Int. J. mech. Sci., Vol. 19, pp. 505–512. Pergamon Press 1977. Printed in Great Britain

A THEORY OF PLASTIC ANISOTROPY BASED ON A YIELD FUNCTION OF FOURTH ORDER (PLANE STRESS STATE)—I

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(*Received* 17 *January* 1977; *in revised form* 14 *June* 1977)

Summary—It is shown from various view points that many of the disadvantages of the conventional theory based on a quadratic yield function can be satisfactorily removed by the use of a yield function of fourth order. Incremental equivalent strain $\overline{d\epsilon_{eq}}$ is defined by $\overline{d\epsilon_{eq}} = \sigma_{ij}\, d\epsilon_{ij}/\bar{\sigma}_{eq}$, and cannot generally be expressed simply by the strain increment components $d\epsilon_{ij}$. In contrast with the conventional theory, coefficients in the yield function f cannot be determined from the r-values only in uniaxial tensile tests, but yield stresses in these tests and for example in an equi-biaxial tension for the same $\bar{\epsilon}_{eq}$ are also required. This fact ensures that the $\bar{\sigma}_{eq} - \bar{\epsilon}_{eq}$ curve for arbitrary loading is uniquely determined by the uniaxial tension curve in the rolling direction (R.D.), and thus such an intrinsic difficulty of the conventional theory as dependence of the $\bar{\sigma}_{eq} - \bar{\epsilon}_{eq}$ curve on types of loading does not arise. Some formulae for the determination of the coefficients in f are given. Relationships between types of earing in axi-symmetrical deep-drawing and the coefficients of f are examined in detail and it is emphasized that only very special cases are included in the conventional yield function and thus use of it is very limited.

INTRODUCTION

FOR describing the anisotropic plasticity of sheet metal, Hill's quadratic yield function g[1] is commonly used for conciseness. However, many disadvantages of this conventional theory have been pointed out. For example; (1) it can only be applied to those materials which form four (or two) ears in an axisymmetrical deep-drawing operation;[1] (2) equivalent stress $\bar{\sigma}_{eq}$ vs equivalent strain $\bar{\epsilon}_{eq}$ curves of various materials by this theory are dependent on the types of loading, though the curves should be intrinsically unique for a given material;[2–4,8] (3) in uniaxial tensile tests, the dependence of uniaxial yield stresses on the tensile directions is little predicted by the theory,[6] though that of r-values is well predicted;[5,6] (4) there exist many sheet metals whose deep-drawability are not well expressed by their r-values (e.g. Ref. 3), and so forth. (2)–(4) are interpreted as being due to the change in the anisotropic coefficients in g owing to deformation. However, formulation of the change is impossible at present and thus such an interpretation seems rather impractical. Furthermore, the difficulties mentioned above do not all seem to be of secondary importance, but their source may be attributed to intrinsic disadvantages of the quadratic yield function.

In this and the following papers the usefulness of a yield function of fourth order (instead of a quadratic one) for rolled sheet metals with orthotropy is clearly proved. The difficulties of the conventional theory (1)–(4) mentioned above are almost completely removed. The problem of the change of anisotropic coefficients due to deformation can be thought to be a secondary one.

In this paper, derivation of a yield function of fourth order, relations between stresses and strain increments, the definition of the equivalent stress $\bar{\sigma}_{eq}$ and the equivalent strain increment $\overline{d\epsilon_{eq}}$, the $\bar{\sigma}_{eq}$ vs $\bar{\epsilon}_{eq}$ curve, the relationship between the coefficients in f (the yield function), the types of ear-formation in an axisymmetrical deep-drawing operation, and the experimental methods of determining the coefficients and so on are presented. "Strain" used in this and the following papers should be understood to be plastic only because of the rigid-plastic idealization.

With regard to a higher order polynomial yield function, Bourne and Hill[7] has already treated a third-degree one (g', say), specifically in connexion with the r-value relationship for a material (brass) that shows 6 ears in axisymmetrical cupping tests, and reported difficulties in the equivalent stress–strain notion. g' contains 5 anisotropic coefficients to be determined and predicts rather well the r-value distribution in a plane, but it is not verified that it also predicts satisfactorily the uniaxial yield stress distribution in experiments. Actually their theory reports that g' fails to predict the directions of ears and hollows. They determined the coefficients in g' by the r-values in uniaxial tensile tests, co-axiality between stress and strain in a tension test at 30 degrees to the rolling direction (R.D.), and the ratio of the strain in R.D. to that in T.D. (the transverse direction) in a compression test normal to the sheet plane, and they did not make any use of the yield stresses. Thus it seems rather doubtful that g' determined by them expresses well the actual distribution of yield stresses. We will use the yield stresses in uniaxial tensile tests and that in an equi-biaxial in plane tension to determine the 8 coefficients contained in the fourth-order polynomial yield function f. As seen later, if we do not avoid a little effort to make a compression test normal to the sheet plane (or a test equivalent to it, e.g. a hydraulic bulging test of a circular blank) added to some uniaxial tensile tests, there does not exist much difference between the efforts in determining the coefficients in g' and in f due to recent development of handy-type electronic computers, if the formulae for that purpose are well established. And when the third-degree polynomial g' is used, we must understand the plastic potential and the yield function to be $|g'|$, not g' itself, because it is an odd function. Moreover, g' does not reduce to the Mises function for isotropy and the conventional quadratic yield function g by Hill for orthotropy as special cases and thus g' cannot be considered to be a direct extension of them to higher-order representations. It seems rather unreasonable from the point of logical consistency. Furthermore, g' does not represent the eight-ears-materials to which annealed commercial pure aluminium and others commonly belong.

DERIVATION OF AN ANISOTROPIC YIELD FUNCTION OF FOURTH ORDER

1. *Derivation of the yield function f*

Let f be written as below, as in Hill's general theory:[1]

$$f = \sum_{i,j,k} A_{ijk}\sigma_x{}^i\sigma_y{}^j\tau_{xy}^{2k}, \qquad i+j+2k \leqq 4, \tag{1}$$

where x and y axes are taken to be coincident with the R.D. and T.D. respectively (i.e. the principal axes of orthotropy), and A_{ijk} are constant coefficients.

The conditions of orthotropy and ear- and hollow-formation in an axisymmetrical deep-drawing due to Hill[1] would require f in (1) to take the following form:

$$f = A_0(\sigma_x+\sigma_y)^2 + [A_1\sigma_x{}^4 + A_2\sigma_x{}^3\sigma_y + A_3\sigma_x{}^2\sigma_y{}^2 + A_4\sigma_x\sigma_y{}^3 + A_5\sigma_y{}^4 + (A_6\sigma_x{}^2 + A_7\sigma_x\sigma_y + A_8\sigma_y{}^2)\tau_{xy}^2 + A_9\tau_{xy}^4]. \tag{2}$$

The first term on the right hand side in (2) may be considered to be a term dependent on the hydrostatic stress and thus can be removed when we assume incompressibility of the material, namely $A_0 = 0$ hereafter.

The condition of ear- and hollow-formation yields the following equations:

$$\cos\alpha \cdot \sin\alpha \cdot F(c) = 0, \qquad c = \cos^2\alpha, \tag{3}$$

and

$$F(c) = 4Ac^3 + 3Bc^2 + 2Cc + D, \tag{4}$$

where α (in degree) indicates the direction of the tensile axis to the R.D. and the coefficients A to D are expressed in terms of the coefficients A_1 to A_9 of f in (2) (see equation 24). As Hill remarks, an ear or a hollow will be formed in the direction $\alpha' = 90 - \alpha$ to the R.D., where α satisfies equation (3), i.e.

$$F(c) = 0, \qquad c = \cos^2\alpha, \tag{5}$$

as well as $\alpha = 0°$ and $90°$.

2. *Formulae of r-value and uniaxial yield stress*

The r-value for the α-direction (r_α) due to the yield function f in (2) reduces to

$$r_\alpha = d\epsilon_b/d\epsilon_t = R_1(c)/R_2(c), \tag{6}$$

$$\left.\begin{aligned} R_1(c) &= 4Ac^4 + C_1c^3 + C_2c^2 + C_3c - A_4 \\ &\text{and} \\ R_2(c) &= C_4c^3 + C_5c^2 + C_6c + 4A_5 + A_4 \end{aligned}\right\}, \tag{7}$$

where $c = \cos^2 \alpha$ and

$$\begin{aligned} C_1 &= -4 - 12A_5 + 6(A_2 + A_6) - 8(A_3 + A_7 + A_9) + 10(A_4 + A_8), \\ C_2 &= 12A_5 - 3A_2 - 2A_6 - 9A_4 - 8A_8 + 6A_3 + 5A_7 + 4A_9, \\ C_3 &= -4A_5 - 2A_3 - A_7 + 4A_4 + 2A_8, \\ C_4 &= 4 - 4A_5 - 2(A_2 + A_6) + 2(A_4 + A_8), \\ C_5 &= 3A_2 - 2A_3 - 3A_4 + 12A_5 + 2A_6 - A_7 - 4A_8, \\ C_6 &= 2A_3 - 12A_5 + A_7 + 2A_8, \qquad A = \text{(see equation 24)}. \end{aligned}$$

$d\epsilon_b$ and $d\epsilon_t$ are breadth and thickness strain increments, respectively.

Next, the uniaxial yield stress σ_α for the α-direction to the R.D. in plane at a certain equivalent strain (which will be defined later) is given from equation (2) as follows:

$$\left.\begin{aligned} \sigma_\alpha &= \sigma_{\mathrm{RD}}/\{G(c)\}^{1/4} \\ &\text{and} \\ G(c) &= Ac^4 + Bc^3 + Cc^2 + Dc + a_5 \end{aligned}\right\}, \tag{8}$$

where σ_{RD} is the uniaxial yield stress in the R.D. and A to D are given in equation (24).

It is noted that r_α due to equation (6) is constant (that is, independent of strain level). And one of the nine coefficients A_1 to A_9 is arbitrary and thus we can put $A_1 = 1$, say, which means $f = \sigma_{\mathrm{RD}}^4$ for any strain level.

STRESS–INCREMENTAL STRAIN RELATION AND STRESS–STRAIN CURVE

Incremental strains are given by use of the function f as the plastic potential as follows:

$$\left.\begin{aligned} d\epsilon_x &= d\lambda[4\sigma_x^3 + 3A_2\sigma_x^2\sigma_y + 2A_3\sigma_x\sigma_y^2 + A_4\sigma_y^3 + (2A_6\sigma_x + A_7\sigma_y)\tau_{xy}^2], \\ d\epsilon_y &= d\lambda[A_2\sigma_x^3 + 2A_3\sigma_x^2\sigma_y + 3A_4\sigma_x\sigma_y^2 + 4A_5\sigma_y^3 + (A_7\sigma_x + 2A_8\sigma_y)\tau_{xy}^2], \\ d\gamma_{xy} &= d\lambda[2(A_6\sigma_x^2 + A_7\sigma_x\sigma_y + A_8\sigma_y^2)\tau_{xy} + 4A_9\tau_{xy}^3], \end{aligned}\right\} \tag{9}$$

where $d\lambda$ is a positive constant dependent on the work-hardening level. $d\gamma_{xy}$ is the engineering shearing strain.

The equivalent stress $\bar{\sigma}_{eq}$ is defined as

$$\bar{\sigma}_{eq} = (f)^{1/4}, \tag{10}$$

and the equivalent strain increment $\overline{d\epsilon_{eq}}$ is defined as

$$dW^p = \bar{\sigma}_{eq}\,\overline{d\epsilon_{eq}} = \sigma_{ij}\,d\epsilon_{ij}, \qquad \text{(summed over } i \text{ and } j), \tag{11}$$

where dW^p is the plastic work increment per unit volume. And thus we have

$$\overline{d\epsilon_{eq}} = (\sigma_x\,d\epsilon_x + \sigma_y\,d\epsilon_y + \tau_{xy}\,d\gamma_{xy})/\bar{\epsilon}_{eq}, \tag{12}$$

and

$$\bar{\epsilon}_{eq} = \int \overline{d\epsilon_{eq}}. \tag{13}$$

From equations (10), (12) and (13) we obtain

$$d\lambda = \overline{d\epsilon_{eq}}/4\bar{\sigma}_{eq}^3. \tag{14}$$

$\overline{d\epsilon_{eq}}$ for the conventional quadratic yield function is expressed by the strain increments $d\epsilon_{ij}$, by eliminating $d\lambda$ and $\bar{\sigma}_{eq}$ from the equations corresponding to equations (9), (10) and (12).[1] In case of the yield function of fourth order, however, $\overline{d\epsilon_{eq}}$ is not generally expressed by $d\epsilon_{ij}$. Namely, if we put

$$\begin{aligned} (\overline{d\epsilon_{eq}})^4 &= E_1\,d\epsilon_x^4 + E_2\,d\epsilon_x^3\,d\epsilon_y + \cdots + E_9\,d\gamma_{xy}^4 \\ &= (4\,d\lambda)^4(\bar{\sigma}_{eq})^{12}, \end{aligned} \tag{15}$$

where the last equation is derived from equation (14), we find that, by substituting equations (9) and (10) into (15) and equating individually the coefficients of the same order terms of σ_{ij} on both sides, far more equations than nine with respect to the nine coefficients E_1 to E_9 to be expressed by A_1 to A_9 are obtained. We can determine them only for the special case where f is equivalent to the square of the conventional quadratic yield function. Thus we should say the conventional theory is restricted to this very special case. Here we adopt equation (12) as the defining equation for $\overline{d\epsilon}_{eq}$.

Now we show several examples of $\overline{d\epsilon}_{eq}$ given by equation (12) which are expressed in terms of the components $d\epsilon_{ij}$.

(i) *Uniaxial tension in α direction to R.D.*

From equations (10) and (12) we have

$$\overline{d\epsilon}_{eq} = \sigma \, d\epsilon_l/\bar{\sigma}_{eq} = d\epsilon_l/\{G(c)\}^{1/4}, \tag{16}$$

where $d\epsilon_l$ is the longitudinal strain increment and $G(c)$ is given in equation (8).

For example, for uniaxial tension in the R.D. we have

$$\bar{\sigma}_{eq} = \sigma, \qquad \overline{d\epsilon}_{eq} = d\epsilon_l. \tag{17}$$

The stress-longitudinal strain curve for the R.D. is the $\bar{\sigma}_{eq} - \bar{\epsilon}_{eq}$ curve of the material, which is due to taking $A_1 = 1$.

(ii) *Equi-biaxial (E.B.) tension*

$$\begin{aligned} \sigma_x &= \sigma_y = \sigma_b, \qquad \tau_{xy} = 0. \\ \bar{\sigma}_{eq} &= (C_b)^{1/4}\sigma_b, \qquad \overline{d\epsilon}_{eq} = (C_b)^{-1/4}|d\epsilon_3|, \\ C_b &= A_1 + A_2 + A_3 + A_4 + A_5. \end{aligned} \tag{18}$$

(iii) *Biaxial normal stress state in x–y directions*

By putting $\sigma_x/\sigma_y = m$, we have

$$\left.\begin{aligned} \bar{\sigma}_{eq} &= |C_m|^{1/4} \cdot |\sigma_y|, \qquad \overline{d\epsilon}_{eq} = |C_m|^{-1/4} \cdot |m \, d\epsilon_x + d\epsilon_y|, \\ C_m &= A_1 m^4 + A_2 m^3 + A_3 m^2 + A_4 m + A_5. \end{aligned}\right\} \tag{19}$$

(iv) *Pure shear in x–y direction*

$$\left.\begin{aligned} \sigma_x &= \sigma_y = 0, \qquad \tau_{xy} = \tau, \qquad d\gamma_{xy} = d\gamma. \\ \bar{\sigma}_{eq} &= (A_9)^{1/4} \cdot |\tau|, \qquad \overline{d\epsilon}_{eq} = (A_9)^{-1/4} \cdot |d\gamma|. \end{aligned}\right\} \tag{20}$$

Now let us illustrate a method of determining the yield stress σ for an arbitrary loading system associated with a certain equivalent strain ϵ_0 (= ln 1·15, say), which is obtained directly from equation (12). First we illustrate the stress–strain curve for the loading system with the stress and strain axes chosen in such a manner that the area below the curve represents the current plastic work per unit volume (e.g. tensile stress and longitudinal strain for uniaxial tension, the stress an in-plane one and the absolute value of thickness strain for the E.B. tension and so on). σ and the corresponding strain ϵ are determined in such a manner as is illustrated in Fig. 1 where the $\bar{\sigma}_{eq} - \bar{\epsilon}_{eq}$ curve of the material is referred to and the shaded areas A and A' are made equal to each other. Conversely, $\bar{\sigma}_{eq}$ and $\bar{\epsilon}_{eq}$ corresponding to the point (σ, ϵ) are given by $\bar{\sigma}^0_{eq}$ and ϵ_0, respectively, where $\bar{\sigma}^0_{eq}$ is the value of $\bar{\sigma}_{eq} = \epsilon_0$; they necessarily lie on the $\bar{\sigma}_{eq} - \bar{\epsilon}_{eq}$ curve. (Of course, we could determine them directly from equations (10), (12) and (13) without any reference to the $\bar{\sigma}_{eq} - \bar{\epsilon}_{eq}$ curve.) Thus we note that the $\bar{\sigma}_{eq} - \bar{\epsilon}_{eq}$ curve should be quite uniquely determined by a representative stress–strain curve (here the $\sigma - \epsilon_l$ curve in the R.D.) of course the choice of the curve is dependent on convenience and the accuracy of its determination. We can adopt, e.g. the E.B. tension curve instead of the R.D. curve. (Then, generally $A_1 \neq 1$ and all other coefficients are multiplied by a certain constant and $A_1 + A_2 + A_3 + A_4 + A_5 = 1$.) However, as is well known, the $\bar{\sigma}_{eq} - \bar{\epsilon}_{eq}$ curves of various materials by conventional theory show evident dependence on the type of loading and, further, those of

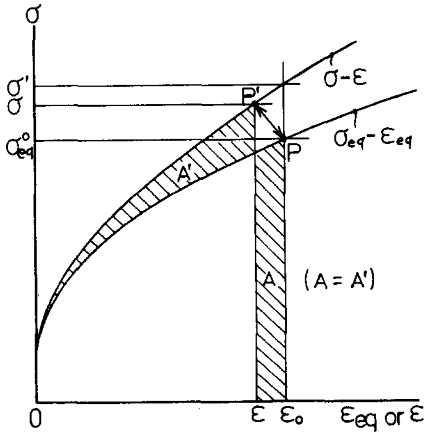

Fig. 1. Correspondence of an arbitrary $(\sigma - \epsilon)$ curve to the $(\bar{\sigma}_{eq} - \bar{\epsilon}_{eq})$ curve.

some materials show stronger dependence than those made under the assumption of isotropy.[1-3] This seems one of the intrinsic disadvantages of the conventional quadratic yield function, the reason for which is rather simple. All coefficients of the conventional yield function are determined by use of r-values only from a few uniaxial tensile tests and none of the information from equation (11) is used. This is a practical advantage for determinating the coefficients but it conceals unreasonableness in that no reference to equation (11) is made. On the other hand, use of equation (11) is made through the use of yield stresses when the yield function of fourth order f is adopted, because as will be seen later, we cannot determine all of them from the r-values only, and uniaxial yield stresses and another yield stress other than them are also needed. The procedure for the determination of the coefficients of f itself would ensure the uniqueness of the $\bar{\sigma}_{eq} - \bar{\epsilon}_{eq}$ curve.

When we adopt a plastic potential different from the yield function, different conclusions would be reached. However, we do not take such a line here, because another problem arises with respect to the choice of the potential function.

RELATIONSHIP BETWEEN THE COEFFICIENTS OF f AND THE TYPES OF EAR-FORMATION

1. *Description of the relationship*

Examinations of equation (6) leads us to the following relations. Of course these are mathematical classifications and thus there may exist such cases that 4-ear-materials identified by the naked eye should be classified as 6- or 8-ear-materials in the following due to ambiguity of ear-formation. We will show such an example for a commercial copper-(1/4)H in Part II.

First, we introduce the following constants where $A_1 = 1$.

$$\left.\begin{aligned}
A &= (A_1 + A_3 + A_5 + A_7 + A_9) - (A_2 + A_4 + A_6 + A_8)\\
B &= (A_2 + 3A_4 + A_6 + 3A_8) - 2(A_3 + 2A_5 + A_7 + A_9)\\
C &= (A_3 + 6A_5 + A_7 + A_9) - 3(A_4 + A_8)\\
D &= A_4 + A_8 - 4A_5\\
E &= 32AC^3 + 108A^2D^2 + 27B^3D - 9BC(12AD + BC)\\
F &= B^3 + 8A^2D - 4ABC, \qquad G = 4A + 3B + 2C + D\\
A^* &= 2A + C/3, \qquad A^{**} = (4A + 2C + D)/3, \qquad B^* = 3B^3 - 8AC.
\end{aligned}\right\} \tag{24}$$

In general it is verified that a hollow forms at $\alpha = 0°$ and $90°$ corresponding to $F(0) > 0$ and $F(1) > 0$, respectively, where the function $F(c)$ is given in equation (4). In the following semi-colons (i.e. ;) should be understood as "and-s" (i.e. "logical products").

(i) *Cases where 8-ears form*

(a) $E \leqq 0;\quad A, C > 0;\quad B, D < 0;\quad \sqrt{(8AC/3)} < -B < \min[4A, A^*, A^{**}].$ (25)

(b) $E \leqq 0;\quad A, C < 0;\quad B, D > 0;\quad \sqrt{(8AC/3)} < B < \min[-4A, -A^*, -A^{**}].$ (26)

($E = 0$ denotes special cases where equation (5) has a double root. Then apparently only four ears form.)

(ii) *Cases where 6-ears form*

(a) $E \leqq 0;\quad A, D > 0;\quad B \geqq 0;\quad C < 0;\quad B > \max[-A^*, -A^{**}].$ (27)

(b) $E \leqq 0;\quad A, D > 0;\quad B \leqq 0;\quad B^2 > 8AC/3;\quad -B < \min[4A, A^*, A^{**}].$ (28)

(c) $E \leqq 0;\quad A, C > 0;\quad B, D < 0;\quad 4A \geqq -B > \max[\sqrt{(8AC/3)}, A^{**}],$ or $-B \geqq 4A$ and $-B > \max[\sqrt{(8AC/3)}, A^*, A^{**}].$ (29)

(d) $E \leqq 0;\quad A, D < 0;\quad B^2 > 8AC/3;\quad 0 \leqq B < \min[-4A, -A^*, -A^{**}].$ (30)

(e) $E \leqq 0;\quad A, D < 0;\quad B \leqq 0;\quad C > 0;\quad B < \min[-A^*, -A^{**}].$ (31)

(f) $E \leqq 0;\quad A, C < 0;\quad B, D > 0;\quad -4A \geqq B > \max[\sqrt{(8AC/3)}, -A^{**}],$ or $B \geqq -4A$ and $B > \max[\sqrt{(8AC/3)}, -A^*, -A^{**}].$ (32)

(g) $A = 0;\quad B, D > 0;\quad \sqrt{(3BD)} \leqq -C < \min[3B, (3B + D)/2].$ (33)

(h) $A = 0;\quad B, D < 0;\quad \sqrt{(3BD)} \leqq C < \min[-3B, -(3B + D)/2].$ (34)

(In (a) to (f) $E = 0$ represents cases where equation (5) has a double root and then apparently weak 6 ears or only two ears may form. Also when equality holds in the last relations of (g) and (h), ears may form in the same manner as above. These are all special cases like those of $A = 0$.)

(iii) *Cases where four ears form*

(a) $3B^2 = 8AC;\quad B^3 = 16A^2D;\quad A > 0$ and $0 < -B < 4A,$ or $A < 0$ and $0 < B < -4A.$ (35)

(This is a very special case where equation (5) has a triple root. Then apparently only four ears may form.)

(b) $AD < 0;\quad AG > 0;\quad B^* \leqq 0,$ or $B^* > 0$ and $E > 0.$ (36)

(c) $A>0;\quad D<0;\quad B^*>0;\quad G>0;\quad B\geqq 0$, or $B\leqq 0$ and $C<0$; $F\geqq 0$ and $E<0$, or $F\leqq 0$ and $E\leqq 0$. (37)

(d) $A>0;\quad D<0;\quad B^*>0;\quad G>0;\quad B\leqq -4A$, or $-4A\leqq B<-A^*$; $F\geqq 0$ and $E\leqq 0$, or $F\leqq 0$ and $E<0$. (38)

(e) $A<0;\quad D>0;\quad B^*>0;\quad G<0;\quad B\leqq 0$, or $B\geqq 0$ and $C>0$; $F\geqq 0$ and $E\leqq 0$, or $F\leqq 0$ and $E<0$. (39)

(f) $A<0;\quad D>0;\quad B^*>0;\quad G<0;\quad B\geqq -4A$, or $-A^*<B\leqq -4A$; $F\geqq 0$ and $E<0$, or $F\leqq 0$ and $E\leqq 0$. (40)

(g) $B^*>0;\quad E<0;\quad AD>0;\quad AG<0.$ (41)

(h) $A=0;\quad C^2-3BD>0;\quad B\neq 0;\quad DG<0.$ (42)

(i) $A=B=0;\quad CD<0;\quad C(2C+D)>0.$ (43)

(iv) *The earless case*

$$A=B=C=D=0. \tag{44}$$

Then we have the following relations with respect to the coefficients:

$$\left.\begin{aligned} &A_1=A_5;\quad B_1=B_3=4A_1;\quad B_2=6A_1;\\ &B_1=A_2+A_6,\quad B_2=A_3+A_7+A_9,\quad B_3=A_4+A_8.\end{aligned}\right\} \tag{45}$$

This earless condition is a necessary but not sufficient condition for planar isotropy, see equation (58).

(v) *Cases where two ears form*

Remainders of (i) to (iv) belong to them, though detail is omitted here.

2. *Cases expressed by the quadratic yield function*

Denoting the conventional quadratic yield function by g, we have

$$g=\sigma_x{}^2-F'\sigma_x\sigma_y+G'\sigma_y{}^2+N'\tau_{xy}^2, \tag{46}$$

where F', G' and N' are the coefficients expressed by r-values. The function $F(c)$ for g is obtained as follows:

$$F(c)=C'c+D'=0, \tag{47}$$

where C' and D' are constants. If we adopt g^2 instead of g corresponding to f, $F(c)$ is given by that in (47) multiplied by g (>0) and thus we obtain a condition equivalent to (47).

According to equation (47), we find that the quadratic yield function can express only four ears for the case (iii)–(i), a few cases of two ears and the earless case of the previous article. The conventional theory can include only very limited and special cases of four and two ears. Thus our ordinary recognition that the anisotropy of four-ear-materials can be expressed by the quadratic yield function may be thought to be optimistic. When we refer to the apparent ambiguity of ear shapes written in the previous section, such a view-point should be further emphasized.

DETERMINATION OF THE COEFFICIENTS OF f

1. *Are only uniaxial tensile tests sufficient for the purpose?*

As is well known, the three coefficients F', G' and N' in the quadratic yield function g (equation 46), are determined by the r-values for $\alpha=0°$, 45° and 90°, namely,

$$\left.\begin{aligned} &F'=2/(1+1/r_0),\quad G'=(1+1/r_{90})/(1+1/r_0),\\ &N'=(1+2r_{45})(1/r_0+1/r_{90})/(1+1/r_0).\end{aligned}\right\} \tag{48}$$

However, the eight coefficients A_2 to A_9 in f cannot be determined by only the r-values, because equation (7) for r_α yields relations dependent on each other with respect to A_3, A_7 and A_9 for an arbitrary number of combinations of α. Thus we must use yield stresses σ_α given by equation (8) as well. However, arbitrary combinations of equations (7) and (8) also yield non-independent relations between A_3, A_7 and A_9. That is to say, we cannot determine two of the three by uniaxial tensile tests of many arbitrary directions alone. If the directions of ears or hollows are known, we can make use of equation (5). However, the function $F(c)$ also contains A_3, A_7 and A_9 in the form $(A_3+A_7+A_9)$ and thus a similar inconvenience arises. We should note that certain information on through thickness property must be added in order to eliminate such an inconvenience. Here we adopt the yield stress from one of the following tests: equi-biaxial (E.B.), tensile test (by the circular hydraulic bulging test—which gives an equi-biaxial stress state to a good approximation even for anisotropic sheet metals[8]), a compression through-thickness test (with several test pieces made to adhere to each other by using some adhesive) or a plane strain compression test of strip, (which we shall call the X-test hereafter).

2. *Determination of yield stresses*

Yield stresses σ_α (uniaxial), σ_b (E.B.-axial) or σ_p (plane strain compression) are determined as described in an earlier part by reference to Fig. 1. For convenience, let us assume an nth power law for the

$$\sigma_{RD} = c_0 \epsilon_0^n \tag{49}$$

is the $\bar{\sigma}_{eq} - \bar{\epsilon}_{eq}$ curve of the material. We note that the ratio of the yield stresses for arbitrary loading and σ_{RD} for the same $\bar{\epsilon}_{eq}$ is assumed constant and independent of $\bar{\epsilon}_{eq}$ when we adopt f as the yield function. (This statement is also true for the conventional quadratic yield function). Thus the exponent n is also assumed constant and it appears more reasonable than that n in equation (49) should be taken as $\bar{n}$, i.e. the average of n in the sheet plane, instead of n_{RD}. Actually, as is well known, dependence of n on the tensile direction α is usually very weak. It is the area below the curve which is in question and thus the strictness of expressions such as equation (49) is not so severely required. Similar to equation (49) we put

$$\sigma_\alpha = c_\alpha \epsilon_\alpha^n, \qquad \sigma_b = c_b |\epsilon_t|^n \qquad \text{and} \qquad |\sigma_p| = c_p |\epsilon_t|^n, \tag{50}$$

where ϵ_α is the longitudinal strain in the α-direction. According to the method mentioned earlier, we have

$$\int_0^{\epsilon_\alpha} \sigma_\alpha \, d\epsilon_\alpha = c_\alpha \epsilon_\alpha^{n+1}/(n+1) = \int_0^{\epsilon_0} \sigma_{RD} \, d\epsilon_0 = c_0 \epsilon_0^{n+1}/(n+1), \tag{51}$$

and thus

$$\epsilon_\alpha = (c_0/c_\alpha)^{1/(n+1)} \cdot \epsilon_0, \qquad \sigma_\alpha = c_\alpha (c_0/c_\alpha)^{n/(n+1)} \cdot \epsilon_0^n. \tag{52}$$

Formulae similar to equation (52) will be obtained with respect to σ_b and σ_p.

3. *Formulae for the determination of the coefficients*

To determine the coefficients of f, uniaxial tensile tests for $\alpha = 0°$, 22·5°, 45° and 90° and the X-test are required. $\alpha = 22{\cdot}5°$ can be replaced by 67·5°; r_0, $r_{22\cdot5}$, r_{45}, r_{90}, σ_{RD}/σ_{TD}, $\sigma_{RD}/\sigma_{22\cdot5}$, σ_{RD}/σ_b (or σ_{RD}/σ_p) give all the information required. In the following equations $c_1 = \cos^2 \alpha_1$ and α_1 is equal to either 22·5 or 67·5, though the numerical equations are given for $\alpha_1 = 22{\cdot}5$. Equations are arranged in the order of calculation. First, A_2 to A_5 are easily found from

$$\left.\begin{aligned} &A_2 = -4\gamma, \qquad A_5 = (\sigma_{RD}/\sigma_{TD})^4, \qquad A_4 = -4\beta A_5, \\ &\gamma = r_0/(1+r_0), \qquad \beta = r_{90}/(1+r_{90}). \end{aligned}\right\} \tag{53}$$

$$X^* = \sigma_b/\sigma_{RD}, \qquad X_p^* = \sigma_p/\sigma_{RD}. \tag{54}$$

$$\left.\begin{aligned} &A_3 = 1/(X^*)^4 - (A_1 + A_2 + A_3 + A_4 + A_5), \\ \text{or} \quad &A_3 = 4/(X_p^*)^4 - (4A_1 + 2A_2 + 0{\cdot}5A_4 + 0{\cdot}25A_5). \end{aligned}\right\} \tag{55}$$

A_6 to A_9 are calculated by the following equations:

$$\left.\begin{aligned}
&A_9 = (16b)\delta + 1 + A_2 + A_3 + A_4 + A_5, \qquad b = (\sigma_{RD}/\sigma_{45})^4, \\
&\delta = r_{45}/(1+r_{45}), \qquad c_1 = \cos^2\alpha_1 = 0{\cdot}85356, \\
&B_8 = -[r_{\alpha 1}\{4c_1^3 + 4(1-c_1)^3 A_5 + c_1^2(3-2c_1)A_2 + (1-c_1)^2(2c_1+1)A_4\} \\
&\quad + 4c_1^3(1-c_1) + 4c_1(1-c_1)^3 A_5 + c_1^2(4c_1^2 - 6c_1 + 3)A_2 \\
&\quad + (1-c_1)^2(4c_1^2 - 2c_1 + 1)A_4]/\{c_1(1-c_1)\} - 2(r_{\alpha 1} + 2c_1^2 - 2c_1 + 1)A_3 + 4c_1(1-c_1)A_9. \\
&\quad = -8[r_{\alpha 1}\{2{\cdot}4875 + 0{\cdot}012561A_5 + 0{\cdot}9419A_2 + 0{\cdot}05805A_4\} \\
&\quad + 0{\cdot}36427 + 0{\cdot}010722A_5 + 0{\cdot}57768A_2 + 0{\cdot}047331A_4] \\
&\quad - 2(r_{\alpha 1} + 0{\cdot}75)A_3 + 0{\cdot}5A_9. \\
&B_7 = (\sigma_{RD}/\sigma_{\alpha 1})^4 - \{c_1^4 + c_1^3(1-c_1)A_2 + c_1^2(1-c_1)^2(A_3 + A_9) \\
&\quad + c_1(1-c_1)^3 A_4 + (1-c_1)^4 A_5\}. \\
&\quad = (\sigma_{RD}/\sigma_{22\cdot5})^4 - \{0.53081 - 0{\cdot}09107A_2 + 0{\cdot}015625(A_3 + A_9) \\
&\quad + 0{\cdot}00268A_4 + 0{\cdot}000460A_5\}. \\
&B_6 = (16b)(1+\delta) - 2A_9. \\
&B_5 = \{B_8 - (r_{\alpha 1} + 4c_1^2 - 4c_1 + 1)B_6\}/(2c_1 - 1) = \sqrt{2}\{B_8 - (r_{\alpha 1} + 0{\cdot}5)B_6\}. \\
&B_4 = \{B_7 - c_1^2(1-c_1)^2 B_6\}/\{c_1(1-c_2)(2c_1 - 1)\} = 8\sqrt{2}(B_7 - 0{\cdot}015625B_6). \\
&D_1 = (2c_1 - 1)(r_{\alpha 1} + 1) = 0{\cdot}70711(r_{\alpha 1} + 1). \\
&A_6 = \{(r_{\alpha 1} + 4c_1 - 3)B_4 - (1-c_1)B_5\}/D_1. \\
&A_8 = \{(r_{\alpha 1} - 4c_1 + 1)B_4 - c_1 B_5\}/D_1. \\
&A_7 = B_6 - (A_6 + A_8).
\end{aligned}\right\} \tag{56}$$

Suffix $\alpha 1$ means α_1.

For six-ear-materials it would be better to select 0, 30, 45, 67·5 and 90 as α-s, where 67·5 is adopted instead of 60, because the combination of 30 and 60 is verified to yield a similar inconvenience to that described in Art. 1 of this section. The formulae are then somewhat modified, though we do not present them here.

When the α's directions of the ears and hollows are known, we can make use of equation (5) for $\alpha_i = 90 - \alpha_i'$. Especially for eight-ear-materials, there exist three α_i''s ($i = 1, 2, 3$) and thus we can calculate all the coefficients of f by use of them and r_0, r_{45}, r_{90}, σ_{RD}/σ_{TD} and X^* (or X_p^*), except for the special case where $\alpha_1 = 90 - \alpha_3$ and $\alpha_2 = 45$. For six- (and eight-) ear-materials there exist two α_i''s ($i = 1, 2$; or two of three for eight-ears). Then α_i and r_0, r_{45}, r_{90}, σ_{RD}/σ_{TD}, σ_{RD}/σ_{45} and X^* (or X_p^*) suffice to determine A_2 to A_9. For four-ear-materials there exists only one α_i' ($i = 1$). Then α_1, r_0, $r_{22\cdot5}$, r_{45}, r_{90}, σ_{RD}/σ_{TD}, σ_{RD}/σ_{45} and X^* (or X_p^*) are required. All formulae for all these cases are omitted here.

4. *Planar isotropy and isotropy*

The condition of planar isotropy (i.e. no in-plane directionality) is equivalent to independence of σ_α in equation (8) and strain increments $d\epsilon_l$, $d\epsilon_b$ and $d\epsilon_t$ (two of them because of incompressibility) on α in uniaxial tension. (In the case of the quadratic yield function the latter suffices to fix the condition.) The former requires $A = B = C = D = 0$ which is equivalent to equation (45), i.e. the earless condition. The latter yeilds

$$A_2 = A_4, \qquad 4A_1 + 2A_2 = 2A_3 + A_7. \tag{57}$$

Combining equations (45) and (57), we obtain the following relations for plana‹ isotropy:

$$\left.\begin{array}{l} A_5 = A_1, \quad A_4 = A_2, \quad A_8 = A_6, \quad A_2 + A_6 = 4A_1, \\ A_3 + A_7 + A_9 = 6A_1, \quad 4A_1 + 2A_2 = 2A_3 + A_7. \end{array}\right\} \tag{58}$$

Next, the condition of isotropy is obtained from equation (58) and the additional conditions imposed in the thickness direction—i.e. $\sigma_b = \sigma_u$ (the uniaxial in-plane yield stress) and $d\epsilon_b = d\epsilon_t = -d\epsilon_l/2$ irrespective of the value of α for uniaxial tension. Finally, we obtain the following relations for isotropy:

$$\left.\begin{array}{l} A_2 = A_4 = -2A_1, \quad A_3 = 3A_1, \quad A_5 = A_1, \quad A_6 = A_8 = -A_7 = 6A_1, \\ A_9 = 9A_1, \quad A_0 = 0. \end{array}\right\} \tag{59}$$

When we put $A_1 = 1$ in (59), the yield function f reduces to the square of the Mises yield function.

CONCLUSIONS

1. When we put $A_1 = 1$ which is the coefficient of the term σ_x^4 in the yield function of fourth order f, the equivalent stress-equivalent strain ($\bar{\sigma}_{eq} - \bar{\epsilon}_{eq}$) curve of the material is uniquely determined by the uniaxial stress-longitudinal strain curve for the rolling direction and thus its dependence on types of loading, which is one of the intrinsic disadvantages of the conventional quadratic yield function g, is removed.

2. The equivalent strain increment $\overline{d\epsilon}_{eq}$ is not generally formulated by the terms of incremental strain components $d\epsilon_{ij}$ excpet for the very special case where f is the square of g. Thus it is noted that the conventional theory is very limited. The definition $\overline{d\epsilon}_{eq} = \sigma_{ij}\, d\epsilon_{ij}/\bar{\sigma}_{eq}$ will be satisfactory in general.

3. Relationships between types of ear-formation in axi-symmetrical deep-drawing operations and the coefficients of f are examined in detail. It is emphasized that g can express only a few types of four- and two-ear-materials.

4. Formulae for determination of the coefficients in f are given. Besides r-values, yield stresses for uniaxial and (e.g.) equi-biaxial tensions are required, where these yield stresses are for the same $\bar{\epsilon}_{eq}$ and determined by virtue of the definition formula for $\overline{d\epsilon}_{eq}$. The procedure for this determination ensures the uniqueness of the $\bar{\sigma}_{eq} - \bar{\epsilon}_{eq}$ curve stated in 2.

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