (u.s.c.). provided that $F'(B)$ is closed for each closed $B\subset Y$.

(2) F is lower semi-continuous (l.s.c.) provided that $F'(V)$ is open for each open $V\subset Y$.

(3) F is continuous provided F is u.s.c. and a l.s.c. mapping.

(4) F is an *us*-quotient (*ls*-quotiont) mapping provided that a subset B of Y is closed (open) if and oniy if $F'(B)$ is a closed (open) subset of X. F is a quotient mapping whenever F is both *us*-quotient mapping and *ls*-quotient mapping (see [2] and [9]).

(5) If P is a property of sets, then a multivalued mapping $F:X\to Y$ is called $Y-P$ $(X-P)$ if and only if $F(x)$ $(F'(y))$ has property P for each $x\in X$ (for each $y\in Y$).

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(6) F is perfect provided that F is a closed, X-compact, Y-compact, u.s.c. multivalued mapping.

2. Some characterizations

Definition 2.1. *Let $f: X \to Y$ be a continuous function.*

(1) *f is bi-quotient (countably by-quotient) if for eash $y \in Y$, every collection (countable collection) of open subsets of X which covers $f^{-1}(y)$ has a finite subcollection whose images cover some neighborhood of y* (see [4] and [5]).

(2) *f is pseudo-open if for each $y \in Y$ and each neighborhood U of $f^{-1}(y)$, $f(U)$ is a neighborhood of y* (see [1]).

Since multivalued mappings behave very much like singlevalued functions, it seems imperative that one considers the exstension of a *bi*-quotient, countably *bi*-quotient and pseudo-open mapping for single-valued functionsto to multivalued mappings.

Definition. 2.2. *Let $F: X \to Y$ be a multivalued mapping. Then F is said to be p_Y-bi-quotient (p_Y-countably bi-quotient) (p_Y-pseudo-open) provided that $p_Y: grF \to Y$ is a bi-quotient (countably bi-quotient) (pseudo-open) function.*

Proposition. 2.3. *If $F: X \to Y$ is a multivalued mapping, then the following properties are equivalent.*

(*a*) *If $y \in Y$ and $\mathcal{U}$ is an open cover of $F'(y)$, then finitely many $F(U)$: $U \in \mathcal{U}$ cover some neighborhood of y in Y.*

(*b*) *if $\mathcal{B}$ is a filter base in Y, and $y \in Y$ adheres to $\mathcal{B}$, then some $x \in F'(y)$ adheres to $F'(\mathcal{B}) = \{F'(B): B \in \mathcal{B}\}$.*

(*c*) *F is p_Y-bi-quotient mapping (p_Y: $grF \to Y$ is a bi-quotient mapping).*

Proof. (*b*) $\Rightarrow$ (*a*). If (*b*) is false, then it is a filter base $\mathcal{B}$ in Y, and a $y \in Y$ adherent to $\mathcal{B}$, such that each $x \in F'(y)$ has an open neighborhood U_x which is disjoint from $F'(B_x)$ for some $B_x \in \mathcal{B}$. Then

$\mathcal{U} = \{U_x\colon x \in F'(y),\ U_x \cap F'(B_x) = \varnothing$ for some $B_x \in \mathcal{B}\}$ covers $F'(y)$. Let U_{x_i}, $i = 1, 2, \ldots, n$ arbitrary finitely in $\mathcal{U}$. Then

$$U_{x_i} \cap F'(B_{x_i}) = \varnothing \Leftrightarrow F(U_{x_i}) \cap B_{x_i} = \varnothing,$$

and we have

$$\left(\bigcup_1^n F(U_{x_i})\right) \cap \left(\bigcup_1^n B_{x_i}\right) = \varnothing.$$

There exists $B \in \mathcal{B}$ such that $B \subset \bigcap_1^n B_{x_i}$. Since $y \in \mathrm{cl}(B)$, $y \in \bigcup_1^n F(U_{x_i})$ and $\left(\bigcup_1^n F(U_{x_i})\right) \cap B = \varnothing$, we have ($y \notin \mathrm{int} \bigcup^n F(U_{x_i})$ and no neighborhood of y in Y is cnvered by finitely many $F(U)$: $U \in \mathcal{U}$. Hence F does not satisfy the condition (*a*).

$(b) \Rightarrow (a)$. If (a) is false there is a $y \in Y$ and an open cover $\mathcal{U}$ of $F'(y)$, such that no neighborhood of y in Y is the union of finitely many $F(U)$: $U \in \mathcal{U}$. Let $\mathcal{B}$ consists of all complements in Y of such finite union. The family $\mathcal{B}$ is a filter base in Y. Let $U_1, \ldots, U_n$ arbitrary finitely in $\mathcal{U}$. Then $y \in \bigcup_1^n F(U_i)$ and $y \notin \operatorname{int}\left(\bigcup_1^n F(U_i)\right)$, and $y \in \operatorname{cl}\left(C\left(\bigcup_1^n F(U_i)\right)\right) = \operatorname{cl}(B)$. Hence $y \in \operatorname{cl}(B)$ for each $B \in \mathcal{B}$, and y adheres to $\mathcal{B}$. If $x \in F'(y)$, then there exists $U_x \in \mathcal{U}$ such that $x \notin CU_x$. Since $\operatorname{cl}(F'(CF(U_x))) = \operatorname{cl}(C\overset{\circ}{F'}(F(U_x))) \subset CU_x = \operatorname{cl}(CU_x))$, we have $x \in \operatorname{cl}(F'(B))$ and no $x \in F'(y)$ adheres to $F'(\mathcal{B})$. Hence (b) is false.

$(c) \Rightarrow (b)$. Let $\mathcal{B}$ be a filter base in Y and a $y \in Y$ adherent to $\mathcal{B}$. There is a $(x, y) \in \overline{p_Y}^1(y)$ such that $(x, y) \in \operatorname{cl}(\overline{p_Y}^1(B))$ for each $B \in \mathcal{B}$. Now, we have

$$(x, y) \in \overline{p_Y}^1(y) \Rightarrow p_X \overline{p_Y}^1(y) = F'(y),$$

and

$$(x, y) \in \operatorname{cl}(\overline{p_Y}^1(B)) \Rightarrow x \in p_X(\operatorname{cl} \overline{p_Y}^1(B)) \subset \operatorname{cl}(p_X \overline{p_Y}^1(B)) = \operatorname{cl} F'(B).$$

Thus, $x \in F'(y)$ adheres to filter base $F'(\mathcal{B})$.

$(b) \Rightarrow (c)$. Suppose that p_Y is not *bi*-quotient. There is a filter base $\mathcal{B}$ in Y and a $y \in Y$ adherent to $\mathcal{B}$, such that each $(x, y) \in \overline{p_Y}^1(y)$ has open neighborhood $U_x^0 \times V_y^0$ disjoint from $\overline{p_Y}^1(B_x)$, for some $B_x \in \mathcal{B}$. Let $\mathcal{B}' = \{B \cap V_y : V_y \in \mathcal{V}_y, B \in \mathcal{B}\}$ ($\mathcal{V}_y$ – base of neighborhood of $y \in Y$). The family $\mathcal{B}'$ is a filter base in Y.

$$(U_x^0 \times V_y^0) \cap \overline{p_Y}^1(B_x) = \varnothing \Rightarrow U_x^0 \cap p_X \overline{p_Y}^1(B_x \cap V_y^0) = \varnothing \Rightarrow$$
$$\Rightarrow U_x^0 \cap F'(B_x \cap V_y^0) = \varnothing \Rightarrow x \notin \operatorname{cl}(F'(B_x \cap V_y^0)).$$

This contradicts the asssumption that F satisfies the condition (b).

The following two propositions can be obtained in the similar way.

Proposition 2.4. *If $F: X \to Y$ is a multivalued mapping, then the following are equivalent:*

(a) If $y \in Y$ and $\{U_n : n \in N\}$ is an increasing open cover of $F'(y)$, there exists an n such that $y \in \operatorname{int} F'(U_n)$.

(b) If $\mathcal{B}$ is a countable filter base in Y and $y \in Y$ adheres to $\mathcal{B}$, then some $x \in F'(y)$ adheres to $F'(\mathcal{B})$.

(c) F is a p_Y-countably bi-quotient mapping.

Proposition 2.5. *If F: $X \to Y$ is a multivalued mapping, then the following are equivalent:*

(a) If U is a neighborhood of $F'(y)$ in X, then $F(U)$ is a neighborhood of y in Y.

(b) F is a p_Y-pseudo-open mapping.

The following diagram ilustratet the implications which exist among the certain classes of multivalued mappings.

open $\to p_Y - bi - q. \to p_Y -$ count. $bi - q. \to p_Y -$ pseudo-open
$\uparrow$
perfect $\to$ closed X-compact

The following result is analogous to a result of A. H. Stone for single--valued mappings ([10] Lemme 1.).

Proposition 2.6. *Let $F: X \to Y$ be a multivalued quotient mapping, X T_1-space and Y T_2-space satisfying the first axiom of countability. Then F is a p_Y-countably bi-quotient mapping.*

Proof. Let y_0 be a point in Y. If y_0 is an isolated point, then the condition (a) of 2.4. is obviously satisfied in y_0 Suppose now that y_0 is not an isolated point, and we have thae $F'(y_0) \neq \varnothing$ ([9] Lemm 3.4).

Let $\{U_n, n \in N\}$ be an increasing open cover of $F'(y_0)$. Then $\overset{\circ}{F}'(y_0) \subset F'(y_0) \subset \cup \{U_n, n \in N\}$, $F'(y_0) \cap U_n \neq \varnothing$.

Let $\{W_n: n \in N\}$ be a basis of neighborhoods of y_0 (we may suppose $W_1 \supset W_2 \supset \ldots$). We show that, for some n, $F(U_n) \supset W_n$. Suppose not; then, for each n, there is a point $y_n \in W_n \setminus F(U_n)$. Denote $B = \{y_n: y_n \in W_n \setminus F(U_n), n \in N\}$. Then we have that $y_0 \notin B$, $y_0 \in \operatorname{cl} B$ and B is not closed. Therefore, from the fact that F is a quotient mapping, $\overset{\circ}{F}'(B)$ is not closed, and there exists a point $x' \in \operatorname{cl} \overset{\circ}{F}'(B) \setminus \overset{\circ}{F}'(B)$. Then $x' \in \overset{\circ}{F}'(\operatorname{cl} B)$ $(\operatorname{cl} \overset{\circ}{F}'(B) \subset \overset{\circ}{F}'(\operatorname{cl}(B)))$ and $F(x') \subset \operatorname{cl} B$. Since $F(x') \not\subset B$, then $y_0 \in F(x')$ i.e. $x' \in F'(y_0)$, and $x' \in U_{n_0}$, for some $n_0 \in N$. The set $U = U_{n_0} - F'(\{y_1, \ldots, y_{0_0}\})$, is an open set. We show that $x' \in U$. Suppose not, then $x' \in F'(\{y_1, \ldots, y_{n_0}\})$ and $F(x) \subset \{y_1, \ldots, y_{n_0}\}$ which contradicts that $y_0 \in F(x')$. Hence $x' \in U$.

Now we show that $U \cap \overset{\circ}{F}'(B) = \varnothing$, contradicting that $x' \in \operatorname{cl} F'(B)$, and this completes the proof.

If $x \in U$, then $F(x) \subset F(U_{n_0})$, $y_{n_0} \notin F(x) \subset F(U_k)$ for $k > n_0$ and $F(x) \cap \cap \{y_k: k \geq n_0\} = \varnothing$. If $x \in \overset{\circ}{F}'(B)$ then $F(x) \subset \{y_1, \ldots, y_{n_0}\}$ (from $F(x) \cap \{y_k: k \geq n_0 = \varnothing$) and $x \in F'(\{y_1, \ldots, y_{n_0}\})$, so $x \notin U$ Hence $U \cap \overset{\circ}{F}'(B) = \varnothing$.

3. Some properties

Proposition. 3.1. *Let $F: X \to Y$ be an u.s.c. Y-compact p_Y-bi-quotient multivalued mapping. If X locally compact T_1-space and Y T_2-space, then Y is a locally compact space.*

Proof. Let $y \in Y$. For each $x \in F'(y)$, let K_x be a compact neighborhood of x in X. Then the family

$$\mathcal{U} = \{\operatorname{int} K_x: x \in F'(y)\}$$

is a covering of $F'(y)$ in X. Since F is u.s.c. and Y-compact, then $F(K_x)$ it a compact set in Y, and since F is a p_Y-bi-quotiient, then the union of finiteiy many $F(K_x)$ is a neighborhood V of y. But this V is compact, so that y has a compact neighborhood, and hence Y it a locally compact space.

The following corollary follows from diagram in Section 2.

Corollary. 3.2. *Let $F: X \to Y$ be a multivalund mapping, X locally compact T_1-space and Y T_2-space. Then*

(a) If F an open, u.s.c. Y-compact mapping, then Y is a locally compact space.

(b) If F is a perfect mapping, then Y is a locally compact space.

Corollary. 3.3. *Let $F: X \to Y$ be a multivalued, Y-compact, X-Lindelöf quotient mapping, X locally compact T_2-space and Y T_2-space satisfying the first axiom fo countability. Then Y is a locally compact space.*

Proof. The mapping F is a p_Y-countably *bi*-quotient mapping (see 2.6.). By 3.1. it suffices to show that F is a p_Y-*bi*-quotient mapping.

Let $y \in Y$ and $\mathcal{U}$ be an open cover of $F'(y)$. Then there exists a countable subcover $\{U_n: n \in N\} \subset \mathcal{U}$ (F is a X-Lindeöf mapping). Let $V_n = \bigcup\{U_i: 1 \leqslant i \leqslant n\}$ for all $n \in N$. Then $\{V_n: n \in N\}$ is an increasing sequence of open subsets of X which covers $F'(y)$, so some $F(V_m)$ is a neighborhood of y in Y, so $F(V_m)$ is covered by $F(U_1), \ldots, F(U_m)$. That completes the proof.

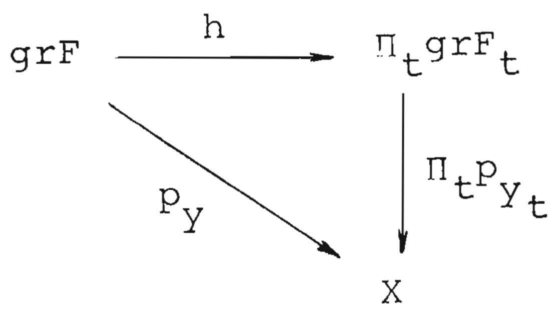
It was proved in [4] that class of *bi*-quotient maps is preserved by arbitrary cartetian product (i.e., if $f_t: X_t \to Y_t$ is *bi*-quotient for all t, so their product $F: \Pi_t X_t \to \Pi_t Y_t$, where $f(x) = (f_t(x_t))$.

Let $F_t: X_t \to Y_t$ be a multivalued mapping for $t \in T$, $X = \Pi_t X_t$, $Y = \Pi_t Y_t$. Then the product mapping $F: X \to Y$ is defined by $F(x) = \Pi_t F_t(x_t)$, for $x = (x_t) \in X = \Pi_t X_t$.

We have the following result.

Proposition. 3.4. *If $F_t: X_t \to Y_t$ is p_{Y_t}-bi-quotient for all $t \in T$, then the product mapping $F: X \to Y$ is p_Y-bi-quotient.*

Proof. The function $h: grF \to \Pi_t grF_t$ defined by $h((x_t), (y_t)) = ((x_t, y_t))$ is a homeomorphism, such that the following diagram commutes



Since $p_{Y_t}: grF_t \to Y_t$ is *bi*-quotient for all $t \in T$, then the product $\Pi_t p_{Y_t}$ is a *bi*-quotient mapping ([4]).

Since, from diagram

$$p_Y = (\Pi_t p_{Y_t}) \cdot h,$$

we have that p_Y is *bi*-quotient, and F is a p_Y-*bi*-quotient mapping.

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O TRI TIPA VIŠEZNAČNIH PRESLIKAVANJA

Momir Stanojević

Razmatraju se takva višeznačna preslikavanja $F:X\to Y$ čija je projekcija $p_Y:grF\to Y$ *bi*-kvocijentno, prebrojivo *bi*-kvocijentno ili pseudo otvoreno preslikavanje, gde je $grF=\{(x,y)\in X\times Y:\ y\in F(x)\}$ a $p_Y(x,y)=y$. Pored ostalog pokazuje se da se lokalna kompaktnost očuvava pri Y-kompaktnom p_Y-*bi*-kvocijentnom polu-nepekidnom odozgo višeznačnom preslikavanju i da je proizvod p_Y-*bi*-kvocijentnih preslikaranja isto takva preslikavanje.

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A THEOREM ON STRONGLY RADIALITY OF MAPPING SPACES

Ljubiša Kočinac

Abstract. In this note a condition is given under which the space $C_k(X, Y)$ of all continuous mappings from X to Y with the compact-open topology is strongly radial at each constant function.

In this note τ is a regular infinite cardinal (= an initial ordinal of the same cardinality). Terminology is standard as in [1].

(1) A *τ-sequence* $s = (x_\alpha : \alpha \in \tau)$ in a topological space X is a function from τ into X.

(2) A topological space X is called *strongly radial at a point* x if there is some cardinal τ such that whenever $\{A_\alpha : \alpha \in \tau\}$ is a decreasing τ-sequence of subsets of X accumulating at x, then there are $x_\alpha \in A_\alpha$ $(\alpha \in \tau)$, such that the τ-sequence $(x_\alpha : \alpha \in \tau)$ converges to x; X is *strongly radial* if it is strongly radial at every point (see [2]).

(3) A (completely regular) space X is called *τ-metrizable* (or linearly uniformizable) if there is a uniformity on X generating the topology of X and having a well-ordered base of order type τ.

(4) Let $\mathcal{U} = \{U_\alpha : \alpha \in \tau\}$ be a τ-sequence of subsets of a space X. A point $x \in X$ is said to be *residual* in $\mathcal{U}$ if there is $\beta \in \tau$ such that $x \in U_\alpha$ for every $\alpha \geq \beta$. If each point $x \in X$ is residual in $\mathcal{U}$, we shall say that X is *residual* in $\mathcal{U}$ (see [3]).

(5) When X is a completely regular space and $(Y, \mathcal{V})$ is a uniform space we use the following notation:

$$W(f, K, V) = \{g \in C(X, Y) : (f(x),\ g(x)) \in V \text{ for each } x \in K\}.$$

Note that in this case the family $\{W(f, K, V) : f \in C(X, Y),\ K \subset X \text{ is compact},\ V \in \mathcal{V}\}$ is a subbase for the compact-open topology on $C(X, Y)$. This space is denoted by $C_k(X, Y)$.

Let X be a completely regular space and let Y be a τ-metrizable space. Let every compact subset of X be »small«, i. e., its cardinality is less than τ. As

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announced in [3], the author can give a condition under which the space $C_k(X,Y)$ is strongly radial at several points; this condition is given in terms of a two-person Telgarsky-type [4] τ-game played on X. This game G is as follows.

The are two players I and II. They choose alternatively subsets of X. If $\alpha \in \tau$ then at αth play I chooses a compact set $K_\alpha \subset X$ and then II chooses an open set $U_\alpha \supset K_\alpha$. We say that player I wins if X is residual in $\{U_\alpha : \alpha \in \tau\}$; otherwise II wins.

Now we shall prove

Theorem. *Let X be a completely regular space with »small« compact sets and let Y be a τ-metrizable space. If I has a winning strategy in the game G played on X, then $C_k(X,Y)$ is strongly radial at each constant function.*

Proof. Let y be a fixed point in Y, let $\mathcal{B}=\{B_\alpha : \alpha \in \tau\}$ be a well-ordered base of the uniformity on Y and let f_0 be the point in $C_k(X,Y)$ such that $f_0(X)=\{y\}$. Suppose that $\{A_\alpha : \alpha \in \tau\}$ is a decreasing τ-sequence of subsets of $C_k(X,Y)$ such that $f_0 \in \cap \{\bar{A}_\alpha : \alpha \in \tau\}$. Assume that at the αth play (of G) I chooses a compact set K_α in X. Let us consider the set $W_\alpha = W(f_0, K_\alpha, B_\alpha)$. Since $f_0 \in \bar{A}_\alpha$, there is a point $f_\alpha \in W_\alpha \cap A_\alpha$. There is no loss of generality in assuming that player II chooses the set

$$U_\alpha = \{x \in X : (f_\alpha(x), f_0(x)) \in B_\alpha\}.$$

Clearly, $U_\alpha \supset K_\alpha$. By the fact that I has a winning strategy in the game G on X, we have that X is residual in $\{U_\alpha : \alpha \in \tau\}$.

We claim that $(f_\alpha : \alpha \in \tau)$ converges to f_0. Let $W(f_0, K, B_\alpha)$ be a standard subbasic neighbourhood of f_0. For every $x \in K$ choose an $\alpha_x \in \tau$ with $x \in U_\beta$ for every $\beta \geqq \alpha_x$. If we put $\lambda = \sup\{\alpha_x : x \in K\}$, then $|K| < \tau$ and the regularity of τ imply $\lambda < \tau$, so that $x \in U_\mu$ for every $\mu \geqq \lambda$ and every $x \in K$. Put $\nu = \sup\{\lambda, \alpha\}$. Then $K \subset U_\eta$ for every $\eta \geqq \nu$. This means that for every $\eta \geqq \nu$, $f_\eta \in W(f_0, K, B_\eta) \subset W(f_0, K, B_\alpha)$ and thus $(f_\alpha : \alpha \in \tau)$ converges to f_0 is established. Therefore, $C_k(X,Y)$ is strongly radial at f_0. Since f_0. was an arbitrary constant function in $C_k(X, Y)$, the theorem is proved.

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TEOREMA O STROGOJ RADIJALNOSTI PROSTORA PRESLIKAVANJA

Ljubiša Kočinac

U noti je dat jedan uslov pri kome je prostor $C_k(X, Y)$ svih neprekidnih preslikavanja iz X u Y sa kompaktno-otvorenom topologijom strogo radijalan u svakoj konstantnoj funkciji.

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SEMIGROUPS OF GALBIATI-VERONESI II

Stojan Bogdanović

Abstract. This paper is the continuation of [1]. Here we consider semigroups which are semilattice of nil-extension of completely simple semigroups (i. e. completely archimede an semigroups) and several subclasses of these semigroups. At the end a problem of T. Tamura ([12]) is solved.

J. L. Galbiati and M. L. Veronesi ([17]) studied π-regular semigroups in which every regular element is completely regular (semigruppi fortemente regolari). These semigroups are completely described bx M. L. Veronesi in [13]. Semigroups which are semilattice of nil-extensions of rectangular groups are discribed by the author in [1].

Throughout this paper, Z^+ will denote the set of all positive integers.

A semigroup S is π-regular if for every $a \in S$ there exists $m \in Z^+$ such that $a^m \in a^m S a^m$. Let us denote by Reg(S) ($G(S)$, $E(S)$) the set of all regular (completely regular, idempotent) elements of a semigroup S. S is a *GV*-semigroup (semigroup of Galbiati-Veronesi) it S is π-regular and Reg$(S) = G_r(S)$ (see [7]).

For undefined notions and notations we refer to [3] and [9].

In our investigations the following result is fundamental (see [13], Theorem 13.1 or [3], Theorem X.1).

Theorem (Veronesi). *S is a semilattice of nil-extensions of completely simple semigroups (completely archimedean semigroups) if and only if S is a GV-semigroup.*

This theorem will be referred to as „Veronesi's theorem".

Theorem 1. *S is a GV-semigroup if and only if*

$$(\forall a, b \in S)(\exists m \in Z^+)\,(ab)^m \in (ab)^m bS(ab)^m. \tag{1}$$

Proof. Let S be a GV-semigroup. Then by Veronesi's theorem S is a semilattice Y of completely archimedean semigroups S_α, $\alpha \in Y$. Assume $a \in S_\alpha$, $b \in S_\beta$, then ab, $ba \in S_{\alpha\beta}$. So by Theorem 1. [4] we have that

$$(ab)^m \in (ab)^m ba\, S_{\alpha\beta}\,(ab)^m \subseteq (ab)^m bS(ab)^m$$

for some $m \in Z^+$.

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Conversely, if (1) holds, then S is π-regular. Assume that $a=axa$. Then there exists $s\in S$ such that

$$a=a(xa)=a(xa)\,asxa=(axa)\,asxa=aasxa=a^2\,sxa.$$

Thus, $\operatorname{Reg}(S)=G_r(S)$, i.e. S is a GV-semigroup.

Theorem 2. *S is a semilattice of nil-extensions of rectangular groups if and only if S is a GV-semigroup and every inverse of an idempotent is an idempotent.*

Proof. Let S be a semilattice of nil-extensions of rectangular groups. Assume that x is an inverse of an idempotent e. Then by Theorem 2.1. [1]. we have that $x=xex^2=xexx=x^2$. That S is a GV-semigroup follows immediately.

Conversely, since S is a GV-semigroup we have by Veronesi's theorem that S is a semilattice of completely archimedean semigroups S_α, $\alpha\in Y$. Assume that K_α is the completely simple kernel of S_α, $\alpha\in Y$. Since every inverse of every idempotent is an idempotdnt we have by Proposition IV. 3.1. [9] that $E(K_\alpha)$ is a subsemigroup of K_α. So by Lemma IV.4.4. [10] we have that K_α is a rectangular group.

Theorem 3. *The following conditions are equivalent on a semigroup S;*

(i) *S is a semilattice of nil-extensions of right groups,*

(ii) *S is a GV-semigroup and for every $e, f\in E(S)$ there exists $n\in Z^+$ such that $(ef)^n=(fef)^n$,*

(iii) $(\forall a, b\in S)(\exists m\in Z^+)\,(ab)^m\in b^{2m}S(ab)^m$.

Proof. (i) $\Leftrightarrow$ (ii). This equivalence is one part of Theorem 2.2. [1].

(i) $\Rightarrow$ (iii). Let S be a semilattice Y of nil-extensions of right groups S_α, $\alpha\in Y$. Assume $a\in S_\alpha$, $b\in S_\beta$. Then ab, $b^ka\in S_{\alpha\beta}$ for every $k\in Z^+$. So by Theorem 2. [4] we obtain

$$(ab)^m\in b^{2m}a(ab)^m S(ab)^m\subseteq b^{2m}S(ab)^m.$$

(iii) $\Rightarrow$ (i). From (iii) we have that for every $a\in S$ there exists $m\in Z^+$ such that $a^{2m}\in a^{2m}Sa^{2m}$. So S is π-regular. Also, from (iii) we have that S is left weakly commutative. Thus by Theorem 2.2. [1] we huve that S is a semilattice of nil-extensions of right groups.

Theorem 4. *S is a semillatice of nil-extensinns of groups if and only if*

(2) $$(\forall a, b\in S)(\exists m\in Z^+)\,(ab)^m\in b^{2m}Sa^{2m}.$$

Proof. Let S be a semilattice Y of nil-extensions of groups S_α, $\alpha\in Y$. Assume $a\in S_\alpha$, $b\in S_\beta$. Then ab, $b^ka^k\in S_{\alpha\beta}$ for every $k\in Z^+$. So by Theorem 4. [4] we obtain

$$(ab)^m\in b^{2m}a(ab)^m S_{\alpha\beta}(ab)^m ba^{2m}\subseteq b^{2m}Sa^{2m}.$$

Convresely, from (2) we have that S is π-regular and weakly commutative. So by Theorem 3.2. [1] we have that S is a semilattice of nil-extensions of groups.

A semigroup S is completely π-regular if for every $a \in S$ there exists $x \in S$ and $m \in Z^+$ such that $a^m = a^m x a^m$ and $a^m x = x a^m$.

Theorem 5. *S is a completely π-regular semigroup aud $E(S) = G_r(S)$ if and only if S is a union of nil-semigroups.*

Proof. Let S be a completely π-regular semigroup and $E(S) = G_r(S)$. Then every subgroup of S is an one-element group. So S is a union of nil-semigroups.

The converse follows immediately.

Problem. Describe the class of π-regular semigroups with $E(S) = G_r(S)$.

Theorem 6. *The following conditions are equivaleot on a semigroups S:*

(i) *S is a semilattice Y of nil-extensions of rectangular bands S_α, $\alpha \in Y$,*

(ii) *S is π-regular and $E(S) = \mathrm{Reg}(S)$,*

(iii) $(\forall a, b \in S)(\exists m \in Z^+)\, (ab)^{2m+1} = (ab)^m ba^2 (ab)^m$.

Proof. (i) $\Rightarrow$ (ii). Assume $a \in \mathrm{Reg}(S)$. Then $a \in S_\alpha$ for some $\alpha \in Y$. For a there exists $x \in S_\beta$, $\beta \in Y$ such that $a = axa \in S_\alpha S_\beta S_\alpha \subseteq S_{\alpha\beta}$. Hence $\alpha\beta = \alpha$. So $a, ax \in S_\alpha$. Assume that K_α be the kernel of S_α. Then $a = (ax)a \in K_\alpha S_\alpha \subseteq K_\alpha \subseteq E(S)$. Therefore, $\mathrm{Reg}(S) = E(S)$.

(ii) $\Rightarrow$ (i). It follows from $E(S) = \mathrm{Reg}(S)$ that $G_r(S) = \mathrm{Reg}(S)$. So by Veronesi's theorem we have that S is a semilattice Y of completely archimedean semigroups S_α, $\alpha \in Y$. But from $E(S) = \mathrm{Reg}(S)$ we have that $E(S) = G_r(S)$ whence by Theorem 5. we have that every subgroup of S is an one-element group. Thus S is a semilattice of nil-extensions of rectangular bands.

(i) $\Rightarrow$ (iii). Assume $a \in S_\alpha$, $b \in S_\beta$. Then $ab, ba^2 \in S_{\alpha\beta}$. Since $S_{\alpha\beta}$ is a nil-extension of a rectangular band $K_{\alpha\beta}$ we have that $(ab)^m = e \in E(K_{\alpha\beta})$ for some $m \in Z^+$. Now by Lemma 1. [4] we have that $(ab)^m ba^2 (ab)^m = e$. Thus (iii) holds.

(iii) $\Rightarrow$ (i). It is clear that for any $a \in S$ there exists $m \in Z^+$ such that $a^{4m+2} = a^{4m+3}$. So by Lemma 4.2. [1] we have that S is a union of nil-semigroups. Assume that $a = axa$. Then

$$a = (ax)a = (ax)(xa^2)axa = ax^2 a^3.$$

Thus, S is a GV-semigroup and since S is a union of nil-semigroups we have that S is a semilattice of nil-extensions of rectangular bands

Theorem 7. *Let S be a semigroup. Then S^2 is a semilattice of right groups if and only if*

$$(\forall x, y \in S)\, xy \in ySxy. \tag{3}$$

Proof. Let S^2 be a semilattice of right groups. Then every regular element from S is completely regular. So S is a GV-semigroup. From this and from Theorem 3. we have that S is a semilattice Y of nil-extensions of right groups S_α, $\alpha \in Y$. Let $x_\alpha \in S_\alpha$, $x_\beta \in S_\beta$. Then $x_\alpha x_\beta \in K_{\alpha\beta}$, where $S^2_{\alpha\beta} = K_{\alpha\beta}$ it a right group. Since $x_\beta x_\alpha \in G_e \subseteq K_{\alpha\beta}$ we have that there exitts in G_e an inverse $(x_\beta x_\alpha)^{-1}$ for $x_\beta x_\alpha$. By Lemma VI 3.1.1. [3] e is a left unity in $K_{\alpha\beta}$. So

$$x_\alpha x_\beta = e x_\alpha x_\beta = x_\beta x_\alpha (x_\beta x_\alpha)^{-1} x_\alpha x_\beta \in x_\beta S x_\alpha x_\beta .$$

Converely, let the condition (3) holds. Then for every $x, y \in S$ there exists $s, s_1, s_2 \in S$ tuch that

$$xy = (yxs)\,xy = (xy)\,(yxs)\,s_1\,(yxs)\,xy = (xy)\,(xy)\,(yxss_1\,yxs)\,s_2\,(yxss_1\,yxs)\,xy.$$

Thus $xy \in (xy)^2 Sxy$, i.e. S^2 is a union of groups. For every e. $f \in E(S)$ there exists $t \in E(S)$ there exists $t \in S$ such that $ef = ftef$. So $ef = fef$. Now by Theorem 2. [8] we have that S^2 is a semilattice of right groups.

Theorem 8. *S is an inflation of a seuilattice of right groups if and only if*

$$(4) \qquad (\forall x, y \in S)\; xy \in ySxSy^2.$$

Proof. Let S be an inflation of a semilattice of right groups. Then by Theorem 1. [2] we have that S^2 is a semilattice of right groups. By Theorem 7. we have that for every $x, y \in S$ there exists $u \in S$ such that $xy = yuxy$. By Theorem 1. [2] it follows that $xy = xyf$, where $y^2 \in G_f$. Therefore

$$xy = yuxy = yuxyf = yuxfy \text{ (Theorem I 4.3. [3],}$$

$$= yux\,(y^2)^{-1}\,y^2\,y \in ySx\,Sy^2.$$

Conversely, let the condition (4) holds. Then for every $x, y \in S$ there exist $u, v, u_1, v_1, u_2, v_2 \in S$ such that

$$xy = yuxyvy^2 = (xy)\,(u_1\,yu)\,(xy)\,(v_1\,(xy)^2\,vy^2)$$

$$= x\,(x \cdot u_2 \cdot yu_1\,yux \cdot v_2 \cdot x^2)\,y \cdot v_1\,(xy)^2\,vy^2.$$

Thus, $xy \in x^2 Sy^2$. So by Theorem 1.[2] S is an inflation of a union of groups. Since for idempotents the following relation $ef = fef$ holds we have by Theorem 2. [8] that S is an inflation of a semilattice of right groups.

Theorem 9. *S^2 is a semilattice of periodic right groups if and only if*

$$(\forall x, y \in S)\,(\exists m \in Z^+)\; xy = (yx)^m\,xy.$$

Proof. Let S^2 be a semilattice of periodic right groups. Then as in Therem 7. we have that S is a semilattice Y of nil-extensions of right groups S_α, $\alpha \in Y$. Let $x_\alpha \in S_\alpha$, $x_\beta \in S_\alpha$, then $x_\alpha x_\beta$, $x_\beta x_\alpha \in K_{\alpha\beta}$, where $S^2_{\alpha\beta} = K_{\alpha\beta}$ is a periodic right group. So there is an idemptotent $e\in K$ such that $(x_\beta x_\alpha)^m = e$ for some $m\in Z^+$. Since every idemptotent in a right group it a left unity of this right group we have that

$$x_\alpha x_\beta = (x_\beta x_\alpha)^m x_\alpha x_\beta.$$

The converse follows by Theorem 7.

Theorem 10. *S is an inflation of a semilattice of periodic right groups if and only if*

$$(\forall x, y\in S)(\exists m, k\in Z^+)\, xy = (yx)^m xy^{k+1}.$$

Proof. By Theorem 2. [2] and by Theorem 9.

Now we consider the problem 2. of T. Tamura, [12]. This it the following: Determine the structure of semigroups satisying an identity of the form

$$(5) \qquad xy = y^{m_1} x^{n_1} \cdots y^{m_{h-1}} x^{n_{h-1}} y^{m_h} = \Phi(x, y),$$

where $m_1, m_h \neq 0$ and $h \geqslant 2$. G. T. Clarke, [6] has shown that if $m_h \geqslant 2$, then a semigroup satisfies this identity if and only if it is an inflation of a semilattice of right groups whose subgroups satisfy the same identity. For the related resultt see [2], [5], [11]. In general (including the case $m_h = 1$) we have the the follvwing:

Theorem 11. *A semigroup S satisfies* (5) *if and only if S^2 is a semilattice of right groups whose subgroups satisfy the same identity.*

Proof. Let the condition (5) holds. If $m_h \geqslant 2$, then the assertion holds by result of Clarke. If $m_h = 1$, then $xy\in ySxy$ and by Theorem 7. we have that S^2 it a semilattice of right groups. Clearly the subgroups of S satisfy (5).

Conversely, let S^2 be a semilattice of right groups whose subgroups satisfy the identity (5). Then by Theorem 3. we have that S is a temilattice Y of nil-extensions of right groups S_α, $\alpha\in Y$. Let $x_\alpha\in S_\alpha$, $x_\beta\in S_\beta$, then $x_\alpha x_\beta\in K_{\alpha\beta} = S^2_{\alpha\beta}$, where $K_{\alpha\beta}$ is a right group. From this we have that $x_\alpha x_\beta \in G_e \subseteq K_{\alpha\beta}$. So

$$(6) \qquad x_\alpha x_\beta = x_\alpha x_\beta e = ex x_\beta.$$

Since $K_{\alpha\beta}$ is a right group and $x_\beta e\in S^2$ we have thet

$$x_\beta e\in S_\beta K_{\alpha\beta} \subseteq S_\beta S_{\alpha\beta} \subseteq S_{\alpha\beta}\cap S^2 = K_{\alpha\beta}.$$

Similarly $x_\alpha e\in K_{\alpha\beta}$. Now by Lemma VI 3.1.1. [3] we obtain that $x_\beta e = e(x_\beta e)$, $x_\alpha e = ex_\alpha e$. By Lemma 1. [4] we have that $x_\alpha e$, $x_\beta e\in G_e$. From this and by (6) we have that

$$x_\alpha x_\beta = x_\alpha e(x_\beta e) = (x_\alpha e)(x_\beta e) = \Phi(x_\alpha e, x_\beta e) = \Phi(x_\alpha, x_\beta) e = \Phi(x_\alpha, x_\beta).$$

Thus S satisfies (5).

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POLUGRUPE GALBIATI-VERONESI II

Stojan Bogdanović

Ovaj rad je nastavak rada [1]. Ovde se razmatraju polugrupe koje su polumreže potpuno arhimedovskih polugrupa i razne podklase ovih. Na kraju, dato je rešenje jednog problema T. Tamure ([12]) u opštem slučaju.

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IMPROVED CORRESPONDENCE BETWEEN SOLUTIONS OF DIFFERENTIAL AND DIFFERENCE EQUATIONS

V. V. Bobkov, Z. Bohte and P. A. Mandrik

Abstract. Some ideas for the construction of new numerical methods for the solution of systems of ordinary differential equations are explained. A few new methods are developed and tested on some numerical examples.

1. Introduction

By means of an example of an initial value problem for simultaneous differential equations of the first order

(1) $$\mathring{u}+f(t, u)=0,$$

where

$$f(t, u)=[f_1(t, u), \ldots, f_n(t, u)]^T,$$

and

$$u=[u_1(t), \ldots, u_n(t)]^T,$$

we shall discuss some approaches to the construction of new numerical methods which possess additional correspondence properties in the qualitative behaviour of the solutions of a differential and the corresponding difference problem.

The implicit Euler method

(2) $$\hat{y}=y-\tau\hat{f},$$

where

$$\hat{f}=f(t+\tau, \hat{y}),\ y\approx u,\ \hat{y}\approx\hat{u}=u(t+\tau),\ \tau>0,$$

ensures good correspondence between the solutions of a differential and difference problems for simultaneous differential equations of the first order with the pro-

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perty of asymptotic stability (see [5]). As a test system for this class we shall consider the system

(3) $$\mathring{u} + Au = 0$$

with a symmetric positive definite matrix A.

Let us pay attention to some important properties of the solutions of the system (3), and also to some notions, used in this work.

It is known that the energy norm

$$\|u\|_D = (Du, u)^{1/2}, \quad D = A^k, \ (u, v) = u^T v,$$

of any nontrivial solution $u(t)$ of (3) is decreasing as $t \to \infty$:

(4) $$\|\hat{u}\|_D < \|u\|_D, \quad u \neq 0.$$

Also, it is possible to show that for any such solution

(5) $$(Du, \mathring{u}) > 0, \quad u \neq 0,$$

and for the solution, satisfying the initial condition

$$u(t) = \xi^m,$$

where ξ^m is an eigenvector of A, corresponding to the eigenvalue λ_m, the equation

(6) $$\hat{u} = \exp(-\lambda_m \tau)\, \xi^m$$

holds. This solution is usually called the m—th harmonics of the system (3).

It is easy to show that the Rayleigh quotient $(Au, u)/(u, u)$ is monotonically decreasing for any nontrivial solution u which is not a harmonics.

A numerical method for the solution of (3) is usually called stable in the energy norm for $\tau < \tau^*$ if, as in (4), the approximate solution satisfies the inequality

$$\|\hat{y}\|_D < \|y\|_D, \quad y \neq 0.$$

We shall call the numerical method monotonic for $\tau < \tau^*$, if for the approximate solution, as in (5),

(7) $$(y, \mathring{y}) > 0, \quad y \neq 0.$$

Similarly, the numerical method is called R-monotonic for $\tau < \tau^*$, if the property of monotone decreasing of the Rayleigh quotient is preserved for the approximate solution

(8) $$\hat{\mu}_0 = (A\hat{y}, \hat{y})/(\hat{y}, \hat{y}) < (Ay, y)/(y, y) = \mu_0, \quad y \neq 0.$$

Let us assess from these aspects some known and some new numerical methods for the solution of systems of ordinary differential equations.

2. An improvement of the implicit Euler method

As it was mentioned above among well-known methods of the first order the implicit Euler method has the best correspondence of the qualitative behaviour between the solutions of the differential and the difference systems.

Applied to (3) the implicit Euler method (2) has the form

(9) $$\hat{y} = y - \tau A\hat{y}.$$

It can be verified that

$$\|\hat{y}\|_D^2 = S\|y\|_D^2$$

where

$$S = (1 + 2\tau\hat{\mu}_k + \tau^2 \hat{\mu}_{k+1}\hat{\mu}_k)^{-1}$$

and

$$\hat{\mu}_i = (A^{i+1}\hat{y}, \hat{y})/(A^i\hat{y}, \hat{y}).$$

As $\hat{\mu}_i$ is positive for any nonzero $\hat{y}$ the energy norm (with the operator D) of the approximate solution is decreasing for any steplength τ. Also, it is easy to show that for any τ, the approximate solution obtained by (9), satisfies (7) and (8).

In accordance with our definitions the implicit Euler method (9) is stable, monotonic and R-monotonic for any τ.

But, putting $y=\xi^m$ into (9), we see that the implicit method (9) is not exact on the harmonics of the system (3).

We shall construct a method free from this defect. Let us consider the implicit method of the first order (see [3])

$$\hat{y} = y - \tau q_1 \hat{f},$$

which applied to (3) has the form

(10) $$\hat{y} = y - \tau q_1 A\hat{y},$$

where

(11) $$q_1 = (\exp(\mu\tau) - 1)/(\mu\tau)$$

and

(12) $$\mu = \mu_0.$$

It can be verified directly that the method (10) — (12) is stable, monotonic, and R-monotonic for any τ and besides, exact on the harmonics of the system (3).

Numerical experiments have shown greater effectiveness of the method (10) — (12) (method II in Table 1) compared to the implicit Euler method (method I).

3. A second order implicit method

Let us consider now the implicit method of the second order (see [4])

(13) $$\hat{y} = y - (\tau/2)(f(t, \hat{y} + \tau\hat{f}) + \hat{f}).$$

It can be shown that this method is stable, monotonic and R-monotonic for any τ but is not exact on the harmonics of the system (3).

As before we shall modify this method with the introduction of the parameter $q_2 = 1 + 0(\tau^2)$

(14) $$\hat{y} = y - (\tau/2)\, q_2 (f(t, \hat{y} + \tau\hat{f}) + \hat{f}).$$

In the case of the system (3) the method (14) has the form

(15) $$\hat{y} = y - \tau q_2 A\hat{y} - (\tau^2/2)\, q_2 A^2 \hat{y}.$$

It is possible to show that the method (15) is stable, monotonic and R-monotonic for any τ.

If we choose

(16) $$q_2 = (\exp(\mu\tau) - 1)/(\mu\tau + \mu^2\tau^2/2),$$

where μ is as in (12), then the method (15) is exact on the harmonics of (3).

The results of a numerical experiment can be found in Table 1 (method III is the method (13), method IV the method (15), (16), (12)). The analysis of the results shows the effectiveness of the regularization of known implicit methods.

4. Some explicit nonlinear methods

All the mentioned methods for the solution of the system of ordinary differential equations are implicit and therefore require at each step the solution of an algebraic system of linear equations. Their usage is therefore limited by the size and the speed of the computer.

Explicit methods do not have these limitations although the requirement for stability imposes severe limitations on the size of τ and are therefore not very practical.

In the following we shall discuss some explicit nonlinear methods which possess the properties of implicit methods for any τ.

Let us return again to the explicit Euler method

(17) $$\hat{y} = y - \tau f,$$

where $f = f(t, y)$.

The requirement for the stability of approximate solutions in the energy space H_D with the norm $\| y \|_D$, $D = A^k$, leads to a severe limitation upon the stepsize

$$\tau < 2/\mu_{k+1},$$

where

$$\mu_i = (A^{i+1} y, y)/(A^i y, y).$$

The requirement for monotonicity and R-monotonicity impose upon τ for explicit Euler method the respective limitations

$$\tau < 1/\mu_0$$

and

$$\tau < 2(\mu_1 - \mu_0)/(\mu_1 \mu_2 - \mu_1 \mu_0).$$

It is also clear that the method (17) is not exact on the harmonics of the system (3).

Together with (17) let us consider the regulated method

$$\hat{y} = y - \tau \rho_1 f, \tag{18}$$

where the factor $\rho_1 = 1 + 0(\tau)$, therefore the order of the method (namely 1), is preserved.

In the case of the system (3) the method (18) has the form

$$\hat{y} = y - \tau \rho_1 A y. \tag{19}$$

Let us choose the parameter ρ_1 so that the method is exact on the harmonics of the system (3). It is easy to verify that this is true for

$$\rho_1 = (1 - \exp(-\mu\tau))/(\mu\tau), \tag{20}$$

where μ is calculated by (12), for instance.

Besides, the first order explicit method (19), (20), (12) is stable in the negative norm $(A^{-1}y, y)^{1/2}$ and monotonic for any steplengths τ.

Using other choices of μ (see [3]) it is possible to obtain additional properties: the stability in Euclidean and other norms and also R-monotonicity for any τ.

In Table 1 there are results of numerical calculations for original explicit Euler method (method V) and the regulated method (19), (20), (12) (method VI). Greater effectiveness of the latter is obvious.

But the drawback of the method VI (the non-fulfilment of R-monotonicity) is apparent as μ does not vary monotonically when trying to augment the stepsize τ, what did not allow the approximate solution for large τ to settle upon the regular regime (see [5]).

5. A new explicit method

Let us now consider an explicit method for the solution of the system of differential equations (3) which allows even closer correspondence between the differential and the corresponding difference problem.

The construction of the new numerical method is based on the extraction and successive inversion of the main part of the differential operator (see [2]).

Let us extract from (3) as the main part the vector μu, where μ is a numerical parameter which is constant on the interval $[t, t+\tau]$. Then we can rewrite the original system (3) in the form

$$\hat{u}=u\exp(-\mu\tau)-\int_{t}^{t+\tau}(A-\mu I)u(x)\exp(-\mu(t+\tau-x))\,dx. \tag{21}$$

The integral in (21) can be approximated by the formula

$$\int_{t}^{t+\tau}(A-\mu I)u(x)\exp(-\mu(t+\tau-x))\,dx\approx\tau\rho_1^*(A-\mu I)y\exp(-\mu\tau),$$

where $\rho_1{}^*$ is a corrective parameter. So, we obtain an approximate rule

$$\hat{y}=\exp(-\mu\tau)(I-\tau\rho_1^*(A-\mu I))y. \tag{22}$$

Note, that selecting various approximations to the integral (21) we can obtain a number of numerical methods, in particular the methods (10), (11) and (19), (20). If we extract from the differential operator polynomials of the independent variable t, it is possible to derive different linear explicit and implicit numerical methods among which are all above methods.

Let us now analyse the properties of the method (22). It is obvious that the method (22) is of the first order if $\rho_1{}^*=1+0(\tau)$. If we select the value of $\mu=\mu_0$ (see (8)) we obtain the method (12). This value minimizes the norm $||Ay-\mu y||_D$, where $D=A^0=I$.

It is easy to show that the numerical method (22), (12) is exact on the harmonics of the system (3), stable in the negative norm and monotonic for any τ, if $\rho_1{}^*$ satisfies the condition

$$0<\rho_1^*\leqq 1.$$

Let

$$\rho_1^*=1/(1+\tau\nu), \tag{23}$$

where $\nu\geqq 0$ is a parameter, which allows us to achieve also R-monotonicity of the method (22), (12), (23) for any τ.

It can be verified directly that it is sufficient to choose

$$\nu\geqq\frac{1}{2}\mu_1(\mu_2-\mu_0)/(\mu_1-\mu_0)-\mu_0. \tag{24}$$

Thus, the explicit method (22), (23), (24), (12) is of the first order, exact on the harmonics of the system (3), stable in the negative norm, monotonic and R-monotonic for any steplength τ.

The results of the numerical experiment (see Table 1) confirm close correspondence between the approximate solution obtained by the method (22), (23), (24), (12) (in Table 1 the method VII) and the exact solution for the test initial value problem.

In a similar manner numerical methods of higher order with similar properties can be constructed.

6. Numerical examples

Let us describe some numerical experiments made on a computer SM-4.

In Table 1 there are the results of the solution of the system of ordinary differential equations of the first order obtained by the longitudinal method of straight lines (the discretization parameter for the space variable $h=0.1$) to the first boundary problem of the homogeneous heat equation

$$u_t - u_{xx} = 0,$$

$$u(0, t) = u(1, t) = 0, \quad 0 \leq t \leq 2,$$

$$u(x, 0) = 100\, x(1-x), \quad 0 \leq x \leq 1.$$

The exact solution $v(t)$ of the approximating system of ordinary differential equations can be written analytically.

Table 1

t_k	method	k_τ	τ_{max}	δ_k	Δ	μ_0
0.01	I	21	5.0E−4	2.10E−2	2.10E−2	9.8159
	II	19	1.0E−3	1.68E−2	1.68E−2	9.8154
	III	7	2.0E−3	2.74E−3	3.25E−3	9.8183
	IV	7	2.0E−3	2.71E−3	3.23E−3	9.8183
	V	23	5.0E−4	2.05E−2	2.05E−2	9.8142
	VI	21	1.0E−3	1.65E−2	1.66E−2	9.8133
	VII	12	1.0E−3	8.13E−3	9.47E−3	9.8183
0.2	I	155	2.0E−3	1.20E−1	1.30E−1	9.7887
	II	43	6.4E−2	8.23E−5	2.27E−2	9.7887
	III	35	8.0E−3	1.27E−2	1.48E−2	9.7887
	IV	18	6.4E−2	2.78E−5	5.34E−3	9.7887
	V	160	2.0E−3	1.20E−1	1.30E−1	9.7887
	VI	59	3.2E−2	2.18E−3	2.10E−2	9.7887
	VII	23	6.4E−2	7.08E−4	9.47E−3	9.7887
2.0	I	238	5.1E−1	1.69E−4	1.30E−1	9.7887
	II	46	5.1E−1	7.9E−13	2.27E−2	9.7887
	III	62	5.1E−1	1.26E−5	1.48E−2	9.7887
	IV	21	5.1E−1	1.0E−11	5.34E−3	9.7887
	V	490	8.0E−3	2.10E−5	1.30E−1	390.21
	VI	219	2.6E−1	1.16E−3	2.10E−2	232.91
	VII	26	5.1E−1	1.4E−11	9.47E−3	9.7887

The steplength τ for the numerical integration was selected on the basis of the local error by comparing the approximate solutions in the maximum norm at every net point obtained by the steplengths τ and $\tau/2$ (the initial $\tau_0=10^{-3}$, the tolerance $\varepsilon=10^{-3}$).

The systems of linear algebraic equations of the order $n=9$ which arose using implicit methods, were solved by the Gauss elimination.

In Table 1 the notations are as follows: t_k — the selected point in the integration interval, k_τ — the number of integration steps, τ_{max} — the maximal steplength, $\delta_k=||v(t_k)-y(t_k)||$, $\Delta=\max\delta_j$, $1\leq j\leq k_\tau$, $\mu_0=(Ay,\ y)/(y,y)$, I — the implicit Euler method (9), II — the method (10), (11), (12), III — the method (13), IV — the method (15), (16), (12), V — the explicit Euler method (17), VI — the method (19), (20), (12), VII — the method (22), (23), (24), (12).

As a conclusion, note that also methods of higher order can be subdued to the method of regularization. In this respect one can orientate oneself to modular methods constructed on the basis of the principle of successive increasing of the order of the method (see [1]). They are very suitable for the method of regularization.

Note that the discussed methods can be applied to nonhomogeneous linear systems and generalized to the stable systems (1) too. They can be used to develop methods for the solution of boundary value problems in partial differential equations as well.

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POBOLJŠANJE USKLAĐENOSTI IZMEĐU REŠENJA DIFERENCIJALNIH I DIFERENČNIH PROBLEMA

V. V. Bobkov, Z. Bohte i P. A. Mandrik

U ovom radu iznose se neke ideje za konstrukciju novih numeričkih metoda za rešavanje običnih deferencijalnih jednačina. Novi metodi su izvedeni i isprobani na numeričkim primerima.

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О ВЕКТОРЕ КРИВИЗНЫ КРИВОЙ В ПОДПРОСТРАНСТВЕ ОБОБЩЕННОГО РИМАНОВА ПРОСТРАНСТВА

Светислав М. Минчич

Резюме. В настоящей работе, прежде всего, доказывается, что вектор кривизны кривой в подпространстве обобщенного риманова пространства (т.е. пространства с несимметрическим основным тензором) выражается тем же способом как это бывает в обыкновенном римановом пространстве. Из-за несимметрии связности, можно определить четыре рода ковариантной производной тензора [2], [3]. Для вектора кривизны кривой получаются две деривационные формулы (2.1′). Используя раньше полученные тождества типпа Риччи [2], [3], получается шесть условий интегрируемости этих деривационных формул. Из этих условий произходят уравнения типпа Гаусса и Петерсона-Кодацци.

В конце работы рассматриваются специальные случаи, когда кривая асимптотическая или геодезическая линия подпространства.

0. Введение

Пусть V_M обобщенное риманово прнстранство с координатами y^α $(\alpha = 1, \dots, N)$ и несимметрическим основным тензором $a_{\alpha\beta}\,(a_{\alpha\beta} \neq a_{\beta\alpha})$. Подпространство V_M пространства $V_N\,(M < N)$ определенное уравнениями (см.[1])

$$y^\alpha = y^\alpha(x^1, \dots, x^M) \quad (\operatorname{rang}(y^\alpha_{,i}) = M < N) \tag{0.1}$$

а его основный тензор g_{ij} тоже несимметрический, при чем

$$a_{\alpha\beta}\, y^\alpha_{,i}\, y^\beta_{,j} = a_{\alpha\beta}\, t^\alpha_i\, t^\beta_j = g_{ij}, \tag{0.2}$$

где запятая обозначает частную производную, на пример, имеем

$$\frac{\partial y^\alpha}{\partial x^i} = y^\alpha_{,i} = t^\alpha_i, \quad \frac{\partial^2 y^\alpha}{\partial x^i\,\partial x^j} = y^\alpha_{,ij} = t^\alpha_{i,j}\,.$$

Символы Кристоффеля пространства V_N

$$\Gamma_{\alpha.\beta\gamma} = \frac{1}{2}(a_{\beta\alpha,\gamma} - a_{\beta\gamma,\alpha} + a_{\alpha\gamma,\beta}), \tag{0.3}$$

$$\Gamma^\alpha_{\beta\gamma} = a^{\alpha\pi}\,\Gamma_{\pi.\beta\gamma}, \tag{0.4}$$

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где $\underline{\alpha\pi}$ обозначает симметрирование по отношению к α, π. В (0.3,4) имеем несимметрию по отношению к индексам β, γ.

Если $N^{\alpha}_{(\rho)}$ единичные взаимно ортогональные друг к другу векторы, которые тоже нормальны к V_M, будет

$$a_{\alpha\beta}\, N^{\alpha}_{(\rho)}\, N^{\beta}_{(\sigma)} = e_{(\rho)}\, \delta_{\rho\sigma}\ \ (e_{(\rho)} = \pm 1),\ \ a_{\underline{\alpha\beta}}\, N^{\alpha}_{(\rho)}\, t^{\beta}_{j} = 0. \tag{0.5}$$

Заметим, что в этой работе греческие индексы принимают значения от 1 до N, а латинские от 1 до M, греческие индексы в скобках принимают значения от $M+1$ до N.

Вследствие несимметричности символов Кристоффеля, можно определить четыре рода ковариантной производной для тензора в обобщенном Римановом пространстве [2], [3]. На пример

$$t^{\alpha}_{i\underset{1}{|}m} = t^{\alpha}_{i,m} + \Gamma^{\alpha}_{\pi\mu}\, t^{\mu}_{m}\, t^{\pi}_{i} - \Gamma^{p}_{im}\, t^{\alpha}_{p}, \tag{0.6a}$$

$$t^{\alpha}_{i\underset{2}{|}m} = t^{\alpha}_{i,m} + \Gamma^{\alpha}_{\mu\pi}\, t^{\mu}_{m}\, t^{\pi}_{i} - \Gamma^{p}_{mi}\, t^{\alpha}_{p}, \tag{0.6b}$$

$$t^{\alpha}_{i\underset{3}{|}m} = t^{\alpha}_{i,m} + \Gamma^{\alpha}_{\pi\mu}\, t^{\pi}_{i} - \Gamma^{p}_{mi}\, t^{\alpha}_{p}, \tag{0.6c}$$

$$t^{\alpha}_{i\underset{4}{|}m} = t^{\alpha}_{i,m} + \Gamma^{\alpha}_{\mu\pi}\, t^{\mu}_{m}\, t^{\pi}_{i} - \Gamma^{p}_{im}\, t^{\alpha}_{p}, \tag{0.6d}$$

$$N^{\alpha}_{(\rho)\underset{1}{|}m} = N^{\alpha}_{(\rho)\underset{3}{|}m} = N^{\alpha}_{(\rho),m} + \Gamma^{\alpha}_{\pi\mu}\, t^{\mu}_{m}\, N^{\pi}_{(\rho)}, \tag{0.7a}$$

$$N^{\alpha}_{(\rho)\underset{2}{|}m} = N^{\alpha}_{(\rho)\underset{4}{|}m} = N^{\alpha}_{(\rho),m} + \Gamma^{\alpha}_{\mu\pi}\, t^{\mu}_{m}\, N^{\pi}_{(\rho)}. \tag{0,7b}$$

Используя четыре рода коваринантой производной получаются четыре рода деривационных формул (см. (16) и (37′) в [4]):

$$t^{\alpha}_{i\underset{0}{|}m} = \underset{0}{\Phi}{}^{p}_{im}\, t^{\alpha}_{p} + \sum_{\rho} \underset{0}{\Omega}_{(\rho)\,im}\, N^{\alpha}_{(\rho)}, \tag{0.8a}$$

$$N^{\alpha}_{(\sigma)\underset{0}{|}m} = -e_{(\sigma)}\, g^{\underline{ps}}\, \underset{0}{\Omega}_{(\sigma)\,sm}\, t^{\alpha}_{p} + \sum_{\rho} \underset{0}{\Psi}_{(\rho\sigma)\,m}\, N^{\alpha}_{(\rho)},\ \ \underset{0}{\Psi}_{(\sigma\sigma)\,m} = 0, \tag{0.8b}$$

где $\theta = 1, 2, 3, 4$ означает род ковариантной производной. Согласно (0.7a,b) утверждается, что

$$\underset{1}{\Omega}_{(\rho)ij} = \underset{3}{\Omega}_{(\rho)ij},\ \ \underset{2}{\Omega}_{(\rho)ij} = \underset{4}{\Omega}_{(\rho)ij}, \tag{0.9a,b}$$

$$\underset{1}{\Psi}_{(\rho\sigma)m} = \underset{3}{\Psi}_{(\rho\sigma)m},\ \ \underset{2}{\Psi}_{(\rho\sigma)m} = \underset{4}{\Psi}_{(\rho\sigma)m}, \tag{0.10a,b}$$

а согласно (48′), (24′) в [4]

$$\underset{2}{\Phi}{}^{h}_{im} = -\underset{1}{\Phi}{}^{h}_{im} = \underset{1}{\Phi}_{mi},\ \ \underset{3}{\Phi}{}^{h}_{im} = \underset{3}{\Phi}{}^{h}_{im} + 2\,\Gamma^{h}_{\underset{\vee}{im}},\ \ \underset{4}{\Phi}{}^{h}_{im} = -\underset{1}{\Phi}{}^{h}_{im} - 2\,\Gamma^{h}_{\underset{\vee}{im}}. \tag{0.11a,b,c}$$

1. Вектор кривизны кривой

Рассмотрим в подпространстве векторное поле v с контравариантныи координатами v^α по отношению к y^α и u^i по отношению к x^i. Тогда

$$v^\alpha = t_i^\alpha u^i. \tag{1.1}$$

Аналогично (0.7a,b) имеем

$$v^\alpha_{\underset{1}{|}j} = v^\alpha_{\underset{3}{|}j}, \quad v^\alpha_{\underset{2}{|}j} = v^\alpha_{\underset{4}{|}j},$$

и потому можно рассматривать $v^\alpha_{\underset{\theta}{|}j}$ для $\theta = 1, 2$. Согласно (1.1) и (0.8a) будет

$$v^\alpha_{\underset{\theta}{|}j} = t^\alpha_{i\underset{\theta}{|}j} u^i + t_i^\alpha u^i_{\underset{0}{|}j} = [\underset{\theta}{\Phi}{}^p_{ij} t^\alpha_p + \sum_\rho \underset{\theta}{\Omega}_{(\rho)ij} N^\alpha_{(\rho)}] u^i + t^\alpha_i u^i_{\underset{\theta}{|}j}. \tag{1.2}$$

Пусть векторное поле v определенное вдоль кривой C в V_M. Умножим предшесвующее уравнение на $\frac{dx^j}{ds}$, где s дуга кривой C и обозначнм

$$\underset{0}{q}{}^\alpha = v^\alpha_{\underset{0}{|}j} \frac{dx^j}{ds}, \quad \underset{0}{p}{}^i = u^i_{\underset{0}{|}j} \frac{dx^j}{ds} \qquad (\theta = 1, 2),$$

т.е.

$$\underset{1}{p}{}^i = u^i_{\underset{1}{|}j} \frac{dx^j}{ds} = \left(u^i_{,j} + \Gamma^i_{pj} u^p\right) \frac{dx^j}{ds} = \frac{du^i}{ds} + \Gamma^i_{pj} u^p \frac{dx^j}{ds}, \tag{1.3a}$$

$$\underset{2}{p}{}^i = u^i_{\underset{2}{|}j} \frac{dx^j}{ds} = \frac{du^i}{ds} + \Gamma^i_{jp} u^p \frac{dx^j}{ds}, \tag{1.3b}$$

$$\underset{1}{q}{}^\alpha = v^\alpha_{\underset{1}{|}j} \frac{dx^j}{ds} = \left(v^\alpha_{,j} + \Gamma^\alpha_{\pi\beta} v^\pi t^\beta_j\right) \frac{dx^j}{ds} = \frac{dv^\alpha}{ds} + \Gamma^\alpha_{\pi\beta} v^\pi \frac{dy^\beta}{ds}, \tag{1.3c}$$

$$\underset{2}{q}{}^\alpha = v^\alpha_{\underset{2}{|}j} \frac{dx^i}{ds} = \frac{dv^\alpha}{ds} + \Gamma^\alpha_{\beta\pi} v^\pi \frac{dy^\beta}{ds}. \tag{1.3d}$$

Теперь уравнение (1.2) можно написать в виде

$$\underset{\theta}{q}{}^\alpha = \underset{0}{\Phi}{}^p_{ij} t^\alpha_p u^i \frac{dx^j}{ds} + \underset{\theta}{p}{}^i t^\alpha_i + \sum_\rho \underset{0}{\Omega}_{(\rho)ij} u^i \frac{dx^j}{ds} N^\alpha_{(\rho)}. \tag{1.4}$$

Векторы $\underset{0}{q}{}^\alpha$, $\underset{0}{p}{}^i$ являются производными векторами от вектора v вдоль C по отношению к V_N т.е. V_M.

Пусть теперь $u^i = \frac{dx^i}{ds}$, т.е. пусть v будет единичный вектор касательной к кривой C. Тогда

$$\underset{\theta}{\Phi}{}^p_{ij} t^\alpha_p u^i \frac{dx^j}{ds} = \underset{\theta}{\Phi}{}^p_{ij} t^\alpha_p \frac{dx^i}{ds} \frac{dx^j}{ds} = 0, \tag{1.5}$$

потому что согласно (24′) в [4] тензоры $\underset{\theta}{\Phi}{}^p_{ij}$ антисимметрические по отношению к i, j.

На том же мнобообразии, на котором определенное обобщенное риманово пространство V_N и его подпространство V_M можно определить обыкновенное риманово пространство $\overline{V}_N$ и его подпространство $\overline{V}_M$ используя как основные тензоры $a_{\underline{\alpha\beta}}$ и $g_{\underline{ij}}$. Символы Кристоффеля для $\overline{V}_N$ и $\overline{V}_M$ будут $\Gamma^{\alpha}_{\underline{\beta\gamma}}$ и $\Gamma^{i}_{\underline{jk}}$. Пусть $K_{(\rho)}$ нормальная кривизна кривой C, соответствующая нормале $N^{\alpha}_{(\rho)}$, $\Omega_{(\rho)ij}$ коэффициенты второй квадратичной формы для $\overline{V}_M$, q^{α}, p^{i} векторы на которые в $\overline{V}_N$, $\overline{V}_M$ сводятся векторы q^{α} соответственно $\underset{0}{p}{}^{i}$.

Для $u^i=\dfrac{dx^i}{ds}$ будет $v^\alpha=t^\alpha_i u^i=\dfrac{\partial y^\alpha}{\partial x^i}\dfrac{dx^i}{ds}=\dfrac{dy^\alpha}{ds}$ и из (1.3a – d) получается

$$(1.6)\qquad \underset{1}{p}{}^i=\frac{d^2x^i}{ds^2}+\Gamma^i_{pj}\frac{dx^p}{ds}\frac{dx^j}{ds}=\frac{d^2x^i}{ds^2}+\Gamma^i_{jp}\frac{dx^p}{ds}\frac{dx^j}{ds}=\underset{2}{p}{}^i=\frac{d^2x^i}{ds}+\Gamma^i_{\underline{pj}}\frac{dx^p}{ds}\frac{dx^j}{ds}=p^i,$$

$$(1.7)\qquad \underset{1}{q}{}^\alpha=\frac{d^2y^\alpha}{ds^2}+\Gamma^\alpha_{\underline{\pi\beta}}\frac{dy^\pi}{ds}\frac{dy^\beta}{ds}=\underset{2}{q}{}^\alpha=q^\alpha,$$

а как согласно (22) в [4] $\underset{0}{\Omega}_{(\rho)ij}=\Omega_{(\rho)ij}$, то получаем

$$(1.8)\qquad \underset{0}{\Omega}_{(\rho)ij}\frac{dx^i}{ds}\frac{dx^j}{ds}=\underset{0}{\Omega}_{(\rho)ij}\frac{dx^i}{ds}\frac{dx^j}{ds}=\Omega_{(\rho)ij}\frac{dx^i}{ds}\frac{dx^j}{ds}=K_{(\rho)}.$$

Следовательно, для $u^i=\dfrac{dx^i}{ds}$, вследствие (1.5-8), уравнение (1.4) получает вид

$$(1.9)\qquad \underset{0}{q}{}^\alpha=t^\alpha_i p^i+\sum_\rho K_{(\rho)}N^\alpha_{(\rho)}=q^\alpha,$$

т.е. получается такое же выражение как в обыкновенном пространстве Римана (см. ур. (12′) в [5], § 92). Напомним, что в (1.9), q^α вектор кривизны кривой, который разлагается на касательнуоı $t^\alpha_i p^i$ (вектор геооезической кривизны) и нормальную компоненту $\sum_\rho K_{(\rho)}N^\alpha_{(\rho)}$ (вектор нормальной кривизныы).

2. Деривационная формула вектора кривизны и условия интегрируемости

2.0. Дифференцируя ковариантно (1.9) и используя (0.8), получим

$$q^\alpha_{\underset{\mu}{|}m}=t^\alpha_{i\underset{\mu}{|}m}p^i+t^\alpha_i p^i_{\underset{\mu}{|}m}+\sum_\rho K_{(\rho)\underset{\mu}{|}m}N^\alpha_{(\rho)}+\sum_\rho K_{(\rho)}N^\alpha_{(\rho)\underset{\mu}{|}m}=$$

$$=[\underset{\mu}{\Phi}{}^s_{im}t^\alpha_s+\sum_\rho\underset{\mu}{\Omega}_{(\rho)im}N^\alpha_{(\rho)}]p^i+t^\alpha_s p^s_{\underset{\mu}{|}m}+\sum_\rho K_{(\rho),m}N^\alpha_{(\rho)}+$$

$$+\sum_\rho K_{(\rho)}[-e_{(\rho)}g^{rs}\underset{\mu}{\Omega}_{(\rho)rm}t^\alpha_s+\sum_\sigma\underset{\mu}{\Psi}{}_{(\sigma\rho)m}N^\alpha_{(\sigma)}],$$

т.е.

$$(2.1)\qquad q^\alpha_{\underset{\mu}{|}m}=[\underset{\mu}{\Phi}{}^s_{im}p^i+p^s_{\underset{\mu}{|}m}-\sum_\rho K_{(\rho)}e_{(\rho)}g^{rs}\underset{\mu}{\Omega}_{(\rho)rm}]t^\alpha_s+$$

$$+\sum_\rho[\underset{\mu}{\Omega}_{(\rho)im}p^i+K_{(\rho),m}+\sum_\sigma K_{(\sigma)}\underset{\mu}{\Psi}{}_{(\rho\sigma)m}]N^\alpha_{(\rho)},$$

где $K_{(\rho)\underset{\mu}{|}m} = K_{(\rho),m} = \dfrac{\partial K_{(\rho)}}{\partial x^m}$ ковариантный вектор. Формулу (2.1) называем *деривационной формулой вектора кривизны кривой* C подпространства V_M. Вводя обозначения

$$\text{(2.2a)} \qquad \underset{\mu}{A}{}^s_m = \underset{\mu}{\Phi}{}^s_{jm}\, p^j + p^s_{\underset{\mu}{|}m} - \sum_\rho e_{(\rho)}\, K_{(\rho)}\, g^{rs}\, \underset{\mu}{\Omega}_{(\rho)rm},$$

$$\text{(22b)} \qquad \underset{\mu}{B}_{(\rho)m} = \underset{\mu}{\Omega}_{(\rho)jm}\, p^j + K_{(\rho),m} + \sum_\sigma K_{(\sigma)}\, \underset{\mu}{\Psi}_{(\rho\sigma)m},$$

уравнение (2.1) получает вид

$$\text{(2.1')} \qquad q^\alpha_{\underset{\mu}{|}m} = \underset{\mu}{A}{}^s_m\, t^\alpha_s + \sum_\rho \underset{\mu}{B}_{(\rho)m}\, N^\alpha_{(\rho)}.$$

Поскольку

$$\text{(2.2c,d)} \qquad q^\alpha_{\underset{3}{|}m} = q^\alpha_{\underset{1}{|}m}, \quad q^\alpha_{\underset{4}{|}m} = q^\alpha_{\underset{2}{|}m},$$

то будет

$$\text{(2.2e-h)} \qquad \underset{3}{A}{}^s_m = \underset{1}{A}{}^s_m, \quad \underset{4}{A}{}^s_m = \underset{2}{A}{}^s_m, \quad \underset{3}{B}_{(\rho)m} = \underset{1}{B}_{(\rho)m}, \quad \underset{4}{B}_{(\rho)m} = \underset{2}{B}_{(\rho)m}.$$

Применяя ковариантную производную рода $\nu \in \{1, 2, 3, 4\}$ по x^n на (2.1') и используя (0.8), получим

$$q^\alpha_{\underset{\mu}{|}m\underset{\nu}{|}n} = \underset{\mu}{A}{}^s_{m\underset{\nu}{|}n}\, t^\alpha_s + \underset{\mu}{A}{}^s_m\, t^\alpha_{s\underset{\nu}{|}n} + \sum_\rho \underset{\mu}{B}_{(\rho)m\underset{\nu}{|}n}\, N^\alpha_{(\rho)} + \sum_\rho \underset{\mu}{B}_{(\rho)m}\, N^\alpha_{(\rho)\underset{\nu}{|}n} =$$

$$= \underset{\mu}{A}{}^s_{m\underset{\nu}{|}n}\, t^\alpha_s + \underset{\mu}{A}{}^s_m \Big[\underset{\nu}{\Phi}{}^p_{sn}\, t^\alpha_p + \sum_\rho \underset{\nu}{\Omega}_{(\rho)sn}\, N^\alpha_{(\rho)}\Big] +$$

$$+ \sum_\rho \underset{\mu}{B}_{(\rho)m\underset{\nu}{|}n}\, N^\alpha_{(\rho)} + \sum_\rho \underset{\mu}{B}_{(\rho)m} \Big[-e_{(\rho)}\, g^{rs}\, \underset{\nu}{\Omega}_{(\rho)rm}\, t^\alpha_s + \sum_\sigma \underset{\nu}{\Psi}_{(\sigma\rho)n}\, N^\alpha_{(\sigma)}\Big],$$

т.е.

$$q^\alpha_{\underset{\mu}{|}m\underset{\nu}{|}n} = \Big[\underset{\mu}{A}{}^s_{m\underset{\nu}{|}n} + \underset{\mu}{A}{}^p_m\, \underset{\nu}{\Phi}{}^s_{pn} - \sum_\rho e_{(\rho)}\, g^{rs}\, \underset{\mu}{B}_{(\rho)m}\, \underset{\nu}{\Omega}_{(\rho)rn}\Big] t^\alpha_s +$$

$$+ \sum_\rho \Big[\underset{\mu}{A}{}^s_m\, \underset{\nu}{\Omega}_{(\rho)sn} + \underset{\mu}{B}_{(\rho)m\underset{\nu}{|}n} + \sum_\sigma \underset{\mu}{B}_{(\sigma)m}\, \underset{\nu}{\Psi}_{(\rho\sigma)n}\Big] N^\alpha_{(\rho)}.$$

Отсюда получается

$$\text{(2.3)} \qquad q^\alpha_{\underset{\mu}{|}m\underset{\nu}{|}n} - q^\alpha_{\underset{\nu}{|}n\underset{\mu}{|}m} = \Big\{\underset{\mu}{A}{}^s_{m\underset{\nu}{|}n} - \underset{\nu}{A}{}^s_{n\underset{\mu}{|}m} + \underset{\mu}{A}{}^p_m\, \underset{\nu}{\Phi}{}^s_{pn} - \underset{\nu}{A}{}^p_n\, \underset{\mu}{\Phi}{}^s_{pm} -$$

$$- \sum_\rho e_{(\rho)}\, g^{rs} \Big[\underset{\mu}{B}_{(\rho)m}\, \underset{\nu}{\Omega}_{(\rho)rn} - \underset{\nu}{B}_{(\rho)n}\, \underset{\mu}{\Omega}_{(\rho)rm}\Big]\Big\} t^\alpha_s +$$

$$+ \sum_\rho \Big\{\underset{\mu}{A}{}^s_m\, \underset{\nu}{\Omega}_{(\rho)sn} - \underset{\nu}{A}{}^s_n\, \underset{\mu}{\Omega}_{(\rho)sm} + \underset{\mu}{B}_{(\rho)m\underset{\nu}{|}n} - \underset{\nu}{B}_{(\rho)n\underset{\mu}{|}m} + \sum_\sigma \Big[\underset{\mu}{B}_{(\sigma)m}\, \underset{\nu}{\Psi}_{(\rho\sigma)n} - \underset{\nu}{B}_{(\sigma)n}\, \underset{\mu}{\Psi}_{(\rho\sigma)m}\Big]\Big\} N^\alpha_{(\rho)}.$$

Применяя тождества типпа Риччи (7), (11), (56) из [2] и (12), (13), (46) из [3] к вектору q^α, имеем

$$q^{\alpha}_{\underset{1}{|}m\underset{1}{|}n} - q^{\alpha}_{\underset{1}{|}n\underset{1}{|}m} = \underset{1}{R}{}^{\alpha}_{\pi\mu\nu}\, q^{\pi}\, t^{\mu}_{m}\, t^{\nu}_{n} - 2\, \Gamma^{u}_{\underset{\vee}{mn}}\, q^{\alpha}_{\underset{1}{|}u}, \tag{2.4}$$

$$q^{\alpha}_{\underset{2}{|}m\underset{2}{|}n} - q^{\alpha}_{\underset{2}{|}n\underset{2}{|}m} = \underset{2}{R}{}^{\alpha}_{\pi\mu\nu}\, q^{\pi}\, t^{\mu}_{m}\, t^{\nu}_{n} + 2\, \Gamma^{u}_{\underset{\vee}{mn}}\, q^{\alpha}_{\underset{2}{|}u}, \tag{2.5}$$

$$q^{\alpha}_{\underset{1}{|}m\underset{2}{|}n} - q^{\alpha}_{\underset{2}{|}n\underset{1}{|}m} = \underset{3}{R}{}^{\alpha}_{\pi mn}\, q^{\pi}, \tag{2.6}$$

$$q^{\alpha}_{\underset{3}{|}m\underset{3}{|}n} - q^{\alpha}_{\underset{3}{|}n\underset{3}{|}m} = \underset{1}{R}{}^{\alpha}_{\pi\mu\nu}\, q^{\pi}\, t^{\mu}_{m}\, t^{\nu}_{n} + 2\, \Gamma^{u}_{\underset{\vee}{mn}}\, q^{\alpha}_{\underset{3}{|}u}, \tag{2.7}$$

$$q^{\alpha}_{\underset{4}{|}m\underset{4}{|}n} - q^{\alpha}_{\underset{4}{|}n\underset{4}{|}m} = \underset{2}{R}{}^{\alpha}_{\pi\mu\nu}\, q^{\pi}\, t^{\mu}_{m}\, t^{\nu}_{n} - \Gamma^{u}_{\underset{\vee}{mn}}\, q^{\alpha}_{\underset{4}{|}u}, \tag{2.8}$$

$$q^{\alpha}_{\underset{3}{|}m\underset{4}{|}n} - q^{\alpha}_{\underset{4}{|}n\underset{3}{|}m} = \underset{4}{R}{}^{\alpha}_{\pi mn}\, q^{\pi}, \tag{2.9}$$

где $\underset{\vee}{mn}$ обозначает антисимметрирование по отношению к m, n. Величины

$$\underset{1}{R}{}^{\alpha}_{\beta\mu\nu} = \Gamma^{\alpha}_{\beta\mu,\nu} - \Gamma^{\alpha}_{\beta\nu,\mu} + \Gamma^{\pi}_{\beta\mu}\Gamma^{\alpha}_{\pi\nu} - \Gamma^{\pi}_{\beta\nu}\Gamma^{\alpha}_{\pi\mu}, \tag{2.10}$$

$$\underset{2}{R}{}^{\alpha}_{\beta\mu\nu} = \Gamma^{\alpha}_{\mu\beta,\nu} - \Gamma^{\alpha}_{\nu\beta,\mu} + \Gamma^{\pi}_{\mu\beta}\Gamma^{\alpha}_{\nu\pi} - \Gamma^{\pi}_{\nu\beta}\Gamma^{\alpha}_{\mu\pi} \tag{2.11}$$

тензоры кривизны пространства V_N, а величины

$$\underset{3}{R}{}^{\alpha}_{\beta mn} = \left(\Gamma^{\alpha}_{\beta\mu,\nu} - \Gamma^{\alpha}_{\nu\beta,\mu} + \Gamma^{\pi}_{\beta\mu}\Gamma^{\alpha}_{\nu\pi} - \Gamma^{\pi}_{\nu\beta}\Gamma^{\alpha}_{\pi\mu}\right) t^{\mu}_{m}\, t^{\nu}_{n} + 2\, \Gamma^{\alpha}_{\underset{\vee}{\beta\mu}}\left(y^{\mu}_{,mn} - \Gamma^{p}_{nm}\, t^{\mu}_{p}\right), \tag{2.12}$$

$$\underset{4}{R}{}^{\alpha}_{\beta mn} = \left(\Gamma^{\alpha}_{\beta\mu,\nu} - \Gamma^{\alpha}_{\nu\beta,\mu} + \Gamma^{\pi}_{\beta\mu}\Gamma^{\alpha}_{\nu\pi} - \Gamma^{\pi}_{\nu\beta}\Gamma^{\alpha}_{\pi\mu}\right) t^{\mu}_{m}\, t^{\nu}_{n} + 2\, \Gamma^{\alpha}_{\underset{\vee}{\beta\mu}}\left(y^{\mu}_{mn} - \Gamma^{p}_{mn}\, t^{\mu}_{p}\right) \tag{2.13}$$

тензоры кривизны пространства V_N по отношению к подпространсту V_M, соответствено третьего и четвертого родов (см. [2], [3]).

2.1. Если правую сторону уравнения (2.4) /учитывая (2.1′)/ приравнить к првой строне уравнения (2.3), предварительно заменяя в последнем $\mu=\nu=1$, получим *первое условие интегрируемости* деривационной формулы (2.1′)

$$\underset{1}{R}{}^{\alpha}_{\pi\mu\nu}\, q^{\pi}\, t^{\mu}_{m}\, t^{\nu}_{m} - 2\, \Gamma^{u}_{\underset{\vee}{mn}}\left[\underset{1}{A}{}^{s}_{u}\, t^{\alpha}_{s} + \sum_{\rho} \underset{1}{B}_{(\rho)u}\, N^{\alpha}_{(\rho)}\right] = \tag{2.14}$$

$$= \Big\{\underset{1}{A}{}^{s}_{m\underset{1}{|}n} - \underset{1}{A}{}^{s}_{n\underset{1}{|}m} + \underset{1}{A}{}^{p}_{m}\, \underset{1}{\Phi}{}^{s}_{pn} - \underset{1}{A}{}^{p}_{n}\, \underset{1}{\Phi}{}^{s}_{pm} - \sum_{\rho} g^{rs}\, e_{(\rho)}\left[\underset{1}{B}_{(\rho)m}\, \underset{1}{\Omega}_{(\rho)rn} - \underset{1}{B}_{(\rho)n}\, \underset{1}{\Omega}_{(\rho)rm}\right]\Big\}\, t^{\alpha}_{s} +$$

$$+ \sum_{\rho}\Big\{\underset{1}{A}{}^{s}_{m}\, \underset{1}{\Omega}_{(\rho)sn} - \underset{1}{A}{}^{s}_{n}\, \underset{1}{\Omega}_{(\rho)sm} + \underset{1}{B}_{(\rho)m\underset{1}{|}n} - \underset{1}{B}_{(\rho)n\underset{1}{|}m} + \sum_{\sigma}\left[\underset{1}{B}_{(\sigma)m}\, \underset{1}{\Psi}_{(\rho\sigma)n} - \underset{1}{B}_{(\sigma)n}\, \underset{1}{\Psi}_{(\rho\sigma)m}\right]\Big\}\, N^{\alpha}_{(\rho)}$$

Умножая предыдущее уравнение на $a_{\underline{\alpha\beta}} t_h^\beta$ и учитывая (0.2,5), получаем

$$(2.15)\quad \underset{1}{R}_{\beta\pi\mu\nu} t_h^\beta q^\pi t_m^\mu t_n^\nu - 2\Gamma^u_{\underset{\vee}{mn}} \underset{1}{A}{}_u^s g_{hs} = \left(\underset{1}{A}{}^s_{m|n} - \underset{1}{A}{}^s_{n|m} + \underset{1}{A}{}^p_m \underset{1}{\Phi}{}^s_{pn} - \underset{1}{A}{}^p_n \underset{1}{\Phi}{}^s_{pm}\right) g_{hs} - \sum_\rho e_{(\rho)} [\underset{1}{B}_{(\rho)m} \underset{1}{\Omega}_{(\rho)hn} - \underset{1}{B}_{(\rho)n} \underset{1}{\Omega}_{(\rho)hm}].$$

Если умножить уравнение (2.14) на $a_{\underline{\alpha\beta}} N^\beta_{(\varphi)}$ и учест (0.5), то получается

$$(2.16)\quad \underset{1}{R}_{\beta\pi\mu\nu} N^\beta_{(\varphi)} q^\pi t_m^\mu t_n^\nu - 2\Gamma^u_{\underset{\vee}{mn}} e_{(\varphi)} \underset{1}{B}_{(\varphi)u}$$
$$= e_{(\varphi)} \left\{\underset{1}{A}{}^s_m \underset{1}{\Omega}_{(\varphi)sn} - \underset{1}{A}{}^s_n \underset{1}{\Omega}_{(\varphi)sm} + \underset{1}{B}_{(\varphi)m|n} - \underset{1}{B}_{(\varphi)n|m} + \sum_\sigma [\underset{1}{B}_{(\sigma)m} \underset{1}{\Psi}_{(\varphi\sigma)n} - \underset{1}{B}_{(\sigma)n} \underset{1}{\Psi}_{(\varphi\sigma)m}]\right\}.$$

2.2. Подставим в (2.3) $\mu = \nu = 2$ и используем (2.5), (2.1′). Таким образом получается *второе условие интегрируемости* деривационной формулы (2.1′)

$$(2.17)\quad \underset{2}{R}{}^\alpha_{\pi\mu\nu} q^\pi t_m^\mu t_n^\nu + 2\Gamma^u_{\underset{\vee}{mn}} [\underset{2}{A}{}_u^s t_s^\alpha + \sum_\rho \underset{2}{B}_{(\rho)u} N^\alpha_{(\rho)}]$$
$$= \left\{\underset{2}{A}{}^s_{m|n} - \underset{2}{A}{}^s_{n|m} + \underset{2}{A}{}^p_m \underset{2}{\Phi}{}^s_{pn} - \underset{2}{A}{}^p_n \underset{2}{\Phi}{}^s_{pm} - \sum_\rho e_{(\rho)} g^{\underline{rs}} [\underset{2}{B}_{(\rho)m} \underset{2}{\Omega}_{(\rho)rn} - \underset{2}{B}_{(\rho)n} \underset{2}{\Omega}_{(\rho)rm}]\right\} t_s^\alpha +$$
$$+ \sum_\rho \left\{\underset{2}{A}{}^s_m \underset{2}{\Omega}_{(\rho)sn} - \underset{2}{A}{}^s_n \underset{2}{\Omega}_{(\rho)sm} + \underset{2}{B}_{(\rho)m|n} - \underset{2}{B}_{(\rho)n|m} + \sum_\sigma [\underset{2}{B}_{(\sigma)m} \underset{2}{\Psi}_{(\rho\sigma)n} - \underset{2}{B}_{(\sigma)n} \underset{2}{\Psi}_{(\rho\sigma)m}]\right\} N^\alpha_{(\rho)}.$$

Если это уравнение умножить на $a_{\alpha\beta} t_h^\beta$, то получиться

$$(2.18)\quad \underset{2}{R}_{\beta\pi\mu\nu} t_h^\beta q^\pi t_m^\mu t_n^\nu + 2\Gamma^u_{\underset{\vee}{mn}} \underset{2}{A}{}_u^s g_{hs} = \left(\underset{2}{A}{}^s_{m|n} - \underset{2}{A}{}^s_{n|m} + \underset{2}{A}{}^p_m \underset{2}{\Phi}{}^s_{pn} - \underset{2}{A}{}^p_n \underset{2}{\Phi}{}^s_{pm}\right) g_{\underline{hs}} - \sum_\rho e_{(\rho)} [\underset{2}{B}_{(\rho)m} \underset{2}{\Omega}_{(\rho)hn} - \underset{2}{B}_{(\rho)n} \underset{2}{\Omega}_{(\rho)hm}],$$

а умножением на $a_{\underline{\alpha\beta}} N^\beta_{(\varphi)}$

$$(2.19)\quad \underset{2}{R}_{\beta\pi\mu\nu} N^\beta_{(\varphi)} q^\pi t_m^\mu t_n^\nu + 2\Gamma^u_{\underset{\vee}{mn}} e_{(\varphi)} \underset{2}{B}_{(\varphi)u} =$$
$$= e_{(\varphi)} \left\{\underset{2}{A}{}^s_m \underset{2}{\Omega}_{(\varphi)sn} - \underset{2}{A}{}^s_n \underset{2}{\Omega}_{(\varphi)sm} + \underset{2}{B}_{(\varphi)m|n} - \underset{2}{B}_{(\varphi)n|m} + \sum_\rho [\underset{2}{B}_{(\sigma)m} \underset{2}{\Psi}_{(\varphi\sigma)n} - \underset{2}{B}_{(\sigma)n} \underset{2}{\Psi}_{(\varphi\sigma)m}]\right\}.$$

2.3. Если в (2.3) заместить $\mu = 1$, $\nu = 2$ и использовать (2.6), получается *третье условие интегрируемости* деривационной формулы (2.1′)

$$(2.20)\quad \underset{3}{R}{}^\alpha_{\pi mn} q^\pi = \left\{\underset{1}{A}{}^s_{m\underset{2}{|}n} - \underset{2}{A}{}^s_{n\underset{1}{|}m} + \underset{1}{A}{}^p_m \underset{2}{\Phi}{}^s_{pn} - \underset{2}{A}{}^p_n \underset{1}{\Phi}{}^s_{pm} - \sum_\rho e_{(\rho)} g^{\underline{rs}} [\underset{1}{B}_{(\rho)m} \underset{2}{\Omega}_{(\rho)rn} - \underset{2}{B}_{(\rho)n} \underset{1}{\Omega}_{(\rho)rm}]\right\} t_s^\alpha +$$
$$+ \sum_\rho \left\{\underset{1}{A}{}^s_m \underset{2}{\Omega}_{(\rho)sn} - \underset{2}{A}{}^s_n \underset{1}{\Omega}_{(\rho)sm} + \underset{1}{B}_{(\rho)m\underset{2}{|}n} - \underset{2}{B}_{(\rho)n\underset{1}{|}m} + \sum_1 [\underset{1}{B}_{(\sigma)m} \underset{2}{\Psi}_{(\rho\sigma)n} - \underset{2}{B}_{(\sigma)n} \underset{1}{\Psi}_{(\rho\sigma)m}]\right\} N^\alpha_{(\rho)}.$$

Умножая это уравнение на $\underline{a_{\alpha\beta}}\, t_h^\beta$, получаем

$$(2.21)\qquad \underset{3}{R}_{\beta\pi mn}\, t_h^\beta q^\pi = \big(\underset{1}{A}{}^s_{m\underset{2}{|}n} - \underset{2}{A}{}^s_{n\underset{1}{|}m} + \underset{1}{A}{}^p_m \underset{2}{\Phi}{}^s_{pn} - \underset{2}{A}{}^p_n \underset{1}{\Phi}{}^s_{pm}\big)\, \underline{g_{hs}} -$$

$$- \sum_\rho e_{(\rho)} \big[\underset{1}{B}_{(\rho)m} \underset{2}{\Omega}_{(\rho)hn} - \underset{2}{B}_{(\rho)n} \underset{1}{\Omega}_{(\rho)hm}\big],$$

а на $\underline{a_{\alpha\beta}}\, N^\beta_{(\varphi)}$

$$(2.22)\qquad \underset{3}{R}_{\beta\pi mn} N^\beta_{(\varphi)} q^\pi = e_{(\varphi)} \big\{\underset{1}{A}{}^s_m \underset{2}{\Omega}_{(\varphi)sn} - \underset{2}{A}{}^s_n \underset{1}{\Omega}_{(\varphi)sm} +$$

$$+ \underset{1}{B}_{(\varphi)m\underset{2}{|}n} - \underset{2}{B}_{(\varphi)n\underset{1}{|}m} + \sum_\rho \big[\underset{1}{B}_{(\sigma)m} \underset{2}{\Psi}_{(\varphi\sigma)n} - \underset{2}{B}_{(\sigma)n} \underset{1}{\Psi}_{(\varphi\sigma)m}\big]\big\}.$$

2.4. Если приравнить правую сторону уравнения (2.7) и правую сторону уравнения (2.3) для $\mu=\nu=3$, получается *четвертое условие интегрируемости* деривационной формулы (2.1′)

$$(2.23)\qquad \underset{1}{R}{}^\alpha_{\pi\mu\nu} q^\pi t_m^\mu t_n^\nu + 2\, \underset{\vee}{\Gamma}{}^\mu_{mn} \big[\underset{3}{A}{}^s_u t_s^\alpha + \sum_\rho \underset{3}{B}_{(\rho)u} N^\alpha_{(\rho)}\big] =$$

$$= \big\{\underset{3}{A}{}^s_{m\underset{3}{|}n} - \underset{3}{A}{}^s_{n\underset{3}{|}m} + \underset{3}{A}{}^p_m \underset{3}{\Phi}{}^s_{pn} - \underset{3}{A}{}^p_n \underset{3}{\Phi}{}^s_{pm} - \sum_\rho e_{(\rho)} \underline{g^{rs}} \big[\underset{3}{B}_{(\rho)m} \underset{3}{\Omega}_{(\rho)rn} - \underset{3}{B}_{(\rho)n} \underset{3}{\Omega}_{(\rho)rm}\big]\big\}\, t_s^\alpha +$$

$$\sum_\rho \big\{\underset{3}{A}{}^s_m \underset{3}{\Omega}_{(\rho)sn} - \underset{3}{A}{}^s_n \underset{3}{\Omega}_{(\rho)sm} + \underset{3}{B}_{(\rho)m\underset{3}{|}n} - \underset{3}{B}_{(\rho)n\underset{3}{|}m} + \sum_\sigma \big[\underset{3}{B}_{(\sigma)m} \underset{3}{\Psi}_{(\rho\sigma)n} - \underset{3}{B}_{(\sigma)n} \underset{3}{\Psi}_{(\rho\sigma)m}\big]\big\} N^\alpha_{(\rho)}.$$

Согласно (0.9a), (0.10a), (0.11b), и (2.2e, g), предыдущее уравнение примет вид

$$\underset{1}{R}{}^\alpha_{\pi\mu\nu} q^\pi t_m^\mu t_n^\nu = \big\{\underset{1}{A}{}^s_{n\underset{3}{|}m} - \underset{3}{A}{}^s_{n\underset{1}{|}m} + \underset{1}{A}{}^p_m \big(\underset{1}{\Phi}{}^s_{pn} + 2\, \underset{\vee}{\Gamma}{}^s_{pn}\big) - \underset{1}{A}{}^p_n \big(\underset{1}{\Phi}{}^s_{pm} + 2\, \underset{\vee}{\Gamma}{}^s_{pm}\big) -$$

$$- \sum_\rho e_{(\rho)} \underline{g^{rs}} \big[\underset{1}{B}_{(\rho)m} \underset{1}{\Omega}_{(\rho)rn} - \underset{1}{B}_{(\rho)n} \underset{1}{\Omega}_{(\rho)rm}\big] + 2\, \underset{\vee}{\Gamma}{}^p_{nm} \underset{1}{A}{}^s_p\big\}\, t_s^\alpha +$$

$$+ \sum_\rho \big\{\underset{1}{A}{}^s_m \underset{1}{\Omega}_{(\rho)sn} - \underset{1}{A}{}^s_n \underset{1}{\Omega}_{(\rho)sm} + \underset{1}{B}_{(\rho)m\underset{2}{|}n} - \underset{1}{B}_{(\rho)n\underset{2}{|}m} + \sum_\sigma \big[\underset{1}{B}_{(\sigma)m} \underset{1}{\Psi}_{(\rho\sigma)n} - \underset{1}{B}_{(\sigma)n} \underset{1}{\Psi}_{(\rho\sigma)m}\big] +$$

$$+ 2\, \underset{\vee}{\Gamma}{}^p_{mn} B_{(\rho)p}\big\} N^\alpha_{(\rho)}.$$

Вычислением отмеченных ковариантных производных и упорядочением предыдущего уравнения, получаем

$$\underset{1}{R}{}^\alpha_{\pi\mu\nu} q^\pi t_n^\nu = \big\{\underset{1}{A}{}^s_{m,n} - \underset{1}{A}{}^s_{n,m} + 2\, \underset{1}{A}{}^p_m \Gamma^s_{pn} - 2\, \underset{1}{A}{}^p_n \Gamma^s_{pm} + \underset{1}{A}{}^p_m \underset{1}{\Phi}{}^s_{pn} - \underset{1}{A}{}^p_n \underset{1}{\Phi}{}^s_{pm} -$$

$$(2.23')\qquad - \underset{1}{A}{}^p_m \Gamma^s_{np} + \underset{1}{A}{}^p_n \Gamma^s_{mp} - \sum_\rho e_{(\rho)} \underline{g^{rs}} \big[\underset{1}{B}_{(\rho)m} \underset{1}{\Omega}_{(\rho)rn} - \underset{1}{B}_{(\rho)n} \underset{1}{\Omega}_{(\rho)rm}\big]\big\}\, t_s^\alpha +$$

$$+ \sum_\rho \big\{\underset{1}{A}{}^s_m \underset{1}{\Omega}_{(\rho)sn} - \underset{1}{A}{}^s_n \underset{1}{\Omega}_{(\rho)sm} + \underset{1}{B}_{(\rho)m,n} - \underset{1}{B}_{(\rho)n,m} + \sum_\rho \big[\underset{1}{B}_{(\sigma)m} \underset{1}{\Psi}_{(\rho\sigma)n} - \underset{1}{B}_{(\sigma)n} \underset{1}{\Psi}_{(\rho\sigma)m}\big]\big\} N^\alpha_{(\rho)}.$$

С другой стороны, из (2.14) таким же способом получается

(2.14′)
$$\underset{1}{R}{}^{\alpha}{}_{\pi\mu\nu}\, q^{\pi}\, t^{\mu}_{m}\, t^{\nu}_{n} = \Big\{\underset{1}{A}{}^{s}_{m,n} - \underset{1}{A}{}^{s}_{n,m} + \Gamma^{s}_{pn}\, \underset{1}{A}{}^{p}_{m} - \Gamma^{s}_{pm}\, \underset{1}{A}{}^{p}_{n} + \underset{1}{A}{}^{p}_{m}\, \underset{1}{\Phi}{}^{s}_{pn} -$$
$$- \underset{1}{A}{}^{p}_{n}\, \underset{1}{\Phi}{}^{s}_{pm} - \sum_{\rho} e_{(\rho)}\, g^{rs} \big[\underset{1}{B}{}_{(\rho)m}\, \underset{1}{\Omega}{}_{(\rho)rn} - \underset{1}{B}{}_{(\rho)n}\, \underset{1}{\Omega}{}_{(\rho)rm}\big]\Big\}\, t^{\alpha}_{s} +$$
$$+ \sum_{\rho} \Big\{\underset{1}{A}{}^{s}_{m}\, \underset{1}{\Omega}{}_{(\rho)sn} - \underset{1}{A}{}^{s}_{n}\, \underset{1}{\Omega}{}_{(\rho)sm} + \underset{1}{B}{}_{(\rho)m,n} - \underset{1}{B}{}_{(\rho)n,m} + \sum_{\sigma} \big[\underset{1}{B}{}_{(\sigma)m}\, \underset{1}{\Psi}{}_{(\rho\sigma)n} - \underset{1}{B}{}_{(\sigma)n}\, \underset{1}{\Psi}{}_{(\rho\sigma)m}\big]\Big\}\, N^{\alpha}_{(\rho)}.$$

Вычитанием (2.14′) от (2.23′) получается

$$\big[\underset{1}{A}{}^{p}_{m}\,(\Gamma^{s}_{pn} - \Gamma^{s}_{np}) - \underset{1}{A}{}^{p}_{n}\,(\Gamma^{s}_{pm} - \Gamma^{s}_{mp})\big]\, t^{\alpha}_{s} = 0$$

т.е.

(2.24)
$$\underset{1}{A}{}^{p}_{m}\, \underset{\vee}{\Gamma}{}^{s}_{pn} = \underset{1}{A}{}^{p}_{n}\, \underset{\vee}{\Gamma}{}^{s}_{pm},$$

а это значить что тензор

(2.25)
$$\underset{1}{A}{}^{i}_{mn} = \underset{1}{A}{}^{p}_{m}\, \underset{\vee}{\Gamma}{}^{i}_{pn}$$

симметрический по отношению к индексам m, n.

Итак, уравнения (2.14) и (2.23) равносильны и в силу (2.24) сводятся к (2.14′).

2.5. Из (2.8) и (2.3) для $\mu = \nu = 4$ получаем *пятое условие интегрируемости* деривационной формулы (2.1′)

(2.26)
$$\underset{2}{R}{}^{\alpha}{}_{\pi\mu\nu}\, q^{\pi}\, t^{\mu}_{m}\, t^{\nu}_{n} - 2\, \underset{\vee}{\Gamma}{}^{p}_{mn} \big[\underset{4}{A}{}^{s}_{p}\, t^{\alpha}_{s} + \sum_{\rho} \underset{4}{B}{}_{(\rho)p}\, N^{\alpha}_{(\rho)}\big] =$$
$$= \Big\{\underset{4}{A}{}^{s}_{m\underset{4}{|}n} - \underset{4}{A}{}^{s}_{n\underset{4}{|}m} + \underset{4}{A}{}^{p}_{m}\, \underset{4}{\Phi}{}^{s}_{pn} - \underset{4}{A}{}^{p}_{n}\, \underset{4}{\Phi}{}^{s}_{pm} - \sum_{\rho} e_{(\rho)}\, g^{rs} \big[\underset{4}{B}{}_{(\rho)m}\, \underset{4}{\Omega}{}_{(\rho)rn} - \underset{4}{B}{}_{(\rho)n}\, \underset{4}{\Omega}{}_{(\rho)rm}\big]\Big\}\, t^{\alpha}_{s} +$$
$$+ \sum_{\rho} \Big\{\underset{4}{A}{}^{s}_{m}\, \underset{4}{\Omega}{}_{(\rho)sn} - \underset{4}{A}{}^{s}_{n}\, \underset{4}{\Omega}{}_{(\rho)sm} + \underset{4}{B}{}_{(\rho)m\underset{4}{|}n} - \underset{4}{B}{}_{(\rho)n\underset{4}{|}m} + \sum_{\sigma} \big[\underset{4}{B}{}_{(\sigma)m}\, \underset{4}{\Psi}{}_{(\rho\sigma)n} - \underset{4}{B}{}_{(\sigma)n}\, \underset{4}{\Psi}{}_{(\rho\sigma)m}\big]\Big\}\, N^{\alpha}_{(\rho)}.$$

Пользуясь (0.9b), (0.10b), (0.11c) и (2.2 f, h), предыдущее уравнение можно написать в форме

$$\underset{2}{R}{}^{\alpha}{}_{\pi\mu\nu}\, q^{\pi}\, t^{\mu}_{m}\, t^{\nu}_{n} = \Big\{\underset{2}{A}{}^{s}_{m\underset{4}{|}n} - \underset{2}{A}{}^{s}_{n\underset{4}{|}m} + \underset{2}{A}{}^{p}_{m}\,\big(-\underset{1}{\Phi}{}^{s}_{pu} - 2\, \underset{\vee}{\Gamma}{}^{s}_{pn}\big) - \underset{2}{A}{}^{p}_{n}\,\big(-\underset{1}{\Phi}{}^{s}_{pm} - 2\, \underset{\vee}{\Gamma}{}^{s}_{pm}\big) -$$
$$- \sum_{\rho} e_{(\rho)}\, g^{rs} \big[\underset{2}{B}{}_{(\rho)m}\, \underset{2}{\Omega}{}_{(\rho)rn} - \underset{2}{B}{}_{(\rho)n}\, \underset{2}{\Omega}{}_{(\rho)rm}\big] + 2\, \underset{\vee}{\Gamma}{}^{p}_{mn}\, \underset{2}{A}{}^{s}_{p}\Big\}\, t^{\alpha}_{s} + \sum_{\rho} \Big\{2\, \underset{\vee}{\Gamma}{}^{p}_{mn}\, \underset{2}{B}{}_{(\rho)p} +$$
$$+ \underset{2}{A}{}^{s}_{m}\, \underset{2}{\Omega}{}_{(\rho)sn} - \underset{2}{A}{}^{s}_{n}\, \underset{2}{\Omega}{}_{(\rho)sm} + \underset{2}{B}{}_{(\rho)m\underset{1}{|}n} - \underset{2}{B}{}_{(\rho)n\underset{1}{|}m} + \sum_{\sigma} \big[\underset{2}{B}{}_{(\sigma)m}\, \underset{2}{\Psi}{}_{(\rho\sigma)n} - \underset{2}{B}{}_{(\sigma)n}\, \underset{2}{\Psi}{}_{(\rho\sigma)m}\big]\Big\}\, N^{\alpha}_{(\rho)},$$

т.с.

$$\underset{2}{R}{}^{\alpha}{}_{\pi\mu\nu}\, q^{\pi}\, t^{\mu}_{m}\, t^{\nu}_{n} = \Big\{ \underset{2}{A}{}^{s}_{m,\,n} - \underset{2}{A}{}^{s}_{n,\,m} + 2\,\Gamma^{s}_{np}\,\underset{2}{A}{}^{p}_{m} - \underset{2}{A}{}^{p}_{m}\,\Gamma^{s}_{pn} - 2\,\Gamma^{s}_{mp}\,\underset{2}{A}{}^{p}_{n} + \underset{2}{A}{}^{p}_{n}\,\Gamma^{s}_{pm} -$$

$$(2.26')\qquad - \underset{2}{A}{}^{p}_{m}\,\underset{1}{\Phi}{}^{s}_{pn} + \underset{2}{A}{}^{p}_{n}\,\underset{1}{\Phi}{}^{s}_{pm} - \sum_{\rho} e_{(\rho)}\, g^{rs}\, [\underset{2}{B}_{(\rho)m}\,\underset{2}{\Omega}_{(\rho)rn} - \underset{2}{B}_{(\rho)n}\,\underset{2}{\Omega}_{(\rho)rm}] \Big\}\, t^{\alpha}_{s} +$$

$$+ \sum_{\rho} \Big\{ \underset{2}{A}{}^{s}_{m}\,\underset{2}{\Omega}_{(\rho)sn} - \underset{2}{A}{}^{s}_{n}\,\underset{2}{\Omega}_{(\rho)sm} + \underset{2}{B}_{(\rho)m,\,n} - \underset{2}{B}_{(\rho)n,\,m} + \sum_{\sigma} [\underset{2}{B}_{(\sigma)m}\,\underset{2}{\Psi}_{(\rho\sigma)n} - \underset{2}{B}_{(\sigma)n}\,\underset{2}{\Psi}_{(\rho\sigma)m}] \Big\}\, N^{\alpha}_{(\rho)}.$$

Если применить предыдущий способ на (2.17), то получается

$$\underset{2}{R}{}^{\alpha}{}_{\pi\mu\nu}\, q^{\pi}\, t^{\mu}_{m}\, t^{\nu}_{n} = \Big\{ \underset{2}{A}{}^{s}_{m,\,n} + \Gamma^{s}_{np}\,\underset{2}{A}{}^{p}_{m} - \underset{2}{A}{}^{s}_{n,\,m} - \Gamma^{s}_{mp}\,\underset{2}{A}{}^{p}_{n} - \underset{2}{A}{}^{p}_{m}\,\underset{1}{\Phi}{}^{s}_{pn} + \underset{2}{A}{}^{p}_{n}\,\underset{1}{\Phi}{}^{s}_{pm} -$$

$$(2.17')\qquad - \sum_{\rho} e_{(\rho)}\, g^{rs}\, [\underset{2}{B}_{(\rho)m}\,\underset{2}{\Omega}_{(\rho)rn} - \underset{2}{B}_{(\rho)n}\,\underset{2}{\Omega}_{(\rho)rm}] \Big\}\, t^{\alpha}_{s} +$$

$$+ \sum_{\rho} \Big\{ \underset{2}{A}{}^{s}_{m}\,\underset{2}{\Omega}_{(\rho)sn} - \underset{2}{A}{}^{s}_{n}\,\underset{2}{\Omega}_{(\rho)sm} + \underset{2}{B}_{(\rho)m,\,n} - \underset{2}{B}_{(\rho)n,\,m} + \sum_{\sigma} [\underset{2}{B}_{(\sigma)m}\,\underset{2}{\Psi}_{(\rho\sigma)n} - \underset{2}{B}_{(\sigma)n}\,\underset{2}{\Psi}_{(\rho\sigma)m}] \Big\}\, N^{\alpha}_{(\rho)}.$$

Вычитая (2.17′) от (2.26′), получим

$$\Big\{ \underset{2}{A}{}^{p}_{m}\,\underset{\vee}{\Gamma}{}^{s}_{np} - \underset{2}{A}{}^{p}_{n}\,\underset{\vee}{\Gamma}{}^{s}_{mp} \Big\}\, t^{\alpha}_{s} = 0,$$

откуда

$$(2.27)\qquad \underset{2}{A}{}^{p}_{m}\,\underset{\vee}{\Gamma}{}^{s}_{np} = \underset{2}{A}{}^{p}_{n}\,\underset{\vee}{\Gamma}{}^{s}_{mp}\,,$$

т.е. тензор

$$(2.28)\qquad \underset{2}{A}{}^{i}_{mn} = \underset{2}{A}{}^{p}_{m}\,\underset{\vee}{\Gamma}{}^{i}_{pn}$$

симметрический по отношению к индексам m, n. Согласно (2.24,27) видно что тензоры

$$\underset{\mu}{A}{}^{i}_{mn} = \underset{\mu}{A}{}^{p}_{m}\,\underset{\vee}{\Gamma}{}^{i}_{pn} \qquad (\mu = 1,\ 2)$$

симметрические по отношению к m, n.

Итак, уравнения (2.16), (2.26) равносильныи и в силу (2.27) сводятся к (2.17′).

2.6. Согласно (2.9) и (2.3) для $\mu = 3$, $\nu = 4$, получается *шестое условие интегрируемости* деривационной формулы (2.1′)

$$\underset{4}{R}{}^{\alpha}{}_{\pi mn}\, q^{\pi} = \Big\{ \underset{1}{A}{}^{s}_{m\,|\,\underset{4}{n}} - \underset{2}{A}{}^{s}_{n\,|\,\underset{3}{m}} + \underset{1}{A}{}^{p}_{m}\,\underset{4}{\Phi}{}^{s}_{pn} - \underset{2}{A}{}^{p}_{n}\,\underset{3}{\Phi}{}^{s}_{pm} -$$

$$(2.29)\qquad - \sum_{\rho} e_{(\rho)}\, g^{rs}\, [\underset{1}{B}_{(\rho)m}\,\underset{2}{\Omega}_{(\rho)rn} - \underset{2}{B}_{(\rho)n}\,\underset{2}{\Omega}_{(\rho)rm}] \Big\}\, t^{\alpha}_{s} +$$

$$+ \sum_{\rho} \Big\{ \underset{1}{A}{}^{s}_{m}\,\underset{2}{\Omega}_{(\rho)sn} - \underset{2}{A}{}^{s}_{n}\,\underset{1}{\Omega}_{(\rho)sm} + \underset{1}{B}_{(\rho)m\,|\,\underset{1}{n}} - \underset{2}{B}_{(\rho)n\,|\,\underset{2}{m}} + \sum_{\sigma} [\underset{1}{B}_{(\sigma)m}\,\underset{2}{\Psi}_{(\rho\sigma)n} - \underset{2}{B}_{(\sigma)n}\,\underset{1}{\Psi}_{(\rho\sigma)m}] \Big\}\, N^{\alpha}_{(\rho)}.$$

Если это уравнение умножить на $a_{\underset{\cdot\cdot}{\alpha\beta}} t_h^\beta$, получается

$$
\begin{aligned}
(2.30)\qquad & \underset{4}{R}_{\beta\pi mn} t_h^\beta q^\pi = \big(\underset{1}{A}{}^s_{m}{}_{|\underset{4}{n}} - \underset{2}{A}{}^s_{n}{}_{|\underset{3}{m}} + \underset{1}{A}{}^p_m \underset{4}{\Phi}{}^s_{pn} - \underset{2}{A}{}^p_n \underset{3}{\Phi}{}^s_{pm}\big) g_{\underline{hs}} - \\
& - \sum_\rho e_{(\rho)} [\underset{1}{B}_{(\rho)m} \underset{2}{\Omega}_{(\rho)hn} - \underset{2}{B}_{(\rho)n} \underset{1}{\Omega}_{(\rho)hm}],
\end{aligned}
$$

а если умножить на $a_{\underset{\cdot\cdot}{\alpha\beta}} N^\beta_{(\varphi)}$

$$
\begin{aligned}
(2.31)\qquad & \underset{4}{R}_{\beta\pi mn} N^\beta_{(\varphi)} q^\pi = e_{(\varphi)} \{\underset{1}{A}{}^s_m \underset{2}{\Omega}_{(\varphi)sn} - \underset{2}{A}{}^s_n \underset{1}{\Omega}_{(\varphi)sm} + \\
& + \underset{1}{B}_{(\varphi)m|\underset{1}{n}} - \underset{2}{B}_{(\varphi)n|\underset{2}{m}} + \sum_\rho [\underset{1}{B}_{(\sigma)m} \underset{2}{\Psi}_{(\varphi\sigma)n} - \underset{2}{B}_{(\sigma)n} \underset{1}{\Psi}_{(\varphi\sigma)m}]\}.
\end{aligned}
$$

Замечание. Уравнения (2.30, 31) как и соответствующие уравнения получающиеся из предыдущих условий интегрируемости — *уравнения типпа Гаусса и Петерсона-Кодаци.*

3. Особые случаи

3.1.0. Если кривая C, вектор кривизны q^α которой рассматриваем, *асимптотическая линия* в V_M, тогда $K_{(\rho)} = 0$, для $\rho = M+1, \ldots, N$ и отсюда и от (1.9) и (2.2) имеем

$$
(3.1a,\ b)\qquad K_{(\rho)} = 0,\quad q^\alpha = t_i^\alpha p^i,
$$

$$
(3.2a,\ b)\qquad \underset{\mu}{A}{}^s_m = \underset{\mu}{\Phi}{}^s_{im} p^i + p^s_{|\underset{\mu}{m}},\ \underset{\mu}{B}_{(\rho)} = \underset{\mu}{\Omega}_{(\rho)im} p^i.
$$

3.1.1. При предыдущих условиях, т.е. для асимпточеской линии, первое условие интегрируемости (2.14) принимает вид

$$
\begin{aligned}
& \underset{1}{R}{}^\alpha_{\pi\mu\nu} t_i^\pi p^i t_m^\mu t_n^\nu - 2\underset{\vee}{\Gamma}{}^u_{mn} \big[\big(\underset{1}{\Phi}{}^s_{iu} p^i + p^s_{|\underset{1}{u}}\big) t_s^\alpha + \sum_\rho \underset{1}{\Omega}_{(\rho)iu} p^i N^\alpha_{(\rho)}\big] = \\
& = \big\{\underset{1}{\Phi}{}^s_{im}{}_{|\underset{1}{n}} p^i + \underset{1}{\Phi}{}^s_{im} + p^i_{|\underset{1}{n}} + p^s_{|\underset{1}{m}|\underset{1}{n}} - \underset{1}{\Phi}{}^s_{in}{}_{|\underset{1}{m}} - \underset{1}{\Phi}{}^s_{in} p^i_{|\underset{1}{m}} - p^s_{|\underset{1}{n}|\underset{1}{m}} + \big(\underset{1}{\Phi}{}^r_{im} p^i + p^r_{|\underset{1}{m}}\big) \underset{1}{\Phi}{}^s_{rn} - \\
& - \big(\underset{1}{\Phi}{}^r_{in} p^i + p^r_{|\underset{1}{n}}\big) \underset{1}{\Phi}{}^s_{rm} - \sum_\rho e_{(\rho)} g^{\underline{rs}} [\underset{1}{\Omega}_{(\rho)im} p^i \underset{1}{\Omega}_{(\rho)rn} - \underset{1}{\Omega}_{(\rho)in} p^i \underset{1}{\Omega}_{(\rho)rm}]\big\} t_s^\alpha + \\
& + \sum_\rho \big\{\big(\underset{1}{\Phi}{}^s_{im} p^i + p^s_{|\underset{1}{m}}\big) \underset{1}{\Omega}_{(\rho)sn} - \big(\underset{1}{\Phi}{}^s_{in} p^i + p^s_{|\underset{1}{n}}\big) \underset{1}{\Omega}_{(\rho)sm} + \\
& + \underset{1}{\Omega}_{(\rho)im|\underset{1}{n}} p^i + \underset{1}{\Omega}_{(\rho)im} p^i_{|\underset{1}{n}} - \underset{1}{\Omega}_{(\rho)in|\underset{1}{m}} p^i - \underset{1}{\Omega}_{(\rho)in} p^i_{|\underset{1}{m}} + \\
& + \sum_\sigma [\underset{1}{\Omega}_{(\sigma)im} p^i \underset{1}{\Psi}_{(\rho\sigma)n} - \underset{1}{\Omega}_{(\sigma)in} p^i \underset{1}{\Psi}_{(\rho\sigma)m}]\big\} N^\alpha_{(\rho)}.
\end{aligned}
$$

Так как согласно (7) в [2]:

$$p^{s}_{\underset{1}{|}m\underset{1}{|}n}-p^{s}_{\underset{1}{|}n\underset{1}{|}m}=\underset{1}{R}{}^{s}_{imn}p^{i}-2\,\underset{\vee}{\Gamma}{}^{r}_{mn}\,p^{s}_{\underset{1}{|}r},$$

то предыдущее уравнение, после вычисления ковариантных производных и упорядочения, примет вид

$$\begin{aligned}\underset{1}{R}{}^{\alpha}_{\pi\mu\nu}\,t^{\pi}_{i}\,t^{\mu}_{m}\,t^{\nu}_{n}\,p^{i}=\Big\{&\underset{1}{R}{}^{s}_{imn}+\underset{1}{\Phi}{}^{s}_{im,n}-\underset{1}{\Phi}{}^{s}_{in,m}+\underset{1}{\Phi}{}^{r}_{im}\Gamma^{s}_{rn}-\underset{1}{\Phi}{}^{r}_{in}\Gamma^{s}_{rm}-\\&-\underset{1}{\Phi}{}^{s}_{rm}\Gamma^{r}_{in}+\underset{1}{\Phi}{}^{s}_{rn}\Gamma^{r}_{im}+\underset{1}{\Phi}{}^{r}_{im}+\underset{1}{\Phi}{}^{s}_{rn}-\underset{1}{\Phi}{}^{r}_{in}\underset{1}{\Phi}{}^{s}_{rm}+\\&+\sum_{\rho}e_{(\rho)}\,g^{rs}\,[\underset{1}{\Omega}_{(\rho)im}\underset{1}{\Omega}_{(\rho)rn}-\underset{1}{\Omega}_{(\rho)in}\underset{1}{\Omega}_{(\rho)rm}]\Big\}\,p^{i}\,t^{\alpha}_{s}+\\&+\sum_{\rho}\Big\{\underset{1}{\Phi}{}^{s}_{im}\underset{1}{\Omega}_{(\rho)sn}-\underset{1}{\Phi}{}^{s}_{in}\underset{1}{\Omega}_{(\rho)sm}+\underset{1}{\Omega}_{(\rho)im,n}-\underset{1}{\Omega}_{(\rho)in,m}+\\&+\Gamma^{r}_{im}\underset{1}{\Omega}_{(\rho)rn}-\Gamma^{r}_{in}\underset{1}{\Omega}_{(\rho)rm}+\sum_{\sigma}[\underset{1}{\Omega}_{(\sigma)im}\underset{1}{\Psi}_{(\rho\sigma)n}-\underset{1}{\Omega}_{(\sigma)in}\underset{1}{\Psi}_{(\rho\sigma)m}]\Big\}\,p^{i}\,N^{\alpha}_{(\rho)}.\end{aligned}\tag{3.3}$$

3.1.2. При условиях (3.1.2) уравнение (2.17) примет вид

$$\begin{aligned}\underset{2}{R}{}^{\alpha}_{\pi\mu\nu}\,t^{\pi}_{i}\,t^{\mu}_{m}\,t^{\nu}_{n}\,p^{i}=\Big\{&\underset{2}{R}{}^{s}_{imn}+\underset{2}{\Phi}{}^{s}_{in,m}-\underset{2}{\Phi}{}^{r}_{im,n}+\underset{2}{\Phi}{}^{r}_{im}\Gamma^{s}_{nr}-\underset{2}{\Phi}{}^{r}_{in}\Gamma^{s}_{mr}-\\&-\underset{2}{\Phi}{}^{s}_{rm}\Gamma^{r}_{ni}+\underset{2}{\Phi}{}^{s}_{rn}\Gamma^{r}_{mi}+\underset{2}{\Phi}{}^{r}_{im}\underset{2}{\Phi}{}^{s}_{rn}-\underset{2}{\Phi}{}^{r}_{in}\underset{2}{\Phi}{}^{s}_{rm}+\\&+\sum_{\rho}e_{(\rho)}\,g^{rs}\,[\underset{2}{\Omega}_{(\rho)im}\underset{2}{\Omega}_{(\rho)rn}-\underset{2}{\Omega}_{(\rho)in}\underset{2}{\Omega}_{(\rho)rm}]\Big\}\,p^{i}\,t^{\alpha}_{s}+\\&+\sum_{\rho}\Big\{\underset{2}{\Phi}{}^{s}_{im}\underset{2}{\Omega}_{(\rho)sn}-\underset{2}{\Phi}{}^{s}_{in}\underset{2}{\Omega}_{(\rho)sm}+\underset{2}{\Omega}_{(\rho)im,n}-\underset{2}{\Omega}_{(\rho)in,m}+\\&+\Gamma^{r}_{mi}\underset{2}{\Omega}_{(\rho)rn}-\Gamma^{r}_{ni}\underset{2}{\Omega}_{(\rho)rm}+\sum_{\sigma}[\underset{2}{\Omega}_{(\sigma)im}\underset{2}{\Psi}_{(\rho\sigma)n}-\underset{2}{\Omega}_{(\sigma)in}\underset{2}{\Psi}_{(\rho\sigma)m}]\Big\}\,p^{i}\,N^{\alpha}_{(\rho)},\end{aligned}\tag{3.4}$$

что доказывается таким же способом как в предыдущем случае.

3.1.3. Уравнение (2.20) при условиях (3.1, 2) будет

$$\begin{aligned}\underset{3}{R}{}^{\alpha}_{\pi mn}\,t^{\pi}_{i}\,p^{i}=\Big\{&\underset{1}{\Phi}{}^{s}_{im\underset{2}{|}n}p^{i}+\underset{1}{\Phi}{}^{s}_{im}\,p^{i}_{\underset{2}{|}n}+p^{s}_{\underset{1}{|}m\underset{2}{|}n}-\underset{2}{\Phi}{}^{s}_{in\underset{1}{|}m}p^{i}-\underset{2}{\Phi}{}^{s}_{in}\,p^{i}_{\underset{1}{|}m}-\\&-p^{s}_{\underset{2}{|}n\underset{1}{|}m}+\big(\underset{1}{\Phi}{}^{r}_{im}p^{i}+p^{r}_{\underset{1}{|}m}\big)\underset{2}{\Phi}{}^{s}_{rn}-\big(\underset{2}{\Phi}{}^{r}_{in}p^{i}+p^{r}_{\underset{2}{|}n}\big)\underset{1}{\Phi}{}^{s}_{rm}-\\&-\sum_{\rho}e_{(\rho)}\,g^{rs}\,[\underset{1}{\Omega}_{(\rho)im}\,p^{i}\,\underset{2}{\Omega}_{(\rho)rn}-\underset{2}{\Omega}_{(\rho)in}\,p^{i}\,\underset{1}{\Omega}_{(\rho)rm}]\Big\}\,t^{\alpha}_{s}+\\&+\sum_{\rho}\Big\{\big(\underset{1}{\Phi}{}^{s}_{im}\,p^{i}+p^{s}_{\underset{1}{|}m}\big)\underset{2}{\Omega}_{(\rho)sn}-\big(\underset{2}{\Phi}{}^{s}_{in}\,p^{i}+p^{s}_{\underset{2}{|}n}\big)\underset{1}{\Omega}_{(\rho)sm}+\underset{1}{\Omega}_{(\rho)im\underset{2}{|}n}\,p^{i}+\end{aligned}\tag{3.5}$$

$$+\underset{1}{\Omega}_{(\rho)im}\,\underset{2}{p}^{i}_{|n} - \underset{2\,1}{\Omega}_{(\rho)in|m}\, p^i - \underset{2}{\Omega}_{(\rho)in}\,\underset{1}{p}^{i}_{|m} +$$

$$+\sum_{\sigma}[\underset{1}{\Omega}_{(\sigma)im}\, p^i\, \underset{2}{\Psi}_{(\rho\sigma)n} - \underset{2}{\Omega}_{(\sigma)in}\, p^i\, \underset{1}{\Psi}_{(\rho\sigma)m}]\}\, N^{\alpha}_{(\rho)}.$$

Так как согласно уравнению (56) в [2]

$$\underset{1\,2}{p^{s}_{|m|n}} - \underset{2\,1}{p^{s}_{|n|m}} = \underset{3}{R}^{s}{}_{rmn}\, p^r,$$

где

(3.6) $$\underset{3}{R}^{i}_{jmn} = \Gamma^{i}_{jm,n} - \Gamma^{i}_{nj,m} + \Gamma^{p}_{jm}\,\Gamma^{i}_{np} - \Gamma^{p}_{nj}\,\Gamma^{i}_{pm} + \Gamma^{p}_{nm}\,(\Gamma^{i}_{pj} - \Gamma^{i}_{jp})$$

тензор кривизны третьего рода подпространства V_M (см. ур. (55b, 52) в [2]), то уравнение (3.5) приводится к виду

(3.5′) $$\underset{3}{R}^{\alpha}{}_{\pi mn}\, t^{\pi}_{i}\, p^i = \{\underset{3}{R}^{s}_{imn} + \underset{1\,2}{\Phi}^{s}_{im|n} - \underset{2\,1}{\Phi}^{s}_{in|m} + \underset{1}{\Phi}^{r}_{im}\,\underset{2}{\Phi}^{s}_{rn} - \underset{2}{\Phi}^{r}_{in}\,\underset{1}{\Phi}^{s}_{rm} -$$

$$-\sum_{\rho} e_{(\rho)}\, g^{rs}\,[\underset{1}{\Omega}_{(\rho)im}\, p^i\, \underset{2}{\Omega}_{(\rho)rn} - \underset{2}{\Omega}_{(\rho)in}\, p^i\, \underset{1}{\Omega}_{(\rho)rm}]\}\, p^i\, t^{\alpha}_{s} +$$

$$+\sum_{\rho}\{\underset{1}{\Phi}^{s}_{im}\,\underset{2}{\Omega}_{(\rho)sn} - \underset{2}{\Phi}^{s}_{in}\,\underset{1}{\Omega}_{(\rho)sm} + \underset{1\,2}{\Omega}_{(\rho)im|n} - \underset{2\,1}{\Omega}_{(\rho)in|m} +$$

$$+\sum_{\sigma}[\underset{1}{\Omega}_{(\sigma)im}\,\underset{2}{\Psi}_{(\rho\sigma)n} - \underset{2}{\Omega}_{(\sigma)in}\,\underset{1}{\Psi}_{(\rho\sigma)m}]\}\, p^i\, N^{\alpha}_{(\rho)}.$$

3.1.4. Как нами установлено выше, в 2.4., уравнения (2.23) и (2.14) равносильны и оба сводятся к уравнению (2.14′), а уравнение (2.14) при условии $K_{(\rho)} = 0$, т.е. при условиях (3.1, 2) сводится к уравнению (3.3). Из этого следует, что в случае асимптотической линии уравнение (2.23) сводится к (3.3).

3.1.5. Тем же способом как в предыдущем случае, заключается, что уравнения (2.17) и (2.26) в случае асимптотической линии сводятся к уравнению (3.4).

3.1.6. Уравнение (2.29) в случае асимптотической линии подпространства V_M приводится к виду

(3.7) $$\underset{4}{R}^{\alpha}{}_{\pi mn}\, t^{\pi}_{i}\, p^i = \{\underset{1\,4}{\Phi}^{s}_{im|n}\, p^i + \underset{1}{\Phi}^{s}_{im}\,\underset{4}{p}^{i}_{|n} + \underset{1\,4}{p}^{s}_{|m|n} - \underset{2}{\Phi}^{s}_{in}\,\underset{3}{p}^{i}_{|m} - \underset{2\,3}{\Phi}^{s}_{in|m}\, p^i - \underset{2\,3}{p}^{s}_{|n|m} +$$

$$+(\underset{1}{\Phi}^{r}_{im}\, p^i + \underset{1}{p}^{r}_{|m})\,\underset{4}{\Phi}^{s}_{rn} - (\underset{2}{\Phi}^{r}_{in}\, p^i + \underset{2}{p}^{r}_{|n})\,\underset{3}{\Phi}^{s}_{rm} - \sum_{\rho} e_{(\rho)}\, g^{rs}\,[\underset{1}{\Omega}_{(\rho)im}\,\underset{2}{\Omega}_{(\rho)rn} -$$

$$-\underset{2}{\Omega}_{(\rho)in}\,\underset{1}{\Omega}_{(\rho)rm}]\, p^i\}\, t^{\alpha}_{s} + \sum_{\rho}\{(\underset{1}{\Phi}^{s}_{im}\, p^i + \underset{1}{p}^{s}_{|m})\,\underset{2}{\Omega}_{(\rho)sn} - (\underset{2}{\Phi}^{s}_{in}\, p^i + \underset{2}{p}^{s}_{|n})\,\underset{1}{\Omega}_{(\rho)sm} +$$

$$+\underset{1\,1}{\Omega}_{(\rho)im|n}\, p^i + \underset{1}{\Omega}_{(\rho)im}\,\underset{1}{p}^{i}_{|n} - \underset{2\,2}{\Omega}_{(\rho)in|m}\, p^i - \underset{2}{\Omega}_{(\rho)in}\,\underset{2}{p}^{i}_{|m} +$$

$$+\sum_{\rho}[\underset{1}{\Omega}_{(\sigma)im}\,\underset{2}{\Psi}_{(\rho\sigma)n} - \underset{2}{\Omega}_{(\sigma)in}\,\underset{1}{\Psi}_{(\rho\sigma)m}]\, p^i\}\, N^{\alpha}_{(\rho)}.$$

Так как согласно уравнению (46) в [3]

$$\underset{1\ 4}{p^{s}_{|m|n}} - \underset{2\ 3}{p^{s}_{|n|m}} = \underset{3\ 4}{p^{s}_{|m|n}} - \underset{4\ 3}{p^{s}_{|n|m}} = \underset{4}{R^{s}}{}_{rmn}\, p^{r},$$

где

$$(3.8) \qquad \underset{4}{R^{i}}{}_{jmn} = \Gamma^{i}_{jm,n} - \Gamma^{i}_{nj,m} + \Gamma^{p}_{jm}\Gamma^{i}_{np} - \Gamma^{p}_{nj}\Gamma^{i}_{pm} + \Gamma^{p}_{mn}\left(\Gamma^{i}_{pj} - \Gamma^{i}_{jp}\right)$$

тензор кривизны четвертого рода подпространства V_M (см. ур. (45), (48) в [3], то принимая во внимание и (0.11), уравнение (3.7) получает вид

$$(3.7') \qquad \underset{4}{R^{\alpha}}{}_{\pi mn}\, t^{\pi}_{i}\, p^{i} = \Big\{ \underset{1\ 4}{\Phi^{s}_{im|n}}\, p^{i} - \underset{2\ 3}{\Phi^{s}_{in|m}}\, p^{i} + \underset{4}{R^{s}}{}_{rmn}\, p^{r} +$$

$$+ \underset{1}{\Phi^{r}_{im}}\, \underset{4}{\Phi^{s}_{rn}}\, p^{i} - \underset{2}{\Phi^{r}_{in}}\, \underset{3}{\Phi^{s}_{rm}}\, p^{i} - 2\, \underset{1}{p^{r}_{|m}}\, \underset{\vee}{\Gamma^{s}_{rn}} - 2\, \underset{2}{p^{r}_{|n}}\, \underset{\vee}{\Gamma^{s}_{rm}} -$$

$$- \sum_{\rho} e_{(\rho)}\, g^{rs} \left[\underset{1}{\Omega_{(\rho)im}}\, \underset{2}{\Omega_{(\rho)rn}} - \underset{2}{\Omega_{(\rho)in}}\, \underset{1}{\Omega_{(\rho)rm}} \right] p^{i} \Big\}\, t^{\alpha}_{s} +$$

$$+ \sum_{\rho} \Big\{ \left(\underset{1}{\Phi^{s}_{im}}\, p^{i} + \underset{1}{p^{s}_{|m}} \right) \underset{2}{\Omega_{(\rho)sn}} - \left(\underset{2}{\Phi^{s}_{in}} + \underset{2}{p^{s}_{|n}} \right) \underset{1}{\Omega_{(\rho)sm}} + \underset{1\ 1}{\Omega_{(\rho)im|n}}\, p^{i} + \underset{1}{\Omega_{(\rho)im}}\, \underset{1}{p^{i}_{|n}} -$$

$$- \underset{2\ 2}{\Omega_{(\rho)in|m}}\, p^{i} - \underset{2}{\Omega_{(\rho)in}}\, \underset{2}{p^{i}_{|m}} + \sum_{\rho} \left[\underset{1}{\Omega_{(\sigma)im}}\, \underset{2}{\Psi_{(\rho\sigma)n}} - \underset{2}{\Omega_{(\sigma)in}}\, \underset{1}{\Psi_{(\rho\sigma)m}} \right] p^{i} \Big\}\, N^{\alpha}_{(\rho)}.$$

3.2.0. Если кривая C геодезическая линия подпространства, тогда $p^{i}=0$ и из (1.9) и (2.2) получается

$$(3.9a,b) \qquad p^{i} = 0, \quad q^{\alpha} = \sum_{\rho} K_{(\rho)}\, N^{\alpha}_{(\rho)},$$

$$(3.10a,b) \qquad \underset{\mu}{A^{s}_{m}} = - \sum_{\rho} e_{(\rho)}\, K_{(\rho)}\, g^{rs}\, \underset{\mu}{\Omega_{(\rho)rm}}, \quad \underset{\mu}{B_{(\rho)m}} = K_{(\rho),m} + \sum_{\sigma} K_{(\sigma)}\, \underset{\mu}{\Psi_{(\rho\sigma)m}}.$$

3.2.1. Под условиях (3.9a,b) (3.10a,b) и так как согласно (14b) в [4] $\underset{\nu}{g^{rs}_{|n}} = 0$ для $\nu \in \{1, 2, 3, 4\}$, уравнение (2.14) примет вид

$$\underset{1}{R^{\alpha}}{}_{\pi\mu\nu}\, t^{\mu}_{m}\, t^{\nu}_{n} \sum_{\rho} K_{(\rho)}\, N^{\pi}_{(\rho)} = \sum_{\rho} e_{(\rho)} \Big\{ g^{rs} \Big[2\, \underset{\vee}{\Gamma^{u}_{nm}}\, \underset{1}{\Omega_{(\rho)ru}} + K_{(\rho)} \Big(2\, \underset{\vee}{\Gamma^{u}_{mn}}\, \underset{1}{\Omega_{(\rho)ru}} - \underset{1}{\Omega_{(\rho)rm,n}} +$$

$$+ \underset{1}{\Omega_{(\rho)rn,m}} + \underset{1}{\Omega_{(\rho)um}}\, \Gamma^{u}_{rn} - \underset{1}{\Omega_{(\rho)un}}\, \Gamma^{u}_{rm} \Big) + \sum_{\sigma} K_{(\sigma)} \left(\underset{1}{\Omega_{(\rho)rm}}\, \underset{1}{\Psi_{(\rho\sigma)n}} - \underset{1}{\Omega_{(\rho)rn}}\, \underset{1}{\Psi_{(\rho\sigma)m}} \right) \Big] +$$

$$+ K_{(\rho)}\, g^{rp} \left[\underset{1}{\Phi^{s}_{pm}}\, \underset{1}{\Omega_{(\rho)rn}} - \underset{1}{\Phi^{s}_{pn}}\, \underset{1}{\Omega_{(\rho)rm}} \right] \Big\}\, t^{\alpha}_{s} + \sum_{\rho} \Big\{ 2\, \underset{\vee}{\Gamma^{s}_{nm}}\, K_{(\rho)s} +$$

$$(3.11)$$

$$+ \sum_{\sigma} \Big[g^{rs}\, e_{(\sigma)}\, K_{(\sigma)} \left(\underset{1}{\Omega_{(\rho)sm}}\, \Omega_{(\sigma)rn} - \underset{1}{\Omega_{(\rho)sn}}\, \underset{1}{\Omega_{(\sigma)rm}} \right) + K_{(\sigma)} \Big(\underset{1}{\Psi_{(\rho\sigma)m,n}} - \underset{1}{\Psi_{(\rho\sigma)n,m}} +$$

$$+ 2\, \underset{1}{\Psi_{(\rho\sigma)s}}\, \underset{\vee}{\Gamma^{s}_{nm}} \Big) + K_{(\sigma)} \sum_{\tau} \left(\underset{1}{\Psi_{(\tau\sigma)m}}\, \underset{1}{\Psi_{(\rho\tau)n}} - \underset{1}{\Psi_{(\tau\sigma)n}}\, \underset{1}{\Psi_{(\rho\tau)m}} \right) \Big] \Big\}\, N^{\alpha}_{(\rho)}.$$

3.2.2. — 6. Используя условия интегрируемости как в случаях 3.1.2. — 3.1.6., для $p^{i}=0$ можно получить соответственные уравнения для геодезической линии подпространства.

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