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Nonlinear elasticity of composite materials

Landau coefficients in dispersions of spherical and cylindrical inclusions

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Abstract. We investigate the elastic properties of model composites, consisting in a dispersion of nonlinear (spherical or cylindrical) inhomogeneities into a linear solid matrix. Both phases are considered isotropic. Under the simplifying hypotheses of small deformation for the material body and of small volume fraction of the embedded phase, we develop a homogenization procedure based on the Eshelby theory, aimed at describing nonlinear features. We obtain the bulk and shear moduli and Landau coefficients of the overall material in terms of the elastic behavior of the constituents and of their volume fractions. The mixing laws for the nonlinear properties describe a complex scenario where possible strong amplifications of the nonlinearities may arise in some given conditions.

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1 Introduction

In recent years the characterization of linear and nonlinear heterogeneous materials (i.e. materials formed by inclusions dispersed into a matrix) has attracted an ever increasing interest. The central problem of considerable technological importance is to evaluate the effective physical properties governing the behavior of such composite materials on the macroscopic scale, taking into account the actual microscale features [1]. At present, we do not benefit of a general procedure providing effective properties by averaging the local ones. In fact, the details of the morphology (or micro-geometry) play a central role in determining the overall features, particularly when the inclusions have highly anysotropic or nonlinear behavior or when there is a large difference in the properties of the constituents. In this context, the primary goal is to understand and classify the relationship between the internal micro-structure and the observed physical properties. Such a relationship may be used for designing and improving materials or, conversely, for interpreting experimental data.

A huge number of theoretical investigations have been developed so far to describe the behavior of composites, when a specific micro-structure is considered. Alternatively, different theories were addressed to the search of general results of broad applicability, without any guess on the actual micro-structure. Among them we quote the classical Hashin-Shtrikman variational bounds theory [2,3], which provides an upper and lower bound for the properties of composites, and the expansion of Brown [4] and Torquato [5,6] which takes into account the spatial correlation function of the constituents.

Dispersions (or suspensions) of inclusions in a matrix are examples of widely studied heterogeneous materials: these media have been extensively analyzed both from the electrical [7,8] and the elastic [9,10] point of view. In particular, the homogenization procedure has been developed for a very dilute concentration of linear elastic spherical inclusions dispersed into a linear solid matrix [11]. In order to extend this approach to arbitrarily large concentrations, the differential method has been applied for spherical [12], cylindrical [13] and ellipsoidal inclusions [14]. Recent works focus on continuous matrices containing inclusions of diverse shapes, properties and orientations [15,16]. The evaluation of the effective elastic properties of a body containing a given distribution of cracks belongs to the field of homogenization techniques as well [17]. Finally, recent investigations consider the effects of the orientational statistical distribution of cracks [18,19].

In heterogeneous materials the nonlinear elastic regime has been investigated under specific conditions [20–22]. For example, the effective energy of nonlinear elastic and conducting composites has been evaluated [23,24] for incompressible dispersions with rigid or liquid inclusions and for particles with a power-law-type shear energy. In this

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case the energy density depends only on the Von Mises equivalent stress, being appropriate for the plasticity theory and for high-temperature creep of metal [24]. In this work we elaborate a more general framework based on a homogenizing procedure and we apply it to two nonlinear composite materials that paradigmatically represent most features of many real systems. Firstly, we consider a dispersion of nonlinear (but isotropic, i.e. amorphous or polycrystalline) spherical inclusions embedded into a linear homogeneous and isotropic matrix. The nonlinearity of the spherical inclusions can be described, at most, by four parameters (the so-called Landau coefficients) measuring the deviation from the linearity. Since the overall behavior of the heterogeneous structure will be elastically nonlinear, the key point is the evaluation of the effective nonlinear properties of the composite material. Secondly, we investigate a distribution of parallel (nonlinear) cylindrical inclusions embedded into a (linear) matrix, a situation mimicking the fiber reinforcement of composites. In both geometries, the most important methodological aspect is given by a useful generalization of the Eshelby theory [25] to nonlinear inhomogeneities.

The layout of the present work is as follows. In Section 2 we introduce the nonlinear constitutive equations adopted to model the embedded inclusions and the nonlinear generalization of the Eshelby theory. In Sections 3 and 4 we develop and discuss our new framework for spherical and cylindrical inclusions, respectively. Finally, in Appendix A we prove a theorem of existence and unicity for a nonlinear Eshelby problem.

2 Formalism

It is known that a nonlinear elastic theory can be developed in two different ways [26]. Nonlinearity can be taken into account by means of the exact relation for the strain (not limited for small deformation) and the exact equilibrium equations for a volume element of the body; this first approach is referred to as geometrical nonlinearity [27]. Alternatively, nonlinear effects can be considered through the arbitrarieness of the (generically not Hookean) stress-strain constitutive relation; this approach is referred to as physical nonlinearity [27]. In this work, we only consider a situation characterized by physical nonlinearity, but geometrically linear. Accordingly, the angles of rotation can be neglected in determining changes of dimension for the line elements and in formulating the equilibrium conditions of a volume element. Therefore, the balance equations are based on the standard small-strain tensor and on the Cauchy stress tensor [26]. However, since we will treat deformations exceeding the Hookean regime, a nonlinear stress-strain constitutive equation will be assumed. This conceptual framework is sometimes referred to as *hypoelasticity*: it is intended to model perfectly reversible nonlinear stress-strain behavior, in the regime of infinitesimal strains. Such a description has been already adopted in the past in order to model nonlinear cubic polycrystals with perturbative and self consistent methods [28].

Following the Cauchy formulation of elasticity, we base our formal device on the existence of a constitutive stress-strain relation $\hat{T} = f(\hat{\epsilon})$ [29,30]. For the following purposes, we are interested in an isotropic nonlinear constitutive equation, expanded up to the second order in the strain components. In order to develop our formalism under the further hypothesis of isotropy, we need to better specify the mechanical behavior under rigid-body rotation. In particular, we remark that the function $f(\hat{\epsilon})$ must satisfy the identity [26]

$$\hat{R}^T f(\hat{\epsilon})\hat{R} = f\left(\hat{R}^T\hat{\epsilon}\hat{R}\right) \tag{1}$$

for all proper orthogonal tensor $\hat{R}$ representing a rotation. A function satisfying the previous identity is known as an isotropic tensor function, and it can be represented in the form [26]

$$\hat{T} = f(\hat{\epsilon}) = q_1\hat{I} + q_2\hat{\epsilon} + q_3\hat{\epsilon}^2 \tag{2}$$

where $\hat{I}$ is the identity operator and q_1, q_2 and q_3 are scalar functions of the invariants $\mathrm{Tr}(\hat{\epsilon})$, $\mathrm{Tr}(\hat{\epsilon}^2)$ e $\mathrm{Tr}(\hat{\epsilon}^3)$ of the strain tensor $\hat{\epsilon}$. The development of equation (2), up to the second order in $\hat{\epsilon}$, provides the following constitutive equation

$$\begin{aligned}\hat{T} = 2\mu\hat{\epsilon} + \lambda\mathrm{Tr}(\hat{\epsilon})\hat{I} + A\hat{\epsilon}^2 + B\mathrm{Tr}\left(\hat{\epsilon}^2\right)\hat{I}\\ +C\left[\mathrm{Tr}(\hat{\epsilon})\right]^2\hat{I} + D\hat{\epsilon}\mathrm{Tr}(\hat{\epsilon})\end{aligned} \tag{3}$$

where μ and λ are the standard linear Lamè moduli and A, B, C and D are the so-called Landau coefficients describing the nonlinear behavior of the material. We remark that by following the Green formulation of the elasticity theory [30,31], we could obtain a constitutive relation similar to equation (3) where $D = 2B$. Therefore, four independent parameters (A, B, C and D) are used in the Cauchy elasticity, while three independent parameters (A, B and C) are used in the Green elasticity.

The Cauchy nonlinear contitutive equation will be used to model the elastic behavior of the inclusions embedded into the linear matrix. In this context, the elastic fields in such inclusions are found by means of a nonlinear generalization of the Eshelby theory [32–36]. A nonlinear isotropic and homogenous ellipsoid can be generically described by the relation $\hat{T} = \hat{\mathcal{C}}^{(2)}(\hat{\epsilon})\,\hat{\epsilon}$ where $\hat{\mathcal{C}}^{(2)}(\hat{\epsilon})$ is an arbitrary nonlinear strain dependent stiffness tensor. Let the embedding matrix be characterized by a stiffness tensor $\hat{\mathcal{C}}^{(1)}$ and let us calculate the strain field inside the inclusion when a uniform field $\hat{T}^\infty = \hat{\mathcal{C}}^{(1)}\hat{\epsilon}^\infty$ is remotely applied to the system. We have recently proved [31] that the internal uniform field satisfy the following equation

$$\hat{\epsilon}^s = \left[\hat{I} - \hat{\mathcal{S}}\left(\hat{I} - \left(\hat{\mathcal{C}}^{(1)}\right)^{-1}\hat{\mathcal{C}}^{(2)}(\hat{\epsilon}^s)\right)\right]^{-1}\hat{\epsilon}^\infty \tag{4}$$

where the Eshelby tensor $\hat{\mathcal{S}}$ was introduced, depending only on the geometry of the ellipsoid (i.e. the semi-axes length) and on the Poisson ratio of the host matrix [25]. If a solution $\hat{\epsilon}^{s*}$ exists for a given $\hat{\epsilon}^\infty$, it means that the nonlinear inclusion could be replaced by a linear one with

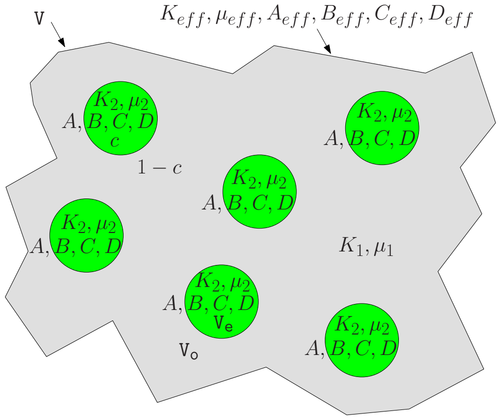

Fig. 1. (Color online) Scheme of a dispersion of nonlinear spheres embedded into a linear matrix.

constant stiffness $\hat{\mathcal{C}}^{(2)} = \hat{\mathcal{C}}^{(2)}\left(\hat{\epsilon}^{s*}\right)$. The calculation of the internal strain field from equation (4) is very complicated and it strongly depends on the actual nonlinear constitutive equation for the inclusion. This task will be accomplished in the following Sections.

To conclude, we have verified the following general statement: if the linear elastic matrix containing an inclusion of ellipsoidal shape is subjected to remote uniform loading, then the stress field inside the inclusion will be uniform independently of the constitutive law for the inclusion, provided that both the matrix and the inclusion are homogeneous bodies.

When the Green approach is adopted it is also possible to verify the existence and the unicity for the solution of equation (4). The proof of this remarkably important result is rather complex; it is given in Appendix A.

3 Dispersion of spherical inclusions

The paradigmatic model of dispersions of nonlinear spherical inclusions is relevant for several applications, ranging from biophysics to advanced technology. For example, the transient elastography has shown its efficiency to map the nonlinear properties of soft tissues or as diagnostic technique [37,38]. In fact, it has been verified that malignant lesions tend to exhibit nonlinear elastic behavior contrary to normal tissues. Another example is given by the self-assembling of semiconductor quantum dots, embedded into a solid matrix. The spherical quantum dots growth, ordering and orientation (occuring during processing) are largely affected by elastic phenomena, even beyond the linear regime [39,40]. Finally, many problems of fracture mechanics in composite materials do contain nonlinear features like, e.g., the interaction between a moving crack and a given inclusion [41].

We consider an assembly of spherical inclusions (see Fig. 1) made by a material described by equation (3) with $\hat{\epsilon} = \hat{\epsilon}^s$, $\hat{T} = \hat{T}^s$ and characterized by moduli μ_2 and λ_2. They are randomly embedded into a linear matrix with stiffness tensor $\hat{\mathcal{C}}^{(1)}$ (moduli λ_1 and μ_1). We also introduce the bulk moduli $K_1 = \lambda_1 + \frac{2}{3}\mu_1$ and $K_2 = \lambda_2 + \frac{2}{3}\mu_2$.

We suppose that the volume fraction c of the embedded phase is small (corresponding to a regime of dilute dispersions). Since the elastic interactions among inclusions can be neglected, each sphere behaves like an isolated one under the effect of a remote load $\hat{T}^\infty = \hat{\mathcal{C}}^{(1)}\hat{\epsilon}^\infty$. The starting point for the evaluation of the induced internal strain $\hat{\epsilon}^s$ is equation (4), which can be usefully rearranged as follows

$$\hat{\epsilon}^s - \hat{\mathcal{S}}\hat{\epsilon}^s + \hat{\mathcal{S}}\left(\hat{\mathcal{C}}^{(1)}\right)^{-1}\hat{T}^s = \hat{\epsilon}^\infty. \quad (5)$$

Here, we have introduced the internal stress given by the relation $\hat{T}^s = \hat{\mathcal{C}}^{(2)}\left(\hat{\epsilon}^s\right)\hat{\epsilon}^s$. We easily obtain that

$$\left(\hat{\mathcal{C}}^{(1)}\right)^{-1}\hat{T}^s = \frac{1}{2\mu_1}\hat{T}^s - \frac{\lambda_1}{2\mu_1\left(2\mu_1+3\lambda_1\right)}\mathrm{Tr}\left(\hat{T}^s\right)\hat{I} \quad (6)$$

while the explicit expression of the Eshelby tensor for this geometry is reported in literature [9,25]

$$\mathcal{S}_{ijkh} = \frac{1}{15(1-\nu_1)}\left[\left(\delta_{ik}\delta_{jh}+\delta_{ih}\delta_{jk}\right)(4-5\nu_1) + \delta_{kh}\delta_{ij}(5\nu_1-1)\right]. \quad (7)$$

By means of equation (7), we get

$$\mathcal{S}_{ijkh}\epsilon^s_{kh} = \frac{2(4-5\nu_1)}{15(1-\nu_1)}\epsilon^s_{ij} + \frac{5\nu_1-1}{15(1-\nu_1)}\epsilon^s_{kk}\delta_{ij} \quad (8)$$

or, equivalently

$$\hat{\mathcal{S}}\hat{\epsilon}^s = \frac{6}{5}\frac{K_1+2\mu_1}{3K_1+4\mu_1}\hat{\epsilon}^s + \frac{1}{5}\frac{3K_1-4\mu_1}{3K_1+4\mu_1}\mathrm{Tr}\left(\hat{\epsilon}^s\right)\hat{I} \quad (9)$$

by taking profit from the standard relation $\nu_1 = \frac{3K_1-2\mu_1}{2(3K_1+\mu_1)}$. By inserting equations (3), (6) and (9) into equation (5) we finally obtain the following important result

$$L\hat{\epsilon}^s + M\mathrm{Tr}\left(\hat{\epsilon}^s\right)\hat{I} + N\left(\hat{\epsilon}^s\right)^2 + O\hat{\epsilon}^s\mathrm{Tr}\left(\hat{\epsilon}^s\right) + P\mathrm{Tr}\left[\left(\hat{\epsilon}^s\right)^2\right]\hat{I} + Q\left[\mathrm{Tr}\left(\hat{\epsilon}^s\right)\right]^2\hat{I} = \hat{\epsilon}^\infty \quad (10)$$

which completely defines the internal strain. The parameters L, M, N, O, P and Q are written in terms of the shear moduli, bulk moduli and nonlinear coefficients as follows

$$L = 1 + \frac{6}{5}\frac{K_1+2\mu_1}{3K_1+4\mu_1}\left(\frac{\mu_2}{\mu_1}-1\right) \quad (11)$$

$$M = \frac{1}{5\left(3K_1+4\mu_1\right)} \times \left[5K_2 - K_1\left(3+2\frac{\mu_2}{\mu_1}\right) - 4\left(\mu_2-\mu_1\right)\right] \quad (12)$$

$$N = \frac{3}{5}\frac{A}{\mu_1}\frac{K_1+2\mu_1}{3K_1+4\mu_1} \quad (13)$$

$$O = \frac{3}{5}\frac{D}{\mu_1}\frac{K_1+2\mu_1}{3K_1+4\mu_1} \quad (14)$$

$$P = \frac{1}{15\left(3K_1+4\mu_1\right)}\left[15B - A\left(1+3\frac{K_1}{\mu_1}\right)\right] \quad (15)$$

$$Q = \frac{1}{15\left(3K_1+4\mu_1\right)}\left[15C - D\left(1+3\frac{K_1}{\mu_1}\right)\right]. \quad (16)$$

We now explicitely take into consideration the dispersion. We define $\mathtt{V}$ as the total volume of the composite material, $\mathtt{V}_e$ as the volume corresponding to the spheres and $\mathtt{V}_\mathtt{o}$ as the volume of the matrix ($\mathtt{V} = \mathtt{V}_\mathtt{o} \cup \mathtt{V}_\mathtt{e}$, see Fig. 1). Since we are working under the hypothesis of small volume fraction c, we can consider the average value of the strain in the matrix to be equal to the externally applied strain $\hat{\epsilon}^\infty$. Therefore, the average strain in the overall system is given by

$$\langle \hat{\epsilon} \rangle = c\hat{\epsilon}^s + (1-c)\hat{\epsilon}^\infty. \tag{17}$$

On the other hand, the average value of the stress can be calculated as follows

$$\begin{aligned} \langle \hat{T} \rangle &= \frac{1}{\mathtt{V}} \int_\mathtt{V} \hat{T} dv = \frac{1}{\mathtt{V}} \hat{\mathcal{C}}^{(1)} \int_{\mathtt{V}_\mathtt{o}} \hat{\epsilon} dv + \frac{1}{\mathtt{V}} \int_{\mathtt{V}_\mathtt{e}} \hat{T} dv \\ &= \frac{1}{\mathtt{V}} \hat{\mathcal{C}}^{(1)} \int_{\mathtt{V}_\mathtt{o}} \hat{\epsilon} dv + \frac{1}{\mathtt{V}} \int_{\mathtt{V}_\mathtt{e}} \hat{T} dv \\ &\quad + \frac{1}{\mathtt{V}} \hat{\mathcal{C}}^{(1)} \int_{\mathtt{V}_\mathtt{e}} \hat{\epsilon} dv - \frac{1}{\mathtt{V}} \hat{\mathcal{C}}^{(1)} \int_{\mathtt{V}_\mathtt{e}} \hat{\epsilon} dv \\ &= \hat{\mathcal{C}}^{(1)} \langle \hat{\epsilon} \rangle + c \left[\hat{T}^s - \hat{\mathcal{C}}^{(1)} \hat{\epsilon}^s \right]. \end{aligned} \tag{18}$$

By substituting equation (10) into equation (17), we obtain the average strain $\langle \hat{\epsilon} \rangle$ in terms of the internal strain $\hat{\epsilon}^s$

$$\begin{aligned} \langle \hat{\epsilon} \rangle = &[c + (1-c)L] \hat{\epsilon}^s \\ &+ (1-c) \Big\{ M \mathrm{Tr}(\hat{\epsilon}^s) \hat{I} + N (\hat{\epsilon}^s)^2 \\ &+ O\hat{\epsilon}^s \mathrm{Tr}(\hat{\epsilon}^s) + P \mathrm{Tr}\left[(\hat{\epsilon}^s)^2\right] \hat{I} + Q \left[\mathrm{Tr}(\hat{\epsilon}^s)\right]^2 \hat{I} \Big\}. \end{aligned} \tag{19}$$

Similarly, by substituting the constitutive relations into equation (18), we obtain the average stress $\langle \hat{T} \rangle$ in terms of $\hat{\epsilon}^s$

$$\begin{aligned} \langle \hat{T} \rangle = &2\mu_1 \langle \hat{\epsilon} \rangle + \left(K_1 - \frac{2}{3}\mu_1 \right) \mathrm{Tr}\langle \hat{\epsilon} \rangle \hat{I} \\ &+ c \Big\{ 2(\mu_2 - \mu_1) \hat{\epsilon}^s \\ &+ \left[K_2 - K_1 - \frac{2}{3}(\mu_2 - \mu_1) \right] \mathrm{Tr}(\hat{\epsilon}^s) \hat{I} \\ &+ A(\hat{\epsilon}^s)^2 + B \mathrm{Tr}\left[(\hat{\epsilon}^s)^2\right] \hat{I} \\ &+ C \left[\mathrm{Tr}(\hat{\epsilon}^s)\right]^2 \hat{I} + D\hat{\epsilon}^s \mathrm{Tr}(\hat{\epsilon}^s) \Big\}. \end{aligned} \tag{20}$$

We remark that equations (19) and (20) implicitely define the macroscopic contitutive equation for the composite system. In order to explicitate such an equation, we rewrite equation (19) in a simpler form

$$\begin{aligned} \langle \hat{\epsilon} \rangle = &L'\hat{\epsilon}^s + M' \mathrm{Tr}(\hat{\epsilon}^s) \hat{I} + N' (\hat{\epsilon}^s)^2 \\ &+ O'\hat{\epsilon}^s \mathrm{Tr}(\hat{\epsilon}^s) + P' \mathrm{Tr}\left[(\hat{\epsilon}^s)^2\right] \hat{I} + Q' \left[\mathrm{Tr}(\hat{\epsilon}^s)\right]^2 \hat{I} \end{aligned} \tag{21}$$

where we have used the definitions $L' = c + (1-c)L$, $M' = (1-c)M$, $N' = (1-c)N$, $O' = (1-c)O$, $P' = (1-c)P$ and $Q' = (1-c)Q$. By means of equation (21), we can straightforwardly calculate all the key quantities $\mathrm{Tr}\langle \hat{\epsilon} \rangle$, $\langle \hat{\epsilon} \rangle^2$, $\langle \hat{\epsilon} \rangle \mathrm{Tr}\langle \hat{\epsilon} \rangle$, $\mathrm{Tr}\left(\langle \hat{\epsilon} \rangle^2\right)$ and $\left[\mathrm{Tr}\langle \hat{\epsilon} \rangle\right]^2$ in terms of the internal strain $\hat{\epsilon}^s$

$$\begin{aligned} \mathrm{Tr}\langle \hat{\epsilon} \rangle &= (L' + 3M')\mathrm{Tr}(\hat{\epsilon}^s) + (N' + 3P')\mathrm{Tr}\left[(\hat{\epsilon}^s)^2\right] \\ &\quad + (O' + 3Q')\left[\mathrm{Tr}(\hat{\epsilon}^s)\right]^2 \\ \langle \hat{\epsilon} \rangle^2 &= L'^2 (\hat{\epsilon}^s)^2 + 2L'M'\hat{\epsilon}^s \mathrm{Tr}(\hat{\epsilon}^s) + M'^2 \left[\mathrm{Tr}(\hat{\epsilon}^s)\right]^2 \hat{I} \\ \langle \hat{\epsilon} \rangle \mathrm{Tr}\langle \hat{\epsilon} \rangle &= L'(L' + 3M')\hat{\epsilon}^s \mathrm{Tr}(\hat{\epsilon}^s) \\ &\quad + M'(L' + 3M')\left[\mathrm{Tr}(\hat{\epsilon}^s)\right]^2 \hat{I} \\ \mathrm{Tr}\left[\langle \hat{\epsilon} \rangle^2\right] &= L'^2 \mathrm{Tr}\left[(\hat{\epsilon}^s)^2\right] + M'(2L' + 3M')\left[\mathrm{Tr}(\hat{\epsilon}^s)\right]^2 \\ \left[\mathrm{Tr}\langle \hat{\epsilon} \rangle\right]^2 &= (L' + 3M')^2 \left[\mathrm{Tr}(\hat{\epsilon}^s)\right]^2. \end{aligned} \tag{22}$$

We remark that equation (22) is valid up to the second order in $\hat{\epsilon}^s$.

3.1 Results

By using equation (20) and by inverting equation (22), we obtain the final form of the constitutive relation for the composite system

$$\begin{aligned} \langle \hat{T} \rangle = &2\mu_{eff} \langle \hat{\epsilon} \rangle + \left(K_{eff} - \frac{2}{3}\mu_{eff} \right) \mathrm{Tr}\langle \hat{\epsilon} \rangle \hat{I} \\ &+ A_{eff} \langle \hat{\epsilon} \rangle^2 + B_{eff} \mathrm{Tr}\left[\langle \hat{\epsilon} \rangle^2\right] \hat{I} \\ &+ C_{eff} \left[\mathrm{Tr}\langle \hat{\epsilon} \rangle\right]^2 \hat{I} + D_{eff} \langle \hat{\epsilon} \rangle \mathrm{Tr}\langle \hat{\epsilon} \rangle \end{aligned} \tag{23}$$

where

$$\mu_{eff} = \mu_1 + c\frac{\mu_2 - \mu_1}{L'} \tag{24}$$

$$K_{eff} = K_1 + c\frac{K_2 - K_1}{L' + 3M'} \tag{25}$$

and

$$A_{eff} = c\frac{A}{L'^2} - 2c\frac{N'(\mu_2 - \mu_1)}{L'^3} \tag{26}$$

$$\begin{aligned} B_{eff} = &2c\frac{(N'M' - L'P')(\mu_2 - \mu_1)}{L'^3 (L' + 3M')} + \\ &- c\frac{(N' + 3P')\left[K_2 - K_1 - \frac{2}{3}(\mu_2 - \mu_1)\right]}{L'^2 (L' + 3M')} + c\frac{B}{L'^2} \end{aligned} \tag{27}$$

$$\begin{aligned} C_{eff} = &\frac{1}{9}\frac{c(9C + 3B + 3D + A)}{(L' + 3M')^2} + \frac{1}{9}\frac{c(A - 3B)}{L'^2} \\ &- \frac{4}{9}\frac{N'(\mu_2 - \mu_1)c}{L'^3} - \frac{1}{9}\frac{c(3D + 2A)}{L'(L' + 3M')} \\ &+ \frac{1}{9}\frac{c(4N' + 6O')(\mu_2 - \mu_1)}{L'^2 (L' + 3M')} \\ &+ \frac{1}{9}\frac{c(3N' + 9P')(K_2 - K_1)}{L'^2 (L' + 3M')} \\ &- \frac{1}{3}\frac{c(9Q' + 3O' + 3P' + N')(K_2 - K_1)}{(L' + 3M')^3} \end{aligned} \tag{28}$$

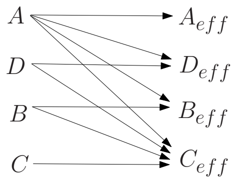

Fig. 2. Mixing scheme for the nonlinear modes.

$$D_{\mathit{eff}} = 2c\frac{(2N'M' - L'O')(\mu_2 - \mu_1)}{L'^3(L' + 3M')} - 2c\frac{M'A}{L'^2(L'+3M')} + c\frac{D}{L'(L'+3M')}. \quad (29)$$

If we use the definitions of the parameters L' and M', we also get

$$\mu_{\mathit{eff}} = \mu_1 + c\frac{\mu_2 - \mu_1}{c + (1-c)\left[1 + \frac{6}{5}\left(\frac{\mu_2}{\mu_1} - 1\right)\frac{K_1 + 2\mu_1}{3K_1 + 4\mu_1}\right]} \quad (30)$$

$$K_{\mathit{eff}} = K_1 + \frac{(3K_1 + 4\mu_1)(K_2 - K_1)c}{3K_2 + 4\mu_1 - 3c(K_2 - K_1)}. \quad (31)$$

These expressions hold in linear regime, as well [11]. Nevertheless, equations (26)–(29) offer a useful generalization of previous results obtained within the framework of the Green elasticity [31].

The general picture outlined above fulfils a series of important universal properties:

1. Equations (26)–(31) are valid also for $c = 1$; it means that if $c = 1$, then $\mu_{\mathit{eff}} = \mu_2$, $K_{\mathit{eff}} = K_2$, $A_{\mathit{eff}} = A$, $B_{\mathit{eff}} = B$, $C_{\mathit{eff}} = C$, $D_{\mathit{eff}} = D$, as expected.
2. The nonlinear elastic moduli A, B, C and D affect the effective nonlinear moduli of the composite material following the universal scheme showed in Figure 2. Therefore, there is a complicated mixing of the nonlinear elastic modes induced by the heterogeneity of the structure.
3. If the linear elastic moduli of the matrix and of the spheres are the very same ($K_1 = K_2$ and $\mu_1 = \mu_2$), we simply obtain $K_{\mathit{eff}} = K_1$, $\mu_{\mathit{eff}} = \mu_1$ and the following special set of results for the nonlinear components

$$A_{\mathit{eff}} = cA \quad (32)$$
$$B_{\mathit{eff}} = cB \quad (33)$$
$$C_{\mathit{eff}} = cC \quad (34)$$
$$D_{\mathit{eff}} = cD \quad (35)$$

equations (32)–(35) imply that the nonlinearity of the overall system is simply proportional to the nonlinearity of the spherical inclusions.
4. If we let $D = 2B$, we move from the Cauchy elasticity to the Green elasticity formalism, assuming the existence of a strain energy function for the inhomogeneities. It is important to remark that the following property holds: if $D = 2B$ then the relation $D_{\mathit{eff}} = 2B_{\mathit{eff}}$ is true for the effective nonlinear moduli. It can be verified by direct calculation and it means that our approach is perfectly consistent with the energy balance of the composite material. In other words, we have verified that if a strain energy function exists for the embedded spherical inclusions, then an overall strain energy function exists for the whole composite structure.
5. If we consider the special value of the Poisson ratio $\nu_1 = \nu_2 = 1/5$ (both for the matrix and the spheres) and different values for the Young moduli $E_1 \neq E_2$, we obtain another interesting result: the effective Poisson ratio assume the same value $\nu_{\mathit{eff}} = 1/5$, the effective Young modulus E_{eff} assumes the value

$$E_{\mathit{eff}} = \frac{E_1(1-c) + E_2(1+c)}{E_1(1+c) + E_2(1-c)}E_1 \quad (36)$$

and the effective nonlinear elastic moduli can be calculated as follows

$$X_{\mathit{eff}} = \frac{8E_1^3 c}{[E_1(1+c) + E_2(1-c)]^3}X \quad (37)$$

where the symbol X represents any modulus A, B, C or D (the four effective parameters exhibit the same behavior). Therefore, we can say that the special value $\nu_1 = \nu_2 = 1/5$ uncouples the behavior of the nonlinear elastic modes (described above), generating a direct correspondence among the nonlinear moduli of the spheres and the effective nonlinear moduli. Furthermore, if we add the condition $E_1 = E_2$, we recover equations (32)–(35). The special value 1/5 for the Poisson ratio comes out in several issues considering a dispersion of spherical inclusions. For example, for linear porous materials (with spherical pores) and for linear dispersions of rigid spheres the value 1/5 is a fixed points for the Poisson ratio: if $\nu_1 = 1/5$, then we have $\nu_{\mathit{eff}} = 1/5$ for all spheres concentrations [14,42]. Moreover, there is another interesting behaviour of the effective Poisson ratio for high volume fraction of pores or rigid spheres: in both cases for $c \to 1$ the effective Poisson ratio converges to the fixed value $\nu_{\mathit{eff}} = 1/5$, irrespective of the matrix Poisson ratio [14,42–44].
6. Finally, we analyze the properties of the dispersion when the embedded spherical inclusions are made of an incompressible material: the constitutive relation equation (3) describes an incompressible medium in the limit $\lambda_2 \to \infty$ (or, equivalently, $K_2 \to \infty$ since $K_2 = \lambda_2 + 2\mu_2/3$); by inverting equation (3), writing the strain tensor in terms of the stress tensor and performing such a limit, we obtain (up to the second order)

$$\hat{\epsilon}^s = \frac{1}{2\mu_2}\hat{T}^s - \frac{1}{6\mu_2}\mathrm{Tr}\left(\hat{T}^s\right)\hat{I} - \frac{A}{8\mu_2^3}\left(\hat{T}^s\right)^2 + \frac{A}{24\mu_2^3}\mathrm{Tr}\left[\left(\hat{T}^s\right)^2\right]\hat{I} - \frac{A}{36\mu_2^3}\left[\mathrm{Tr}\left(\hat{T}^s\right)\right]^2\hat{I} + \frac{A}{12\mu_2^3}\hat{T}^s\mathrm{Tr}\left(\hat{T}^s\right) \quad (38)$$

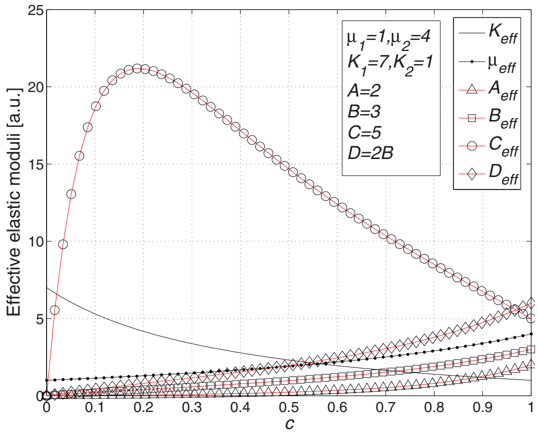

Fig. 3. (Color online) Linear and nonlinear effective elastic moduli of a dispersion of spheres in terms of the volume fraction c. We have used the values $\mu_1 = 1, \mu_2 = 4, K_1 = 7, K_2 = 1, A = 2, B = 3, C = 5, D = 2B$ in arbitrary units.

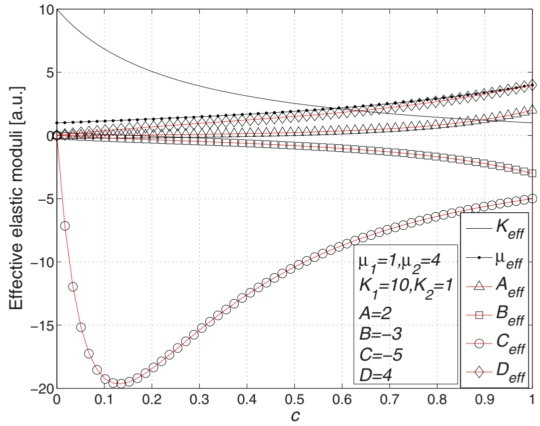

Fig. 4. (Color online) Linear and nonlinear effective elastic moduli of a dispersion of spheres in terms of the volume fraction c. We have used the values $\mu_1 = 1, \mu_2 = 4, K_1 = 10, K_2 = 1, A = 2, B = -3, C = -5, D = 4$ in arbitrary units.

which describes a nonlinear isotropic and incompressible material. We remark that only the nonlinear modulus A appears in such a constitutive equation and that equation (38) imposes $\mathrm{Tr}(\hat{\epsilon}^s) = 0$, as requested by the incompressibility. In this limiting condition, as for the effective linear moduli, we observe that equation (30) for μ_{eff} remains unchanged and equation (31) leads to

$$K_{eff} = K_1 + \left(K_1 + \frac{4}{3}\mu_1\right)\frac{c}{1-c}. \tag{39}$$

On the other hand, the nonlinear elastic moduli have been eventually found as

$$A_{eff} = 125A\theta \tag{40}$$

$$B_{eff} = -\frac{125}{3}A\theta \tag{41}$$

$$C_{eff} = \frac{250}{9}A\theta \tag{42}$$

$$D_{eff} = -\frac{250}{3}A\theta \tag{43}$$

where

$$\theta = \frac{c\,(3K_1 + 4\mu_1)^3\,\mu_1^3}{\psi^3} \tag{44}$$

$$\psi = 6\,(K_1 + 2\mu_1)\,[c\mu_1 + (1-c)\mu_2] + \mu_1\,(9K_1 + 8\mu_1). \tag{45}$$

One can observe that, as expected, the effective nonlinear elastic moduli depend only on the modulus A describing the nonlinearity of the spheres, as shown in equation (38). Moreover, we remark that a single modulus A for the spheres can generate four different effective nonlinear moduli, as predicted by the scheme in Figure 2.

To conclude we present some numerical results obtained by the implementation of equations (26)–(31). They are reported in Figures 3 and 4 where the Green and Cauchy elasticity cases are investigated, respectively. The effective elastic moduli are reported as function of the volume fraction c of the spherical inclusions. In both Green and Cauchy cases we observe a consistent amplification (in absolute value) of the nonlinear effective modulus C_{eff}. We have verified that such a phenomenon is always exhibited when $K_1 \gg K_2$ (i.e. when the matrix is much more incompressible than the spheres) and that the higher values of C_{eff} appear for small values of the volume fraction c, belonging to the range of applicability of the present theory.

As it is well known, simple bounds for the linear effective moduli exist

$$\frac{1}{\frac{1-c}{K_1} + \frac{c}{K_2}} \le K_{eff} \le (1-c)\,K_1 + cK_2 \tag{46}$$

$$\frac{1}{\frac{1-c}{\mu_1} + \frac{c}{\mu_2}} \le \mu_{eff} \le (1-c)\,\mu_1 + c\mu_2. \tag{47}$$

The lower bounds in equations (46) and (47) are referred to as the Voigt bounds, and the upper bounds are designated as the Reuss bounds [10]. More refined limitations have been derived by Hashin and Shtrikman [3]. From our numerical results, we observe that the nonlinear properties, contrarily to the linear ones, are not bounded. Rather, they show a strong amplification under suitable conditions, leading to nonlinear effective moduli much greater than those ones of the constituent materials. This point is important in the topic of designing materials with desired properties and functions.

4 Dispersion of parallel cylindrical inclusions

The nonlinear elastic features of fiber-reinforced media are relevant in many materials science problems. Recently, arrays of parallel carbon nanotubes were produced and embedded into a homogeneous matrix [45]. The high level of ordering and uniformity in these arrays is useful for applications in reinforcing techniques and sensors designing. Composite films with homogeneously dispersed single wall nanotubes showed an extraordinary reinforcing effect: the addition of 1.0% of fibers tripled the tensile strength of the original tapes [46]. Moreover, an effective functionalization method was investigated to take full advantage of the exceptional performance of both carbon nanotubes and epoxy polymer for composite application. The elastic modulus of the nanocomposite was enhanced 24.6% with only 0.5% loading of functionalized carbon nanotubes, in contrast to the 3.2% increase of unfunctionalized carbon nanotube reinforced composite [47]. Alternatively, the mechanical reinforcement of optically functional materials is of significant interest because of the rapid expansion of active displays. Finally, transparent polymeric composites with enhanced properties have been developed by using cellulose nanofibers as mechanical reinforcing agents [48,49].

In earlier works the linear analysis for a parallel distribution of fibres has been developed by means of the Eshelby methodology and of the differential effective medium theory [13,50]. Moreover, the mechanical response of elastic and inelastic fibre-strengthened materials has been investigated, also with self-consistent models [51–53]. Here we take into consideration an assembly of parallel cylinders, as represented in Figure 5, described by an arbitrary Cauchy constitutive relation. The cylindrical inclusions are randomly embedded into a linear matrix with elastic moduli K_1 and μ_1. This is a simple but meaningful model of a nonlinear fibrous material. As in the previous Section, we suppose that the volume fraction c of the embedded phase is small (dilute dispersion). By considering that the system shows a transverse isotropic symmetry (uniaxial symmetry), we assume the plane strain condition on an arbitrary plane π (see Fig. 5) orthogonal to the cylinders. We therefore can elaborate our problem by two-dimensional elasticity. Moreover, in plane strain condition, it is a common choice to introduce the two dimensional elastic moduli $\mu^{2D} = \mu$ and $K^{2D} = K + \mu/3$, where K and μ are the customarily used three-dimensional moduli [50]. For sake of simplicity, throughout this section we rename $K^{2D} = K$ and $\mu^{2D} = \mu$. The linear matrix is described by

$$\hat{T} = 2\mu_1\hat{\epsilon} + (K_1 - \mu_1)\,\mathrm{Tr}\,(\hat{\epsilon})\,\hat{I} \tag{48}$$

while the cylindrical inclusions are described by the Cauchy constitutive relation

$$\begin{aligned}\hat{T}^s =& 2\mu_2\hat{\epsilon}^s + (K_2 - \mu_2)\,\mathrm{Tr}\,(\hat{\epsilon}^s)\,\hat{I} + A\,(\hat{\epsilon}^s)^2 \\ &+ B\mathrm{Tr}\left[(\hat{\epsilon}^s)^2\right]\hat{I} + C\left[\mathrm{Tr}\,(\hat{\epsilon}^s)\right]^2\hat{I} + D\hat{\epsilon}^s\mathrm{Tr}\,(\hat{\epsilon}^s)\end{aligned} \tag{49}$$

where any strain or stress tensor is now represented by a square matrix of order two. We remark that equation (4) or, equivalently, equation (5) are correct for any geometry. Therefore, they can be directly used in the present analysis. Nevertheless, in order to use equation (5) we need to recall that

$$\left(\hat{\mathcal{C}}^{(1)}\right)^{-1}\hat{T}^s = \frac{1}{2\mu_1}\hat{T}^s - \frac{K_1 - \mu_1}{4\mu_1 K_1}\mathrm{Tr}\left(\hat{T}^s\right)\hat{I} \tag{50}$$

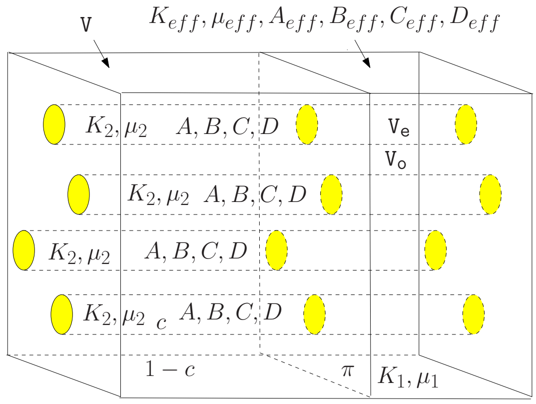

Fig. 5. (Color online) Scheme of a dispersion of nonlinear parallel cylinders embedded into a linear matrix.

and

$$\hat{\mathcal{S}}\hat{\epsilon}^s = \frac{1}{2}\frac{K_1 + 2\mu_1}{K_1 + \mu_1}\hat{\epsilon}^s + \frac{1}{4}\frac{K_1 - 2\mu_1}{K_1 + \mu_1}\mathrm{Tr}\,(\hat{\epsilon}^s)\,\hat{I} \tag{51}$$

where $\hat{\mathcal{S}}$ is the Eshelby tensor for the cylindrical inclusion [25]. By inserting equations (49)–(51) into equation (5), a tedious calculation leads to

$$\begin{aligned}&L\hat{\epsilon}^s + M\mathrm{Tr}\,(\hat{\epsilon}^s)\,\hat{I} + N\,(\hat{\epsilon}^s)^2 + O\hat{\epsilon}^s\mathrm{Tr}\,(\hat{\epsilon}^s) \\ &\qquad + P\mathrm{Tr}\left[(\hat{\epsilon}^s)^2\right]\hat{I} + Q\left[\mathrm{Tr}\,(\hat{\epsilon}^s)\right]^2\hat{I} = \hat{\epsilon}^\infty\end{aligned} \tag{52}$$

which fully characterizes the internal strain induced in a nonlinear cylindrical inclusion by the uniform externally applied deformation $\hat{\epsilon}^\infty$. The parameters L, M, N, O, P and Q have been calculated as

$$L = 1 + \frac{1}{2}\frac{K_1 + 2\mu_1}{K_1 + \mu_1}\left(\frac{\mu_2}{\mu_1} - 1\right) \tag{53}$$

$$M = \frac{1}{4\,(K_1 + \mu_1)} \times \left[2K_2 - K_1\left(1 + \frac{\mu_2}{\mu_1}\right) - 2\,(\mu_2 - \mu_1)\right] \tag{54}$$

$$N = \frac{A}{4\mu_1}\frac{K_1 + 2\mu_1}{K_1 + \mu_1} \tag{55}$$

$$O = \frac{D}{4\mu_1}\frac{K_1 + 2\mu_1}{K_1 + \mu_1} \tag{56}$$

$$P = \frac{1}{8\,(K_1 + \mu_1)}\left(4B - A\frac{K_1}{\mu_1}\right) \tag{57}$$

$$Q = \frac{1}{8\,(K_1 + \mu_1)}\left(4C - D\frac{K_1}{\mu_1}\right). \tag{58}$$

To further proceed, we follow once again a procedure similar to that described in the previous section: we make use of equations (17) and (18) for the average values of stress and strain in the composite structure; then, we use the constitutive equations of the materials and we derive the following relations

$$\begin{aligned}
\mathrm{Tr}\,\langle\hat{\epsilon}\rangle &= (L'+2M')\mathrm{Tr}\,(\hat{\epsilon}^s) + (N'+2P')\mathrm{Tr}\left[(\hat{\epsilon}^s)^2\right] \\
&\quad +(O'+2Q')\left[\mathrm{Tr}\,(\hat{\epsilon}^s)\right]^2 \\
\langle\hat{\epsilon}\rangle^2 &= L'^2\,(\hat{\epsilon}^s)^2 + 2L'M'\hat{\epsilon}^s\mathrm{Tr}\,(\hat{\epsilon}^s) + M'^2\left[\mathrm{Tr}\,(\hat{\epsilon}^s)\right]^2\hat{I} \\
\langle\hat{\epsilon}\rangle\,\mathrm{Tr}\langle\hat{\epsilon}\rangle &= L'(L'+2M')\hat{\epsilon}^s\mathrm{Tr}\,(\hat{\epsilon}^s) \\
&\quad +M'(L'+2M')\left[\mathrm{Tr}\,(\hat{\epsilon}^s)\right]^2\hat{I} \\
\mathrm{Tr}\left[\langle\hat{\epsilon}\rangle^2\right] &= L'^2\mathrm{Tr}\left[(\hat{\epsilon}^s)^2\right] + 2M'(L'+M')\left[\mathrm{Tr}\,(\hat{\epsilon}^s)\right]^2 \\
\left[\mathrm{Tr}\,\langle\hat{\epsilon}\rangle\right]^2 &= (L'+2M')^2\left[\mathrm{Tr}\,(\hat{\epsilon}^s)\right]^2 \qquad (59)
\end{aligned}$$

in order to obtain the constitutive equation of the fibrous system. Here we have used again the definitions $L' = c + (1-c)\,L$, $M' = (1-c)M$, $N' = (1-c)N$, $O' = (1-c)O$, $P' = (1-c)P$ and $Q' = (1-c)Q$.

4.1 Results

The constitutive equation is expressed in terms of the effective linear and nonlinear elastic moduli as follows

$$\begin{aligned}
\langle\hat{T}\rangle &= 2\mu_{eff}\langle\hat{\epsilon}\rangle + (K_{eff}-\mu_{eff})\,\mathrm{Tr}\langle\hat{\epsilon}\rangle\hat{I} \\
&\quad +A_{eff}\langle\hat{\epsilon}\rangle^2 + B_{eff}\mathrm{Tr}\left[\langle\hat{\epsilon}\rangle^2\right]\hat{I} \\
&\quad +C_{eff}\left[\mathrm{Tr}\langle\hat{\epsilon}\rangle\right]^2\hat{I} + D_{eff}\langle\hat{\epsilon}\rangle\mathrm{Tr}\langle\hat{\epsilon}\rangle \qquad (60)
\end{aligned}$$

where

$$\begin{aligned}
\mu_{eff} &= \mu_1 + c\frac{\mu_2-\mu_1}{L'} \\
&= \mu_1 + c\frac{\mu_2-\mu_1}{c+(1-c)\left[1+\frac{1}{2}\left(\frac{\mu_2}{\mu_1}-1\right)\frac{K_1+2\mu_1}{K_1+\mu_1}\right]} \qquad (61)
\end{aligned}$$

$$\begin{aligned}
K_{eff} &= K_1 + c\frac{K_2-K_1}{L'+2M'} \\
&= K_1 + c\frac{K_2-K_1}{c+(1-c)\frac{\mu_1+K_2}{\mu_1+K_1}}. \qquad (62)
\end{aligned}$$

We remark that both μ_{eff} and K_{eff} are consistent with results published elsewhere [54]. As for the effective nonlinear elastic moduli, we get

$$A_{eff} = \frac{Ac}{L'^2} - 2c\frac{N'\,(\mu_2-\mu_1)}{L'^3} \qquad (63)$$

$$\begin{aligned}
B_{eff} &= \frac{c\left[N'\,(\mu_2-\mu_1)+BL'\right]}{L'^3} \\
&\quad -\frac{c\,(2P'+N')\,(K_2-K_1)}{L'^2\,(L'+2M')} \qquad (64)
\end{aligned}$$

$$\begin{aligned}
C_{eff} &= c\frac{4C+2B+2D+A}{4\,(L'+2M')^2} + c\frac{A-2B}{4L'^2} \\
&\quad +c\frac{2\,(O'+N')\,(\mu_2-\mu_1)+(2P'+N')\,(K_2-K_1)}{2L'^2\,(L'+2M')} \\
&\quad -\frac{c\,(2P'+N'+4Q'+2O')\,(K_2-K_1)}{2\,(L'+2M')^3} \\
&\quad -\frac{cN'\,(\mu_2-\mu_1)}{L'^3} - c\frac{A+D}{2L'\,(L'+2M')} \qquad (65)
\end{aligned}$$

$$\begin{aligned}
D_{eff} &= 2\frac{(2N'M'-L'O')\,(\mu_2-\mu_1)\,c}{L'^3\,(L'+2M')} \\
&\quad -2c\frac{M'A}{L'^2\,(L'+2M')} + \frac{cD}{L'\,(L'+2M')}. \qquad (66)
\end{aligned}$$

They represent the thorough nonlinear characterization of the random dispersion of parallel cylindrical inclusions. It is interesting to observe that all the properties described in the previous section for the dispersion of spherical inclusions can be easily verified also for the present case. In particular, the scheme represented in Figure 2 remains valid.

We analyze in more detail the special case with $\nu_1 = \nu_2 = 1/4$ (corresponding to the two-dimensional Poisson ratio $\nu_{2D} = \nu_{3D}/(1-\nu_{3D}) = 1/3$ [50]) and $E_1 \neq E_2$. In this case, the effective Poisson ratio is $\nu_{eff} = 1/4$ and the effective Young modulus E_{eff} is given by

$$E_{eff} = \frac{E_1\,(1-c)+E_2\,(2+c)}{E_1\,(1+2c)+2E_2\,(1-c)}E_1 \qquad (67)$$

while the effective nonlinear elastic moduli are calculated as follows

$$X_{eff} = \frac{27E_1^3c}{\left[E_1\,(1+2c)+2E_2\,(1-c)\right]^3}X \qquad (68)$$

where the symbol X represents any modulus A, B, C or D (the four effective parameters exhibit the same behavior). Therefore, as before, we can say that for the special value $\nu_1 = \nu_2 = 1/4$ the nonlinear elastic modes are decoupled, generating a direct correspondence among the nonlinear moduli of the spheres and the effective nonlinear moduli.

Finally, we have numerically implemented equations (61)–(66) in order to shown some explicit results for both Green (see Fig. 6) and Cauchy (see Fig. 7) elasticity. As in the previous section, we observe a consistent amplification (in absolute value) of the nonlinear effective modulus C_{eff}. We have also verified that such a phenomenon is exhibited when $K_1 \gg K_2$ (i.e. when the matrix is much more incompressible than the spheres) and that the higher values of C_{eff} appear for small values of the volume fraction c, belonging to the range of applicability of the present theory. The enhancement of C_{eff} represents therefore a general feature of nonlinear composite systems.

5 Conclusions

In this work we have considered the linear and nonlinear elastic behavior of a composite material. In particular, we have taken into account a dispersion of isotropic nonlinear inclusions (spherical or cylindrical) embedded into a linear isotropic host matrix.

We have obtained the expressions of the four nonlinear effective elastic moduli of the composite medium with inclusions described by Cauchy constitutive equations. Then, as a particular case, we have considered the Green elasticity to describe the nonlinear behavior of the

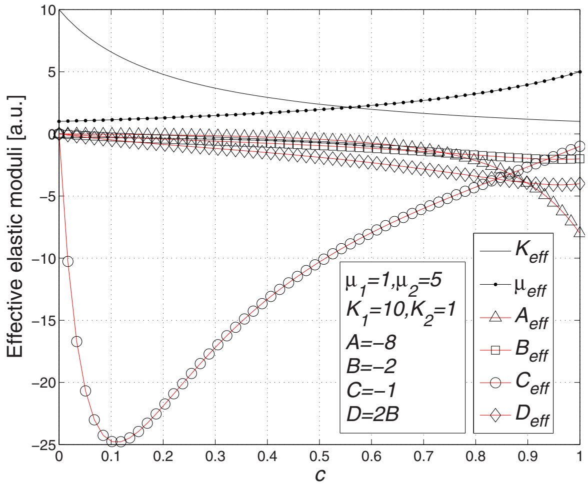

Fig. 6. (Color online) Linear and nonlinear effective elastic moduli for a dispersion of cylinders in terms of the volume fraction c. We have used the values $\mu_1 = 1, \mu_2 = 5, K_1 = 10, K_2 = 1, A = -8, B = -2, C = -1, D = 2B$ in arbitrary units

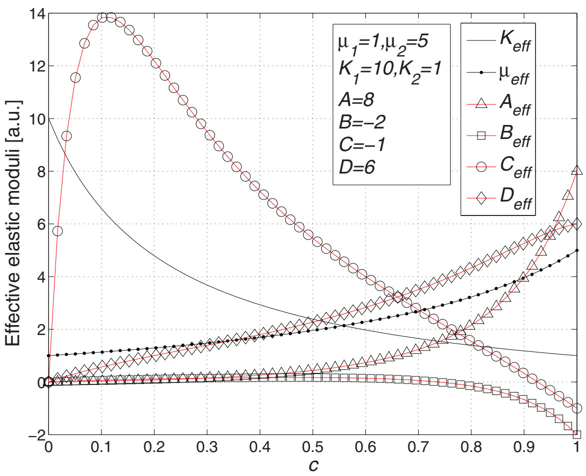

Fig. 7. (Color online) Linear and nonlinear effective elastic moduli for a dispersion of cylinders in terms of the volume fraction c. We have used the values $\mu_1 = 1, \mu_2 = 5, K_1 = 10, K_2 = 1, A = 8, B = -2, C = -1, D = 6$ in arbitrary units.

inclusions. In this case we have verified that if a strain energy function exists for the inhomogeneities, then an overall strain energy function exists for the whole composite structure.

Moreover, we have observed that the nonlinear effective elastic moduli, contrarily to the linear ones, are not subjected to any bound limiting their values. We have indeed found some large amplifications of the nonlinear behavior in certain given conditions.

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Appendix A: Nonlinear Eshelby theory within Green elasticity

In this Appendix we provide the complete proof of the existence and unicity of the solution of equation (4) (within Green elasticity). This is a key result, underlying all the formal developments described in this paper. At first, we briefly outline the concepts of eigenstrain and inclusion in order to introduce the adopted notation and to recall the most important equations of the Eshelby theory [25,32,33].

We suppose to consider an infinite medium with stiffness tensor $\hat{\mathcal{C}}^{(1)}$; moreover, we consider an embedded ellipsoidal inclusion $\mathtt{V}$ described by the constitutive equation $\hat{T} = \hat{\mathcal{C}}^{(1)}(\hat{\epsilon} - \hat{\epsilon}^*)$. The strain $\hat{\epsilon}^*$ is called eigenstrain (or stress-free strain). In these conditions the following relations describe the strain inside and outside the region $\mathtt{V}$ [25]

$$\hat{\epsilon}(\boldsymbol{x}) = \begin{cases} \hat{\mathcal{S}}\hat{\epsilon}^* & \text{if } \boldsymbol{x} \in \mathtt{V} \\ \hat{\mathcal{S}}^{\infty}(\boldsymbol{x})\,\hat{\epsilon}^* & \text{if } \boldsymbol{x} \notin \mathtt{V} \end{cases} \tag{A.1}$$

where $\hat{\mathcal{S}}$ is the internal Eshelby tensor and $\hat{\mathcal{S}}^{\infty}$ is the external Eshelby tensor.

We suppose now to consider an infinite medium with stiffness tensor $\hat{\mathcal{C}}^{(1)}$ in $\Re^3\diagdown\mathtt{V}$ (matrix) and $\hat{\mathcal{C}}^{(2)}$ in the ellipsoidal region $\mathtt{V}$ (inclusion). We remotely load the system with a uniform strain $\hat{\epsilon}^{\infty}$ or, equivalently, with the uniform stress $\hat{T}^{\infty}$. Of course we have $\hat{T}^{\infty} = \hat{\mathcal{C}}^{(1)}\hat{\epsilon}^{\infty}$. This configuration can be analyzed by means of the Eshelby equivalence principle [32]: the system can be described by the superimposition of two simpler cases (see Fig. 8) [25]. The first situation A concerns a homogeneous medium with stiffness $\hat{\mathcal{C}}^{(1)}$ uniformly deformed by means of the remote loads $\hat{\epsilon}^{\infty}$ or T^{∞}. The second situation B, without remote loads, is represented by an eigenstrain $\hat{\epsilon}^*$ embedded into a medium, characterized everywhere by $\hat{\mathcal{C}}^{(1)}$. The eigenstrain must be imposed searching for the equivalence between the original problem and the superimposition $A+B$. The following relation hold on inside the region $\mathtt{V}$ (s means inside $\mathtt{V}$)

$$\begin{aligned} \hat{\epsilon}^s &= \hat{\epsilon}^{A,s} + \hat{\epsilon}^{B,s} = \hat{\epsilon}^{\infty} + \hat{\mathcal{S}}\hat{\epsilon}^* \\ \hat{T}^s &= \hat{T}^{A,s} + \hat{T}^{B,s} = \hat{\mathcal{C}}^{(1)}\hat{\epsilon}^{\infty} + \hat{\mathcal{C}}^{(1)}\left(\hat{\epsilon}^{B,s} - \hat{\epsilon}^*\right) \\ &= \hat{\mathcal{C}}^{(1)}\hat{\epsilon}^{\infty} + \hat{\mathcal{C}}^{(1)}\left(\hat{\mathcal{S}}\hat{\epsilon}^* - \hat{\epsilon}^*\right). \end{aligned} \tag{A.2}$$

In the inclusion we have $\hat{T}^s = \hat{\mathcal{C}}^{(2)}\hat{\epsilon}^s$ and therefore

$$\underbrace{\hat{\mathcal{C}}^{(1)}\hat{\epsilon}^{\infty} + \hat{\mathcal{C}}^{(1)}\left(\hat{\mathcal{S}}\hat{\epsilon}^* - \hat{\epsilon}^*\right)}_{T^s} = \hat{\mathcal{C}}^{(2)}\underbrace{\left(\hat{\epsilon}^{\infty} + \hat{\mathcal{S}}\hat{\epsilon}^*\right)}_{\hat{\epsilon}^s}. \tag{A.3}$$

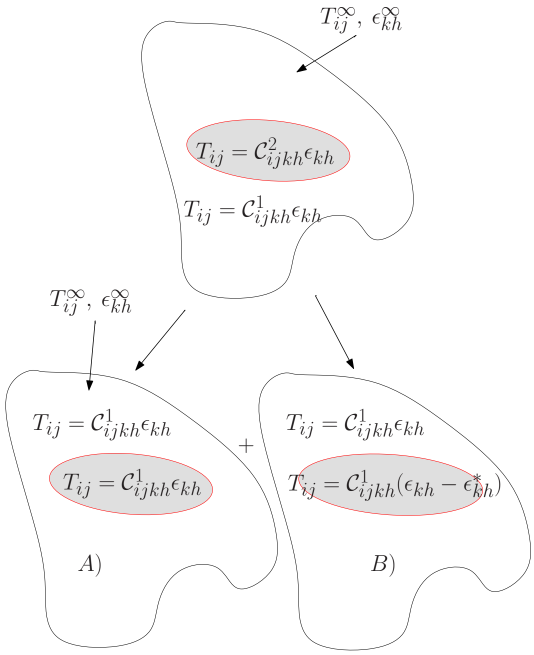

Fig. 8. Scheme of an ellipsoidal inclusion and the Eshelby equivalence principle.

The following relations can be finally obtained for the eigenstrain and for the actual strain in $\mathtt{V}$

$$\hat{\epsilon}^* = \left[\left(\hat{I} - \left(\hat{\mathcal{C}}^{(1)}\right)^{-1}\hat{\mathcal{C}}^{(2)}\right)^{-1} - \hat{\mathcal{S}}\right]^{-1}\hat{\epsilon}^{\infty} \tag{A.4}$$

$$\hat{\epsilon}^s = \left(\hat{I} - \left(\hat{\mathcal{C}}^{(1)}\right)^{-1}\hat{\mathcal{C}}^{(2)}\right)^{-1}\hat{\epsilon}^* \tag{A.5}$$

$$\hat{\epsilon}^s = \left[\hat{I} - \hat{\mathcal{S}}\left(\hat{I} - \left(\hat{\mathcal{C}}^{(1)}\right)^{-1}\hat{\mathcal{C}}^{(2)}\right)\right]^{-1}\hat{\epsilon}^{\infty} \tag{A.6}$$

if $\hat{\mathcal{C}}^{(2)} = 0$ (void) we obtain

$$\hat{\epsilon}^* = \hat{\epsilon}^s = \left[\hat{I} - \hat{\mathcal{S}}\right]^{-1}\hat{\epsilon}^{\infty}. \tag{A.7}$$

We now move towards the demonstration of the existence and unicity for the nonlinear Eshelby problem. To this aim, it is convenient to adopt the Green formulation of the elasticity theory where a strain energy function $U(\hat{\epsilon})$ defines the constitutive equation $\hat{T}(\hat{\epsilon}) = \frac{\partial U(\hat{\epsilon})}{\partial \hat{\epsilon}}$ of the inclusion, and it drives to $\hat{T}(\hat{\epsilon}) = \hat{\mathcal{C}}^{(2)}(\hat{\epsilon})\,\hat{\epsilon}$. In these conditions, the existence and unicity of a solution for equation (4) can be exactly proved under the sole hypotesis of convexity for the strain energy function $U(\hat{\epsilon})$. To prove this statement, we rearrange equation (4) as follows

$$\left\{\hat{I} - \hat{\mathcal{S}}\left[\hat{I} - \left(\hat{\mathcal{C}}^{(1)}\right)^{-1}\hat{\mathcal{C}}^{(2)}(\hat{\epsilon}^s)\right]\right\}\hat{\epsilon}^s = \hat{\epsilon}^{\infty}$$

$$\left[\hat{I} - \hat{\mathcal{S}}\right]\hat{\epsilon}^s + \hat{\mathcal{S}}\left(\hat{\mathcal{C}}^{(1)}\right)^{-1}\frac{\partial U(\hat{\epsilon}^s)}{\partial \hat{\epsilon}^s} = \hat{\epsilon}^{\infty}$$

$$\hat{\mathcal{C}}^{(1)}\left[\hat{\mathcal{S}}^{-1} - \hat{I}\right]\hat{\epsilon}^s - \hat{\mathcal{C}}^{(1)}\hat{\mathcal{S}}^{-1}\hat{\epsilon}^{\infty} + \frac{\partial U(\hat{\epsilon}^s)}{\partial \hat{\epsilon}^s} = 0. \tag{A.8}$$

The first linear term can be converted to the gradient of a quadratic form and the second constant term can be converted to the gradient of a linear form. We therefore observe that the internal strain field must satisfy the following relation

$$\frac{\partial}{\partial \hat{\epsilon}}\left\{\frac{1}{2}\hat{\epsilon}\hat{\mathcal{C}}^{(1)}\left[\hat{\mathcal{S}}^{-1} - \hat{I}\right]\hat{\epsilon} - \hat{\epsilon}\hat{\mathcal{C}}^{(1)}\hat{\mathcal{S}}^{-1}\hat{\epsilon}^{\infty} + U(\hat{\epsilon})\right\} = 0 \tag{A.9}$$

which is equivalent to equation (4). The first term represents a symmetric and positive definite quadratic form in $\hat{\epsilon}$ (see below) while the second term is a linear function of $\hat{\epsilon}$. Therefore, the sum of these two terms is a convex functional with relative minimum at $\left[\hat{I} - \hat{\mathcal{S}}\right]\hat{\epsilon}^{\infty}$. This value represents the strain field in a void ($\hat{\mathcal{C}}^{(2)}(\hat{\epsilon}) = 0$ in equation (4) or $U(\hat{\epsilon}) = 0$ in equation (A.9)) embedded into the matrix with stiffness $\hat{\mathcal{C}}^{(1)}$. If $U(\hat{\epsilon})$ is a convex functional (with $U(0) = 0$) the brackets in equation (A.9) contain the sum of two convex terms: they result in an overall convex functional with a unique minimal extremum at $\hat{\epsilon}^s$.

The next step consists in proving that the tensor given by $\hat{\mathcal{C}}^{(1)}\left[\hat{\mathcal{S}}^{-1} - \hat{I}\right]$ is symmetric. We consider the same region $\mathtt{V}$ with two different values for the eigenstrain $\hat{\epsilon}^*$ and $\hat{\epsilon}^{**}$ embedded into the material defined by $\hat{\mathcal{C}}^{(1)}$. The symmetry of the tensor can be established by means of a revised version of the Betti's reciprocal theorem [29]. We define $\hat{T}^* = \hat{\mathcal{C}}^{(1)}\hat{\epsilon}^*$ and $\hat{T}^{**} = \hat{\mathcal{C}}^{(1)}\hat{\epsilon}^{**}$. The first situation is described by the fields $\hat{T}', \hat{\epsilon}', \boldsymbol{u}'$ and the second one by $\hat{T}'', \hat{\epsilon}'', \boldsymbol{u}''$ everywhere in the space. The preliminary symmetry of the tensor $\hat{\mathcal{S}}\left[\hat{\mathcal{C}}^{(1)}\right]^{-1}$ is proved. We begin by considering the following relation (Σ is the boundary of $\mathtt{V}$ and $\boldsymbol{n}$ is the external normal unit vector)

$$\mathtt{V}\hat{T}^*\hat{\mathcal{S}}\left[\hat{\mathcal{C}}^{(1)}\right]^{-1}\hat{T}^{**} = \mathtt{V}\hat{T}^*\hat{\mathcal{S}}\hat{\epsilon}^{**} = \mathtt{V}\hat{T}^*\hat{\epsilon}'' = \hat{T}^*\int_{\mathtt{V}}\hat{\epsilon}''dv$$
$$= \hat{T}^*\int_{\mathtt{V}}\frac{\partial \boldsymbol{u}''}{\partial \boldsymbol{x}}dv = \hat{T}^*\int_{\Sigma}\boldsymbol{u}''\boldsymbol{n}dS = \hat{\mathcal{C}}^{(1)}\hat{\epsilon}^*\int_{\Sigma}\boldsymbol{u}''\boldsymbol{n}dS. \tag{A.10}$$

At the interface Σ we have $\hat{T}'\boldsymbol{n}|_{\Sigma^-} = \hat{T}'\boldsymbol{n}|_{\Sigma^+}$ (sign + indicates the external side of Σ and sign − indicates its internal side). Recalling the definition of eigenstrain we simply obtain $\hat{\mathcal{C}}^{(1)}(\hat{\epsilon}' - \hat{\epsilon}^*)\,\boldsymbol{n}|_{\Sigma^-} = \hat{\mathcal{C}}^{(1)}\hat{\epsilon}'\boldsymbol{n}|_{\Sigma^+}$ and, finally, we get $\hat{\mathcal{C}}^{(1)}\hat{\epsilon}'\boldsymbol{n}|_{\Sigma^-} - \hat{\mathcal{C}}^{(1)}\hat{\epsilon}'\boldsymbol{n}|_{\Sigma^+} = \hat{\mathcal{C}}^{(1)}\hat{\epsilon}^*\boldsymbol{n}$. We use it in equation (A.10), obtaining

$$\mathtt{V}\hat{T}^*\hat{\mathcal{S}}\left[\hat{\mathcal{C}}^{(1)}\right]^{-1}\hat{T}^{**} = \int_{\Sigma}\left[\hat{\mathcal{C}}^{(1)}\hat{\epsilon}'\boldsymbol{n}|_{\Sigma^-} - \hat{\mathcal{C}}^{(1)}\hat{\epsilon}'\boldsymbol{n}|_{\Sigma^+}\right]\boldsymbol{u}''dS. \tag{A.11}$$

On Σ^- we have $\hat{T}' = \hat{\mathcal{C}}^{(1)}\left(\hat{\epsilon}' - \hat{\epsilon}^*\right)$ and on Σ^+ we have $\hat{T}' = \hat{\mathcal{C}}^{(1)}\hat{\epsilon}'$, therefore

$$
\begin{aligned}
&\mathtt{V}\hat{T}^*\hat{\mathcal{S}}\left[\hat{\mathcal{C}}^{(1)}\right]^{-1}\hat{T}^{**}\\
&= \int_{\Sigma^-}\left(\hat{T}' + \hat{T}^*\right)\boldsymbol{n}\boldsymbol{u}''dS - \int_{\Sigma^+}\hat{T}'\boldsymbol{n}\boldsymbol{u}''dS\\
&= \int_{\mathtt{V}}\frac{\partial}{\partial \boldsymbol{x}}\left[\left(\hat{T}' + \hat{T}^*\right)\boldsymbol{u}''\right]dv + \int_{\Re^3\setminus\mathtt{V}}\frac{\partial}{\partial \boldsymbol{x}}\left[\hat{T}'\boldsymbol{u}''\right]dv\\
&= \int_{\mathtt{V}}\left(\hat{T}' + \hat{T}^*\right)\hat{\epsilon}''dv + \int_{\Re^3\setminus\mathtt{V}}\hat{T}'\hat{\epsilon}''dv\\
&= \int_{\mathtt{V}}\left[\hat{\mathcal{C}}^{(1)}\left(\hat{\epsilon}' - \hat{\epsilon}^*\right) + \hat{T}^*\right]\hat{\epsilon}''dv + \int_{\Re^3\setminus\mathtt{V}}\hat{T}'\hat{\epsilon}''dv\\
&= \int_{\mathtt{V}}\hat{\epsilon}'\hat{\mathcal{C}}^{(1)}\hat{\epsilon}''dv + \int_{\Re^3\setminus\mathtt{V}}\hat{\epsilon}'\hat{\mathcal{C}}^{(1)}\hat{\epsilon}''dv\\
&= \int_{\Re^3}\hat{\epsilon}'\hat{\mathcal{C}}^{(1)}\hat{\epsilon}''dv.
\end{aligned}
\tag{A.12}
$$

We have thus obtained a symmetric form (since $\hat{\mathcal{C}}^{(1)}$ is symmetric). Therefore, the following dual relation is valid and it can be verified as above

$$
\mathtt{V}\hat{T}^{**}\hat{\mathcal{S}}\left[\hat{\mathcal{C}}^{(1)}\right]^{-1}\hat{T}^* = \int_{\Re^3}\hat{\epsilon}'\hat{\mathcal{C}}^{(1)}\hat{\epsilon}''dv. \tag{A.13}
$$

By comparison of equations (A.12) and (A.13) we obtain

$$
\mathtt{V}\hat{T}^*\hat{\mathcal{S}}\left[\hat{\mathcal{C}}^{(1)}\right]^{-1}\hat{T}^{**} = \mathtt{V}\hat{T}^{**}\hat{\mathcal{S}}\left[\hat{\mathcal{C}}^{(1)}\right]^{-1}\hat{T}^* \tag{A.14}
$$

which establishes the symmetry of $\hat{\mathcal{S}}\left[\hat{\mathcal{C}}^{(1)}\right]^{-1}$. The inverse tensor $\left\{\hat{\mathcal{S}}\left[\hat{\mathcal{C}}^{(1)}\right]^{-1}\right\}^{-1} = \hat{\mathcal{C}}^{(1)}\hat{\mathcal{S}}^{-1}$ is again symmetric and, finally, the quantity $\hat{\mathcal{C}}^{(1)}\left[\hat{\mathcal{S}}^{-1} - \hat{I}\right]$ is symmetric since it is a sum of symmetric tensors.

We further proceed by demonstrating that the tensor $\hat{\mathcal{C}}^{(1)}\left[\hat{\mathcal{S}}^{-1} - \hat{I}\right]$ is positive definite. We consider two similar situations as described in Figure 9. The first corresponds to a homogeneous medium with displacement prescribed on the boundary, while the second case considers the addition of an inclusion without changing the fixed displacements on the external surface. No body forces are present in both schemes. We begin searching for the difference between the elastic energy stored in the two cases

$$
\Delta E = \frac{1}{2}\int_{\Omega}\left(\hat{\epsilon}_b\hat{T}_b - \hat{\epsilon}_a\hat{T}_a\right)dv. \tag{A.15}
$$

It is easy to prove that

$$
\int_{\Omega}\hat{\epsilon}_a\hat{T}_a dv = \int_{\Omega}\hat{\epsilon}_b\hat{T}_a dv \tag{A.16}
$$

$$
\int_{\Omega}\hat{\epsilon}_a\hat{T}_b dv = \int_{\Omega}\hat{\epsilon}_b\hat{T}_b dv. \tag{A.17}
$$

In order to verify equation (A.16) we write the relation

$$
\int_{\Omega}\left(\hat{\epsilon}_a - \hat{\epsilon}_b\right)\hat{T}_a dv = \int_{\Omega}\left(\frac{\partial \boldsymbol{u}_a}{\partial \boldsymbol{x}}\hat{T}_a - \frac{\partial \boldsymbol{u}_b}{\partial \boldsymbol{x}}\hat{T}_a\right)dv \tag{A.18}
$$

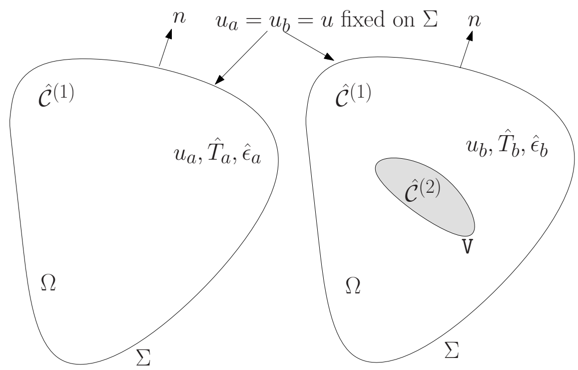

Fig. 9. Schemes of a homogeneous region and a heterogeneous one with an inclusion V. The boundary conditions prescribe the same displacement on the external surface.

where $\frac{\partial \boldsymbol{u}_a}{\partial \boldsymbol{x}}\hat{T}_a = \frac{\partial \boldsymbol{u}_a\hat{T}_a}{\partial \boldsymbol{x}}$ since $\frac{\partial \hat{T}_a}{\partial \boldsymbol{x}} = 0$ at equilibrium and similarly $\frac{\partial \boldsymbol{u}_b}{\partial \boldsymbol{x}}\hat{T}_a = \frac{\partial \boldsymbol{u}_b\hat{T}_a}{\partial \boldsymbol{x}}$. Therefore, we obtain

$$
\begin{aligned}
\int_{\Omega}\left(\hat{\epsilon}_a - \hat{\epsilon}_b\right)\hat{T}_a dv &= \int_{\Omega}\left(\frac{\partial \boldsymbol{u}_a\hat{T}_a}{\partial \boldsymbol{x}} - \frac{\partial \boldsymbol{u}_b\hat{T}_a}{\partial \boldsymbol{x}}\right)dv\\
&= \int_{\Sigma}\left(\boldsymbol{u}_a\hat{T}_a - \boldsymbol{u}_b\hat{T}_a\right)\boldsymbol{n}dS = 0
\end{aligned}
\tag{A.19}
$$

since $\boldsymbol{u}_a = \boldsymbol{u}_b$ on Σ. The dual relation given in equation (A.17) can be verified with the same method.
By inserting equations (A.16) and (A.17) into equation (A.15) we obtain

$$
\begin{aligned}
2\Delta E &= \int_{\Omega}\left(\hat{\epsilon}_b\hat{T}_b - \hat{\epsilon}_a\hat{T}_a\right)dv = \int_{\Omega}\left(\hat{\epsilon}_a\hat{T}_b - \hat{\epsilon}_b\hat{T}_a\right)dv\\
&= \int_{\Omega\setminus\mathtt{V}}\left(\hat{\epsilon}_a\hat{T}_b - \hat{\epsilon}_b\hat{T}_a\right)dv + \int_{\mathtt{V}}\left(\hat{\epsilon}_a\hat{T}_b - \hat{\epsilon}_b\hat{T}_a\right)dv\\
&= \int_{\Omega\setminus\mathtt{V}}\left(\hat{\epsilon}_a\hat{\mathcal{C}}^{(1)}\hat{\epsilon}_b - \hat{\epsilon}_b\hat{\mathcal{C}}^{(1)}\hat{\epsilon}_a\right)dv + \int_{\mathtt{V}}\left(\hat{\epsilon}_a\hat{T}_b - \hat{\epsilon}_b\hat{T}_a\right)dv.
\end{aligned}
\tag{A.20}
$$

Since the stiffness tensor $\hat{\mathcal{C}}^{(1)}$ is symmetric, we obtain the following general expression for the energy difference

$$
\Delta E = \frac{1}{2}\int_{\mathtt{V}}\left(\hat{\epsilon}_a\hat{T}_b - \hat{\epsilon}_b\hat{T}_a\right)dv. \tag{A.21}
$$

We now suppose that the prescribed displacement on Σ imposes a uniform strain in the first case of Figure 9; therefore, the second situation can be described by the Eshelby solution. With this additional hypothesis the energy difference can be rearranged as follows

$$
\begin{aligned}
\Delta E &= -\frac{1}{2}\int_{\mathtt{V}}\left(\hat{T}_a\hat{\epsilon}_b - \hat{\epsilon}_a\hat{T}_b\right)dv\\
&= -\frac{1}{2}\int_{\mathtt{V}}\left(\hat{T}_a\hat{\epsilon}_b - \hat{T}_a\left(\hat{\mathcal{C}}^{(1)}\right)^{-1}\hat{\mathcal{C}}^{(2)}\hat{\epsilon}_b\right)dv\\
&= -\frac{1}{2}\int_{\mathtt{V}}\hat{T}_a\left(\hat{I} - \left(\hat{\mathcal{C}}^{(1)}\right)^{-1}\hat{\mathcal{C}}^{(2)}\right)\hat{\epsilon}_b dv\\
&= -\frac{1}{2}\int_{\mathtt{V}}\hat{T}_a\hat{\epsilon}^* dv
\end{aligned}
\tag{A.22}
$$

having used equation (A.5). Utilizing equation (A.4) we obtain

$$\Delta E = -\frac{1}{2}\int_{\mathrm{V}} \hat{\epsilon}_a \hat{\mathcal{C}}^{(1)} \left[\left(\hat{I} - \left(\hat{\mathcal{C}}^{(1)}\right)^{-1} \hat{\mathcal{C}}^{(2)}\right)^{-1} - \hat{\mathcal{S}}\right]^{-1} \hat{\epsilon}_a dv. \tag{A.23}$$

From now on we suppose that the embedded inclusion is a void ($\hat{\mathcal{C}}^{(2)} = 0$). Accordingly, we obtain

$$\Delta E = E_b\left(\hat{\epsilon}_b\right) - E_a\left(\hat{\epsilon}_a\right) = -\frac{1}{2}\int_{\mathrm{V}} \hat{\epsilon}_a \hat{\mathcal{C}}^{(1)} \left[\hat{I} - \hat{\mathcal{S}}\right]^{-1} \hat{\epsilon}_a dv. \tag{A.24}$$

If we take into account a body without body forces and with prescribed displacements on the whole external surface, then the variational formulation of the elasticity theory leads to the minimum potential energy principle [29,30]. We apply this principle to the second case of Figure 9 (with a void). If the fields $\boldsymbol{u}_b, \hat{\epsilon}_b, \hat{T}_b$ correspond of the actual elastic fields in such a case, we have $E_b\left(\boldsymbol{u}_b, \hat{\epsilon}_b, \hat{T}_b\right) \leqslant E_b\left(\boldsymbol{u}, \hat{\epsilon}, \hat{T}\right)$ where the fields $\boldsymbol{u}, \hat{\epsilon}, \hat{T}$ correspond to any displacement $\boldsymbol{u}$ matching the prescribed boundary. In particular, we have $E_b\left(\hat{\epsilon}_b\right) \leqslant E_b\left(\hat{\epsilon}_a\right)$, where $\hat{\epsilon}_a$ is the strain in the first case of Figure 9. Moreover, we write

$$\begin{aligned} E_b\left(\hat{\epsilon}_a\right) &= \frac{1}{2}\int_{\Omega\setminus\mathrm{V}} \hat{\epsilon}_a \hat{\mathcal{C}}^{(1)} \hat{\epsilon}_a dv + \frac{1}{2}\int_{\mathrm{V}} \hat{\epsilon}_a \hat{\mathcal{C}}^{(2)} \hat{\epsilon}_a dv \\ &= \frac{1}{2}\int_{\Omega\setminus\mathrm{V}} \hat{\epsilon}_a \hat{\mathcal{C}}^{(1)} \hat{\epsilon}_a dv = E_a\left(\hat{\epsilon}_a\right) - \frac{1}{2}\int_{\mathrm{V}} \hat{\epsilon}_a \hat{\mathcal{C}}^{(1)} \hat{\epsilon}_a dv \end{aligned} \tag{A.25}$$

so that

$$\begin{aligned} E_b\left(\hat{\epsilon}_b\right) &\leqslant E_b\left(\hat{\epsilon}_a\right) \\ E_b\left(\hat{\epsilon}_b\right) &\leqslant E_a\left(\hat{\epsilon}_a\right) - \frac{1}{2}\int_{\mathrm{V}} \hat{\epsilon}_a \hat{\mathcal{C}}^{(1)} \hat{\epsilon}_a dv \\ E_b\left(\hat{\epsilon}_b\right) - E_a\left(\hat{\epsilon}_a\right) &\leqslant -\frac{1}{2}\int_{\mathrm{V}} \hat{\epsilon}_a \hat{\mathcal{C}}^{(1)} \hat{\epsilon}_a dv. \end{aligned} \tag{A.26}$$

Since $\hat{\epsilon}_a$ is uniform, combining equations (A.24) and (A.26), we obtain

$$\hat{\epsilon}_a \hat{\mathcal{C}}^{(1)} \left[\hat{I} - \hat{\mathcal{S}}\right]^{-1} \hat{\epsilon}_a - \hat{\epsilon}_a \hat{\mathcal{C}}^{(1)} \hat{\epsilon}_a \geqslant 0. \tag{A.27}$$

or

$$\hat{T}_a \left[\hat{I} - \hat{\mathcal{S}}\right]^{-1} \left[\hat{\mathcal{C}}^{(1)}\right]^{-1} \hat{T}_a - \hat{T}_a \left[\hat{\mathcal{C}}^{(1)}\right]^{-1} \hat{T}_a \geqslant 0. \tag{A.28}$$

So, the tensor $\left[\hat{I} - \hat{\mathcal{S}}\right]^{-1} \left[\hat{\mathcal{C}}^{(1)}\right]^{-1} - \left[\hat{\mathcal{C}}^{(1)}\right]^{-1}$ is positive definite.

For any tensor it is true that $[I - A]^{-1} = I + \left[A^{-1} - I\right]^{-1}$ and therefore we obtain

$$\left[\hat{I} - \hat{\mathcal{S}}\right]^{-1} \left[\hat{\mathcal{C}}^{(1)}\right]^{-1} - \left[\hat{\mathcal{C}}^{(1)}\right]^{-1} = \left[\hat{\mathcal{S}}^{-1} - \hat{I}\right]^{-1} \left[\hat{\mathcal{C}}^{(1)}\right]^{-1}. \tag{A.29}$$

Finally, the tensor $\left[\hat{\mathcal{S}}^{-1} - \hat{I}\right]^{-1} \left[\hat{\mathcal{C}}^{(1)}\right]^{-1}$ and its inverse $\hat{\mathcal{C}}^{(1)} \left[\hat{\mathcal{S}}^{-1} - \hat{I}\right]$ are symmetric and positive definite.

It is interesting to observe that all the results given in Appendix A and in Section 2 exactly apply also for an anisotropic and homogeneous ellipsoidal inclusion embedded into an anisotropic and homogeneous matrix [31]. In this case, the Eshelby tensor $\hat{\mathcal{S}}$ depends on the geometry and on $\hat{\mathcal{C}}^{(1)}$ [25].

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