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Surface Transport in Continuum Mechanics

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Abstract: A moving surface is considered as a 3-dimensional submanifold in the 4-dimensional space-time setting. Elementary differential geometry is used to identify parameter-time and parameter independent normal-time derivatives, and their differences. A few elementary observations are made concerning surface transport theorems.

Key Words: surface transport, surface gradient, parameter-time derivative, normal-time derivative, 4D representation

1. INTRODUCTION

Analogous to the classical Reynold's transport theorem in continuum mechanics, the surface transport theorem is essential in the study of thin films undergoing large deformations, in epitaxial growth and in the study of phase boundary evolution. It is also important in the modeling of a singular surface which carries a certain structure of its own as it migrates through and interacts with a material body. In the last few of years, researchers have paid a fair amount of attention to this topic, e.g., [1–3]. See also the monograph [4]. However, in reading these works, we find that the origin of some results is enhanced by using a different approach. Here, we consider the issues of surface transport from an elementary differential geometry point of view and then make a few elementary observations on surface transport theorems.

2. NOTATION AND DEFINITIONS

Throughout this paper, bold face miniscules, both Roman and Greek, are used to denote multi-tuples. For example, $\mathbf{x} = (x_1, \ldots, x_n)$ is an n-tuple which is a point in the vector space $\mathbb{R}^n$. We shall use $\mathbb{E}^n$ to denote $\mathbb{R}^n$ with an inner product defined, i.e., a Euclidean vector

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space of dimension n. Most often, we shall write simply $\mathbf{x}$ to denote an element of one of these vector spaces and the dimension n will be clear from the context. A *basis* for $\mathbb{R}^n$ is a set of n linearly independent elements and a *frame* is a basis in $\mathbb{E}^n$. Note that a frame is defined only in $\mathbb{E}^n$, where length and angle are meaningful concepts.[1]

Given two sets $\mathcal{D}$ and $\mathcal{I}$, the ordered tuple $(\mathcal{D}, \mathcal{I})$ will be used to indicate their Cartesian product $\mathcal{D} \times \mathcal{I}$. Hence, given points $\boldsymbol{\lambda} \in \mathcal{D}$ and $t \in \mathcal{I}$, then the set $(\mathcal{D}, t) = \mathcal{D} \times \{t\}$ and $(\boldsymbol{\lambda}, t) = \{\boldsymbol{\lambda}\} \times \{t\}$. Moreover, if f is a mapping defined on $\mathcal{D} \times \mathcal{I}$, then its range is written as $f(\mathcal{D}, \mathcal{I})$. We also, alternatively, write $f_t(\mathcal{D}) \equiv f(\mathcal{D}, t)$ to indicate the range $\{f(\boldsymbol{\lambda}, t) | \boldsymbol{\lambda} \in \mathcal{D}\}$.

An n-dimensional manifold $\mathcal{M}$ is a topological set which can be locally parameterized by $\mathbb{R}^n$. Of course, it is an open set. Throughout this paper, we will assume that $\mathcal{M}$ is a Riemannian manifold. Therefore, at each point p the tangent space to $\mathcal{M}$, denoted by $T_p\mathcal{M}$, can be identified with the Euclidean space $\mathbb{E}^n$. Given the vector space structure of $T_p\mathcal{M}$, then at the same point p on $\mathcal{M}$ there is an associated cotangent vector space $T_p^*\mathcal{M}$ such that each $\omega \in T_p^*\mathcal{M}$ denotes a linear transformation of $T_p\mathcal{M} \to \mathbb{R}$. Also, $T_p^*\mathcal{M}$ can be identified with the Euclidean space $\mathbb{E}^n$. Equipped with these two identifications, it follows that we need not make a distinction between elements of the tangent space $T_p\mathcal{M}$ and the cotangent space $T_p^*\mathcal{M}$; elements in each correspond to vectors in the Euclidean space $\mathbb{E}^n$.

If $\mathcal{S} \subset \mathbb{E}^n$ is an $(n-1)$-dimensional differentiable manifold, it is called a hypersurface embedded in $\mathbb{E}^n$. For $n = 3$, the prefix "hyper" is omitted and $\mathcal{S}$ is simply called a surface in $\mathbb{E}^3$.

3. REPRESENTATIONS OF A MOVING SURFACE

In usual practice, a moving surface is often viewed as a one(time)-parameter family of surfaces in $\mathbb{E}^3$. We briefly introduce this approach below in the first part of this section. However, for added clarity in describing surface transport theorems we find it most convenient to introduce a space–time 4D representation of a moving surface in $\mathbb{E}^4$. We do this in the second part of this section.

3.1. 3D Representation

Let $\mathcal{D} \subset \mathbb{R}^2$ be an open set in the parameter space $\mathbb{R}^2$ with points $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \in \mathcal{D}$ and let $\mathcal{I} \subset \mathbb{R}$ be an open interval of time. Then, a mapping $\bar{\mathbf{x}} : \mathcal{D} \times \mathcal{I} \to \mathbb{E}^3$, such that $\mathbf{x} = \bar{\mathbf{x}}(\boldsymbol{\lambda}, t) = \bar{\mathbf{x}}_t(\boldsymbol{\lambda})$, represents a regular moving surface in $\mathbb{E}^3$ provided its Jacobian matrix

$$J_{i\alpha}(\boldsymbol{\lambda}, t) = \frac{\partial}{\partial \lambda_\alpha} \bar{x}_i(\boldsymbol{\lambda}, t) \quad \text{for } i = 1, 2, 3 \text{ and } \alpha = 1, 2 \tag{3.1}$$

is rank-2 for any $(\boldsymbol{\lambda}, t) \in \mathcal{D} \times \mathcal{I}$. An element $\boldsymbol{\lambda} \in \mathcal{D}$ represents the parametric value of a particular surface point. At each given time t, the range $\bar{\mathbf{x}}_t(\mathcal{D})$ is a smooth orientable surface $\mathcal{S}_t$ in $\mathbb{E}^3$. Hence, at each point $\mathbf{x} = \bar{\mathbf{x}}_t(\boldsymbol{\lambda}) \in \mathcal{S}_t$, we may define the following *coordinate basis vectors*[2] in $\mathbb{E}^3$:

$$\mathbf{e}_1(\mathbf{x},t)|_{\mathbf{x}=\bar{\mathbf{x}}_t(\lambda)} = \bar{\mathbf{e}}_1(\boldsymbol{\lambda},t) \equiv \frac{\partial}{\partial\lambda_1}\bar{\mathbf{x}}_t(\boldsymbol{\lambda}),$$

$$\mathbf{e}_2(\mathbf{x},t)|_{\mathbf{x}=\bar{\mathbf{x}}_t(\lambda)} = \bar{\mathbf{e}}_2(\boldsymbol{\lambda},t) \equiv \frac{\partial}{\partial\lambda_2}\bar{\mathbf{x}}_t(\boldsymbol{\lambda}). \tag{3.2}$$

Clearly, the set $\{\mathbf{e}_1(\mathbf{x},t), \mathbf{e}_2(\mathbf{x},t)\}$ spans the tangent plane $T_{\mathbf{x}}\mathcal{S}_t \subset \mathbb{E}^3$, which is a Euclidean space of dimension 2, so the set may be called a frame. Using the relations $\mathbf{e}^\alpha \cdot \mathbf{e}_\beta = \delta^\alpha_\beta$, we introduce the associated covectors $\mathbf{e}^1(\mathbf{x},t)$ and $\mathbf{e}^2(\mathbf{x},t)$, which span the cotangent plane $T^*_{\mathbf{x}}\mathcal{S}_t$, also identified with the same Euclidean space of dimension 2; these covectors simply define another frame. We define the orientation of $\mathcal{S}_t$ through the unit normal field

$$\mathbf{n}(\mathbf{x},t) \equiv \frac{\mathbf{e}_1(\mathbf{x},t) \times \mathbf{e}_2(\mathbf{x},t)}{|\mathbf{e}_1(\mathbf{x},t) \times \mathbf{e}_2(\mathbf{x},t)|}. \tag{3.3}$$

Finally, the partial derivative of $\bar{\mathbf{x}}_t(\boldsymbol{\lambda})$ with respect to time t defines the velocity field $\mathbf{w} = \mathbf{w}(\mathbf{x},t) = \bar{\mathbf{w}}_t(\boldsymbol{\lambda})$ at $\mathbf{x} = \bar{\mathbf{x}}_t(\boldsymbol{\lambda}) \in \mathcal{S}_t$ of the moving surface according to the equivalent alternative expressions

$$\mathbf{w}_t(\mathbf{x})|_{\mathbf{x}=\bar{\mathbf{x}}_t(\lambda)} \equiv \frac{\partial}{\partial t}\bar{\mathbf{x}}_t(\boldsymbol{\lambda}) = \bar{\mathbf{w}}(\boldsymbol{\lambda},t) = \bar{\mathbf{w}}_t(\boldsymbol{\lambda}). \tag{3.4}$$

In (3.4), $\mathbf{w}_t(\bar{\mathbf{x}}_t(\boldsymbol{\lambda}))$ is tangent to the trajectory $\bar{\mathbf{x}}_t(\boldsymbol{\lambda})$ in $\mathbb{E}^3$ at time $t \in \mathcal{I}$, but not generally tangent to $\mathcal{S}_t$.

3.2. 4D Representation

Here, we introduce $\mathbf{z} = (\mathbf{x},t)$ as a point in $\mathbb{R}^4$, and we define a mapping $\bar{\mathbf{z}} : \mathcal{D} \times \mathcal{I} \to \mathbb{E}^4$ such that

$$\bar{z}_i(\boldsymbol{\lambda},t) = \bar{x}_i(\boldsymbol{\lambda},t) \quad \text{for } i = 1,2,3, \quad \bar{z}_4(\boldsymbol{\lambda},t) = t. \tag{3.5}$$

The image set of $\bar{\mathbf{z}}$ is denoted by $\Omega = \bar{\mathbf{z}}(\mathcal{D},\mathcal{I})$ which represents a regular hypersurface in $\mathbb{E}^4$ because (3.1) guarantees a rank-3 property of the Jacobi matrix $\partial\bar{z}_I/\partial\lambda_j$, $I = 1,2,3,4$ and $j = 1,2,3$. The following notational exchanges will be used below:

$$(\bar{\mathbf{x}}_t(\boldsymbol{\lambda}),t) = (\mathbf{x},t) = \mathbf{z} = \bar{\mathbf{z}}(\boldsymbol{\lambda},t).$$

At a given point $\mathbf{z} = \bar{\mathbf{z}}(\boldsymbol{\lambda},t)$ on Ω, we define, in $\mathbb{E}^4$, the coordinate base vectors

$$\mathbf{b}_1(\mathbf{z})|_{\mathbf{z}=\bar{\mathbf{z}}(\lambda,t)} = \bar{\mathbf{b}}_1(\boldsymbol{\lambda},t) \equiv \frac{\partial}{\partial\lambda_1}\bar{\mathbf{z}}(\boldsymbol{\lambda},t) = (\mathbf{e}_1(\mathbf{x},t)|_{\mathbf{x}=\bar{\mathbf{x}}_t(\lambda)}, 0) = (\bar{\mathbf{e}}_1(\boldsymbol{\lambda},t), 0),$$

$$\mathbf{b}_2(\mathbf{z})|_{\mathbf{z}=\bar{\mathbf{z}}(\lambda,t)} = \bar{\mathbf{b}}_2(\boldsymbol{\lambda},t) \equiv \frac{\partial}{\partial\lambda_2}\bar{\mathbf{z}}(\boldsymbol{\lambda},t) = (\mathbf{e}_2(\mathbf{x},t)|_{\mathbf{x}=\bar{\mathbf{x}}_t(\lambda)}, 0) = (\bar{\mathbf{e}}_2(\boldsymbol{\lambda},t), 0), \tag{3.6}$$

$$\mathbf{b}_3(\mathbf{z})|_{\mathbf{z}=\bar{\mathbf{z}}(\lambda,t)} = \bar{\mathbf{b}}_3(\boldsymbol{\lambda},t) \equiv \frac{\partial}{\partial t}\bar{\mathbf{z}}(\boldsymbol{\lambda},t) = (\mathbf{w}(\mathbf{x},t)|_{\mathbf{x}=\bar{\mathbf{x}}_t(\lambda)}, 1) = (\bar{\mathbf{w}}(\boldsymbol{\lambda},t), 1).$$

Clearly, the set $\{\mathbf{b}_1(\mathbf{z}), \mathbf{b}_2(\mathbf{z}), \mathbf{b}_3(\mathbf{z})\}$ spans the tangent hyperplane $T_{\mathbf{z}}\Omega \subset \mathbb{E}^4$ and it is easy to show that at the same point $\mathbf{z} = \bar{\mathbf{z}}(\boldsymbol{\lambda}, t)$ on Ω the covectors, associated with $\{\mathbf{b}_1(\mathbf{z}), \mathbf{b}_2(\mathbf{z}), \mathbf{b}_3(\mathbf{z})\}$ thru $\mathbf{b}^i \cdot \mathbf{b}_j = \delta^i_j$, are given by

$$\begin{aligned} \mathbf{b}^1(\mathbf{z}) &= \left(\mathbf{e}^1(\mathbf{x}, t), -w^1(\mathbf{x}, t)\right), \quad \mathbf{b}^2(\mathbf{z}) = \left(\mathbf{e}^2(\mathbf{x}, t), -w^2(\mathbf{x}, t)\right), \\ \mathbf{b}^3(\mathbf{z}) &= \gamma^{-1}\left(w_n \mathbf{n}(\mathbf{x}, t), 1\right), \end{aligned} \tag{3.7}$$

where

$$w_n \equiv \mathbf{w} \cdot \mathbf{n}, \quad w^1 \equiv \mathbf{w} \cdot \mathbf{e}^1, \quad w^2 \equiv \mathbf{w} \cdot \mathbf{e}^2, \quad \gamma \equiv 1 + w_n^2. \tag{3.8}$$

4. PARAMETER-TIME DERIVATIVE OF A SCALAR FIELD ON A MOVING SURFACE

Corresponding to the 2D and 3D representations of a moving surface introduced in Section 3, there are also two ways to track a scalar field on a moving surface.

At a given time t, let $f_t : \mathcal{S}_t \to \mathbb{R}$ be a scalar function defined on $\mathcal{S}_t$ and, for $\mathbf{x} = \bar{\mathbf{x}}_t(\boldsymbol{\lambda}) \in \mathcal{S}_t$, consider the composition

$$f_t(\mathbf{x})|_{\mathbf{x}=\bar{\mathbf{x}}_t(\boldsymbol{\lambda})} = (f_t \circ \bar{\mathbf{x}}_t)(\boldsymbol{\lambda}) \equiv \bar{f}_t(\boldsymbol{\lambda}) = \bar{f}(\boldsymbol{\lambda}, t). \tag{4.1}$$

Note that $\bar{f} : \mathcal{D} \times \mathcal{I} \to \mathbb{R}$. Special attention must be paid here in calculating the parameter-time derivative[3] by using the chain rule because the partial derivative of $f_t(\mathbf{x})$ with respect to t is not well-defined; this requires that $\mathbf{x} = \bar{\mathbf{x}}_t(\boldsymbol{\lambda}) \in \mathcal{S}_t$ be held fixed while t varies and it is not.[4] However, by redefining this scalar field on the hypersurface $\Omega \subset \mathbb{E}^4$, it is easy to overcome this apparent difficulty. To do this, first note that if $g : \Omega \to \mathbb{R}$ is a scalar field defined on Ω, then

$$g(\mathbf{z})|_{\mathbf{z}=\bar{\mathbf{z}}(\boldsymbol{\lambda},t)} = (g \circ \bar{\mathbf{z}})(\boldsymbol{\lambda}, t) \equiv \bar{g}(\boldsymbol{\lambda}, t), \tag{4.2}$$

where the composite mapping $\bar{g} : \mathcal{D} \times \mathcal{I} \to \mathbb{R}$.

Comparing (4.2) with (4.1), we see that by identifying

$$g(\mathbf{z})|_{\mathbf{z}=\bar{\mathbf{z}}(\boldsymbol{\lambda},t)} \equiv f_t(\mathbf{x})|_{\mathbf{x}=\bar{\mathbf{x}}_t(\boldsymbol{\lambda})} \quad \forall\, (\boldsymbol{\lambda}, t) \in \mathcal{D} \times \mathcal{I} \tag{4.3}$$

we reach

$$\bar{g} \equiv \bar{f} \quad \text{on} \quad \mathcal{D} \times \mathcal{I}. \tag{4.4}$$

Now, the parameter-time derivative of $f_t(\mathbf{x})$, herein denoted as $\overset{\circ}{f}_t(\mathbf{x})$, is defined by

$$\overset{\circ}{f}_t(\mathbf{x})\bigg|_{\mathbf{x}=\bar{\mathbf{x}}_t(\boldsymbol{\lambda})} \equiv \frac{\partial}{\partial t}\bar{f}(\boldsymbol{\lambda}, t) = \frac{\partial}{\partial t}\bar{g}(\boldsymbol{\lambda}, t), \tag{4.5}$$

which represents the change of $\bar{f}$ with respect to time t, carried by the parameter point $\boldsymbol{\lambda} \in \mathcal{D}$.[5]

4.1. Definition of Surface Gradient

To compute the parameter-time derivative defined in (4.5) by using the chain rule, some knowledge of surface gradients is essential.

Let $\mathcal{S}$ be a hypersurface in $\mathbb{E}^n$, parameterized by a mapping $\bar{\mathbf{x}} : \mathcal{D} \to \mathbb{E}^n$, where $\mathcal{D}$ is an open set in the parameter space $\mathbb{R}^{n-1}$. Given an open interval $\mathcal{I} \in \mathbb{R}$, consider a mapping $\mathbf{y} : \mathcal{I} \to \mathbb{E}^n$ such that $\mathbf{y}(\mathcal{I}) \subset \mathcal{S}$ is a curve on $\mathcal{S}$ with $\mathbf{y}'(c) \neq 0$ for all $c \in \mathcal{I}$. Clearly, $\mathbf{y}'(c)$ is tangent to $\mathcal{S}$ at the point $\mathbf{y}(c) \in \mathcal{S}$.

Let $\upsilon : \mathcal{S} \to \mathcal{F}$ be a field[6] defined on $\mathcal{S}$. Hence, the composite mapping $\bar{\upsilon} = \upsilon \circ \bar{\mathbf{x}} : \mathcal{D} \to \mathcal{F}$ is a field defined on the parameter domain $\mathcal{D} \subset \mathbb{R}^{n-1}$.

Definition 4.1. Let $\mathcal{S}$ be a hypersurface in $\mathbb{E}^n$ and $\upsilon : \mathcal{S} \to \mathcal{F}$ a field on $\mathcal{S}$. The *surface gradient* of υ at a point $\mathbf{x} \in \mathcal{S}$ is an element(unique), $\operatorname{grad}_{\mathcal{S}}\upsilon(\mathbf{x})$ of $\mathcal{F} \otimes T_{\mathbf{x}}\mathcal{S}$, such that for all C^1 curves $\mathbf{y}(\mathcal{I})$ in $\mathcal{S}$ with $\mathbf{y}(c) = \mathbf{x}$ for $c \in \mathcal{I}$ and $\mathbf{x} \in \mathcal{S}$ the following chain rule holds:

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\upsilon(\mathbf{y}(\alpha))\bigg|_{\alpha=c} = (\operatorname{grad}_{\mathcal{S}}\upsilon(\mathbf{x}))\mathbf{y}'(c). \tag{4.6}$$

Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_{n-1}) \in \mathcal{D}$ be a typical point in the parameter space $\mathbb{R}^{n-1}$ and fix the values of $(\lambda_2, \ldots, \lambda_{n-1})$. Then, the mapping $\bar{\mathbf{x}}(\cdot, \lambda_2, \ldots, \lambda_{n-1})$ is a λ_1-coordinate curve on $\mathcal{S}$ and the coordinate base vector

$$\mathbf{e}_1(\mathbf{x})|_{\mathbf{x}=\bar{\mathbf{x}}(\lambda)} = \bar{\mathbf{e}}_1(\lambda_1, \ldots, \lambda_{n-1}) \equiv \frac{\partial}{\partial\lambda_1}\bar{\mathbf{x}}(\lambda_1, \ldots, \lambda_{n-1})$$

is its natural tangent vector. Hence, at $\mathbf{x} = \bar{\mathbf{x}}(\boldsymbol{\lambda}) \in \mathcal{S}$ we see, using Definition 4.1, that

$$\begin{aligned}
\frac{\partial}{\partial\lambda_1}\bar{\upsilon}(\boldsymbol{\lambda}) &= \frac{\partial}{\partial\lambda_1}(\upsilon \circ \bar{\mathbf{x}})(\lambda_1, \ldots, \lambda_{n-1}) \\
&= (\operatorname{grad}_{\mathcal{S}}\upsilon(\mathbf{x}))\frac{\partial}{\partial\lambda_1}\bar{\mathbf{x}}(\lambda_1, \ldots, \lambda_{n-1}) \\
&= (\operatorname{grad}_{\mathcal{S}}\upsilon(\mathbf{x}))\mathbf{e}_1(\mathbf{x}) \equiv \left(\operatorname{grad}_{\mathcal{S}}\upsilon(\mathbf{x})\right)_1.
\end{aligned} \tag{4.7}$$

Similar relations can be obtained for $\partial\bar{\upsilon}(\boldsymbol{\lambda})/\partial\lambda^\alpha$, $\alpha = 2, \ldots, n-1$. Therefore, at $\mathbf{x} = \bar{\mathbf{x}}(\boldsymbol{\lambda}) \in \mathcal{S}$ we have the following representation of the surface gradient as an element of $\mathcal{F} \otimes T_{\mathbf{x}}\mathcal{S}$:

$$\operatorname{grad}_{\mathcal{S}}\upsilon(\mathbf{x}) = \left(\operatorname{grad}_{\mathcal{S}}\upsilon(\mathbf{x})\right)_j \otimes \mathbf{e}^j(\mathbf{x}) = \left(\frac{\partial}{\partial\lambda_j}\bar{\upsilon}(\boldsymbol{\lambda})\right) \otimes \mathbf{e}^j(\mathbf{x}), \tag{4.8}$$

where $\{\mathbf{e}^i(\mathbf{x})|\, i = 1, \dots, n-1\}$ is the set of covectors naturally associated with the set of base vectors $\{\mathbf{e}_i(\mathbf{x})|\, i = 1, \dots, n-1\}$. Note that, if v (hence $\partial\bar{v}/\partial\lambda_j$ for each j) is a scalar field, the tensor product symbol $\otimes$ is normally omitted following common conventions.

Definition 4.2. The surface divergence of a vector field $\mathbf{v} : \mathcal{S} \to \mathbb{E}^n$ at point $\mathbf{x} = \bar{\mathbf{x}}(\boldsymbol{\lambda}) \in \mathcal{S}$ is the contraction of its surface gradient in the sense that

$$\mathrm{div}_{\mathcal{S}}\mathbf{v}(\mathbf{x}) = \left(\frac{\partial}{\partial\lambda_j}\bar{\mathbf{v}}(\boldsymbol{\lambda})\right) \cdot \bar{e}^j(\mathbf{x}), \tag{4.9}$$

where the composite mapping $\bar{\mathbf{v}} = \mathbf{v} \circ \bar{\mathbf{x}} : \mathbb{R}^{n-1} \to \mathbb{E}^n$ is a vector field on the parameter domain $\mathcal{D}$.

Surface gradient of the scalar field g of (4.2) on $\Omega \subset \mathbb{E}^4$. We have noted in Section 3.2 that a moving surface in $\mathbb{E}^3$ also can be regarded as a 3-dimensional hypersurface $\boldsymbol{\Omega} \in \mathbb{E}^4$. Using (4.8) and the notation of Section 3.2, the surface gradient of a scalar field $g : \ \Omega \to \mathbb{R}$ at $\mathbf{z} = \bar{\mathbf{z}}(\boldsymbol{\lambda}, t)$, $\boldsymbol{\lambda} \in \mathcal{D} \subset \mathbb{E}^2$, lies in the tangent hyperplane $T_{\mathbf{z}}\boldsymbol{\Omega}$ and has the representation

$$\mathrm{grad}_{\Omega}g(\mathbf{z}) = \frac{\partial}{\partial\lambda_1}\bar{g}(\boldsymbol{\lambda}, t)\mathbf{b}^1(\mathbf{z}) + \frac{\partial}{\partial\lambda_2}\bar{g}(\boldsymbol{\lambda}, t)\mathbf{b}^2(\mathbf{z}) + \frac{\partial}{\partial t}\bar{g}(\boldsymbol{\lambda}, t)\mathbf{b}^3(\mathbf{z}), \tag{4.10}$$

where, recall (4.2), the composite mapping $\bar{g} = g \circ \bar{\mathbf{z}} : \ \mathcal{D} \times \mathcal{I} \to \mathbb{R}$.

Now, returning to the scalar function $f_t : \ \mathcal{S}_t \to \mathbb{R}$ defined on the surface $\mathcal{S}_t \subset \mathbb{E}^3$ at each t, and using the identification (4.3) and (4.4), we see from (4.5) that the parameter-time derivative of $f_t(\mathbf{x})$ is then

$$\left.\overset{\circ}{f}_t(\mathbf{x})\right|_{\mathbf{x}=\bar{\mathbf{x}}_t(\lambda)} = \frac{\partial}{\partial t}\bar{g}(\boldsymbol{\lambda}, t) = \left(\mathbf{b}_3(\mathbf{z}) \cdot \mathrm{grad}_{\Omega}g(\mathbf{z})\right)\big|_{\mathbf{z}=\bar{\mathbf{z}}(\lambda,t)}. \tag{4.11}$$

Surface gradient of the scalar field f_t on $\mathcal{S}_t \subset \mathbb{E}^3$. At any given time t, the image $\mathcal{S}_t = \bar{\mathbf{x}}_t(\mathcal{D})$ is a regular surface in $\mathbb{E}^3$ and the surface gradient at a given point $\mathbf{x} = \bar{\mathbf{x}}_t(\boldsymbol{\lambda}, t) \in \mathcal{S}_t$ of the scalar field $f_t : \ \mathcal{S}_t \to \mathbb{R}$ is a tangent vector $\mathrm{grad}_{\mathcal{S}_t} f_t(\mathbf{x})$ in the tangent plane $T_{\mathbf{x}}\mathcal{S}_t$. According to (4.8), it has the representation

$$\mathrm{grad}_{\mathcal{S}_t} f_t(\mathbf{x}) = \frac{\partial}{\partial\lambda_1}\bar{f}(\boldsymbol{\lambda}, t)\mathbf{e}^1(\mathbf{x}, t) + \frac{\partial}{\partial\lambda_2}\bar{f}(\boldsymbol{\lambda}, t)\mathbf{e}^2(\mathbf{x}, t). \tag{4.12}$$

Recalling the velocity field $\mathbf{w}_t : \ \mathcal{S}_t \to \mathbb{E}^3$ defined in (3.4) and, correspondingly, the composite field $\bar{\mathbf{w}}_t = \mathbf{w}_t \circ \bar{\mathbf{x}}_t : \ \mathcal{D} \to \mathbb{E}^3$, we see, using Definition 4.2, that the surface divergence of $\mathbf{w}_t(\mathbf{x})$ at $\mathbf{x} = \bar{\mathbf{x}}(\boldsymbol{\lambda}) \in \mathcal{S}_t$ is

$$\mathrm{div}_{\mathcal{S}_t}\mathbf{w}_t(\mathbf{x}) = \left(\frac{\partial}{\partial\lambda_i}\bar{\mathbf{w}}_t(\boldsymbol{\lambda})\right) \cdot \mathbf{e}^i(\mathbf{x}, t). \tag{4.13}$$

4.2. Orthogonal Decomposition of the Parameter-Time Derivative and the Definition of Normal-Time Derivative

First, let us recall from (3.6) that $\mathbf{b}_3(\mathbf{z}) = (\mathbf{w}(\mathbf{x}, t), 1) \in T_{\mathbf{z}}\Omega$ and note that $\mathbf{w}(\mathbf{x}, t) \notin T_{\mathbf{x}}\mathcal{S}_t$. It will be helpful to introduce the orthogonal decomposition

$$\mathbf{b}_3(\mathbf{z}) = \mathbf{b}_3^{\parallel}(\mathbf{z}) + \mathbf{b}_3^{\perp}(\mathbf{z}), \quad \mathbf{b}_3^{\parallel}(\mathbf{z}) \equiv (\mathbf{w}^{\parallel}(\mathbf{x}, t), 0), \quad \mathbf{b}_3^{\perp}(\mathbf{z}) \equiv (w_n \mathbf{n}(\mathbf{x}, t), 1). \tag{4.14}$$

Here, $\mathbf{w}^{\parallel} \equiv (\mathbf{e}_1 \otimes \mathbf{e}^1 + \mathbf{e}_2 \otimes \mathbf{e}^2)\mathbf{w}$ is the projection of $\mathbf{w}$ onto $T_{\mathbf{x}}\mathcal{S}_t$ in $\mathbb{E}^3$. It is easy to verify, using (4.10) and (3.7), that

$$\left(\left(\mathbf{b}_3^{\parallel}(\mathbf{z}) \cdot \mathrm{grad}_{\Omega} g(\mathbf{z})\right)\right)\Big|_{\mathbf{z}=\bar{\mathbf{z}}(\boldsymbol{\lambda},t)} = \left(\mathbf{w}^{\parallel}(\mathbf{x}, t) \cdot \mathrm{grad}_{\mathcal{S}_t} f_t(\mathbf{x})\right)\big|_{\mathbf{x}=\bar{\mathbf{x}}_t(\boldsymbol{\lambda})}. \tag{4.15}$$

Hence, using the decomposition (4.14), (4.11) and the definition (4.5), we have

$$\overset{\circ}{f}_t(\mathbf{x})\Big|_{\mathbf{x}=\bar{\mathbf{x}}_t(\boldsymbol{\lambda})} = \overset{\square}{f}_t(\mathbf{x})\Big|_{\mathbf{x}=\bar{\mathbf{x}}_t(\boldsymbol{\lambda})} + \left(\mathbf{w}^{\parallel}(\mathbf{x}, t) \cdot \mathrm{grad}_{\mathcal{S}_t} f_t(\mathbf{x})\right)\big|_{\mathbf{x}=\bar{\mathbf{x}}_t(\boldsymbol{\lambda})}, \tag{4.16}$$

where

$$\overset{\square}{f}_t(\mathbf{x})\Big|_{\mathbf{x}=\bar{\mathbf{x}}_t(\boldsymbol{\lambda})} = \left(\mathbf{b}_3^{\perp}(\mathbf{z}) \cdot \mathrm{grad}_{\Omega} g(\mathbf{z})\right)\big|_{\mathbf{z}=\bar{\mathbf{z}}(\boldsymbol{\lambda},t)}. \tag{4.17}$$

This represents an alternative interpretation of the normal-time derivative at $\mathbf{x} = \bar{\mathbf{x}}_t(\boldsymbol{\lambda}) \in \mathcal{S}_t$ as introduced in [2] and [5]. Clearly, recalling the decomposition (4.14), we see that the right hand side in (4.17) is independent of the tangential velocity $\mathbf{w}^{\parallel}$ of $\mathcal{S}_t$.

Further interpretations of the normal-time and parameter-time derivatives. Now, we suppose that $f_t : \mathcal{S}_t \to \mathbb{R}$ is the restriction of a function $\hat{f}_t : \mathbb{E}^3 \to \mathbb{R}$. Then, the identification (4.3) shows that $g : \Omega \to \mathbb{R}$ is the restriction of a function $\hat{g} : \mathbb{E}^4 \to \mathbb{R}$ and we have

$$\hat{g}(\mathbf{z}) = \hat{f}_t(\mathbf{x}), \quad \mathbf{z} = (\mathbf{x}, t) \in \mathbb{E}^4, \quad \forall \mathbf{x} \in \mathbb{E}^3, \quad t \in \mathcal{I}. \tag{4.18}$$

This implies that

$$\mathrm{grad}\hat{g}(\mathbf{z}) = \left(\mathrm{grad}\hat{f}_t(\mathbf{x}), \frac{\partial}{\partial t}\hat{f}_t(\mathbf{x})\right), \tag{4.19}$$

where "grad" denotes the gradient operation in the domains of definition of the respective functions $\hat{g} : \mathbb{E}^4 \to \mathbb{R}$ and $\hat{f}_t : \mathbb{E}^3 \to \mathbb{R}$. But, because $\mathbf{b}_3(\mathbf{z})$ and $\mathbf{b}_3^{\perp}(\mathbf{z})$ are members of $T_{\mathbf{z}}\Omega$, we then see, using (4.14) and (4.17), that for all $\mathbf{z} = \bar{\mathbf{z}}(\boldsymbol{\lambda}, t) \in \Omega$ and correspondingly $\mathbf{x} = \bar{\mathbf{x}}_t(\boldsymbol{\lambda}) \in \mathcal{S}_t$,

$$\begin{aligned}
\overset{\square}{f}_t(\mathbf{x}) &= \mathbf{b}_3^{\perp}(\mathbf{z}) \cdot \mathrm{grad}_{\Omega} g(\mathbf{z}) \\
&= \mathbf{b}_3^{\perp}(\mathbf{z}) \cdot \mathrm{grad}\hat{g}(\mathbf{z}) \\
&= (w_n \mathbf{n}(\mathbf{x}, t), 1) \cdot \mathrm{grad}\hat{g}(\mathbf{z}) \\
&= w_n \mathbf{n}(\mathbf{x}, t) \cdot \mathrm{grad}\hat{f}_t(\mathbf{x}) + \frac{\partial}{\partial t}\hat{f}_t(\mathbf{x}),
\end{aligned} \tag{4.20}$$

and

$$\begin{aligned}
\overset{\circ}{f}_t(\mathbf{x}) &= \mathbf{b}_3(\mathbf{z}) \cdot \mathrm{grad}_{\Omega} g(\mathbf{z}) \\
&= \mathbf{b}_3(\mathbf{z}) \cdot \mathrm{grad}\hat{g}(\mathbf{z}) \\
&= (\mathbf{w}(\mathbf{x}, t), 1) \cdot \mathrm{grad}\hat{g}(\mathbf{z}) \\
&= \mathbf{w}(\mathbf{x}, t) \cdot \mathrm{grad}\hat{f}_t(\mathbf{x}) + \frac{\partial}{\partial t}\hat{f}_t(\mathbf{x}).
\end{aligned} \tag{4.21}$$

Perhaps (4.20) is the origin of the name "normal-time derivative" because it represents the sum of the rate of change at $\mathbf{x} = \bar{\mathbf{x}}_t(\boldsymbol{\lambda}) \in \mathcal{S}_t$ due to the normal motion of $\mathcal{S}_t$ through a spatially variable external field (i.e., normal convection) plus that due to the temporally changing field, itself.

5. REMARKS CONCERNING SURFACE TRANSPORT THEOREMS

We begin by recalling two well-known results in surface theory. The first concerns the parameter-time derivative of the area measure of a surface element. The area element at $\mathbf{x} = \bar{\mathbf{x}}_t(\boldsymbol{\lambda})$ on the moving surface $\mathcal{S}_t$ has area measure $da_t(\mathbf{x}) = \bar{\mu}(\boldsymbol{\lambda}, t) d\lambda_1 d\lambda_2$, where

$$\begin{aligned}
\bar{\mu}(\boldsymbol{\lambda}, t) &\equiv |\bar{\mathbf{e}}_1(\boldsymbol{\lambda}, t) \times \bar{\mathbf{e}}_2(\boldsymbol{\lambda}, t)| \\
&= |\mathbf{e}_1(\mathbf{x}, t) \times \mathbf{e}_2(\mathbf{x}, t)|\big|_{\mathbf{x}=\bar{\mathbf{x}}_t(\boldsymbol{\lambda})} \\
&= \mu(\mathbf{x}, t)|_{\mathbf{x}=\bar{\mathbf{x}}_t(\boldsymbol{\lambda})} .
\end{aligned} \tag{5.1}$$

It is well known that

$$\frac{\partial}{\partial t}\bar{\mu}(\boldsymbol{\lambda}, t) = \bar{\mu}(\boldsymbol{\lambda}, t)\ \mathrm{div}_{\mathcal{S}_t}\mathbf{w}_t(\mathbf{x})\big|_{\mathbf{x}=\bar{\mathbf{x}}_t(\boldsymbol{\lambda})} . \tag{5.2}$$

The second is the following surface divergence theorem.

Theorem 5.1 (Surface Divergence Theorem). *Let* $\mathbf{f} : \mathcal{S} \to \mathbb{E}^3$ *be a vector field defined on a smooth surface* $\mathcal{S} \subset \mathbb{E}^3$. *Then,*

$$\int_{\mathcal{S}} \mathrm{div}_{\mathcal{S}}\mathbf{f} \mathrm{d}a = \int_{\partial\mathcal{S}} \mathbf{f} \cdot \boldsymbol{\nu} \mathrm{d}s - \int_{\mathcal{S}} \kappa \mathbf{f} \cdot \mathbf{n} \mathrm{d}a, \tag{5.3}$$

where $\kappa : \mathcal{S} \to \mathbb{R}$ *is the total (twice the mean) curvature of the surface and* $\mathbf{v} : \partial\mathcal{S} \to \mathbb{E}^3$ *is the unit vector field which is tangent to the surface* $\mathcal{S}$*, perpendicular to its boundary curve* $\partial\mathcal{S}$ *and directed "outer" to* $\mathcal{S}$*.*

Equipped with these results, it is then straightforward to show the following surface transport theorem.

Theorem 5.2 (Surface Transport Theorem). *Let* $f_t(\cdot) : \mathcal{S}_t \to R$ *be a scalar field defined on the moving surface* $\mathcal{S}_t$*. Then,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{S}_t} f_t \,\mathrm{d}a_t = \int_{\mathcal{S}_t}\left(\overset{\circ}{f}_t + f_t \mathrm{div}_{\mathcal{S}_t}\mathbf{w}\right)\mathrm{d}a_t \tag{5.4}$$

$$= \int_{\mathcal{S}_t}\left(\overset{\square}{f}_t - f_t\kappa w_n\right)\mathrm{d}a_t + \int_{\partial\mathcal{S}_t} f_t \mathbf{w}\cdot\mathbf{v}\mathrm{d}s_t \tag{5.5}$$

where in (5.5) $\mathbf{w} = \mathbf{w}_t(\mathbf{x})|_{\mathbf{x}\in\partial\mathcal{S}_t}$ *is the velocity of the boundary curve* $\partial\mathcal{S}_t$*, this set being the limit set of* $\mathcal{S}_t$ *having length measure* ds_t *inherited from* $\mathcal{S}_t$*.*

Remark 5.1. Written in the form of (5.4), the detailed representation of the two separate terms in the integrand of the right hand side depend upon the particular choice of parametrization of the surface $\bar{\mathbf{x}}_t(\boldsymbol{\lambda})$, $\boldsymbol{\lambda} \in \mathcal{D}$. Of course, together this dependence cancels out. However, in the form (5.5) each of the three separate integrand terms on the right hand side is independent of this parametrization; the two integrands of the surface integral each depend only on the normal speed of the moving surface, and the integrand of the boundary integral depends only on the normal speed of the limiting edge of the surface, the limit being from within $\mathcal{S}_t$ and the normal being the "edge-normal".

Remark 5.2. Gurtin, Struthers, and Williams in their paper [2] give a somewhat lengthy and cumbersome argument to show directly that, if the evolving surface $\mathcal{S}_t$ is identified as the intersection with a fixed region region $\mathcal{R} \subset E^3$, then (5.5) holds with the boundary integral term replaced by[7]

$$-\int_{\partial\mathcal{S}_t} \phi w_n p(1-p^2)^{-1/2}\mathrm{d}s_t \tag{5.6}$$

where $p = \mathbf{n}\cdot\mathbf{m}$ and $\mathbf{m}$ is the outer unit normal vector to the fixed region $\mathcal{R}$. Following their geometric setting, it is straightforward to show that (5.6) reduces to the boundary integral in (5.5). Thus, the complications encountered in the direct derivation of (5.6) in [2] are not essential. To see this reduction, we first observe that since $\partial\mathcal{S}_t \subset \partial\mathcal{R}$ for all t, we must have $\mathbf{w}\cdot\mathbf{m} = 0$. Then, because $\boldsymbol{\sigma}\cdot\mathbf{n} = \boldsymbol{\sigma}\cdot\mathbf{m} = \boldsymbol{\sigma}\cdot\mathbf{v} = 0$, where $\boldsymbol{\sigma} = \mathbf{n}\times\mathbf{v}$ is the unit tangent vector to $\partial\mathcal{S}_t$, the following four vectors $\{\mathbf{n}, \mathbf{m}, \mathbf{v}, \mathbf{w} - (\mathbf{w}\cdot\boldsymbol{\sigma})\,\boldsymbol{\sigma}\}$ must be coplanar. Finally, because $\mathbf{n}\cdot\mathbf{v} = 0$ it is easy to obtain the following trigonometric relations:

$$\tan\theta = -\frac{\mathbf{w}\cdot\mathbf{n}}{\mathbf{w}\cdot\mathbf{v}}, \quad \cos\theta = \mathbf{n}\cdot\mathbf{m} = p. \tag{5.7}$$

Hence,

$$\mathbf{w} \cdot \mathbf{v} = -(\mathbf{w} \cdot \mathbf{n}) \cot\theta = -w_n p(1-p)^{-1/2} \tag{5.8}$$

which is the basis of our claim.

5.1. Special Case: Migrating Patch on a Material Surface

As an interesting application, let us consider $\mathcal{B}_t$ to be, say, a ferromagnetic thin film moving and deforming in $\mathbb{E}^3$ and let $\mathcal{S}_t \subset \mathcal{B}_t$ be a phase domain migrating in the thin film. The "body" of the thin film corresponds to an open set $\mathcal{B} \subset \mathbb{R}^2$ in the parameter space $\mathbb{R}^2$ and a typical particle of $\mathcal{B}$ is denoted by the parameter pair $\boldsymbol{\beta} = (\beta_1, \beta_2)$. The motion of the thin film is given by $\boldsymbol{\chi}$: $\mathcal{B} \times \mathcal{I} \to \mathbb{E}^3$ and the velocity field of the thin film is denoted by $\mathbf{v}(\mathbf{x}, t)|_{\mathbf{x}=\boldsymbol{\chi}(\boldsymbol{\beta},t)}$. It is clear that the unit normal vector field for $\mathcal{B}_t = \boldsymbol{\chi}(\mathcal{B}, t)$ with a given orientation, $\mathbf{n}(\mathbf{x}, t)|_{\mathbf{x}=\boldsymbol{\chi}(\boldsymbol{\beta},t)}$, is also the unit normal vector field for $\mathcal{S}_t$ in the same orientation. Moreover, at a common point $\mathbf{x} = \bar{\mathbf{x}}(\boldsymbol{\lambda}, t) = \boldsymbol{\chi}(\boldsymbol{\beta}, t)$ of both $\mathcal{S}_t$ and $\mathcal{B}_t$ for $\lambda \in \mathcal{D}$, $\boldsymbol{\beta} \in \mathcal{B}$, the normal speeds satisfy $\mathbf{w}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}, t) = \mathbf{v}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}, t)$.

Corollary 5.3. *Suppose $\mathcal{S}_t \equiv \bar{\mathbf{x}}(\mathcal{D}, t) \subset \mathcal{B}_t$ and let $\mathcal{P} \subset \mathcal{B}$ be such that $\mathcal{P}_t \equiv \boldsymbol{\chi}(\mathcal{P}, t) = \mathcal{S}_t$ at time t. Then,*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{S}_t} f_t(\mathbf{x}) \mathrm{d}a_t = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{P}_t} f_t(\mathbf{x}) \mathrm{d}a_t + \int_{\mathcal{C}_t} f_t (\mathbf{w} - \mathbf{v}) \cdot \boldsymbol{\nu} \mathrm{d}s_t, \tag{5.9}$$

where $\mathcal{C}_t = \partial\mathcal{S}_t = \partial\mathcal{P}_t$ at time t.

Proof. In (5.5), set $\mathbf{w} = \mathbf{v} + (\mathbf{w} - \mathbf{v})$, and use $w_n = \mathbf{w} \cdot \mathbf{n} = \mathbf{v} \cdot \mathbf{n} = v_n$ to reach the fact that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{S}_t} f_t(\mathbf{x}) \mathrm{d}a_t = \int_{\mathcal{S}_t} \left(\overset{\square}{f}_t - f_t \kappa v_n \right) \mathrm{d}a_t + \int_{\partial\mathcal{S}_t} f_t \mathbf{v} \cdot \boldsymbol{\nu} \mathrm{d}s_t + \int_{\partial\mathcal{S}_t} f_t (\mathbf{w} - \mathbf{v}) \cdot \boldsymbol{\nu} \mathrm{d}s_t. \tag{5.10}$$

Now, as a key observation we recall that $\overset{\square}{f}_t$ depends only on the normal speed $w_n = v_n$. Then, because instantaneously $\mathcal{S}_t = \mathcal{P}_t$, we may write the first two integrals on the right-hand side as

$$\begin{aligned} & \int_{\mathcal{S}_t} \left(\overset{\square}{f}_t - f_t \kappa v_n \right) \mathrm{d}a_t + \int_{\partial\mathcal{S}_t} f_t \mathbf{v} \cdot \boldsymbol{\nu} \mathrm{d}s_t \\ = & \int_{\mathcal{P}_t} \left(\overset{\square}{f}_t - f_t \kappa v_n \right) \mathrm{d}a_t + \int_{\partial\mathcal{P}_t} f_t \mathbf{v} \cdot \boldsymbol{\nu} \mathrm{d}s_t \\ = & \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{P}_t} f_t(\mathbf{x}) \mathrm{d}a_t, \end{aligned} \tag{5.11}$$

the latter following because of the surface transport theorem Theorem 5.2, as applied to $\mathcal{P}_t = \chi(\mathcal{P}, t)$. Thus, we have (5.9). □

Remark 5.3. Although possible, it is more problematic to introduce explicitly the parameter-time derivative in proving this corollary because the parameter-time derivative in $\mathcal{S}_t$ is different than that in $\mathcal{B}_t$, even at the instantaneous common points of intersection, $\mathcal{S}_t = \mathcal{P}_t$, of both surfaces. Of course, this is because the two surfaces have different tangential velocity. Thus, version (5.5) of the surface transport theorem, rather than (5.4), was employed in this proof. The elementary nature of the steps in (5.11) are a reflection of this choice.

Remark 5.4. The field f_t of this corollary is fundamentally carried by the phase domain $\mathcal{S}_t$, which has a pointwise tangential velocity different than that of the magnetic thin film $\mathcal{B}_t$. The first term on the right-hand side of (5.9) expresses the total time rate of change in the phase domain from the point of view of the magnetic thin film. The second boundary integral term expresses the total additional change due to the "slippage" that takes place between the two surfaces at the edge of $\mathcal{S}_t$. It is noteworthy that the pointwise difference in tangential velocities manifests itself only at the edge.

NOTES

1. Two frames $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ and $\{\mathbf{a}_1^*, \ldots, \mathbf{a}_n^*\}$ are isometrically equivalent if lengths and angles are preserved under the linear transformation

$$\mathbf{a}_i^* = \mathbf{Q}\mathbf{a}_i, \quad i = 1, \ldots, n, \quad \mathbf{Q} : \mathbb{E}^n \to \mathbb{E}^n. \tag{2.1}$$

This requires that $\mathbf{Q} \in$ Orth, where Orth denotes the set of all orthogonal transformations of $\mathbb{E}^n \to \mathbb{E}^n$. Of course, all points $\mathbf{x} \in \mathbb{E}^n$ may be identified according to

$$\mathbf{x} \equiv x^i \mathbf{a}_i, \quad \forall (x^1, \ldots, x^n) \in \mathbb{R}^n, \tag{2.2}$$

but they may be equally identified by fixing one point $\mathbf{c} \in \mathbb{E}^n$ and identifying all the points according to

$$\begin{aligned} \mathbf{x}^* &\equiv x^i \mathbf{a}_i^* + \mathbf{c}, \quad \forall (x^1, \ldots, x^n) \in \mathbb{R}^n, \\ &= \mathbf{Q}\mathbf{x} + \mathbf{c} \quad \forall \mathbf{x} \in \mathbb{E}^n. \end{aligned} \tag{2.3}$$

Clearly, the zero $\mathbf{x} = \mathbf{0}$ does not correspond to the zero $\mathbf{x}^* = \mathbf{0}$; the correct correspondence is $\mathbf{x} = -\mathbf{Q}^T\mathbf{c} \Rightarrow \mathbf{x}^* = \mathbf{0}$. The transformation (2.3) represents an isometric transformation of $\mathbb{E}^n$ into itself and is called a Euclidean transformation of $\mathbb{E}^n \to \mathbb{E}^n$. In continuum mechanics such a transformation is considered to be a Euclidean reframing of space and is referred to briefly as a 'change of frame'; in many applications $\mathbf{c} = \mathbf{c}(t)$ and $\mathbf{Q} = \mathbf{Q}(t)$ are functions of time t.
2. We emphasize that the domain of $\mathbf{e}_\alpha$ is $\cup_{t \in \mathcal{I}} (\mathcal{S}_t, t)$ and not $\mathcal{S}_t \times \mathcal{I}$. At different times t and t', the domains of the vector fields $\mathbf{e}_\alpha(\,\cdot\,, t)$ and $\mathbf{e}_\alpha(\,\cdot\,, t')$ are, respectively, $\mathcal{S}_t$ and $\mathcal{S}_{t'}$; these are different subsets of $\mathbb{E}^3$.
3. If $\boldsymbol{\lambda}$ parametrically defines a material point on a membrane, this derivative is called the material-time derivative in the continuum mechanics literature.
4. The same issue arises in classical continuum mechanics when $f_t(\mathbf{x})$ is a function defined for $\mathbf{x} \in \mathcal{B}_t$, where $\mathcal{B}_t$ is the configuration of a body at time t. There, it is common, though often tacit, to consider f_t to be defined not on $\mathcal{B}_t$, but, rather on a fixed "control volume" $\mathcal{V} \in \mathbb{E}^3$ large enough to contain $\mathcal{B}_t$

during an interval of time $\mathcal{I}$. This, then, allows the partial time derivative to be taken at any $t \in \mathcal{I}$, and it represents the time rate of change of the field $f_t(\mathbf{x})$ at the fixed point $\mathbf{x} \in \mathcal{V}$ which various particles of the body occupy during the interval $\mathcal{I}$.

5. This is referred to as the "parameter-dependent time derivative" in [1], but is denoted differently therein as $\frac{\delta}{\delta t} f_t(\mathbf{x})$; see their Equation (3.7).
6. It can be a scalar field, a vector field, or a tensor field, etc. Moreover, its image does not necessarily lie in the tangent hyperplane of $\mathcal{S}$.
7. We have translated their notation into ours. Their derivation is completely independent of any surface parametrization as they introduce only the normal speed of the surface. The approach in [2] is non-classical and interesting but it does introduce certain complications. In the end it does not explicitly identify the edge normal speed as an important parametrization independent quantity in the edge integral contribution to the surface transport theorem, as is shown in (5.5). In [3], however, Gurtin and Struthers do introduce the edge normal speed and include it explicitly in the boundary integral term of their transport theorem, equation (2.38). They reference [2] for the proof of the theorem, itself.

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