VARIATIONAL FORMULATION OF THE EQUATIONS OF MOTION IN CONTINUUM MECHANICS

Tudor S. Ratiu
Section de Mathématiques

and

Bernoulli Center
Ecole Polytechnique Fédérale de Lausanne, Switzerland

With F. Gay-Balmaz and J. Marsden. One of Jerry's last papers.

ICIAM Vancouver, July 2011

PLAN OF THE PRESENTATION

- *Continuum mechanical setup*
- *The heavy top*
- *Affine semidirect product Lagrangian reduction*
- *Body and spatial equations for the heavy top*
- *Fixed boundary barotropic fluids*
- *Elasticity*
- *Free boundary fluids*

Continuum mechanical setup

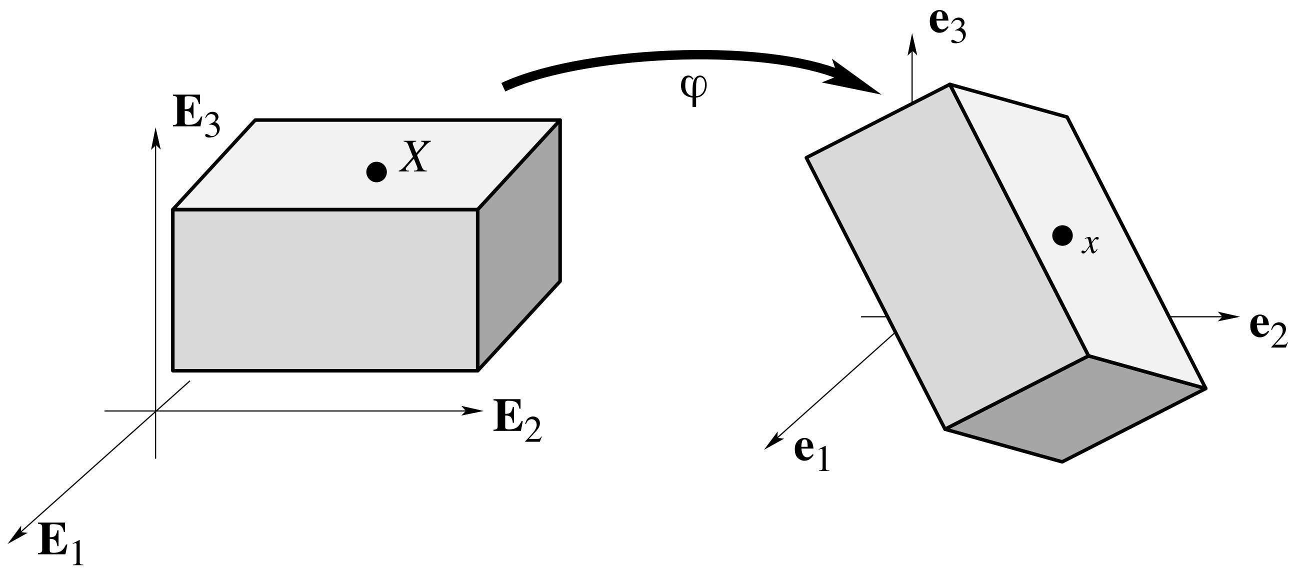
Reference configuration: $(\mathcal{B}, \mathbf{G})$ oriented Riemannian manifold
Usually $\mathcal{B} \subset \mathbb{R}^3 = \{\mathbf{X} = (X^1, X^2, X^3)\}$; $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$ orthonormal

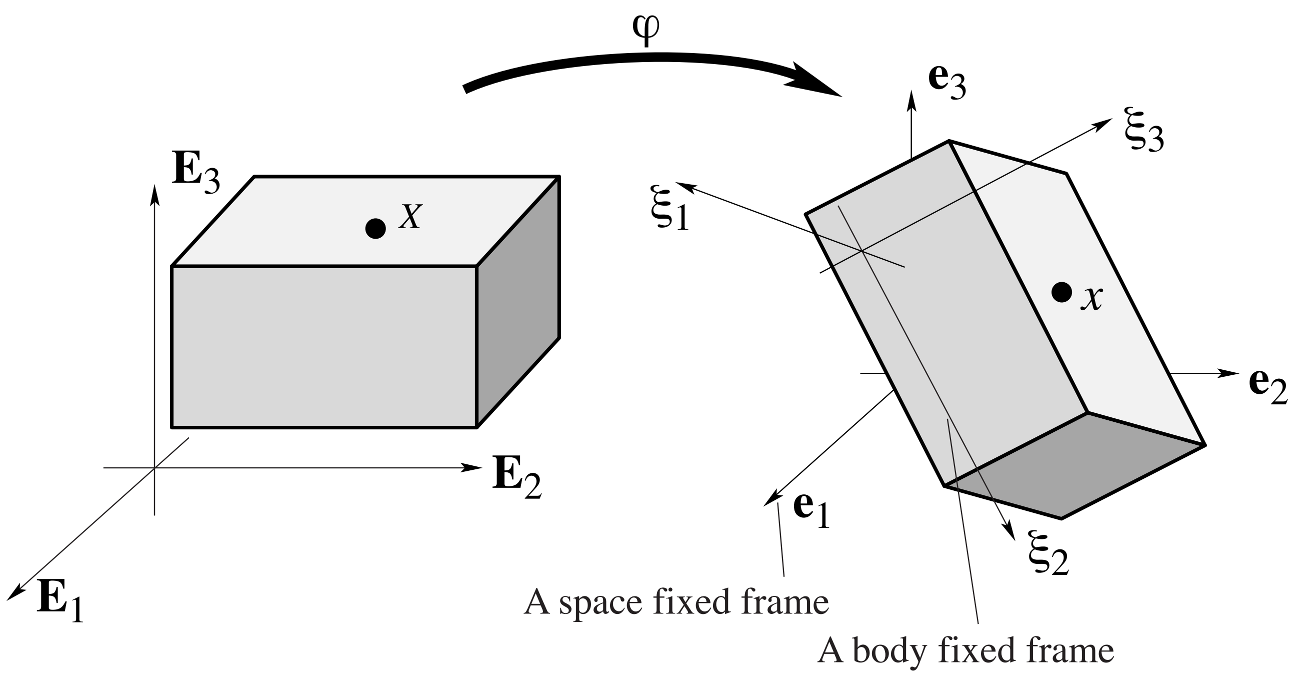
Spatial configuration: $(\mathcal{S}, \mathbf{g})$ oriented Riemannian manifold
Usually $\mathcal{S} = \mathbb{R}^3 = \{\mathbf{x} = (x^1, x^2, x^3)\}$; $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ orthonormal

Configuration: orientation preserving embedding $\varphi : \mathcal{B} \to \mathcal{S}$, so the ***configuration space*** is $\mathrm{Emb}_+(\mathcal{B}, \mathcal{S})$

Motion: $\varphi_t(\mathbf{X}) = \mathbf{x}(\mathbf{X}, t)$ time dependent family of configurations

Time dependent basis anchored in the body moving together with it: $\boldsymbol{\xi}_i := \varphi_t(\mathbf{E}_i)$, $i = 1, 2, 3$. ***Body*** or ***convected coordinates***: coordinates relative to $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3$.


φ
$\mathbf{E}_3$
X
$\mathbf{E}_2$
$\mathbf{E}_1$
$\mathbf{e}_3$
x
$\mathbf{e}_2$
$\mathbf{e}_1$


φ
$\mathbf{E}_3$
X
$\mathbf{E}_2$
$\mathbf{E}_1$
$\mathbf{e}_3$
ξ_3
ξ_1
x
$\mathbf{e}_2$
$\mathbf{e}_1$
ξ_2
A space fixed frame
A body fixed frame

The ***material*** or ***Lagrangian velocity*** is defined by

$$\mathbf{V}(\mathbf{X},t) := \frac{\partial \mathbf{x}(\mathbf{X},t)}{\partial t} = \frac{\partial}{\partial t}\varphi_t(\mathbf{X}).$$

The ***spatial*** or ***Eulerian velocity*** is defined by

$$\mathbf{v}(\mathbf{x},t) := \mathbf{V}(\mathbf{X},t) \Longleftrightarrow \mathbf{v}_t \circ \varphi_t = \mathbf{V}_t.$$

The ***body*** or ***convective velocity*** is defined by

$$\mathcal{V}(\mathbf{X},t) := -\frac{\partial \mathbf{X}(\mathbf{x},t)}{\partial t} = -\frac{\partial}{\partial t}\varphi_t^{-1}(\mathbf{x}) \Longleftrightarrow \mathcal{V}_t = T\varphi_t^{-1} \circ \mathbf{V}_t = \varphi_t^* \mathbf{v}_t$$

The ***particle relabeling group*** $\mathrm{Diff}(\mathcal{B})$ acts on the *right* on $\mathrm{Emb}_+(\mathcal{B},\mathcal{S})$. The ***material frame indifference group*** $\mathrm{Diff}(\mathcal{S})$ acts on the *left* on $\mathrm{Emb}_+(\mathcal{B},\mathcal{S})$.

In continuum mechanics it is important to keep all options open and always have three descriptions available. They serve different purposes and the interactions between them gives interesting physical insight.

HEAVY TOP

Material or ***Lagrangian description:*** $\mathcal{B} \subset \mathbb{R}^3$ compact region with non-empty interior, $\mathbf{G}_{ij} = \delta_{ij}$. Relative to an orthonormal basis $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$ we get ***material/Lagrangian coordinates*** X^1, X^2, X^3

Spatial or ***Eulerian description:*** $\mathcal{S} = \mathbb{R}^3$, $\mathbf{g}_{ij} = \delta_{ij}$. Relative to an orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ we get ***spatial/Eulerian coordinates*** x^1, x^2, x^3

Body: Time dependent orthonormal basis anchored in the body moving together with it: $\boldsymbol{\mathcal{E}}_i := A(t)\mathbf{E}_i$, $i = 1, 2, 3$. Relative to the orthonormal basis $\boldsymbol{\mathcal{E}}_1, \boldsymbol{\mathcal{E}}_2, \boldsymbol{\mathcal{E}}_3$ we get the ***body/convected coordinates*** $\mathcal{X}^1, \mathcal{X}^2, \mathcal{X}^3$

Components of a vector $\mathbf{U}$ in the basis $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$ are the same as the components of the vector $A(t)\mathbf{U}$ in the basis $\boldsymbol{\mathcal{E}}_1, \boldsymbol{\mathcal{E}}_2, \boldsymbol{\mathcal{E}}_3$.
Note that the body coordinates of $\mathbf{x}(\mathbf{X}, t) = A(t)\mathbf{X}$ are X^1, X^2, X^3.

Euler angles

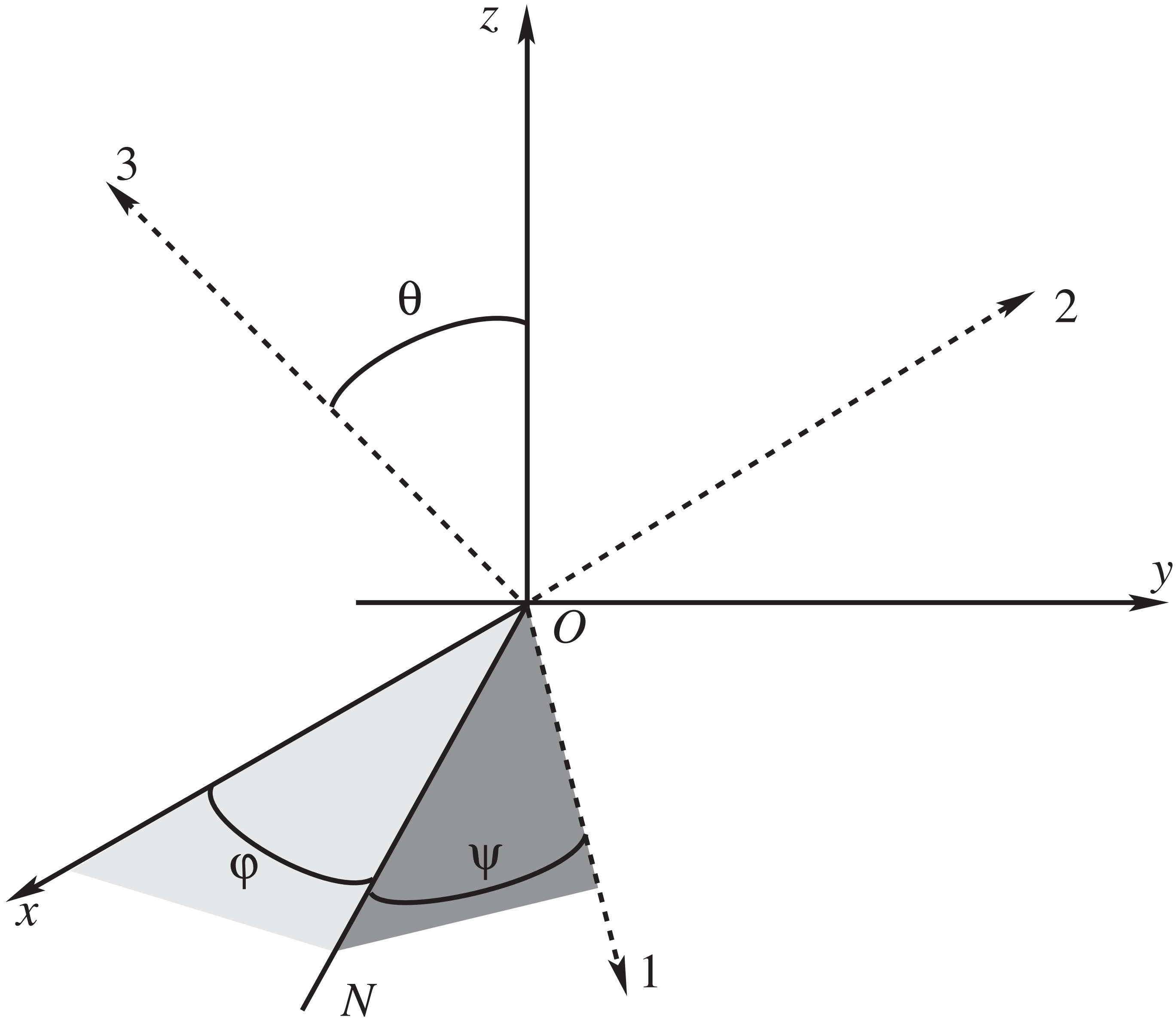
Passage from orthonormal spatial basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to orthonormal basis in the body $\boldsymbol{\mathcal{E}}_1, \boldsymbol{\mathcal{E}}_2, \boldsymbol{\mathcal{E}}_3$ by three consecutive counterclockwise rotations (specific order): first rotate around the axis $\mathbf{e}_3$ by the angle φ and denote the resulting position of $\mathbf{e}_1$ by ON (line of nodes), then rotate about ON by the angle θ and denote the resulting position of $\mathbf{e}_3$ by $\boldsymbol{\mathcal{E}}_3$, finally rotate about $\boldsymbol{\mathcal{E}}_3$ by the angle ψ.

By construction: $0 \leq \varphi, \psi < 2\pi$, $0 \leq \theta < \pi$. Get a bijection between $\{(\varphi, \psi, \theta)\}$ and SO(3). It is *not* a chart: its differential vanishes at $\varphi = \psi = \theta = 0$. But for $0 < \varphi, \psi < 2\pi$, $0 < \theta < \pi$ the ***Euler angles*** (φ, ψ, θ) do form a chart.

The resulting linear map has matrix relative to the bases $\boldsymbol{\mathcal{E}}_1, \boldsymbol{\mathcal{E}}_2, \boldsymbol{\mathcal{E}}_3$ and $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ equal to

$$A =$$

$$\begin{bmatrix} \cos\psi\cos\varphi - \cos\theta\sin\varphi\sin\psi & \cos\psi\sin\varphi + \cos\theta\cos\varphi\sin\psi & \sin\theta\sin\psi \\ -\sin\psi\cos\varphi - \cos\theta\sin\varphi\cos\psi & -\sin\psi\sin\varphi + \cos\theta\cos\varphi\cos\psi & \sin\theta\cos\psi \\ \sin\theta\sin\varphi & -\sin\theta\cos\varphi & \cos\theta \end{bmatrix}$$


z
3
θ
2
y
O
φ
ψ
x
N
1

For the rigid body moving about a fixed point, the motions are rotations: $\mathbf{x}(\mathbf{X},t) := A(t)\mathbf{X}$, where $A(t) \in SO(3)$.

The ***material*** or ***Lagrangian velocity***

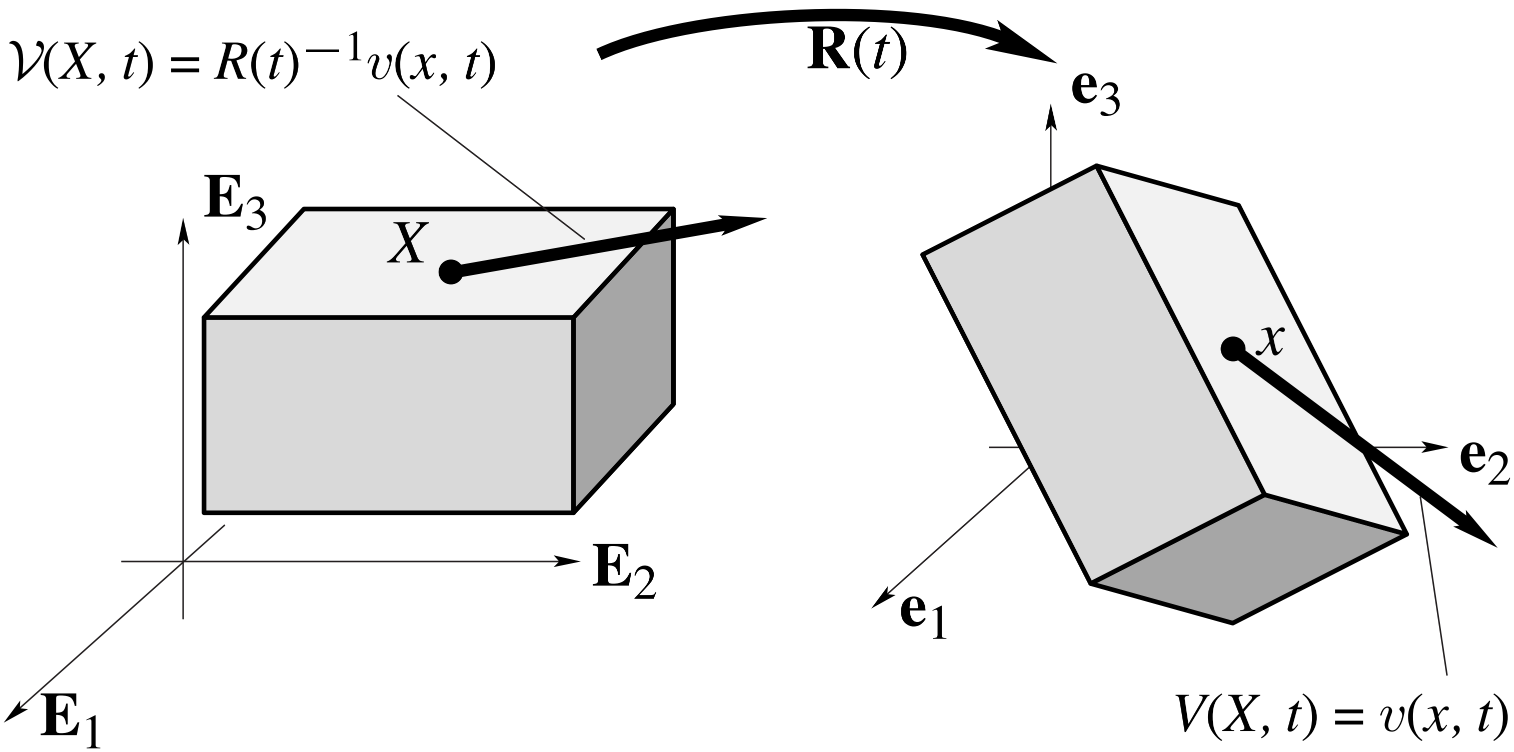
$$\mathbf{V}(\mathbf{X},t) := \frac{\partial \mathbf{x}(\mathbf{X},t)}{\partial t} = \dot{A}(t)\mathbf{X}.$$

The ***spatial*** or ***Eulerian velocity***

$$\mathbf{v}(\mathbf{x},t) := \mathbf{V}(\mathbf{X},t) = \dot{A}(t)\mathbf{X} = \dot{A}(t)A(t)^{-1}\mathbf{x}.$$

The ***body*** or ***convective velocity***

$$\begin{aligned}\mathcal{V}(\mathbf{X},t) : &= -\frac{\partial \mathbf{X}(\mathbf{x},t)}{\partial t} = A(t)^{-1}\dot{A}(t)A(t)^{-1}\mathbf{x} = A(t)^{-1}\dot{A}(t)\mathbf{X} \\ &= A(t)^{-1}\mathbf{V}(\mathbf{X},t) = A(t)^{-1}\mathbf{v}(\mathbf{x},t).\end{aligned}$$



Material velocity $\mathbf{V}$, spatial velocity $\mathbf{v}$, and body velocity $\mathcal{V}$.

Kinetic energy

ρ_0 density in the reference configuration. The kinetic energy at time t in material, spatial, and convective representation:

$$
\begin{aligned}
K(t) &= \frac{1}{2}\int_{\mathcal{B}} \rho_0(\mathbf{X})\|\mathbf{V}(\mathbf{X},t)\|^2 d^3\mathbf{X} && \textit{material}\\
&= \frac{1}{2}\int_{A(t)\mathcal{B}} \rho_0(A(t)^{-1}\mathbf{x})\|\mathbf{v}(\mathbf{x},t)\|^2 d^3\mathbf{x} && \textit{spatial}\\
&= \frac{1}{2}\int_{\mathcal{B}} \rho_0(\mathbf{X})\|\mathcal{V}(\mathbf{X},t)\|^2 d^3\mathbf{X}. && \textit{body}
\end{aligned}
$$

Define $\quad \widehat{\boldsymbol{\omega}}(t) := \dot{A}(t)A(t)^{-1}, \qquad \widehat{\boldsymbol{\Omega}}(t) := A(t)^{-1}\dot{A}(t), \quad$ then

$$
\mathbf{v}(\mathbf{x},t) = \boldsymbol{\omega}(t) \times \mathbf{x}, \qquad \mathcal{V}(\mathbf{X},t) = \boldsymbol{\Omega}(t) \times \mathbf{X},
$$

$\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ are ***spatial*** and ***body angular velocities***; $\boldsymbol{\omega}(t) = A(t)\boldsymbol{\Omega}(t)$.
$\widehat{\ }: (\mathbb{R}^3, \times) \to (\mathfrak{so}(3), [\,,\,])$ is the Lie algebra isomorphism $\hat{\mathbf{u}}\mathbf{v} = \mathbf{u} \times \mathbf{v}$.
Useful formula: $A\hat{\mathbf{u}}A^{-1} = \widehat{A\mathbf{u}}$. Inverse: $\mathfrak{so}(3) \ni \boldsymbol{\xi} \mapsto \boldsymbol{\xi}^\vee \in \mathbb{R}^3$.

So $\quad K(t) = \frac{1}{2}\int_{\mathcal{B}} \rho_0(\mathbf{X})\|\boldsymbol{\Omega}(t) \times \mathbf{X}\|^2 d^3\mathbf{X} =: \frac{1}{2}\langle\!\langle \boldsymbol{\Omega}(t), \boldsymbol{\Omega}(t) \rangle\!\rangle$

which is the quadratic form of the bilinear symmetric map on $\mathbb{R}^3$

$$
\langle\!\langle \mathbf{a}, \mathbf{b} \rangle\!\rangle := \int_{\mathcal{B}} \rho_0(\mathbf{X})(\mathbf{a} \times \mathbf{X}) \cdot (\mathbf{b} \times \mathbf{X}) d^3\mathbf{X} = \mathbb{I}\mathbf{a} \cdot \mathbf{b},
$$

where $\mathbb{I} : \mathbb{R}^3 \to \mathbb{R}^3$ is the symmetric isomorphism (relative to the dot product) whose components are $\mathbb{I}_{ij} := \mathbb{I}\mathbf{E}_j \cdot \mathbf{E}_i = \langle\!\langle \mathbf{E}_j, \mathbf{E}_i \rangle\!\rangle$, i.e.,

$$\mathbb{I}_{ij} = -\int_{\mathcal{B}} \rho_0(\mathbf{X}) X^i X^j d^3\mathbf{X} \quad \text{if} \quad i \neq j$$
$$\mathbb{I}_{ii} = \int_{\mathcal{B}} \rho_0(\mathbf{X}) \left(\|\mathbf{X}\|^2 - (X^i)^2 \right) d^3\mathbf{X}.$$

So $\mathbb{I}$ is the ***moment of inertia tensor***. ***Principal axis body frame***: basis in which $\mathbb{I}$ is diagonal; diagonal elements I_1, I_2, I_3 of $\mathbb{I}$ are the ***principal moments of inertia*** of the top. From now on, choose $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$ to be a principal axis body frame.

$\langle\!\langle \mathbf{\Omega}, \cdot \rangle\!\rangle \in (\mathbb{R}^3)^*$ is identified with the ***angular momentum in the body frame*** $\mathbf{\Pi} := \mathbb{I}\mathbf{\Omega} \in \mathbb{R}^3$, so

$$K(\mathbf{\Pi}) = \frac{1}{2}\mathbf{\Pi} \cdot \mathbb{I}^{-1}\mathbf{\Pi} \quad \text{or} \quad K(\mathbf{\Omega}) = \frac{1}{2}\mathbf{\Omega} \cdot \mathbb{I}\mathbf{\Omega}$$

where $\mathbb{I} = \text{diag}(I_1, I_2, I_3)$.

This is the expression of the kinetic energy in the body representation, either as a function of $\mathbf{\Omega}$ or $\mathbf{\Pi}$.

So the kinetic energy $K : T\,\mathrm{SO}(3) \to \mathbb{R}$ in material representation

$$K(A, \dot{A}) = \frac{1}{2}\mathbb{I}\left(A^{-1}\dot{A}\right)^{\vee} \cdot \left(A^{-1}\dot{A}\right)^{\vee} = \frac{1}{2}(A\mathbb{I}A^{-1})A\left(A^{-1}\dot{A}\right)^{\vee} \cdot A\left(A^{-1}\dot{A}\right)^{\vee}$$
$$= \frac{1}{2}(A\mathbb{I}A^{-1})\left(\dot{A}A^{-1}\right)^{\vee} \cdot \left(\dot{A}A^{-1}\right)^{\vee} = \frac{1}{2}(A\mathbb{I}A^{-1})\boldsymbol{\omega} \cdot \boldsymbol{\omega}$$

is *left invariant* (action is $B \cdot (A, \dot{A}) := (BA, B\dot{A})$). It is the kinetic energy of the left invariant Riemannian metric on $SO(3)$ obtained by left translating the inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$.

Define the ***spatial moment of inertia tensor*** $\mathbb{I}_S(t) := A(t)\mathbb{I}A(t)^{-1}$. Since $\boldsymbol{\Omega} = A^{-1}\boldsymbol{\omega}$ it follows that the ***spatial angular momentum*** is $\boldsymbol{\pi} := \mathbb{I}_S\boldsymbol{\omega} = A\boldsymbol{\Pi}$, so

$$K(\boldsymbol{\omega}, \mathbb{I}_S) = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbb{I}_S\boldsymbol{\omega}$$

This is the expression of the kinetic energy in the spatial representation.

Note that a major complication has arisen: the new dynamic variable $\mathbb{I}_S$ has been introduced.

Potential energy

The potential energy U is determined by the height of the center of mass over the horizontal plane in the spatial representation.

• ℓ length of segment from fixed point to center of mass
• $\boldsymbol{\chi}$ unit vector from origin on this segment
• $M = \int_{\mathcal{B}} \rho_0(\mathbf{X}) d^3\mathbf{X}$ total mass of the body
• g magnitude of gravitational acceleration
• $\boldsymbol{\Gamma}(t) := Mg\ell A(t)^{-1}\mathbf{e}_3$, spatial Oz unit vector viewed in body description
• $\boldsymbol{\lambda}(t) := Mg\ell A(t)\boldsymbol{\chi}$, unit vector on the line connecting the origin with the center of mass viewed in the spatial description

$$
\begin{aligned}
U &= Mg\ell\mathbf{e}_3 \cdot A(t)\boldsymbol{\chi} && \textit{material} \\
&= \mathbf{e}_3 \cdot \boldsymbol{\lambda} && \textit{spatial} \\
&= \boldsymbol{\Gamma} \cdot \boldsymbol{\chi} && \textit{body}
\end{aligned}
$$

New complications appear: **There are new variables, depending on the representation; λ in the spatial and Γ in the body representation**

AFFINE SEMIDIRECT PRODUCT LAGRANGIAN REDUCTION

Give the general theory on the *right* because it is useful for fluids. For heavy top, everything is on the *left*, so there are sign changes.

$\rho : G \to \mathrm{Aut}(V)$ *right* Lie group representation. Form $S = G \circledS V$

$$(g_1, v_1)(g_2, v_2) = (g_1 g_2, v_2 + \rho_{g_2}(v_1)).$$

The Lie algebra $\mathfrak{s} = \mathfrak{g} \circledS V$ of S has bracket

$$\mathrm{ad}_{(\xi_1, v_1)}(\xi_2, v_2) = [(\xi_1, v_1), (\xi_2, v_2)] = ([\xi_1, \xi_2], v_1\xi_2 - v_2\xi_1),$$

where $v\xi$ denotes the induced action of $\mathfrak{g}$ on V, that is,

$$v\xi := \left.\frac{d}{dt}\right|_{t=0} \rho_{\exp(t\xi)}(v) \in V.$$

If $(\xi, v) \in \mathfrak{s}$ and $(\mu, a) \in \mathfrak{s}^*$ we have

$$\mathrm{ad}^*_{(\xi, v)}(\mu, a) = (\mathrm{ad}^*_\xi \mu + v \diamond a, a\xi),$$

where $a\xi \in V^*$ and $v \diamond a \in \mathfrak{g}^*$ are given by

$$a\xi := \frac{d}{dt}\Big|_{t=0} \rho^*_{\exp(-t\xi)}(a) \quad \text{and} \quad \langle v \diamond a, \xi \rangle_{\mathfrak{g}} := -\langle a\xi, v \rangle_V,$$

$\langle \cdot, \cdot \rangle_{\mathfrak{g}} : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ and $\langle \cdot, \cdot \rangle_V : V^* \times V \to \mathbb{R}$ are the duality parings.

$c \in \mathcal{F}(G, V^*)$ a right one-cocycle: $c(fg) = \rho^*_{g^{-1}}(c(f)) + c(g)$, $\forall f, g \in V^*$. So $c(e) = 0$ and $c(g^{-1}) = -\rho^*_g(c(g))$. Form the affine *right* representation

$$\theta_g(a) = \rho^*_{g^{-1}}(a) + c(g).$$

Note that

$$\frac{d}{dt}\Big|_{t=0} \theta_{\exp(t\xi)}(a) = a\xi + \mathbf{d}c(\xi).$$

and

$$\langle a\xi + \mathbf{d}c(\xi), v \rangle_V = \langle \mathbf{d}c^{\mathsf{T}}(v) - v \diamond a, \xi \rangle_{\mathfrak{g}},$$

where $\mathbf{d}c : \mathfrak{g} \to V^*$ is defined by $\mathbf{d}c(\xi) := T_e c(\xi)$, and $\mathbf{d}c^{\mathsf{T}} : V \to \mathfrak{g}^*$ is defined by

$$\langle \mathbf{d}c^{\mathsf{T}}(v), \xi \rangle_{\mathfrak{g}} := \langle \mathbf{d}c(\xi), v \rangle_V.$$

- $L : TG \times V^* \to \mathbb{R}$ right G-invariant under the affine action $(v_h, a) \mapsto (T_h R_g(v_h), \theta_g(a)) = (T_h R_g(v_h), \rho^*_{g^{-1}}(a) + c(g))$.

- So, if $a_0 \in V^*$, define $L_{a_0} : TG \to \mathbb{R}$ by $L_{a_0}(v_g) := L(v_g, a_0)$. Then L_{a_0} is right invariant under the lift to TG of the right action of $G^c_{a_0}$ on G, where $G^c_{a_0} := \{g \in G \mid \theta_g(a_0) = a_0\}$.

- Right G-invariance of L permits us to define $l : \mathfrak{g} \times V^* \to \mathbb{R}$ by

$$l(T_g R_{g^{-1}}(v_g), \theta_{g^{-1}}(a_0)) = L(v_g, a_0).$$

- For a curve $g(t) \in G$, let $\xi(t) := TR_{g(t)^{-1}}(\dot{g}(t))$ and define the curve $a(t)$ as the unique solution of the following affine differential equation with time dependent coefficients

$$\dot{a}(t) = -a(t)\xi(t) - \mathbf{d}c(\xi(t)),$$

with initial condition $a(0) = a_0$. The solution can be written as $a(t) = \theta_{g(t)^{-1}}(a_0)$.

i With a_0 held fixed, Hamilton's variational principle

$$\delta \int_{t_1}^{t_2} L_{a_0}(g(t), \dot{g}(t))dt = 0,$$

holds, for variations $\delta g(t)$ of $g(t)$ vanishing at the endpoints.

ii $g(t)$ satisfies the Euler-Lagrange equations for L_{a_0} on G.

iii The constrained variational principle

$$\delta \int_{t_1}^{t_2} l(\xi(t), a(t))dt = 0,$$

holds on $\mathfrak{g} \times V^*$, upon using variations of the form

$$\delta\xi = \frac{\partial \eta}{\partial t} - [\xi, \eta], \quad \delta a = -a\eta - \mathbf{d}c(\eta),$$

where $\eta(t) \in \mathfrak{g}$ vanishes at the endpoints.

iv The affine Euler-Poincaré equations hold on $\mathfrak{g} \times V^*$:

$$\frac{\partial}{\partial t}\frac{\delta l}{\delta \xi} = -\,\mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi} + \frac{\delta l}{\delta a} \diamond a - \mathbf{d}c^{\mathsf{T}}\left(\frac{\delta l}{\delta a}\right).$$

HEAVY TOP

QUESTION: The parameters are $\mathbf{e}_3, \boldsymbol{\chi} \in \mathbb{R}^3$, $\mathbb{I} \in \mathrm{Sym}_2$, $Mg\ell \in \mathbb{R}$. This is the representation space V^*. There is no cocycle, so $c = 0$ in theorem.

$$L(A, \dot{A}, \mathbf{e}_3, \mathbb{I}, \boldsymbol{\chi}) = \frac{1}{2}\mathbb{I}\left(A^{-1}\dot{A}\right)^{\vee} \cdot \left(A^{-1}\dot{A}\right)^{\vee} - Mg\ell\mathbf{e}_3 \cdot A\boldsymbol{\chi}$$

Left SO(3)-***representation***: $B \cdot (\mathbf{e}_3, \mathbb{I}, \boldsymbol{\chi}) := (B\mathbf{e}_3, \mathbb{I}, \boldsymbol{\chi})$. Note

$$L(BA, B\dot{A}, B\mathbf{e}_3, \mathbb{I}, \boldsymbol{\chi}) = L(A, \dot{A}, \mathbf{e}_3, \mathbb{I}, \boldsymbol{\chi})$$

So, general theory says that we have Euler-Poincaré equations and associated variational principles for the **body Lagrangian**

$$L_B(\boldsymbol{\Omega}, \boldsymbol{\Gamma}, \mathbb{I}, \boldsymbol{\chi}) := L(I, A^{-1}\dot{A}, A^{-1}\mathbf{e}_3, \mathbb{I}, \boldsymbol{\chi}) = \frac{1}{2}\boldsymbol{\Omega} \cdot \mathbb{I}\boldsymbol{\Omega} - \boldsymbol{\Gamma} \cdot \boldsymbol{\chi}$$

Since $\frac{\delta L_B}{\delta \boldsymbol{\Omega}} = \mathbb{I}\boldsymbol{\Omega} = \boldsymbol{\Pi}$ and $\frac{\delta L_B}{\delta \boldsymbol{\Gamma}} = -\boldsymbol{\chi}$, we get the equations

$$\dot{\boldsymbol{\Pi}} = \boldsymbol{\Pi} \times \boldsymbol{\Omega} + \boldsymbol{\Gamma} \times \boldsymbol{\chi}, \quad \dot{\boldsymbol{\Gamma}} = \boldsymbol{\Gamma} \times \boldsymbol{\Omega}, \quad \dot{\mathbb{I}} = 0, \quad \dot{\boldsymbol{\chi}} = 0$$

Right SO(3)-***representation***: $(\mathbf{e}_3, \mathbb{I}, \boldsymbol{\chi}) \cdot B := (\mathbf{e}_3, B^{-1}\mathbb{I}B, B^{-1}\boldsymbol{\chi})$.

$$L(AB, \dot{A}B, \mathbf{e}_3, B^{-1}\mathbb{I}B, B^{-1}\boldsymbol{\chi}) = L(A, \dot{A}, \mathbf{e}_3, \mathbb{I}, \boldsymbol{\chi})$$

So, general theory says that we have Euler-Poincaré equations and associated variational principles for the **spatial Lagrangian**

$$L_S(\boldsymbol{\omega}, \mathbf{e}_3, \mathbb{I}_S, \boldsymbol{\lambda}) := L(I, \dot{A}A^{-1}, \mathbf{e}_3, A\mathbb{I}A^{-1}, A\boldsymbol{\chi}) = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbb{I}_S\boldsymbol{\omega} - \mathbf{e}_3 \cdot \boldsymbol{\lambda}$$

Since $\frac{\delta L_S}{\delta \boldsymbol{\omega}} = \mathbb{I}_S\boldsymbol{\omega} = \boldsymbol{\pi}$, $\frac{\delta L_S}{\delta \boldsymbol{\lambda}} = -\mathbf{e}_3$, $\frac{\delta L_S}{\delta \mathbb{I}_S} = \boldsymbol{\omega} \otimes \boldsymbol{\omega}$, we get (Holm, Marsden, TR 1986, CRM Montreal Volume):

$$\dot{\boldsymbol{\pi}} = \mathbf{e}_3 \times \boldsymbol{\lambda}, \quad \dot{\mathbf{e}}_3 = 0, \quad \dot{\mathbb{I}}_S = [\mathbb{I}_S, \widehat{\boldsymbol{\omega}}], \quad \dot{\boldsymbol{\lambda}} = \boldsymbol{\omega} \times \boldsymbol{\lambda}$$

Remark: In body representation, we have equations on $\mathfrak{se}(3)^* = \mathbb{R}^3 \times \mathbb{R}^3$. Four dimensional generic orbits; Casimirs are $\mathbf{\Pi} \cdot \mathbf{\Gamma}$, $\|\mathbf{\Gamma}\|^2$.

In spatial representation, equations are on the dual of the semidirect product $\mathfrak{so}(3) \circledS (\mathrm{Sym}^2 \times \mathbb{R}^3)$. This is 12 dimensional. It has 6 Casimirs: the three invariants of $\mathbb{I}_S$, $\|\boldsymbol{\lambda}\|^2$, $(\mathbb{I}_S\boldsymbol{\lambda}) \cdot \boldsymbol{\lambda}$, $\|\mathbb{I}_S\boldsymbol{\lambda}\|^2$. The coadjoint orbit is symplectomorphic to $(T^*\mathrm{SO}(3), can)$. One more integral: $\boldsymbol{\pi} \cdot \mathbf{e}_3$. Reduce and get to 4 dimensions $(TS^2, magnetic)$.

Special case: Free or Euler top

$\ell = 0$, so $L(A, \dot{A}) = K(A, \dot{A}) = \frac{1}{2}\left(\mathbb{I}A^{-1}\dot{A}\right) \cdot \left(A^{-1}\dot{A}\right)$, i.e., we study geodesic motion on SO(3) for the left invariant metric whose value at I is $\langle\langle \mathbf{a}, \mathbf{b} \rangle\rangle = \mathbb{I}\mathbf{a} \cdot \mathbf{b}$. The equations in body representation decouple:

$$\dot{\mathbf{\Pi}} = \mathbf{\Pi} \times \mathbf{\Omega}, \quad \mathbf{\Gamma} = 0,$$

Geodesic equations on $\mathrm{SO}(3) \times \mathbb{R}^3 \cong T\,\mathrm{SO}(3)$ (left trivialized) are

$$\dot{\mathbf{\Pi}} = \mathbf{\Pi} \times \mathbf{\Omega}, \quad \dot{A} = A\widehat{\mathbf{\Omega}}$$

The left action induces a momentum map $\mathbf{J}_L : T\,\mathrm{SO}(3) \to \mathbb{R}^3$ which is conserved. Recall $\mathbf{J}_L(\alpha_A) = T_I^* R_A(\alpha_A)$ which after the identifications becomes $\mathbf{J}_L(A, \mathbf{\Pi}) = A\mathbf{\Pi}$. Direct verification:

$$\dot{\boldsymbol{\pi}} = \dot{A}\mathbf{\Pi} + A\dot{\mathbf{\Pi}} = A\widehat{\mathbf{\Omega}}\mathbf{\Pi} + A(\mathbf{\Pi} \times \mathbf{\Omega}) = A(\mathbf{\Omega} \times \mathbf{\Pi} + \mathbf{\Pi} \times \mathbf{\Omega}) = 0$$

We shall see a similar phenomenon for fluids.

*Momentum map of SU(2)-action on $\mathbb{C}^2$, the Cayley-Klein parameters, the Kustaanheimo-Stiefel coordinates, and the family of Hopf fibrations on concentric three-spheres in $\mathbb{C}^2$ are **the same map**.*

FIXED BOUNDARY BAROTROPIC FLUIDS

group = Diff($\mathcal{D}$), $V^* = |\Omega^n(\mathcal{D})| \times S_2(\mathcal{D})$, Riemannian metrics $G = g$ on $\mathcal{B} = \mathcal{S}$

$$L_{(\bar{\varrho},g)}(V_\eta) = \frac{1}{2}\int_{\mathcal{D}} g(\eta(X))(V_\eta(X), V_\eta(X))\bar{\varrho}(X) - \int_{\mathcal{D}} E(\bar{\varrho}(X), g(\eta(X)), T_X\eta)\bar{\varrho}(X), \qquad \textit{material}$$

$$\ell_{spat}(\mathbf{v}, \bar{\rho}, g) = \frac{1}{2}\int_{\mathcal{D}} g(x)(\mathbf{v}(x), \mathbf{v}(x))\bar{\rho}(x) - \int_{\mathcal{D}} e(\rho(x))\bar{\rho}(x), \qquad \textit{spatial}$$

$$\ell_{conv}(\mathcal{V}, \bar{\varrho}, C) = \frac{1}{2}\int_{\mathcal{D}} C(\mathcal{V}, \mathcal{V})\bar{\varrho} - \int_{\mathcal{D}} \mathcal{E}(\bar{\varrho}, C)\bar{\varrho}, \qquad \textit{convective}$$

- $\bar{\varrho}(X) =: \varrho(X)\mu(g)(X) := (\eta^*\bar{\rho})(X), \qquad \bar{\rho}(x) := \rho(x)\mu(g)(x)$ mass density
- $C := \eta^* g$ Cauchy-Green tensor
- $E(\bar{\varrho}(X), g(\eta(X)), T_X\eta) := e\left(\frac{\bar{\varrho}(X)}{\mu(\eta^* g)(X)}\right) = e\left(\frac{\bar{\varrho}(X)}{\mu(C)(X)}\right) =: \mathcal{E}(\bar{\varrho}(X), C(X))$ internal energy density

L is *right*-invariant under the action of $\varphi \in \mathrm{Diff}(\mathcal{D})$ given by

$$(V_\eta, \bar{\varrho}, g) \mapsto (V_\eta \circ \varphi, \varphi^* \bar{\varrho}, g)$$

and the reduction map

$$(V_\eta, \bar{\varrho}, g) \mapsto (\mathbf{v}, \bar{\rho}, g) := (V_\eta \circ \eta^{-1}, \eta_* \bar{\varrho}, g)$$

induces the spatial Lagrangian $\ell_{spat}(\mathbf{v}, \rho, g)$ because

$$E(\bar{\varrho}, g \circ \eta, T\eta) \mapsto E(\varphi^* \bar{\varrho}, g \circ \eta \circ \varphi, T\eta \circ T\varphi) = E(\bar{\varrho}, g \circ \eta, T\eta) \circ \varphi$$

when $(\eta, \bar{\varrho}) \mapsto (\eta \circ \varphi, \varphi^* \bar{\varrho})$. g is not acted on by $\mathrm{Diff}(\mathcal{D})$.
L is *left*-invariant under the action of $\psi \in \mathrm{Diff}(\mathcal{D})$ given by

$$(V_\eta, \bar{\varrho}, g) \mapsto (T\psi \circ V_\eta, \bar{\varrho}, \psi_* g).$$

and the reduction map

$$(V_\eta, \bar{\varrho}, g) \mapsto (\mathcal{V}, \bar{\varrho}, C) := (T\eta^{-1} \circ V_\eta, \bar{\varrho}, \eta^* g),$$

induces the convective Lagrangian $\ell_{conv}(\mathcal{V}, \bar{\varrho}, C)$ because

$$E(\bar{\varrho}, g \circ \eta, T\eta) \mapsto E(\bar{\varrho}, \psi_* g \circ (\psi \circ \eta), T\psi \circ T\eta) = E(\bar{\varrho}, g \circ \eta, T\eta)$$

when $(\eta, g) \mapsto (\psi \circ \eta, \psi_* g)$. $\bar{\varrho}$ is not acted on by $\mathrm{Diff}(\mathcal{D})$.

General semidirect product reduction gives spatial equations

$$\begin{cases} \partial_t \mathbf{v} + \nabla_{\mathbf{v}} \mathbf{v} = -\frac{1}{\rho} \operatorname{grad}_g p, \quad p = \rho^2 \frac{\partial e}{\partial \rho} \\ \partial_t \rho + \operatorname{div}_g(\rho \mathbf{v}) = 0, \quad \mathbf{v} \| \partial \mathcal{D}, \end{cases}$$

and convective equations

$$\begin{cases} \bar{\varrho}\,(\partial_t \mathcal{V} + \nabla_{\mathcal{V}} \mathcal{V}) = 2 \operatorname{Div}_C \left(\frac{\partial \mathcal{E}}{\partial C} \bar{\varrho} \right) \\ \partial_t C - \pounds_{\mathcal{V}} C = 0, \quad \mathcal{V} \| \partial \mathcal{B}, \end{cases}$$

right hand side is related to the spatial pressure p by the formula

$$2 \frac{\partial \mathcal{E}}{\partial C} \bar{\varrho} = -(p \circ \eta) \mu(C) C^\sharp, \quad \text{so} \quad 2 \operatorname{Div}_C \left(\frac{\partial \mathcal{E}}{\partial C} \bar{\varrho} \right) = -\operatorname{grad}_C (p \circ \eta) \mu(C),$$

$C^\sharp \in S^2(\mathcal{D})$ is the cometric, grad_C is the gradient relative to C.

Special case: ideal homogeneous incompressible fluid. Group is $\operatorname{Diff}_\mu(g) := \{\eta \in \operatorname{Diff}(\mathcal{D}) \mid \eta^* \mu(g) = \mu(g)\}$, $V^* = S_2(\mathcal{D})$.
Lagrangian in spatial and convective rep. (suppose $H^1(\mathcal{D}, \mathbb{R}) = 0$):

$$\ell_{spat}(\mathbf{v}, g) = \frac{1}{2}\int_{\mathcal{D}} g(x)(\mathbf{v}(x), \mathbf{v}(x))\mu(g)(x)$$

$$\ell_{conv}(\mathcal{V}, \bar{\varrho}, C) = \frac{1}{2}\int_{\mathcal{D}} C(X)(\mathcal{V}(X), \mathcal{V}(X))\mu(g)(X)$$

In spatial representation: if $\mathfrak{X}_{div,\|}(\mathcal{D})^* = \mathfrak{X}_{div,\|}(\mathcal{D}) \Longrightarrow$

$$\partial_t \mathbf{v} + \nabla_{\mathbf{v}}\mathbf{v} = -\operatorname{grad} p \qquad \textit{Euler equations}$$

if $\mathfrak{X}_{div,\|}(\mathcal{D})^* = \mathbf{d}\Omega^1_{\delta,\|}(\mathcal{D}) := \{\mathbf{d}\mathbf{v}^{\flat_g} \mid \mathbf{v} \in \mathfrak{X}_{div,\|}(\mathcal{D})\} = \Omega^2_{ex}(\mathcal{D}) \Longrightarrow$

$$\partial_t \omega + \boldsymbol{\pounds}_{\mathbf{v}}\omega = 0, \text{ where } \omega := \mathbf{d}\mathbf{v}^{\flat_g} \qquad \textit{vorticity advection}$$

In convective representation: if $\mathfrak{X}_{div,\|}(\mathcal{D})^* = \Omega^1_{\delta,\|}(\mathcal{D}) \Longrightarrow$

$$\partial_t \mathbb{P}\left(\mathcal{V}^{\flat_C}\right) = 0 \quad \text{and} \quad \partial_t C - \boldsymbol{\pounds}_{\mathcal{V}} C = 0.$$

$\mathbb{P} : \Omega^1(\mathcal{D}) \to \Omega^1_{\delta,\|}(\mathcal{D})$ orthogonal Hodge projector for the metric g

if $\mathfrak{X}_{div,\|}(\mathcal{D})^* = \Omega^2_{ex}(\mathcal{D}) \Longrightarrow$

$$\partial_t \Omega = 0 \quad \text{and} \quad \partial_t C - \boldsymbol{\pounds}_{\mathcal{V}} C = 0.$$

where $\Omega := \mathbf{d}\mathcal{V}^{\flat_C}$ is the convective vorticity.

ELASTICITY

Euler-Poincaré theory does not apply; do by hand with EP as guide.
BC: Displacement (η given on part of $\partial\mathcal{B}$); traction ($\mathbf{P}\cdot\mathbf{N}_C|_{\partial\mathcal{B}} = \tilde{\boldsymbol{\tau}}$).
Configuration space Emb$(\mathcal{B},\mathcal{S})$. Material Lagrangian:

$$L(V_\eta, \bar{\varrho}, g, G) = \frac{1}{2}\int_{\mathcal{B}} g(\eta(X))(V_\eta(X), V_\eta(X))\bar{\varrho}(X) - \int_{\mathcal{B}} W(g(\eta(X)), T_X\eta, G(X))\bar{\varrho}(X).$$

Material frame indifference: the *material stored energy function* W is invariant under the transformations

$$(\eta, g) \mapsto (\psi\circ\eta, \psi_* g), \quad \psi \in \text{Diff}(\mathcal{S}), \qquad i.e.,$$

$$W\left(\psi_* g(\psi(\eta(X))), T_{\eta(X)}\psi\circ T_X\eta, G(X)\right) = W\left(g(\eta(X)), T_X\eta, G(X)\right).$$

$\forall \eta \in$ Emb$(\mathcal{B},\mathcal{S})$, $\forall\psi : \eta(\mathcal{B}) \to \mathcal{B}$

So can define the *convective stored energy* $\mathcal{W}$ by

$$\mathcal{W}(C(X), G(X)) := W\left(\eta^* g(X), \mathbf{I}, G(X)\right) = W\left(g(\eta(X)), T_X\eta, G(X)\right).$$

Convective quantities: $C := \eta^* g$ Cauchy-Green tensor,

$(\mathcal{V}, \bar{\varrho}, C, G) := \left(T\eta^{-1} \circ V_\eta, \bar{\varrho}, \eta^* g, G\right) \in \mathfrak{X}(\mathcal{B}) \times |\Omega^n(\mathcal{B})| \times S_2(\mathcal{B}) \times S_2(\mathcal{B})$,

$$\ell_{conv}(\mathcal{V}, \varrho, C, G) = \frac{1}{2}\int_{\mathcal{B}} C(\mathcal{V}, \mathcal{V})\varrho - \int_{\mathcal{B}} \mathcal{W}(C, G)\varrho.$$

Cannot apply Euler-Poincaré reduction since the Lagrangian is not defined on the tangent bundle of the symmetry group. Compute the variations by hand: $\eta_\varepsilon \in \mathrm{Emb}(\mathcal{B}, \mathcal{S})$ deformation of the embedding $\eta_0 := \eta \Longrightarrow \delta\mathcal{V} \in T_{\mathcal{V}}\mathfrak{X}_{bdry}(\mathcal{B})$ is

$$\delta\mathcal{V} = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} T\eta_\varepsilon^{-1} \circ \dot{\eta}_\varepsilon = \frac{d}{dt}\zeta + T\mathcal{V} \circ \zeta - T\zeta \circ \mathcal{V} = \dot{\zeta} - [\mathcal{V}, \zeta],$$

where $\zeta := T\eta^{-1} \circ \delta\eta \in \mathfrak{X}_{bdry}(\mathcal{B})$. The variation δC is

$$\delta C = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \eta_\varepsilon^* g = \eta^* \pounds_{\delta\eta\circ\eta^{-1}} g = \pounds_{T\eta^{-1}\circ\delta\eta}\eta^* g = \pounds_\zeta C.$$

Variational principle for $\ell_{conv} \Rightarrow$ convective equations of motion:

$$\varrho\left(\partial_t\mathcal{V} + \nabla_{\mathcal{V}}\mathcal{V}\right) = 2\,\mathrm{Div}_C\left(\frac{\partial\mathcal{W}}{\partial C}\varrho\right), \qquad \partial_t C - \pounds_{\mathcal{V}} C = 0.$$

So, elasticity has always a convective representation. Spatial rep.?

Isotropy: Need invariance under the *right* action of $\mathrm{Diff}(\mathcal{B})$:

$$(V_\eta, \varrho, g, G) \mapsto (V_\eta \circ \varphi, \varphi^* \varrho, g, \varphi^* G), \quad \varphi \in \mathrm{Diff}(\mathcal{B})$$

Kinetic energy is right-invariant. So sufficient condition is

$$W\Big(g(\eta(\varphi(X))), T_X(\eta \circ \varphi), \varphi^* G(X)\Big) = \Big(W(g(\eta(\_)), T_\_ \eta, G(\_)) \circ \varphi\Big)(X),$$

for all $\varphi \in \mathrm{Diff}(\mathcal{B})$. This is equivalent to

$$\mathcal{W}(\varphi^* C, \varphi^* G) = \mathcal{W}(C, G) \circ \varphi, \qquad \forall \varphi \in \mathrm{Diff}(\mathcal{B})$$

This is *material covariance* which implies isotropy.

Spatial quantities: $c := \eta_* G \in \mathcal{S}_2(D_\Sigma)$ Finger deformation tensor

$$\mathbf{u} := \dot{\eta} \circ \eta^{-1} \in \mathfrak{X}(D_\Sigma), \quad \bar{\rho} := \eta_* \bar{\varrho} \in |\Omega^n(D_\Sigma)|,$$

$\Sigma = \eta(\partial \mathcal{B})$ boundary of *current configuration* $D_\Sigma := \eta(\mathcal{B}) \subset \mathcal{S}$,

$$w_\Sigma(c, g) := \mathcal{W}(\eta^* g, \eta^* c) \circ \eta^{-1}$$

spatial stored energy function. w_Σ, $\mathcal{W}$, and W are related by

$$(w_\Sigma(c,g)\circ\eta)(X) = \mathcal{W}(\eta^* g(X), \eta^* c(X)) = W\left(g(\eta(X)), T_X\eta, \eta^* c(X)\right).$$

Doyle-Ericksen formula for the Cauchy stress tensor

$$\boldsymbol{\sigma} = 2\rho\frac{\partial w_\Sigma}{\partial g} \in S^2(D_\Sigma)$$

Reduced Lagrangian

$$\ell_{spat}(\Sigma, \mathbf{v}, \bar{\rho}, g, c) = \frac{1}{2}\int_{D_\Sigma} g(\mathbf{v},\mathbf{v})\bar{\rho} - \int_{D_\Sigma} w_\Sigma\,(c,g)\,\bar{\rho},$$

variables defined on current configuration D_Σ; note Σ is a variable.

$$\delta\mathbf{v} := \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \dot{\eta}_\varepsilon \circ \eta_\varepsilon^{-1} = \dot{\xi} + T\xi\circ\mathbf{v} - T\mathbf{v}\circ\xi = \dot{\xi} + [\mathbf{v},\xi],$$

$$\delta\Sigma := g(x)\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \eta_\varepsilon\circ\eta^{-1}, \mathbf{n}_g\right) = g(\xi, \mathbf{n}_g)$$

$$\delta\bar{\rho} := \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} (\eta_\varepsilon)_*\bar{\varrho} = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} (\eta_\varepsilon)_*\eta^*\eta_*\bar{\varrho} = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} (\eta\circ\eta_\varepsilon^{-1})^*\bar{\rho} = -\pounds_\xi\bar{\rho}$$

$$\delta c := \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} (\eta_\varepsilon)_* G = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} (\eta_\varepsilon)_*\eta^*\eta_* G = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} (\eta\circ\eta_\varepsilon^{-1})^* c_\Sigma = -\pounds_\xi c$$

where $\xi := \delta\eta\circ\eta^{-1} \in \mathfrak{X}(D_\Sigma)$ is an arbitrary curve with vanishing end-points, $\mathbf{n}_g$ is the outward-pointing unit normal vector field relative to g. Constrained variational principle

$$\delta \int_{t_0}^{t_1} \ell_{spat}(\Sigma, \mathbf{v}, \bar{\rho}, g, c) dt = 0$$

for the variations give above yield the spatial equations of motion:
(BC) $\mathbf{v}|_{\Sigma_d} = 0, \quad \boldsymbol{\sigma} \cdot \mathbf{n}_g|_{T\Sigma_\tau} = 0$

$$\rho\left(\partial_t \mathbf{v} + \nabla_{\mathbf{v}} \mathbf{v}\right) = \operatorname{Div}_g(\boldsymbol{\sigma}), \qquad \partial_t c + \pounds_{\mathbf{v}} c = 0, \qquad \partial_t \bar{\rho} + \pounds_{\mathbf{v}} \bar{\rho} = 0,$$
$$\partial_t \Sigma = g(\mathbf{v}, \mathbf{n}_g)$$

First Piola-Kirchhoff tensor: two-point tensor over η defined by

$$\mathbf{P}(\alpha_X, \beta_x) := J_\eta(X) \boldsymbol{\sigma}(\eta(X))(T^*\eta^{-1}(\alpha_X), \beta_x), \quad x = \eta(X),$$

$\alpha_X \in T^*\mathcal{B}$, $\beta_x \in T^*\mathcal{S}$, J_η Jacobian of η relative to the metrics g and G, i.e., $\eta^*\mu(g) = J_\eta \mu(G)$. We thus have the relations

$$\begin{aligned} \mathbf{P}(\alpha_X, \beta_x)\mu(G) &= \boldsymbol{\sigma}(\eta(X)) \left(T^*\eta^{-1}(\alpha_X), \beta_x\right) \mu(C) \\ &= \boldsymbol{\Sigma}(X) \left(\alpha_X, T^*\eta(\beta_x)\right) \mu(C) = 2\varrho \left(\frac{\partial W}{\partial(T\eta)}\right)^{\sharp_g} \end{aligned}$$

(Doyle-Ericksen), $\sharp_g$ is g-index raising operator.

FREE BOUNDARY FLUIDS

Configuration space Emb$(\mathcal{B}, \mathcal{S})$. Material Lagrangian:

$$L_{(\bar{\varrho},g)}(V_\eta) = \frac{1}{2}\int_{\mathcal{B}} g(V_\eta, V_\eta)\bar{\varrho} - \int_{\mathcal{B}} E(\bar{\varrho}(X), g(\eta(X)), T_X\eta)\bar{\varrho} - \tau \int_{\partial\mathcal{B}} \gamma(\eta^* g),$$

E internal energy density related to the spatial energy e as before, τ a constant. Third term proportional to area of current configuration and represents the potential energy associated with surface tension; $\gamma(\eta^* g)$ boundary volume form of Riemannian volume form for $\eta^* g$.

Convective representation: L left Diff$(\mathcal{S})$-invariant, so produces

$$\ell_{conv}(\mathcal{V}, \bar{\varrho}, C) = \frac{1}{2}\int_{\mathcal{B}} C(\mathcal{V}, \mathcal{V})\bar{\varrho} - \int_{\mathcal{B}} \mathcal{E}(\bar{\varrho}, C)\bar{\varrho} - \tau \int_{\partial\mathcal{B}} \gamma(C).$$

Convective equations of motion

$$\begin{cases} \bar{\varrho}\,(\partial_t \mathcal{V} + \nabla_{\mathcal{V}}\mathcal{V}) = 2\,\mathrm{Div}_C\left(\dfrac{\partial \mathcal{E}}{\partial C}\bar{\varrho}\right) \\ \partial_t C - \pounds_{\mathcal{V}} C = 0, \end{cases}$$

(BC) $p \circ \eta|_{\partial\mathcal{B}} = \tau\kappa_C$, κ_C mean curvature of $\partial\mathcal{B}$ relative to C and p is the spatial pressure. In terms of p, the right hand side of the motion equation reads $-\,\mathrm{grad}_C(p \circ \eta)\mu(C)$.

Spatial representation: L right Diff($\mathcal{B}$)-invariant: $(V_\eta, \bar{\varrho}, g) \mapsto (V_\eta \circ \varphi, \varphi^* \bar{\varrho}, g)$, $\forall \varphi \in$ Diff($\mathcal{B}$). This leads to the spatial Lagrangian:

$$\ell_{spat}(\Sigma, \mathbf{v}, \bar{\rho}, g) = \frac{1}{2}\int_{D_\Sigma} g(\mathbf{v}, \mathbf{v})\bar{\rho} - \int_{D_\Sigma} e(\rho)\bar{\rho} - \tau \int_\Sigma \gamma(g)$$

and the spatial equations of motion

$$\begin{cases} \rho\left(\partial_t \mathbf{v} + \nabla_{\mathbf{v}} \mathbf{v}\right) = -\operatorname{grad}_g p \\ \partial_t \bar{\rho} + \pounds_{\mathbf{v}} \bar{\rho} = 0 \end{cases} \quad \text{on} \quad \Sigma$$

with the boundary condition and boundary movement

$$p|_\Sigma = \tau \kappa_g, \quad \partial_t \Sigma = g(\mathbf{v}, \mathbf{n}_g).$$

Can generalize to a large class of continua that include both elasticity and free boundary fluids:

$$L(V_\eta, \bar{\varrho}, g, G) = \frac{1}{2}\int_{\mathcal{B}} g(\eta(X))(V_\eta(X), V_\eta(X))\bar{\varrho}(X)$$
$$- \int_{\mathcal{B}} U(g(\eta(X)), T_X\eta, G(X), \bar{\varrho}(X)) - \tau \int_{\partial\mathcal{B}} \gamma(\eta^* g),$$

U density on $\mathcal{B}$. This form is more general than the Lagrangian for free boundary fluids and for elastic materials considered before.