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Thermoelastic-Plastic Flow in Solids

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PREFACE

The aim of this monograph is to develop the theory of elastic-plastic flow in solids, with application to a particular class of processes, namely those in which the dissipation of plastic work cannot be neglected. Examples of such processes are high-speed impact phenomena and cratering, shock compression, often shock release as well, and explosive deformation and welding. An important part of the development is to include the anisotropic elastic properties of a solid, and for this reason the theory applies to solids in general. On the other hand, the prototype solid which the author has specifically in mind is metals, and so it is possible that some of the results will have limited validity for nonmetals. The theory is valid in principle for complex flow, compression and/or tension, and unstable flow as well, although the presence and growth of cracks and voids is not explicitly included. A notational conflict arose, because in continuum mechanics the extensive quantities are taken per unit mass, while in thermodynamics they are usually per unit volume. The continuum mechanics normalization is used throughout, with the result that uncommon factors of density show up in the thermodynamic equations.

There is nothing particularly new or profound about the present work. However, there is a need to write it down in something like the present form, because it crosses such a broad range of disciplines. In certain areas of this study, the author encountered substantial difficulties in drawing a clear understanding from references in the field. The material is therefore presented in a tutorial fashion, with the intention of helping the student, or the interested reader, to comprehend the principles easily. All of the important results are derived from basic principles. Certain conceptual errors which appear in the literature are quoted without giving references, on the grounds that clarifying the issue is the only thing to be accomplished. The exercises are an integral part of the logical development, and they should at least be read; solving the exercises is guaranteed to be educational.

This work was started in January of 1977. Along the way, a great many people have taken the time to help the author with a great many details. Those who have made a substantial contribution to this work, through years of collaboration and encouragement, are James N. Johnson, Galen Straub, Paul Follansbee, and Davis Tonks. The author is happy to commend Barbara Forrest for generous assistance in all phases of the manuscript preparation, including expert word processing.

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CHAPTER I

MATERIAL RESPONSE

In this chapter, the basic material-response concepts underlying the entire theory are described in words. In the next three chapters, the corresponding system of equations is constructed. Any individual who makes it through the whole monograph will find it worthwhile to read this first chapter again. Some of the observations on time-rate effects, and in support of local thermodynamic equilibrium during plastic flow, were originally published in Physical Review.¹

1. Equilibrium and Nonequilibrium

A system is a quantity of material whose behavior we want to study, e.g. a cubic centimeter of gold, or a beaker of water. An isolated system is one for which *nothing* flows in or out of the system. If a system remains isolated, it will presumably reach a state which remains constant in all its macroscopic properties. This is an equilibrium state. Let us for the moment consider only states for which the forces applied to the system are isotropic, and make a partial list of the macroscopic properties of the equilibrium state:

$$\begin{aligned} V &= \text{volume,} \\ P &= \text{pressure,} \\ U &= \text{internal energy,} \\ T &= \text{temperature,} \\ S &= \text{entropy.} \end{aligned} \quad (1.1)$$

Equilibrium thermodynamics is the study of processes by which a material (or several materials simultaneously) can pass from one equilibrium state to another, along paths which are constrained to pass only through equilibrium states. The primary physical laws invoked are conservation of energy, and the existence of an exact differential dS ; the rest of equilibrium thermodynamics is (almost entirely) mathematics. When the stress is isotropic pressure, there are only two independent variables, which means two variables completely specify an equilibrium state of a given material. These variables can be any two from the above list, or two combinations of them, or other extensions. Take for example V and T as the independent variables. Then for all possible equilibrium states of a material, relations of the following form hold:

$$\begin{aligned} P &= P(V, T), \\ U &= U(V, T), \\ S &= S(V, T). \end{aligned} \quad (1.2)$$

The term "equation of state" is sometimes used to denote the above equation for the pressure. A more

general usage of the term, and that which will be followed in the present work, is to denote any or all of the information contained in the set of equations (1.2).

Definition The equation of state is any or all of the set of equilibrium thermodynamic equations for the dependent variables.

To enter the realm of nonequilibrium states and processes, it is helpful to think in terms of statistical mechanics. Consider a monatomic nearly-ideal gas; mentally subdivide the space occupied by the gas into a large number of volume elements, each with the same volume. Each element contains a large number of atoms, and can be treated as a statistical subsystem. The important statistical measure of a subsystem is the distribution $f(\mathbf{p})$ of the atomic momenta \mathbf{p} . The equilibrium distribution is Maxwellian, with a temperature T :

$$f(\mathbf{p}) \propto e^{-\beta \mathbf{p}^2/2M} \quad (1.3)$$

where $\beta = 1/kT$, k is Boltzmann's constant, and M is the atomic mass. When the gas is in equilibrium, the mass and temperature are the same for each element. When the gas is not in equilibrium, the momentum distribution can be anything. If the distribution is not at least approximately Maxwellian, then the temperature cannot be defined. However, even when equilibrium-thermodynamic quantities such as temperature are *not defined*, mechanical quantities are *always defined*. The most important macroscopic mechanical quantities representing each volume element, which in fact are just the zeroth, first, and second moments of $f(\mathbf{p})$, are the total mass, total linear momentum, and total energy. If the gas is thought of as a continuum, these quantities translate into local fields representing density, fluid velocity, and energy density.

We can now define nonequilibrium states which are "close to equilibrium." Any element of the gas is close to equilibrium if its momentum distribution is close to that given by (1.3); specifically this means

$$f(\mathbf{p}) \propto e^{-\beta \mathbf{p}^2/2M} + \delta f(\mathbf{p}) \quad (1.4)$$

where the only restriction on $\delta f(\mathbf{p})$ is that it is small enough to be treated as a perturbation. But this will be the case only if the spatial and temporal variations of the *mechanical quantities*, the density, fluid velocity, and energy density, are sufficiently small. It is important always to differentiate between mechanical and thermodynamic quantities, and to remember that equilibrium thermodynamic quantities can be defined only for states which are close to equilibrium.

Exercise Why is it that the mechanical energy is always defined, while the thermodynamic internal energy is defined only for states close to equilibrium?

Exercise In what ensemble is the number of atoms N a dynamical variable, and what is the ensemble parameter conjugate to N ? In what ensemble is V a dynamical variable, and what is its conjugate ensemble parameter?

Irreversible thermodynamics is the study of processes which pass through nonequilibrium states, but only those which are close enough to equilibrium states so that the equilibrium-thermodynamic quantities can be meaningfully defined. Irreversible-thermodynamic processes are characterized by the following properties.

(a) A driving force is present, which causes a flow that tries to cancel the driving force. The force-flow relation is called the constitutive relation, and the material properties which enter this relation are called constitutive properties. The constitutive relation couples to, and alters parts of, the equation of state.

(b) Dissipation is present (generation of entropy). The work done by the driving force against the flow is always at least partially dissipated, and usually it is totally dissipated.

Consider a material in which there is a local temperature gradient; heat flows, in an attempt to cancel out the temperature gradient. A common constitutive relation sets the heat current proportional to the temperature gradient. The coefficient of proportionality, the thermal conductivity, is a constitutive property of the material. Consider a beaker of water, stirred gently with a stirring rod, and then isolated. Viscous stresses are present, working against the velocity gradients, and the macroscopic mechanical motion of the water gradually dies away, as it is turned into heat. The linear relation between viscous stresses and velocity gradients is the constitutive relation, and the viscosity coefficients are constitutive properties of the water.

In the literature, the term "constitutive" is sometimes used to include equation-of-state information. Here, the term is restricted to nonequilibrium properties, because we want to emphasize that these properties are separate from equilibrium properties. It is important to remember that equation-of-state data does not contain any information about constitutive behavior of a material.

Definition The constitutive relation, or constitutive behavior, of a material describes the dissipative force-flow part of an irreversible-thermodynamic process.

The present work is concerned only with processes in the irreversible-thermodynamic regime: equilibrium-

thermodynamic quantities can always be defined, at least approximately. A point of usage needs to be addressed. It seems that physicists are raised to think of adiabatic, when used in the thermodynamic sense, as meaning isentropic. Engineers generally use adiabatic to mean "without heat flow." We could simply abandon this word in the present work, except that we will have much need for the adiabatic elastic moduli, which of course are isentropic elastic moduli. The following definition will therefore be followed.

Definition Adiabatic is used to mean the same as isentropic.

2. Elastic Response and Plastic Response

The significant property of condensed matter, which makes it condensed, is the dominance of forces within the material. For an ordinary solid or fluid composed of atoms, or more accurately, composed of ions and electrons, the forces derive from effective potentials between the ions. These forces are elastic forces. Since the effective ion-ion potentials operate through the electrons, they are "instantaneous" potentials, as far as the present work is concerned. There are also forces arising from thermal energy in a material, e.g. the thermal excitations of electrons and phonons. Thermal forces can remain close to equilibrium as long as their spatial and temporal variations are slow compared to relaxation lengths and times within the electron-phonon system. Such relaxation lengths and times are quite short, being measured in terms of lattice spacings and picoseconds, respectively. The elastic forces and thermal forces taken together are called thermoelastic forces, and for solids we have the following conclusion.

Notice In a solid, stresses are supported by thermoelastic forces, and local thermoelastic equilibrium will be a valid approximation up to very high spatial and/or temporal rates of change.

When an anisotropic stress is applied to a solid material, a multitude of processes begin, all of them acting to reduce the stress. Some of these processes can be effective only on geological time scales; others are important in minutes or days. In the present work, we are interested in rather fast processes, say things which happen in one second or less. We will not be concerned with any stress relaxation mechanism which operates on a slower timescale. Further, in solids, whatever stress relaxation mechanism is important on our timescales will be called plastic flow. Plastic flow does not have to be due to

dislocations; however, it usually is, and we will often use dislocation concepts in examining the nature of plastic flow.

Consider a small region of a solid material, and suppose there is an anisotropic stress in the region, which results from forces applied by the surrounding material. There may be a dislocation in the region, or one might be generated; in any case the dislocation moves in such a way as to reduce the local anisotropic stress. Transforming this picture to the language of irreversible thermodynamics, we say that the stress drives plastic flow, and that for a given material in a given state, the material constitutive equation specifies the plastic response to any applied stress. We have thus arrived at the most important point of logic in the present theoretical construction, stated as follows.

Notice Within a solid material, at all times and locations, the stresses satisfy two separate conditions: the equilibrium thermoelastic equations, and the plastic constitutive equation.

There is a subtle discord between the nature of plastic flow and the customary "textbook picture" of irreversible processes. This can be illustrated with a simple example. Suppose a shear stress τ is applied to a solid, and the solid deforms elastically for $\tau < \tau_0$, and at τ_0 the solid begins to flow plastically. The irreversible-thermodynamic driving force is τ_0 . This constitutive behavior cannot be represented by a linear phenomenological law, according to which the driving force is zero in equilibrium, and is a linear function of some measure of the departure from equilibrium. The *essential nonlinearity* of plastic constitutive behavior has led to the statement that "plastic flow cannot be treated by irreversible thermodynamics." This statement can safely be ignored. But let us continue with the example, and ask about the connection between time-rates and reversibility. The customary picture is that if you make a process slow enough, it will be arbitrarily close to equilibrium; that slow is reversible. Is this really true? In an ordinary solid, adiabatic elastic waves can be transmitted at very high frequencies, under near-equilibrium conditions. On the other hand, in driving plastic flow, the shear stress can be adjusted so that the plastic strainrate is arbitrarily small, yet the process is still irreversible: the driving force τ_0 is finite for an arbitrarily slow process. This result allows the following important conclusion to be made.

Notice The thermodynamic reversibility of a process is not determined by its rate.

So far, we have ignored the heterogeneous nature of plastic flow. In fact, plastic flow is intrinsically heterogeneous, and what is worse, it is heterogeneous on several different length and time scales. The finest scale of heterogeneity is that of a single dislocation. A dislocation is a line defect, surrounded by a nonuniform elastic strain field. When a dislocation moves, it presumably acts as a source of mechanical excitations, and with the appropriate relaxation time, this mechanical energy becomes thermalized to heat. Larger scales of heterogeneity are associated with dislocation substructure, the networks and cells and so on. On this large scale, plastic instabilities may develop during the course of a process. For example, shear bands may form, within which the plastic flow may tend to become localized. The question arises, how is the heterogeneous nature of plastic flow to be addressed by irreversible thermodynamics?

First we will simply assume that the finest scale of heterogeneity can be ignored; that single-dislocation effects can be averaged for *any* mass element, whether the mass element contains only a few dislocations, or is located within an active shear band. There are two ways in which larger scale inhomogeneities can be treated. The first way is simply to resolve the inhomogeneities which occur in a given flow process. After all, the continuum-mechanics and irreversible-thermodynamic theory is a local field theory, and applies in principle to spatially and temporally inhomogeneous processes. The study of inhomogeneous and unstable flow in fluids is currently quite popular, and there is no doubt that similar problems in elastic-plastic flow will be studied in the future.

Notice The theory of this monograph is applicable to inhomogeneous elastic-plastic flow, in particular to problems in which shear bands or other plastic instabilities appear and grow.

An alternate procedure, which entails a sacrifice of resolution on a certain scale, is to replace a certain type of inhomogeneity by a homogeneous model. In this way, an additional field variable, or parameter, is introduced into the continuum theory, and the modeled inhomogeneity formally disappears. It should always be remembered, however, that the model has to be consistent with the basic thermoelastic properties of the solid. For example, consider a single crystal with a single straight dislocation; apply a shear stress to the crystal and the dislocation moves. Now replace the whole slip plane of the dislocation by a shear band, and apply the same shear stress. The stress is supported elastically, with the same elastic stress-strain relation in both cases;

the only difference in the two cases is the plastic constitutive relation. Hence, the appearance and growth of shear bands can be modeled homogeneously, by using a combination of plastic constitutive relations, together with a local field variable denoting what part of the total plastic flow is due to the shear bands. Another example is the growth of voids in a metal under tension. A single void contributes to the flow problem in the following way: plastic flow proceeds around the void's surface, as the void grows, and the cross section of the void does not support stress. These effects have been modeled in a manner consistent with the thermoelastic properties of a metal, in a calculation of necking and ductile fracture.²

3. Fluids

All fluids are presumably viscoelastic. This means the fluid response is viscous at low and moderate strainrates, but the response is elastic at high strainrates. For a simple monatomic fluid, the elastic regime begins at strainrates around the inverse mean-atomic-vibration time, or 10^{13} s^{-1} at ordinary temperatures and pressures. The elastic response in a fluid is due to the interatomic potentials, and does not occur in a gas, where all interactions are represented simply by two-particle elastic collisions. That a fluid and a gas are *essentially* different is demonstrated by the behavior of the viscosity, which decreases with temperature for a fluid, and increases with temperature for a gas. The onset of elastic response in a fluid at high strainrates suggests that viscous stresses cannot be arbitrarily large; this is certainly the case in the shock process, as will be discussed in Section 17.

When we speak of a viscous fluid, we mean a real fluid (a viscoelastic fluid) which is operating in the

viscous regime. The most universally popular myth in high-strainrate materials response today is that an elastic-plastic solid is equivalent to, or approximately equivalent to, a viscous fluid. The myth is more than popular; it is believed with dedication and with fervor.

Caution No amount of desire will make a wrong theory right.

There exists in the literature a "proof" that the constitutive equation for an elastic-plastic solid is the same as that for a viscous fluid. The proof is false. The difference in constitutive behavior between an elastic-plastic solid and a viscous fluid is not trivial, it is vital. The following statement is quantitatively accurate.

Notice In constitutive behavior, an elastic-plastic solid is as different from a viscous fluid as it is from an inviscid fluid.

Experiments have been done to measure the viscosity of a solid. Any experiment performed on a solid, and interpreted with viscous fluid theory, will indicate an enormous "viscosity," and one which depends intimately on the experiment itself. To add to the confusion, there is also the popular "viscoelastic solid." Just because this term is contained in a textbook, in a list of constitutive models, does not mean that any solid on earth behaves that way.

CHAPTER II
CONTINUUM MECHANICS

Continuum mechanics is a standard engineering discipline, and only the merest introduction is given in the present chapter. An extended material is considered, which is continuous, and which has continuously varying densities and currents. The content of the theory is to relate the local variations in densities and currents through the laws of mechanics, namely conservation of mass, momentum, and energy. There is no thermodynamics in continuum mechanics. A treatise on continuum mechanics which the present author has found very helpful is that of Prager.³

4. Deformation

The material is divided into infinitesimal mass elements, sometimes called material particles. There are no cracks or voids between the mass elements, each mass element contains a fixed amount of mass for all time, and exchange of matter between mass elements is allowed but need not be explicitly considered. At any time t , the location of a given mass element is $\mathbf{x}(t)$ in laboratory coordinates. An initial, or reference, configuration is denoted by subscript a . The location of a given mass element at t_a is $\mathbf{x}_a = \mathbf{x}(t_a) = \mathbf{X}$, so that \mathbf{X} is the Lagrangian coordinate of the mass element. The Lagrangian coordinate provides a label for each and every mass element in the material. At any time t , the mass element \mathbf{x} is located at $\mathbf{x}(t)$, and has velocity

$$\mathbf{v} = \left. \frac{\partial \mathbf{x}}{\partial t} \right)_{\mathbf{X}} \quad (4.1)$$

The equations of continuum mechanics are local, which means they hold for all locations and times. Quantities defined locally for the material are fields. The independent variables for the fields can be taken as the set $\{\mathbf{x}, t\}$, or the set $\{\mathbf{X}, t\}$. The velocity field, for example, can be expressed in "Eulerian form" as $\mathbf{v}(\mathbf{x}, t)$, or in "Lagrangian form" as $\mathbf{v}(\mathbf{X}, t)$. Finally, we will also use \mathbf{X} or \mathbf{x} as symbolic designations for an entire configuration, e.g. "the configuration \mathbf{X} ."

A multitude of different strain measures are commonly used, but here we will define only two. The first is the matrix α , which is the transformation from the initial configuration \mathbf{X} to any other configuration \mathbf{x} . This may be written $\alpha(\mathbf{X} \rightarrow \mathbf{x})$, or $\alpha(\mathbf{x})$ for short, or just α . If we introduce a Cartesian coordinate system, with indices i, j, k, l standing for Cartesian coordinates, then

$$\alpha_{ij} = \left. \frac{\partial x_i}{\partial X_j} \right)_{\mathbf{t}} \quad (4.2)$$

Obviously $\alpha_{ij}(\mathbf{X}) = \delta_{ij}$. Differentiating (4.2) with respect to time gives

$$\left. \frac{\partial \alpha_{ij}}{\partial t} \right)_{\mathbf{X}} = \left. \frac{\partial v_i}{\partial X_j} \right)_{\mathbf{t}} \quad (4.3)$$

Because the material is continuous, the differentials defining α always exist; α is a continuous point transformation. α defines the transformation of vectors: an incremental vector $d\mathbf{X}$, between two material particles at t_a , becomes $d\mathbf{x}$ at t , where

$$dx_i = \alpha_{ij} dX_j \quad (4.4)$$

Note repeated indices i, j, k, l, m, n are always summed. The vector transformation is used in differential form because, in general, α varies over space at any given time. If α were constant over the whole system at any given time, but still a function of time, (4.4) could be integrated over space to $x_i = \alpha_{ij} X_j$, which is the transformation for a homogeneous deformation.

The second strain measure we will use is the displacement. Let us name two configurations in a suggestive manner:

\mathbf{x} = current configuration,
 \mathbf{y} = next configuration.

The displacement field is $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$, where for a given mass element,

$$u_i = y_i - x_i \quad (4.5)$$

The displacement gradients are

$$u_{ij} = \left. \frac{\partial u_i}{\partial x_j} \right)_{\mathbf{t}} = \left. \frac{\partial y_i}{\partial x_j} \right)_{\mathbf{t}} - \delta_{ij} \quad (4.6)$$

The displacement gradients are not in general symmetric. The symmetric part of u_{ij} is ϵ_{ij} , sometimes called pure strain:

$$\epsilon_{ij} = \frac{1}{2} (u_{ij} + u_{ji}) \quad (4.7)$$

The antisymmetric part of u_{ij} is ω_{ij} , often called pure rotation:

$$\omega_{ij} = \frac{1}{2} (u_{ij} - u_{ji}) \quad (4.8)$$

In order to relate the different strain measures, it is necessary to consider the two transformations $\alpha(\mathbf{X} \rightarrow \mathbf{x})$ and $\alpha(\mathbf{x} \rightarrow \mathbf{y})$, and the displacement $\mathbf{u}(\mathbf{x} \rightarrow \mathbf{y})$. It then follows, from the definitions,

$$\alpha_{ij}(\mathbf{y}) - \alpha_{ij}(\mathbf{x}) = u_{ik}(\mathbf{x})\alpha_{kj}(\mathbf{x}) \quad (4.9)$$

If the displacement is incremental, i.e. if $\mathbf{y} = \mathbf{x} + d\mathbf{x}$, then the displacement gradients are incremental, $u_{ij} = du_{ij}$, and (4.9) becomes

$$d\alpha_{ij} = du_{ik}\alpha_{kj} \quad (4.10)$$

Finally, this can be inverted with the aid of the inverse matrix α^{-1} :

$$du_{ij} = d\alpha_{ik}(\alpha^{-1})_{kj} \quad (4.11)$$

Note that α is in general nonsymmetric. Even if du is symmetric for all dt , it still does not follow that $d\alpha$ is symmetric for all dt , unless α and du commute. If α and du commute for all times, then they are simultaneously diagonal in a fixed coordinate system, and the description of strain is simplified. An example of this simplification is uniaxial flow, which is discussed in Chapter V.

Exercise Do the algebra required to arrive at equations (4.9), (4.10), and (4.11).

One more bit of algebra is needed before getting on with the physics, and that is relating Eulerian and Lagrangian derivatives. Let g be any field, expressible as $g(\mathbf{x}, t)$ or as $g(\mathbf{X}, t)$. Chain rule differentiation gives

$$\left(\frac{\partial g}{\partial x_i}\right)_t = \left(\frac{\partial g}{\partial X_j}\right)_t (\alpha^{-1})_{ji} \quad (4.12)$$

where we used $(\alpha^{-1})_{ji} = (\partial X_j / \partial x_i)_t$, which follows from (4.2). To relate Eulerian and Lagrangian time derivatives, write

$$\begin{aligned} dg &= \left(\frac{\partial g}{\partial t}\right)_x dt + \left(\frac{\partial g}{\partial x_i}\right)_t dx_i \\ &= \left(\frac{\partial g}{\partial t}\right)_X dt + \left(\frac{\partial g}{\partial X_i}\right)_t dX_i \end{aligned}$$

Take $dX_i = 0$ and divide by dt to get the result

$$\left(\frac{\partial g}{\partial t}\right)_X = \left(\frac{\partial g}{\partial t}\right)_x + v_i \left(\frac{\partial g}{\partial x_i}\right)_t \quad (4.13)$$

To derive the equation for conservation of mass, we simply have to calculate the volume of a mass element in configurations \mathbf{X} and \mathbf{x} . Because the mass element is infinitesimal, the transformation α can be taken as constant over the volume, and the shape of the mass element is a parallelepiped in all configurations. Let dL_{ij} and dl_{ij} be the j Cartesian component of the i edge of the mass element in configurations \mathbf{X} and \mathbf{x} , respectively. Then denoting the volume dV , and using the notation $\|\alpha\|$ or $\|\alpha_{ij}\|$ for the determinant of α ,

$$dV(\mathbf{x}) = \|dl_{ij}\| = \|\alpha_{ik}dL_{kj}\| = \|\alpha\|dV(\mathbf{X})$$

This relation is a statement of conservation of mass. In continuum mechanics, it is convenient to work interchangeably with two fields representing mass, namely the local density ρ and the local volume V :

$$\begin{aligned} \rho &= \text{mass density,} \\ V &= \rho^{-1} = \text{volume per unit mass.} \end{aligned} \quad (4.14)$$

The conservation of mass can then be expressed in the form

$$\frac{V}{V_a} = \frac{\rho_a}{\rho} = \|\alpha\| \quad (4.15)$$

The differential form of this is

$$d \ln V = d\epsilon_{ii} \quad (4.16)$$

where the repeated index is to be summed. In deriving this last equation, we made use of Jacobi's identity,

$$\frac{\partial \|\alpha\|}{\partial \alpha_{ij}} = (\alpha^{-1})_{ji} \|\alpha\| \quad (4.17)$$

The customary continuum-mechanics expression for conservation of mass is obtained by differentiating (4.15). Holding the mass element fixed gives the Lagrangian form,

$$\left(\frac{\partial \rho}{\partial t}\right)_X = -\rho(\alpha^{-1})_{ji} \left(\frac{\partial v_i}{\partial X_j}\right)_t \quad (4.18)$$

Transforming with the aid of (4.12) and (4.13) gives the Eulerian form,

$$\left. \frac{\partial \rho}{\partial t} \right|_x + \left. \frac{\partial \rho v_i}{\partial x_i} \right|_t = 0 \quad (4.19)$$

Exercise Prove Jacobi's identity.

Exercise Do the algebra required to arrive at equations (4.16), (4.18), and (4.19).

5. Stresses

Stress, like strain, is a local field; it is a point property of a medium. Stresses are surface forces: at any point in the material draw a surface element with normal in the $+i$ direction; the material on the $+i$ side of the surface exerts a force on the material on the $-i$ side, the force per unit area being the vector τ_i . The j component of τ_i is τ_{ij} . Here it appears that the order of the subscripts is significant, but in fact it is not, because τ is a symmetric tensor, as will be shown below. The material on the $-i$ side exerts an equal and opposite force across the surface on the material on the $+i$ side. The forces at any point within the material are completely specified by the stress vectors τ_i on each of three mutually orthogonal planes; that is, by the stress-tensor components $\tau_{ij} = \tau_{ij}(x,t)$. Note body forces are neglected here, and throughout this work.

We now have to clarify a subtle point. First note that the same word stress is used to mean either *mechanical forces* or *thermodynamic forces*, depending on whether or not the material state is close to equilibrium. The word pressure, however, will be used only for a thermodynamic force. When a system is in thermodynamic equilibrium, in the presence of an externally applied pressure P , the stress tensor is constant throughout the system, and is

$$\tau_{ij} = -P\delta_{ij} \quad (5.1)$$

In the case of local thermodynamic equilibrium, (5.1) holds locally wherever the stress is isotropic. The minus sign is present because stresses are outward forces, while pressure is a force inward on a body. The mean compressive stress \bar{P} is defined without regard to equilibrium:

$$\bar{P} = -\frac{1}{3} \tau_{ii} \quad (5.2)$$

The important point is that P is not "the pressure," not even when the material is locally, or globally, in thermodynamic equilibrium. The pressure is uniquely determined by the independent thermodynamic

variables, which means, for example, that $P(V,T)$ is single valued, while the mean compressive stress is not a single-valued function of V and T , but can be made to take on a continuous range of values at fixed V and T , by varying components of the stress tensor. Hence, when the stress tensor is not isotropic, the pressure is not defined.

Notice When the stress tensor is not isotropic, the mean compressive stress \bar{P} is not the pressure, and in fact the pressure is not defined.

The anisotropic stress components, or stress deviators s_{ij} , are

$$s_{ij} = \tau_{ij} + \bar{P}\delta_{ij} \quad (5.3)$$

Obviously, $s_{ii} = 0$, because of (5.2). The set of components \bar{P} and s_{ij} is equivalent to the original set of components τ_{ij} .

To derive the equation for conservation of linear momentum, consider a mass element of fixed mass m , whose volume w is enclosed by the surface s . A surface element is ds , and the force acting across ds , exerted on the material inside the surface, is $df = \tau_i ds_i$. The total force acting on the material inside the surface is f , where

$$f_j = \int_s df_j = \int_s \tau_{ij} ds_i \quad (5.4)$$

Gauss' theorem is

$$\int_s g_i(x) ds_i = \int_w \frac{\partial g_i(x)}{\partial x_i} dw \quad (5.5)$$

so we have

$$f_j = \int_w \frac{\partial \tau_{ij}}{\partial x_i} dw \quad (5.6)$$

Since the volume is infinitesimal, the integrand can be taken constant. Then eliminating f with Newton's law, $f = m(\partial v/\partial t)_x$, and using $\rho = m/w$, gives the expression of conservation of linear momentum,

$$\rho \left. \frac{\partial v_j}{\partial t} \right|_x = \left. \frac{\partial \tau_{ij}}{\partial x_i} \right|_t \quad (5.7)$$

This is a mixed Eulerian-Lagrangian form, which is useful for some computer calculations. Transformation with the aid of equations (4.12) and (4.13) gives the Lagrangian form

$$\rho \left(\frac{\partial v_j}{\partial t} \right)_{\mathbf{X}} = (\alpha^{-1})_{ki} \left(\frac{\partial \tau_{ij}}{\partial X_k} \right)_{\mathbf{t}}, \quad (5.8)$$

or the Eulerian form

$$\rho \left(\frac{\partial v_j}{\partial t} \right)_{\mathbf{x}} + \rho v_i \left(\frac{\partial v_j}{\partial x_i} \right)_{\mathbf{t}} = \frac{\partial \tau_{ij}}{\partial x_i} \Big|_{\mathbf{t}}. \quad (5.9)$$

To examine the consequence of conservation of angular momentum, we introduce the Levi-Civita anti-symmetric symbol ϵ_{ijk} , which is $+1(-1)$ if i, j, k is an even (odd) permutation of 1,2,3, and which is zero otherwise. Important properties of ϵ_{ijk} are, for any vectors \mathbf{a} and \mathbf{b} , $(\mathbf{a} \times \mathbf{b})_i = \epsilon_{ijk} a_j b_k$, and for a tensor \mathbf{c} , $\epsilon_{ijk} c_{jk} = 0$ if and only if \mathbf{c} is symmetric. Consider again the mass element with mass m , whose volume w is enclosed by the surface s . With respect to the (arbitrary) origin of coordinates, the angular momentum \mathbf{M} of the mass element has components

$$M_i = \int (\mathbf{x} \times \rho \mathbf{v})_i dw = \int \epsilon_{ijk} x_j \rho v_k dw, \quad (5.10)$$

and the torque \mathbf{L} applied to the mass element has components

$$L_i = \int_s (\mathbf{x} \times d\mathbf{f})_i = \int_s \epsilon_{ijk} \left(\frac{\partial x_j \tau_{ik}}{\partial x_l} \right)_{\mathbf{t}} dw, \quad (5.11)$$

where $d\mathbf{f}$ in (5.11) is the same as in (5.4). Since the volume is infinitesimal, the integrands can be taken outside the integrals. Then the law of conservation of angular momentum, $(\partial M_i / \partial t)_{\mathbf{X}} = L_i$, gives the equation

$$\epsilon_{ijk} \left[m \left(\frac{\partial x_j v_k}{\partial t} \right)_{\mathbf{X}} - w \left(\frac{\partial x_j \tau_{ik}}{\partial x_l} \right)_{\mathbf{t}} \right] = 0. \quad (5.12)$$

With conservation of linear momentum, equation (5.7), this reduces to

$$\epsilon_{ijk} (\rho v_j v_k - \tau_{jk}) = 0. \quad (5.13)$$

The first term vanishes because $v_j v_k$ is symmetric, and the resulting equation requires τ to be symmetric for all times and locations:

$$\tau_{ij} = \tau_{ji}. \quad (5.14)$$

Exercise Carry out the algebra involved in going from equations (5.10) and (5.11) to equation (5.14).

Since τ is a real-symmetric matrix, it can be diagonalized by an orthogonal transformation (a rotation). This defines principal axes, or principal directions:

Definition Principal axes are the Cartesian axes in which the stress tensor is locally diagonal.

Since τ is diagonal in principal axes, it has only three nonzero elements, namely τ_{11} , τ_{22} , and τ_{33} (note repeated number indices are never summed). These are the principal stresses. They are the components of a vector in the principal axis system. Hence, while specification of stress at a given point requires by definition the specification of force vectors acting on each of three mutually orthogonal planes, it can also be done by specifying the principal axes and only one vector. In general, of course, the principal axes vary with location and time.

6. Energy

The energy belonging to each mass element is divided into two parts:

- (a) Translational kinetic energy, $\frac{1}{2} v^2$ per unit mass;
- (b) Center-of-mass energy, E per unit mass.

To calculate the translational kinetic energy, take the mass element to be rigid. The net force on it is \mathbf{f} , in a time dt the mass element moves $d\mathbf{x}$, so the translational work done is

$$\mathbf{f} \cdot d\mathbf{x} = f_j v_j dt. \quad (6.1)$$

The increase in translational kinetic energy of the mass element is

$$d(\frac{1}{2} m v^2) = m v_j \left(\frac{\partial v_j}{\partial t} \right)_{\mathbf{X}} dt. \quad (6.2)$$

These two expressions must be equal, and in fact they are because of conservation of linear momentum, or what is equivalent, by Newton's law.

The center-of-mass energy is again divided into two parts, work done and heat transported. These will be calculated in incremental form. Define W as the work done on the material, per unit mass, in the center-of-mass frame. In configuration x , a mass element has mass m , and volume w enclosed in a surface s . We carry out a homogeneous deformation of the mass element, i.e. a deformation with spatially constant strains, in which the point x moves to y , where

$$y_j - x_j = u_{jk}x_k \quad (6.3)$$

This is obtained by integrating (4.6), with the u_{jk} being constants. The work done on the mass element is $mW(x \rightarrow y)$.

$$mW(x \rightarrow y) = \int_s df_j(y_j - x_j) \quad (6.4)$$

and with df_j from (5.4) this is

$$mW(x \rightarrow y) = \int_s \tau_{ij} u_{jk} x_k ds_i \quad (6.5)$$

With τ_{ij} and u_{jk} constants, this is transformed to a volume integral by Gauss' theorem, and yields

$$mW(x \rightarrow y) = \tau_{ij} u_{jk} w \quad (6.6)$$

Finally, we let the deformation $x \rightarrow y$ be infinitesimal, to transform (6.6) to a differential form. Because τ_{ij} is symmetric, only the symmetric part ϵ_{ij} of u_{ij} contributes to the sum in (6.6), and we have

$$\rho dW = \tau_{ij} d\epsilon_{ij} \quad (6.7)$$

Let us take a brief aside to examine some aspects of rotational motion. In the last section, and the present one, forces arising from rotation, and rotational energy, have not been explicitly recognized. It is well known, however, that in a high-speed flywheel the centrifugal force can reach the ultimate tensile strength of a solid material. Further, in turbulent fluid motion, the rotational energy can be significant.

Exercise How are the centrifugal force and the rotational energy included in the present formulation of continuum mechanics?

Exercise for the advanced student Can you justify neglecting centrifugal force and rotational energy which arise within a mass element? What is the implication for computer calculations which use finite mass elements?

In order to provide a mental picture for heat transport, it is useful to establish a conceptual definition of "heat," as follows. Heat is the excitation of a system, or a mass element, at fixed configuration. For an atomic solid or fluid, it is the motional energy of the atoms in the center-of-mass frame; in a metal, the excitation of conduction electrons is also included. Heat is defined without regard to thermodynamic equilibrium. Also, there is no (significant) transport of mass or momentum associated with heat transport. It should be recognized, however, that the real operational definition of heat is not contained in these words, but is contained in the constitutive equation for heat transport which one uses in any given calculation. Some common forms for this equation are discussed in Section 12.

Q is the quantity of heat per unit mass, and J is the heat flux. For a mass element with mass m , the rate of increase of heat is

$$m \left. \frac{\partial Q}{\partial t} \right|_X$$

This must be equal to the rate at which heat flows into the mass element through its surface, which is

$$- \int_s J_i ds_i = - \int_s \left. \frac{\partial J_i}{\partial x_i} \right|_t dw$$

Taking the integrand constant over the volume w , and equating the two expressions, gives the continuity equation for heat transport,

$$\rho \left. \frac{\partial Q}{\partial t} \right|_X = - \left. \frac{\partial J_i}{\partial x_i} \right|_t \quad (6.8)$$

It is often convenient to write this in the incremental form, for a given mass element,

$$\rho dQ = - \nabla \cdot J dt \quad (6.9)$$

The operator ∇ always signifies Eulerian differentiation.

In differential form, for a given mass element, conservation of center-of-mass energy is

$$dE = dW + dQ \quad (6.10)$$

Converting this to Lagrangian time derivatives, and using (6.7) and (6.8), gives the usual continuum-mechanic form,

$$\rho \left(\frac{\partial E}{\partial t} \right)_{\mathbf{X}} = \tau_{ij} \left(\frac{\partial \varepsilon_{ij}}{\partial t} \right)_{\mathbf{X}} - \left(\frac{\partial J_i}{\partial x_i} \right)_{\mathbf{t}} \quad (6.11)$$

An important point about the energy equation needs to be emphasized. First, recall the meaning of the incremental work dW : this is work done on a mass element, by its surroundings, through the action of stresses. Nothing was said about dW being conservative, or reversible, and in fact it generally is not. The applied stresses can drive plastic flow in a solid, or viscous flow in a fluid, and these are both dissipative processes.

Notice The work term in the energy conservation equation, (6.10) or (6.11), can be partially or totally dissipative.

Because of this, it is not possible to keep track of the heat content of a given mass element by simply integrating the term dQ ; dQ represents the net heat transported into the mass element, but dW acts partially or totally as a source term for heat as well.

The continuum-mechanic conservation equations are often called the equations of motion. Sometimes the term "equation-of-motion" means only the equation for conservation of momentum. The equations derived in the present chapter are valid for any continuum material. However, the student should be warned that, in many texts on fluid mechanics, the procedure for constructing equations of motion, and sometimes for interpreting them as well, is often different from the present approach. That procedure is to first write down the equations for an ideal fluid, for which the stress tensor is defined to be isotropic pressure, and then try to "correct the equations of motion," to account for nonideal behavior. For example, conservation of linear momentum for an ideal fluid is Euler's equation. Starting with Euler's equation, it is necessary to add a correction to

account for viscous stresses, and in that way the Navier-Stokes equation is obtained. The situation is still more confusing with regard to energy. To explain this, we neglect heat transport, so $dQ = 0$ in (6.10), then take processes close to equilibrium, so the center-of-mass energy E is to be identified with the thermodynamic internal energy U . Conservation of energy then reads

$$dU = dW \quad (6.12)$$

This particular equation startles many people, but it is correct for the stated conditions. However, if the stress is isotropic pressure P , then $dW = -PdV$, and then (6.12) implies $dS = 0$. This result has led to the statement, "the equations of motion are valid only for isentropic flow." In the study of shocks, the further argument has been made, "since the equations of motion are valid only for isentropic flow, and since a shock is known to be dissipative, then a shock must be a real mathematical discontinuity."

In the present approach, the equations of motion will never need to be "corrected." Further, since entropy is a *thermodynamic quantity*, it has nothing to do with the equations of motion. To make certain there is no question about these and related points, we will summarize the most important facts, which hold within the assumption of a continuous material and the laws of mechanics.

Notice The continuum-mechanic conservation equations are correct as written; they hold for any material with any constitutive behavior; they apply to dissipative processes; and no process is discontinuous.

Finally, note that we have made it all the way through the foundations of continuum mechanics without having to worry about strain compatibility conditions. This is not to be taken as evidence that the student can ignore strain compatibility at will, in any future endeavor.

CHAPTER III

THERMOELASTICITY

The physical content of thermoelasticity is the equilibrium thermodynamics of a solid of arbitrary crystal symmetry in the presence of anisotropic stresses. In contrast to continuum mechanics, this subject is not so widely known; the elasticity part is well known, but the coupling to thermodynamics is not. The present chapter is a separate part of the overall theory, entirely separate from continuum mechanics or plastic flow. Recommended references for thermoelasticity are a review article,⁴ and Chapter 1 of the book.⁵

7. Isotropic-Pressure Thermodynamics

To illustrate the basic idea, we will take the simple case where the stress is isotropic (hydrostatic) pressure P :

$$\tau_{ij} = -P\delta_{ij} \quad (7.1)$$

We think of an equilibrium-thermodynamic process, and we want to integrate the appropriate differential equations to calculate the thermodynamic quantities of interest throughout the process. The process will thus be represented by a sequence of equilibrium states. While time may be thought of as an implicit variable of the process, time does not explicitly enter the equations, because equilibrium thermodynamics has no dependence on rates. Also, equilibrium thermodynamics customarily applies to a macroscopic system which is homogeneous, i.e. one in which thermodynamic variables are constant throughout. Alternatively, we can use the concept of local thermodynamic equilibrium, in which the thermodynamic variables are fields, but all the equilibrium relations hold locally. The equations of the present chapter can be viewed in either way: as global or local. Also, note that thermodynamics operates entirely in the rest frame of the material, globally or locally. And finally, so as not to leave the reader with the wrong idea, we recall from Chapter I that when irreversible thermodynamics is coupled into the system of equations, one or more of the equilibrium relations will be altered.

To proceed, we assume that all thermodynamic quantities of interest are known in the "current state." Since the stress is pressure, only two thermodynamic variables are needed to specify a state, and these will be taken as V and S . At each current state, we are given the proper dV and dS to increment to the next state. With this information, and the thermodynamic differential equations, it is possible to integrate to find all the thermodynamic quantities of interest in the next state. The source of information for the increments dV and dS

will in fact be the equations of motion and the irreversible thermodynamics, but this point is not relevant in setting up the thermoelastic theory. The easiest way to construct the thermodynamic equations is to start with a state function, from which all quantities can be derived. When V and S are the independent variables, the state function is the internal energy. We write

$$U = U(V,S) \quad (7.2)$$

and the pressure and temperature are first derivatives,

$$P = - \left(\frac{\partial U}{\partial V} \right)_S \quad (7.3)$$

$$T = \left(\frac{\partial U}{\partial S} \right)_V \quad (7.4)$$

Second derivatives of U produce the quantities

$$\begin{aligned} B &= \text{adiabatic bulk modulus,} \\ \gamma &= \text{Grüneisen parameter,} \\ C &= \text{constant-volume heat capacity.} \end{aligned} \quad (7.5)$$

These quantities are defined by the relations

$$\left(\frac{\partial^2 U}{\partial V^2} \right)_S = - \left(\frac{\partial P}{\partial V} \right)_S = \rho B \quad (7.6)$$

$$\left(\frac{\partial^2 U}{\partial V \partial S} \right)_{SV} = - \rho \gamma T \quad (7.7)$$

$$\left(\frac{\partial^2 U}{\partial S^2} \right)_V = \left(\frac{\partial T}{\partial S} \right)_V = \frac{T}{C} \quad (7.8)$$

Now the differentials of the dependent variables can be constructed. For the internal energy, we write

$$dU = \left(\frac{\partial U}{\partial V} \right)_S dV + \left(\frac{\partial U}{\partial S} \right)_V dS \quad (7.9)$$

and with (7.3) and (7.4) this is

$$dU = - PdV + TdS \quad (7.9)$$

In the same way dP and dT are expressed as

$$dP = - \rho B dV + \rho \gamma T dS \quad (7.10)$$

$$dT = -\rho\gamma T dV + \frac{TdS}{C} \quad (7.11)$$

This procedure can be continued to higher orders; obviously the dependent variables will continue to be coupled from one order to the next.

Notice Thermodynamics consists of an infinite hierarchy of coupled differential equations.

Let us consider how these equations can be used to integrate the equilibrium-thermodynamic process under study. First, dV and dS are given for every current state, hence V and S are always available, by integration from the initial state. U is obtained by integrating dU ; dU is calculated from (7.9) if P and T are known; P and T are obtained by integrating dP and dT ; these are calculated from (7.10) and (7.11) if B , γ , and C are known; and so on. The hierarchy has to be broken in order to do any real calculation. It can be broken at the level of (7.10) and (7.11), if B , γ , and C are regarded as known functions of V and S . This is generally the approach taken in the present work. On the other hand, the hierarchy can be broken one level earlier, namely at the level of (7.9), if P and T are regarded as known functions of V and S .

The independent variables do not have to be V and S . When thermodynamics is coupled to continuum mechanics, it is often convenient to choose V and U as independent variables. However, it should be recognized that whatever the independent variables, the hierarchic nature of the thermodynamic equations is still present. It will always be necessary to break the hierarchy, by using equation-of-state information, or by introducing an approximation. A simple example, which represents common practice in hydrodynamic computer codes, is as follows. The independent variables are V and U . In (7.9), TdS is approximated by $-q dV$, where q is the artificial viscosity:

$$dU = -(P + q)dV. \quad (7.12)$$

The increment dV is supplied from the equations of motion, P is a known function of V and U (the equation of state), and q is a known function of V , \dot{V} and U , where \dot{V} is the Lagrangian time derivative of V . Hence, dU can be calculated, and the system of equations is closed.

Incidentally, another useful form of (7.7) is the common expression for γ ,

$$\rho\gamma = \left. \frac{\partial P}{\partial U} \right)_V \quad (7.13)$$

This and all the other thermodynamic relations of the present section can be found in Chapter 1 of Reference 5.

8. Anisotropic Stresses

The concepts here, and the procedure, will be essentially the same as in the last section. With anisotropic stresses, an equilibrium thermodynamic state can be specified by the elastic configuration, or by the stress tensor, plus one more variable such as entropy or temperature. The different sets of independent variables have different advantages, and for any given problem, the practicing theorist should be able to transform from one set to another. Here the independent variables will be taken to be the elastic configuration and the entropy. Since it will eventually be necessary to differentiate between elastic and plastic strains, we will use a superscript e to denote elastic strains. The incremental elastic strain gradients du_{ij}^e carry the current configuration x into the next configuration $x + dx$. The incremental entropy dS represents entropy from all sources. Thus, the independent incremental variables, which are presumed given for each current state, are the set $\{du_{ij}^e, dS\}$. The symmetric elastic strains de_{ij}^e are

$$de_{ij}^e = \frac{1}{2} (du_{ij}^e + du_{ji}^e) \quad (8.1)$$

The rotations are to be thought of as rigid, hence rotation increments are denoted simply by $d\omega_{ij}$, where

$$d\omega_{ij} = \frac{1}{2} (du_{ij}^e - du_{ji}^e) \quad (8.2)$$

Also, with (8.1) and (8.2), the set of independent variables can be taken as de_{ij}^e , $d\omega_{ij}$, and dS :

$$\{du_{ij}^e, dS\} = \{de_{ij}^e, d\omega_{ij}, dS\} \quad (8.3)$$

In order to simplify the notation, we will adopt the practice of omitting the indication, in partial derivatives, of which variables are to be held constant. This can be done because, once the set of independent variables is specified, any partial derivative is to be carried out with all the *remaining* independent variables held constant. For example, for any function f , $\partial f / \partial S$ means all ϵ_{ij}^e and ω_{ij} are held constant. Also, $\partial f / \partial \epsilon_{ij}^e$ means S , all ω_{ij} , and all *other* ϵ_{ij}^e are held constant. A subtle point regards the enumeration of independent variables. All nine ϵ_{ij}^e , and all nine ω_{ij} , are regarded as independent variables, and all sums over i contain three terms, all sums over both i and j contain nine terms, and

so on. The symmetry relations among the ϵ_{ij}^e and the ω_{ij} are regarded as constraints. This is how the algebra is done. Returning to the matter of notation, the practice of omitting the indication of which variables are held constant is common in calculus, but it is not common in thermodynamics. Obviously, it will not work when one is trying to calculate a change from one set of independent variables to another. The practice will be followed, when possible, for the remainder of this work.

The combined first- and second-laws of thermodynamics is

$$dU = dW^e + TdS \quad (8.4)$$

The work done on a material by stresses is given by (6.7), and this is elastic work dW^e when the strains are elastic. Hence (8.4) is

$$pdU = \tau_{ij}d\epsilon_{ij}^e + \rho TdS \quad (8.5)$$

The customary expressions for stresses and temperature, as energy derivatives, follow directly from (8.5):

$$\tau_{ij} = \rho \frac{\partial U}{\partial \epsilon_{ij}^e} \quad (8.6)$$

$$T = \frac{\partial U}{\partial S} \quad (8.7)$$

Also (8.5) implies that U is invariant under rigid rotations:

$$\frac{\partial U}{\partial \omega_{ij}} = 0 \quad (8.8)$$

The next step is to construct thermoelastic equations for increments in the dependent variables which appear in the equation for dU , namely τ_{ij} and T .

Another subtle point arises. It goes without saying that both sides of any thermodynamic equation are evaluated at the same state. In the case of (8.7), the equation is valid for any configuration, and hence it can be differentiated with respect to strain. Equation (8.6) is also valid for any configuration, but the equation has already been evaluated at $\epsilon_{ij}^e = \omega_{ij} = 0$. This means that (8.6) cannot be differentiated with respect to strain. Alternatively, if you calculate $\partial \tau_{ij} / \partial \epsilon_{kr}^e$, or $\partial \tau_{ij} / \partial \omega_{kr}$, by differentiating (8.6), you will get the wrong answer.

Notice Thermodynamic equations containing strain increments, or strain derivatives, usually cannot be differentiated with respect to strain.

To calculate correct strain derivatives of the stress tensor is merely a matter of algebra. It is done in Reference 4, equations (6.9) and (6.11), and the result is

$$\frac{\partial \tau_{ij}}{\partial \epsilon_{kr}^e} = B_{ijkre} \quad (8.9)$$

$$\frac{\partial \tau_{ij}}{\partial \omega_{kr}} = \frac{1}{2} (\tau_{ir}\delta_{jk} - \tau_{ik}\delta_{je} + \tau_{je}\delta_{ik} - \tau_{jk}\delta_{ie}), \quad (8.10)$$

where B_{ijkre} is an adiabatic stress-strain coefficient. The letter B is in respect of F. Birch. Since we want to establish, within this monograph, the foundations for the most important parts of the theory, we will justify the above two equations. The first one is to be considered a definition of the stress-strain coefficients. If from any equilibrium state one changes the stress by $d\tau_{ij}$, at constant S , and the elastic strain $d\epsilon_{kr}^e$ results, the incremental stress-strain relation constitutes a definition of B_{ijkre} :

$$d\tau_{ij} = B_{ijkre}d\epsilon_{kr}^e$$

To relate this definition to the case of isotropic pressure, by analogy, it is the same as introducing the adiabatic bulk modulus by means of its stress-strain definition, namely

$$B = -V(\partial P / \partial V)_S$$

instead of as a second derivative of the energy [see equation (1.6)].

The significance of (8.10) is as follows. When an element of mass undergoes pure elastic strain $d\epsilon_{ij}^e$, the corresponding $d\tau_{ij}$ is given by (8.9). This thermoelastic relation is rotationally invariant, i.e. it does not know whether or not the mass element has rotated. If the mass element has in fact rotated, then the stress has rotated with it. Hence, components of stress in the laboratory system are changed, and it is this change which equation (8.10) gives. In detail, assume that a mass element rotates by $d\omega_{ij}$, and the stress tensor rotates with it. Then τ_{ij} in the laboratory system is given by $(R\tau R^{-1})_{ij}$, where $R = 1 + d\omega$, to order $d\omega$. Let us denote the change in the stress components, in the laboratory system, due to the rotation, as $d\tau_{ij}^{\omega}$:

$$d\tau_{ij}^{\omega} = (R\tau R^{-1})_{ij} - \tau_{ij} \quad (8.11)$$

Straightforward evaluation of this gives

$$d\tau_{ij}^{\omega} = -(\tau_{ik}d\omega_{kj} + \tau_{jk}d\omega_{ki}) \quad (8.12)$$

But this is precisely the result obtained from (8.10), when $d\tau_{ij}^{\omega}$ is calculated from the expression

$$d\tau_{ij}^{\#} = \frac{\partial \tau_{ij}}{\partial \omega_{kl}} d\omega_{kl} .$$

$$(8.13) \quad \frac{\partial T}{\partial \omega_{ij}} = 0 .$$

$$(8.20)$$

This justifies (8.10).

Exercise Calculate equation (8.12) from (8.11), and also from (8.13). With equation (8.12), show that the trace of the stress tensor is rotationally invariant.

Let us continue the program of calculating derivatives required to construct equations for $d\tau_{ij}$ and dT . The anisotropic Grüneisen parameters γ_{ij} are defined by

$$\frac{\partial \tau_{ij}}{\partial S} = -\rho T \gamma_{ij} . \quad (8.14)$$

They are expressed as energy derivatives by differentiating (8.6):

$$T \gamma_{ij} = - \frac{\partial^2 U}{\partial S \partial \epsilon_{ij}} . \quad (8.15)$$

The γ_{ij} represent the thermal energy contribution, or heat contribution, to the stresses, as can be seen by rewriting (8.14) in the form

$$\rho \gamma_{ij} = - \left. \frac{\partial \tau_{ij}}{\partial U} \right)_{\eta} . \quad (8.16)$$

The subscript η means at constant elastic configuration; its use is historical.⁵ The heat capacity at constant elastic configuration is C_{η} , defined by

$$\frac{\partial T}{\partial S} = \frac{T}{C_{\eta}} , \quad (8.17)$$

and with (8.7), this is expressed as an energy derivative,

$$\frac{T}{C_{\eta}} = \frac{\partial^2 U}{\partial S^2} . \quad (8.18)$$

Finally, from (8.7), and with (8.8) and (8.15), the strain derivatives of the temperature are

$$\frac{\partial T}{\partial \epsilon_{ij}} = - T \gamma_{ij} , \quad (8.19)$$

Equations for $d\tau_{ij}$ and dT can at last be written down, with the help of various preceding equations. The results are:

$$d\tau_{ij} = B_{ijkl} d\epsilon_{kl} + d\tau_{ij}^{\#} - \rho \gamma_{ij} T dS , \quad (8.21)$$

$$dT = - T \gamma_{ij} d\epsilon_{ij} + \frac{T dS}{C_{\eta}} . \quad (8.22)$$

This is the second level of the thermoelastic hierarchy, following equation (8.5), which is the first level. We break the hierarchy at this level, by assuming the coefficients C_{η} , γ_{ij} , and B_{ijkl} are known functions of the independent variables.

Exercise for the advanced student For a single crystal of tetragonal or lower symmetry, describe the experiment you would do to measure C_{η} , and also the experiment to measure γ . Show that a measurement of γ is contained in the first experiment, and a measurement of γ_{ij} is contained in the second experiment. Derive an equation for $C_{\eta} - C$.

For abbreviation, it is often convenient to use Voigt notation, in which one Greek-letter index replaces a pair ij , according to the following scheme:

$$\begin{array}{cccccccc} ij & = & 11 & 22 & 33 & 32 & \text{or } 23 & 31 & \text{or } 13 & 21 & \text{or } 12 \\ \alpha & = & 1 & 2 & 3 & 4 & & 5 & & 6 & \\ & & & & & & & & & & (8.23) \end{array}$$

Obviously, this replacement is allowed whenever a coefficient is symmetric in ij , as are τ_{ij} and γ_{ij} ; each of these matrices has only six independent components. From the definition (8.9), the B_{ijkl} are symmetric in ij , and also in kl , so they can be written in Voigt notation: $B_{ijkl} \rightarrow B_{\alpha\beta}$. Complete Voigt symmetry, or simply Voigt symmetry, means a set of coefficients is symmetric in Voigt indices. The $B_{\alpha\beta}$ do not have Voigt symmetry in general, since in general $B_{\alpha\beta} \neq B_{\beta\alpha}$. The use of Voigt notation introduces some tricky counting problems; sums on repeated indices, or tensor contractions, acquire counting factors when expressed in Voigt notation. We will not use Voigt notation in algebraic manipulations, but only in expressing the results.

The student should be aware of the fact that there are different sets of "elastic coefficients," with different

properties. Three sets in particular are important. The elastic constants C_{ijkl} are symmetric-strain derivatives of a state function, of the internal energy for the adiabatic constants, and of the Helmholtz free energy for the isothermal constants. The C_{ijkl} always have Voigt symmetry. The coefficients $A_{ijk\ell}$ are nonsymmetric derivatives, whose physical significance is that they govern small-amplitude wave propagation in solids, for example, ultrasonic waves. The $A_{ijk\ell}$ never have Voigt symmetry, unless the stresses are zero. Finally, the $B_{ijk\ell}$ are the coefficients in stress-strain relations, and they have Voigt symmetry when the stress is isotropic pressure. These three sets of coefficients differ from one another by terms linear in stresses. The three sets are the same at zero stress, but their stress derivatives are never the same.

A common circumstance in treatises on elastic-plastic strain is that "equation number one" is Hooke's law. In the present work, the first term on the right of (8.21) is Hooke's law, in differential form. Omitting consideration of the rotation term, there is still another contribution to (8.21), and that is the term in TdS . That term is necessary for the specification of $d\tau_{ij}$. Even if one changes variables, the equivalent of the TdS term will always be present on the right side of (8.21). In other words, stress can never be expressed in terms of strain alone.

Notice Hooke's law is not a complete thermoelastic equation.

To complete the present discussion, we will write the equation for the adiabatic stress-strain coefficients, as strain derivatives of the internal energy.

$$B_{ijk\ell} = \rho \frac{\partial^2 U}{\partial \epsilon_{ij}^c \partial \epsilon_{k\ell}^c} + \frac{1}{4} (\tau_{ik} \delta_{j\ell} + \tau_{i\ell} \delta_{jk} + \tau_{j\ell} \delta_{ik} + \tau_{j\ell} \delta_{ki} - 4 \tau_{ij} \delta_{kl}) \quad (8.24)$$

Exercise for the advanced student Derive equation (8.10) by differentiating an equation for the stress tensor. Then derive equation (8.24).

9. Approximation of Small Anisotropy

Thermoelasticity was developed in the first place to describe experiments, especially stress-strain experi-

ments and wave-propagation experiments, on single crystals in the presence of applied stresses. Applications of thermoelasticity can become quite complicated, especially for crystals of low symmetry. On the other hand, in plastic flow experiments, it is often possible to consider the solid under study as nearly isotropic. A polycrystalline aggregate, for example, is approximately isotropic for many purposes. It is therefore useful to simplify the thermoelastic equations for the case of an isotropic solid.

Definition An isotropic solid is one which is isotropic in any equilibrium-thermodynamic state in the presence of arbitrary applied pressure.

Of course, as soon as an anisotropic stress is applied, the solid undergoes anisotropic elastic strain, and in this state the solid is not isotropic. But if the anisotropy of the elastic strain is small, then the material anisotropy will be small, and this will be the basis of our approximation. Ultimately, we rely on plastic flow to keep the anisotropy of elastic strains small, or equivalently, to keep the stress deviators small. There will always be some solids for which the approximation is not acceptable.

Let us begin by making a formal expansion of thermodynamic functions in anisotropic elastic strains. The initial isotropic state $\{V, S\}$ of a solid goes at constant entropy and at constant volume to the anisotropic state $\{x, S\}$, by the elastic strains ϵ_{ij}^c . Then for any thermodynamic function f ,

$$f(x, S) = f(V, S) + \frac{\partial f}{\partial \epsilon_{ij}^c} \epsilon_{ij}^c + \dots \quad (9.1)$$

where \dots represents terms of second and higher order in the ϵ_{ij}^c , and where

$$V(x) = V \quad (9.2)$$

Obviously, (9.1) cannot be applied across a phase boundary. Otherwise, however, (9.1) is not an approximation, and it gives a way to calculate thermodynamic functions in anisotropic states. This expansion will be used in Chapter V to calculate stress-strain coefficients in biaxially strained states. For the present purpose, (9.1) allows us to make a precise definition of the small-anisotropy approximation.

Definition The small-anisotropy approximation consists of replacing the second-order thermoelastic coefficients C_{ij} , γ_{ij} , and B_{ijkl} in the thermoelastic hierarchy, by their leading terms in the expansion (9.1).

The approximation is easily visualized in terms of the equilibrium thermodynamic process of the preceding section. While the process goes along a line in anisotropic-state space, having volume $V(x)$ and entropy S at any point, a *unique* image point moves along a line in isotropic-state space, always having the same V and S . The second-order coefficients are evaluated at the image point.

The approximation of small anisotropy leads to the following replacements:

$$C_{ij} \rightarrow C \quad , \quad (9.3)$$

$$\gamma_{ij} \rightarrow \gamma \delta_{ij} \quad , \quad (9.4)$$

and the B_{ijkl} are replaced by the isotropic-solid $B_{\alpha\beta}$, which have the following symmetry:

$$\left(\begin{array}{ccc|ccc} B_{11} & B_{12} & B_{12} & & & \\ B_{12} & B_{11} & B_{12} & & & \\ B_{12} & B_{12} & B_{11} & & & \\ \hline & & & B_{44} & 0 & 0 \\ 0 & & & 0 & B_{44} & 0 \\ & & & 0 & 0 & B_{44} \end{array} \right) \quad (9.5)$$

There are only two independent $B_{\alpha\beta}$, because of the additional relation

$$B_{44} = \frac{1}{2} (B_{11} - B_{12}) \quad . \quad (9.6)$$

An equivalent pair of stress-strain coefficients are the adiabatic bulk and shear moduli, B and G respectively, related to the $B_{\alpha\beta}$ according to

$$B_{11} = B + \frac{4}{3} G \quad , \quad (9.7)$$

$$B_{44} = G \quad . \quad (9.8)$$

Still another equivalent pair are the Lamé coefficients λ and μ , which are not stress-strain coefficients, but are elastic constants:

$$B_{11} = \lambda + 2\mu - P \quad , \quad (9.9)$$

$$B_{44} = \mu - P \quad , \quad (9.10)$$

where P is the pressure of the isotropic state: $P = P(V,S)$. It is also useful to relate the $B_{\alpha\beta}$ to wave-propagation velocities. If c_l and c_t are, respectively, the velocities of longitudinal and transverse adiabatic waves, then

$$B_{11} = \rho c_l^2 \quad , \quad (9.11)$$

$$B_{44} = \rho c_t^2 \quad . \quad (9.12)$$

It is common practice to define a "bulk sound velocity" c_B , according to the relation

$$B = \rho c_B^2 - \frac{4}{3} \rho c_t^2 = \rho c_B^2 \quad . \quad (9.13)$$

However, no elastic wave actually travels at this velocity in an isotropic solid.

The approximation of small anisotropy is to be used in the equations for dU , $d\tau_{ij}$, and dT , which are (8.5), (8.21), and (8.22). It will be convenient to change stress variables, from τ_{ij} to the set \bar{P}, s_{ij} , defined in (5.2) and (5.3). We will also use conservation of mass, equation (4.16), for the elastic strains,

$$d \ln V = d\epsilon_{11}^e + d\epsilon_{22}^e + d\epsilon_{33}^e \quad . \quad (9.14)$$

This equation introduces a new notation, in which Cartesian indices are written as numbers 1,2,3, and repeated numbers are *not* summed. There are times when the algebra is simplified by enumerating the Cartesian indices, and explicitly writing out all the terms in a calculation. Results for the thermoelastic equations in the small-anisotropy approximation are as follows:

$$dU = -\bar{P}dV + TdS + V s_{ij} d\epsilon_{ij}^e \quad , \quad (9.15)$$

$$d\bar{P} = -B d \ln V + \rho \gamma T dS \quad , \quad (9.16)$$

$$dT = -\gamma T d \ln V + \frac{T dS}{C} \quad , \quad (9.17)$$

$$ds_{11} = 2G(d\epsilon_{11}^e - \frac{1}{3} d \ln V) + ds_{11}^0 \quad , \quad (9.18)$$

$$ds_{12} = G d\epsilon_{12}^e + ds_{12}^0 \quad , \quad (9.19)$$

where the remaining diagonal and off-diagonal stress deviators are given by obvious relabeling of (9.18) and (9.19), respectively. The rotation terms are

$$ds_{11}^0 = 2(s_{12} d\omega_{12} + s_{13} d\omega_{13}) \quad , \quad (9.20)$$

$$ds_{12}^0 = (s_{22} - s_{11})d\omega_{12} + s_{23}d\omega_{13} + s_{13}d\omega_{23} \quad , \quad (9.21)$$

and the remaining rotation terms are given by cyclic permutation of indices in (9.20) and (9.21).

Exercise Derive equations (9.15) through (9.21).

Once again, consider the equilibrium thermodynamic process, in which the solid moves along a line whose points are specified by $\{x, S\}$, while its image moves along a corresponding line specified by $\{V, S\}$. The image is undergoing a process in isotropic-state space, and hence is governed by the equations of Section 7. Let us use the subscript I, for isotropic, to denote variables belonging to the image. Then comparison of the small-anisotropy equations (9.15) through (9.17) with the isotropic equations (7.9) through (7.11), and remembering that $V(x) = V$, gives the results

$$\bar{P}(x, S) = P_I(V, S) \quad , \quad (9.22)$$

$$T(x, S) = T_I(V, S) \quad , \quad (9.23)$$

$$U(x, S) = U_I(V, S) + U_A(x, S) \quad , \quad (9.24)$$

where U_A is the anisotropic elastic energy of the solid, and is given by

$$U_A(x, S) = \int V s_{ij} d\epsilon_{ij}^0 \quad . \quad (9.25)$$

The integral is along the path of the process, from the initial state to the current state. The above equations show that the approximation of small anisotropy reduces the thermoelastic process to a "minimally anisotropic" one: the mean compressive stress and the temperature can be obtained from isotropic equation-of-state data, and the presence of anisotropy is contained only in the existence of nonzero stress deviators, and of the anisotropic elastic energy. Of course, when thermoelasticity is coupled to continuum mechanics, the presence of these anisotropic terms will affect the course of the process, at each timestep.

Just how good the approximation of small anisotropy is will depend largely on your point of view. From the solid state physics point of view, where elastic coefficients are standard precision tools of the trade, the approximation is not acceptable. In calculating processes in the explosive deformation of metals, the small-anisotropy approximation may very well be the best one can do.

Exercise For a given polycrystalline solid, how would you make the best currently-possible estimates of the quantities B , G , γ , and C , for pressures up to 100 kbar, and for temperatures from room temperature to melting? Make the estimates in terms of whatever variables are convenient, and then transform to V and S as independent variables.

CHAPTER IV

PLASTIC FLOW AND IRREVERSIBLE THERMODYNAMICS

The first two sections of this chapter describe the simplest theory of plastic flow for an isotropic solid, namely the Prandtl-Reuss-von Mises theory. Some standard textbooks on this subject are Hill,⁶ Mendelson,⁷ and Kachanov.⁸ Techniques for computer calculation of elastic-plastic flow problems in one and two dimensions are discussed by Wilkins.⁹ The constitutive equation for heat transport is discussed in the third section. In the last section, the entropy production is specified, and the complete irreversible-thermodynamic theory of elastic-plastic flow is summarized.

10. Plastic Yield

To fully appreciate the criterion for plastic yielding, it is helpful to take a brief look at the kinematics of stress in an isotropic solid. Under rotation R , τ transforms as $R\tau R^{-1}$. A real-symmetric 3×3 matrix has three rotational invariants, and for the stress tensor they are denoted T_1 , T_2 , and T_3 :

$$T_1 = \text{Tr } \tau = \tau_{11} + \tau_{22} + \tau_{33} \quad (10.1)$$

$$T_2 = \tau_{12}^2 + \tau_{23}^2 + \tau_{31}^2 - (\tau_{11}\tau_{22} + \tau_{22}\tau_{33} + \tau_{33}\tau_{11}) \quad (10.2)$$

$$T_3 = \|\tau\| = \tau_{11}\tau_{22}\tau_{33} + 2\tau_{12}\tau_{23}\tau_{31} - (\tau_{11}\tau_{23}^2 + \tau_{22}\tau_{31}^2 + \tau_{33}\tau_{12}^2) \quad (10.3)$$

The rotational invariants of the real-symmetric matrix of stress deviators are denoted S_1 , S_2 , and S_3 , and with the above three equations, they can be expressed as

$$S_1 = \text{Tr } s = 0 \quad (10.4)$$

$$S_2 = T_2 + \frac{1}{3} T_1^2 \quad (10.5)$$

$$S_3 = T_3 + \frac{1}{3} T_1 T_2 + \frac{2}{27} T_1^3 \quad (10.6)$$

S_2 is an important measure of the stress anisotropy, and it is useful to express S_2 in alternate forms. A commonly appearing form is

$$S_2 = \frac{1}{6} [(\tau_{11} - \tau_{22})^2 + (\tau_{22} - \tau_{33})^2 + (\tau_{33} - \tau_{11})^2 + (\tau_{12}^2 + \tau_{23}^2 + \tau_{31}^2)] \quad (10.7)$$

In terms of the stress deviators, S_2 is simply

$$S_2 = \frac{1}{2} s_{ij} s_{ij} \quad (10.8)$$

In principal axes, both τ and s are diagonal, and (10.8) reduces to

$$S_2 = \frac{1}{2} (s_{11}^2 + s_{22}^2 + s_{33}^2) \quad (10.9)$$

Exercise Prove that the quantities (10.1) through (10.3) are invariant under infinitesimal rotation.

Exercise Verify the relations (10.4) through (10.8).

Given a plane, the stress acting on it can be resolved into a *normal* component and a *tangential* component. For the plane 1, for example, the normal stress is τ_{11} , and components of the tangential stress are τ_{12} and τ_{13} . We will consider normal and tangential stresses in the principal axis system. The tangential stresses have local minima, with the value zero, on the principal planes. They have local maxima on planes which bisect the principal axes, and the maximum values are $\pm \frac{1}{2}(\tau_{11} - \tau_{22})$, $\pm \frac{1}{2}(\tau_{22} - \tau_{33})$, $\pm \frac{1}{2}(\tau_{33} - \tau_{11})$. These are called the principal tangential stresses. In principal axes, the spherical stress line (also called the hydrostatic line) is the octahedral line $\tau_{11} = \tau_{22} = \tau_{33}$. Planes perpendicular to this line (octahedral planes) are planes of constant spherical stress, which is constant mean compressive stress in our notation. Recall that an arbitrary stress is represented by a vector in principal axes. Hence, an arbitrary stress is decomposable into a projection onto the spherical stress line, and a projection onto the octahedral plane. In our notation, these projections are proportional to the mean compressive stress, and the shear stress. Specifically, the stress vector in principal axes has the following properties:

$$\sqrt{\tau_{11}^2 + \tau_{22}^2 + \tau_{33}^2} = \text{length of stress vector,}$$

$$- \sqrt{3} \bar{P} = \text{length of spherical stress projection,}$$

$$\sqrt{2S_2} = \text{length of octahedral-plane projection.} \quad (10.10)$$

Exercise Draw a figure showing the principal axes, the spherical stress line and the octahedral plane, the stress vector and its projections onto the spherical stress line and the octahedral plane. Prove the results stated in equation (10.10).

The concept of yield is as follows. If the stresses applied to an elastic-plastic solid are increased from zero, the response of the solid is initially elastic, and when the anisotropic stresses reach a certain "point," plastic flow begins. The point reached is a point on the yield surface. The yield surface depends on the state of the solid when the experiment is performed; it depends, for example, on the temperature, the pressure, and the dislocation density. For the moment, let us consider merely the stress dependence: the yield surface is given by an expression of the form $f(\tau_{ij}) = 0$. For an isotropic solid, the yield surface should depend on the stress only through the rotational invariants, which means $f(T_1, T_2, T_3) = 0$. Changing variables with the aid of (10.4) through (10.6) gives $f(\bar{P}, S_2, S_3) = 0$. But from the decomposition of the stress in principal axes, and arguing that the yield surface should not depend on \bar{P} , because it is the *anisotropic part* of the stress that drives plastic flow, one expects the yield surface to depend only on S_2 . This is the approximation of von Mises:¹⁰ yield occurs when the component of tangential stress on the octahedral plane reaches a fixed value. von Mises introduced the constant K , and wrote the yield condition in principal axes as

$$\frac{1}{4}(\tau_{11}-\tau_{22})^2 + \frac{1}{4}(\tau_{22}-\tau_{33})^2 + \frac{1}{4}(\tau_{33}-\tau_{11})^2 = 2K^2 \quad (10.11)$$

This is the same as

$$S_2 = \frac{4}{3}K^2 \quad (10.12)$$

In the present work, we define a rotationally invariant effective shear stress τ , according to

$$\tau^2 = \frac{3}{4}S_2 = \frac{3}{8}s_{ij}s_{ij} \quad (10.13)$$

The von Mises yield condition is then $\tau = K$. In the next section, we will generalize this to a flow rule.

It is worthwhile to compare our notation with that of the textbooks. Hill⁶ defines a generalized stress, or effective stress, or equivalent stress, which he calls δ , and which is 2τ . Mendelson⁷ defines an octahedral shear stress τ_{oct} , which is $\sqrt{8/9}\tau$. Kachanov⁸ defines a tangential stress intensity T , which is $\sqrt{4/3}\tau$. Wilkins⁹ writes the yield condition in principal axes as

$$(\tau_{11}-\tau_{22})^2 + (\tau_{22}-\tau_{33})^2 + (\tau_{33}-\tau_{11})^2 = 2(Y^0)^2 \quad .$$

Y^0 is the yield in simple tension, because for simple tension $\tau_{11} = \tau_{22} = 0$, and the yield condition is then $\tau_{33} = Y^0$. Mendelson⁷ and Kachanov⁸ use the same condition as Wilkins, but with Y^0 replaced by σ_0 and σ_s , respectively.

11. Plastic Flow

The plastic strain (or flow) will be denoted by a superscript p , as $d\epsilon_{ij}^p$. For a process in which elastic and plastic strains are going on simultaneously, it is possible to add the two kinds of strain in infinitesimal form. The total strain $d\epsilon_{ij}$ is then

$$d\epsilon_{ij} = d\epsilon_{ij}^e + d\epsilon_{ij}^p \quad (11.1)$$

Since the rotation is rigid, involving neither elastic nor plastic strain, increments of the displacement gradients are

$$du_{ij} = d\epsilon_{ij}^e + d\epsilon_{ij}^p + d\omega_{ij} \quad (11.2)$$

Now a problem arises. Consider the transformation α , which transforms the initial configuration X to the current configuration x . Increments $d\alpha_{ij}$ of the *total* α are defined by (4.10), together with (11.2), and these increments can be integrated along the process to give $\alpha(X \rightarrow x)$ for any configuration x . Thus, the *total* configuration is known, for every mass element and for every time. Furthermore, it is possible to write equations for $d\alpha_{ij}^e$ and $d\alpha_{ij}^p$ separately, under the condition that $\alpha_{ij} = \alpha_{ik}^e \alpha_{kj}^p$. The total α has the same meaning as before, but the component matrices α^e and α^p have no meaning, because they do not commute. In other words:

Notice When elastic and plastic strains are going on simultaneously, it is not possible to integrate the strains along the process to find separately the current elastic and plastic configurations.

Exercise What does commutativity of α^e and α^p have to do with the determination of elastic and plastic configurations?

The problem is resolved as follows. First, the incremental strains $d\epsilon_{ij}^e$ and $d\epsilon_{ij}^p$ are perfectly good differential variables for the coupled differential equations of a flow process; these equations can still be integrated step by step. However, we do not have the elastic configuration available as an integrated state variable. But the integrated stresses τ_{ij} can serve as state variables for thermoelasticity. These and one other variable, say S , completely specify a thermoelastic state, including the elastic configuration. For the plastic strain, we will introduce a non-negative scalar measure of the total plastic strain increment, and the integral of this will provide a measure of the total plastic strain which has occurred in the material.

The simplification of plastic strain is accomplished by the Prandtl-Reuss approximation: it is assumed that plastic strain increments are proportional to the corresponding stress deviators.^{11,12} That is, $d\epsilon_{ij}^p = s_{ij}d\lambda$, where $d\lambda$ is some scalar infinitesimal. To identify an effective plastic strain in $d\lambda$, we divide by the effective shear stress τ , and write the Prandtl-Reuss approximation as

$$d\epsilon_{ij}^p = \frac{3s_{ij}}{4\tau} d\psi \quad (11.3)$$

where $d\psi$ is the effective plastic strain increment. With the Prandtl-Reuss approximation, there is no volume change due to plastic strain, i.e.

$$d\ln V^p = d\epsilon_{ii}^p = 0 \quad (11.4)$$

because $s_{ii} = 0$. This property is in accord with the experimental observation that plastic strain is volume conserving.

Another property of the effective plastic strain results from the physical nature of the process, namely that plastic strain is a stress-relaxing process. For any incremental process which occurs at constant total configuration, i.e. $d\epsilon_{ij} = 0$, the set of plastic increments $d\epsilon_{ij}^p$ must be such as to reduce the magnitude of each and every s_{ij} . With the Prandtl-Reuss approximation, this will be the case if and only if $d\psi > 0$. The simplest way to show this is to calculate the incremental plastic work done on a mass element. This is given by equation (6.7), with the total strains replaced by plastic strains:

$$pdW^p = \tau_{ij}d\epsilon_{ij}^p \quad (11.5)$$

Thus, the plastic work is work done by the stresses against the plastic strains (there is no restriction on the total $d\epsilon_{ij}$ here). With τ_{ij} replaced by $s_{ij} - \bar{P}\delta_{ij}$, and with the Prandtl-Reuss approximation (11.3), the plastic work is

$$pdW^p = \frac{3}{4\tau} (s_{ij} - \bar{P}\delta_{ij})s_{ij}d\psi \quad (11.6)$$

From this it is seen that the mean compressive stress does no work in plastic strain, because $s_{ii} = 0$. With the definition of τ , equation (10.13), the plastic work becomes

$$pdW^p = 2\tau d\psi \quad (11.7)$$

Now in any mass element, plastic strain cannot proceed in such a way as to do work on the surroundings. Therefore, we must have $dW^p \geq 0$, and since $\tau \geq 0$,

$$d\psi \geq 0 \quad (11.8)$$

This condition is not to be regarded as a constraint, because it should be automatically satisfied in any correct calculation. As a result of (11.8), for any mass element, the integrated plastic strain ψ is a nondecreasing function of time.

Exercise For a process at constant total configuration, with the Prandtl-Reuss approximation and with the approximation of small anisotropy, prove that the magnitude of every s_{ij} decreases if and only if $d\psi > 0$.

With the Prandtl-Reuss approximation, the constitutive behavior of a solid is almost completely specified. It remains only to specify the behavior of the effective shear stress τ . This alone is an extremely difficult undertaking, because of the complications of plastic flow behavior in real solids. We will merely try to list the variables which control the behavior of τ , and that for an isotropic solid only. The von Mises constant K is generalized to a flow surface, such that $\tau = K$ when plastic flow is going on. K should depend on the thermoelastic state, which for an isotropic solid is specified by V and S . A dependence on pressure is implied here. In contrast to von Mises' original point of view, which neglects the dependence of K on \bar{P} , because \bar{P} does not drive plastic flow, we may want to apply the theory at pressures high enough to alter the material properties. Hence the dependence on pressure is kept.

The effect of work hardening is represented by a dependence of K on the plastic strain ψ , and also K should depend on the plastic strainrate $\dot{\psi}$, which is the Lagrangian time derivative of ψ ,

$$\dot{\psi} = \left. \frac{\partial \psi}{\partial t} \right)_X \quad (11.9)$$

Hence, in the plastic flow regime, i.e. when $\dot{\psi} > 0$,

$$\tau = K(\psi, \dot{\psi}, V, S) \quad (11.10)$$

When the effective shear stress is *inside* the flow surface, the plastic flow ceases and the solid is in the elastic regime, with $\dot{\psi} = 0$:

$$\tau \leq K(\psi, 0, V, S) \quad (11.11)$$

These two equations represent a common way of expressing the plastic constitutive behavior of an isotropic solid. However, the relations are presumably unique, which means invertible, and it is logically simpler to think of the plastic strainrate as the dependent variable. This approach also has better stability properties for numerical calculations. Hence, (11.10) and (11.11) together are written

$$\dot{\psi} = \dot{\psi}(\tau, \psi, V, S) \quad (11.12)$$

This is the plastic constitutive equation.

In recent years, Los Alamos has undertaken significant work to extend our understanding of plastic constitutive behavior of metals. A great amount of experimental work on plastic response to biaxial loading was reviewed by Hecker.¹³ Extensive experimental studies have been done on very large deformations at low strainrates.¹⁴⁻¹⁸ A new generation of high-strainrate experiments, with $\dot{\psi}$ in the range 10^3 to 10^7 s⁻¹, has been undertaken.¹⁹ Effort is also being directed at the problem of rationalizing experimental data. A method is being developed, based on Taylor factors,²⁰ to account for slip on single-crystal planes in a polycrystalline aggregate.²¹ A case has been made for the importance of dislocation drag at high strainrates.²² And finally, following the argument of Fred Kocks, one will eventually want to replace ψ in the plastic constitutive equation by a parameter which more accurately represents the internal plastic state of a solid.^{23,24}

12. Constitutive Equation for Heat Transport

The continuity equation (6.9) for heat transport is

$$\rho dQ = - \nabla \cdot \mathbf{J} dt \quad (12.1)$$

To obtain a closed system of equations for heat transport alone, it is customary to eliminate Q in favor of T . Let us assume, for the sake of discussion, that the process considered is at constant pressure. Then $dQ = C_p dT$, where C_p is the heat capacity per unit mass at constant pressure, and (12.1) reads

$$\rho C_p \frac{\partial T}{\partial t} + \nabla \cdot \mathbf{J} = 0 \quad (12.2)$$

This is to be coupled to a constitutive equation for the heat current \mathbf{J} . An equation which includes the rate-dependence of \mathbf{J} is

$$\theta \frac{\partial \mathbf{J}}{\partial t} + \mathbf{J} = - \kappa \nabla T \quad (12.3)$$

where θ is a relaxation time and κ is the thermal conductivity. The meaning of θ is as follows: if ∇T is suddenly changed, \mathbf{J} changes in response, with a relaxation time θ . With the relaxation equation for \mathbf{J} , the temperature obeys a dissipative wave equation,

$$\frac{\partial^2 T}{\partial t^2} + \frac{1}{\theta} \frac{\partial T}{\partial t} - \frac{\kappa}{\theta \rho C_p} \nabla^2 T = 0 \quad (12.4)$$

This equation describes the propagation of thermal signals. The maximum propagation velocity which can be achieved, according to (12.4), is

$$v_m = \sqrt{\frac{\kappa}{\theta \rho C_p}} \quad (12.5)$$

For a metal, θ should be roughly the electron-phonon relaxation time, say about 10^{-14} s, and v_m should be several cm/ μ s.

For most problems of thermoelastic-plastic flow in solids, the *propagation* of thermal energy is not important. Heat is generated locally, by plastic dissipation, and the important process of heat transport is heat *conduction*. In other words, the relaxation term in (12.3) should be negligible for most problems. The condition that this term is negligible is simply

$$d\mathbf{J} \ll \mathbf{J} \text{ in a time } \theta \quad (12.6)$$

With this approximation, the constitutive equation for heat transport reduces to Fourier's law of heat conduction:

$$\mathbf{J} = - \kappa \nabla T \quad (12.7)$$

When this is coupled to the continuity equation (12.2), the result is a diffusion equation for the temperature:

$$\frac{\rho C_P}{\kappa} \frac{\partial T}{\partial t} - \nabla^2 T = 0 \quad (12.8)$$

The diffusion equation will not describe propagation of a thermal signal. It is consistent only with an infinite velocity of propagation. The reason for this is in the approximation (12.6), which will always fail when J is sufficiently close to zero. If this defect in the theory does not cause a problem in an actual computation, then one can usually justify use of the conduction equation (12.7).

13. Irreversible Thermodynamics

Consider processes which are sufficiently close to equilibrium so that the total center-of-mass energy can be identified as the thermodynamic internal energy. Then $dE = dU$ in the continuum-mechanics equation for conservation of energy, (5.10) or (6.11). Writing dW as the sum of elastic and plastic contributions then puts the energy equation in the form

$$dU = dW^e + dW^p + dQ \quad (13.1)$$

The key step in completing the irreversible thermodynamic description is the identification of the entropy production. Heat transport produces entropy dS^k , according to the usual relation

$$TdS^k = dQ \quad (13.2)$$

Also, the plastic work is assumed to be totally dissipated:

$$TdS^v = dW^p \quad (13.3)$$

The total dS is $dS^k + dS^v$, and the energy equation (13.1) reads

$$\rho dU = \tau_{ij} de_{ij}^e + \rho TdS \quad (13.4)$$

Equation (13.4) is now identical to the thermoelastic energy equation (8.5). This identity is a necessary condition for a consistent theory. With (11.7) for the plastic work dW^p , the total entropy increment is given by

$$\rho TdS = \rho dQ + 2\tau d\psi \quad (13.5)$$

At this point, it is useful to make a list of the complete set of equations which describe the irreversible-thermodynamic elastic-plastic flow process in a solid.

The Complete Set of Equations

Conservation of mass	(4.16) or (4.18) or (4.19)
Conservation of linear momentum	(5.7) or (5.8) or (5.9)
Conservation of energy	(13.4)
Entropy production	(13.5)
Stress tensor increments	(8.21)
Temperature increment	(8.22)
Plastic constitutive	(11.12)
Heat-transport constitutive	(12.1) with (12.3) or (12.7)

For most problems, this system can be simplified considerably. Further, it is always possible to eliminate the elastic strains de_{ij}^e , in favor of the total strains de_{ij} and the plastic strain $d\psi$. This is done merely by writing $de_{ij}^e = de_{ij} - de_{ij}^p$, and then using the Prandtl-Reuss approximation to replace de_{ij}^p with $d\psi$. This is a useful step, because the equations of motion determine the total de_{ij} . Let us make a sketch of how the system of equations works in a numerical calculation. Suppose that all the field functions are known in the current state $\{x, S\}$. Given a time increment dt , we want to evaluate the functions at the next state $\{x+dx, S+dS\}$. The equations of motion (conservation of mass and momentum) give the increments de_{ij} and $d\tau_{ij}^p$. The plastic constitutive equation gives $d\psi = \dot{\psi}dt$. Then the heat transport equations and the entropy production equation give the total dS . With these increments, the changes in the internal energy, the stresses, and the temperature can be calculated, and all the functions are then determined in the next state.

It should be emphasized that the present theory can also be used to advantage in analyzing experiments. For most experimental conditions, all the material properties in the theory are reasonably well known, except for the plastic constitutive behavior. Hence, the complete set of equations, together with a well-designed set of experimental measurements, will determine the plastic constitutive behavior of a given solid. This procedure was used in analyzing nonsteady shock profiles, to determine the plastic behavior of an aluminum alloy, for plastic strains up to 5%, and for plastic strain rates up to 10^7 s^{-1} .

In assigning the entropy production, it was assumed that the plastic work is totally dissipated. Experimental support for this in metals goes back to the work of Farren and Taylor,²⁵ and of Taylor and Quinney,²⁶ who found that approximately 90% of the plastic work is dissipated, for strains greater than a few percent. Subsequent research on the energy stored in cold working was

reviewed by Tichener and Bever.²⁷ This stored energy goes into the defect structure of the solid, most notably the dislocation structure, and the energy is in fact recoverable. However, this energy is not included in ordinary thermoelastic theory, and an explicit accounting of it would require a redefinition of the thermoelastic quantities. For example, the presence of stored energy will give rise to a negative contribution to the heat capacity, which is strongly dependent on the heating rate, because of annealing. Hence, the assumption of total plastic dissipation, besides being a good approximation as far as the total plastic work is concerned, also simplifies the complete theory. In the same vein, the remaining thermoelastic coefficients, namely the anisotropic Grüneisen parameters and the stress-strain

coefficients, are presumed to be independent of the defect structure introduced by plastic flow. To the extent of the author's knowledge, this is in accord with experimental observation. On the other hand, the significant effect of the stored part of the plastic work, namely work hardening, is contained in the theory, through the presence of the plastic strain ψ in the plastic constitutive equation. Finally, an observation can be made regarding experiments designed to increase our understanding of high-strainrate processes in solids. If one is studying a process in which the temperature rise due to dissipation is important, then a measurement of the temperature can be helpful in extracting plastic constitutive behavior from the experiment.

CHAPTER V

APPLICATION TO UNIAXIAL COMPRESSION

The purpose of this chapter is to further illustrate the general theory. The example of uniaxial flow is chosen because of its geometrical simplicity, and also because it includes planar wave processes. The restriction to compression is done merely to avoid worrying about the sign of the shear stress: it is always positive in compression. The case of rarefaction, or tension, requires changing a few signs in the equations of the first two sections of this chapter. That the general theoretical construction includes any kind of material in a natural way is illustrated for the case of a viscous fluid in the last section. While the theory of the shock process is beyond the scope of this monograph, the basic equations underlying recent progress in shock theory are constructed, both for solids and for viscous fluids.

14. Uniaxial Compression in an Isotropic Solid

In a Cartesian coordinate system, material motion is in the x -direction only. For a given mass element, the transverse surfaces do not move. A subtle point arises immediately. For uniaxial compression, when plastic flow occurs, atoms actually move in transverse directions. But this transverse motion of atoms does not give rise to a net transport of mass, or a net momentum, so the transverse motion is not seen in the continuum-mechanic equations for conservation of mass and momentum. The occurrence of plastic flow is accounted for by the continuum variable ψ ; the boundary conditions on the material motion, in the present case that transverse surfaces do not move, are instrumental in controlling the amount of plastic flow which takes place. The same observations apply to continuum elastic-plastic flow in other geometries as well.

Exercise Is the energy involved in transverse motion of atoms during plastic flow contained in the continuum-mechanic equation for conservation of energy?

For uniaxial flow, the first step is to write down the simplified list of continuum-mechanic variables. The student should verify that these results follow from the definitions of Chapter II, and the condition of uniaxial flow.

x = laboratory coordinate,

$X = x_a =$ Lagrangian coordinate,

$$v = \left. \frac{\partial x}{\partial t} \right|_X = \text{material or particle velocity.} \quad (14.1)$$

The transformation matrix is

$$\begin{aligned} \alpha_{xx} &= \left. \frac{\partial x}{\partial X} \right|_t = \frac{\rho_a}{\rho} , \\ \alpha_{yy} &= \alpha_{zz} = 1 , \\ \alpha_{ij} &= 0 \text{ for } i \neq j . \end{aligned} \quad (14.2)$$

No summation is implied by repeated indices x , y , or z . The second equality for α_{xx} follows because $\|\alpha\| = \rho_a/\rho$, from (4.15). The symmetric infinitesimal strains are

$$\begin{aligned} d\epsilon_{xx} &= d\ln V , \\ \text{all other } d\epsilon_{ij} &= 0 . \end{aligned} \quad (14.3)$$

The stress tensor has only two independent components, the normal compressive stress σ , and the shear stress τ , where

$$\begin{aligned} \sigma &= -\tau_{xx} , \\ (\sigma - 2\tau) &= -\tau_{yy} = -\tau_{zz} , \\ \tau_{ij} &= 0 \text{ for } i \neq j . \end{aligned} \quad (14.4)$$

The components σ and τ are both positive in compression. Also note there is no rotation in the case of uniaxial flow.

Exercise Prove that τ defined in equation (14.4) agrees with the definition (10.13) for the effective shear stress which drives plastic flow.

The Lagrangian equation (4.18) for conservation of mass reduces to

$$\left. \frac{\rho_a}{\rho^2} \frac{\partial \rho}{\partial t} \right|_X + \left. \frac{\partial v}{\partial X} \right|_t = 0 . \quad (14.5)$$

Conservation of linear momentum in Lagrangian form, equation (5.8), becomes

$$\rho_n \left(\frac{\partial v}{\partial t} \right)_X + \left(\frac{\partial \sigma}{\partial X} \right)_t = 0 \quad (14.6)$$

To proceed to the thermoelastic equations, we need to find expressions for the elastic and plastic components of strain. We will make repeated use of the symmetry in y and z . The stress deviators follow from (14.4),

$$s_{xx} = -\frac{4}{3} \tau \quad (14.7)$$

$$s_{yy} = \frac{2}{3} \tau$$

and the Prandtl-Reuss approximation (11.3) gives

$$de_{xx}^p = -d\psi$$

$$de_{yy}^p = \frac{1}{2} d\psi \quad (14.8)$$

Then, solving the total strain increments (14.3) for the elastic parts gives

$$de_{xx}^e = d\ln V + d\psi$$

$$de_{yy}^e = -\frac{1}{2} d\psi \quad (14.9)$$

Hence, in uniaxial compression with plastic flow, there are only two independent strain measures, which we will take to be $d\ln V$ and $d\psi$. The energy equation (13.4) now reduces to

$$\rho dU = -\sigma d\ln V - 2\tau d\psi + \rho T dS \quad (14.10)$$

and the entropy production is

$$\rho T dS = -\left(\frac{\partial J}{\partial x} \right)_t dt + 2\tau d\psi \quad (14.11)$$

The heat current J is in the x -direction. The quantity of heat Q has been eliminated in favor of J , since this generally simplifies computations. A tricky point should be noted and remembered: elastic strain and plastic strain are not *separately apparent* in our equations, because they have been coupled through the boundary condition. In (14.10), for example, the first two terms on

the right, involving both $d\ln V$ and $d\psi$, are just the elastic work $\tau_{ij} de_{ij}^e$.

An isotropic solid under biaxial elastic strain has tetragonal symmetry. The adiabatic stress-strain coefficients $B_{\alpha\beta}$ have the following symmetry:

$$\begin{pmatrix} B_{11} & B_{12} & B_{12} & & & \\ B_{21} & B_{22} & B_{23} & & & \\ B_{21} & B_{23} & B_{22} & & & \\ \hline & & & B_{44} & 0 & 0 \\ & & & 0 & B_{66} & 0 \\ & & & 0 & 0 & B_{66} \end{pmatrix} \quad (14.12)$$

The coefficients B_{44} and B_{66} do not enter the equations for the stresses and temperature, because $de_{ij}^e = 0$ for $i \neq j$. Voigt notation will also be used for the anisotropic Grüneisen parameters: $\gamma_{ij} = \gamma_{\beta}$. The tetragonal symmetry is accounted for by writing

$$\gamma_{xx} = \gamma_1$$

$$\gamma_{yy} = \gamma_{zz} = \gamma_2 \quad (14.13)$$

With these symmetries, the equations (8.21) for the stresses, and (8.22) for the temperature, are evaluated to give

$$d\sigma = \rho \gamma_1 T dS - B_{11} d\ln V - (B_{11} - B_{12}) d\psi \quad (14.14)$$

$$d\tau = \frac{1}{2} \rho (\gamma_1 - \gamma_2) T dS - \frac{1}{2} (B_{11} - B_{21}) d\ln V$$

$$- \frac{1}{2} (B_{11} + \frac{1}{2} B_{22} + \frac{1}{2} B_{23} - B_{12} - B_{21}) d\psi \quad (14.15)$$

$$dT = -\gamma_1 T d\ln V - (\gamma_1 - \gamma_2) T d\psi + \frac{T dS}{C_{\eta}} \quad (14.16)$$

Exercise Derive equations (14.14), (14.15), and (14.16).

The complete set of equations for uniaxial compression of an isotropic solid has now been constructed. The set consists of conservation of mass and momentum, (14.5) and (14.6); the energy and entropy equations, (14.10) and (14.11); equations for the stresses and temperature, (14.14) through (14.16); and the constitutive equations for the heat current and the plastic

strainrate. The energy equation is uncoupled from the rest, and can be discarded. Further, whenever the transport of heat can be neglected, then $\rho T dS = 2\tau d\psi$, and dS can be eliminated from the set of equations. It was this latter set of equations, with $J = 0$, which was used to extract plastic constitutive data from weak-shock profiles for an aluminum alloy.²⁸ Following that, it was possible to calculate equation-of-state data from the shock measurements, explicitly accounting for the nonsteady nature of the shock profiles.²⁹ This procedure gives more accurate equation-of-state information than does the customary method of using Hugoniot jump conditions for the shock analysis, since the jump conditions hold only for steady waves (see Section 16).

15. Stress-Strain Expansions

The expansions to be constructed will be helpful in evaluating stress-strain coefficients in uniaxial flow problems. We will have to introduce the elastic constants, because the key equation, namely the elastic-strain expansion of second-order elastic coefficients, has been derived specifically for the elastic constants. The general relation between stress-strain coefficients B_{ijkl} and elastic constants C_{ijkl} is equation (6.10) of Reference 4:

$$B_{ijkl} = C_{ijkl} + \frac{1}{2} (\tau_{ik}\delta_{jl} + \tau_{il}\delta_{jk} + \tau_{jk}\delta_{il} + \tau_{jl}\delta_{ik} - 2\tau_{ij}\delta_{kl}) \quad (15.1)$$

Let us begin with an isotropic solid in the presence of an arbitrary pressure P , in the state $\{V, S\}$. Elastic constants and stress-strain coefficients in this state will be denoted by overhead bars, as $\bar{C}_{\alpha\beta}$ and $\bar{B}_{\alpha\beta}$. There are two independent second-order elastic constants, given in terms of the Lamé constants λ and μ [Reference 4, equation (14.13)]:

$$\begin{aligned} \bar{C}_{11} &= \lambda + 2\mu \quad , \\ \bar{C}_{12} &= \lambda \quad . \end{aligned} \quad (15.2)$$

There are three independent third-order elastic constants, given in terms of the Murnaghan constants ζ , ξ , and ν [Reference 4, equation (14.14)]:

$$\begin{aligned} \bar{C}_{111} &= 2\zeta + 4\xi \quad , \\ \bar{C}_{112} &= 2\zeta \quad , \\ \bar{C}_{123} &= 2\zeta - 2\xi + \nu \quad . \end{aligned} \quad (15.3)$$

Note for a polycrystalline aggregate, these are polycrystal, not single-crystal, third-order elastic constants. The two independent stress-strain coefficients follow from (15.1) and (15.2), and they were already used in Section 9:

$$\begin{aligned} \bar{B}_{11} &= \lambda + 2\mu - P \quad , \\ \bar{B}_{12} &= \lambda + P \quad . \end{aligned} \quad (15.4)$$

Exercise How would you evaluate λ and μ for a specific material in polycrystalline form? There are two ways to obtain ζ , ξ , and ν for a polycrystalline solid, namely by calculating aggregate averages of single-crystal third-order elastic constants, or by directly measuring ζ , ξ , and ν for the aggregate. Which way is the best?

In making elastic strain expansions, it is convenient to change the concept from differential relations, involving the strain differentials $d\epsilon_{ij}$, to power-series expansions in the small but not infinitesimal strains ϵ_{ij} . One can then work to first order in the strains ϵ_{ij} , or to second order, or higher. From the isotropic state $\{V, S\}$, we carry out an adiabatic elastic strain to the state $\{V + \epsilon_{ij}, S\}$. The strain has the biaxial symmetry of the elastic strain which occurs in uniaxial flow, namely the symmetry given by equation (14.9). There are only two independent nonzero strain components, which are ϵ_{xx}^e , and $\epsilon_{yy}^e = \epsilon_{zz}^e$. The adiabatic strain expansion of the second-order elastic constants (from an arbitrary initial configuration) is given by equation (8.20) of Reference 4, and in the present notation this is

$$C_{ijkl} = \bar{C}_{ijkl} + [\bar{C}_{ijkm}\epsilon_{lm}^e + \bar{C}_{ijml}\epsilon_{km}^e + \bar{C}_{imkl}\epsilon_{jm}^e + \bar{C}_{mjkl}\epsilon_{im}^e - \bar{C}_{ijk\ell mn}\epsilon_{mn}^e] \quad (15.5)$$

Here, C_{ijkl} is evaluated at the state $\{V + \epsilon_{ij}, S\}$, and the expansion is correct only to first order in the elastic strains. Evaluating (15.5) with the elastic constants in (15.2) and (15.3) tells us that there are four independent second-order elastic constants in the state $\{V + \epsilon_{ij}, S\}$, and that they have the following values to first order in elastic strains:

$$\begin{aligned} C_{11} &= (\lambda + 2\mu) + (3\lambda + 6\mu + 2\zeta + 4\xi)\epsilon_{xx}^e \\ &\quad + (-2\lambda - 4\mu + 4\xi)\epsilon_{yy}^e \quad , \\ C_{12} &= \lambda + (\lambda + 2\zeta)\epsilon_{xx}^e + (4\zeta - 2\xi + \nu)\epsilon_{yy}^e \quad , \end{aligned}$$

$$\begin{aligned}
C_{22} &= (\lambda + 2\mu) + (-\lambda - 2\mu + 2\zeta)\epsilon_{xx}^e \\
&\quad + (2\lambda + 4\mu + 4\zeta + 4\xi)\epsilon_{yy}^e, \\
C_{23} &= \lambda + (-\lambda + 2\zeta - 2\xi + \nu)\epsilon_{xx}^e \\
&\quad + (2\lambda + 4\zeta)\epsilon_{yy}^e. \tag{15.6}
\end{aligned}$$

Exercise Derive the equations (15.6).

Equations (15.6) will be used as the basis for two different expansions. In the first case, the elastic strain is taken at constant volume. Then $\epsilon_{yy}^e = -\frac{1}{2}\epsilon_{xx}^e$, and the equations (15.6) reduce to

$$\begin{aligned}
C_{11} &= \lambda + 2\mu + (4\lambda + 8\mu + 4\xi)\epsilon_{xx}^e, \\
C_{12} &= \lambda + (\lambda + \xi - \frac{1}{2}\nu)\epsilon_{xx}^e, \\
C_{22} &= \lambda + 2\mu - (2\lambda + 4\mu + 2\xi)\epsilon_{xx}^e, \\
C_{23} &= \lambda - (2\lambda + 2\xi - \nu)\epsilon_{xx}^e. \tag{15.7}
\end{aligned}$$

Now to evaluate the $B_{\alpha\beta}$, from equation (15.1), we need to construct the stress tensor in the state $\{V + \epsilon_{ij}^e, S\}$. From the definition of the stress-strain coefficients, equation (8.9), and since the stress tensor is $-P\delta_{ij}$ in the state $\{V, S\}$, we can write

$$\tau_{ij} = -P\delta_{ij} + \bar{B}_{ijkl}\epsilon_{kl}^e, \tag{15.8}$$

to first order in the elastic strains. For the volume-conserving elastic strain under consideration, and with \bar{B}_{ijkl} from (15.4), the result is

$$\begin{aligned}
\tau_{xx} &= -P + 2(\mu - P)\epsilon_{xx}^e, \\
\tau_{yy} &= \tau_{zz} = -P - (\mu - P)\epsilon_{xx}^e, \\
\tau_{ij} &= 0 \text{ for } i \neq j. \tag{15.9}
\end{aligned}$$

The stress-strain coefficients can now be evaluated at $\{V + \epsilon_{ij}^e, S\}$, with the elastic constants (15.7) and the stresses (15.9). In doing this, it is useful to change the expansion variable, from the elastic strain ϵ_{xx}^e to the ratio τ/G , where τ is shear stress in the elastically-strained state, and G is the adiabatic shear modulus in the state $\{V, S\}$. G is given by [see equations (9.8) and (9.10)]

$$G = \mu - P. \tag{15.10}$$

Then from the definition of τ , equation (14.4), and from equations (15.9), we have

$$\frac{\tau}{G} = -\frac{3}{2}\epsilon_{xx}^e. \tag{15.11}$$

Finally, the particular combinations of $B_{\alpha\beta}$ which appear in the uniaxial flow equations (14.14) through (14.16) are evaluated, and the results are:

$$\begin{aligned}
B_{11} &= (\lambda + 2\mu - P) \\
&\quad - \frac{4}{3}(2\lambda + 5\mu + 2\xi - P)\frac{\tau}{G}, \\
\frac{1}{2}(B_{11} - B_{12}) &= (\mu - P) \\
&\quad - (\lambda + 4\mu + \xi + \frac{1}{6}\nu - \frac{4}{3}P)\frac{\tau}{G}, \\
\frac{1}{2}(B_{11} - B_{21}) &= (\mu - P) \\
&\quad - (\lambda + 3\mu + \xi + \frac{1}{6}\nu - \frac{1}{3}P)\frac{\tau}{G}, \\
\frac{1}{3}(B_{11} + \frac{1}{2}B_{22} + \frac{1}{2}B_{23} - B_{12} - B_{21}) \\
&= (\mu - P) - 2(\mu + \frac{1}{6}\nu - \frac{1}{3}P)\frac{\tau}{G}. \tag{15.12}
\end{aligned}$$

These relations are an extension of the approximation of small anisotropy. They give the adiabatic stress-strain coefficients in the anisotropic state $\{x, S\}$, in terms of coefficients evaluated in the isotropic state $\{V, S\}$, where $V(x) = V$. Equations (15.12) are correct to first order in τ/G , and are equivalent to the general expansion (9.1), to first order. A result of (15.12) is the first-order relation

$$\frac{1}{2}(B_{12} - B_{21}) = \tau. \tag{15.13}$$

Equations (15.12) and (15.13) were first published in Reference 1.

Exercise Derive equations (15.9) and (15.12).

We now want to make another expansion, which is useful for analyzing weak shock profiles. The expansion

will be made at constant S , from the initial state ahead of the shock, which has volume V_a , and $P_a = 0$. In other words, we are expanding on the initial-state adiabat, $S = S_a$. The small but not infinitesimal plastic strain is ψ , and the elastic strains are obtained from (14.9),

$$\begin{aligned}\varepsilon_{xx}^e &= \ln(V/V_a) + \psi, \\ \varepsilon_{yy}^e &= -\frac{1}{2}\psi.\end{aligned}\quad (15.14)$$

The variable commonly used in shock theory to represent compression is not $\ln(V/V_a)$, but is ε , given by

$$\varepsilon = 1 - \frac{V}{V_a}.\quad (15.15)$$

Expansion of $\ln(V/V_a)$ for small ε gives

$$\ln(V/V_a) = -(\varepsilon + \frac{1}{2}\varepsilon^2 + \dots).\quad (15.16)$$

Correct to first order in ε and ψ , the expansions (15.6) are

$$\begin{aligned}C_{11} &= (\lambda + 2\mu) - (3\lambda + 6\mu + 2\zeta + 4\xi)\varepsilon \\ &\quad + (4\lambda + 8\mu + 4\xi)\psi, \\ C_{12} &= \lambda - (\lambda + 2\zeta)\varepsilon + (\lambda + \xi - \frac{1}{2}v)\psi, \\ C_{22} &= (\lambda + 2\mu) + (\lambda + 2\mu - 2\zeta)\varepsilon \\ &\quad - (2\lambda + 4\mu + 2\xi)\psi, \\ C_{23} &= \lambda + (\lambda - 2\zeta + 2\xi - v)\varepsilon \\ &\quad - (2\lambda + 2\xi - v)\psi.\end{aligned}\quad (15.17)$$

As usual, the stresses at constant S are obtained from the definition of the stress-strain coefficients. With the coefficients (15.4), and noting $P_a = 0$, and with the strains (15.14), the stresses to first order in ε and ψ are

$$\begin{aligned}\tau_{xx} &= -(\lambda + 2\mu)\varepsilon + 2\mu\psi, \\ \tau_{yy} &= \tau_{zz} = -\lambda\varepsilon - \mu\psi, \\ \tau_{ij} &= 0 \text{ for } i \neq j.\end{aligned}\quad (15.18)$$

The $B_{\alpha\beta}$ are then calculated from (15.1), with (15.17) and (15.18):

$$\begin{aligned}B_{11} &= (\lambda + 2\mu) - (4\lambda + 8\mu + 2\zeta + 4\xi)\varepsilon \\ &\quad + (4\lambda + 10\mu + 4\xi)\psi, \\ B_{12} &= \lambda + (2\mu - 2\zeta)\varepsilon \\ &\quad + (\lambda - 2\mu + \xi - \frac{1}{2}v)\psi, \\ B_{21} &= \lambda - 2\zeta\varepsilon + (\lambda + \mu + \xi - \frac{1}{2}v)\psi, \\ B_{22} &= (\lambda + 2\mu) + (2\mu - 2\zeta)\varepsilon \\ &\quad - (2\lambda + 5\mu + 2\xi)\psi, \\ B_{23} &= \lambda + (2\lambda - 2\zeta + 2\xi - v)\varepsilon \\ &\quad - (2\lambda - \mu + 2\xi - v)\psi.\end{aligned}\quad (15.19)$$

Equations (15.19), which are at constant S , can be used to calculate the normal stress σ , and the shear stress τ , to second order in strains at constant S . This is done simply by omitting the TdS terms from equations (14.14) and (14.15) for $d\sigma$ and $d\tau$, respectively, and by putting in the expansions (15.19), and integrating on $d\varepsilon$ and $d\psi$. The initial conditions are $\varepsilon_a = \psi_a = 0$. The calculation gives

$$\begin{aligned}\sigma &= (\lambda + 2\mu)\varepsilon - 2\mu\psi \\ &\quad - (\frac{3}{2}\lambda + 3\mu + \zeta + 2\xi)\varepsilon^2 + (4\lambda + 10\mu + 4\xi)\varepsilon\psi \\ &\quad - (\frac{3}{2}\lambda + 6\mu + \frac{3}{2}\xi + \frac{1}{4}v)\psi^2; \\ \tau &= \mu\varepsilon - \frac{3}{2}\mu\psi \\ &\quad - (\lambda + \frac{3}{2}\mu + \xi)\varepsilon^2 \\ &\quad + (\frac{3}{2}\lambda + \frac{9}{2}\mu + \frac{3}{2}\xi + \frac{1}{4}v)\varepsilon\psi \\ &\quad - (\frac{9}{4}\mu + \frac{3}{8}v)\psi^2.\end{aligned}\quad (15.20)$$

Once again, these equations hold on the adiabat at $S = S_a$. The physical path of the weak shock process lies "above" the adiabat, since entropy is generated during the shock process. If we neglect heat transport in the weak shock process, (14.11) gives for the entropy

$$\rho T dS = 2\tau d\psi.\quad (15.22)$$

This together with (15.21) shows that the entropy change is of *second order* in the strains. Hence, to calculate entropy contributions to the stresses, working to the same order as (15.20) and (15.21), we only need to know the coefficients of TdS to zeroth order, in the equations for $d\sigma$ and dt . Those coefficients are the anisotropic Grüneisen parameters, and to zeroth order they simply have to be evaluated in state a:

$$\begin{aligned} \gamma_1 &= \gamma_a, \\ \gamma_1 - \gamma_2 &= 0. \end{aligned} \quad (15.23)$$

Therefore, along the physical path of the weak shock process, the entropy contribution to σ , correct to second order in the strains, is from (14.14)

$$2\gamma_a \int_0^{\epsilon, \psi} \tau d\psi. \quad (15.24)$$

The integral has to be evaluated along the physical path. Also, from (14.15), the entropy contribution to τ is formally of third order in the strains. These results were first published in Reference 28.

Exercise Derive equations (15.19), (15.20), and (15.21). Having done this, you are certified to extract plastic constitutive data from weak shock profiles.

16. Steady Shock in an Isotropic Solid

The risetime of weak shocks in metals is long, and the nonsteady wave profile can be observed in detail with VISAR techniques.³⁰ As the shock strength increases, the risetime decreases, until at a hundred kbar or so, the risetime can no longer be resolved. However, as far as it is known experimentally, a planar shock always travels at a constant velocity.³¹ Hence, for moderately strong shocks, specifically for overdriven shocks, one expects a shock to propagate as a steady wave. The steady-wave condition allows the equations of motion to be integrated, which profoundly simplifies the complete set of equations for the flow process.

The symmetry is that of uniaxial compression, as described in Section 14. For any function $g(x,t)$, or $g(X,t)$, the relations between Lagrangian and Eulerian derivatives, equations (4.12) and (4.13), reduce to

$$\left. \frac{\partial g}{\partial X} \right|_t = \frac{\rho_a}{\rho} \left. \frac{\partial g}{\partial x} \right|_t, \quad (16.1)$$

$$\left. \frac{\partial g}{\partial t} \right|_X = \left. \frac{\partial g}{\partial t} \right|_x + v \left. \frac{\partial g}{\partial x} \right|_t. \quad (16.2)$$

A steady wave is a wave which travels at constant velocity without changing its shape. By this we mean that for any material property $g(x,t)$, the graph of g vs t through the wave profile is the same for all x . Hence $g(x,t)$ depends on only a single variable z , the "laboratory steady-wave variable,"

$$z = x - Dt, \quad (16.3)$$

where D is the wave velocity. The steady-wave condition is

$$g(x,t) = g(z). \quad (16.4)$$

Partial differentiation yields

$$\left. \frac{\partial g}{\partial x} \right|_t = \frac{dg}{dz}, \quad (16.5)$$

$$\left. \frac{\partial g}{\partial t} \right|_x = -D \frac{dg}{dz}. \quad (16.6)$$

With (16.1) and (16.2), the Lagrangian derivatives are found to be

$$\left. \frac{\partial g}{\partial X} \right|_t = \frac{\rho_a}{\rho} \frac{dg}{dz}, \quad (16.7)$$

$$\dot{g} = \left. \frac{\partial g}{\partial t} \right|_X = (v - D) \frac{dg}{dz}. \quad (16.8)$$

Exercise Prove, using conservation of mass, that the steady-wave condition $g(x,t) = g(z)$ implies, and is implied by, the condition $g(X,t) = g(Z)$, where $Z = X - Dt$ is the Lagrangian steady-wave variable.

The state ahead of the shock is assumed to be a thermodynamic equilibrium state, characterized by zero particle velocity v , zero normal stress σ , and zero heat current J :

$$v_a = \sigma_a = J_a = 0. \quad (16.9)$$

The state behind the shock is the Hugoniot state, denoted by subscript H, and it is also assumed to be a thermodynamic equilibrium state:

$$J_H = 0 \quad (16.10)$$

In the steady-wave analysis, it is convenient to use the compression variable ε , defined by (15.15). In terms of ε , equation (14.5) for conservation of mass is

$$\left(\frac{\partial \varepsilon}{\partial t}\right)_X + \left(\frac{\partial v}{\partial X}\right)_t = 0 \quad (16.11)$$

With (16.7) and (16.8), and noting that $\varepsilon_a = 0$, this integrates to

$$\varepsilon = \frac{v}{D} \quad (16.12)$$

In the same way, equation (14.6) for conservation of linear momentum integrates to

$$\sigma = \rho_a D v \quad (16.13)$$

The curve of normal stress vs compression through the shock process is called the Rayleigh line, and from the last two equations this curve is a straight line:

$$\sigma = \rho_a D^2 \varepsilon \quad (16.14)$$

It is important to recognize that the Rayleigh line is a straight line as a result of conservation of mass, conservation of momentum, and the steady-wave condition; no more and no less.

Exercise Derive equations (16.12) and (16.13).

The energy and entropy equations (14.10) and (14.11) are already in total differential form, except for the heat current term. With (16.3), (16.5), and (16.12), this term can be written, for a steady wave,

$$-\left(\frac{\partial J}{\partial x}\right)_t dt = \frac{\rho dJ}{\rho_a D} \quad (16.15)$$

Then the energy and entropy equations become

$$dU = \sigma v_a d\varepsilon + \frac{dJ}{\rho_a D} \quad (16.16)$$

$$T dS = \frac{dJ}{\rho_a D} + 2V_t d\psi \quad (16.17)$$

With σ replaced by (16.14), and with the initial conditions (16.9), the integral of dU is

$$U - U_a = \frac{1}{2} D^2 \varepsilon^2 + \frac{J}{\rho_a D} \quad (16.18)$$

The Hugoniot is the locus of equilibrium states behind shocks of varying strengths; the Hugoniot exists for steady or nonsteady waves. The shock velocity D serves as a parameter specifying the shock strength. The Hugoniot jump conditions are the statements of conservation of mass, momentum, and energy, across a steady-wave shock. From the preceding integrals of the equations of motion, the Hugoniot jump conditions for a given shock velocity D are

$$v_H = D\varepsilon_H \quad (16.19)$$

$$\sigma_H = \rho_a D v_H \quad (16.20)$$

$$U_H - U_a = \frac{1}{2} D^2 \varepsilon_H^2 \quad (16.21)$$

It is important to remember that these conditions hold only for a *steady wave*. Since $\dot{\psi}_H = 0$, the shear stress τ_H is presumably on the (static) yield surface. If one sets $\tau_H = 0$, as an approximation, then σ_H becomes the pressure P_H , and the equations (16.19) through (16.21) become the Hugoniot jump conditions for a fluid. For a fluid, the steady-wave Hugoniot can be constructed from the jump conditions together with the equation of state.

Exercise For a given D , prove that the jump conditions for a fluid, together with the equation of state, have a unique solution for ε_H , v_H , P_H , and U_H .

As a result of the steady-wave condition, space and time dependence has been eliminated from the equations of motion. This cannot be done for the constitutive equations, in general, but the dependence can be reduced to the single variable z . The heat conduction equation (12.7) becomes

$$J = -\kappa \frac{dT}{dz} \quad (16.22)$$

and the plastic constitutive equation (11.12) becomes

$$(v - D) \frac{d\psi}{dz} = \psi(\tau, \psi, V, S) \quad (16.23)$$

We can now make a list of the complete set of equations, called the Rayleigh-line equations, which govern the steady shock process.

Rayleigh-Line Equations for an Isotropic Solid	
Rayleigh line	(16.14)
Entropy production	(16.17)
Normal stress	(14.14)
Shear stress	(14.15)
Temperature	(14.16)
Heat-transport constitutive	(16.22)
Plastic constitutive	(16.23)

In these equations, one can think of ϵ as the independent variable, and of course V is equivalent to ϵ . The equations listed are seven equations in the seven dependent variables σ , τ , S , T , J , ψ , and z . Hence, for any given D , the steady shock process can be calculated from the seven equations. In fact, the set of equations can be reduced algebraically. With the equation for the Rayleigh line itself, σ is trivially eliminated from the set. Also, one can use the energy equation to replace S by U , if this change of variables is desired. The real space and time dependence of the process can be calculated from $z(\epsilon)$. For example, at a constant x , say $x = 0$,

$$t(\epsilon) = - \frac{z(\epsilon)}{D} \quad (16.24)$$

and at a constant t , say $t = 0$,

$$x(\epsilon) = z(\epsilon) \quad (16.25)$$

Exercise Prove $dZ = (\rho/\rho_s)dz$. If $z(\epsilon)$ is known, how can you calculate $t(\epsilon)$ at a constant X , and $X(\epsilon)$ at a constant t ?

Note that there are two rate-dependent processes going on simultaneously in the shock, namely the transport of heat, and plastic flow. At any point in the shock, the time dependence of the profile must be simultaneously consistent with both of the dissipative processes. In other words, the shock risetime is consistent with both processes at once.

The above list of Rayleigh-line equations was used for a detailed analysis of the process of overdriven shocks in solids.³² In doing this, the plastic constitutive equation was purposefully removed from the set, on the grounds that it is a totally unknown quantity under such shock conditions. With one equation removed from the set, it is still possible to learn a great deal about the steady shock process, and to establish several theorems regarding the existence of solutions. For metals, the theory predicts that the shock risetime will decrease to around 10^{-12} s, and will not decrease further, as the shock strength increases.³³ It is hoped that this prediction can be checked by experiment within the lifetime of the present author. Finally, note that for shocks stronger than a few Mbar in metals, irreversible thermodynamics breaks down, and a new theory has to be constructed.

17. Viscous Fluids

We will treat a compressible viscous fluid; a standard reference for this subject is Landau and Lifshitz.³⁴ The stress tensor is taken to be the fluid equilibrium pressure $P = P(V, S)$, plus the nonequilibrium viscous stress τ_{ij}^v :

$$\tau_{ij} = - P\delta_{ij} + \tau_{ij}^v \quad (17.1)$$

The argument for constructing the form of τ_{ij}^v is as follows. τ_{ij}^v depends on the relative motion between various parts of the fluid, so it depends on the velocity gradients \dot{u}_{ij} , which are

$$\dot{u}_{ij} = v_{ij} = \left(\frac{\partial v_i}{\partial x_j} \right)_t \quad (17.2)$$

τ_{ij}^v vanishes when the velocity gradients vanish, and so for small velocity gradients it should be linear in them. τ_{ij}^v cannot depend on pure rotation, so it depends only on the symmetric parts $\dot{\epsilon}_{ij}$. The most general second-rank tensor of the form prescribed has a term in $\dot{\epsilon}_{ij}$, and one in $\dot{\epsilon}_{kk}\delta_{ij}$. By convention, this is written in the form

$$\tau_{ij}^v = 2\eta_s \dot{\epsilon}_{ij} + (\eta_v - \frac{2}{3}\eta_s)\dot{\epsilon}_{kk}\delta_{ij} \quad (17.3)$$

Here $\eta_s = \eta_s(V, S)$ is the shear viscosity, $\eta_v = \eta_v(V, S)$ is the bulk viscosity, and η_s and η_v are never negative. From (17.1) and (17.3), it is seen that the mean compressive stress is $P - \eta_v \dot{\epsilon}_{kk}$. The Navier-Stokes equation is conservation of linear momentum for a viscous fluid, and it follows at once from (5.7) or (5.9), together with the stress tensor (17.1). Finally, to complete the specification of the irreversible behavior, we assume that the viscous work dW^v , which is the work done by

viscous stresses against total strains, is completely dissipated:

$$\rho dW^v = \tau_{ij}^v de_{ij} = TdS^v \quad (17.4)$$

Exercise Write the Navier-Stokes equation, in the manner described above, and compare it with whatever is called the Navier-Stokes equation in your favorite text.

To construct the Rayleigh-line equations for a viscous fluid shock, we first have to specialize the strains and stresses to the geometry of uniaxial compression. The strains are given by (14.3), which may be expressed as Lagrangian time derivatives:

$$\dot{\epsilon}_{xx} = \frac{\dot{V}}{V} \quad ,$$

$$\text{all other } \dot{\epsilon}_{ij} = 0 \quad (17.5)$$

The stress tensor (17.1), with the viscous part (17.3), is now evaluated with the above components of strainrate. The stress reduces to the symmetry (14.4), with the normal compressive stress σ , and the shear stress τ , given by

$$\sigma = P + \left(\frac{4}{3} + \frac{\eta_v}{\eta_s} \right) \tau \quad (17.6)$$

$$\tau = - \eta_s \frac{\dot{V}}{V} \quad (17.7)$$

For a steady wave, the equations of motion are the same as in Section 16, because continuum mechanics is not concerned with whether a material is solid or fluid. In the steady-wave entropy equation (16.17), the heat current contribution remains the same, and the plastic dissipation is to be replaced by the viscous dissipation, giving

$$T dS = \frac{dJ}{\rho_a D} + \left(\frac{4}{3} + \frac{\eta_v}{\eta_s} \right) V_a \tau d\epsilon \quad (17.8)$$

In the shear stress, the Lagrangian time derivative \dot{V} can be reduced to a z derivative, by (16.8), to give

$$\tau = - (v - D) \frac{\eta_s}{V} \frac{dV}{dz} \quad (17.9)$$

Finally, the pressure and temperature on the Rayleigh line are presumed to be given by the equilibrium thermodynamic relations for an isotropic material, equations (7.10) and (7.11), respectively. The complete set of equations which governs the steady shock process is given by the following list.

Rayleigh-Line Equations for a Viscous Fluid

Rayleigh line	(16.14)
Entropy production	(17.8)
Normal stress	(17.6)
Shear stress	(17.9)
Pressure	(7.10)
Temperature	(7.11)
Heat-transport constitutive	(16.22)

As usual, ϵ is the independent variable, and V is equivalent to ϵ . The Rayleigh-line equations consist of seven equations in the seven dependent variables σ , τ , J , S , P , T , and z . The set of equations can be reduced algebraically. First of all, the variable z can be eliminated by combining the equations for heat conduction and for the shear stress, to obtain

$$\tau(\epsilon) = \frac{\eta_s dJ(\epsilon)}{\kappa(dT/d\epsilon)} \quad (17.10)$$

where the derivative $dT/d\epsilon$ is on the Rayleigh line. Secondly, σ and S can be eliminated quite simply, to leave four equations in the variables τ , J , P , and T . At this point, the equation for dP can be solved for dJ , and dJ can be integrated on the Rayleigh line. Or else P can be eliminated, leaving only three equations in τ , J , and T .

Exercise Derive equation (17.10).

Exercise for the advanced student Prove that it is not possible to represent the behavior of an elastic-plastic solid with viscous fluid theory.

We now want to introduce the property of rate-dependent response of a fluid. This will generalize a viscous fluid to a viscoelastic fluid. The important physical concept is that viscous flow is driven by applied forces, and hence it must proceed in such a way as to reduce the applied forces. With this concept, we can construct a quantitatively correct description of the

constitutive behavior of a fluid, for a geometrically simple example. To an element of mass inside a fluid with zero initial stresses, apply a small (infinitesimal) planar compressive stress σ in a time Δt , such that the rate $\dot{\sigma} = \sigma/\Delta t$ is constant throughout Δt . Impose the boundary condition of uniaxial flow, so there is no motion of the transverse surfaces of the mass element. This experiment is to be done repeatedly, with σ constant, but with variable Δt . We will inquire about the compression ϵ , and the shear stress τ , at the end of the time Δt . To order ϵ , the compression is adiabatic. The fluid response is governed by the frequency-dependent adiabatic elastic moduli B_ω and G_ω , where $G_0 = 0$, and by the transport coefficients, of which we will keep only η_s , and neglect κ and η_v . Finally, there is a shear relaxation time θ_s , which is not independent of η_s , as will be shown. If $\dot{\sigma}$ is very high, i.e. if $\Delta t \ll \theta_s$, there is not time for viscous flow to begin, so the fluid response is elastic and ϵ is given by

$$\sigma = (B_s + \frac{4}{3} G_s)\epsilon \quad (17.11)$$

The corresponding shear stress is

$$\tau = G_s \epsilon \quad (17.12)$$

If now Δt is increased, the elastic equations hold and ϵ is constant until $\Delta t \approx \theta_s$. At this point, shear viscous flow becomes noticeable and ϵ begins to increase, while τ begins to decrease. As Δt is further increased, through the ranges $\Delta t > \theta_s$ and $\Delta t \gg \theta_s$, viscous flow is fully operative and the elastic moduli have reached their $\omega = 0$ limits:

$$\sigma = B_0 \epsilon + \frac{4}{3} \tau \quad (17.13)$$

$$\tau = \eta_s \dot{\epsilon} \quad (17.14)$$

where $\dot{\epsilon} = \epsilon/\Delta t$. The limiting (largest) value of ϵ results when $\Delta t \rightarrow \infty$, which gives the thermodynamic equilibrium process

$$\sigma = P = B_0 \epsilon \quad (17.15)$$

Since ϵ is a continuous function of Δt , the change from elastic to viscous behavior at $\Delta t \approx \theta_s$ implies

$$(B_s + \frac{4}{3} G_s)\epsilon \approx B_0 \epsilon + \frac{4}{3} \eta_s \dot{\epsilon} \quad , \quad \text{at } \Delta t \approx \theta_s.$$

Consistent with neglecting the bulk viscosity, we set $B_s \approx B_0$, and the crossover condition tells us

$$\theta_s \approx \frac{\eta_s}{G_s} \quad (17.16)$$

Exercise For an ordinary liquid metal, give numbers to the following estimates: $\theta_s \approx$ the mean atomic vibration time; $G_s \approx G$ of the solid just below melting; then η_s from equation (17.16) should be ≈ 0.01 poise.

Reviewing the above discussion, it is seen that the shear stress at the end of Δt is not constant, and reflects the nature of the response. The maximum τ is the elastically supported τ of (17.12). The viscous τ is always less than the elastic τ , because the viscous relation (17.14) holds only for $\Delta t > \eta_s/G_s$.

The complete viscoelastic constitutive equation is given by Litovitz and Davis.³⁵ With this constitutive equation, the process of steady shocks in dense fluids was studied in detail, and several theorems were established regarding the existence of solutions.³⁶ In fact, one of these theorems becomes obvious from the preceding results. The normal stress is given by two equations, namely the Rayleigh line, $\sigma = \rho_a D^2 \epsilon$, and as the sum of pressure and viscous terms in (17.6). For a fixed small ϵ , $\sigma(\epsilon)$ increases as D increases. But $\tau(\epsilon)$ is limited to the elastic-response limit, equation (17.12), so $P(\epsilon)$ has to increase as D increases. Again, P has elastic and entropy contributions, according to (7.10), and the only mechanism for increasing $P(\epsilon)$ at a fixed ϵ is by increasing the entropy. Hence the theorem: For an overdriven shock in a fluid, no solution is possible without heat transport.³⁶

Exercise In interstellar space, two sparse clouds of hydrogen atoms collide with a relative velocity of $4(10^5)$ cm/s. Are the gases viscous?

Exercise In interstellar space, two dense masses of metallic-hydrogen fluid collide with a relative velocity of $4(10^5)$ cm/s. How is this different from the collision of gas clouds?

REFERENCES

1. D.C. Wallace, *Phys. Rev.* **B22**, 1477 (1980).
2. J.N. Johnson, *J. Appl. Mech.* **105**, 593 (1983).
3. W. Prager, *Introduction to Mechanics of Continua* (Ginn, Boston, 1961).
4. D.C. Wallace, in *Solid State Physics*, edited by H. Ehrenreich, F. Seitz, and D. Turnbull (Academic, New York, 1970), Vol 25, p. 3C.
5. D.C. Wallace, *Thermodynamics of Crystals* (Wiley, New York, 1972).
6. R. Hill, *The Mathematical Theory of Plasticity* (Clarendon Press, Oxford, 1950).
7. A. Mendelson, *Plasticity: Theory and Application* (Macmillan, New York, 1968).
8. L.M. Kachanov, *Foundations of the Theory of Plasticity* (North-Holland, Amsterdam, 1971).
9. M.L. Wilkins, in *Methods in Computational Physics*, edited by B. Alder, S. Fernbach, and M. Rotenberg (Academic, New York, 1964), Vol. 3, p. 211.
10. R. von Mises, *Gött. Nachrichten, Math-Phys.* 582 (1913).
11. L. Prandtl, in *Proceedings of the First International Congress for Applied Mechanics*, edited by C. Biezeno and J. Burgers (Waltman, Delft, 1925), p. 43.
12. A. Reuss, *Z. Angew. Math. Mech.* **10**, 266 (1930).
13. S.S. Hecker, in *Constitutive Equations in Viscoplasticity*, edited by J.A. Stricklin and K. J. Saczalski (ASME, New York, 1976), p.1.
14. J.E. Hockett, P.S. Gilman, and O.D. Sherby, *J. Nuc. Materials* **64**, 231 (1977).
15. P.E. Armstrong, J.E. Hockett, and O.D. Sherby, *J. Mech. Phys. Solids* **30**, 37 (1982).
16. S.S. Hecker, M.G. Stout, and D.T. Eash, in *Plasticity of Metals at Finite Strain*, edited by E.H. Lee and R.L. Mallett (Quicksilver, Palo Alto, 1982), p. 162.
17. M.G. Stout and S.S. Hecker, in *29th Sagamore Army Materials Research Conference*, edited by J. Mescall and V. Weiss (Plenum, New York, 1983), p. 39.
18. S.S. Hecker and M.G. Stout, in *ASM Seminar on Deformation Processing and Structure*, edited by G. Krauss (ASM, Metals Park, Ohio, 1984).
19. J.A. Morgan, R.R. Karpp, R.H. Warnes, P.S. Follansbee, P.E. Armstrong, and C.E. Frantz, experiments in progress.
20. G.I. Taylor, *J. Inst. Metals* **62**, 307 (1938).
21. U.F. Kocks, G.R. Canova, and M.G. Stout, work in progress.
22. P.S. Follansbee, G. Regazzoni, and U.F. Kocks, *Inst. Phys. Conf. Ser. No. 70*, 71 (1984).
23. U.F. Kocks, in *Constitutive Equations in Plasticity*, edited by A.S. Argon (MIT Press, Cambridge, 1975), p. 81.
24. U.F. Kocks, A.S. Argon, and M.F. Ashby, *Progress in Materials Science* **19**, 1 (1975).
25. W.S. Farren and G.I. Taylor, *Proc. Roy. Soc. (London)* **A107**, 422 (1925).
26. G.I. Taylor and H. Quinney, *Proc. Roy. Soc. (London)* **A143**, 307 (1934).
27. A.L. Tichener and M.B. Bever, *Prog. Metal Phys.* **7**, 247 (1958).
28. D.C. Wallace, *Phys. Rev.* **B22**, 1487 (1980).
29. D.C. Wallace, *Phys. Rev.* **B22**, 1495 (1980).
30. J.N. Johnson and L.M. Barker, *J. Appl. Phys.* **40**, 4321 (1969).
31. Information supplied by R. G. McQueen.
32. D.C. Wallace, *Phys. Rev.* **B24**, 5597 (1981).
33. D.C. Wallace, *Phys. Rev.* **B24**, 5607 (1981).
34. L.D. Landau and E.M. Lifshitz, *Fluid Mechanics* (Pergamon, London, 1959).
35. T.A. Litovitz and C.M. Davis, in *Physical Acoustics*, edited by W.P. Mason (Academic, New York, 1965), Vol. IIA, p. 281.
36. D.C. Wallace, *Phys. Rev.* **A25**, 3290 (1982).