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SOME APPLICATIONS OF THE METHOD OF FORCING

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Stevo Todorchevich and Ilijas Farah

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Some Applications of the Method of Forcing. During the Fall Semester of 1991 Stevo Todorchevich (Todorčević) gave a course on applications of the method of forcing at the Mathematical Institute in Belgrade. This text contains material presented in the course, as well as some additional closely related results included for completeness. The method of forcing, i.e. the method of adding a generic object to a given structure, is frequently used to get independent results, that is, results showing that certain statements cannot be proved (or disproved) in ZFC or some other similar theory. The main purpose of these notes is to present to a general mathematical audience a number of applications of the method of forcing to other branches of mathematics, such as general topology and measure theory. Most of the presented results do not require any additional axioms of set theory, but use standard set-theoretical and forcing constructions, such as Suslin tree, generic models, Cohen and random reals, etc. Among topics included are Borel Equivalence Relations, Halpern-Laüchli Theorem, the Open Coloring Axiom and the Proper Forcing Axiom. For more ambitious readers there are a few exercises scattered throughout the text, and a list of yet unsolved problems is included.

During the Fall Semester of 1991 Stevo Todorčević gave a course on applications of the method of forcing at the Mathematical Institute in Belgrade. This text contains material presented in the course, as well as some additional closely related results included here for completeness. I wish to thank Zoran Marković, the director of the Institute, who gave us the idea of writing this booklet, Aleksandar Jovanović whose handwritten notes from the course I have used in the preparation of this text, Aleksandar Kron who has had the patience to carefully read the first version of this text and greatly improve its English, and Žarko Mijajlović for conversations, literature, and some TEXnical help. I also wish to thank Nenad Batoćanin, Darko Bulat, and Dragan Cvetković for their help with E-mail. Of course, I owe my deepest gratitude to Stevo Todorčević who conceived and held the course, who had written larger part of the text, and who had helped me to get a deeper understanding of the material during our conversations after the lectures. My job was in taking and organizing the set of notes and in asking for more details in the proofs whenever I felt that this would make the text more accessible.

Ilijas Farah
Belgrade, July 1993.

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Ilijas Farah
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Some Applications of the Method of Forcing

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CONTENTS

Dedicated to Professor Đuro Kurepa

0. INTRODUCTION

The method of forcing, i.e. the method of adding a generic object to a given structure, is frequently used to get *independence results*, that is, results showing that certain statements cannot be proved (or disproved) in ZFC or some other similar theory. In these notes we shall be concerned mostly with the use of forcing in proving theorems of ordinary mathematics, i.e. results that do not require any additional axioms. Sections are independent of each other and can be read in any order, with the following exceptions: §1 and §2 should be read first, §8 depends on §7, and §11 depends on §10. Here is a short survey of the presented material.

1. The notion of the generic set is introduced, as well as some examples showing how the existence of certain generic objects effects the additivity of measure and category, and the continuity of functions. Posets for adding Cohen and random reals are defined. It is also proved that all presented posets satisfy the countable chain condition.

2. We define the forcing relation "$\Vdash$", giving its basic properties. We consider models of ZFC, specifically the way in which a generic object can be used to extend the model. The description of Cohen and random reals over a fixed model without mention of any forcing is given. Finally, the properties of the forcing relation are employed to prove that the additivity of the Mokobodzki ideal is nothing more than $\aleph_1$.

3. We define a mapping from $\mathbb{R}$ into the set of all ω_1-trees, showing that every Cohen generic real maps to a Suslin tree. This is the proof that the existence of a Suslin tree is consistent with ZFC, but it is also a ZFC construction of an object that under certain circumstances becomes very special. (Compare this with the standard proof that Con(ZFC + $\neg$SH), e.g. in [Kunen]). We also sketch there a way of how this object can be used in finding a set of reals that does not have the property of Baire (of course, under some additional set-theoretic assumptions).

4. Like in §3, to every real r we adjoin a topological space Y_r such that the cellularity of Y_r^2 is uncountable, and show that if the real r is random then the cellularity of Y_r is countable. However, this time we go one step further then in the previous chapter, showing that an additional set-theoretic assumption (RVM) implies the existence of such a real in the ground model. The presented absoluteness argument is interesting for its own sake. Using the same ideas, we show how the cellularities of the topological powers Y^k ($k < \omega$) of some compact space Y can essentially be made arbitrary modulo the obvious monotonicity requirement.

5. We show how certain properties of the forcing relation can be used in proving the well-known theorem of Silver that the number of equivalence classes of a Borel equivalence relation is either countable or of size continuum. Similar proof is used to show that if the equivalence relation is analytic then the number of equivalence classes can in addition be $\aleph_1$ and nothing else.

6. The Halpern–Laüchli Theorem (HL), a deep combinatorial theorem dealing with partitions of products of perfect trees, is introduced; because of its length, this section is divided into three subsections. In the first one, by viewing the order type of rationals, η, as a perfect tree, we use HL to obtain some partition relations for η. In the second subsection we apply the properties of the poset for adding many Cohen reals to prove HL. The third subsection is devoted to applications of HL to perfect sets of reals. These applications can be considered as reformulations of HL into the language of Real Analysis perhaps making it closer to a wider mathematical audience.

The text in §7–§13 partly overlaps with the text of [Todorčević 1989]. It is given here not only for the sake of completeness, but also because the proofs are adapted for the wider mathematical audience.

7. The notion of a Suslin partition is defined, as well as the associated cardinal invariant $\mathfrak{m}$. Roughly speaking, cardinals less than $\mathfrak{m}$ in many ways behave like $\aleph_0$. This fact is demonstrated by using partitions of §1; e.g. it is shown that additivities of measure and category are at least $\mathfrak{m}$. It is also shown that the assumption $\mathfrak{m} > \omega_1$ (also known as MA($\aleph_1$)) decides some questions in General Topology independent of ZFC. This is an example of a *forcing axiom*: an assertion that assures the existence of a generic object. Remaining sections also deal with assumptions of this kind. At the end of §7 we present a theorem that, in some sense, parallels to constructions in §3 and §4: in connection with $\mathfrak{m} > \omega_1$ it gives us interesting consistency results, but it also gives us a better insight into the behavior of cardinal functions on topological spaces.

8. Continuing in the direction of the previous section, the inequality $\mathfrak{m} > \omega_1$ is applied to solve a problem from the Topological Measure Theory.

9. The notion of a gap in $\langle [\omega]^\omega, \subseteq^* \rangle$, needed in later sections is introduced. A Suslin pre-gap is defined, and it is shown how this kind of object gives us a Suslin partition with an unusual property. Using a construction that parallels the one of §3, a set of reals without the property of Baire is obtained.

10. A Ramsey-type statement for sets of reals, OCA, is introduced. It is shown that various generalizations of its present formulation are false. This interesting statement has found and is likely to find many applications in problems dealing with structures in close relationship with the set of reals. For example, we shall show that it essentially determines the structure of gaps in $\langle \omega^\omega, <^* \rangle$. We also introduce a version of OCA more suitable for dealing with the definable sets of reals and give some of its applications.

11. OCA is applied to give a partial solution to a problem in Topology.

12. PFA, a forcing axiom with some large cardinal strength and a few of its consequences are presented—all $\aleph_1$-dense sets of reals are isomorphic, OCA, Automatic Continuity in Banach Algebras, ...

13. A combinatorial concept of a groupwise dense set in $[\omega]^\omega$ and the related cardinal $\mathfrak{g}$ are introduced. The assumption that there is a cardinal κ such that the intersection of κ groupwise dense sets is always nonempty and there is an ultrafilter base (on ω) of size κ, shortly $\mathfrak{u} < \mathfrak{g}$, is a powerful forcing axiom. This axiom has a strong influence on the structures of filters on ω. We prove some of its consequences, particularly that $\mathfrak{u} < \mathfrak{g}$ implies "Filter Dichotomy" implies "Near Coherence of Filters".

14. A short list of open problems is given.

The five sections of the Appendix may provide the necessary background for this text.

Throughout the text there are few scattered exercises, but every sentence beginning with "It is trivial (evident, obvious, easily seen, etc). that ... " defines an exercise. Needles to say, frequently a nontrivial one.

The main purpose of this text is to present a number of applications of the method of Forcing to other branches of mathematics rather than to reach a frontier of mathematical research. Although there are few previously unpublished results of the first author, the text consists mainly of the known material. We haven't always tried to identify the authors of the presented results or proofs. But in every instance, we have included a reference where the (frequently lengthy) discussion on the authorship is presented.

1. SOME EXAMPLES OF GENERIC OBJECTS

Consider a partially ordered set (*poset*) $\mathcal{P}$ with the relation $\leq_{\mathcal{P}}$. The set $\mathcal{P}$ is said to be *a set of conditions*, and for $p \leq_{\mathcal{P}} q$ we also say that p *bears more information than* q, or that p is an *extension of* q. Usually we shall omit the subscript $\mathcal{P}$ and write $\leq$ for $\leq_{\mathcal{P}}$. Two elements p and q of the poset $\mathcal{P}$ are *incomparable* iff neither $p \leq q$ nor $q \leq p$. They are *incompatible* ($p \perp q$) iff there is no $r \in \mathcal{P}$ such that $r \leq p$ and $r \leq q$. Observe that "compatible" is weaker than "comparable".

If $\mathcal{P}$ and Q are posets, the *product* $\mathcal{P} \times Q$ is a poset $\{\langle p,q\rangle : p \in \mathcal{P}, q \in Q\}$ ordered as follows: $\langle p,q\rangle \leq \langle r,s\rangle$ iff $p \leq_{\mathcal{P}} r$ and $q \leq_Q s$.

DEFINITION 1.1 *A set G is said to be a filter iff*

(1) *For any $p \in G$ and $q \geq p$ we have $q \in G$, and*
(2) *for all p and q from G there is an $r \in G$ such that $r \leq p$ and $r \leq q$.*

DEFINITION 1.2 a) *A set $D \subseteq \mathcal{P}$ is dense iff for every $p \in \mathcal{P}$ there is a q in D that bears more information than p, i.e. $(\forall p \in \mathcal{P})(\exists q \in \mathcal{D}) q \leq p$. Similarly, D is dense below p iff it is dense in the poset $\{q \in \mathcal{P} : q \leq p\}$.*
b) *If $\mathcal{D}$ is a family of dense sets and G is a filter that intersects every D in $\mathcal{D}$ we say that G is $\mathcal{D}$-generic. A filter is simply generic iff it intersects all dense sets.*

Observe that for a countable $\mathcal{D}$ a $\mathcal{D}$-generic filter always exists; it can be constructed by an easy induction.

EXERCISE An *atom* is an element p of a poset $\mathcal{P}$ such that the set $\{q \in \mathcal{P} : q \leq p\}$ is linearly ordered by $\leq$. Prove that there is a generic filter G intersecting *all* dense sets in $\mathcal{P}$ iff there is an atom in $\mathcal{P}$. (Hint: consider the set $\{q \in \mathcal{P} : (\exists p \in G) p \perp q\}$).

EXAMPLE 1.1 (The poset for adding a single Cohen real) From now on, we shall frequently use the *Cantor cube* (the set $\{0,1\}^\omega$ with the product topology) as our working copy of the set of reals, $\mathbb{R}$. The fact that the first space is zero-dimensional and the second is connected will never be essential for what we are doing (see Appendix C). Let $\mathcal{C}_\omega$ denote the set of all finite partial functions from ω into $\{0,1\}$, with the ordering $p \leq q$ iff $p \supseteq q$. Let n be a natural number, consider the following subset of $\mathcal{C}_\omega$:

$$D_n = \{p \in \mathcal{C}_\omega : n \in \mathrm{dom}(p)\}.$$

This set is dense, because for every function $p \in C_\omega$ we have either $p \in D_n$ or $p \cup \{(n,0)\} \in C_\omega$. Notice that every $\{D_n : n \in \omega\}$-generic filter gives us a total function from ω into $\{0,1\}$. We shall only deal with families of dense subsets of C_ω which contain this basic family $\{D_n : n \in \omega\}$ of dense sets. In this way the union of a generic filter will be a member of the Cantor set which will be called a *Cohen real*. For every $p \in C_\omega$ we say that $[p] = \{f \in \{0,1\}^\omega : f \supset p\}$ is an *interval determined by p*. Note that all such intervals form a base for the Cantor cube $\{0,1\}^\omega$.

The following easy fact gives us the correspondence between dense subsets of C_ω and dense open subsets of the Cantor cube.

LEMMA 1.1 A subset F of the Cantor cube is nowhere dense iff the set $D_F = \{p \in C_\omega : [p] \cap F = \emptyset\}$ is dense in C_ω. □

Hence, if $\mathcal{F}$ is a family of nowhere dense subsets of the Cantor cube then a $\{D_F : F \in \mathcal{F}\}$-generic filter of C_ω gives us a Cohen real which avoids every element of $\mathcal{F}$.

EXAMPLE 1.2 (The poset for adding a random real) Consider the Cantor space $\{0,1\}^\omega$ with the Haar measure (hence, $\mu(\mathbb{R}) = 1$; see Appendix C.3) Let $\mathcal{R}_\omega$ denote a set of all measurable sets of a positive measure. Order $\mathcal{R}_\omega$ by the inclusion relation; hence, the smaller set bears more information than the bigger one. Let $\mathcal{D}$ be a family of dense sets which, in particular, contains each of the following dense sets:

$$D_n = \{p \in \mathcal{R}_\omega : p \text{ is compact and } \mu(p) < 1/(n+1)\}, \quad (n \in \omega)$$

If G is a $\mathcal{D}$-generic filter, then $\bigcap G$ contains exactly one point. Let r be its unique element. We say that r is a *random real*.

We say that a set of reals is *null* iff it has measure zero. For a null set N, let $D_N = \{p \in \mathcal{R}_\omega : p \cap N = \emptyset\}$. Then D_N is dense in $\mathcal{R}_\omega$; so if $\mathcal{F}$ is a family of null sets of reals, then every $(\{D_n : n < \omega\} \cup \{D_N : N \in \mathcal{F}\})$-generic filter gives us a random real which avoids every element of $\mathcal{F}$.

EXAMPLE 1.3 (The Additivity of Measure) Let $\mathcal{P}_3 = \mathcal{A}_\epsilon$ be a set of all open subsets of $\mathbb{R}$ of measure strictly smaller than ϵ, and for A and B in $\mathcal{A}_\epsilon$ let $A \le B$ iff $A \supseteq B$. For any null set N let

$$D_N = \{A \in \mathcal{A}_\epsilon : N \subseteq A\}.$$

Notice that every D_N is dense in $\mathcal{A}_\epsilon$. So if $\mathcal{F}$ is a family of null sets and if G is a $\{D_N\}_{N\in\mathcal{F}}$-generic filter, then $\bigcup\mathcal{F} \subseteq \bigcup G$ and $\mu(\bigcup G) \le \epsilon$; it follows that $\mu(\bigcup\mathcal{F}) \le \epsilon$. So if we can obtain such a filter in $\mathcal{A}_\epsilon$ for every $\epsilon > 0$ this would give us that $\mu(\bigcup\mathcal{F}) = 0$.

DEFINITION 1.3 *A set of reals is meager, or of first category iff it is a countable union of nowhere dense sets.*

EXAMPLE 1.4 (The Additivity of Category) A poset $\mathcal{P}_4$ is defined as the set of all $p = \langle f_p, \mathcal{F}_p\rangle$, where

(1) $f_p : n_p \to C_\omega$, (here $n_p \in \omega$ is identified with the set $\{0, 1, \ldots, n_p - 1\}$),
(2) $\mathrm{dom}(f_p(i)) \cap \mathrm{dom}(f_p(j)) = \emptyset$ for all $i \ne j$, and
(3) $\mathcal{F}_p$ is a finite family of nowhere dense subsets of $\{0,1\}^\omega$.

The ordering $\le$ of $\mathcal{P}_4$ is defined by $p \le q$ iff:

(4) $f_p \supseteq f_q$
(5) $\mathcal{F}_p \supseteq \mathcal{F}_q$
(6) $(\forall i \in \mathrm{dom}(f_p) \setminus \mathrm{dom}(f_q))\,(\forall F \in \mathcal{F}_q)\,[f_p(i)] \cap F = \emptyset$.

For a nowhere dense set F define $D_F = \{p \in \mathcal{P}_4 : F \in \mathcal{F}_p\}$. Then D_F is dense in $\mathcal{P}_4$, because for a given $p \in \mathcal{P}_4$ the condition $q = \langle f_p, \mathcal{F}_p \cup F\rangle$ is in D_F and $q \le p$. Now define $D_i = \{p \in \mathcal{P}_4 : i \in \mathrm{dom}(f_p)\}$ for $i \in \omega$. Then we claim that each D_i is dense in $\mathcal{P}_4$. To see this fix $p \in \mathcal{P}_4$. Without loss of generality suppose that $i = n_p$. Define $q \le p$ by $q = \langle f_p \cup \{(i,s)\}, \mathcal{F}_p\rangle$, where s is an element of C_ω such that

(1) $\mathrm{dom}(s) \cap \mathrm{dom}(f_p(n)) = \emptyset$ for all $n < i$,
(2) $[s] \cap F = \emptyset$ for all $F \in \mathcal{F}_p$.

The proof that such an s exists is left to the reader as an exercise. Now suppose that we have a family $\mathcal{F}$ of nowhere dense sets and a $\{D_F : F \in \mathcal{F}\} \cup \{D_i : i \in \omega\}$-generic filter G in $\mathcal{P}_4$. The set G intersects every D_i, so it defines a total function

$$f_G = \bigcup_{p\in G} f_p, \qquad f_G : \omega \to C_\omega$$

such that

$$(*) \qquad \left(\bigcap_{n<\omega}\bigcup_{m\ge n}[f_G(m)]\right) \cap \left(\bigcup\mathcal{F}\right) = \emptyset,$$

(in other words, we have produced a dense G_δ set disjoint from all members of $\mathcal{F}$). To see this choose an $F \in \mathcal{F}$. Then $G \cap D_F \ne \emptyset$, so we can fix an element p from this intersection. Now (*) follows from the following fact.

CLAIM For every $n \ge n_p$ we have $[f_G(n)] \cap F = \emptyset$, thus $\bigcup_{n\ge n_p}[f_G(n)] \cap F = \emptyset$.

PROOF For a given n choose $q \in G \cap D_n$. Then there is an $r \in G$ such that $r \le p$ and $r \le q$; hence, $n \in \mathrm{dom}(f_r)$ and $f_r(n) = f_G(n)$. Since $r \le p$ the condition (6) gives us that $[f_G(n)] \cap F = \emptyset$, since F is an element of $\mathcal{F}_p$. □

EXAMPLE 1.5 (Continuity of functions) Recall the well-known result of Luzin that every measurable real function is continuous on a set of positive measure, or the well-known fact that every Borel function is continuous on a dense G_δ-subset of $\mathbb{R}$ (see also Appendix, Theorem C.2.2). We shall now consider this phenomenon without any assumption on the given real function f.

For any two distinct elements x and y of the Cantor cube, set

$$\Delta(x,y) = \min\{i : x(i) \ne y(i)\}. \qquad \text{(cf. Fig. 1)}$$

Notice that this mapping Δ determines the metric by $d(x,y) = 1/(\Delta(x,y)+1)$ and that the topology induced by this metric coincides with the standard product topology. Also, the *lexicographical ordering* of the Cantor cube is defined via Δ as follows: $x <_{\mathrm{Lex}} y$ iff $x(\Delta(x,y)) < y(\Delta(x,y))$.

Recall that for a given set X by $[X]^n$ we denote the set of all subsets of X of cardinality n, while by $[X]^{<\omega}$ we denote the set of all finite subsets of X. (This

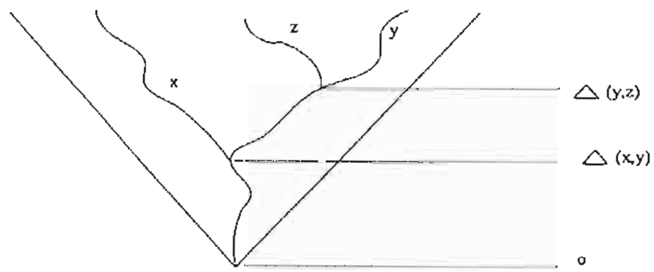

Fig. 1

definition naturally extends to infinite cardinals; see also Appendix D). Now for a given subset $X \subseteq \{0,1\}^\omega$ and $f: X \to \{0,1\}^\omega$ define a partition $[X]^3 = K_0 \cup K_1$ as follows: $\{x, y, z\} \in K_0$ iff

(†) $(\forall a, b, c \in \{x, y, z\})\, \Delta(a, c) \neq \Delta(b, c) \;\to\; \Delta(f(a), f(c)) \neq \Delta(f(b), f(c))$.

Let the poset $\mathcal{P}_5$ be the set of all finite subsets F of X *homogeneous for* K_0 (or 0-homogeneous)—i.e., finite subsets F of X such that $[F]^3 \subseteq K_0$. We order $\mathcal{P}_5$ by letting $F \leq F'$ iff $F \supseteq F'$.

CLAIM If a set Y is K_0-homogeneous, then $f \restriction Y$ is continuous (cf. Fig. 2).

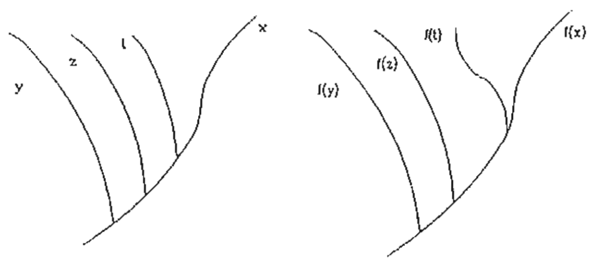

Fig. 2

PROOF If $f \restriction Y$ is not continuous at some $a \in Y$ then we can find a sequence a_n $(n < \omega)$ of elements of Y converging to a such that $f(a_n)$ $(n < \omega)$ doesn't converge fo $f(a)$. Hence one of the sets $A_n = \{i < \omega : \Delta(f(a), f(a_i)) = n\}$ is infinite and we can find an $n < m$ such that $\Delta(a, a_n) < \Delta(a, a_m)$ but $\Delta(f(a), f(a_n)) = \Delta(f(a), f(a_m))$. Then $\{a, a_n, a_m\}$ is a triple of $[Y]^3$ which doesn't satisfy the condition (†) of being a member of K_0, a contradiction. □

It follows that every filter of our poset $\mathcal{P}_5$ would give us a subset $Y \subseteq X$ such that $f \restriction Y$ is continuous, i.e. that our poset $\mathcal{P}_5$ forces large continuous restrictions of the given function f.

The elements of a given poset $\mathcal{P}$ can be considered as approximations to the "generic" object which we hope to introduce using $\mathcal{P}$. It is therefore not surprising that the properties of the generic object are very dependent of the set $\mathcal{P}$ of its approximations. The following property of a partially ordered set $\mathcal{P}$ is shared by all the posets introduced so far. It is one of the most important and most frequently used restrictions on $\mathcal{P}$.

DEFINITION 1.4 *An antichain of* $\mathcal{P}$ *is a set of mutually incompatible elements of* $\mathcal{P}$. *The poset* $\mathcal{P}$ *has the Suslin property (or* $\mathcal{P}$ *is ccc, or* $\mathcal{P}$ *satisfies the Countable Chain Condition) iff every antichain of* $\mathcal{P}$ *is countable. In general, for a cardinal* θ *a poset* $\mathcal{P}$ *has a* θ*-cc iff every family of mutually incompatible elements is of cardinality less than* θ *(so ccc is the same as the* $\aleph_1$*-cc).*

FACT Every countable poset is ccc. So, in particular, $\mathcal{C}_\omega$ is ccc. □

Very frequently we shall need to show that some poset is ccc. This is usually accomplished by starting with some uncountable family of its elements and then successively refining it, by usually using one of the following two means:

(1) Counting arguments, through partitioning an uncountable set into countably many subsets and than choosing one of them (an uncountable one).
(2) The Δ-system Lemma (Lemma 1.2).

DEFINITION 1.5 *An uncountable family of sets* $\mathcal{F}$ *forms a* Δ*-system with root* r *iff for every two distinct* $x, y \in \mathcal{F}$ *we have* $x \cap y = r$.

LEMMA 1.2 (Δ-SYSTEM LEMMA) For every uncountable family of finite sets there is an uncountable subfamily forming a Δ-system.

PROOF We may suppose that all sets are of the same size n and proceed by induction on n. For $n = 1$ the statement is trivial. Suppose that it is true for some n, and that we have a family $\mathcal{F} = \{F_\xi : \xi < \omega\}$ of sets of size $n+1$. If there is an x and an uncountable $\mathcal{F}_x \subseteq \mathcal{F}$ such that $x \in F$ for all $F \in \mathcal{F}_x$, apply the induction hypothesis to $\mathcal{F}'_x = \{F \setminus \{x\} : F \in \mathcal{F}_x\}$. Otherwise, $\mathcal{F}_x$ is countable for any $x \in \bigcup \mathcal{F}$, and we shall construct a disjoint subfamily $\mathcal{F}'$ inductively on $\xi < \omega_1$. We set $F'_0 = F_0$, and $\mathcal{F}_1 = \mathcal{F} \setminus \bigcup\{F_x : x \in F_0\}$. Take F'_1 to be the first element (the element with the least index) of $\mathcal{F}_1$ and continue in this way ω_1 steps. □

We shall frequently skip these refinements when proving that a given poset is ccc by simply saying "we may without loss of generality suppose that the family A is such and such".

LEMMA 1.3 The poset $\langle \mathcal{R}_\omega, \subseteq \rangle$ is ccc.

PROOF Suppose that there is an uncountable family $\mathcal{F}$ of sets of positive measure such that for all $A, B \in \mathcal{F}$ we have $\mu(A \cap B) = 0$. Then we can find an $n \in \omega$ such that the set $\{A \in \mathcal{F} : \mu(A) > 1/n\}$ is infinite, but this contradicts the additivity of the measure μ. □

LEMMA 1.4 The poset $\mathcal{A}_\epsilon$ is ccc.

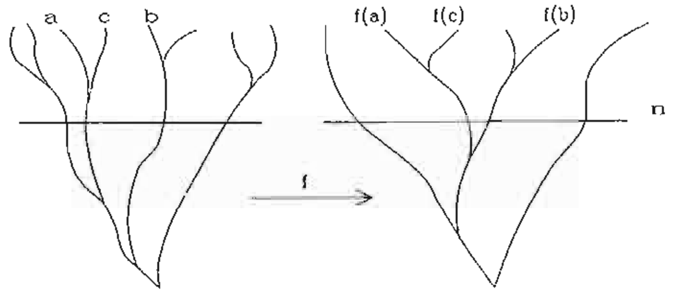

Fig. 3

PROOF Let $\mathcal{F}$ be an uncountable family of elements of $\mathcal{A}_\epsilon$. Then $\mu(A) < \epsilon$ for all A in $\mathcal{F}$ so there is an uncountable $\mathcal{F}' \subseteq \mathcal{F}$ and $n < \omega$ such that

$$\mu(A) < \epsilon - 1/n \quad \text{for all } A \text{ in } \mathcal{F}.$$

For each A in $\mathcal{F}$ choose an open subset $I_A \subseteq A$ which can be written as a disjoint union of intervals with rational endpoints such that $\mu(A \setminus I_A) < 1/(2n)$. Since the family of finite unions of rational intervals is countable there exists an uncountable $\mathcal{F}'' \subseteq \mathcal{F}'$ such that for some I and all A in $\mathcal{F}''$, $I_A = I$. It follows that for every A and B in $\mathcal{F}''$

$$\mu(A \cup B) \leq \mu(I) + \mu(A \setminus I) + \mu(B \setminus I) < \epsilon - \frac{1}{n} + \frac{1}{2n} + \frac{1}{2n} = \epsilon;$$

i.e., $A \cup B$ is an element of $\mathcal{A}_\epsilon$. □

LEMMA 1.5 The poset $\mathcal{P}_4$ is ccc.

PROOF Notice that in a typical element $p = \langle f_p, \mathcal{F}_p \rangle$ there exist only countably many possibilities for the finite function f_p. On the other hand, if p and q are such that $f_p = f_q$ then $\langle f_p, \mathcal{F}_p \cup \mathcal{F}_q \rangle$ is their common extension. So $\mathcal{P}_4$ has a strong form of ccc property. □

LEMMA 1.6 The poset $\mathcal{P}_5$ is ccc.

PROOF Recall that

$$\mathcal{P}_5 = \{F \in [X]^{<\omega} : [F]^3 \subseteq K_0\},$$

ordered by $\supseteq$. For every F in $\mathcal{P}_5$, let n_F be the minimal integer n such that

(1) $\Delta(x, y) < n$ for all $x \neq y$ in F,
(2) $\Delta(f(x), f(y)) < n$ for $x, y \in F$ with $f(x) \neq f(y)$.

CLAIM If F and F' are two elements of $\mathcal{P}_5$ such that $n_F = n_{F'} = n$, $F \restriction n = F' \restriction n$ and $f \restriction (F \restriction n) = f \restriction (F' \restriction n)$, then they are compatible in $\mathcal{P}_5$, or equivalently $[F \cup F']^3 \subseteq K_0$.

[Here $F \restriction n = \{x \restriction n : x \in F\}$ and $f \restriction (F \restriction n)$ is the function which maps $x \restriction n$ to $f(x) \restriction n$.]

PROOF We need to check the condition (2) for $F \cup F'$ (cf. Fig. 3). For suppose there exist $\{a, b, c\} \in [F \cup F']$ such that $\Delta(a, b) \neq \Delta(a, c)$ but $\Delta(f(a), f(b)) = \Delta(f(a), f(c)) = m$. From Fig. 3 it should be clear that $m < n$. Moreover, we may assume that two elements of the triple, say a and b, are in the same set, say F. We know that $c \in F' \setminus F$ and that there is a $d \in F$ such that

$$c \restriction n = d \restriction n \quad \text{and} \quad f(c) \restriction n = f(d) \restriction n.$$

Note that the assumption $d \notin \{a, b\}$ would contradict the condition (2) for F since we would have

$$\Delta(a, b) = \Delta(a, c) = \Delta(a, d) \quad \text{and}$$
$$\Delta(f(a), f(b)) = \Delta(f(a), f(c)) = \Delta(f(a), f(d)).$$

Notice that $d \neq b$ since otherwise we would have $\Delta(a, b) = \Delta(a, c)$. So we must have that $d = a$. But in this case we would have $\Delta(f(a), f(c)) > n$, a contradiction. □

Since there exist only countably many possibilities for $\langle n_F, F \restriction n_F, f \restriction (F \restriction n_F) \rangle$ the Claim shows that $\mathcal{P}_5$ satisfies a strong form of the ccc property. □

REMARK The reader might have already noticed that the posets $\mathcal{C}_\omega$, $\mathcal{R}_\omega$, $\mathcal{A}_\epsilon$, $\mathcal{P}_4$ and $\mathcal{P}_5$ have the ccc property of "different strengths". Being able to notice these differences is frequently a crucial step in many forcing arguments. Proving that a given poset is ccc can sometime involve quite deep combinatorial arguments as will be seen in later chapters of this text.

2. THE FORCING RELATION

The forcing relation is a very useful tool in applications of the method of forcing. It relates the elements of the poset $\mathcal{P}$ (approximations to the generic object) with various statements about the generic object. In this section we give a formal definition of this relation relativized to any given structure M, but we shall not rely on reader's full understanding of this matter. All we expect from the reader is to get the intuitive idea about this relation and to learn few of its basic properties exposed in some of the examples.

We shall work with a transitive models of a sufficiently large fragment of ZFC, and M will usually denote such a model (for more details on models of ZFC see Appendix B). If we have a poset $\mathcal{P} \in M$ and a filter G in $\mathcal{P}$ that intersects all sets dense in $\mathcal{P}$ which belong to the model M, we say that G is *M-generic*.

EXERCISE Prove that a filter G is M-generic iff it intersects all maximal antichains in $\mathcal{P}$ iff it intersects all dense open sets in $\mathcal{P}$. (A set $\mathcal{D} \subseteq \mathcal{P}$ is *open* iff $p \in \mathcal{D}$ and $q \leq p$ implies $q \in \mathcal{D}$ for all $p, q \in \mathcal{P}$).

It turns out that there exists the minimal transitive extension of M which contains G, and we denote it by $M[G]$. We say that M is the *ground model*, while $M[G]$ is the *generic model*. To describe $M[G]$ more precisely, we introduce the set of *names* for elements of $M[G]$. Note that the following description is "inner"—it takes place in the model M.

First fix a poset $\mathcal{P}$, and describe a set of $\mathcal{P}$-names (denoted by $V^{\mathcal{P}}$) regardless of a model M.

DEFINITION 2.1 *A set τ is a $\mathcal{P}$-name iff*

(1) $\sigma \subset V \times \mathcal{P}$,
(2) *For every $\langle \sigma, p\rangle \in \tau$ set σ is a $\mathcal{P}$-name.*

The $\mathcal{P}$-names are usually marked by a dot, like $\dot{\tau}$. For a model M containing $\mathcal{P}$ put $M^{\mathcal{P}} = M \cap V^{\mathcal{P}}$. If we have M-generic G, an *interpretation* of a $\mathcal{P}$-name τ, denoted by $\mathrm{int}_G(\tau)$, is defined by the recursion (see Appendix, Theorem A.2.2) on the rank of τ:

$$\mathrm{int}_G(\tau) = \{\mathrm{int}_G(\sigma) : (\exists p \in G)\langle \sigma, p\rangle \in \tau\}.$$

The domain of a model $M[G]$ is the set of all $\mathrm{int}_G(\tau)$ for $\tau \in M^{\mathcal{P}}$. To see that the generic set G is in $M[G]$ notice that we have a $\mathcal{P}$-name for it:

$$\dot{G} = \{\langle p, p\rangle : p \in \mathcal{P}\},$$

and for any $x \in M$ we have a $\mathcal{P}$-name $\check{x} = \{\langle \check{y}, p\rangle : y \in x, p \in \mathcal{P}\}$. (Observe that $\mathcal{P}$-names in general are marked with the dot ($\dot{}$) and that the names for the objects of V are marked with the check ($\check{}$)). It can be proved that $M[G]$ satisfies a large part of the ZFC axioms which depends on the fragment of ZFC which M satisfies (see [Kunen, VII.2]). For example, if x and y are elements of $M[G]$ and σ and τ are their $\mathcal{P}$-names, then $\{\langle \sigma, p\rangle, \langle \tau, p\rangle : p \in \mathcal{P}\}$ is a $\mathcal{P}$-name for the unordered pair $\{x, y\}$.

REMARK (See Appendix, Definition B.2.1) Assuming that the formula "$x \in \mathcal{P}$" is absolute, one can prove that the formula "τ is a $\mathcal{P}$-name" is also absolute.

Now we define the forcing relation $\Vdash$.

DEFINITION 2.2 *For $p \in \mathcal{P}$, formula ϕ, and $\mathcal{P}$-names $\tau_0, \ldots, \tau_{n-1}$ we say that p forces $\phi(\tau_0, \ldots, \tau_{n-1})$ (in symbols $p \Vdash \phi(\tau_0, \ldots, \tau_{n-1})$) iff for every generic G containing p the following is true:*

$$M[G] \models \phi(\mathrm{int}_G(\tau_0), \ldots, \mathrm{int}_G(\tau_{n-1})).$$

If every $p \in \mathcal{P}$ forces $\phi(\tau_0, \ldots, \tau_{n-1})$, we omit p and write $\Vdash \phi(\tau_0, \ldots, \tau_{n-1})$.

FACT If p does not force ϕ, then there is a $q \leq p$ such that $q \Vdash \neg\phi$. □

REMARK Suppose that conditions in $\mathcal{P}$ are sets of reals, and the intersection of a generic filter contains a single generic real. Then for all $a \in \mathcal{P}$ and a formula $\phi(x)$: $\phi(r)$ for all $r \in a$ iff $a \Vdash \phi(\dot{c})$ ($\dot{c}$ stands for a name for the generic real).

The forcing relation can be internally defined, without mentioning the model $M[G]$ or a generic set G (see [Kunen VII.3], the relation $\Vdash^*$. We will not make the distinction between relations "$\Vdash$" and "$\Vdash^*$"). For example, $p \Vdash \dot{\tau}_0 = \dot{\tau}_1$ iff for all $\langle \sigma_0, p_0\rangle$ from $\dot{\tau}_0$ the set $\{q \in \mathcal{P} : q \leq p_0 \rightarrow \exists\langle \sigma_1, p_1\rangle \in \tau_1\ (q \leq p_1\ \&\ q \Vdash \sigma_0 = \sigma_1)\}$ is dense below p. The relation $\mathcal{P} \Vdash \dot{\tau}_0 \in \dot{\tau}_1$ is defined in a similar manner, and the recursion continues on the complexity of ϕ. The importance of the internal definition is seen in the following Lemmas.

LEMMA 2.1 With the notation from the previous paragraph, the set

$$\{p \in \mathcal{P} : p \Vdash \phi(\tau_0, \ldots, \tau_{n-1})\}$$

is in the model M. □

LEMMA 2.2 (Truth Lemma of forcing extensions, for proof see [Kunen VII.3.5]). With the notation from the previous paragraph

$$M[G] \models \phi(\mathrm{int}_G(\tau_0), \ldots, \mathrm{int}_G(\tau_{n-1})) \qquad \text{iff} \qquad (\exists p \in G)\ p \Vdash \phi(\tau_0, \ldots, \tau_{n-1}). \quad \square$$

Let us now return to the Example 1.1, the poset $\mathcal{C}_\omega$. Notice that if M is a transitive model for a sufficiently large fragment of ZFC, then $\mathcal{C}_\omega \in M$ (see Appendix B.3). Suppose that G is a M-generic filter on $\mathcal{C}_\omega$. Then

$$c : \omega \rightarrow 2, \quad \text{where} \quad c = \bigcup G$$

since the dense sets $D_n = \{p \in \mathcal{C}_\omega : n \in \mathrm{dom}(p)\}$ are all members of M. The real c is called a *Cohen real over M*. Notice that G can be reconstructed from c as follows:

$$G = \{p \in \mathcal{C}_\omega : p \subseteq c\}.$$

The generic extension $M[G]$ is frequently also denoted by $M[c]$. To give an analysis of the set of reals in this extension we need a notion of a code for a set of reals. Let s_i $(i < \omega)$ be a recursive enumeration of $\mathcal{C}_\omega$. (Thus $\langle s_i : i < \omega\rangle$ will be in any transitive model of a large enough fragment of ZFC). A *code for an open set* $U \subseteq \{0,1\}^\omega$ is the real $x \in \{0,1\}^\omega$ such that for $n < \omega$

$$x(n) = 1 \qquad \text{iff} \qquad [s_n] \subseteq U.$$

We shall say that x codes U. Notice that if a code of U is in M then for all practical reasons M "knows" U even though U may not be an element of M, although $U^M = U \cap M$ is an element of M. Simple facts about U are "absolute". For example, if M thinks U^M is dense open then U itself must be dense in $\mathbb{R}$. This is so because the denseness of U really depends on the relationship between U and the basic open sets which M knows. Similarly, M correctly computes the measure of U, in symbols $\mu(U) = \epsilon$ iff $M \models \mu(U^M) = \epsilon$. Clearly, if we know U then we know its complement, so we take x to be also the code of the closed set $\mathbb{R} \setminus U$. A *code for a G_δ-set U* is a sequence of reals $\langle x_n : n < \omega\rangle$ such that each x_n codes an open set U_n and

$$U = \bigcap_{n<\omega} U_n.$$

So similar "absoluteness" works for G_δ-sets, ... etc. We have already mentioned that every Borel function $f : \mathbb{R} \rightarrow \mathbb{R}$ ($\mathbb{R} = \{0,1\}^\omega$) is continuous on a dense G_δ-set U_f. Then we let the code of f be a pair of sequences $\langle\langle x_n : n < \omega\rangle, \langle y_m : m < \omega\rangle\rangle$ where $\langle x_n : n < \omega\rangle$ codes U_f while for each $m < \omega$, y_m codes an open set V_m such that

$$f^{-1}([s_m]) \cap U_f = V_m \cap U_f.$$

Thus $\langle\{x_n\}, \{y_m\}\rangle$ determines f (up to the dense G_δ-set U_f) in any transitive model M containing the code. We shall frequently abuse the notation and write $f \in M$ when we really mean $\langle\{x_n\}, \{y_m\}\rangle \in M$. Thus for every x in $(U_f)^M$, the model M can correctly compute the value $f(x)$ by letting it to be the unique y such that for all $m < \omega$,

$$s_m \subseteq y \qquad \text{iff} \qquad x \in (V_m)^M.$$

Going to complements, all this can be used to define a code for an F_σ-set and a continuous map defined on an F_σ-set. Here is a typical application of these ideas.

THEOREM 2.1 A real c is Cohen over M iff c is an element of every dense open set with code in M.

PROOF ($\Rightarrow$) If U is a dense open set of reals with code in M, then

$$D_U = \{p \in \mathcal{C}_\omega : [p] \subseteq U\}$$

is a dense open subset of $\mathcal{C}_\omega$ which is a member of M. So there is a $p \in D_U$ such that $p \subseteq c$, so c in an element of U.
($\Leftarrow$) Let D be a dense open subset of $\mathcal{C}_\omega$ such that $D \in M$. Then $U = \bigcup_{p\in D}[p]$ is a dense open set of reals with code in M. [To see this define $x \in \{0,1\}^\omega$ by: $x(n) = 1$ iff $s_n \in D$.] So c is a member of U so there exist $p \in D$ such that $p \subseteq c$, i.e. the generic filter determined by c intersects D. $\square$

A completely analogous argument gives the following fact for the random real poset $\mathcal{R}_\omega$, where we say that r is *random over* M if $\{r\} = \bigcap G$ for some M-generic filter G of $\mathcal{R}_\omega$.

THEOREM 2.2 A real r is random over M iff r is a member of every F_σ-set of measure 1 coded in M. $\square$

The following fact shows that reals of the Cohen-real generic extension $M[c]$ can be named by Borel functions from M.

THEOREM 2.3 Let c be a Cohen real of M. Then for every real x of $M[c]$ there is a Borel map $f \in M$ such that $f(c) = x$.

PROOF Choose a name $\tau \in M^{\mathcal{C}_\omega}$ such that $x = \mathrm{int}_G(\tau)$, where G is the M-generic filter determined by c. For $n < \omega$ and $i < 2$ let

$$D_n^i = \{p \in \mathcal{C}_\omega : p \Vdash \tau(\check{n}) = \check{i}\}.$$

Notice that every $D_n = D_n^0 \cup D_n^1$ is dense open in $\mathcal{C}_\omega$ and that $\langle D_n^i : n < \omega, i < 2\rangle \in M$. For $n < \omega$ and $i < 2$ set

$$U_n^i = \bigcup_{p\in D_n^i}[p].$$

Then each $U_n = U_n^0 \cup U_n^1$ is dense open in $\mathbb{R}$ and therefore $U = \bigcap_{n<\omega} U_n$ is dense G_δ. Define $f: \mathbb{R} \to \mathbb{R}$ by letting f to be equal to 0 outside U while for y in U and $n < \omega$ set

$$f(y)(n) = i \quad \text{iff} \quad y \in U_n^i.$$

(notice that $U_n^0 \cap U_n^1 = \emptyset$, so this is well-defined). Clearly f is a Borel map coded in M since a code can be easily constructed from $\{D_n^i\}$. By Theorem 2.1 the Cohen real c is a member of U so the nontrivial definition of $f(c)$ applies. We claim that, in fact, $f(c) = x$. For suppose that $x(n) = i$ for some n and i. Then by the Truth Lemma of forcing extensions there exist $p \in G$ (i.e. $p \subseteq c$) such that $p \Vdash \tau(\check{n}) = \check{i}$. This means that $p \in D_n^i$ and therefore that $c \in U_n^i$ which by the definition of f means that $f(c)(n) = i$. $\square$

Again a completely analogous argument gives the following fact about the random real.

THEOREM 2.4 Let r be a random real over M. Then for every real x of $M[r]$ there is a Borel map $f \in M$ such that $f(r) = x$. $\square$

Notice that different functions f and g may denote the same real in $M[c]$, respectively $M[r]$. This will be always true if the set

$$X = \{x \in \mathbb{R} : f(x) = g(x)\}$$

is *comeager* (= contains a dense open set) respectively, has measure 1. This gives us a quite convenient way of translating facts about reals true in the forcing extension $M[c]$, respectively $M[r]$, into facts about Borel maps in M, and conversely. Remarkably, many classical results of Real Analysis and Measure Theory translate into interesting properties of $M[c]$ or $M[r]$, and conversely, the analysis of $M[c]$ and $M[r]$ has produced meaningful results in these two fields of mathematics. To present one such result we need a definition.

DEFINITION 2.3 *Let $\mathcal{I}_M$ be the ideal of subsets $A \subseteq \mathbb{R}^2$ such that for every $\epsilon > 0$ there is an open set U of the plane containing A such that all vertical sections of U have measure $< \epsilon$. (Vertical sections of U are sets of the form*

$$U_x = \{y \in \mathbb{R} : \langle x, y\rangle \in U\},$$

where $x \in \mathbb{R}$).

The ideal $\mathcal{I}_M$ was introduced in [Mokobodzki] in the course of studying versions of the meager or the measure-zero ideal in the plane. We shall see later (§7) that additivities of Lebesgue measure and Baire category can be quite large so it is natural to ask whether the same is true about the *Mokobodzki ideal*. We shall now see that the answer to this question is negative and that it comes out of an analysis of the forcing extension $M[c]$.

THEOREM 2.5 There exists $\aleph_1$ many elements of $\mathcal{I}_M$ whose union is not in $\mathcal{I}_M$.

PROOF The "plane" for our purpose here will be the product of the irrationals and the unit interval $I = [0,1]$, and the irrationals will be naturally identified with the Baire space which we denote by $\mathbb{R}$, and which itself will be considered as a set of all mappings $x: \{0,1\}^{<\omega} \to \omega$ rather than the mappings $x: \omega \to \omega$. This leads us to look at the Cohen poset $\mathcal{C}_\omega$ as a set of all finite maps p such that $\mathrm{dom}(p) \subseteq \{0,1\}^{<\omega}$ and $\mathrm{range}(p) \subseteq \omega$. Clearly, in all these identifications we are only ignoring the nature of the index set or ignoring countably many reals which obviously cannot be crucial in any of the properties of $\mathcal{I}_M$.
For $n < \omega$ and $k < 2^{n+1}$, let

$$I_k^n = \{x \in [0,1] : k/2^{n+1} < x < (k+1)/2^{n+1}\}.$$

For $k \geq 2^{n+1}$, let $I_k^n = I_0^n$. For $r \in \{0,1\}^\omega$ and $a \in \mathbb{R}$, let $r_a: \omega \to \omega$ be determined by

$$r_a(n) = a(r \restriction a(r \restriction n)) \quad (n < \omega).$$

Finally, for every $r \in \{0,1\}^\omega$ we can define

$$B_r^n = \{\langle a, x\rangle \in \mathbb{R} \times I : x \in I^n_{r_a(n)}\},$$
$$A_r = \{\langle a, x\rangle \in \mathbb{R} \times I : x \in I^n_{r_a(n)} \quad \text{for infinitely many } n < \omega\}$$
$$\left(= \bigcap_{m<\omega} \bigcup_{n \geq m} B_r^n\right).$$

(Notice that B_n^r is open and A_r is G_δ).

CLAIM 1 $A_r \in \mathcal{I}_M$ for every $r \in \{0,1\}^\omega$.

PROOF Given $\epsilon > 0$ pick m such that $1/2^m < \epsilon$. Let

$$U = \{\langle a, x\rangle : x \in I^n_{r_a(n)} \quad \text{for some } n \geq m\}$$
$$= \bigcup_{n \geq m} B_r^n.$$

Then U is an open subset of the plane which covers A_r such that for every $a \in I$,

$$\mu(U_a) \leq \sum_{n=m+1}^{\infty} \mu(I^n_{r_a(n)}) = \sum_{n=m+1}^{\infty} \frac{1}{2^n} = \frac{1}{2^m} < \epsilon. \qquad \square$$

The following claim finishes the proof of Theorem 2.5.

CLAIM 2 $A_X = \bigcup_{r \in X} A_r \notin \mathcal{I}_M$ for every uncountable $X \subseteq \{0,1\}^\omega$.

PROOF Every A_r is a G_δ subset of the plane so we may consider it in any model containing its code (=the above description). So, in particular, we can consider these sets or their vertical sections in the forcing extension of C_ω.

Let U be an open subset of the plane which contains A_X. It is enough to show that

$$B = \{a \in \mathbb{R} : \mu(U_a) > 1/2\}$$

is nonempty. In fact, we shall show that its complement F is nowhere dense. For suppose that F is not nowhere dense; then, since it is closed, it contains a basic open set $[p]$ for some $p \in C_\omega$. It follows that

(1) $$p \Vdash \mu(U_{\dot c}) \leq 1/2.$$

(Here $\dot c$ is the canonical name for the Cohen real). To get a contradiction it suffices to show that (remember that $\bigcup_{r \in X} A_r \subseteq U$):

(2) $$p \Vdash [0,1] \setminus \mathbb{Q} \subseteq \bigcup_{r \in X} (A_r)_{\dot c} \subseteq U_{\dot c}$$

if (2) fails then there is a name $\dot x$ for a real in $[0,1] \setminus \mathbb{Q}$ and an extension p^* of p such that for all r in X

(3) $$p^* \Vdash \dot x \notin I^n_{r_{\dot c}(n)} \quad \text{for all but finitely many } n.$$

Therefore, for every $r \in X$ we can find a condition $q_r \leq p^*$ and an $m_r < \omega$ such that for all $n \geq m_r$,

(4) $$q_r \Vdash \dot x \notin I^n_{r_{\dot c}(n)}$$

Since C_ω is countable there is an uncountable $Y \subseteq X$ such that for some $q \leq p^*$ and $m < \omega$, $q_r = q$ and $m_r = m$ for all $r \in Y$. Choose $n \geq m$ such that $\mathrm{dom}(q) \subseteq \{0,1\}^{<n}$ (remember that the elements of C_ω are now finite partial functions from $\{0,1\}^{<\omega}$ into ω). Let r^k $(k < 2^{n+1})$ be a 1-1 sequence of elements of Y such that for some $t \in \{0,1\}^n$

(5) $$r^k \restriction n = t \quad \text{for all} \quad k < 2^{n+1},$$

and pick an $\ell > n$ such that

(6) $$\Delta(r^i, r^j) < \ell \quad \text{for all} \quad i < j < 2^{n+1}.$$

Extend q to q^* with the domain

$$\mathrm{dom}(q) \cup \{t\} \cup \{r^k \restriction \ell : k < 2^{n+1}\}$$

such that

(7) $$q^*(t) = \ell \quad \text{and} \quad q^*(r^k \restriction \ell) = k \quad \text{for} \quad k < 2^{n+1}.$$

Then

$$q^* \Vdash \bigcup_{k<2^{n+1}} I^n_{r^k_{\dot c}(n)} \supseteq I \setminus \{i/2^{n+1} : 0 \leq i \leq 2^{n+1}\}$$

contradicting (3). This finishes the proof. $\square$

REMARK For more about subsets of the plane with similar properties the reader is referred to [Pawlikowski].

3. THE SUSLIN HYPOTHESIS

In this section we make a connection between Cohen reals and the Suslin Hypothesis. Let us first recall the basic definitions about trees which are relevant to the Suslin Hypothesis (for more about this see Appendix D).

DEFINITION 3.1 *A tree is a partially ordered set T such that for every $x \in T$ the set $\{y \in T : y < x\}$ is well-ordered. A maximal linearly ordered subset of T is called a branch. The height of some $x \in T$ (denoted by $\mathrm{ht}(x)$) is the order type of the set $\{y \in T : y < x\}$. The height (or length) of a branch $B \subseteq T$ is the ordinal $\{\mathrm{ht}(x) : x \in B\}$. The height of a tree is defined in the same way. For an ordinal α the αth level of T is the set $T(\alpha) = \{x \in T : \mathrm{ht}(x) = \alpha\}$. An ω_1-tree is a tree of height ω_1 having countable levels. A tree is an Aronszajn tree iff it is an ω_1-tree without uncountable branches. An antichain in a tree is the set of mutually incomparable elements. A tree is Suslin iff it is an ω_1-tree without uncountable chains or antichains. A tree is special iff it is representable as a countable union of its antichains. The Suslin Hypothesis (SH) is the statement that every linearly ordered continuum satisfying the Suslin condition is isomorphic to $\mathbb{R}$.*

It is well-known (Kurepa) that this statement is equivalent to the statement that there are no Suslin trees.

LEMMA 3.1 There is a function $\rho_1 : [\omega_1]^2 \to \omega$ such that for all $\alpha \leq \beta < \omega_1$ and $n < \omega$:

(1) $F_n(\alpha) = \{\xi < \alpha : \rho_1(\xi, \alpha) \leq n\}$ is finite,
(2) $D(\alpha, \beta) = \{\xi < \alpha : \rho_1(\xi, \alpha) \neq \rho_1(\xi, \beta)\}$ is finite.

PROOF For $\alpha < \omega_1$ fix an unbounded set $C_\alpha \subseteq \alpha$ such that $C_\alpha = \{\xi\}$ if $\alpha = \xi + 1$ and $\mathrm{tp}(C_\alpha) = \omega$ for limit α. Recursively define $\rho_1 : [\omega_1]^2 \to \omega$ by

$$\rho_1(\alpha, \beta) = \max\{|C_\beta \cap \alpha|, \rho_1(\alpha, \min(C_\beta \setminus \alpha))\},$$

where we put $\rho_1(\alpha, \alpha) = 0$. Let us inductively check the conditions (1) and (2).

(PROOF OF 1) Suppose that $F_n(\beta)$ is infinite and let δ be its first limit point. Let $\gamma = \min(C_\beta \setminus \delta)$. Notice that (because of the first factor in the definition of

ρ_1), we must have that $\delta < \beta$ and therefore that $\gamma < \beta$. Notice also that by the definition of ρ_1

(3) $\rho_1(\xi,\beta) = \rho_1(\xi,\gamma)$ for every ξ in the interval $(\max(C_\beta \cap \delta), \delta)$ such that $\rho_1(\xi,\gamma) > n$.

[Since $C_\beta \cap \delta$ is finite and δ is a limit ordinal, the interval $(\max(C_\beta \cap \delta), \delta)$ is infinite.] It follows that $F_n(\gamma)$ must also be infinite, contradicting the inductive assumption.

(PROOF OF 2) Suppose now that $D(\alpha,\beta)$ is infinite and let δ be its first limit point. Let

$$n = |C_\beta \cap \delta|, \qquad \eta = \max(C_\beta \cap \delta) \quad \text{and} \quad \gamma = \min(C_\beta \setminus \delta)$$

($\max\emptyset = 0$). Then $\gamma < \beta$, and

(4) $\rho_1(\xi,\gamma) = \rho_1(\xi,\beta)$ for all $\xi \in (\eta,\delta)$ such that $\rho_1(\xi,\gamma) > n$.

It follows that

$$(D(\alpha,\beta) \setminus F_n(\gamma)) \cap (\eta,\delta) \subseteq D(\alpha,\gamma).$$

Since $F_n(\gamma)$ is finite, this means that $D(\alpha,\gamma)$ must be infinite contradicting the inductive hypothesis. □

LEMMA 3.2 There is a function $e: [\omega_1]^2 \to \omega$ such that for all $\alpha < \beta < \gamma < \omega_1$:

(a) $e(\alpha,\gamma) \neq e(\beta,\gamma)$, and
(b) $\{\xi < \alpha : e(\xi,\alpha) \neq e(\xi,\beta)\}$ is finite.

PROOF The function ρ_1 does not necessarily satisfy the condition (a), so we need to stretch it up as follows:

$$e(\alpha,\beta) = 2^{\rho_1(\alpha,\beta)}(2|F_{\rho_1(\alpha,\beta)}(\alpha)| + 1).$$

Then it is easily seen that the conditions (a) and (b) are satisfied. □

Now we show how to construct an ω_1-tree using the function e of Lemma 3.2.

DEFINITION 3.2 *For a given function $a: [\omega_1]^2 \to \omega$ and $\beta < \omega_1$ let a_β denote the function $a(\cdot,\beta)$, and set*

$$T(a) = \{a_\beta \restriction \alpha : \alpha \leq \beta < \omega_1\}.$$

This $T(a)$ is a tree under the inclusion ordering.

EXERCISE Prove that every ω_1-tree is of this kind for some function a.

DEFINITION 3.3 *For a real $r \in \omega^\omega$ define $e_r: [\omega_1]^2 \to \omega$ by the formula*

$$e_r(\alpha,\beta) = r(e(\alpha,\beta)).$$

Notice that this gives us an ω_1-tree $T(e_r)$ for every real r. This correspondence has the following extreme cases:

(1) If r is a constant function, then $T(e_r) \cong (\omega_1, <)$.
(2) If r is an identity function, then $T(e_r) \cong T(e)$; hence in particular it is an Aronszajn tree.

For the purpose of this section let the *Baire space* ω^ω play the role of the reals and let now C_ω be the poset of all finite partial functions mapping ω into ω (rather than into $\{0,1\}$ as in Example 1.1). Clearly, this is an unessential change (see Appendix C) and a Cohen real is now a member of the Baire space rather than the Cantor cube.

THEOREM 3.1 If c is a Cohen real, then $T(e_c)$ is a Suslin tree.

PROOF To prove that $T(e_c)$ has the Suslin property, we have to prove that in every uncountable set $A \subseteq \omega_1$ there are $\alpha < \beta \in A$ such that $(e_c)_\alpha$ and $(e_c)_\beta$ are comparable in $T(e_c)$, i.e. that $(e_c)_\alpha \subseteq (e_c)_\beta$. It suffices to prove that the set

$$D_A = \{p \in C_\omega : (\exists\alpha,\beta \in A)\ p \Vdash (e_c)_\alpha \subseteq (e_c)_\beta\}$$

is dense in C_ω. Notice that for $\alpha < \beta$ and $p \in C_\omega$, $p \Vdash (e_c)_\alpha \subseteq (e_c)_\beta$ if

$$(*) \quad (\forall\xi < \alpha)\ e(\xi,\alpha) \neq e(\xi,\beta)$$
$$\to (e(\xi,\alpha) \in \mathrm{dom}(p)\ \&\ e(\xi,\beta) \in \mathrm{dom}(p)\ \&\ p(e(\xi,\alpha)) = p(e(\xi,\beta))).$$

Fix some $p \in C_\omega$. We may suppose that $\mathrm{dom}(p) = \{0,1,\ldots,n\}$. For $\alpha \in A$ set

$$F_n(\alpha) = \{\xi < \alpha : e(\xi,\alpha) \leq n\}.$$

Apply the Δ-system Lemma on the set $\{F_n(\alpha) : \alpha \in A\}$ and get an uncountable set $A' \subseteq A$ such that $\{F_n(\alpha) : \alpha \in A'\}$ forms a Δ-system with the root F. We may choose $\alpha < \beta$ from A' so that for all $\xi \in F$ we have $e(\xi,\alpha) = e(\xi,\beta)$. Now fix $q \supseteq p$ so that $\mathrm{dom}(q)$ includes all $\xi < \alpha$ such that $e(\xi,\alpha) \neq e(\xi,\beta)$ (remember that this set is finite) and such that it corrects the finite disagreement of $e(\cdot,\alpha)$ and $e(\cdot,\beta)$ i.e. satisfies (*). □

Recall that a set of reals A has the *Property of Baire* iff there is an open set U such that $A \Delta U$ is meager (see also Appendix C.2).

EXERCISES Let

$$A = \{r \in \omega^\omega : T(e_r) \text{ is an Aronszajn tree}\}$$
$$S = \{r \in \omega^\omega : T(e_r) \text{ is a special Aronszajn tree}\}$$
$$B = \{r \in \omega^\omega : T(e_r) \text{ has an uncountable chain}\}.$$

Clearly, $S \subseteq A$, $A \cup B = \omega^\omega$, and $A \cap B = \emptyset$.

(1) What are the complexities of A, B and S relative to the complexity of the function e?

(2) Show that if ω_1 is not inaccessible in the constructible universe then one can choose e in such a way that S and B are Σ^1_3 sets of reals and that A is Π^1_3 set of reals. Thus, if moreover every Aronszajn tree is special then $A = S$ and B are Δ^1_3 sets of reals. (It is known that every Aronszajn tree is special if for example we assume $\mathrm{MA}(\aleph_1)$; for the definition of $\mathrm{MA}(\aleph_1)$ see §7 of this text).

(3) Notice that Theorem 3.1 and (2) together show that $\mathrm{MA}(\aleph_1)$ implies that every nonmeager set with the property of Baire intersects both A and B. It follows that in this case the sets A and B do not have the property of Baire. Find a minimal assumption which implies that A and B do not have the property of Baire.

4. RVM AND RANDOM REALS

In this section we shall use an additional axiom to prove that there is a compact space X of countable cellularity whose square does not have this property.

DEFINITION 4.1 *Let RVM (Real-Valued Measure) denote the hypothesis that there is a σ-additive extension of the Lebesgue measure to all sets of reals.*

RVM has its origin in Banach's reformulation of a problem posed by Lebesgue early in this century after Vitaly proved that no such extension can be translation invariant. It is known (see [Jech]) that this axiom is relatively consistent with the usual axioms of Set Theory.

DEFINITION 4.2 *The cellularity, $c(X)$, of a topological space X is the supremum of cardinalities of all disjoint families of open sets of X. A topological space is Suslin (or ccc) iff it has countable cellularity. Thus, a space is Suslin iff the poset of its open sets ordered by $\subseteq$ is Suslin.*

EXAMPLE 4.1 Let T be a tree without terminal nodes and let X be the set of all maximal chains of T. For $x \in T$ define $U_x = \{b \in X : x \in b\}$. Let the topology of X be generated by the sets U_x for $x \in T$. Every antichain in T naturally corresponds to a family of disjoint open sets of X and conversely. It follows that if there is a Suslin tree then there is a ccc space X such that X^2 is not ccc. To see this, for every x in T choose two incomparable successors x_0 and x_1 and consider the family $U_{x_0} \times U_{x_1}$ $(x \in T)$.

If r is a real, then for $X \subseteq \mathbb{R} = \{0,1\}^\omega$ define a partition $[X]^2 = K_0^r \cup K_1^r$ by

$$\{x, y\} \in K_i^r \quad \text{iff} \quad r(\Delta(x, y)) = i \qquad (i = 0, 1).$$

The sets K_0^r and K_1^r are both open, and thus clopen. We say that a set A is 0-r-homogeneous iff $[A]^2 \subseteq K_0^r$. Now for a set of reals X define another set of reals

$$\mathcal{F}_X = \{r \in \mathbb{R} : \text{there is a partition } X = \bigcup_{n \in \omega} X_n$$
$$\text{and every } X_n \text{ is closed and 0-}r\text{-homogeneous.}\}$$

REMARK (A digression) It is interesting that this object can also be used in proving the following three well-known results of [Shelah]. Their proofs can be found e.g. in [Bekkali].

THEOREM 4.1 If X is well-orderable and uncountable, then $\mathcal{F}_X$ is not measurable. □

Thus, the existence of a well-orderable uncountable set of reals is the only form of AC needed to obtain a nonmeasurable set of reals.

THEOREM 4.2 If $X = L \cap \mathbb{R}$ (X is the set of all constructible reals) then either

(1) There is a Σ^1_2 nonmeasurable set of reals, or
(2) The set $\mathcal{F}_X$ is nonmeasurable. □

The set $\mathcal{F}_X$ is obviously Σ^1_3, so we have:

COROLLARY If every Σ^1_3 set of reals is measurable, then ω_1 is inaccessible in the constructible universe. □ (End of digression).

Returning to the partition $[X]^2 = K^r_0 \cup K^r_1$, we define

$$T^r_i(X) = \{A \subseteq X : [A]^2 \subseteq K^r_i\} \qquad (i = 0, 1).$$

Now $T^r_i(X) \subseteq \{0,1\}^X$, and in the product topology induced from $\{0,1\}^X$ both $T^r_i(X)$ are closed and therefore compact spaces. We shall prove that they are Suslin spaces, too, if r is a random real (Lemma 4.2). On the other hand we have that

LEMMA 4.1 For $T^r_0(X)$ and $T^r_1(X)$ as defined, the space $T^r_0(X) \times T^r_1(X)$ is never Suslin. Moreover, cellularity of this space equals $|X|$.

PROOF The only assumption on K^r_0 and K^r_1 that we shall use is that they are disjoint. Define

$$[x]_i = \{A \subseteq X : x \in A,\ A^2 \subseteq K^r_i\} \qquad (i = 0, 1).$$

Now notice that for any two distinct reals x and y a pair $\{x, y\}$ is either in K^r_0 or in K^r_1. So either $[x]_0 \cap [y]_0$ is empty or $[x]_1 \cap [y]_1$ is empty and

$$(\forall x, y \in \mathbb{R})\, [x]_0 \times [x]_1 \cap [y]_0 \times [y]_1 = \emptyset;$$

hence the family $\{[x]_0 \times [x]_1 : x \in X\}$ is a family of disjoint clopen sets of cardinality $|X|$. We have proved that $c(T^r_0(X) \times T^r_1(X)) \geq |X|$. But the weight of the space 2^X is equal to $|X|$, consequently $c(T^r_0(X) \times T^r_1(X)) \leq |X|$. □

It is consistent with ZFC that the product of two ccc spaces is always ccc (see Corollary to Theorem 7.2). Also there is a bound for the cellularity of a product of two ccc spaces (see [Kurepa]):

THEOREM 4.3 The product of two ccc spaces has cellularity of at most $\mathfrak{c}$ (the cardinality of the continuum).

[Thus setting $|X| = \mathfrak{c}$ we get the maximal possible cellularity.]

PROOF Suppose that there is a family $\mathcal{F}$ of $\mathfrak{c}^+$ mutually nonintersecting basic open sets in $X \times Y$. By taking subsets, we may assume that $\mathcal{F} = \{X_\xi \times Y_\xi\}_{\xi < \mathfrak{c}^+}$, where each X_ξ (Y_ξ) is open in X (in Y). Define a partition $p: [\mathcal{F}]^2 \to \{0, 1\}$ by

$$p(X_\xi \times Y_\xi, X_\eta \times Y_\eta) = 0 \qquad \text{iff} \qquad X_\xi \cap X_\eta \neq \emptyset.$$

Now by the partition theorem $\mathfrak{c}^+ \to (\aleph_1)^2_2$ (see Definition 6.2 and Theorem 6.5) there is an uncountable homogeneous set U for c. It is easy to check that if $p''[U]^2 = \{0\}$ then Y is not ccc, and that in the other case X is not ccc. □

REMARK It is known that in general the cellularity is not preserved in products of topological spaces (see [Todorčević 1986]).

LEMMA 4.2 If r is a random real, then the spaces $T^r_i(X)$ ($i = 0, 1$) are Suslin.

REMARK Notice that this is not true for every r—for example, it is easy to see that if r is rational, then one of the spaces $T^r_i(X)$ is Suslin while the other is not.

PROOF The base of a space $T^r_i(X)$ is the set of intervals $\{[p] : [p]^2 \subseteq K^r_i, |p| < \omega\}$; hence $T^r_i(X)$ is Suslin iff a poset

$$\mathcal{P}^r_i = \{A \subseteq X : [A]^2 \subseteq K^r_i \ \&\ A \text{ is finite}\}$$

is Suslin. Therefore we have to prove that for every uncountable family $\mathcal{F} \subseteq \mathcal{P}^r_i$ there are $F, G \in \mathcal{F}$ such that $[F \cup G]^2 \subseteq K^r_i$. By applying Δ-system Lemma and throwing out the root, we see that it is enough to prove that this holds for every disjoint family $\mathcal{F}$. After translating this fact to the language of forcing ($\mathcal{R}_\omega$ is the family of Lebesgue-measurable sets of reals of a positive measure), we get:

CLAIM For every $a \in \mathcal{R}_\omega$ and for every $\mathcal{R}_\omega$-name $\dot{\mathcal{F}}$ for an uncountable family of finite i-homogenous subsets of X ($i = 0, 1$) there are $b \leq a$, $\dot{F}$ and $\dot{G}$ such that

$$b \Vdash \dot{F}, \dot{G} \in \dot{\mathcal{F}} \quad \text{and} \quad \dot{F} \neq \dot{G} \quad \text{and} \quad [\dot{F} \cup \dot{G}]^2 \subseteq K^{\dot{r}}_i.$$

PROOF Fix $i = 0$ to simplify the notation. First notice that we may find a finite set F of elements of X and a set $a_F \subseteq a$ of positive measure such that $a_F \Vdash \check{F} \in \dot{\mathcal{F}}$. To see this, consider any extension $M[G]$ and the interpretation of some $\tau \in \mathcal{F}$; then apply Lemma 2.2. Moreover, we may inductively construct an uncountable disjoint family $\mathcal{F}_0$ of finite sets $F \subset X$ and sets of positive measure $a_F \subseteq a$ such that $a_F \Vdash \check{F} \in \dot{\mathcal{F}}$ for each $F \in \mathcal{F}_0$. Using the counting arguments, we may suppose that every $F \in \mathcal{F}_0$ is of the same size, say m. Thus we may represent a typical F as $F = \langle \alpha^F_j : j < m \rangle$. We need to find F and G in $\mathcal{F}_0$ such that the measure of the set

$$b = a_F \cap a_G \cap \{z \in \{0,1\}^\omega : (\forall x \in F)(\forall y \in G)\, z(\Delta(x, y)) = 0\}$$

is positive.

LEMMA 4.3 Let P be a separable metric space (in our case a subspace of some $\mathbb{R}^m$). If a family $\{a_x : x \in P\}$ of sets in $\mathcal{R}_\omega$ is uncountable then there is a Cauchy sequence $\{x_n\}_{n \in \omega}$ in P such that the set $\bigcap_{n \in \omega} a_{x_n}$ is in $\mathcal{R}_\omega$.

PROOF (Sketch) Use the fact that the metric space of Lebesgue measurable sets modulo the ideal of null sets is separable (where $d(A, B) = \mu(A\Delta B)$), look at its product with P and find there a converging sequence inside $\{\langle a_x, x\rangle : x \in P\}$. $\square$

Use the Lemma to get a sequence $\{F_n\}_{n\in\omega}$ of elements of $\mathcal{F}_0$ such that $b_0 = \bigcap_{n\in\omega} a_{F_n}$ has positive measure and

$$\lim_{n\to\infty} \langle \alpha_0^{F_n}, \ldots, \alpha_{m-1}^{F_n}\rangle = \langle \alpha_0, \ldots, \alpha_{m-1}\rangle$$

for some fixed $\langle \alpha_0, \ldots, \alpha_{m-1}\rangle$ from $\mathbb{R}^m$. Remember that $\mathcal{F}_0$ is a disjoint family, and we may suppose that $\alpha_j^{F_n} \neq \alpha_j$ for all $j < m$ and all $n < \omega$. We consider the following two cases:

1^0 If we suppose that all α_i's are different, then there is an $N \in \omega$ such that

$$\Delta(\alpha_i^{F_k}, \alpha_j^{F_l}) = \Delta(\alpha_i, \alpha_j) \qquad \text{for all } k, l > N \text{ and all distinct } i, j < m.$$

[Take N large enough to assure that $\Delta(\alpha_i^{F_k}, \alpha_i) > \max_{i<j<n} \Delta(\alpha_i, \alpha_j)$ for all $k > N$; see also Fig. 4] It is sufficient to find $k, l > N$ and $b \in \mathcal{R}_\omega$ such that

$$b \Vdash \dot{r}(\Delta(\check{\alpha}_j^{F_k}, \check{\alpha}_j^{F_l})) = 0 \qquad \text{for all } j < m.$$

Moreover, it is sufficient to find a $k > N$ and a $b \in \mathcal{R}_\omega$ such that

$$b \Vdash \dot{r}(\Delta(\check{\alpha}_j^{F_k}, \check{\alpha}_j)) = 0 \qquad \text{for all } j < m;$$

this is because there always exists an $\ell > k$ such that

$$\Delta(\alpha_j^{F_\ell}, \alpha_j) > \Delta(\alpha_j^{F_k}, \alpha_j) - \Delta(\alpha_j^{F_k}, \alpha_j^{F_\ell}) \qquad \text{for all } j < m\text{; see Fig. 4.}$$

Define $S_n \subseteq \omega$ by

$$S_n = \{\Delta(\alpha_j^{F_n}, \alpha_j) : j < m\}.$$

By convergence we have that $\lim_{n\to\infty} \min(S_n) = \infty$; so we may assume that S_n's are actually disjoint. For each n, let p_n be the element of $\mathcal{C}_\omega$ such that $\mathrm{dom}(p_n) = S_n$ and $p_n(j) = 0$ for all $j \in S_n$. Let $[p_n]$ be the corresponding basic open set. Then $\mu([p_n]) = 1/2^m$ for every $n < \omega$. Moreover, $[p_n]$'s are *independent* (i.e. for every finite sequence $n_0 < \cdots < n_{k-1} < \omega$,

$$\mu\left(\bigcap_{i=0}^{k-1} [p_{n_i}]\right) = \frac{1}{2^{m\cdot k}} = \prod_{i=0}^{k-1} \mu([p_{n_i}])).$$

By the Borel–Cantelli lemma it follows that the set $\bigcap_{k=0}^{\infty}\bigcup_{n=k}^{\infty}[p_n]$ has measure 1. In particular, there exist arbitrarily large n such that $[p_n] \cap b_0$ has positive measure. It is now clear that for every such n, $[p_n] \cap b_0$ forces that $[\dot{F}_n \cap \dot{F}_k] \subseteq \dot{K}_0^{\dot{r}}$ for every large enough k. This finishes the discussion of this case.

2^0 The second case, when there are equal numbers among α_j's (say, two equal numbers), is proved analogously to the first case. $\square$ (Claim)

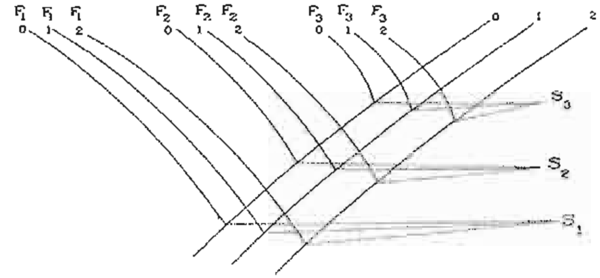

Fig. 4

Lemma 4.2 immediately follows. $\square$

REMARK Note that the same proof gives us that the space $(T_1^r(X))^n$ is Suslin for all $n \in \omega$. Note also that the disjoint sum of spaces $T_0^r(X)$ and $T_1^r(X)$ is Suslin, while its square is not.

We have two Suslin spaces $T_0^r(X)$ and $T_1^r(X)$ whose product is not Suslin, but this takes place in a forcing extension. Now we show how this can be pulled down to ground model, using the assumption RVM. So, let μ be a σ-additive extension of the usual probability measure of $\mathbb{R} = \{0,1\}^\omega$ to all sets of reals. Let

$$I_\mu^+ = \{A \subseteq \mathbb{R} : \mu(A) > 0\}$$

We consider I_μ^+ as a partially ordered set ordered by the inclusion. It is clear that this is a ccc poset and using a rather deep result of Measure Theory (Maharam's Theorem, see [Fremlin 89]) it can be shown that the forcing with I_μ^+ is in fact the same as the forcing with the measure algebra of some Cantor cube $\{0,1\}^\theta$ equipped with its Haar measure. We shall not use this fact here.

DEFINITION 4.3 *Let $\dot{r}$ be the I_μ^+-name for the element of $\{0,1\}^\omega$ such that for every $n < \omega$, $m < 2$ and $B \in I_\mu^+$,*

$$B \Vdash \dot{r}(\check{n}) = \check{m} \qquad \text{iff} \qquad \mu(\{x \in B : x(n) \neq m\}) = 0.$$

Since μ extends the usual measure of $\mathbb{R}$ it is easily seen that every condition of I_μ^+ forces that $\dot{r}$ avoids every G_δ measure zero set coded in the ground model, i.e. that $\dot{r}$ is a random real. In fact, by essentially rewriting the proof of Lemma 4.2 we get the following fact about any forcing extension of the poset I_μ^+ and any ground model set of reals X.

LEMMA 4.2* The poset I_μ^+ forces that spaces $T_0^{\dot{r}}(X)$ and $T_1^{\dot{r}}(X)$ are Suslin. $\square$

THEOREM 4.4 (RVM) *If X is of size smaller than the first weakly inaccessible cardinal, then there is a real r such that both spaces T_0^r and T_1^r are Suslin.*

COROLLARY RVM implies that the countable cellularity is not invariant under products of topological spaces. In fact, not even the first weakly inaccessible cardinal can bound the cellularity of the product of any two Suslin spaces. $\square$

PROOF Suppose that such r does not exist for some "small" uncountable $X \subseteq \mathbb{R}$.

Let θ be a large enough cardinal number such that the structure H_θ (see Definition B.1.1) is correct about all statements of our interest here. (For example, $\theta = (2^{\mathfrak{c}})^+$ works). We shall actually consider the extended structure $(H_\theta, \in, <_w)$, where $<_w$ is a well-ordering of H_θ. (This well-ordering will enable us to take the "Skolem hull", Hull(A), of a given subset containing A in the sense that every existential statement is witnessed by a $<_w$-minimal element of H_θ which must be put inside Hull(A) whenever its parameters are there; see Appendix B.2).

Let $M_0 = \mathrm{Hull}(X \cup \{X, \mu\})$, where μ is a fixed measure defined on all sets of reals and extending the usual probability measure on $\mathbb{R} = \{0,1\}^\omega$. Then the size of M_0 is equal to $|X| + \aleph_0$ (see Lemma B.2.4) and the smallness of X is chosen to mean in particular that the additivity of μ is much bigger than the size of X. So the intersection

$$(*) \qquad \bigcap \{A \subseteq \mathbb{R} : A \in M_0 \ \&\ \mu(A) = 1\}$$

is nonempty and we can fix a real r in it. Let

$$G = \{A \in I_\mu^+ : A \in M_0 \ \&\ r \in A\}.$$

Clearly, G is a filter of the poset $I_\mu^+ \cap M_0$.

FACT 1 If $\mathcal{D} \in M_0$ is a dense open subset of I_μ^+ then $G \cap \mathcal{D} \neq \emptyset$.

PROOF Note that I_μ^+ is a ccc poset so there is $\langle A_n : n < \omega\rangle \in M_0$ such that $\{A_n : n < \omega\}$ is a maximal antichain of I_μ^+ which is included in $\mathcal{D}$. Let

$$B = \mathbb{R} \setminus \bigcup_{n<\omega} A_n.$$

Then $B \in M_0$ and $\mu(B) = 0$; hence $r \notin B$ and therefore there is an n such that $r \in A_n$. It follows that $A_n \in \mathcal{D} \cap G$. □

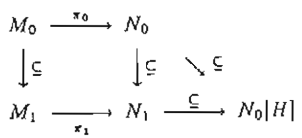

Fig. 5

Let N_0 be a transitive collapse of M_0 (see Theorem A.4.2) and let $\pi_0 : M_0 \to N_0$ be the collapsing map (see Fig. 5). By Fact 1,

$$H = \{\pi_0(A) : A \in G\}$$

is a N_0-generic filter of the poset $\pi_0(I_\mu^+)$. hence, we can form the generic extension

$$N_0[H] = \{\mathrm{int}_H(\tau) : \tau \text{ is a } \pi_0(\mathcal{I}_\mu^+)\text{-name in } N_0.\}$$

FACT 2 (With $\dot r$ from Definition 4.3 and r from the set (*)) $\mathrm{int}_H(\pi_0(\dot r)) = r$.

PROOF Let $s = \mathrm{int}_H(\pi_0(\dot r))$ and suppose that for some $n < \omega$ (say) $s(n) = 0$ but $r(n) = 1$. By the definition of the name $\dot r$ and the fact that π_0 is an isomorphism, it follows that

$$B = \{x \in \pi_0(\mathbb{R}) : x(n) = 0\}$$

is an element of H, so its preimage $A = \pi_0^{-1}(B)$ is an element of G, which means that r is an element of A. But notice that (since π_0 is an isomorphism)

$$A = \{x \in \mathbb{R} : x(n) = 0\},$$

a contradiction. □

Notice that $\pi_0(X) = X$, so by Lemma 4.2* (applied to N_0 as the ground model and the poset $\pi_0(I_\mu^+) = J_{\pi_0(\mu)}^+$), we have that

$$N_0[H] \models \text{“}T_0^r(X) \text{ and } T_1^r(X) \text{ are Suslin.”}$$

Consider now the following submodel of H_θ:

$$M_1 = \mathrm{Hull}(M_0 \cup \{r\})$$

Let $\pi_1 : M_1 \to N_1$ be its transitive collapse.

CLAIM $N_1 \subseteq N_0[H]$.

PROOF By Lemma B.2.4, for every element x of M_1 there is a formula ϕ and a sequence $\bar a$ of elements of M_0 such that $\phi(x, r, \bar a)$ holds and x is the unique element satisfying this. Let f_x be the function mapping $\mathbb{R}$ to M_0 defined by

$$f_x(s) = y \quad \text{iff} \quad \phi(y, s, \bar a).$$

REMARK Notice that ϕ and $\bar a$ are not unique; on the other hand, we need an uniform way of defining f_x for each $x \in M_1$. But formulas can be coded by elements of V_ω (see e.g. [Kunen]) and we can always take the $<_w$-least pair $\langle \phi, \bar a\rangle$ in M_0.

Then f_x is clearly an element of M_0 since its defining parameters $\bar a$ also are from M_0. Note that for any x and y in M_1, the following four (also another four in parentheses) conditions are equivalent:

a. $x \in y$ (respectively $x = y$),
b. $f_x(r) \in f_y(r)$ (respectively $f_x(r) = f_y(r)$),
c. $\{s \in \mathbb{R} : f_x(s) \in f_y(s)\} \in G$ (respectively, $\{s \in \mathbb{R} : f_x(s) = f_y(s)\} \in G$), and
d. $\{s \in \pi_0(\mathbb{R}) : \pi_0(f_x)(s) \in \pi_0(f_y)(s)\} \in H$
(respectively, $\{s \in \pi_0(\mathbb{R}) : \pi_0(f_x)(s) = \pi_0(f_y)(s)\} \in H$).

[To prove b. ⇔ c. notice that the set in c. is, by RVM, measurable; also consider the remark after Definition 2.2.]

This means that $(M_1, \in)$ is isomorphic to some structure $\langle U, E\rangle$ where $U \subseteq N_0$ and $E \in N_0[H]$, which is (at least), locally definable in $N_0[H]$; so its transitive collapse, and therefore the transitive collapse of $(M_1, \in)$ must be included in $N_0[H]$ (see Theorem A.4.3).) □

Note that $\pi_1(r) = r$ $(= \mathrm{int}_H(\pi_0(\dot r)))$, and that $\pi_1(X) = \pi_0(X) = X$, so the partition

$$[X]^2 = K_0^r \cup K_1^r$$

is an element of N_1. Since $N_0[H]$ is a ccc forcing extension of N_0, it follows that $\omega_1^{N_0}(= \omega_1^{N_1}) = \omega_1^{N_0[H]}(= \omega_1)$, so both pieces of this partition must be Suslin also in the smaller model N_1. So

$$N_1 \models \text{“}T_0^r(X) \text{ and } T_1^r(X) \text{ are Suslin”}$$

and, therefore

$$M_1 \models \text{“}T_0^r(X) \text{ and } T_1^r(X) \text{ are Suslin”},$$

contradicting the fact that M_1 is an elementary submodel of $(H_\theta, \in, <_w)$ which thinks that there is no real r such that $T_0^r(X)$ and $T_1^r(X)$ are both Suslin. This finishes the proof. □

Here is another consequence of RVM related to Theorem 4.4:

THEOREM 4.5 (RVM) There exists a compact Suslin space Y such that the cellularity of the space Y^{n+1} is $\aleph_n$ for all $n \in \omega$.

The proof is very similar to the proof of Theorem 4.4. Firstly we prove the following Lemma:

LEMMA 4.4 (In the presence of a random real r) For every natural number n there exists a compact space Y such that Y^k is Suslin for all $k \leq n$, while $c(Y^{n+1}) = \aleph_n$.

PROOF (Lemma) Now we identify $\mathbb{R}$ with the set $(n+1)^\omega$. Using the fact that if RVM holds then the continuum is quite large (e.g. by Ulam's Theorem the additivity of the extension measure is a weakly inaccessible cardinal), we choose for A_n a set of reals of cardinality $\aleph_n$. We define the following partition (r is the random real):

$$[A_n]^2 = \bigcup_{i \leq n} K_i^r \qquad \text{with}$$

$$\{x_0, x_1, \ldots, x_n\} \in K_i^r \qquad \text{iff} \qquad r(\Delta(x, y)) \neq i.$$

[Notice that $K_i^r \cap K_j^r$ need not be empty even for distinct i and j.] Define an i-r-homogeneous set in the obvious way, and let T_i^r (for $i = 0, 1, \ldots, n$) be a set of all i-r-homogeneous subsets of A_n.

CLAIM 1 The product $\prod_{j<k} T_{i_j}^r$ ($i_j \leq n$ for all j) is Suslin if there is $m < n$ such that $m \neq i_j$ for all $j < k$. Specifically, it is Suslin if $k < n$.

PROOF (Sketch) The proof of Lemma 4.2 works here with some minor modifications—for example, Lemma 4.3 is applied with $M = \mathbb{R}^{k \cdot n}$, while all the refinements of a family $\mathcal{F}$ are done in the same way. Use the fact that there is an $m \leq n$ such that $m \neq i_j$ for all $j < k$. Define an analogue of the set S_n, and find a sufficiently large n to make the set

$$\{z \in b_0 : (\forall j \in S_n)\ z(j) = m\}$$

of positive measure. The rest of the proof is the same as the proof of Lemma 4.2. □

CLAIM 2 The product $\prod_{i \leq n} T_i^r$ is of cellularity $\aleph_n$.

PROOF (cf. Lemma 4.1) For every $x \in A_n$ and $i \leq n$ define the following subset of $\prod_{i \leq n} T_i^r$:

$$[x]_i = \{A \subseteq A_n : x \in A,\ [A]^2 \subseteq K_i^r\}.$$

These sets are open, and the family

$$\{[x]_0 \times \cdots \times [x]_n : x \in A_n\}$$

is disjoint and has cardinality $\aleph_n$. □

Take for Y_n the disjoint sum of spaces T_i^r ($i = 0, 1, \ldots, n$). The space Y_n^k is the disjoint finite union of spaces of the form $\prod_{j<k} T_{i_j}^r$ ($i_j \leq n$, with possible repetitions). By Claims 1 and 2, Y_n satisfies the statement of the lemma. □

PROOF (Theorem 4.5) Now we need $\aleph_0$ many random reals, $\langle r_i : i \in \omega \rangle$. Consider the space $\{0,1\}^{\omega \cdot \omega}$ with the Haar measure. Define reals r_n in the following way (f_G denotes generic function):

$$r_n(k) = f_G(\omega \cdot n + k), \quad \text{for } k < \omega$$

or, in other words, r_n is the isomorphic copy of $f_G \restriction [\omega \cdot n, \omega \cdot (n+1))$.

REMARK (A digression) The sequence $\langle r_i : i \in \omega \rangle$ consists of *independent* random reals iff each r_i is a random real over $M[r_0][r_1] \ldots [r_{i-1}]$. It is not difficult to prove that the sequence just defined has the required property. Notice that the space $\{0,1\}^{\omega \cdot \omega}$ is isomorphic to $\{0,1\}^\omega$; thus in any generic extension generated by adding a random real there are countably many independent random reals.

Define Y_n as in Lemma 4.4, taking care that all A_n be disjoint. Take for Y the disjoint sum of spaces Y_n, $n \in \omega$. We already know that Y is Suslin and by Claim 2 the cellularity of Y^{n+1} is at least $\aleph_n$. To see that it is not greater than $\aleph_n$ it suffices to check that each product of the kind

$$\prod_{j<k} T_{i_j}^{r_j} \qquad i_j \in \omega \quad \text{and} \quad r_j \in \{r_n : n \in \omega\} \qquad \text{is}$$

(1) Suslin, if there is no $n \in \omega$ such that all $T_i^{r_n}$ ($i = 0, 1, \ldots, n$) occur in the product, and
(2) of cellularity $\aleph_k$, if k is the greatest n such that all $T_i^{r_n}$ ($i = 0, 1, \ldots, n$) occur in the product.

But this is just another variant of the Lemma 4.2 with no new ideas involved.

So we have shown that the spaces $T_i^{r_n}$ exist in any generic extension of I_μ^+. It remains to consider the sentence

“There exists a real r that encodes a sequence of reals $\langle r_n : n \in \omega \rangle$ such that for every $k \in \omega$ the cellularity of a product $\prod_{j<k} T_{i_j}^{r_j}$ is determined by (1) and (2)”,

and proceed as in the proof of Theorem 4.4. □

REMARK The reader has certainly noticed that we have proved more than was stated in Theorem 4.6: For every nondecreasing function $f:\omega \to \omega$ there is a Suslin space Y such that $c(Y^{n+1}) = \aleph_{f(n)}$ for every positive integer n.

REMARK In the light of Example 4.1 presented above it is interesting that RVM does not imply that there is a Suslin tree, so the examples of this section had to come from a combinatorially quite different source.

5. BOREL EQUIVALENCE RELATIONS

In this section we give an application of forcing to an area of Descriptive Set-theory which has some interesting interpretations in other mathematical disciplines such as Ergodic Theory and Operator Algebra (see [Kechris]).

DEFINITION 5.1 *A relation* E *on* $\mathbb{R}$ *is Borel (analytic, coanalytic, etc). Borel (analytic, coanalytic, ...) if it is a Borel (analytic, coanalytic, ...) subset of* $\mathbb{R}$.

A typical example of a Borel equivalence relation on the Cantor set $\{0,1\}^\omega$ is the following:

$$x \mathrm{E}_0 y \quad \text{iff} \quad (\exists m)(\forall n \geq m) x(n) = y(n).$$

Another natural way to generate a Borel equivalence relation is to take an arbitrary sequence $\{B_n\}$ of Borel subsets of $\mathbb{R}$ and define

$$x \mathrm{E}_1 y \quad \text{iff} \quad \forall n (x \in B_n \leftrightarrow y \in B_n).$$

How typical are these two examples can be seen from the following dichotomy result which says that every other Borel equivalence relation is "in between" these two (see [Harrington–Kechris–Louveau]).

THEOREM 5.1 If E is a Borel equivalence relation on $\mathbb{R}$ then either

(1) there is a sequence $\{B_n\}$ of Borel sets such that for every x and y in $\mathbb{R}$, $x \mathrm{E}\, y$ iff $\forall n(x \in B_n \leftrightarrow y \in B_n)$, or
(2) there is a continuous 1-1 function f from the Cantor set into $\mathbb{R}$ such that for every x and y in $\{0,1\}^\omega$, $x \mathrm{E}_0 y \leftrightarrow f(x) \mathrm{E}\, f(y)$. □

In this section we shall prove the following famous corollary of this result originally due to J. Silver (see [Kechris]).

THEOREM 5.2 Every Borel equivalence relation on $\mathbb{R}$ either has countably many equivalence classes or there is a perfect set of pairwise nonequivalent elements.

To see that Theorem 5.1 is stronger than Theorem 5.2, notice that the alternative 5.1(2) gives us a perfect set of nonequivalent elements since it is easy to see that

$\{0,1\}^\omega$ contains a perfect set of pairwise non-E_0-equivalent elements. If 5.1(1) holds, consider the map $H : \mathbb{R} \to \{0,1\}^\omega$ defined by

$$H(x)(n) = 1 \quad \text{iff} \quad x \in B_n.$$

Clearly, H is a Borel map. In §10, Theorem 10.2, we shall see that every such map either has a countable range in which case E has countably many equivalence classes, or there is a perfect set of reals on which H is 1-1, which gives us the second alternative of Theorem 5.2.

The proof of Theorem 5.2 requires a nice representation of analytic subsets on $\mathbb{R}$ and $\mathbb{R}^2$. It will also be convenient if we take the Baire space ω^ω to be our copy of $\mathbb{R}$. (It should be clear that Theorem 5.2 doesn't depend on which of the standard Polish spaces $\mathbb{R}$, $[0,1]$, $\{0,1\}^\omega$ or ω^ω we take as our copy of "the reals").

Let $\omega^{<\omega} \otimes \omega^{<\omega}$ be the set

$$\{\langle s,t\rangle \in \omega^{<\omega} \times \omega^{<\omega} : |s| = |t|\},$$

i.e. the set of all pairs of finite sequences of integers having equal lengths. Notice that $\omega^{<\omega} \otimes \omega^{<\omega}$ is a tree under the ordering of coordinatewise inclusions. Of course, the higher products $\omega^{<\omega} \otimes \omega^{<\omega} \otimes \omega^{<\omega} \otimes \ldots$ are defined analogously. By a "subtree" of $\omega^{<\omega} \otimes \omega^{<\omega}$ or of any of the higher powers we consider a downward closed subset of the corresponding power. To such a tree $T \subseteq \omega^{<\omega} \otimes \omega^{<\omega}$ we associate the set

$$X_T = \{x \in \omega^\omega : (\exists y \in \omega^\omega)(\forall n)(x \restriction n, y \restriction n) \in T\}.$$

In other words, X_T is the set of all x in ω^ω for which the tree

$$T_x = \{t \in \omega^{<\omega} : \langle x \restriction |t|, t\rangle \in T\}$$

has an infinite branch. The set X_T is usually denoted by $p[T]$, the *projection of T*. This is the *tree representation of analytic sets*.

The class of subsets of $\mathbb{R}$ which can be represented in this way coincides with the class of analytic sets. For example, take the classical fact (see [Kuratowski–Mostowski]) that every analytic set $X \subseteq \mathbb{R}$ is a result of the *$\mathcal{A}$-operation* applied to a family $B = \{B_s : s \in \omega^{<\omega}\}$ of closed sets with the property that $B_s \subseteq B_t$ if $t \subseteq s$, i.e.

$$X = \mathcal{A}(B) = \bigcup_{y \in \omega^\omega} \bigcap_{n<\omega} B_{y \restriction n}.$$

To see that the second representation gives the first, let

$$T_X = \{\langle x \restriction |t|, t\rangle \in \omega^{<\omega} \otimes \omega^{<\omega} : B_t \cap [x \restriction |t|] \neq \emptyset\}.$$

(For $s \in \omega^{<\omega}$, $[s]$ is the basic clopen set $\{y \in \omega^\omega : s \subseteq y\}$). To see that the tree representation of analytic sets gives the classical one, for $s \in \omega^{<\omega}$ set

$$B_s = \{y \in \omega^\omega : \langle s, y \restriction |s|\rangle \in T\}.$$

Since every Borel set is analytic (check this!) it follows that every Borel set allows a tree representation.

A subset Y of $\mathbb{R}$ is *coanalytic* if its complement $\mathbb{R} \setminus Y$ is analytic. So fix a tree $T \subseteq \omega^{<\omega} \otimes \omega^{<\omega}$ such that

$$\mathbb{R} \setminus Y = p[T].$$

Thus we have

$$x \in Y \quad \text{iff} \quad x \notin \mathbb{R} \setminus Y \quad \text{iff} \quad T_x \text{ is well-founded}$$

of coanalytic sets (i.e. T_x has no infinite branches). This is the *tree representation of coanalytic sets*. Since the class of Borel sets is closed under the complementation, it follows that every Borel set of reals also has such representation. At this point it seems also appropriate to give a more useful form of the statement "T_x is well-founded". Suppose that T is a "well-founded subtree" of $\omega^{<\omega}$. This means that the poset $\langle T, \supseteq\rangle$ is well-founded in the usual sense, i.e. that it has no infinite decreasing sequences. So there is a uniquely inductively defined rank function $\mathrm{rk} : T \to \mathrm{Ord}$:

$$\mathrm{rk}(t) = \{\mathrm{rk}(s) + 1 : s \supsetneq t\}$$

which is *order-reversing*, i.e. it has the property that $\mathrm{rk}(t) < \mathrm{rk}(s)$ whenever $s \subsetneq t$ are in T. Thus a *terminal node* (i.e. a maximal element in $\langle T, \leq\rangle$) gets rank $0, \ldots$ etc.

Of course, it makes sense talking about tree-representation of Borel, analytic, or coanalytic relations of $\mathbb{R}$ with the obvious adjustments (especially because $(\omega^\omega)^2 \cong \omega^\omega$). Thus, for example, every coanalytic relation $E \subseteq \mathbb{R}^2$ has a subtree $T \subseteq \omega^{<\omega} \otimes \omega^{<\omega} \otimes \omega^{<\omega}$ such that for every x and y in $\mathbb{R}$

$$x E y \quad \text{iff} \quad T_{xy} \text{ is well-founded},$$

where (as expected)

$$T_{xy} = \{u \in \omega^{<\omega} : \langle x \restriction |u|, y \restriction |u|, u\rangle \in T\}.$$

(See Fig. 6. The "real" picture should be four-dimensional, but x and y are represented in the same dimension).

Another technical fact about subtrees T of $\omega^{<\omega}$ one frequently needs is the notion of "*the lexicographically least infinite branch*" of T (if there is one at all). This branch b is defined as follows: Let

$$T^* = \{t \in T : t \text{ is a contained in an infinite branch of } T\}.$$

The branch b will be a subset of T^*, and it will be equal to $\{t_n : n < \omega\}$, where $|t_n| = n$ for all $n < \omega$. Set $t_0 = \emptyset$ and suppose that we know t_n. Let k be the minimal integer such that

$$t_n \cup \{\langle n, k\rangle\} \in T^*,$$

and set $t_{n+1} = t_n \cup \{\langle n, k\rangle\}$. The branch b, or for that matter any other branch, will be identified with the element of ω^ω it determines.

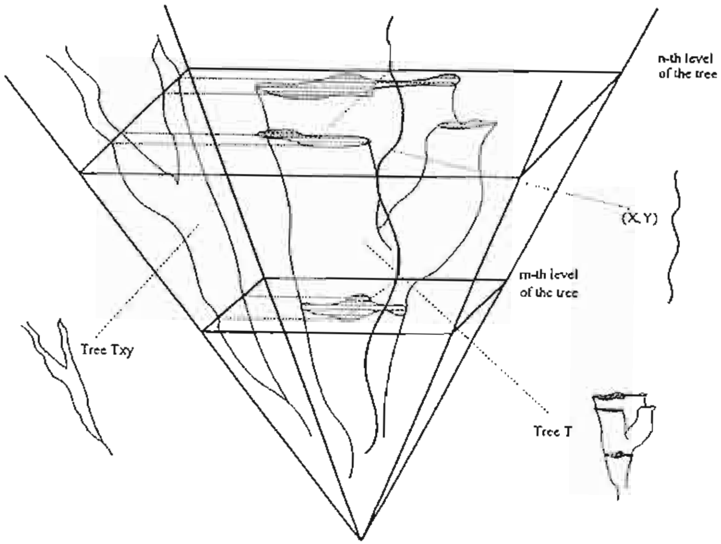

Fig. 6

PROOF (Theorem 5.2) Suppose that E is a Borel equivalence relation and that the set of equivalence classes is uncountable. Fix an uncountable set $A \subseteq \mathbb{R}$ such that $x \not\mathrm{E}\, y$ for every pair $x \neq y$ of elements of A. Let T be a tree representation of E.

Now we are ready to define our poset $\mathcal{P}$ as as the set of all uncountable subsets of A with the inclusion ordering. To a generic filter G on $\mathcal{P}$ we assign a generic real r_G in the following way:

$$r_G(m) = n \quad \text{iff} \quad \{x \in A : x(m) = n\} \in G.$$

The real r_G is well-defined, for the set

$$D_m = \{X \subseteq A : |X| \geq \aleph_1 \ \& \ (\forall x, y \in X) x(m) = y(m)\}$$

is dense for every $m \in \omega$. We say that a real corresponding to a generic filter is generic. Now Theorem 5.2 reduces to the following Lemma about the forcing relation $\Vdash$ of the poset $\mathcal{P} \times \mathcal{P}$.

LEMMA 5.1 For every two uncountable subsets X, Y of A we have that

$$\langle X, Y\rangle \Vdash r_{G_0} \not\mathrm{E}\, r_{G_1}$$

(Here $G_0 \times G_1$ is a generic filter of $\mathcal{P} \times \mathcal{P}$ containing $\langle X, Y\rangle$. The dots above G_0 and G_1 indicating that, in fact, we are dealing with the names for these objects, are omitted to keep the notation simple, as there is no danger of confusion).

PROOF Suppose that there are uncountable X and Y in $\mathcal{P}$ such that $\langle X, Y\rangle \Vdash r_{G_0} \mathrm{E}\, r_{G_1}$ ($\Leftrightarrow$ the tree $T_{r_{G_0} r_{G_1}}$ is well-founded). We fix a bijection $f: X \to X$ such that $x \neq f(x)$ for all $x \in X$. This function naturally defines a real $r_{f(G_0)}$ defined by

$$r_{f(G_0)}(m) = n \quad \text{iff} \quad \{x : f^{-1}(x)(m) = n\} \in G_0.$$

CLAIM 1 $\langle r_{f(G_0)}, r_{G_1}\rangle$ is generic.

PROOF Function $f^*: \mathcal{P}_{\subseteq X} \times \mathcal{P}_{\subseteq Y} \to \mathcal{P}_{\subseteq X} \times \mathcal{P}_{\subseteq Y}$ defined by $f^*(\langle U, V\rangle) = \langle f''U, V\rangle$ is an isomorphism; hence $G_0 \times G_1$ is generic iff $f(G_0) \times G_1$ is generic, and the conclusion follows (where $f(G_0) = \{f''D : D \in G_0\}$). □

From the Claim 1 it follows that $r_{f(G_0)} \mathrm{E}\, r_{G_1}$.

CLAIM 2 $\langle X, Y\rangle \Vdash r_{G_0} \not\mathrm{E}\, r_{f(G_0)}$.

PROOF Otherwise we can find $X_0 \subseteq X$, $Y_0 \subseteq Y$, an ordinal β and a name $\dot{H}$ for a function such that

$$\langle X_0, Y_0\rangle \Vdash \dot{H}: T_{r_{G_0} r_{f(G_0)}} \to \check{\beta}$$

and $\dot{H}$ is decreasing, namely

$$\forall s, t \in T_{r_{G_0} r_{f(G_0)}} (s \subset t \to \dot{H}(s) > \dot{H}(t)).$$

Notice that for every $x \in X_0$ we have $x \not\mathrm{E}\, f(x)$, i.e. the tree $T_{x f(x)}$ is not well-founded; hence we have the lexicographically least branch $b_{x f(x)}$ of this tree. Thus, we can build recursively $\langle X_0, Y_0\rangle \supseteq \langle X_1, Y_1\rangle \supseteq \ldots$, $s_0 \subset s_1 \subset \ldots$ and $\alpha_0, \alpha_1, \ldots < \beta$ such that

$$\langle X_{i+1}, Y_{i+1}\rangle \Vdash \dot{H}(s_i) = \check{\alpha}_i$$

and

$$(\forall x \in X_{i+1}) b_{x f(x)} \upharpoonright (i+1) = s_i.$$

This is clearly a contradiction, since we must have $\alpha_i > \alpha_{i+1}$ (as $\langle X_{i+2}, Y_{i+2}\rangle$ forces it) for all $i < \omega$. □

Thus we have produced three $\mathcal{P} \times \mathcal{P}$ names r_{G_0}, r_{G_1} and $r_{f(G_0)}$ such that $\langle X, Y\rangle$ forces $r_{G_0} \mathrm{E}\, r_{G_1}$, $r_{f(G_0)} \mathrm{E}\, r_{G_1}$ but $r_{G_0} \not\mathrm{E}\, r_{f(G_0)}$. Fix names $\dot{H}_0$ and $\dot{H}_1$ for decreasing functions such that

$$\langle X, Y\rangle \Vdash \dot{H}_0: T_{r_{G_0} r_{G_1}} \to \mathrm{Ord}, \quad \text{and}$$
$$\langle X, Y\rangle \Vdash \dot{H}_1: T_{r_{f(G_0)} r_{G_1}} \to \mathrm{Ord}.$$

Starting with $\langle X, Y\rangle$, we build recursively $\langle X, Y\rangle = \langle X_0, Y_0\rangle \supseteq \langle X_1, Y_1\rangle \supseteq \ldots$ determining more and more of G_0, G_1 and $r_{f(G_0)}$, and more and more of H_0 and H_1, and (as in the previous proof) for some $s_0 \subset s_1 \subset \ldots$ we have $b_{x f(x)} \upharpoonright (i+1) = s_i$ for all $x \in X_{i+1}$. Eventually we get three reals r_0, r_1 and r_f such that $T_{r_0 r_1}$ and $T_{r_f r_1}$ are both well-founded but $T_{r_0 r_f}$ is not, since it contains the infinite branch determined by the s_i's. In other words, $r_0 \mathrm{E}\, r_1$, $r_f \mathrm{E}\, r_1$ and $r_0 \not\mathrm{E}\, r_f$, contradicting the transitivity of E. □ (Lemma 5.1)

REMARK Note that we have just proved the slightly more general fact: that the Borel equivalence relations remain transitive in forcing extensions. This could have also been deduced from a rather deep fact, known under the name of Shoenfield's Absoluteness Theorem, but we have decided for the more direct way in order to make this section as self-contained as possible.

We return to the proof of Theorem 5.2. Remember that $x \not\!E\, y$ iff T_{xy} has an infinite branch. For every pair $x, y \in A$ denote by b_{xy} the lexicographically least infinite branch of T_{xy}. Now we construct a family of uncountable subsets of A,

$$\{X_\sigma : \sigma \in \{0,1\}^{<\omega}\}$$

such that for all σ and τ in $\{0,1\}^{<\omega}$:

(1) $\sigma \subset \tau \to X_\sigma \supset X_\tau$,
(2) for each σ of length n there is $s_\sigma \in \omega^n$ such that $x \restriction n = s_\sigma$ for all x in X_σ,
(3) If $\sigma \neq \tau$ and $|\sigma| = |\tau| = n$ then $s_\sigma \neq s_\tau$, and
(4) if $\sigma \neq \tau$ and $|\sigma| = |\tau| = n$ then there exists $t_{\sigma\tau} \in \omega^n$ such that $b_{xy} \restriction n = t_{\sigma\tau}$ for all $x \in X_\sigma$ and $y \in X_\tau$.

To see how the construction goes on, suppose that all these objects have been determined up to some level n. For each $\sigma \in \{0,1\}^n$ pick uncountable $X'_{\sigma 0}, X'_{\sigma 1} \subseteq X_\sigma$ such that for some distinct $s_{\sigma 0}$ and $s_{\sigma 1}$ in ω^{n+1} we have that $X'_{\sigma 0} \subseteq [s_{\sigma 0}]$ and $X'_{\sigma 1} \subseteq [s_{\sigma 1}]$. Now shrink each X'_σ ($\sigma \in \{0,1\}^{n+1}$) successively to obtain uncountable $X_\sigma \subseteq X'_\sigma$ ($\sigma \in \{0,1\}^{n+1}$) such that for every $\sigma \neq \tau$ in $\{0,1\}^{n+1}$ there is $t_{\sigma\tau}$ in ω^{n+1} such that

$$\langle X_\sigma, X_\tau \rangle \Vdash b_{r_{G_0} r_{G_1}} \restriction (n+1) = t_{\sigma\tau}.$$

(These sets can be found since by the key Lemma 5.1, for every such two σ and τ, $\langle X'_\sigma, X'_\tau \rangle$ forces that $r_{G_0} \not\!E\, r_{G_1}$; so, in particular, it forces that the infinite branch $b_{r_{G_0} r_{G_1}}$ of $T_{r_{G_0} r_{G_1}}$ exists). Notice also that

$$\langle X_\sigma, X_\tau \rangle \Vdash r_{G_0} \restriction (n+1) = s_\sigma \quad \text{and} \quad r_{G_1} \restriction (n+1) = s_\tau$$

so the condition (4) remains satisfied.

For $f \in \{0,1\}^\omega$, set

$$x_f = \bigcup_{n<\omega} s_{f \restriction n}.$$

Notice that by (1), (2) and (4), $P = \{x_f : f \in \{0,1\}^\omega\}$ is a perfect set isomorphic with the Cantor set via $f \mapsto x_f$. By the construction, for every $f \neq g$ in $\{0,1\}^\omega$

$$b_{fg} = \bigcup_{n<\omega} t_{f \restriction n g \restriction n}$$

is an element of ω^ω which is, moreover, an infinite branch of $T_{x_f x_g}$. Hence $x_f \not\!E\, x_g$ for every $f \neq g$ in $\{0,1\}^\omega$, so we are done. □

The reader must have already noticed that we have proved Theorem 5.2 for the wider class of (coanalytic) equivalence relations than claimed. The restriction to coanalytic or Borel equivalence relations in Theorem 5.2 is essential, which is seen from the following classical example (see [Kuratowski–Mostowski]).

EXAMPLE 5.1 (Lebesgue) There is an analytic equivalence relation with exactly $\aleph_1$ equivalence classes. Canonically enumerate $\mathbb{Q}$ as $\{r_n : n \in \omega\}$, and let $x \mathrm{E}_l y$ iff

(1) $\mathbb{Q}_x = \{r_n : x(n) = 1\}$ or $\mathbb{Q}_y = \{r_n : y(n) = 1\}$ is not well–ordered in $\langle \mathbb{Q}, \leq \rangle$, or
(2) $\mathbb{Q}_x$ and $\mathbb{Q}_y$ are well–ordered and isomorphic as ordered sets.

Since both conditions (1) and (2) involve one existential quantifier over subsets of $\mathbb{Q}$, E_l is an analytic subset of $\mathbb{R}^2$. Notice that there are $\aleph_1$ many equivalence classes, for the equivalence classes are in 1-1 correspondence to the set of countable ordinals.

The following result tells us that this must be the case with any other analytic equivalence relation.

THEOREM 5.3 Let E be an analytic equivalence relation; then either $\mathbb{R}/\mathrm{E}$ is of cardinality at most $\aleph_1$ or there is a perfect set of mutually nonequivalent elements.

PROOF This proof is very similar to the proof of Theorem 5.1, with all main lemmas being the same, but with somewhat different proofs. We suppose that there is a set A of mutually nonequivalent elements of size $\aleph_2$. Let

$$\mathcal{P} = \{Y \subseteq A : |Y| = \aleph_2\}.$$

If G is a generic filter on $\mathcal{P}$ then we define a real r_G by:

$$r_G(n) = m \qquad \text{iff} \qquad \{x \in A : x(n) = m\} \in G.$$

By using the fact that E is analytic we find a tree $T \subseteq \omega^{<\omega} \times \omega^{<\omega} \times \omega^{<\omega}$ such that

$$x \mathrm{E} y \qquad \text{iff} \qquad T_{xy} \text{ is not well–founded.}$$

LEMMA 5.2 For every pair $X, Y \in \mathcal{P}$ we have that $\langle X, Y \rangle \Vdash r_{G_0} \not\!E\, r_{G_1}$.

PROOF Suppose that some $\langle X, Y \rangle$ forces $r_{G_0} \mathrm{E}\, r_{G_1}$. Fix a 1-1 function $f : X \to X$ such that $f(x) \neq x$ for all $x \in X$. For all $x \in X$ the tree $T_{xf(x)}$ is well–founded, so we fix an ordinal $\alpha_x < \omega_1$ and a strictly decreasing function $H_x : T_{xf(x)} \to \alpha_x$. By shrinking X assume further that for some α and all x in X, $\alpha_x = \alpha$.

CLAIM 1 $\langle r_{f(G_0)}, r_{G_1} \rangle$ is generic.

PROOF Same as the proof of a similar Claim in the proof of Theorem 5.1. □

CLAIM 2 $\langle X, Y \rangle \Vdash r_{G_0} \not\!E\, r_{f(G_0)}$.

PROOF Note that since the real $r_{f(G_0)}$ is determined by using only G_0 and f, this is really a statement about the forcing extension by G_0, i.e. the Claim is equivalent to showing that $X \Vdash r_{G_0} \not\!E\, r_{f(G_0)}$. So suppose that this is false, and fix a $\mathcal{P}$-name $\dot{b}$ for the lexicographically least infinite branch of $T_{r_{G_0} r_{f(G_0)}}$. Fix some positive integer i and an uncountable subset of X, which we call again X. Then we can always refine X to decide what $\dot{b} \restriction (i+1)$ is. Moreover, we have $\aleph_2$ elements in any such refinement, but only countably many possibilities below α for $H_x(\dot{b} \restriction (i+1))$. Hence we can find an $Y \subseteq X$ of size $\aleph_2$ and $\beta < \alpha$ such that

$$H_x(\dot{b} \restriction (i+1)) = \beta, \quad \text{for all } x \in Y.$$

It follows that, by a successive application of this refining procedure, we can construct an infinite decreasing chain of sets $X \supset X_1 \supset \cdots \supset X_i \supset \ldots$ such that:

(1) $|X_i| = \aleph_2$, X_i determines $\dot{b} \restriction i$, and
(2) $H_x(\dot{b} \restriction i) = \alpha_i$ for all $x \in X_i$ and some ordinal α_i.

But then the sequence of ordinals α_i ($i < \omega$) would be strictly decreasing – a contradiction. □

The reals r_0, r_1 and r_f are constructed and used in finishing the proof similarly as in Lemma 5.1. □

The construction of a family $\{X_\sigma : \sigma \in \{0,1\}^{<\omega}\}$ is the same as in the proof of Theorem 5.1. □

Theorems 5.2 and 5.3 state that:

(1) Every Σ^1_0-definable equivalence relation has either $\leq \aleph_0$ or $2^{\aleph_0}$ equivalence classes.
(2) Every Σ^1_1-definable equivalence relation has either $\leq \aleph_1$ or $2^{\aleph_0}$ equivalence classes.

Much of the further research has been concentrated in extending this list to the higher levels of the projective hierarchy. In particular, there is much research on the following hypothesis which indicates what should be the behavior of the next level of the projective hierarchy.

HYPOTHESIS Every Σ^1_2-definable equivalence relation has either $\leq \aleph_2$ or $2^{\aleph_0}$ equivalence classes.

A variation of the presented method is used to prove following results from the Borel Combinatorics (see [Harrington–Marker–Shelah]).

THEOREM 5.4 If X is a Borel subset of $\mathbb{R}$ linearly ordered by some Borel relation ρ, then there is a strictly increasing Borel function $f: \langle X, \rho\rangle \to \langle \{0,1\}^\alpha, <\rangle$ for some countable ordinal α. □

Thus the sets $\{0,1\}^\omega$, $\{0,1\}^{\omega+1}, \ldots, \{0,1\}^\alpha$ ($\alpha < \omega_1$) are not only typical, but in some sense universal Borel linearly ordered sets. Of course, there is a considerable structure (still unknown) of Borel linearly ordered sets which are embeddable into some $\{0,1\}^\alpha$.

THEOREM 5.5 If $\langle X, \rho\rangle$ is a Borel poset (i.e. X and ρ are Borel and ρ partially orders X) then either X may be covered by countably many chains that are Borel sets or there is a perfect antichain in X. □

The last Theorem may be viewed as a generalization of the well-known result of [Dilworth]:

THEOREM 5.6 If n is a positive integer then every partially ordered set which has no antichain of size $n+1$ can be decomposed into $\leq n$ of its chains. □

Recently the following Borel version of Dilworth's Theorem has been proved by a student of the University of Paris:

THEOREM 5.7 If n is a positive integer then every Borel poset X with no antichain of size $n+1$ can be decomposed into $\leq n$ of its chains which are, moreover, Borel subsets of X. □

6. THE HALPERN–LAÜCHLI THEOREM

The Halpern–Laüchli Theorem (HL) is a deep combinatorial fact obtained as a by-product of the proof of Con(ZF+¬AC+BP), (BP is the statement that there exists a prime ideal in every Boolean algebra, a consequence of the Axiom of Choice equivalent to Compactness Theorem for the First–Order Predicate Calculus). It has been noticed since then that the lemma of the proof might be of independent interest; it is interesting that the story of the Ramsey's Theorem is quite similar. We shall give a number of applications of the HL Theorem, as well as a proof of the theorem using the method of forcing.

DEFINITION 6.1 a) *For trees T and T' let*

$$T \otimes T' = \{\langle t, t'\rangle \in T \times T' : l(t) = l(t')\}$$

with the ordering $\leq$ defined by $\langle s, s'\rangle \leq \langle t, t'\rangle$ iff $s \leq t$ and $s' \leq t'$. We similarly define $\bigotimes_{i<d} T_i$ for a given sequence of trees $T_0, T_1, \ldots, T_{d-1}$.
b) *If T is a tree of height ω and $A \in [\omega]^\omega$, let*

$$T \restriction A = \{t \in T : ht(t) \in A\},$$

with the ordering inherited from T.
c) *A tree is perfect iff every of its elements has at least two incomparable successors. For $x <_T y \in T$ we say that y is an immediate successor of x iff x and y belong to consecutive levels of T. A node is splitting iff it has two distinct immediate successors. A subtree is a substructure of a tree that is closed downwards.*

Now we are ready to formulate the HL theorem.

THEOREM 6.1 (HL_d) Let d be a natural number, let T_i be a perfect tree of height ω for every $i < d$, and let k be a natural number. For every function $f : \bigotimes_{i<d} T_i \to k$ there is an $A \in [\omega]^\omega$ and a perfect subtree $U_i \subseteq T_i$ for every $i < d$ such that the function

$$f \restriction \left(\bigotimes_{i<d}(U_i \restriction A)\right)$$

is constant, or in other words, $\bigotimes_{i<d}(U_i \restriction A)$ is *homogeneous for f*.

This interesting result has found many applications in a rather wide range of problems. For example, one of the first applications of HL_d was in Model Theory, where it was used to prove the two-cardinal theorem $(\aleph_\omega, \aleph_0) \Rightarrow (2^{\aleph_0}, \aleph_0)$ (see [Shelah]). In Model Theory the HL Theorem is frequently used in order to set the so-called *tree indiscernibles*. It is also quite useful in proving partition relations for ordered sets, results about continuous functions on the Hilbert cube, etc.

6.A. PARTITION RELATIONS FOR ORDERED SETS.

DEFINITION 6.2 *The order type* $\mathrm{tp}(X)$ *of a linearly ordered set* X *is the class of all sets order-isomorphic to* X. *The symbol* η *stands for the order type of the set of rational numbers,* $\mathrm{tp}(\mathbb{Q}) = \eta$. *We do not make difference between ordinals and their order types.*

If X and Y are ordered sets, then $\mathrm{tp}(X) \leq \mathrm{tp}(Y)$ means "X is order-isomorphic to a subset of Y". If x and y are order types, then $x + y$ is the type that we get when we paste y at the end of x; formally it is the type of $\{0\} \times x \cup \{1\} \times y$ ordered lexicographically. (The $\leq$ relation is a quasi-ordering (i.e. it is reflexive and transitive) on the class of linearly ordered sets. To see that it is not antisymmetric, consider e.g. types η and $\eta + 1$).

DEFINITION 6.3 *If* α *and* β *are order types and* m, n *are natural numbers then*

$$\alpha \to (\beta)^n_m$$

means "for every X *such that* $\mathrm{tp}(X) = \alpha$ *and a coloring* $f: [X]^n \to m$ *there is an* $Y \subseteq X$ *such that* $\mathrm{tp}(Y) = \beta$ *and* Y *is homogeneous for* f ($f \restriction [Y]^n$ *is constant*). "

If α *and* β *are order types, then*

$$\alpha \to (\beta)^m_{<\omega/n}$$

means "for every coloring of the set $[\alpha]^m$ *into finitely many colors there is a subset* X *of* α *with* $\mathrm{tp}(X) = \beta$ *and* $|f''[X]^m| = n$". *If* $\alpha = \beta$ *and the last sentence applies to a given coloring then we say that* f *is reducible to* n *colors.*

THEOREM 6.2 Let $f: \mathbb{Q}^n \to k$ be a coloring into finitely many colors. There are subsets X_i $(i < n)$ of $\mathbb{Q}$ such that $\mathrm{tp}(X_i) = \eta$ for all $i < n$ and $\left|f'' \prod_{i<n} X_i\right| \leq n!$.

This is the best possible bound for the number of colors, as the following example shows.

EXAMPLE 6.1 There is a function $f: \mathbb{Q}^n \to n!$ such that for every sequence X_0, $X_1, \ldots X_{n-1}$ of infinite subsets of $\mathbb{Q}$, the function f takes all its values on the product $X_0 \times X_1 \times \cdots \times X_{n-1}$. Fix a well-ordering $\leq_\omega$ of $\mathbb{Q}$ in order type ω. The function f is defined by (see Fig. 7):

$f(x_0, \ldots, x_{n-1}) = k$ iff $x_0, \ldots, x_{n-1}$ is the k-th permutation of the strictly $\leq_\omega$-increasing reenumeration of $(x_0, \ldots, x_{n-1})$.

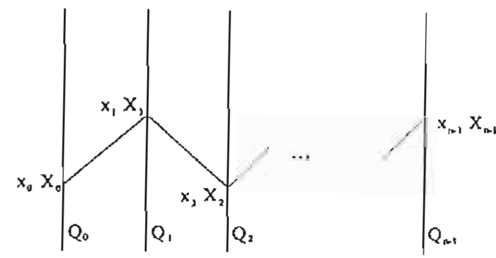

Fig. 7

We shall prove a Lemma that immediately implies the Theorem 6.2. In the following proof, "for almost every" means "for all but finitely many". Notice that in the previous example, X_i's may be chosen so that for a fixed color k, we have

(***)
for almost every $x_0 \in X_0$,
for almost every $x_1 \in X_1$,
$\vdots$
for almost every $x_{n-1} \in X_{n-1}$,
$f(x_0, x_1, \ldots, x_{n-1}) = k$.

(Notice that (***) is much weaker than "For almost every $\langle x_0, \ldots, x_{n-1}\rangle \in X_0 \times \cdots \times X_{n-1}$"). We denote the i-th copy of $\mathbb{Q}$ in the product $\mathbb{Q}^n$ by $\mathbb{Q}_i$ and we assume that it is well-ordered in type ω by the relation $<_i$. Notice that this enables us to define an *increasing n-tuple* $\langle x_0, \ldots, x_{n-1}\rangle$ in $\mathbb{Q}_0 \times \cdots \times \mathbb{Q}_{n-1}$ with respect to this ordering in the obvious way (see Fig. 7).

LEMMA 6.1 For every $f: \mathbb{Q}^n \to k$ there are $X_i \subseteq \mathbb{Q}$ such that $\mathrm{tp}(X_i) = \eta$ for all $i < n$ and a color $r < k$ such that for almost every $x_0 \in X_0$, ..., for almost every $x_{n-1} \in X_{n-1}$ we have $f(x_0, \ldots, x_{n-1}) = r$.

PROOF (Theorem 6.2) Apply the Lemma $n!$ times to every partition induced by f by permuting the copies of $\mathbb{Q}$ in the product $\mathbb{Q} \times \cdots \times \mathbb{Q}$. □

PROOF (Lemma 6.1) First identify $\mathbb{Q}$ with the set

$$\{0,1\}^{<\omega} = \bigcup_{n<\omega}^{\infty} \{0,1\}^n, \text{ with the lexicographical ordering, } <_{\mathrm{Lex}}.$$

Denote the tree $\{0,1\}^{<\omega}$ corresponding to $\mathbb{Q}_i$ by P_i. Notice that for every perfect subtree P of P_i and every infinite $A \subseteq \omega$, $P \restriction A$ has order type $\geq \eta$ (and also $\leq \eta$) with respect to the ordering inherited from $\mathbb{Q}$. The proof of the Lemma now goes by induction on n:

$n = 1$ – easy, by HL_1.

Suppose that the Lemma is true for $n-1$. To any $s = \langle s_0, \dots, s_{n-1}\rangle \in \bigotimes_{i<n} P_i$ we assign $n-1$ sets $X^i_{s_i} \subseteq P_i$ $(i = 1, \dots, n)$ defined by

$$X^i_{s_i} = \{x \in P_i : s_i \le x\},$$

and a function

$$g\colon X^1_{s_1} \times X^2_{s_2} \times \cdots \times X^{n-1}_{s_{n-1}} \to k$$

defined by

$$g(x_1, \dots, x_{n-1}) = f(s_0, x_1, \dots, x_{n-1}).$$

Applying the induction hypothesis on g we get $Z^i_{s_i} \subseteq X^i_{s_i}$ for $i = 1, \dots, n-1$ and a color $h(s_0, \dots, s_{n-1})$ such that for almost every $x_1 \in Z^1_{s_1}$, …, for almost every $x_{n-1} \in Z^{n-1}_{s_{n-1}}$,

$$f(s_0, x_1, \dots, x_{n-1}) = h(s_0, \dots, s_{n-1}).$$

By HL_n applied to $\bigotimes_{i<n} P_i$ and h we get n perfect subtrees $U_i \subseteq P_i$ $(i < n)$, an infinite set $A \subseteq \omega$ and a color $r < n$ such that $h(s_0, \dots, s_{n-1}) = r$ for all $\langle s_0, \dots, s_{n-1}\rangle \in (\bigotimes_{i<n} U_i) \restriction A$. Finally, let $X_0 = U_0 \restriction A$ and $X_i = \min_{<_i}\{Z^i_{s_i} : s \in U_i \restriction A\}$ for $1 \le i < n$. For every X_i we have that $\mathrm{tp}(X_i) \ge \eta$, and this completes the proof of the Lemma. □

What happens if we are coloring unordered pairs of elements of $\mathbb{Q}$? The following result gives the optimal answer in the two-dimensional case.

THEOREM 6.3 $(\eta \to (\eta)^2_{<\omega/2})$ For every $f\colon [\mathbb{Q}]^2 \to k$ (k is finite) there are $X \subseteq \mathbb{Q}$ with $\mathrm{tp}(X) = \eta$ and $i_0, i_1 < k$ such that $f''[X]^2 = \{i_0, i_1\}$.

PROOF Again we identify $\mathbb{Q}$ and $\{0,1\}^{<\omega}$ and fix a well-ordering $<_1$ on $\mathbb{Q}$ such that $\mathbb{Q}$ is of type ω with the respect to $<_1$. For $s \in 2^n$ by $s^\frown 0$ ($s^\frown 1$) we denote $t \in 2^{n+1}$ such that $t \restriction n = s$ and $t(n) = 0$ ($t(n) = 1$). To every $s \in 2^n$ assign the set $X_s \subseteq \{0,1\}^{<\omega}$ with $X_s = \{t : s \subseteq t\}$. We shall construct a subset $A = \{a_s : s \in \{0,1\}^n,\ n \in \omega\}$ of $\{0,1\}^{<\omega}$ such that $a_t < a_s$ iff $t \subset s$. It is easy to see that such A forms a perfect tree with respect to the order inherited from $\{0,1\}^{<\omega}$ (notice that A need not be a subtree of $\{0,1\}^{<\omega}$ in the sense of Definition 6.1). The construction goes by induction as follows:

Set $A_0 = \{0,1\}^{<\omega}$, and $a_\emptyset = \emptyset$. In the nth step we choose $a_s \in A_n$ for all $s \in \{0,1\}^{n+1}$ and define a set $A_{n+1} \subseteq A_n$ taking care that A_{n+1} has perfect intersections with both $X_{a_s{}^\frown 0}$ ($a_s{}^\frown 0$, not $a_{s^\frown 0}$) and $X_{a_s{}^\frown 1}$ for every $s \in 2^n$ (cf. Fig. 8). For every $a_s \in A_n$ apply Theorem 6.2 to sets $X_{a_s{}^\frown 0} \cap A_n$ and $X_{a_s{}^\frown 1} \cap A_n$ and get sets $Z_s \subseteq X_{a_s{}^\frown 0} \cap A_n$, $Y_s \subseteq X_{a_s{}^\frown 1} \cap A_n$ and natural numbers $i^s_0, i^s_1 < k$ such that

$$f(\{y, z\}) = \begin{cases} i^s_0, & \text{if } y <_1 z, \\ i^s_1, & \text{if } z <_1 y. \end{cases}$$

Choose for $a_{s^\frown 0}$ ($a_{s^\frown 1}$) the minimal element of Z_s (Y_s). Let A_{n+1} be the union of $Y_s \cup Z_s$ $(s \in \{0,1\}^{<\omega})$.

When A is constructed, consider a function $g\colon \{0,1\}^{<\omega} \to k \times k$ defined by $g(s) = \langle i^s_0, i^s_1\rangle$ and apply HL_1 to get a perfect subtree $U \subseteq \{0,1\}^{<\omega}$, infinite $D \subseteq \omega$, and $i_0, i_1 < k$ such that $i^s_0 = i_0$ and $i^s_1 = s_1$ for all $s \in U \restriction D$. Obviously, the set $\{a_{s^\frown 0}, a_{s^\frown 1} : s \in U \restriction D\}$ verifies the Theorem. □

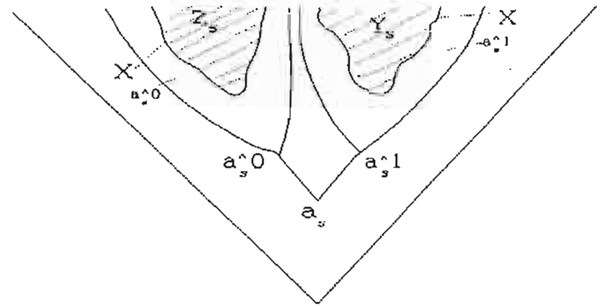

Fig. 8

To see that this is an optimal result, consider an enumeration $\{q_n\}_{n=0}^\infty$ of $\mathbb{Q}$ and define $[\mathbb{Q}]^2 = S_0 \cup S_1$ by $\{q_n, q_m\} \in S_0$ iff $n < m$ is equivalent to $q_n < q_m$. This is so-called *Sierpiński's partition* associated to the enumeration (well-ordering) $\{q_n\}_{n=0}^\infty$ of $\mathbb{Q}$. Clearly, $[X]^2 \cap S_i \ne \emptyset$ for every $i < 2$ and every $X \subseteq \mathbb{Q}$ order-isomorphic to $\mathbb{Q}$. Note that Theorem 6.3 is saying that for any other partition $[\mathbb{Q}]^2 = K_0 \cup K_1$ witnessing $\eta \not\to (\eta)^2_2$ there is an $X \subseteq \mathbb{Q}$ order-isomorphic to $\mathbb{Q}$ such that the partition agrees with Sierpiński's partition on $[X]^2$ modulo only a possible interchange of indexes 0 and 1. To see this, apply Theorem 6.3 to the partition

$$[\mathbb{Q}]^2 = \bigcup_{i,j<2} K_i \cap S_j,$$

and find an $X \subseteq \mathbb{Q}$ order-isomorphic to $\mathbb{Q}$ whose square intersects at most two of the four classes. A similar comparison argument can be used in the following exercise.

EXERCISE Use Theorem 6.3 to show that for every one-to-one $f\colon \mathbb{Q} \to \mathbb{R}$ there is an $X \subseteq \mathbb{Q}$ of order type η such that either f is monotonic on X, or $f''X$ has order type ω or ω^*.

Theorem 6.3 admits the following generalization to higher dimensions. A generalization of the preceding result is in order.

THEOREM 6.4 $(\eta \to (\eta)^n_{<\omega/\tan^{(2n-1)}(0)})$ For every coloring of $[\mathbb{Q}]^n$ in finitely many colors there is an $X \subseteq \mathbb{Q}$ with $\mathrm{tp}(X) = \eta$ and $|f''[X]^n| \le \tan^{(2n-1)}(0)$. □

Joint generalization of Theorems 6.2 and 6.4 as well as the proof of the former can be found in [Devlin]. The bound given for a number of colors is the best possible one, as Example 6.2 shows. We shall not prove Theorem 6.4, although an interested reader may guess that the main idea in the proof is borrowed from the Theorem 6.3 and uses a generalization of patterns from Example 6.2. The proof itself is long and tedious.

EXAMPLE 6.2 The sequence $\{\tan^{(2n-1)}(0)\}$ of tangent numbers starts as follows:

n	1	2	3	4	5	6	7	…
$\tan^{(2n-1)}(0)$	1	2	16	272	7 936	353 792	22 368 256	…

We have already remarked that $\tan^{(2n-1)}(0)$ is the smallest number satisfying Theorem 6.4. To see this in cases $n = 2$ and $n = 3$ we shall take $\{0,1\}^{<\omega}$ with the lexicographical ordering as our copy of $\mathbb{Q}$. (The length of some $s \in \{0,1\}^{<\omega}$ is denoted by $|s|$. This length coincides with the cardinality of the set s, hence no confusion may arise). Define $f_2: [\{0,1\}^{<\omega}]^2 \to 2$ by letting $f_2(\{s,t\}) = 0$ iff $s <_{\text{Lex}} t$ implies $|s| < |t|$. The claim is that both *patterns*:

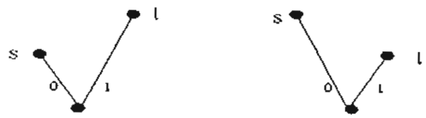

Fig. 9

occur in any $X \subseteq \{0,1\}^{<\omega}$ of order type η. The checking that this is indeed so is left to the reader as an easy exercise. [Hint: Every $X \subseteq \{0,1\}^{<\omega}$ of order type $\geq \eta$ contains a subset Y which forms a perfect tree with the inclusion ordering induced from $\{0,1\}^{<\omega}$.] The sixteen patterns in the case $n = 3$ are obtained by taking patterns represented in Fig. 10 and their mirror images from Fig. 11 . In other words, for $x <_{\text{Lex}} y <_{\text{Lex}} z$, $f_3(\{x,y,z\})$ codes the way the numbers

$$\{\Delta(x,y), \Delta(y,z), \Delta(x,z), |x|, |y|, |z|\}$$

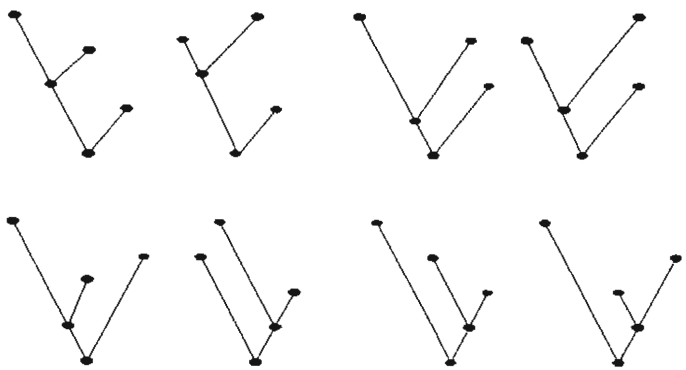

Fig. 10

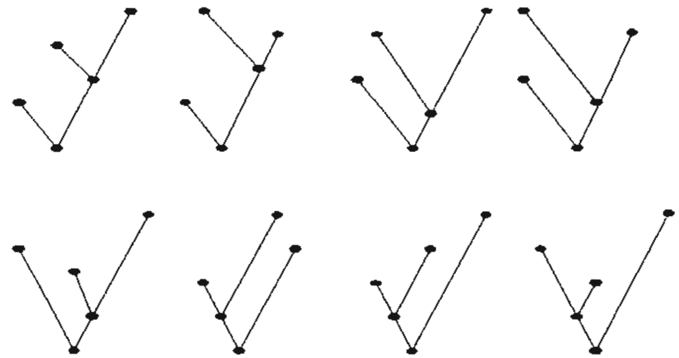

Fig. 11

are related to each other (ignoring the possibilities $\Delta(x,y) = \Delta(y,z) = \Delta(x,z)$, or $|x| = |y|$, or $|x| = |z|$, or $|y| = |z|$, or the possibility that one of the first three numbers is equal to one of the last three). It should be clear that for every $X \subseteq \{0,1\}^{<\omega}$ of order type η all of the sixteen patterns are realized in $[X]^3$. We also hope that starting from this case the reader will not have much difficulty in showing that the following recursive formula counts the number of patterns in general case $n \geq 1$:

$$t_n = \sum_{i=1}^{n-1} \binom{2n-2}{2i-1} t_i \cdot t_{n-i},$$

with the initial value $t_1 = 1$.

6.B. THE PROOF.

Before we start proving HL, let us give a variant of Definition 6.3 which will be needed in the proof.

DEFINITION 6.4 *If κ and λ are cardinals and m, n are natural numbers, then $\kappa \to (\lambda)^n_m$ means "for every X of size κ and a coloring $f: [X]^n \to m$ there is an $Y \subseteq X$ of size λ such that Y is homogeneous for f."*

We shall give a proof of HL_d using forcing. Our forcing is the same one from Cohen's original construction, so we will use this opportunity to present Cohen's proof that ZF+¬CH is consistent. Define C_θ to be the poset of all finite partial functions from θ into 2, ordered by $\subseteq$.

CLAIM C_θ is ccc.

PROOF Two elements $p, q \in C_\theta$ are compatible iff they coincide on $\text{dom}(p) \cap \text{dom}(q)$. Let $\mathcal{F}$ be an uncountable subset of C_θ. Applying Δ-system Lemma on the set $\{\text{dom}(p) : p \in C_\theta\}$, we refine $\mathcal{F}$ so that $\text{dom}(p) \cap \text{dom}(q) = r$ for all $p, q \in \mathcal{F}$ and some fixed finite $r \subset \theta$. There are only finitely many functions from r to $\{0,1\}$, so there must be two different members of $\mathcal{F}$ with the same restriction to r. □

For every ordinal $\alpha < \theta$ let $\dot{r}_\alpha$ be a C_θ-name for a real such that

$$p \Vdash \dot{r}_\alpha(\check{m}) = \check{n} \quad \text{iff} \quad \omega\alpha + m \in \operatorname{dom}(p) \quad \text{and} \quad p(\omega\alpha + m) = n.$$

(Of course, both multiplication and addition in the formula $\omega\alpha+m$ are ordinal multiplication and addition). Let $\dot{f}_\theta$ be the name for $\bigcup \dot{G}_\theta$, where $\dot{G}_\theta$ is the canonical name for the generic filter of C_θ.

Lemma 6.2 $\Vdash_{C_\theta}$ "$\dot{r}_\alpha$ is a Cohen's real number".

Proof Let $\mathcal{D}$ be a given dense open subset of C_ω. Let $\mathcal{D}^* = \{p^* \in C_\theta :$ there is a $p \in \mathcal{D}$ such that for every $n, m < \omega$,

$$m \in \operatorname{dom} p \;\&\; p(m) = n \quad \text{implies} \quad \omega\alpha + m \in \operatorname{dom}(p^*) \;\&\; p^*(\omega\alpha + m) = n\}.$$

Then $\mathcal{D}^*$ is dense open in C_θ. So f_G extends a member of $\mathcal{D}^*$ which implies that $\dot{r}_\alpha$ extends a member of $\mathcal{D}$. □

Lemma 6.3 For $\alpha \neq \beta < \theta$ we have that $\Vdash_{C_\theta} \dot{r}_\alpha \neq \dot{r}_\beta$. Thus we are adding at least θ reals.

Proof We prove that the set of all q's from C_θ such that $q \Vdash \dot{r}_\alpha \neq \dot{r}_\beta$ is dense in C_θ. Let p be in C_θ; we shall construct $q \supset p$ as required. Remember that p is finite, so there is an $n \in \omega$ such that neither $\omega\alpha + n$ nor $\omega\beta + n$ is in $\operatorname{dom}(p)$. Now we simply set

$$q = p \cup \{\langle \omega\alpha + n, 0\rangle, \langle \omega\beta + n, 1\rangle\},$$

hence $q \Vdash \dot{r}_\alpha(\check{n}) = \check{0}$ and $q \Vdash \dot{r}_\beta(\check{n}) = \check{1}$ and we are done. □

This still does not mean that CH is false in a generic model. We have just added θ many reals. We still must prove that there is no bijection between θ and ω_1 in a model V^{C_θ}. There are posets forcing the existence of such functions—e.g. $\mathcal{P} = \{p : p$ is a countable partial function mapping θ to $\omega_1\}$, ordered by $\subseteq$. We say that a cardinal κ is preserved by $\mathcal{P}$ iff forcing by $\mathcal{P}$ does not add a function mapping an ordinal strictly less than κ onto κ.

Definition 6.5 *Let $\mathcal{B}_\theta$ be the smallest σ-algebra of subsets of $\{0,1\}^\theta$ which includes the basic clopen sets. The algebra $\mathcal{B}_\theta$ is usually called the algebra of Baire sets subsets of $\{0,1\}^\theta$.*

Of course, if θ is countable then $\mathcal{B}_\theta$ is just the algebra of Borel subsets of $\{0,1\}^\theta$. If $\theta \geq \omega_1$ this is no longer true, since for example the points of $\{0,1\}^\theta$ are not in $\mathcal{B}_\theta$.

Lemma 6.4 For every C_θ-name $\dot{r}$ for a Cohen real there is a *Baire function* $H: \{0,1\}^\theta \to \mathbb{R}$ (i.e. $H^{-1}(I) \in \mathcal{B}_\theta$ for every open $I \subseteq \mathbb{R}$) such that $\Vdash_{C_\theta} \dot{r} = \check{H}(\dot{f}_\theta)$.

Proof Similar to the proof of Lemma 2.3. □

Notice that every Baire subset B of $\{0,1\}^\theta$ depends only on countably many coordinates in θ, i.e. there is a countable set $A \subseteq \theta$ such that for every $x, y \in \{0,1\}^\theta$ with $x \restriction A = y \restriction A$, $x \in B$ iff $y \in B$. Call such A a *support of* B. Notice that for a countable $A \subseteq \theta$ there exist only continuum many Baire subsets of $\{0,1\}^\theta$ with their support included in A. So the following Lemma gives the bound on the size of the continuum in V^{C_θ}.

Lemma 6.5 Let $H: \{0,1\}^\theta \to \mathbb{R}$ be a Baire function; then there is a countable $A \subseteq \theta$ and a function $H_A: \{0,1\}^A \to \mathbb{R}$ such that for all x and y from $\{0,1\}^\theta$

$$x \restriction A = y \restriction A \quad \text{iff} \quad H(x) = H(y) = H_A(x)$$

Proof Fix some enumeration of open rational intervals $\langle I_n : n \in \omega\rangle$. The function H is determined by the sets $B_n = H^{-1}(I_n)$. These sets are elements of $\mathcal{B}_\theta$, so for each B_n there is a countable $A_n \subseteq \theta$ such that $x \restriction A_n$ decides whether x is in B_n. Set $A = \bigcup_{n\in\omega} A_n$ and $H_A = H \restriction A$. □

According to preceding lemmas, $\Vdash_{C_\theta} |\dot{\mathbb{R}}| \leq \check{\kappa}$, where $\kappa = \theta^{\aleph_0}$. We want to prove, if $\theta \geq \aleph_2$, that CH does not hold in a generic model. We do have at least θ many reals in a generic model, but this still does not mean that CH does not hold in a generic model.

Lemma 6.6 The cardinals and cofinalities are preserved in any generic extension of C_θ.

Proof Let us illustrate this by showing only that ω_1 remains uncountable. Suppose the contrary, namely that there is a $p \in C_\theta$ and a name $\dot{f}$ for a function such that $p \Vdash \dot{f}: \check{\omega} \xrightarrow{\text{onto}} \check{\omega}_1$. Let A_n be the set

$$\{\alpha : (\exists q < p) q \Vdash \dot{f}(\check{n}) = \check{\alpha}\}.$$

For each α in A_n choose q_α forcing $\dot{f}(\check{n}) = \check{\alpha}$). Then $q_\alpha \perp q_\beta$ for $\alpha \neq \beta$ in A_n. Hence each A_n is countable by the ccc property of C_θ. The cofinality of ω_1 is greater than ω; so there is a $\xi < \omega_1$ such that ξ is greater than any ordinal from the union of A_n's. But then $p \Vdash \operatorname{range}(\dot{f}) \subseteq \check{\xi}$, a contradiction. □

Corollary If $\kappa = \theta^{\aleph_0}$ then $\Vdash_{C_\theta} |\dot{\mathbb{R}}| = \check{\kappa}$; so, in particular, if $\theta \geq \omega_2$ then $\Vdash_{C_\theta} \neg$CH. □

We shall prove a reformulation of $\mathbf{HL}_d$ (see Theorem 6.5) that allows a direct finitization (see Theorem 6.6) after two more definitions.

Definition 6.6 *Let T be a tree. A set $A \subseteq T$ is n-dense iff there is an $m > n$ such that $A \subseteq T(m)$ and for every $s \in T(m)$ there exists a $t \in A$ such that $t > s$. Similarly, if $T_0, \ldots, T_{d-1}$ are trees and $A_i \subseteq T_i(m)$ for all $i = 0, \ldots, d-1$, then the sequence $\langle A_0, \ldots, A_{d-1}\rangle$ is n-dense in $\langle T_0, \ldots, T_{d-1}\rangle$ iff for every $i < d$ the set A_i is n-dense in T_i.*

Definition 6.7 *If T is a tree and $t \in T$, then $T[t]$ denotes a subtree of all $s \in T$ comparable with t, $T[t] = \{s \in T : s \leq t$ or $t \leq s\}$.*

Theorem 6.5 (The Dense Set Version of HL_d Theorem) For every function $f: \bigotimes_{i<d} T_i \to \{0,1\}$ we have one of the following two possibilities:

(1) For every $n \in \omega$ there is an n-dense set $\langle A_0, \ldots, A_{d-1}\rangle$ in $\langle T_0, \ldots, T_{d-1}\rangle$ such that $f'' A_0 \times \cdots \times A_{d-1} = \{0\}$ (cf. Fig. 12), or

(2) There is a $\bar{t} \in T_0 \times \cdots \times T_{d-1}$ such that for all $n \in \omega$ there is an n-dense set $\langle B_0, \ldots, B_{d-1}\rangle$ in $\langle T_0[t_0], \ldots T_{d-1}[t_{d-1}]\rangle$ such that $f'' B_0 \times \cdots \times B_{d-1} = \{1\}$ (cf. Fig. 13).

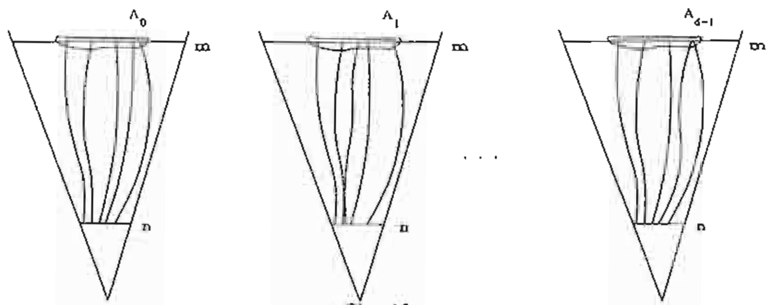

Fig. 12

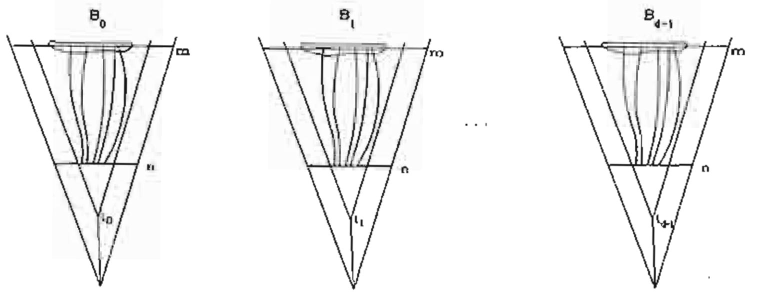

Fig. 13

The first version of HL Theorem is easily proved by induction from the dense set reformulation. We have also this finite version that can be formalized and proved within PA.

THEOREM 6.6 (HL THEOREM, FINITE VERSION) For every pair $d, n \in \omega$ there is a $k \in \omega$ such that for every n-ary tree T of height k and every coloring $f: \bigotimes_{i<d} T \to \{0,1\}$ there is a $\vec{t} \in \bigotimes_{i<d} T$ of height l and a $(l+1)$-dense set $\langle A_0, \ldots, A_{d-1} \rangle$ in $\langle T_0[t_0], \ldots T_{d-1}[t_{d-1}] \rangle$ such that $f \restriction A_0 \times \cdots \times A_{d-1}$ is constant. □

It might be of interest to find a stronger version of the finite form of HL, strong enough to give the full result of HL; for some metamathematical aspects of what may happen when some partition statements get finitized, see [Paris].

PROOF (Theorem 6.5) We start with few simple facts about the compatibility relation of the poset C_θ. Two elements p and q of C_θ are said to be *isomorphic* iff there is an order-preserving map e from dom(p) to dom(q) such that $p(\xi) = q(e(\xi))$ for all ξ in dom(p). Clearly, this is an equivalence relation on C_θ and the class of an element p of C_θ will be called the *isomorphism type of* p. We shall identify the isomorphism type of p with the (unique) element of $\{0,1\}^{<\omega}$ isomorphic to p.

LEMMA 6.7 If $m = 2^n + 1$ then for every sequence $p_0, \ldots, p_{m-1}$ of elements of C_θ of size n (i.e. of some type in $\{0,1\}^n$) there exist $k < \ell < m$ such that p_k and p_ℓ are compatible in C_θ (i.e. $p_k \cup p_\ell \in C_\theta$).

PROOF Let μ be the product measure of $\{0,1\}^\theta$ (see Appendix C). Then

$$\mu([p_\ell]) = 2^{-n} \quad \text{for every } \ell < m.$$

Since $p_k \perp p_\ell$ iff $[p_k] \cap [p_\ell] = \emptyset$, the conclusion follows from the additivity of μ. □

REMARK Notice that $m = 2^n + 1$ is the minimal integer satisfying Lemma 6.7.

LEMMA 6.8 For every type t in $\{0,1\}^{<\omega}$ and integers m and d there is an integer $M = M(t, m, d)$ such that for every sequence $p_{\vec{x}}$ ($\vec{x} \in M^d$) of elements of C_θ of type t there exists $H_i \subseteq M$ ($i < d$) all of size m such that all $p_{\vec{x}}$ ($\vec{x} \in \bigotimes_{i<d} H_i$) are pairwise compatible.

PROOF We shall prove this by induction on d (for all m and t). Clearly, the case $d = 1$ is contained in the previous Lemma modulo an application of (finite) Ramsey' Theorem. So suppose $d > 1$. Let

$$k = 2^{|t| \cdot m^{d-1}} \cdot M(s, m, 1) + 1,$$

where s is any element of $\{0,1\}^{|t| \cdot m^{d-1}}$ (notice that $M(t, m, d)$ depends only on the length $|t|$ of t). Let $F: \omega \to \omega$ be defined by

$$F(\ell) = M(t, \ell, d-1).$$

Let F^k be the k-th iterate of F and $M = F^k(m)$. We claim that this M works. So let $p_{\vec{x}}$ ($\vec{x} \in M^d$) be a given sequence of elements of C_θ of type t. By the choice of M there exists H_i^0 ($1 \le i < d$) all of the size $F^{k-1}(m)$ such that

$$p_{0 \frown \vec{x}} \quad \text{are pairwise compatible for } \vec{x} \in \bigotimes_{i<d} H_i^0.$$

For the same reason there exists $H_i^1 \subseteq H_i^0$ ($1 \le i < d$) of size $F^{k-2}(m)$ such that

$$p_{1 \frown \vec{x}} \quad \text{are pairwise compatible for } \vec{x} \in \bigotimes_{i<d} H_i^1.$$

Proceeding in this way we construct $H_i = H_i^{k-1}$ ($1 \le i < d$) of size m such that for all $\ell < k$

$$p_{\ell \frown \vec{x}} \quad \text{are pairwise compatible for } \vec{x} \in \bigotimes_{i<d} H_i^\ell.$$

Let p_ℓ be the union of $p_{\ell \frown \vec{x}}$ for $\vec{x} \in \bigotimes_{i<d} H_i$. Then by the choice of k there exists $I \subseteq k$ of size $H(s, m, 1)$ such that p_ℓ ($\ell \in I$) all have the same type s for some s of length at most $|t|m^{d-1}$. So applying the one-dimensional case, there exists $H_0 \subseteq I$ of size m such that p_ℓ ($\ell \in H_0$) are compatible. It is easily checked that

$$p_{\vec{x}} \quad \text{are pairwise compatible for all } \vec{x} \in \bigotimes_{i<d} H_i.$$

This finishes the proof. □

Now we are ready to start the proof of the dense set version of HL. So let $1 \leq d < \omega$ be a fixed dimension and let θ_d be the minimal cardinal θ such that for every $f: \theta^d \to \omega$ there exists an infinite $H_i \subseteq \theta$ $(i < d)$ such that f is constant on the product $H_0 \times \cdots \times H_{d-1}$.

REMARK Notice that, by the partition relation $\beth_{d-1}^+ \to (\aleph_0)_2^d$ (see [Kunen] or [Erdös–Hajnal–Maté–Rado]), we have $\theta_d \leq \beth_{d-1}^+$ for all d; but the function θ_d in general might behave differently then the exponentiation function. For example, it can be shown that the statement $\theta_2 = \omega_2$ is independent of the Continuum Hypothesis.

Let

$$\bigotimes_{i<d} T_i = K_0 \cup K_1$$

be a given partition. For the sake of the simplicity of the notation only we shall consider the special case when T_i is equal to the complete binary tree $\{0,1\}^{<\omega}$ for all i. It will be obvious from the proof that this restriction is unessential.

The poset C_θ for $\theta = \theta_d$ adds θ many Cohen reals r_α $(\alpha < \theta)$ which we consider to be branches of $\{0,1\}^{<\omega}$ in the natural way. Let $\dot{\mathcal{U}}$ be a fixed C_θ-name for a nonprincipal ultrafilter on ω (i.e. ultrafilter containing only infinite sets). Then for every $\vec{\alpha} \in \theta^d$ either

(1) every condition of C_θ forces that $\dot{X}_{\vec{\alpha}} = \{n : \dot{r}_{\vec{\alpha}} \restriction n \in \check{K}_0\} \in \dot{\mathcal{U}}$, or
(2) there is a $p_{\vec{\alpha}}$ in C_θ forcing that $\dot{Y}_{\vec{\alpha}} = \{n : \dot{r}_{\vec{\alpha}} \restriction n \in \check{K}_1\} \in \dot{\mathcal{U}}$.

(Here $r_{\vec{\alpha}} \restriction n$ denotes $\langle \dot{r}_{\alpha_0} \restriction n, \dot{r}_{\alpha_1} \restriction n, \ldots, \dot{r}_{\alpha_{d-1}} \restriction n \rangle$). For $\xi < \theta$ let I_ξ be the interval $[\omega\xi, \omega(\xi+1))$. If the second alternative happens extend $p_{\vec{\alpha}}$ (if necessary) so that $p_{\vec{\alpha}} \restriction I_{\alpha_i}$ $(i < d)$ have all the same size, say $\ell(\vec{\alpha})$.

By the choice of θ there exist infinite $H_i \subseteq \theta$ $(i < d)$ such that either

(1) the first alternative happens for all $\vec{\alpha}$ in $\bigotimes_{i<d} H_i$, or
(2) there exist t in $\{0,1\}^{<\omega}$, $\ell \in \omega$, and t_i $(i < d)$ in $\{0,1\}^\ell$ such that for all $\vec{\alpha}$ in $\bigotimes_{i<d} H_i$, the second alternative happens; moreover, $p_{\vec{\alpha}}$ has type t, $\ell = \ell(\vec{\alpha})$, and $p_{\vec{\alpha}} \restriction I_{\alpha_i}$ has type t_i for all $i < d$.

Since the consideration of (2) includes all the arguments needed for the first alternative, we shall deal only with (2). For $i < d$ let

$$U_i = \{s \in \{0,1\}^{<\omega} : s \subseteq t_i \quad \text{or} \quad t_i \subseteq s\} \qquad (\text{i.e. } U_i = T_i[s]).$$

We need to show that for every $n < \omega$ there is an n-dense set in $\bigotimes_{i<d} U_i$ whose product is included in K_1. Let

$$a = |U_i \cap \{0,1\}^n|, \quad \text{for some (i.e. all) } i < d.$$

By Lemma 6.8 choose disjoint $\bar{H}_i \subseteq H_i$ $(i < d)$ of size a such that

$$p_{\vec{\alpha}}, \quad \vec{\alpha} \in \bigotimes_{i<d} \bar{H}_i$$

are pairwise compatible. For $i < d$, let $\alpha_i : U_i(n) \to \bar{H}_i$ be a fixed bijection, where $U_i(n) = U_i \cap \{0,1\}^n$. Let $\alpha : \bigotimes_{i<d} U_i(n) \to \prod_{i<d} \bar{H}_i$ be defined by

$$\alpha(\vec{s}) = \langle \alpha_0(s_0), \ldots, \alpha_{d-1}(s_{d-1}) \rangle, \quad \text{if} \quad \vec{s} = \langle s_0, \ldots, s_{d-1} \rangle.$$

Assuming without loss of generality that $n > |t_i|$ $(i < d)$, extend each $p_{\alpha(\vec{s})}$ (for $\vec{s} \in \bigotimes_{i<d} U_i(n)$) to the element $q_{\alpha(\vec{s})}$ of C_θ such that $q_{\alpha(\vec{s})} \restriction I_{\alpha_i(s_i)}$ has type s_i for all $i < d$. Notice that all

$$q_{\alpha(\vec{s})} \quad \text{for } \vec{s} \in \bigotimes_{i<d} U_i(n)$$

are still pairwise compatible, so we can let q be their union. Then q forces that

$$\bigcap \{Y_{\alpha(\vec{s})} : \vec{s} \in \bigotimes_{i<d} U_i(n)\} \in \dot{\mathcal{U}}$$

so we can find an extension r of q and $m > n$ such that r forces that m is an element of this intersection. Extending r if necessary, assume that for all β in $\bigcup_{i<d} H_i$

$$\{\omega\beta + \ell : \ell < m\} \subseteq \mathrm{dom}(r)$$

Let $t_\beta \in \{0,1\}^m$ be the type of $r \restriction \{\omega\beta + \ell : \ell < m\}$. For $i < d$ let

$$A_i = \{t_\beta : \beta \in \bar{H}_i\}.$$

Then $\langle A_i : i < d \rangle$ is n-dense in $\langle U_i : i < d \rangle$. (For $s \in U_i(n)$ let $\beta = \alpha_i(s)$; then $t_\beta > s$). We claim that

$$A_0 \times \cdots \times A_{d-1} \subseteq K_1$$

which will finish the proof. To see this, pick a $\vec{\beta}$ in $\bigotimes_{i<d} \bar{H}_i$, and let $s_i = \alpha_i^{-1}(\beta_i)$ for $i < d$. Then $\alpha(\vec{s}) = \vec{\beta}$ so q (and therefore r) forces that m is an element of $\dot{Y}_{\vec{s}}$, i.e. that

$$\langle \dot{r}_{\beta_0} \restriction m, \ldots, \dot{r}_{\beta_{d-1}} \restriction m \rangle = \langle t_{\beta_0}, \ldots, t_{\beta_{d-1}} \rangle,$$

and we are done. □

In the general case we replace C_θ by the poset $\mathcal{P}_\theta$ of all finite functions p such that

(1) $\mathrm{dom}(p) \subseteq \theta \times d$,
(2) $p(\alpha, i) \in T_i$ for all $i < d$.

The ordering of $\mathcal{P}_\theta$ is defined by letting $p \leq q$ iff

(3) $\mathrm{dom}(p) \supseteq \mathrm{dom}(q)$,
(4) $q(\alpha, i) <_{T_i} p(\alpha, i)$ for all $(\alpha, i) \in \mathrm{dom}(q)$.

Thus $\mathcal{P}_\theta$ adds θ many generic branches to each T_i, so the proof of the general case is only notationally different. For example, notice that $\mathcal{P}_\theta$ and C_θ have isomorphic regular open algebras (see e.g. [Jech]) so they give us the same forcing extension.

6.C. HL AND PERFECT SETS OF REALS.

We now return to applications of HL, and also present some assertions equivalent to HL. If T is a subset of $\{0,1\}^{<\omega}$ (not necessarily closed downwards) then define

$$[T] = \{x \in \{0,1\}^{\omega} : (\forall n)(\exists s \in T) x \restriction n \subseteq s\}.$$

In some sense, this is the set of all reals that are in the closure of T considered as the set of rationals.

FACT If P is any set of reals and $T = \{s : (\exists x \in P) s \subseteq x\}$ then $[T]$ is the closure of the set P. □

FACT T is a perfect tree iff the set $[T]$ is perfect. □

DEFINITION 6.8 *A family of nonempty perfect sets $\mathcal{P} = \{P_s : s \in \{0,1\}^{<\omega}\}$ is a fusion sequence iff it satisfies the following conditions:*

(1) *if $s \subseteq t$ then $P_s \supseteq P_t$,*
(2) *for all $s \neq t$ of the same length, sets P_s and P_t are disjoint, and*
(3) *the diameter of P_s converges to 0 as the length of s tends to infinity.*

The set

$$\bigcup_{f \in \{0,1\}^{\omega}} \bigcap_{n \in \omega} P_{f \restriction n} = \bigcap_{n \in \omega} \bigcup_{s \in 2^n} P_s$$

is called the fusion of the family $\mathcal{P}$.

FACT (*Fusion Lemma*) Fusion is always a perfect set. □

Notice that by (1) the family $\mathcal{P}$ forms a perfect tree, and that the fusion of every its perfect subtree is also a perfect set.

Let $\mathcal{S}$ denote the set of all perfect (nonempty) subsets of $\mathbb{R}$ ordered by $\subseteq$. Then $\mathcal{S}$ is the *Perfect-set Forcing* and if $\mathcal{G} \subseteq \mathcal{S}$ is a sufficiently generic filter then the unique element of its intersection is called *Sacks real*. The Fusion Lemma is quite instrumental in proving properties of the forcing $\mathcal{S}$. For example, one may prove that for every subset X of ω in the forcing extension by $\mathcal{S}$ there is an infinite A in the ground model such that either $A \subseteq X$ or $A \cap X = \emptyset$. In fact, this is true about any product $\mathcal{S}^d$ $(d \leq \omega)$ of the Perfect-set Forcing (with the coordinatewise ordering) but it is a much deeper fact essentially equivalent to HL_d. To see the equivalence one needs to extend the notion of a fusion sequence to higher dimensions as follows.

DEFINITION 6.9 *A family P_s^i $(s \in \{0,1\}^{<\omega}, i < d)$ is a fusion sequence in $\mathcal{S}^d$ iff for every $i < d$, P_s^i $(s \in \{0,1\}^{<\omega})$ is a fusion sequence in the sense of the previous definition except that P_s^i have the constant value $P_\emptyset^i$ when s has length $\leq i$. The sequence $\langle P^i : i < d\rangle$ of perfect sets defined by*

$$P^i = \bigcup_{f \in \{0,1\}^{\omega}} \bigcap_{n < \omega} P^i_{f \restriction n}$$

is called the fusion of the sequence $\{P_s^i\}$.

Now, if we have an $\mathcal{S}^d$-name τ for a subset of ω we first construct a fusion sequence P_s^i $(s \in \{0,1\}^{<\omega}, i < d)$ such that for every $n < \omega$ and every sequence $\langle s_i : i < d\rangle$ from $(\{0,1\}^n)^d$,

$$\langle P^i_{s_i} : i < d\rangle \Vdash \check{n} \in \tau \quad \text{or} \quad \langle P^i_{s_i} : i < d\rangle \Vdash \check{n} \notin \tau.$$

By HL_d, there exist perfect trees $T_i \subseteq \{0,1\}^{<\omega}$ $(i < \omega)$ and an infinite $A \subseteq \omega$ such that for every

$$\langle s_i : i < d\rangle \in \bigotimes_{i<d} T_i \restriction A$$

one of the two alternatives holds. Let

$$P_i = \bigcup_{f \in [T_i]} \bigcap_{n < \omega} P_{f \restriction n} \quad (i < d).$$

Then each P_i is a perfect set of reals and $\langle P_i : i < d\rangle$ forces that either $A \subseteq \mathrm{int}_G(\tau)$ or $A \cap \mathrm{int}_G(\tau) = \emptyset$.

To deduce HL_d from the forcing statement, look at the natural $\mathcal{S}^d$-name τ associated to a given coloring $f: \{0,1\}^{<\omega} \otimes \{0,1\}^{<\omega} \otimes \cdots \to \{0,1\}$ (i.e. $\langle P^i_{s_i} : i < d\rangle \Vdash \check{n} \in \tau$ iff $f(\langle s_i : i < d\rangle) = 1$ for $\langle s_i : i < d\rangle \in \{0,1\}^n \times \{0,1\}^n \times \ldots$ and $n < \omega$).

There is even a finer preservation result for the forcing extension of $\mathcal{S}^d$ $(d \leq \omega)$. It involves the following notion.

DEFINITION 6.10 *A nonprincipal ultrafilter $\mathcal{U}$ on ω is selective if for every sequence A_n $(n < \omega)$ of elements of $\mathcal{U}$ there is an A in $\mathcal{U}$ such that $A \setminus n \subseteq A_n$ for every $n \in A$. Such A is said to be a diagonalization of the sequence A_n $(n < \omega)$.*

Selective ultrafilters are quite useful objects in Analysis even though their existence requires some additional set-theoretical assumptions such as CH. This is so because most of the statements from Analysis involve only reals and therefore are absolute with respect to any forcing extension which do not add new reals. It is clear that CH can be made true in such an extension (look at the forcing notion of all countable partial mappings from countable ordinals into $\mathbb{R}$; see the beginning of §12). In the rest of this section we consider the statements that involve only reals, or can be coded in such a way; e.g. statements involving continuous functions from $\mathbb{R}$ into $\mathbb{R}$. Therefore, without loss of generality we may assume the existence of a selective ultrafilter $\mathcal{U}$. Selective ultrafilters are sometimes called *Ramsey ultrafilters* because of the following straightforward fact.

FACT A nonprincipal ultrafilter $\mathcal{U}$ on ω is selective iff for every partition $p: [\omega]^2 \to \{0,1\}$ there exists $i < 2$ and $A \in \mathcal{U}$ such that $p''[A]^2 = \{i\}$. □

In fact, many Ramsey-theoretic statements involving ω have their "selective analogues", not just the Ramsey Theorem. For example, the well-known combinatorial theorems of Nash-Williams, Galvin-Prikry, Silver and others about "Ramsey subsets" of $[\omega]^{\omega}$ have their selective analogues such as the following (see [Mathias]).

THEOREM 6.7 If $\mathcal{U}$ is a selective ultrafilter on ω and if $\mathcal{A}$ is an analytic subset of $[\omega]^{\omega}$ then there is an $A \in \mathcal{U}$ such that $[A]^{\omega} \subseteq \mathcal{A}$ or $[A]^{\omega} \cap \mathcal{A} = \emptyset$. □

What is interesting is that Theorem 6.7 and other "selective analogues" can, in fact, be rather easily deduced from original theorems (which state the same thing except that X is only an infinite subset of ω rather than member of some selective ultrafilter) using a forcing argument.

DEFINITION 6.11 *If $\mathcal{U}$ is an ultrafilter on ω let $HL_d(\mathcal{U})$ be HL_d with the requirement that the set A be a member of $\mathcal{U}$.*

THEOREM 6.8 For every $d \leq \omega$ and every selective ultrafilter $\mathcal{U}$ on ω, $\mathrm{HL}_d(\mathcal{U})$ is true.

PROOF One may go either through the above proof for $d < \omega$ or the proof in [Laver] for $d = \omega$ to get the conclusion. But we may also prove $\mathrm{HL}_d(\mathcal{U})$ using the following general reasoning. Let T_i $(i < d)$ be a given sequence of perfect trees (say, subtrees of $\{0,1\}^{<\omega}$) and let

$$p\colon \bigotimes_{i<d} T_i \to \{0,1\}$$

be a given coloring. If $d = \omega$, we further assume, without loss of generality, that if s_i is the minimal splitting node of T_i for $i < \omega$, then the length of s_i increases with i. Let

$$\mathcal{A} = \{A \in [\omega]^\omega : (\forall i < d)(\exists \text{ perfect } U_i \subseteq T_i)(\exists \epsilon < 2)p''\bigotimes_{i<d} U_i \restriction A = \{\epsilon\}\}.$$

Then $\mathcal{A}$ is clearly an analytic subset of $[\omega]^\omega$ so by Theorem 6.7 there exists $A \in \mathcal{U}$ such that either

$$[A]^\omega \subseteq \mathcal{A} \quad \text{or} \quad [A]^\omega \cap \mathcal{A} = \emptyset.$$

Note that by HL_d the second alternative does not happen, so the Theorem follows. □

It should be clear that the argument presented just before the Definition 6.10 shows that Theorem 6.8 has the following equivalent formulation.

THEOREM 6.8* Every selective ultrafilter $\mathcal{U}$ generates a selective ultrafilter $\mathcal{U}^*$ in the forcing extension of $\mathcal{S}^\omega$.

PROOF The only thing left is to show that

$$\mathcal{U}^* = \{\mathrm{int}_G(\tau) \subseteq \omega : (\exists A \in \mathcal{U})A \subseteq \mathrm{int}_G(\tau)\}$$

is selective, i.e. that every sequence $\langle \dot{A}_n : n < \omega\rangle$ of elements of $\mathcal{U}^*$ (which can actually be taken from $\mathcal{U}$ itself) has a diagonalization in $\mathcal{U}^*$. (This is not immediate since even though each member of the sequence $\langle \dot{A}_n : n < \omega\rangle$ is from the ground model, the sequence itself is not). But this is also easy to show considering a fusion sequence P_s^i $(s \in \{0,1\}^{<\omega}, i < \omega)$ such that for every $n < \omega$ and $\vec{s} = \langle s_i : i < \omega\rangle$ in $(\{0,1\}^n)^\omega$ the condition $\langle P_{s_i}^i : i < \omega\rangle$ forces that $\dot{A}_n$ is equal to some element $B_{\vec{s}}$ of $\mathcal{U}$. Let $\langle P_i : i < \omega\rangle$ be the fusion of this sequence and let $B \in \mathcal{U}$ be such that $B \setminus n \subseteq B_{\vec{s}}$ for every $n \in B$ and $\vec{s} \in (\{0,1\}^n)^\omega$ (note that for a given $n < \omega$ the set $\{B_{\vec{s}} : \vec{s} \in (\{0,1\}^n)^\omega\}$ is of the size at most 2^n). This finishes the proof. □

As an illustration we give the following application of these ideas to a problem from Real Analysis.

THEOREM 6.9 For every sequence $\{f_n\}$ of continuous functions from the Hilbert cube $[0,1]^\omega$ into the interval $[0,1]$ there is a subsequence $\{f_n'\}$ of $\{f_n\}$ and a sequence $\{P_n\}$ of perfect subsets of $[0,1]$ such that $\{f_n'\}$ monotonically (and therefore uniformly) converges to a continuous function on the product $P_0 \times P_1 \times P_2 \times \ldots$.

PROOF In the forcing extension of $\mathcal{S}^\omega$ look at the sequence of reals

$$f_n(\langle \dot{s}_i : i < \omega\rangle),$$

where $\langle \dot{s}_i : i < \omega\rangle$ is the canonical name for the generic sequence of Sacks reals. Since $\mathcal{U}^*$ is a selective (Ramsey) ultrafilter apply the above fact to the partition, $p\colon [\omega]^2 \to \{0,1\}$ defined by

$$p(\{m,n\}) = 0 \quad \text{iff} \quad f_n(\langle \dot{s}_i : i < \omega\rangle) \leq f_m(\langle \dot{s}_i : i < \omega\rangle)$$

and get a member A of $\mathcal{U}^*$ (actually of $\mathcal{U}$) such that

$$(1) \qquad f_n(\langle \dot{s}_i : i < \omega\rangle) \quad (n \in A)$$

is monotonic. Fix $\langle P_i : i < \omega\rangle \in \mathcal{S}^\omega$ deciding the set A and the fact that the sequence is, say, increasing. Let

$$F = \{\langle x_i : i < \omega\rangle \in \prod_{i<\omega} P_i : (\forall m < n \in A) f_m(\langle x_i : i < \omega\rangle) \leq f_n(\langle x_i : i < \omega\rangle)\}$$

Notice that F is a closed subset of $\prod_{i<\omega} P_i$. Since every nonempty open subset of this product contains a product of a sequence $\langle P_i' : i < \omega\rangle$ of perfect sets which must also force that the sequence (1) is increasing, the set F must be equal to the product $\prod_{i<\omega} P_i$. Let

$$g\colon \prod_{i<\omega} P_i \to [0,1]$$

be the limit of f_n $(n \in A)$. Since g is of Baire class one, by shrinking P_i's we may assume that g is in fact continuous. This finishes the proof. □

For $1 \leq d < \omega$ we get the following form of Theorem 6.9.

THEOREM 6.10 For every finite d and a sequence $\{f_n\}$ of continuous functions from the cube $[0,1]^d$ into $[0,1]$ there is a single perfect set $P \subseteq [0,1]$ and a subsequence $\{f_n'\}$ of $\{f_n\}$ which uniformly converges on P^d.

PROOF Set P will be the result of a fusion sequence P_s $(s \in \{0,1\}^{<\omega})$ constructed together with a decreasing sequence of sets A_n $(n < \omega)$ such that for every $n < \omega$ and every sequence s_i $(i < d)$ of distinct elements of $\{0,1\}^n$ the subsequence $\{f_k : k \in A_n\}$ when restricted to the product

$$\prod_{i<d} P_{s_i}$$

monotonically (and therefore uniformly) converges to a continuous function. Clearly, there is no problem in constructing this fusion using the previous result. It should also be clear that if $\{k_n\}_{n<\omega}$ is a strictly increasing sequence such that $k_n \in A_n$ for every $n < \omega$ then $\{f_{k_n}\}_{n<\omega}$ and the set P satisfy the conclusion of the theorem. This finishes the proof. □

Another equivalent reformulation of HL is the following partition property of perfect sets of reals resembling to Theorem 6.4. Symbol $\{x_0, x_1, \dots, x_{n-1}\}_{<_{Lex}}$ denotes an n-tuple such that $x_0 <_{Lex} x_1 <_{Lex} \cdots <_{Lex} x_{n-1}$. We shall need the following lemma.

LEMMA 6.9 For every meager $M \subseteq [\mathbb{R}]^n$ there is a perfect set $P \subseteq \mathbb{R}$ such that $[P]^n \cap M = \emptyset$.

PROOF Write M as an increasing union of nowhere dense sets F_i $(i < \omega)$. The set P is constructed as a fusion $\bigcap_{n\in\omega} \bigcup_{s\in\{0,1\}^n} P_s$, where

(1) every P_s is a nontrivial closed interval,
(2) for all $i \in \omega$ and all $s, t \in \{0,1\}^i$ if $s <_{Lex} t$ then $P_s < P_t$ (this means that $x < y$ for all $x \in P_s$ and all $y \in P_t$),
(3) $P_s \supset P_t$ whenever $s \subset t$,
(4) $\bigotimes_{i<n} P_{s_i} \cap F_k = \emptyset$ for every k and every $<_{Lex}$-increasing n-tuple $\langle s_i : i < n\rangle$ of elements of $\{0,1\}^k$.

The condition (4) is easily arranged using the fact that each F_k is nowhere dense. □

THEOREM 6.11 For every finite n, perfect $P \subseteq [0,1]$ and Borel $c: [P]^n \to \omega$ with finite range there is a perfect $P' \subseteq P$ such that $|c''[P']^n| \leq (n-1)!$.

PROOF Since for every $i \in \text{range}(c)$ the set $c^{-1}(i)$ is Borel and therefore has the property of Baire, by Lemma 6.9 we may shrink P to a perfect set P' such that the mapping c is continuous on $[P']^n$. So, we may assume that c is in fact continuous. Let T be a perfect subtree of $\{0,1\}^{<\omega}$ such that $[T] = P$. Refining T to a perfect subtree, we may assume that each level of T contains at most one splitting node.

CLAIM There is a perfect $T' \subseteq T$ such that for every n-tuple $\bar{t} = \langle t_0, \dots, t_{n-1}\rangle$ of distinct elements of the same level of T the function c is constant on the set

$$(*) \qquad [T[t_0]] \times [T[t_1]] \times \cdots \times [T[t_{n-1}]].$$

PROOF We construct a decreasing sequence of trees $T = T_0 \supseteq T_1 \supseteq T_2 \supseteq \dots$ such that for some increasing sequence of integers k_i $(i < \omega)$ the following holds:

(1) $T_{i+1} \cap \{0,1\}^{k_i} = T_i \cap \{0,1\}^{k_i}$,
(2) T_{i+1} has exactly one splitting node v_{i+1} such that $|v_{i+1}| \in [k_i, k_{i+1})$,
(3) for each node $y \in T_i \cap \{0,1\}^{k_i}$ there is a $j \in \omega$ such that the splitting node v_j is above y,
(4) For all $i \in \omega$ and all $t \in T \cap (\{0,1\}^{k_i})^n$ the mapping c is constant on $[t]$.

It is clear that conditions (1)–(3) assure that the tree $\bigcap_{i\in\omega} T_i$ is perfect, and that it naturally corresponds to a fusion as defined in Definition 6.11. In order to settle (3), enumerate T as t_i $(i < \omega)$ and take care about t_i at the i-th stage of the fusion argument. To deal with (4), note that since c is continuous, for every $b \in [T]^n$ there is a $t \in b$ such that $c''[t] = c(b)$. □

DEFINITION 6.12 *Let $\{x_0, x_1, \dots, x_{n-1}\}_{<_{Lex}}$ be a given element of $[T]^n$. The pattern of $\{x_0, x_1, \dots, x_{n-1}\}_{<_{Lex}}$ is a permutation σ of $\{0, 1, \dots, n-2\}$ determined by*

$$i <_\sigma j \quad \text{iff} \quad \Delta(x_i, x_{i+1}) < \Delta(x_j, x_{j+1}).$$

Notice that the set of all n-tuples with pattern σ is open in $[T]^n$, because the pattern of an n-tuple is determined by its restriction to some large enough integer k. Two patterns for $n = 3$ and six patterns for $n = 4$ are displayed on Fig. 14 (the corresponding permutation is written underneath the pattern):

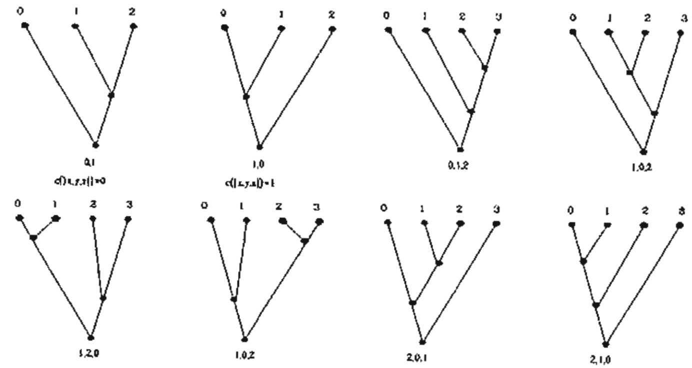

Fig. 14

It is clear that Theorem 6.11 follows the following lemma:

LEMMA 6.10 For every perfect T, continuous $c: [T]^n \to$ some_finite_space, and every pattern $\sigma \in (n-1)!$ there is perfect $U \subseteq T$ such that every $\{x_0, x_1, \dots, x_{n-1}\}$ in $[U]^n$ of pattern σ gets the same value under c.

PROOF Induction on n. Let l be the last element of $\{0, 1, \dots, n-2\}$ in the ordering induced by σ. We construct a tree U as an intersection of a fusion sequence of trees $T = T_0 \supseteq T_1 \supseteq T_2 \supseteq \dots$ satisfying conditions (1)–(3). Fix a well-ordering $<_w$ of T of order type ω. Suppose that we have k_i and T_i. We shall first construct a finite decreasing chain of trees $T_i = T_i^{(0)} \supseteq T_i^{(1)} \supseteq \cdots \supseteq T_i^{(m)}$ such that $T_i^{(j)} \restriction k_i = T_i \restriction k_i$ for all $j \leq m$. Fix $s \in T_i \cap \{0,1\}^{k_i}$. Choose an $(n-1)$-tuple $t_0, \dots, t_l = s, t_{l+2}, \dots, t_{n-1}$ of distinct elements of $T_i \cap \{0,1\}^{k_i}$ such that the pattern of any n-tuple $x_0, \dots, x_{n-1}$ of elements of $[T_i]^n$ such that (see Fig. 15):

$$(**) \qquad \begin{array}{ll} t_i \subseteq x_i & \text{for } i < n-1, \text{ and } i \neq l, l+1 \\ t_l \subseteq x_l, x_{l+1} & \end{array}$$

is equal to σ. We say that the *l-pattern* of such an $(n-1)$-tuple is σ.

For $j < n$ and $j \neq l+1$ let $P_j = T[t_j]$ and define a coloring $f: \bigotimes_{\substack{j<n \\ j\neq l+1}} P_j \to$ finite in the following way: Choose an $(n-1)$-tuple $\langle u_0, \dots, u_l, u_{l+2}, \dots, u_{n-1}\rangle$ of the product $\bigotimes_{\substack{j<n \\ j\neq l+1}} P_j$ such that all u_i's have the same height m. Let $k \leq m$ be the maximal integer such that $u_l \restriction k$ is a splitting node of T_i. If $k \geq n$, let $f(u_0, \dots, u_l, u_{l+2}, \dots, u_{n-1})$ be the constant value of c on

$$[T[u_0 \restriction (k+1)]] \times \cdots \times [T[(u_l \restriction k)^\frown 0]] \times [T[(u_l \restriction k)^\frown 1]] \times \cdots \times [T[u_{n-1} \restriction (k+1)]];$$

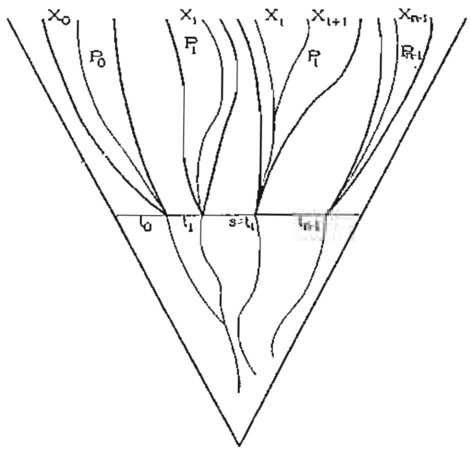

Fig. 15

otherwise, let $f(u_0, \dots, u_l, u_{l+2}, \dots, u_{n-1})$ be an arbitrary value of c.

By HL_{n-1} we choose a perfect $P_j' \subseteq P$ (for $j < n$, $j \neq l+1$) and an infinite $A \subseteq \omega$ such that f is constant on $\left(\bigotimes_{\substack{j<n \\ j\neq l+1}} P_j\right) \restriction A$. What this does is that we get a tree T_i' that is slightly smaller than T_i (and still has the same intersection with $\{0,1\}^{k_i}$) such that for every n-tuple $x_0, \dots, x_{n-1}$ in $[T_i']^n$ satisfying (**) the color $c(x_0, \dots, x_{n-1})$ does not depend on x_{l+1}; in other words, for any $x_{l+1}' \supseteq s$ distinct from x_l we have that $c(x_0, \dots, x_l, x_{l+1}, \dots, x_{n-1}) = c(x_0, \dots, x_l, x_{l+1}', \dots, x_{n-1})$.

Now apply the described procedure to every choice of s and $n-1$-tuple $t_0, \dots, t_l = s, t_{l+2}, \dots, t_{n-1}$ having the l-pattern σ, and get the corresponding chain $T_i = T_i^{(0)} \supseteq T_i^{(1)} \supseteq \cdots \supseteq T_i^{(m)}$ of refinements. Set $T_{i+1} = T_i^{(m)}$, then for every $\{x_0, \dots, x_{n-1}\} \in [T_{i+1}]^n$ of pattern σ such that

$$(***) \qquad \begin{array}{ll} \Delta(x_j, x_{j+1}) < k_i & \text{for all } j \neq l, \text{ and} \\ \Delta(x_l, x_{l+1}) \geq k_i & \end{array}$$

the value $c(x_0, \dots, x_{n-1})$ does not depend on the choice of x_{l+1}. For the next element $t_i = s$ of $T_i \cap \{0,1\}^{k_i}$ with no splitting node in $T_i \cap \{0,1\}^{k_i}$ extending it find a splitting node v_{i+1} of T_{i+1} such that $v_{i+1} > s$, set $k_{i+1} = |v_{i+1}| + 1$ and refine T_{i+1} to satisfy (2). This describes the construction of T_{i+1}.

Let $U = \bigcap_{i\in\omega} T_i$. Then for every n-tuple $\{x_0, \dots, x_{n-1}\}_{<_{Lex}}$ of elements of $[U]$ of pattern σ the value $c(x_0, \dots, x_{n-1})$ does not depend on x_{l+1}, so we are done by the induction hypothesis. This finishes the proof. ☐ ☐

Again the number $(n-1)!$ is the smallest integer with this property. By Theorem 6.11 for $n = 2$ we have the ordinary Ramsey property, i.e. c is constant on $[P']^2$

while for $n = 3$ this cannot be always achieved as the two patterns and the mapping c_3 defined on Fig. 15 show. What Theorem 6.11 is saying in case $n = 3$ is that for an arbitrary Borel mapping $c: [\{0,1\}^\omega]^3 \to$ some_finite_space there is a perfect set $P' \subseteq \{0,1\}^\omega$ such that $c \restriction [P']^3$ is either constant or isomorphic to $c_3 \restriction [P']^3$. Thus Theorem 6.11 is in some sense a reduction principle i.e. for every $n \geq 1$ there is a finite canonical list $\mathcal{C}_n$ of partitions of $[\{0,1\}^\omega]^n$ into at most $(n-1)!$ Borel pieces such that for an arbitrary $c: [\{0,1\}^\omega]^n \to$ some_finite_space there is a perfect set $P' \subseteq \{0,1\}^\omega$ and a $c' \in \mathcal{C}_n$ such that $c \restriction [P']^n$ is isomorphic to $c' \restriction [P']^n$. That is, there is a bijection b of their ranges making the diagram

$$\begin{array}{ccc} [P']^n & \xrightarrow{\;c\;} & c''[P']^n \\ & {\scriptstyle c'}\searrow & \uparrow{\scriptstyle b} \\ & & (c')''[P']^n \end{array}$$

commute. The set $\mathcal{C}_n$ can naturally be considered as the set of all equivalence relations defined on a set of size $(n-1)!$. For example, if ρ is an equivalence relation on $\{0, 1, \dots, (n-1)! - 1\}$ the member c_ρ of $\mathcal{C}_n$ that we associate to ρ is obtained from the partition c_n (see Fig. 15 for the definition in case $n = 3$) by joining the pieces if their indexes are ρ-equivalent. This gives us a way to compute the size of $\mathcal{C}_n$ for small n. For example, $|\mathcal{C}_2| = 1$, $|\mathcal{C}_3| = 2$, $|\mathcal{C}_4| = 203, \dots$.

Remark It has been already mentioned that the infinitary version, HL_ω, of Halpern–Läuchli Theorem is a more recent result of Laver. However, the infinitary form of the dense-set version of HL is still an open problem (see [Laver]). This entire section is largely based on this paper of Laver where the reader can find more informations as well as the references and historical remarks concerning this interesting subject.

7. INTERNAL FORCING—SUSLIN PARTITIONS

In this section we present some *forcing axioms*—axioms that postulate the existence of generic objects with some reduced degree of genericity. This is the method of *internal forcing*.

DEFINITION 7.1 *A partition $[S]^{<\omega} = K_0 \cup K_1$ is a Suslin partition (or a ccc partition) iff the following is true:*

(1) *$\{x\} \in K_0$, for every $x \in S$.*
(2) *For all $F, G \subseteq S$, if $F \subsetneq G \in K_0$ then $F \in K_0$.*
(3) *For every uncountable $\mathcal{F} \subseteq K_0$ there are $F \neq G$ in $\mathcal{F}$ such that $F \cup G \in K_0$.*

If m is finite, a partition $[S]^m = K_0 \cup K_1$ is ccc iff it satisfies (1), (2) and

(4) *For every uncountable family $\mathcal{F}$ of finite 0-homogeneous sets there are F and G in $\mathcal{F}$ such that $F \cup G$ is 0-homogeneous.*

DEFINITION 7.2 *Let $\mathfrak{m}_R$ be the least cardinal θ such that there is a ccc partition $[\theta]^{<\omega} = K_0 \cup K_1$ such that θ is not the union of countably many 0-homogeneous sets.*

PROPOSITION 7.1 $\omega < \mathrm{cf}\, \mathfrak{m}_R$ and $\mathfrak{m}_R \leq \mathfrak{c}$.

PROOF Suppose that $\mathrm{cf}\, \mathfrak{m}_R = \omega$, i.e. that $\mathfrak{m}_R = \sup_{i \in \omega} \theta_i$ for some sequence θ_i $(i < \omega)$ of smaller cardinals. If $[\mathfrak{m}_R]^{<\omega} = K_0 \cup K_1$ is ccc, then $[\theta_i]^{<\omega} = K_0 \restriction [\theta_i]^{<\omega} \cup K_1 \restriction [\theta_i]^{<\omega}$ is also ccc. It follows that each θ_i can be written as a countable union of 0-homogeneous sets. Hence, θ can also be represented as such a union, a contradiction.

For the other inequality we shall need to consider only the three-dimensional ccc-partitions. (In fact, with a more subtle reasoning it suffices to consider only two-dimensional partitions to prove that $\mathfrak{c} \geq \mathfrak{m}_R$; see Lemma 10.3). We first identify $\mathbb{R}$ with ω^ω, then define a partition $[\mathbb{R}]^3 = K_0 \cup K_1$ by letting $\{x, y, z\}$ in K_0 iff x, y and z do not branch at the same place, or more formally,

$$\{x, y, z\} \in K_0 \quad \text{iff} \quad |\{\Delta(x, y), \Delta(y, z), \Delta(x, z)\}| > 1.$$

To prove that this partition is ccc, notice that for every $F \in [K_0]^{<\omega}$ there is a $k_F \in \omega$ such that $\Delta(x, y) < k_F$ for all $x, y \in F$. If $\mathcal{F}$ is uncountable family of finite

0-homogeneous sets, we may assume that there is a finite k such that $k_F < k$ for all $F \in \mathcal{F}$. We can also assume that $F \restriction k$ is the same for all $F \in \mathcal{F}$. It follows that if x and y are two distinct elements of $\bigcup \mathcal{F}$ then $\Delta(x, y) \geq k$ can happen only when x and y are not elements of a single F in $\mathcal{F}$. So the union of every two elements of $\mathcal{F}$ is 0-homogeneous.

Clearly, every interval in $\mathbb{R}$ contains three reals x, y, z such that $\Delta(x, y) = \Delta(y, z) = \Delta(x, z)$; therefore every 0-homogeneous set is nowhere dense, so $\mathbb{R}$ is not equal to a countable union of 0-homogeneous sets. □

REMARK It is known that the following facts are (separately) consistent with ZFC: $\mathfrak{c} = \mathfrak{m}_R > \omega_1$, $\mathfrak{m}_R > \mathrm{cf}\, \mathfrak{m}_R = \omega_1$, $\mathfrak{m}_R = \omega_1 < \mathfrak{c}$, and $\mathfrak{m}_R = \mathfrak{c} = \omega_1$.

Let us return to the question of finding, for a given real function f, a "large" set $X \subseteq \mathbb{R}$ such that f is continuous on X.

PROPOSITION 7.2 For every real function f and every cardinal $\theta < \mathfrak{m}_R$ there is a set $X \subseteq \mathbb{R}$ of power θ such that $f \restriction X$ is continuous.

PROOF Identify $\mathbb{R}$ with 2^ω and define a partition $[\mathbb{R}]^3 = K_0 \cup K_1$ so that $\{x, y, z\} \in K_0$ iff

$$(\forall a, b, c \in \{x, y, z\})\, \Delta(a, c) \neq \Delta(b, c) \rightarrow \Delta(f(a), f(c)) \neq \Delta(f(b), f(c))$$

Then f is continuos on every 0-homogeneous set (cf. Example 1.5). The partition is ccc; this may be proved using the argument similar to the one in the proof of the Proposition 7.1. By the definition of $\mathfrak{m}_R$ there is a 0-homogeneous set X of cardinality θ if the cofinality of θ is $> \omega$. If $\mathrm{cf}\,\theta = \omega$, split X into intervals of large-enough cardinalities and apply the previous argument to each of them. □

PROPOSITION 7.3 (cf. Example 1.3) The Lebesgue measure is $\mathfrak{m}_R$-additive.

PROOF Let $\{A_\xi : \xi < \theta\}$ be a sequence of null subsets of $[0, 1]$ of length $\theta < \mathfrak{m}_R$. We shall prove that the union of $\{A_\xi : \xi < \theta\}$ is of measure zero. So, without loss of generality suppose that $A_\xi \subseteq A_\eta$ for all $\xi < \eta < \theta$. Fix $\epsilon > 0$, and for every $\xi < \theta$ choose a compact $C_\xi \subseteq [0, 1]$ such that $C_\xi \cap A_\xi = \emptyset$ and $\mu(C_\xi) > 1 - \epsilon$. Now define $[\theta]^{<\omega} = K_0 \cup K_1$ by:

$$F \in K_0 \quad \text{iff} \quad \mu\left(\bigcap_{\xi \in F} C_\xi\right) > 1 - \epsilon.$$

CLAIM This is a ccc partition.

PROOF For any $F \in K_0$ define $C_F = \bigcap_{\xi \in F} C_\xi$. Fix an uncountable $\mathcal{F} \subseteq K_0$. We may assume that there is a $\delta > 0$ such that $\mu(C_F) > 1 - \epsilon + \delta$ for all $F \in \mathcal{F}$. Remember that the metric space of the algebra of Lebesgue-measurable subsets of $[0, 1]$ modulo the ideal of measure zero sets with the metric $d(A, B) = \mu(A \Delta B)$ is separable since the finite unions of disjoint rational intervals form a countable dense set. So shrinking $\mathcal{F}$ to an uncountable subfamily, we may assume that there is a finite union I of rational intervals such that for all F in $\mathcal{F}$ we have $\mu(C_F \Delta I) < \delta/2$. It follows that $\mu(C_F \cap C_G) > 1 - \epsilon$ for every F and G from $\mathcal{F}$. □

By the definition of $\mathfrak{m}_R$ there is a set of indexes $H \subseteq \theta$ of size $\mathfrak{m}_R$ and such that $\{C_\xi\}_{\xi \in H}$ is 0-homogeneous. Let C be $\bigcap_{\xi \in H} C_\xi$. By compactness we have $\mu(C) \geq 1 - \epsilon$ and $A_\xi \cap C = \emptyset$ for all $\xi < \theta$. This shows that $\bigcup_{\xi < \theta} A_\xi$ can be covered by an open set of measure $\leq \epsilon$. Since ϵ can be an arbitrary number > 0, we are done. □

EXERCISE Prove that the ideal of meager sets of reals is $\mathfrak{m}_R$-additive (see also Example 1.4).

DEFINITION 7.3 *The cardinal* $\mathfrak{m}$ *is defined to be equal to* $\min\{\theta :$ *there is a ccc poset* $\mathcal{P}$ *with a family* $\{\mathcal{D}_\xi\}_{\xi < \theta}$ *of dense subsets of* $\mathcal{P}$ *such that there is no* $\{\mathcal{D}_\xi : \xi < \theta\}$*-generic filter* $G \subseteq \mathcal{P}\}$.

MA(κ) is the assumption that $\mathfrak{m} > \kappa$. Martin's Axiom (MA) is the assumption that $\mathfrak{m} = \mathfrak{c}$. This is weaker than the Continuum Hypothesis.

DEFINITION 7.3 *(Topological version)* $\mathfrak{m} = \min\{\theta :$ *there is a ccc compact space* X *with the family* $\{U_\xi\}_{\xi < \theta}$ *of open dense sets such that* $\bigcap_{\xi < \theta} U_\xi = \emptyset\}$. *(In other words,* $\mathfrak{m}$ *is the least cardinal* κ *such that the* κ*-variant of the Baire Category Theorem fails).*

For the proof that these two definitions are equivalent see e.g. [Kunen, Theorem I-1.3.4]. This proof uses the third version of Definition 7.3—the Boolean-algebraic version. The next result shows that Martin's Axiom is a Ramsey-type statement.

THEOREM 7.1 The cardinals $\mathfrak{m}$ and $\mathfrak{m}_R$ are equal. □

DEFINITION 7.4 *A family of sets* $\{U_\xi\}_{\xi \in \theta}$ *is centered iff for every* $F \in [\theta]^{<\omega}$ *the set* $\bigcap_{\xi \in F} U_\xi$ *is not empty.*

DEFINITION 7.5 *Let* θ *be a cardinal. A topological space* X *has* θ *as a precaliber (caliber) iff for every family* $\{U_\xi\}_{\xi < \theta}$ *of open sets in* X *there is a* $A \in [\theta]^\theta$ *such that* $\{U_\xi : \xi \in A\}$ *is centered (has nonempty intersection).*

An analogous definition can be provided for posets, Boolean algebras, etc. The reader may examine which ccc posets that we had encountered so far have precaliber $\aleph_1$. (See proofs in the second part of § 1).

FACT X is separable $\Rightarrow X$ has $\aleph_1$ as a caliber $\Rightarrow X$ has $\aleph_1$ as a precaliber $\Rightarrow X$ is ccc, and no implication is reversible. □

Now we shall see that these properties are equivalent under some additional set-theoretical assumptions.

THEOREM 7.2 ($\mathfrak{m} > \omega_1$)

(1) Every ccc space X has precaliber $\aleph_1$.
(2) Every compact ccc space X has caliber $\aleph_1$.

PROOF (1) Let $\{U_\xi\}_{\xi < \omega_1}$ be a family of open sets in X. Define a partition $[\omega_1]^{<\omega} = K_0 \cup K_1$ by

$$F \in K_0 \quad \text{iff} \quad \bigcap_{\xi \in F} U_\xi \neq \emptyset.$$

To see that this partition is ccc fix an uncountable $\mathcal{F} \subseteq K_0$. To every $F \in \mathcal{F}$ we associate the open set $U_F = \bigcap_{\xi \in F} U_\xi$. Then there are $F, G \in \mathcal{F}$ such that $U_F \cap U_G \neq \emptyset$, since X is ccc. Then $F \cup G \in K_0$. A family of open sets is 0-homogeneous iff it has a finite intersection property, and by the definition of $\mathfrak{m}_R$, there is an uncountable 0-homogeneous set. To prove (2) apply the previous argument to a family $\{V_\xi\}_{\xi<\omega_1}$ such that $\overline{V_\xi} \subseteq U_\xi$ for all ξ, and apply the compactness of X. □

COROLLARY ($\mathfrak{m} > \omega_1$) A product of two ccc spaces is ccc.

(The reader may compare this Corollary with the results in §4).

PROOF Let X and Y be two ccc spaces, and let $\{U_\xi \times V_\xi\}_{\xi<\omega_1}$ be an uncountable family of disjoint open sets. By the Theorem 7.2 there is an uncountable set of indexes $A \in [\omega_1]^{\aleph_1}$ such that $U_\xi \cap U_\eta \neq \emptyset$ for all $\xi, \eta \in A$. There is also an uncountable set of indexes $B \in [A]^{\aleph_1}$ such that $V_\xi \cap V_\eta \neq \emptyset$ for all $\xi, \eta \in B$. It follows that $(U_\xi \times V_\xi) \cap (U_\eta \times V_\eta) \neq \emptyset$ for every ξ and η in B. This contradicts the assumption that $\{U_\xi \times V_\xi\}_{\xi<\omega_1}$ is an antichain. □

COROLLARY If $\mathfrak{m} > \omega_1$ then SH holds.

PROOF If the Suslin Hypothesis fails, the ccc property is not productive (see Example 4.1). □

REMARK Theorems of this kind (that is, with some additional assumptions, like $\mathfrak{m} > \omega_1$) may not be always widely appreciated among mathematicians. So we shall give a generalization of Theorem 7.2 (see Theorem 7.4 and Theorem 7.5), which doesn't use any such assumption but which have been discovered (and perhaps could only have been discovered) through the lengthy detour involving the cardinal $\mathfrak{m}$.

DEFINITION 7.6 *A topological space X has* countable tightness *(or X is* countably tight*) iff for all x in X, for every $A \subseteq X \setminus \{x\}$ such that $x \in \overline{A}$ there is a countable $B \subseteq A$ such that $x \in \overline{B}$.*

The following is a version of Theorem 7.2 (2) where the additional axiomatic assumption is replaced to a restriction on the compact space X.

THEOREM 7.3 Let X be a compact countably tight topological space; then X is separable iff X has caliber $\aleph_1$.

PROOF This follows immediately from Theorem 7.4 below. □

DEFINITION 7.7 *A family $\mathcal{B}$ of nonempty open subsets of X forms a* π-base *for X iff for every nonempty open $U \subseteq X$ there exists a $B \in \mathcal{B}$ such that $B \subseteq U$. The minimal cardinality of a π-base in X is called a* π-weight *and denoted by $\pi w(X)$.*

THEOREM 7.4 Every countably tight compact space has a *point-countable* π-*base*, i.e. a π-base $\mathcal{B}$ such that $\mathcal{B}_x = \{B \in \mathcal{B} : x \in B\}$ is countable for every x in X.

PROOF This Theorem itself will immediately follow from Theorem 7.5 and Proposition 7.4 below. □

A nice example of a space with easily "readable" cardinal functions is that of a *linearly ordered continuum*. (A *continuum* is a topological space that is connected, compact and Hausdorff, an example of a linearly ordered continuum is $[0,1]$). To "read" cardinal functions of X one needs to consider only an arbitrary partition tree T of intervals of X (splitting intervals in two in an inductive procedure until we reach the singletons). The interiors of elements of T will then be a π-base of T; thus we are left with the problem of computing the size of T. To this end, we need to know the width and the height of T. The former is bounded by the cellularity of X and the latter is determined by the sizes of the chains of T. To generalize this to arbitrary compact space Y one encounters several substantial difficulties. For example, what should play the role of the "partition tree" of Y, and even if we know this, what is the analogue of a "chain" of the partition tree? We shall now show how to overcome these difficulties. To this end, we firstly introduce the following notion which substitutes the notion of a clopen set in a situation where such sets don't necessarily exist.

DEFINITION 7.8 *A pair $\langle F, G\rangle$ is a* regular pair *iff G is open, F closed and $F \subseteq G$.*

Thus, the intention is that in a non-0-dimensional space, a regular pair is a "substitute" for a clopen set. The next definition gives us the desired analog—the notion of a free sequence of regular pairs—that should play the role of a chain in a partition tree.

DEFINITION 7.9 *A* free sequence of regular pairs *is a family $\{\langle F_\xi, G_\xi\rangle\}_{\xi<\theta}$ such that for every $K, L \in [\theta]^{<\omega}$, if $K < L$, then the set $\bigcap_{\xi\in K} F_\xi \cap \bigcap_{\xi \in L}(X \setminus G_\xi)$ is nonempty.*

We are now ready to state and prove the promised theorem which does not use any additional set-theoretical assumptions but which has Theorems 7.2–7.4 as its corollaries.

THEOREM 7.5 Let X be a topological space and let π be the minimal cardinality of a π-base for X; then there is a sequence $\{\langle F_\xi, G_\xi\rangle\}_{\xi<\pi}$ of regular pairs such that $\{\operatorname{int} F_\xi : \xi < \pi\}$ forms a π-base and for every $A \subseteq \pi$: if a family $\{F_\xi : \xi \in A\}$ is centered, then $\{\langle F_\xi, G_\xi\rangle : \xi \in A\}$ is a free sequence of regular pairs.

REMARK To prove Theorems 7.3 and 7.4 by using Theorem 7.5, we need the following fact whose proof gives us some explanation of the notion of free sequence.

PROPOSITION 7.4 A compact countably tight space X cannot have uncountable free sequence of regular pairs.

PROOF Suppose that $\langle F_\xi, G_\xi\rangle$ ($\xi < \omega_1$) is an uncountable free sequence of regular pairs of a compact space X. Then for each $\alpha < \omega_1$ the family

$$\mathcal{F}_\alpha = \{F_\xi : \xi < \alpha\} \cup \{X \setminus G_\xi : \alpha \leq \xi < \omega_1\}$$

of closed subsets of X has the finite intersection property. So, by compactness, $\bigcap \mathcal{F}_\alpha \neq \emptyset$ and we can fix a point x_α in this intersection. Let x be a complete accumulation point (see Appendix E) of $A = \{x_\alpha : \alpha < \omega_1\}$ in X. Then $x \in \overline{A}$ but $x \notin \overline{B}$ for every countable $B \subseteq A$. To see this, notice that for every $\xi < \omega_1$:

$$\{x_\alpha : \alpha \leq \xi\} \subseteq X \setminus G_\xi \quad \text{and} \quad \{x_\alpha : \xi < \alpha < \omega_1\} \subseteq F_\xi.$$

Thus, in particular, G_ξ is a neighborhood of x which does not intersect the initial part of the sequence A. It follows that X is not a countably tight space. □

PROOF (Theorem 7.5) Let $\pi = \pi w(X)$. We shall assume that X is "πw-homogeneous", i.e. that $\pi w(U) = \pi$ for every nonempty open set $U \subseteq X$. This assumption can be made since the following construction can be done separately below each member of a maximal disjoint family of πw-homogeneous open subsets of X. The family of regular pairs will be constructed recursively on ξ, with every F_ξ being closed G_δ with the nonempty interior and every G_ξ being open F_σ. Fix a π-base $\{U_\xi : \xi < \pi\}$ of X. Suppose that we have constructed a sequence $\{\langle F_\xi, G_\xi\rangle\}_{\xi<\alpha}$ for some $\alpha < \pi$. Consider the family

$$\mathcal{F}_\alpha = \{\bigcap_{\xi\in K} F_\xi \cap \bigcap_{\xi\in L}(X \setminus G_\xi) : K < L \text{ are finite subsets of } \alpha\}.$$

Note that the family $\mathcal{F}_\alpha$ is of cardinality at most $|\alpha| + \aleph_0 < \pi$. Note that $\mathcal{F}_\alpha$ is a family of closed G_δ subsets of X, and so by compactness and our assumption that $\pi w(U_\alpha) = \pi$ there must be nonempty open $G_\alpha \subseteq U_\alpha$ which includes no nonempty member of $\mathcal{F}_\alpha$. Clearly, we can take G_α to be also an F_σ-subset of X. For F_α choose any closed G_δ-subset of G_α having nonempty interior. The family constructed in this way has the required properties; the only nontrivial thing to check is that if F_ξ $(\xi \in A)$ is a centered family for some $A \subseteq \pi$, then $\bigcap_{\xi\in K} F_\xi \cap \bigcap_{\xi\in L}(X \setminus G_\xi) \neq \emptyset$ for all finite subsets $K < L$ of A. This is an easy induction on the size of L and is left to the reader. □

REMARK For more applications of Theorem 7.5 and for other historical remarks about this interesting subject the reader is referred to [Todorčević 1990].

8. REGULAR RADON MEASURES

In this section we shall apply the inequality $\mathfrak{m} > \omega_1$ to solve a problem from the Topological Measure Theory (see [Schwartz]). We shall prove that if $\mathfrak{m} > \omega_1$ then every regular Radon measure is σ-finite.

DEFINITION 8.1 *Let $\langle X, \mu, \mathrm{Borel}(X)\rangle$ be a given topological measure space ($\mathrm{Borel}(X)$ is the family of all Borel subsets of X). The measure μ is called a regular Radon measure iff*

(1) $\mu\left(\overline{\{x\}}\right) = 0$ *for every* $x \in X$,
(2) *For every x in X there is an open U such that $x \in U$ and $\mu(U) < \infty$,*
(3) $\mu(B) = \inf\{\mu(U) : B \subseteq U$ *and U is open\} for every Borel subset B of X (the outer regularity of μ),*
(4) $\mu(B) = \sup\{\mu(K) : K \subseteq B$ *and K is compact and Borel\} for every Borel subset B of X (the inner regularity of μ).*

Such a measure is σ-finite iff there is a family of Borel sets $\{X_n\}_{n\in\omega}$ such that $X = \bigcup_{n\in\omega} X_n$ and $\mu(X_n) < \infty$ for all $n \in \omega$.

LEMMA 8.1 Let B be a Boolean algebra with a strictly positive measure μ such that $\mu(1_B)$ is finite; then for every sequence $\{a_\alpha : \alpha < \omega_1\} \subseteq B$ there is an uncountable set $I \subseteq \omega_1$ such that $\mu(a_\alpha \cap a_\beta) > 0$ for every α and β in I.

PROOF Without loss of generality we could assume that $\mu(1_B) = 1$ and that there is an $\epsilon > 0$ such that for every α, $\mu(a_\alpha) > \epsilon$. Fix N such that $N \cdot \epsilon > 1$ and define a partition $[\omega_1]^2 = K_0 \cup K_1$ by $\{\alpha, \beta\} \in K_0$ iff $\mu(a_\alpha \cap a_\beta) > 0$. By the partition relation $\omega_1 \to (\omega_1, N)^2$ (for every finite N; see Appendix D—Theorem D.2.1 has this as a corollary), it is sufficient to show that there is no 1-homogeneous J of cardinality N; otherwise we would have:

$$1 = \mu(1_B) > \mu\bigcup_{b\in J} b = \sum_{b\in J} \mu b > N \cdot \epsilon > 1. \qquad \square$$

In this section we assume that $\langle X, \mu, \mathrm{Borel}(X)\rangle$ is a given topological space with a regular Radon measure μ defined on the family of Borel subsets of X. By $K \subseteq_\mu H$ we denote the fact that K is a subset of H in the sense of the measure μ, i.e. $\mu(K \setminus H) = 0$.

6 Зак. 2290

LEMMA 8.2 There is a $\subseteq$-maximal disjoint family $\mathcal{K}$ of compact subsets of X of positive measure such that for every $K \in \mathcal{K}$ every open set U is either disjoint from K or its intersection with K has positive measure. (Sets with this property will be called *μ-supporting*).

PROOF Let $\mathcal{K}_0$ be any $\subseteq$-maximal disjoint family of compact subsets of X. For every K in $\mathcal{K}$ let

$$K' = K \setminus \bigcup\{U : U \text{ is open and } \mu(U \cap K) = 0\},$$

and let $L_K = K \setminus K'$.

CLAIM Every compact $H \subseteq L_K$ has measure zero.

PROOF (Claim) The family $\{H \cap U : \mu(U \cap K) = 0\}$ forms an open cover of H, so it has a finite subcover. □

Thus, by the inner regularity of the measure μ, we have that $\mu(L_K) = 0$. The required family $\mathcal{K}$ is $\{K' : K \in \mathcal{K}_0\}$. □

REMARK Notice that if $\mathcal{K}$ is the family of Lemma 8.2 then every open set $U \subseteq X$ of finite measure intersects at most countably many elements of $\mathcal{K}$. So the above proof cannot be used to deduce that $\mathcal{K}$ is, in fact, countable, the conclusion which we are really interested in here.

The next lemma give us a characterization of the σ-finiteness of the measure μ.

LEMMA 8.3 The measure μ is σ-finite iff $\mathcal{K}$ is countable.

PROOF ($\Rightarrow$)We assume that there is a family of Borel sets $\{X_n : n \in \omega\}$ of finite measure such that $X = \bigcup_{n\in\omega} X_n$. We may also assume that every X_n is open and $X_n \subseteq X_{n+1}$ for all n. For every $K \in \mathcal{K}$ there is an $n \in \omega$ such that $K \subseteq X_n$, because K is compact. Every X_n is of finite measure, so the set $\mathcal{K}_{X_n} = \{K \in \mathcal{K} : K \subseteq X_n\}$ is at most countable for every n. [This is easily seen by noticing that the set $\{K \in \mathcal{K}_{X_n} : \mu(K) > 1/k\}$ (for $k, n < \omega$) is of the cardinality less than $n \cdot \mu(X)$, and thus finite.] Hence, $\mathcal{K}$ is countable as a countable union of finite sets.

($\Leftarrow$) By the maximality of $\mathcal{K}$ we have $\mu(X \setminus \bigcup\mathcal{K}) = 0$. □

LEMMA 8.4 Let $\mathcal{K}$ be a family of compact μ-supporting sets and suppose that for every K in $\mathcal{K}$ we have picked an open set U_K of finite measure containing K. If $\mathcal{K}$ is uncountable, then we cannot choose a point x_K in K for each K in $\mathcal{K}$ in such a way that $x_K \notin U_{K'}$ for $K \neq K'$ in $\mathcal{K}$.

PROOF Suppose that the choice is possible and let $Y = \{x_K : K \in \mathcal{K}\}$ and

$$B = \overline{Y} \cap \bigcup_{K\in\mathcal{K}} U_K.$$

Then B is a Borel subset of X, so μB is defined.

FACT The inner measure of B is equal to 0.

PROOF Suppose that $C \subseteq B$ is compact. Then C is discrete, and therefore finite; hence $\mu(C) = 0$. □

FACT The outer measure of B is ∞.

PROOF Suppose that $C \supseteq B$ is open. We have $\mu(U \cap K) > 0$ for every $K \in \mathcal{K}$, so by the above remark $\mu(U) = \infty$. □

These two facts, of course, contradict our assumption that μ is a regular Radon measure. □

Without the assumption that $\mathfrak{m} > \omega_1$ one cannot prove that $\mathcal{K}$ is countable—the best possible bound without any additional assumptions is given by the following Lemma (cf. remark after the proof of Lemma 8.4).

LEMMA 8.5 Suppose $\mathcal{K}$ is a disjoint family of compact subsets of X of positive measure. Then $\mathcal{K}$ has cardinality $\leq \aleph_1$.

PROOF As in the proof of Lemma 8.2, by shrinking every element of $\mathcal{K}$ if necessary, we may assume that every K in $\mathcal{K}$ is μ-supporting. For every K in $\mathcal{K}$ choose an open U_K of finite measure such that $U_K \supseteq K$. Now define $F : \mathcal{K} \to [\mathcal{K}]^{\leq\aleph_0}$ by

$$F(K) = \{H \in \mathcal{K} : H \neq K,\ H \cap U_K \neq \emptyset\}$$

Note that $F(K)$ is countable for every K in $\mathcal{K}$, so if $|\mathcal{K}| > \aleph_1$ then there is an uncountable set $\mathcal{K}' \subseteq \mathcal{K}$ such that $\mathcal{K}'$ is *free for* F—that is, $H \notin F(K)$ for all $H, K \in \mathcal{K}'$ (see [Erdös–Hajnal–Máté–Rado], Theorem 44.1). For every $K \in \mathcal{K}$ choose any $x_K \in K$ and notice that this family contradicts Lemma 8.4. □

REMARK Note that there is an $F : \omega_1 \to [\omega_1]^\omega$ without uncountable free sets. For example, take $F(\alpha) = \{\beta : \beta < \alpha\}$. So the above proof cannot be used to deduce that $\mathcal{K}$ is, in fact, countable – the conclusion which we are really interested in here.

THEOREM 8.1 ($\mathfrak{m} > \omega_1$) Every regular Radon measure is σ-finite.

PROOF Suppose that $|\mathcal{K}| = \aleph_1$. Define U_K for $K \in \mathcal{K}$ as before. For $p \in [\mathcal{K}]^{<\omega}$ and for $K \in \mathcal{K}$ define

$$U_p = \bigcup_{K\in p} U_K, \qquad \text{and} \qquad K_p = K \setminus U_{p\setminus\{K\}}.$$

The key partition $[\mathcal{K}]^{<\omega} = \mathcal{P}_0 \cup \mathcal{P}_1$ can now be defined by letting p into $\mathcal{P}_0$ iff K_p has a positive measure for every $K \in p$.

CLAIM $[\mathcal{K}]^{<\omega} = \mathcal{P}_0 \cup \mathcal{P}_1$ is a Suslin partition.

PROOF Fix $\{p_\alpha\}_{\alpha<\omega_1} \subseteq \mathcal{P}_0$. By the Δ-system Lemma we may assume that $\{p_\alpha\}_{\alpha<\omega_1}$ is a Δ-system with root q; hence $p_\alpha = q \cup r_\alpha$ with the r_α's being disjoint.

SUBCLAIM Without loss of generality we may assume that the root q is empty.

PROOF Fix a K in q and consider the family

$$\{K_{p_\alpha} : \alpha < \omega_1\}$$

of subsets of K of positive measure. By Lemma 8.1 there is an uncountable $I_0 \subseteq \omega_1$ such that $K_{p_\alpha} \cap K_{p_\beta}$ has positive measure for all α and β in I_0. Thus, refining the sequence $\{p_\alpha\}_{\alpha<\omega_1}$ successively for every element of the root q we can find uncountable $I \subseteq \omega_1$ such that $\mu(K_{p_\alpha} \cap K_{p_\beta}) > 0$ for all K in q and α and β in I. It is clear now that for α and β in I, $p_\alpha \cup p_\beta \in \mathcal{P}_0$ iff $r_\alpha \cup r_\beta \in \mathcal{P}_0$. □

Remember that every open U of finite measure intersects at most countably many K's from $\mathcal{K}$; hence we may assume without loss of generality that $U_K \cap K' = \emptyset$ for all $K \in p_\alpha$, $K' \in p_\beta$ and $\alpha < \beta < \omega_1$.

We may further assume that there are $\epsilon > 0$ and an integer M such that

$$(\forall\alpha)(\forall K \in p_\alpha)\, \mu(K_{p_\alpha}) > \epsilon \quad \text{and} \quad \mu\left(\bigcup_{H \in p_\alpha} U_H\right) < M.$$

Now define a function $I: \omega_1 \to [\omega_1]^{<\aleph_0}$ by

$$I(\beta) = \left\{\alpha < \beta : (\exists K \in p_\alpha) K \subseteq_\mu \bigcup_{H \in p_\beta} U_H\right\}.$$

(Recall that $A \subseteq_\mu B$ iff $A \setminus B$ is of measure zero.) If n is such that $n \cdot \epsilon > M$, then we have $|I(\beta)| < n$ for all β. Hence, there is an $A \in [\omega_1]^{\aleph_1}$ such that for all $\alpha \neq \beta$ in A the set $I(\beta)$ does not contain α. Therefore, for any α, β in A the set $p_\alpha \cup p_\beta$ is in $\mathcal{P}_0$, and the partition is ccc. □ (Claim)

By the Claim and the assumption that $\omega_1 < \mathfrak{m}$ there is an uncountable 0-homogenous $\mathcal{K}' \subseteq \mathcal{K}$. For all $K \in \mathcal{K}'$ define

$$\mathcal{F}_K = \{K \setminus U_{K'} : K' \in \mathcal{K}', K' \neq K\}$$

This is a family of closed subsets of K with the finite intersection property. Therefore we can choose an x_K in $\bigcap \mathcal{F}_K$ for every K in $\mathcal{K}'$. So we are again in a situation which contradicts Lemma 8.4. □ (Theorem)

9. SOME FACTS ABOUT GAPS IN $[\omega]^\omega$

When studying the power-set algebra of the integers modulo the ideal of finite sets, i.e. when studying various compactifications of the integers, one frequently needs to consider "the gaps", or the discontinuities of this structure. To introduce these objects, we need few definitions.

For two infinite subsets a and b of ω define the relations

$$a \subseteq^* b \quad \text{iff} \quad a \setminus b \text{ is finite.}$$
$$a =^* b \quad \text{iff} \quad a \subseteq^* b \text{ and } b \subseteq^* a,$$

and say that a and b are *almost disjoint* if the set $a \cap b$ is finite.

DEFINITION 9.1 *If κ and λ be two regular cardinals, then $\{a_\xi : \xi < \kappa\}$ and $\{b_\eta : \eta < \lambda\}$ form a gap in $[\omega]^\omega$ iff*

(1) The sequence a_η is $\subseteq^*$ increasing, while the sequence b_η is $\subseteq^*$ decreasing,
(2) $a_\xi \subseteq^* b_\eta$ for all $\xi < \kappa$ and $\eta < \lambda$, and
(3) There is no $c \subseteq \omega$ such that $a_\xi \subseteq^* c$ and $c \subseteq^* b_\eta$ for all $\xi < \kappa$ and $\eta < \lambda$.

(Sequences a_ξ and b_ξ satisfying (1) and (2) form a so-called *pre-gap*. If there is a c as in (3), it is said that such c *fills* (or *splits*) the pre-gap, and the pre-gap is thus *fillable* (or *splittable*). We say that (κ, λ^*) is the *type* of this gap.

Remember that if $(\alpha, <)$ is an ordered set, then by α^* we denote the order type of $(\alpha, >)$. So saying that the type of a gap is (κ, λ^*) instead of (κ, λ) suggests a way of visualizing the gap as set linearly ordered by $\subseteq^*$ in order type $\kappa + \lambda^*$. There is still another way to look at a given gap—if we consider the sets $\omega \setminus b_\eta$ instead of b_η $(\eta < \lambda)$, then the new gap consists of two $\subseteq^*$-increasing families such that no upper bound of the first one is almost disjoint with all sets of the second. We shall use both variants interchangeably. The gap points to a missing element in $[\omega]^\omega$, so the existence of gaps might be interpreted as "$[\omega]^\omega$ is not continuous" (or "is not saturated"). In many cases the conditions asserting the existence and the nonexistence of forcing notions which add the missing element (i.e. fill the gap) turn out to be rather natural conditions that can be put on a given gap. This explains why the knowledge about gaps in $[\omega]^\omega$ is useful in applications of Forcing, and conversely, why the Method of Forcing can help us to discover a great deal about the "discontinuities" in $[\omega]^\omega$. This will become more clear in latter sections.

Recall that in a partially ordered set $(X,\leq)$ a set $A\subseteq X$ is *cofinal* (*coinitial*) in a set $B\subseteq X$ if for every $b\in B$ there is an $a\in A$ such that $a\geq b$ $(a\leq b)$.

DEFINITION 9.2 *Let $(a_\xi,b_\eta)_{\xi<\kappa,\eta<\lambda}$ and $(c_\xi,d_\eta)_{\xi<\kappa,\eta<\lambda}$ be two pre-gaps of the same type. The first pre-gap is cofinal in the second iff $\{a_\xi:\xi<\kappa\}$ is cofinal in $\{c_\xi:\xi<\kappa\}$ and $\{b_\xi:\xi<\lambda\}$ is coinitial in $\{d_\xi:\xi<\lambda\}$. Two pre-gaps are equivalent iff they are cofinal in each other.*

It is easy to check that the property of being fillable is invariant under the equivalence of pre-gaps. For functions $f,g\in\omega^\omega$ define

$$f<^* g \qquad \text{iff} \qquad \{i\in\omega: f(i)>g(i)\} \quad \text{is finite.}$$

A gap in ω^ω is defined analogously to a gap in $[\omega]^\omega$ as the subset $\{a_\alpha:\alpha<\kappa\}\cup\{b_\alpha:\alpha<\lambda\}$ of ω^ω such that $\mathrm{tp}((\{a_\alpha:\alpha<\kappa\},<^*)=\kappa$, $\mathrm{tp}((\{b_\alpha:\alpha<\lambda\},<^*)=\lambda^*$, and there is no $c\in\omega^\omega$ such that $a_\alpha<^* c<^* b_\beta$ for all $\alpha<\kappa$ and $\beta<\lambda$. The proof of the following fact is left to the reader as an exercise:

FACT For every pair of cardinals κ,λ, a gap of type (κ,λ^*) exists in ω^ω iff it exists in $[\omega]^\omega$. □

We shall use this fact and deal always with the structure that is easier to handle in a given context. The existence and nonexistence of various types of gaps have proved to be very interesting questions relevant to a wide variety of problems some of which will be mentioned below.

THEOREM 9.1 There is an (ω_1,ω_1^*) gap in $[\omega]^\omega$.

PROOF See e.g. [Bekkali; p. 96]. □

The (ω_1,ω_1^*)-gaps are usually called *Hausdorff* gaps in honor of a mathematician who first constructed them (see [Hausdorff]). The following cardinal is connected with the gaps in ω^ω:

$\mathfrak{b}=\min\{|\mathcal{F}|:\mathcal{F}\subseteq\omega^\omega$ and $\mathcal{F}$ is $<^*$ unbounded$\}$.

It is easy to prove that $\mathfrak{b}>\omega$ using ordinary diagonalization. It is also not difficult to prove that $\mathfrak{b}\geq\mathfrak{m}$.

THEOREM 9.2 There are $(\mathfrak{b},\omega)$ and $(\omega,\mathfrak{b}^*)$ gaps in $[\omega]^\omega$. □

The following proposition shows that $\mathfrak{b}$ is, in fact, the minimal cardinal κ for which there is a (κ,ω^*) gap in $[\omega]^\omega$.

PROPOSITION 9.1 There is no (ω,ω^*)-gap in $[\omega]^\omega$. In fact, there are neither (κ,ω^*) nor (ω,κ^*) gaps in $[\omega]^\omega$ for any $\kappa<\mathfrak{b}$.

PROOF Fix a pre-gap $(a_i,b_i)_{i<\omega}$ and define a poset $\mathcal{P}$ as the set of all $p=(s_p,n_p,A_p)$ such that:

(1) s_p and A_p are finite subsets of ω, and
(2) n_p is in ω and $s_p\subseteq n_p$.

Ordering is defined by: $p\leq q$ iff

(3) $s_p\cap n_q=s_q$,
(4) $n_p\geq n_q$ and $A_p\supseteq A_q$,
(5) $a_i\cap[n_q,n_p)\subseteq s_p\cap[n_q,n_p)\subseteq b_i\cap[n_q,n_p)$ for all $i\in A_q$.

Sets $\mathcal{D}_n=\{p\in\mathcal{P}:n\in s_p\}$ and $\mathcal{E}_n=\{p\in\mathcal{P}:n\in A_p\}$ are dense in $\mathcal{P}$, and if G is $\{\mathcal{D}_n,\mathcal{E}_n:n<\omega\}$-generic, then $c=\bigcup_{p\in G}s_p$ fills the pre-gap.

To prove the general statement consider, for example, a (κ,ω^*)-pre-gap a_ξ $(\xi<\kappa)$, b_n $(n<\omega)$, where $\kappa<\mathfrak{b}$. For a given $\xi<\kappa$, let f_ξ be the element of ω^ω defined by

$$f_\xi(n)=\min\{m: a_\xi\subseteq b_n\cup m\}.$$

Pick an increasing mapping g in ω^ω such that $f_\xi<^* g$ for all ξ and set

$$c=\bigcap_{n<\omega}(b_n\cup g(n)).$$

Then it is easily checked that c splits the pre-gap. □

This combinatorial characterization of gaps in ω^ω will be needed later:

LEMMA 9.1 The following conditions are equivalent for every (ω_1,ω_1^*)-pre-gap $(a_\xi,b_\xi)_{\xi<\omega_1}$ in ω^ω:

(1) $(a_\xi,b_\xi)_{\xi<\omega_1}$ is a gap.
(2) For every pair X,Y of uncountable subsets of ω_1 and for every $\bar{n}\in\omega$ there are $\xi\in X$, $\eta\in Y$ and $n\geq\bar{n}$ such that $a_\xi(n)>b_\eta(n)$.

PROOF $\neg(2)\rightarrow\neg(1)$ Fix X, Y and $\bar{n}$ such that $a_\xi(n)<b_\eta(n)$ for all $\xi\in X$, $\eta\in Y$ and $n\geq\bar{n}$; then the function $c\in\omega^\omega$ defined by

$$c(n)=\begin{cases}0, & \text{for } n<\bar{n}\\ \sup_{\xi\in X}\{a_\xi(n)\}, & \text{for } n\geq\bar{n}, \text{ and}\end{cases}$$

fills the gap.

$\neg(1)\rightarrow\neg(2)$ Suppose that $c\in\omega^\omega$ fills the gap; then for every α there is an $n_\alpha\in\omega$ such that

$$(\forall n\geq n_\alpha)\ a_\alpha(n)<c(n)<b_\alpha(n). \quad \text{(cf. Fig. 16)}$$

Hence, there is an $\bar{n}\in\omega$ such that the set $X=\{\xi<\omega_1:n_\xi=\bar{n}\}$ is uncountable. Set $Y=X$ and check that $\neg(2)$ holds. □

COROLLARY 9.1 Suppose that the pre-gap $(a_\xi,b_\xi)_{\xi<\omega_1}$ has the property that $a_\xi(n)\leq b_\xi(n)$ for all $\xi<\omega_1$ and $n<\omega$. Then if for all $\xi<\eta\in\omega_1$ there exists an $n\in\omega$ such that $a_\xi(n)>b_\eta(n)$, then the pair $(a_\xi,b_\xi)_{\xi<\omega_1}$ forms a gap. □

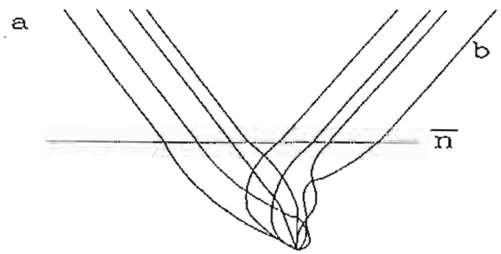

Fig. 16

Let us call a gap $\langle a_\xi, b_\xi\rangle_{\xi<\omega_1}$ with the property stated in the Corollary a *special gap*. We shall see later (Theorem 12.6) that for every (ω_1, ω_1^*)-gap $\langle a_\xi, b_\xi\rangle_{\xi<\omega_1}$ in ω^ω the poset of all finite $p \subseteq \omega_1$ such that for all $\xi < \eta$ in p there is an n such that $a_\xi(n) > b_\xi(n)$ is ccc. Thus, assuming $\mathfrak{m} > \omega_1$, every gap in ω^ω can be refined to (and thus is equivalent to) a special gap.

To avoid a trivial discussion, we shall always implicitly assume that $a_\xi \subseteq b_\xi$ for all ξ in the case when we are dealing with gaps in $[\omega]^\omega$, or that $a_\xi(n) \leq b_\xi(n)$ for all ξ and n when dealing with (ω_1, ω_1^*)-gaps in ω^ω.

DEFINITION 9.3 *Let $\langle a_\xi, b_\xi\rangle_{\xi<\omega_1}$ be a given gap in $[\omega]^\omega$. For a real $c \subseteq \omega$ let $\langle a_\xi^c, b_\xi^c\rangle_{\xi<\omega_1}$ denote the pre-gap $\langle a_\xi \cap c, b_\xi \cap c\rangle_{\xi<\omega_1}$*

DEFINITION 9.4 *Call a pre-gap $\langle a_\xi, b_\xi\rangle_{\xi<\omega_1}$ a Suslin pre-gap iff the partition $[\omega_1]^2 = K_0 \cup K_1$ defined by $\{\alpha, \beta\} \in K_0$ iff $a_\alpha \subseteq b_\beta$ and $a_\beta \subseteq b_\alpha$ is Suslin.*

It is easy to see that the property of being Suslin is not invariant under the equivalence of pre-gaps.

FACT Every fillable pre-gap is equivalent to a Suslin pre-gap.

PROOF By going to an equivalent pre-gap, we may assume that $a_\xi \subseteq b_\xi$ for all $\xi < \omega_1$. Suppose that $c \subseteq \omega$ fills the pre-gap $\langle a_\xi, b_\xi\rangle_{\xi<\omega_1}$. Let $\mathcal{F} = \{F_\xi : \xi < \omega_1\}$ be a family of finite 0-homogeneous sets. We may suppose that all $F \in \mathcal{F}$ are of the same length n, and so for every ξ we can fix an enumeration $F_\xi = \{\xi^i : i < n\}$. Let $a_{\xi^i} \setminus c = s_{\xi^i}$ and $c \setminus b_{\xi^i} = t_{\xi^i}$. Set $k_\xi = \max \bigcup_{i<n} s_{\xi^i} \cup t_{\xi^i}$. We may suppose without loss of generality that for some $k \in \omega$, some $u_i, v_i, s_i, t_i \subseteq k$ (for $i < n$) and all $\xi < \omega_1$:

(1) $k_\xi = k$,
(2) $F_\xi \restriction k = \langle u_i, v_i : i < n\rangle$ (this means that $a_{\xi^i} \cap k = u_i$ and $b_{\xi^i} \cap k = v_i$) for all $i < n$, and
(3) $s_{\xi^i} = s_i$ and $t_{\xi^i} = t_i$, for all $i < n$.

Notice that the following is true for all $\xi, \eta < \omega$ and all $i, j < n$:

(4) $s_i \subseteq t_j$ (because $a_{\xi^i} \subseteq b_{\xi^j}$),
(5) $a_{\xi^i} \setminus c = s_i \subseteq k$,

(6) $c \setminus b_{\eta^i} = t_i \subseteq k$; hence
(7) $a_{\xi^i} \setminus k \subseteq c \setminus k \subseteq b_{\eta^i} \setminus k$.

It follows now that for all $\xi, \eta < \omega_1$ and all $i, j < n$ we have that

$$\begin{aligned} a_{\xi^i} \setminus b_{\eta^j} &= ((a_{\xi^i} \setminus b_{\eta^j}) \cap k) \cup ((a_{\xi^i} \setminus b_{\eta^j}) \setminus k) \\ &= (s_i \setminus t_j) \cup ((a_{\xi^i} \setminus k) \setminus (b_{\eta^j} \setminus k)) \\ &= \emptyset. \end{aligned}$$

Therefore $F_\eta \cup F_\xi$ is homogeneous for all $\eta, \xi < \omega_1$, and this ends the proof. □

The converse is not always true as we shall see soon. The following Lemma is of the independent interest—the property of being Suslin depends only on the behavior of singletons, which is a very unusual phenomenon in the realm of Suslin partitions.

LEMMA 9.2 Let $\langle a_\xi, b_\xi\rangle_{\xi<\omega_1}$ be a pre-gap; then the following two conditions are equivalent:

(1) $\langle a_\xi, b_\xi\rangle_{\xi<\omega_1}$ is equivalent to a Suslin pre-gap,
(2) $\langle a_\xi, b_\xi\rangle_{\xi<\omega_1}$ is equivalent to a pre-gap $\langle a'_\xi, b'_\xi\rangle_{\xi<\omega_1}$ such that for every uncountable $X \subseteq \omega_1$ there exist $\xi < \eta$ in X such that $a'_\xi \subseteq b'_\eta$ and $a'_\eta \subseteq b'_\xi$.

PROOF We prove only the nontrivial direction. Suppose that (2) is true and, to simplify the notation, assume that, in fact, the pre-gap $\langle a_\xi, b_\xi\rangle_{\xi<\omega_1}$ has the property stated in (2). [Recall that we are also implicitly assuming that $a_\xi \subseteq b_\xi$ for all ξ.] Fix an uncountable family $\mathcal{F} = \{F_\eta : \eta < \omega_1\}$ of 0-homogeneous sets. We may assume that for some n, $|F_\eta| = n$ for all $\eta < \omega$, so we can increasingly enumerate each F_η as $\{\eta^i : i < n\}$. Refining $\mathcal{F}$, we may assume to have a $k \in \omega$ such that for all $\eta, \xi < \omega_1$:

(3) $F_\eta \restriction k = F_\xi \restriction k$ (this means that there are $s^i, t^i \subseteq k$ for all $i < n$ such that $a_{\eta^i} \cap k = s^i$ and $b_{\eta^i} \cap k = t^i$ for all $i < n$),
(4) $a_{\eta^i} \setminus k \subseteq a_{\eta^j} \setminus k$ and $b_{\eta^j} \setminus k \subseteq b_{\eta^i} \setminus k$ for all $i < j < n$.

Remember that also $a_{\xi^i} \subseteq b_{\xi^j}$ for all $i, j < n$, by the 0-homogeneity of the set F_ξ. Now applying (2) on the set $\{\eta^{n-1} : \eta < \omega_1\}$ we get an η and a ξ such that $a_{\eta^{n-1}} \subseteq b_{\xi^{n-1}}$ and $a_{\xi^{n-1}} \subseteq b_{\eta^{n-1}}$.

CLAIM The set $F_\eta \cup F_\xi$ is 0-homogeneous.

PROOF We have to prove that $a_{\xi^i} \subseteq b_{\eta^j}$ for all $i, j < n$. But

$$\begin{aligned} a_{\xi^i} \setminus b_{\eta^j} &= ((a_{\xi^i} \setminus b_{\eta^j}) \cap k) \cup ((a_{\xi^i} \setminus b_{\eta^j}) \setminus k) \\ &= (a_{\xi^i} \setminus b_{\eta^j}) \setminus k \\ &\subseteq (a_{\xi^{n-1}} \setminus b_{\eta^{n-1}}) \setminus k \\ &= \emptyset. \quad \square \end{aligned}$$

This ends the proof of Lemma 9.2. □

The following Theorem is an analogue of Theorem 3.1.

THEOREM 9.3 Let $\langle a_\xi, b_\xi\rangle_{\xi<\omega_1}$ be a Hausdorff gap in $[\omega]^\omega$ such that $a_\eta \subseteq b_\eta$ for all η. If c is a Cohen real, then $\langle a_\xi^c, b_\xi^c\rangle_{\xi<\omega_1}$ is a Suslin gap. Therefore it is consistent that there is a Suslin gap.

PROOF Let $p \in \{0,1\}^{<\omega}$ be a given condition and let $\dot{X}$ be a Cohen name for an uncountable subset of ω_1. Since $\dot{X}$ is a name for an uncountable set, we can find an uncountable set $Y \subseteq \omega_1$ and for each $\xi \in Y$ a condition $q_\xi \leq p$ forcing $\check{\xi} \in \dot{X}$. By shrinking Y we may assume that $q_\xi = q$ for some q and all $\xi \in Y$, and that $\mathrm{dom}(q) = n$ for some $n \in \omega$. Shrinking further we assume that for some $s, t \subseteq n$ and all $\xi \in Y$ we have $a_\xi \cap n = s$ and $b_\xi \cap n = t$ (note that $s \subseteq t$).

Pick $\xi < \eta$ from Y arbitrarily and let $u = a_\xi \setminus b_\eta \cup a_\eta \setminus b_\xi$; then u is a finite set disjoint from n; thus we can find a $\bar{q} \leq q$ including u in its domain such that $q(i) = 0$ for all $i \in u$. Hence, $\bar{q} \Vdash a_\xi^c \subseteq b_\eta^c$ and $a_\eta^c \subseteq b_\xi^c$ and so we are done by Lemma 9.2. Similarly one proves that $\langle a_\xi^c, b_\xi^c\rangle_{\xi<\omega_1}$ is a gap by invoking Lemma 9.1 instead of Lemma 9.2. □

EXERCISES (1) Let $\langle a_\xi, b_\xi\rangle_{\xi<\omega_1}$ be a fixed Hausdorff gap. Set

$$A = \{c \subseteq \omega : \langle a_\xi^c, b_\xi^c\rangle_{\xi<\omega_1} \text{ is still a Hausdorff gap}\}$$

Looking at A as a set of reals (i.e. a subset of $\{0,1\}^\omega$), analyze whether A has the Property of Baire. For example, show that if $\mathfrak{m} > \omega_1$ then A does not have the Property of Baire.

(2) Is Theorem 9.3 also true for the random real?

(3) What is the complexity of A relative to $\langle a_\xi, b_\xi\rangle_{\xi<\omega_1}$?

10. THE OPEN COLORING AXIOM, OCA

In this section we introduce a Ramsey-type hypothesis about sets of real numbers which can be considered as a two-dimensional perfect-set property and which is likely to have influence on any structure in close relationship with the set of reals.

DEFINITION 10.1 *Let X be a topological space. We say that the set $K \subseteq [X]^2$ is open iff the set $\{\langle x,y\rangle : \{x,y\} \in K, x \neq y\}$ is open in X^2. A partition $[X]^2 = K \cup L$ such that K is open is called an open partition.*

For a given topological space X, consider the following "Open Coloring Axiom" about partitions of $[X]^2$:

DEFINITION 10.2 *OCA(X): For every partition $[X]^2 = K_0 \cup K_1$, if K_0 is an open subset of $[X]^2$ then either*

(1) *There exists an uncountable 0-homogeneous Y, or*

(2) *There is a family of sets X_i for $i \in \omega$ such that $X = \bigcup_{i\in\omega} X_i$ and every X_i is 1-homogeneous.*

For which spaces X the statement OCA(X) is true? This is still a widely open problem. But consider the following restrictions on X:

a. The space X does not have uncountably many isolated points.

b. The space X is *second countable*, i.e. it has a countable basis.

FACT OCA(X) implies that the set of isolated points of X is at most countable.

PROOF Suppose that X has an uncountable set D of isolated points. Assume further that D has the size of at most continuum. Let $<_w$ be a well-ordering of D, and let $<_r$ be a linear ordering of D such that $\langle D, <_r\rangle$ is isomorphic with a set of reals. Let

$$K_0 = \{\langle x,y\rangle \in [X]^2 : x, y \in D \;\&\; (x <_w y \leftrightarrow x <_r y)\}.$$

Since points of D are isolated in X, this is an open subset of $[X]^2$. Note that if there is an uncountable 0-homogeneous set the orderings $<_w$ and $<_r$ would agree on an uncountable subset of D which is impossible. To see that the second alternative also fails assume $X = \bigcup_{n<\omega} X_n$ such that $[X_n]^2 \subseteq K_1$ for all n. Since D is uncountable, there must be n such that $D_n = D \cap X_n$ is uncountable. It follows that $[D_n]^2 \subseteq K_1$, so D_n is an uncountable conversely well-ordered subset of $\langle D, <_r\rangle$, a contradiction. □

It follows that the restriction a. must be assumed in any form of the Open Coloring Axiom about topological spaces. Of course, the restriction b. is much stronger. We shall now see that, in fact, it is the same restriction as the assumption that X is a set of reals.

LEMMA 10.1 Let X and Y be topological spaces such that Y is T_0 and there is a continuous function $f\colon X \xrightarrow{\text{onto}} Y$; then OCA($X$) implies OCA($Y$).

PROOF For an open partition $[Y]^2 = L_0 \cup L_1$ define a partition $[X]^2 = K_0 \cup K_1$ by

$$(*) \qquad \{x, y\} \in K_0 \quad \text{iff} \quad f(x) \neq f(y) \quad \text{and} \quad \{f(x), f(y)\} \in L_0$$

It is clear that this is an open partition of $[X]^2$ and that the image of a 0- (1-) homogeneous set is a 0- (1-) homogeneous set. □

Recall that every T_2 second countable space is a 1-1 continuous image of a set of reals (i.e. a subset of $\{0,1\}^\omega$). It follows that OCA for sets of reals implies OCA for second countable spaces. It is known that OCA for sets of reals is a consistent statement and also that it follows from the Proper Forcing Axiom which will be considered later (see also corollary to Theorem 12.4 and [Todorčević 1989]).

We now give a variant of OCA(X) which is more relevant to the study of definable sets of reals. Recall that a family $\mathcal{F}$ of sets of reals has the *Perfect Set Property* (PSP) iff for every $X \in \mathcal{F}$ either

(1) X contains a perfect set, or
(2) X is countable.

DEFINITION 10.3 *OCA*(X) is the following statement: Let X be a subset of $\mathbb{R}$ with the induced topology. For every partition $[X]^2 = K_0 \cup K_1$ such that K_0 is an open subset of $[X]^2$, either*

(1) there is a perfect (i.e. a nonempty compact subset without isolated points) 0-homogeneous $Y \subseteq X$, or
(2) X is the union of countably many 1-homogeneous sets.

Thus, OCA* can be considered as a two-dimensional version of the PSP. It is interesting that the obvious generalizations of OCA or OCA* to higher dimensions are false, as the next example shows. Of course, this should not prevent us from looking for a version of OCA which makes sense in all finite dimensions.

FACT The three-dimensional version of OCA* (and also the three-dimensional version of OCA) is false.

PROOF Notice that for any $x, y, z \in 2^\omega$ the set $\{\Delta(x,y), \Delta(y,z), \Delta(x,z)\}$ has exactly two members (cf. Fig. 1 or Fig. 17). Now define a partition of $[2^\omega]^3$ into K_0 and K_1 by $\{x, y, z\} \in K_0$ iff the set

$$\{x(\Delta(x,y) - 1), y(\Delta(y,z) - 1), z(\Delta(x,z) - 1)\}$$

has exactly two members (we accept the convention that $x(-1) = 0$). This partition is open.

CLAIM 1 There is no infinite 0-homogeneous set.

PROOF Suppose that H is 0-homogeneous and infinite. Let y be an accumulation point of H in $\{0,1\}^\omega$. Then we can choose x_0, x_1, x_2 and x_3 in H such that if $n_i = \Delta(x_i, y)$ ($i \leq 3$) then $n_0 < n_1 < n_2 < n_3$. Note that (see Fig. 17):

$$n_0 = \Delta(x_0, x_i), \quad \text{for all } 1 \leq i \leq 3,$$
$$n_1 = \Delta(x_1, x_i), \quad \text{for all } 2 \leq i \leq 3, \text{ and}$$
$$n_2 = \Delta(x_2, x_3).$$

Choose $i < j < 3$ such that $y(n_i - 1) = y(n_j - 1)$. It follows that $\{x_i, x_j, x_3\} \in K_1$, a contradiction. □

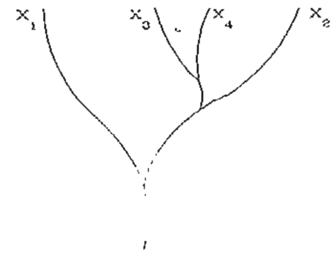

Fig. 17

CLAIM 2 Every 1-homogeneous subset H is nowhere dense in $\{0,1\}^\omega$.

PROOF Fix $s \in \{0,1\}^\omega$. We have to find $t \supset s$ such that $[t] \cap H = \emptyset$. Choose x_0, x_1 and x_2 in $[s]$ such that $\{x_0, x_1, x_2\} \in K_0$. Let $n < \omega$ be such that $\Delta(x_i, x_j) < n$ for all $i < j < 3$. Let $s_i = x_i \upharpoonright n$ for $i < 3$. It follows that there must be some $i < 3$ such that $[s_i] \cap H = \emptyset$, and so we are done. □

The fact that $\{0,1\}^\omega$ is not the union of countably many 1-homogeneous sets follows now from Claim 2 and the Baire Category Theorem. □

Notice that Lemma 10.1 is true also for OCA*, and recall that a set of reals is analytic iff it is a continuous image of ω^ω. Thus we have:

COROLLARY OCA*(ω^ω) implies OCA*(analytic sets). □

THEOREM 10.1 OCA*(X) is true for every analytic set of reals X.

PROOF By the Corollary it suffices to prove OCA*(ω^ω). Let us fix an open partition $[\omega^\omega]^2 = K_0 \cup K_1$ such that ω^ω can't be covered by countably many 1-homogeneous sets. Recursively on $\sigma \in \{0,1\}^{<\omega}$ construct a perfect subtree $\{t_\sigma : \sigma \in \{0,1\}^{<\omega}\} \subseteq \omega^{<\omega}$ such that:

(1) $t_\sigma \subseteq t_\tau$ for all $\sigma \subseteq \tau$.
(2) $t_{\sigma^\frown 0}$ and $t_{\sigma^\frown 1}$ are two incomparable successors of t_σ.
(3) If σ and τ are two different sequences of the same length then $[t_\sigma] \times [t_\tau] \subseteq K_0$.
(4) No interval $[t_\sigma]$ can be covered by countably many 1-homogeneous sets.

(Of course, this subtree need not be downward closed). To take care of (3) and (4) in the induction step, we need

CLAIM If I is an interval such that I^2 is not covered by countably many homogeneous sets, then there are clopen intervals $I_0, I_1 \subseteq I$ such that $I_0 \times I_1 \subseteq K_0$ and I_i $(i = 0, 1)$ can not be covered by countably many 1-homogeneous sets.

PROOF If $\mathcal{F}$ is the family of all basic clopen subintervals of I which can be covered by countably many 1-homogeneous sets then $A = I \setminus \bigcup \mathcal{F}$ is nonempty. This set is not 1-homogeneous (for $\mathcal{F}$ is countable), so fix $\{x, y\} \in [A]^2$ such that $\{x, y\} \in K_0$. The partition is open; hence there is a 0-homogeneous clopen base set $I_0 \times I_1$ containing this point. Note that the intervals I_0 and I_1 are not in $\mathcal{F}$, so this completes the proof. □

Let $P = \bigcap_{n<\omega} \bigcup_{\sigma \in \{0,1\}^n} [I_\sigma]$; then P is a perfect set and $[P]^2 \subseteq K_0$. □

FACT If $X \subseteq \mathbb{R}$, then OCA*(X) implies PSP(X).

PROOF Consider the partition $[X]^2 = K_0 \cup K_1$ such that $K_0 = [X]^2$ and $K_1 = \emptyset$. □

COROLLARY Every analytic set of reals is either countable or it contains a nonempty perfect subset. □

Here is a typical application of OCA* for analytic sets proved above.

THEOREM 10.2 If $f: \mathbb{R} \to \mathbb{R}$ is a Borel map then either f has a countable range or else there is a perfect $P \subseteq \mathbb{R}$ such that $f \restriction P$ is 1-1 (continuous and strictly increasing or strictly decreasing).

PROOF Let

$$X = \{\langle x, f(x) \rangle : x \in \mathbb{R}\}.$$

Then X is a Borel subset of $\mathbb{R}^2$ (see Appendix, Theorem C.1.1), so OCA*(analytic) applies to X. Consider the partition $[X]^2 = K_0 \cup K_1$ defined by

$$\{\langle x, y \rangle, \langle x', y' \rangle\} \in K_0 \quad \text{iff} \quad y \neq y'.$$

It is clear that K_0 is open. If the first alternative of OCA*(X) holds, i.e. if there is a perfect set $P \subseteq X$ such that $[P]^2 \subseteq K_0$, let P_0 be the projection of P to the first coordinate. Then $f \restriction P_0$ is 1-1. Shrinking P_0 further we may assume that the restriction is continuous (strictly increasing or strictly decreasing). Suppose now that the second alternative of OCA*(X) holds, i.e. that X can be covered by countably many sets $\{X_n\}$ such that $[X_n]^2 \subseteq K_1$ for all n. Then for every n, the projection of X_n to the second coordinate is a singleton, and so f has a countable range. □

In fact, the proof of Theorem 10.2 gives that following higher dimensional version of this result.

THEOREM 10.2* If A is an analytic subset of some finite power $\mathbb{R}^n$ of $\mathbb{R}$, then either:

(a) there is a perfect set $P \subseteq A$ such that $x_i \neq y_j$ for all $x \neq y$ in P and $i, j < n$, or

(b) A can be covered by a sequence A_i $(i < n)$ of sets such that for every $i < n$, the i-th projection of A_i is at most countable.

PROOF Define $[A]^2 = K_0 \cup K_1$, by $\{x, y\} \in K_0$ iff $x_i \neq y_j$ for all $i, j \neq n$, and consider the alternatives given by OCA*. Thus a perfect 0-homogeneous set gives the alternative (a). So suppose A can be covered by a sequence B_k $(k < \omega)$ of 1-homogeneous sets. For each $k < \omega$ fix an arbitrary $b_k = \langle b_{ki} : i < n \rangle$ in B_k and set $C = \{b_{ki} : k < \omega, i < n\}$. Then for every x in A, one of its projections must be in C, i.e. $\bigcup_{i<n} \pi_i^{-1} C$ covers A. □

REMARK The conclusion of Theorem 10.2* is also true for analytic subsets of the infinite power $\mathbb{R}^\omega$ with essentially the same application of OCA*. For more about this type of problems as well as for some historical remarks and references the reader is referred to [Hohti].

It is interesting that the dual of OCA does not hold, as the following proposition shows.

PROPOSITION 10.1 There is an open partition $[\mathbb{R}]^2 = K_0 \cup K_1$ such that there is no uncountable 1-homogeneous set and $\mathbb{R}$ is not a countable union of 0-homogeneous sets.

PROOF We identify $\mathbb{R}$ with the set ω^ω. For each $f \in \mathbb{R}$ define a sequence of reals $\{f_i\}_{i\in\omega}$ such that $\lim_{i\to\infty} f_i = f$ in the following way:

(1) Let $n_0 < n_1 < \ldots$ be the (finite or infinite) sequence of all $n \in \omega$ such that $f(2n + 1) \neq 0$

(2) Let $k_i(f) = \min\{k : f(2n_0 + 1) + f(2n_1 + 1) + \cdots + f(2n_k + 1) > i\}$,

(3) Define a sequence $\{f_i\}_{i\in\omega}$ of elements of $\mathbb{R}$ as follows:

If $k_i(f)$ exists then

(4) $f_i \restriction n_{k_i(f)} = f \restriction n_{k_i(f)}$, and

(5) $f_i(n_{k_i(f)} + j) = f(2^{i+1}(2n_{k_i(f)} + 2j + 1))$;

otherwise set $f_i = f$.

A partition $[\mathbb{R}]^2 = K_0 \cup K_1$ is defined by:

$$\{f, g\} \in K_0 \quad \text{iff} \quad \forall i (f \neq g_i \;\&\; g \neq f_i)$$

CLAIM 1 This partition is open.

PROOF By definition, if $f \neq g_i$ and $g \neq f_i$ for every $i \in \omega$ for some reals f, g, then the reason for that is captured on some finite level n—that is, any pair of reals that extends $\langle f \restriction n, g \restriction n \rangle$ is also in K_0. This n defines a clopen neighborhood of $\langle f, g \rangle$ that lies in K_0 (cf. Fig. 18). □

CLAIM 2 There is no uncountable 1-homogeneous set Y.

PROOF Suppose that there is an uncountable 1-homogeneous set Y. Then we can find uncountable $Y_0, Y_1 \subseteq Y$ such that for some $n < \omega$,

(6) $\Delta(f, g) = n$ for all $f \in Y_0$ and $g \in Y_1$.

Since for every f, the set $\{f_i : i < \omega\}$ is either finite or a sequence converging to f, we have that for every $\epsilon < 2$,

(7) $\{f_i : i < \omega\} \cap Y_{1-\epsilon}$ is finite for every $f \in Y_\epsilon$.

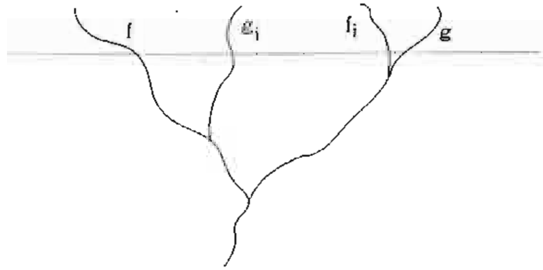

Fig. 18

So for every f in Y_0 there is a basic open interval I_f such that $Y_1 \cap I_f$ is uncountable and disjoint from $\{f_i : i < \omega\}$. So there is uncountable $Z_0 \subseteq Y_0$ such that for some fixed I we have $I_f = I$ for all f in Z_0. Similarly we can get an interval J such that $J \cap Z_0$ is uncountable and disjoint from $\{g_i : i < \omega\}$ for every g in some uncountable subset Z_1 of $Y_1 \cap I$. Take an f in $J \cap Z_0$ and a g in Z_1. Then $\{f, g\} \in K_0$ contradicting the initial assumption that the set Y is 1-homogeneous. □

CLAIM 3 $\mathbb{R}$ can't be covered by a countable family $\{X_i\}_{i\in\omega}$ of 0-homogeneous sets.

PROOF Let X_i $(i < \omega)$ be a given sequence of 0-homogeneous sets. We shall define (by a sequence of partial approximations) an element f of $\mathbb{R}$ avoiding all X_i's. Pick $f^0 \in X_0$ arbitrarily and set:

$$f(0) = f^0(0), \quad f(1) = 1, \quad f(2) = f^0(2) + 1, \quad \text{and}$$
$$f(2(2 \cdot 0 + 2j + 1)) = f^0(0 + j) \quad (j < \omega).$$

Let $t_0 = f \restriction 3$. If $X_1 \cap [t_0] \neq \emptyset$, choose f^1 in the intersection and set:

$$f(3) = 1, \quad f(4) = f^1(4) + 1, \quad \text{and}$$
$$f(2^2(2 \cdot 1 + 2j + 1)) = f^1(1 + j), \quad (j < \omega).$$

If $X_1 \cap [t_0] = \emptyset$, choose f^1 in $[t_0]$ arbitrarily and use it to partially determine f as above. Let $t_1 = f \restriction 5$ and ask if $X_2 \cap [t_1] \neq \emptyset$ or not, ... etc. Clearly this procedure will determine an element f of $\mathbb{R}$ such that $f_i = f^i$ for all i.

SUBCLAIM $f \notin X_i$ for all i.

PROOF Suppose that f is an element of X_i for some i. (Notice that since $f^0 \in X_0$ and $f^0 \neq f$ our i must be > 0). At the i-th stage of the construction the answer to the question whether $X_i \cap [t_{i-1}] \neq \emptyset$ or not was positive, so f^i was chosen in this intersection. Moreover, at that stage of the construction we have made sure that f differs from f^i at some coordinate. It follows that $\{f, f^i\}$ is an element of $[X_i]^2 \cap K_1$, a contradiction. □

This concludes the proof of the Claim and the proof of Proposition 10.1. □ □

REMARK It is interesting that by a well-known set-mapping theorem (see e.g. in [Erdős–Hajnal–Máté–Rado, Theorem 44.1]) applied to the set mapping $f \mapsto \{f_i : i < \omega\}$ defined in the proof of Proposition 10.1, it follows that $\mathbb{R}$ can be represented as the union of $\aleph_1$ 0-homogeneous sets.

Now we give an application of OCA. An ideal $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ is a *P-ideal* iff for every countable set $\{a_n\}_{n\in\omega} \subseteq \mathcal{I}$ there is a $b \in \mathcal{I}$ such that $a_n \subseteq^* b$ for all $n \in \omega$. For some ideal $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ the family of functions $\mathcal{F} = \{f_a : a \to \mathbb{N}\}_{a\in\mathcal{I}}$ is *trivial* iff there is $g: \mathbb{N} \to \mathbb{N}$ such that $f_a =^* g \restriction a$. It is said that such g *trivializes* $\mathcal{F}$.

THEOREM 10.3 (OCA) If $\mathcal{F} = \{f_a : a \to \mathbb{N}\}_{a\in\mathcal{I}}$ is a family indexed by some P-ideal $\mathcal{I}$ then $\mathcal{F}$ is trivial iff every subfamily of $\mathcal{F}$ of power $\aleph_1$ is trivial.

PROOF ($\Leftarrow$, the other direction is trivial) First identify every a in $\mathcal{I}$ with the graph of the function f_a, that is, the set $\{(i, f_a(i)) : i \in a\}$. Thus we may assume that $\mathcal{I}$ is a subspace of $\{0,1\}^{\mathbb{N}\times\mathbb{N}}$. Define a partition $[\mathcal{I}]^2 = K_0 \cup K_1$ by:

$$(*) \qquad \{a, b\} \in K_0 \quad \text{iff} \quad (\exists i \in a \cap b)\, f_a(i) \neq f_b(i).$$

To verify that the partition is open, for a pair $\{a, b\} \in K_0$ consider the basic open set $U \subseteq K_0$ of all pairs $\{f, g\}$ in $\mathcal{F}$ such that $f(i) = f_a(i)$ but $g(i) \neq f_a(i)$.

By the OCA, we have the following alternatives:

1^0 There is a 0-homogeneous family $\mathcal{K}$ of cardinality $\aleph_1$. Suppose that $\{f_a : a \in \mathcal{K}\}$ is trivial, i.e. that we have $g: \mathbb{N} \to \mathbb{N}$ such that $f_a =^* g$ for all $a \in \mathcal{K}$. Use the counting argument to get a $\mathcal{K}' \subseteq \mathcal{K}$ of power $\aleph_1$ and $n \in \mathbb{N}$ such that $f_a \restriction (a \setminus n) = g \restriction (a \setminus n)$ and $f_a \restriction n = f_b \restriction n$ for all $a, b \in \mathcal{K}'$. It follows that f_a and f_b agree on $a \cap b$ for all a and b in $\mathcal{K}$ contradicting the fact that $\mathcal{K}$ is 0-homogeneous.

2^0 There is a family $\{\mathcal{I}_n : n \in \omega\}$ such that $\mathcal{I} = \bigcup_{n\in\omega} \mathcal{I}_n$ and $[\mathcal{I}_n]^2 \subseteq K_1$ for all n in ω. There is an $n \in \mathbb{N}$ such that for every $a \in \mathcal{I}$ there is a $b \in \mathcal{I}_n$ such that $a \subseteq^* b$ (i.e. $\mathcal{I}_n$ is $\subseteq^*$-*cofinal* in $\mathcal{I}$), for $\mathcal{I}$ is a P-ideal. In this case $g = \bigcup_{a\in\mathcal{I}_n} f_a$ trivializes $\mathcal{F}$. □

Theorem 10.3 allows another formulation in the theory of chain conditions of partially ordered sets. To state this result, let us say that a poset $\mathcal{P}$ is a *poset of partial functions from* $\mathbb{N}$ *into* $\mathbb{N}$ if every element f of $\mathcal{P}$ is a function such that $\mathrm{dom}(f) \subseteq \mathbb{N}$ and $\mathrm{range}(f) \subseteq \mathbb{N}$ and the ordering of $\mathcal{P}$ is $\supseteq$. For example, the Cohen poset $\mathcal{C}_\omega$ considered in previous sections is such a poset with all elements having finite domains. The poset $\mathcal{C}_\omega$ is ccc for the trivial reasons that it is countable. Finding a nontrivial class of ccc posets of possibly infinite partial functions is a rather difficult matter. The following two variants of Theorem 10.3 explains why this is so.

THEOREM 10.3* (OCA) Let $\mathcal{P}$ be a poset of partial functions from $\mathbb{N}$ into $\mathbb{N}$. Then $\mathcal{P}$ is ccc iff $\mathcal{P}$ is σ-centered. □

THEOREM 10.3** The following are equivalent for every analytic poset of partial functions from $\mathbb{N}$ into $\mathbb{N}$:

(1) $\mathcal{P}$ is ccc,
(2) $\mathcal{P}$ is σ-centered,
(3) $\mathcal{P}$ contains no perfect set of pairwise incompatible elements. □

(Here, "analytic" means that if we identify $\mathcal{P}$ with a set of reals as in the proof of Theorem 10.3 then $\mathcal{P}$ becomes an analytic set of reals).

We shall now give an application of Theorem 10.3 which shows that OCA has a strong influence on gaps in $[\omega]^\omega$. Comparing it with the results cited in §9, we see that the set of types of gaps in ω^ω is very restricted under OCA.

THEOREM 10.4 (PFA, almost OCA) Every gap in $(\omega^\omega, <^*)$ is either of type (ω_1, ω_1^*), or (ω_2, ω^*), or (ω, ω_2^*).

Theorem 10.4 follows from the following Lemma and Theorem 10.5; recall that (κ, λ^*) exists in ω^ω iff it exists in $[\omega]^\omega$.

LEMMA 10.2 (OCA) Every (κ, λ^*) gap in $([\omega]^\omega, \subseteq^*)$ where both κ and λ have uncountable cofinalities, is of type (ω_1, ω_1^*).

PROOF Let $\{a_\xi : \xi < \kappa\}$, $\{b_\eta : \eta < \lambda\}$ be a given gap in $[\omega]^\omega$, i.e.:

(1) the sequence of a's is $\subseteq^*$ increasing, $a_\xi \subseteq^* a_\eta$ for all $\xi < \eta < \kappa$,
(2) the sequence of b's is $\subseteq^*$ increasing, $b_\xi \subseteq^* b_\eta$ for all $\xi < \eta < \lambda$,
(3) a_ξ and b_η are almost disjoint, $|a_\xi \cap b_\eta| < \omega$ for all $\xi < \kappa$ and $\eta < \lambda$
(4) there is no $c \subseteq \omega$ such that $a_\xi \subseteq^* c$ and $|b_\eta \cap c| < \omega$ for all $\xi < \kappa$ and $\eta < \lambda$.

Let $\mathcal{I}$ be an ideal in $\mathcal{P}(\omega)$ generated by the family $Y = \{a_\xi \cup b_\eta : \xi < \kappa, \eta < \lambda\}$. For $a_\xi \cup b_\xi$ in Y, let $f_{a_\xi \cup b_\eta} : (a_\xi \cup b_\eta) \to \{0,1\}$ be such that $f^{-1}(0) = a_\xi$ and $f^{-1}(1) =^* b_\eta$. By our assumptions that $\mathrm{cf}(\kappa) > \omega$ and $\mathrm{cf}(\lambda) > \omega$, the ideal $\mathcal{I}$ is a P-ideal. Note that $\mathcal{F} = \{f_y : y \in Y\}$ is coherent but it is not trivial, because if g were a trivializing function on $\mathcal{F}$, the set $c = g^{-1}(0)$ would fill the gap.

CLAIM Suppose that there are $A \in [\kappa]^{\aleph_1}$ and $B \in [\lambda]^{\aleph_1}$ such that the family $\mathcal{F}_0 = \{f_{a_\xi \cup b_\eta} : \xi \in A, \eta \in B\}$ is not trivial; then A is cofinal in κ and B is cofinal in λ.

PROOF Suppose that A is not cofinal in κ; then there is a ξ_0 such that $a_{\xi_0} \supseteq^* a_\xi$ for all $\xi \in A$. Then the characteristic function of $\omega \setminus a_{\xi_0}$ would trivialize $\mathcal{F}_0$. □

By the Claim, our gap is of type (ω_1, ω_1^*).. □

REMARK Note that the above proof shows that Theorem 10.3** can be used to considerably determine the structure of *Borel gaps* in $[\omega]^\omega$ or ω^ω.

THEOREM 10.5 If $\mathfrak{b} > \omega_2$, then there is a (ω_2, λ^*)-gap for some λ such that $\mathrm{cf}(\lambda) > \omega$.

PROOF Fix $A = \{a_\xi : \xi < \omega_2\}$, a $<^*$-strictly increasing family of elements of ω^ω. Let $\mathcal{F}$ be a maximal family that is linearly ordered by $<^*$ and extends A, and let $\mathcal{F}^+$ be the set $\{b \in \mathcal{F} : (\forall \xi < \mathfrak{b}) a_\xi <^* b\}$. Pick a coinitial subset B of $\mathcal{F}^+$ of order type λ^* for some regular cardinal λ. Then (A, B) forms a gap, and λ cannot be 0, 1 or ω by the Proposition 9.1 (see also [Rothberger]). □

LEMMA 10.3 Let $A = \{f_\xi : \xi < \mathfrak{b}\}$ be a $<^*$-increasing $<^*$-unbounded sequence of increasing mappings from ω^ω. Then there is an open partition $[A]^2 = K_0 \cup K_1$ without a homogeneous set of size $\mathfrak{b}$.

PROOF Put $\{f, g\}$ in K_0 iff $f(i) > g(i)$ and $f(j) < g(j)$ for some $i, j < \omega$. Clearly, K_0 is open in $[A]^2$. The fact that there are no homogeneous sets of size $\mathfrak{b}$ follows from Lemma 0.7 of [Todorčević 1989]. □

COROLLARY OCA implies that $\mathfrak{b} = \omega_2$.

PROOF By Theorem 10.5 and Lemma 10.2 we have that $\mathfrak{b} \le \omega_2$. The inequality $\mathfrak{b} > \omega_1$ follows from Lemma 10.3. □

PROOF (Theorem 10.4) By Lemma 10.2 we have to take care only of the gaps of type (κ, ω^*). By Proposition 10.1 the gaps of type (ω_1, ω^*) do not exist since $\mathfrak{b} > \omega_1$ while the gaps of type (κ, ω^*) for some $\kappa > \omega_2$ do not exist simply because PFA implies that $\mathfrak{c} = \aleph_2$ (see [Bekkali]). □

REMARK It is not known whether Theorem 10.4 can be proved using OCA alone. The reader may investigate the connection between Theorem 10.3 and Theorem 8.7 in [Todorčević 1989]—these two theorems are instances of a more general statement. He may also compare the proof of Theorem 10.4 (more precisely, the proof of Lemma 10.2) with the proof of Theorem 8.7 there.

There is another version of Theorem 10.3 which might be worth pointing out. It deals with the distance between two partial functions f and g from ω into $\mathbb{R}$:

$$\|f - g\| = \sup_{n \in a \cap b} |f(n) - g(n)|,$$

where $a = \mathrm{dom}(f)$, $b = \mathrm{dom}(g)$, and where we let $\|f - g\| = 0$ if $a \cap b = \emptyset$. Let a *ball of partial functions* with center h and radius $0 < \epsilon \le \infty$ be the set

$$B_\epsilon(h) = \{f : \|f - h\| < \epsilon\}.$$

This set will also be called an *ϵ-ball around h*. Note that if f and g are two partial functions whose domains are included in a domain of a third partial function h then

$$\|f - g\| \le \|f - h\| + \|h - g\|.$$

Thus, if $\mathcal{F}$ is a family of partial functions such that $\mathcal{F} \subseteq B_\epsilon(h)$ for some total $h : \omega \to \mathbb{R}$ and $0 < \epsilon \le \infty$ then $\|f - g\| < \infty$ for all f and g in $\mathcal{F}$. The following result gives a necessary and sufficient condition for the existence of a total function h and $0 < \epsilon \le \infty$ such that $\mathcal{F} \subseteq B_\epsilon(h)$.

THEOREM 10.6 (OCA) Let $\mathcal{F} = \{f_a : a \to \mathbb{R}\}_{a \in \mathcal{I}}$ be a family indexed by a P-ideal $\mathcal{I}$. Then $\mathcal{F}$ is included in a ball around a total function iff this is true for every $\mathcal{F}_0 \subseteq \mathcal{F}$ of size at most $\aleph_1$.

PROOF ($\Leftarrow$, the other direction is trivial) Clearly, we may assume that $\mathcal{I}$ is not of the form $\{a : a \subseteq^* b\}$ for some $b \subseteq \omega$, since in this case the result is trivial. Define the partition $[\mathcal{I}]^2 = K_0 \cup K_1$ by

$$\{a, b\} \in K_0 \quad \text{iff} \quad \|f_a - f_b\| > \Delta(a, b)$$

(here, $\Delta(a,b) = \min(a\Delta b)$). As in the proof of Theorem 10.3, there is a separable metric topology on $\mathcal{I}$ for which K_0 is an open subset of $[\mathcal{I}]^2$. Thus, by OCA we may consider the two alternatives:

1^0 There is an $\mathcal{X} \subseteq \mathcal{I}$ of size $\aleph_1$ such that $[\mathcal{X}]^2 \subseteq K_0$. By the assumption, pick $h: \omega \to \mathbb{R}$ such that for some $0 < \epsilon \leq \infty$,

$$\{f_a : a \in \mathcal{X}\} \subseteq B_\epsilon(h).$$

Thus, in particular $\|f_a - h\| < \infty$ for all $a \in \mathcal{X}$. So there is an $M < \infty$ and an uncountable $\mathcal{X}_0 \subseteq \mathcal{X}$ such that $\|f_a - h\| < M$ for all a in $\mathcal{X}$. It follows that

$$\|f_a - f_b\| < 2M \quad \text{for all } a, b \in \mathcal{X}_0$$

Choose a and b in $\mathcal{X}_0$ such that $\Delta(a,b) > 2M$. [The set $\{a \cap 2M : a \in \mathcal{X}_0\}$ is finite, hence the required a and b do exist.] Then $\{a,b\} \in K_0$ implies

$$\|f_a - f_b\| > \Delta(a,b) > 2M,$$

a contradiction.

2^0 $\mathcal{I} = \bigcup_{n<\omega} \mathcal{I}_n$ such that $[\mathcal{I}_n]^2 \subseteq K_1$ for all $n < \omega$. Then, since $\mathcal{I}$ is a P-ideal, one of the $\mathcal{I}_n$'s (say $\mathcal{I}_0$) must be $\subseteq^*$-cofinal in $\mathcal{I}$. For the same reasons, for each $k < \omega$ there is an $s \subseteq k$ such that

$$\mathcal{I}_0^s = \{a \in \mathcal{I}_0 : a \cap k = s\}$$

is cofinal in $\mathcal{I}$.

CLAIM There is a $k < \omega$ and distinct $s, t \subseteq k$ such that both $\mathcal{I}_0^s$ and $\mathcal{I}_0^t$ are cofinal in $\mathcal{I}$.

PROOF If not, then for every k there is the unique $s_k \subseteq k$ for which $\mathcal{I}_0^{s_k}$ is cofinal. So for every $t \subseteq k$ distinct from s_k we can choose an $a(t,k) \in \mathcal{I}$ such that $a(t,k) \not\subseteq a$ for every $a \in \mathcal{I}_0^t$. Choose $b \in \mathcal{I}_0$ such that

$$a(t,k) \subseteq^* b \quad \text{for all } k < \omega \text{ and } t \subseteq k \text{ such that } t \neq s_k.$$

Then for every $k < \omega$ we must have that $b \cap k = s_k$. Choose a $c \in \mathcal{I}_0$ such that $b \subseteq^* c$ and $c \neq b$. Let k be such that $c \cap k = t \neq b \cap k = s_k$. Then $c \in \mathcal{I}_0^t$ and $a(t,k) \subseteq^* c$ contradicting the choice of $a(t,k)$. □

Fix k, s and t as obtained by Claim, and let $h: \omega \to \mathbb{R}$ be defined by

$$h(n) = \min\{f_a(n) : a \in \mathcal{I}_0^s, n \in a\}$$

($\min \emptyset = 0$). We claim that $\mathcal{F} \subseteq B_\infty(h)$, i.e. that $\|f_a - h\| < \infty$ for all $a \in \mathcal{I}$. Since $\mathcal{I}_0^t$ is cofinal, it suffices to check this for a in $\mathcal{I}_0^t$. Suppose that this is not true and pick an $a \in \mathcal{I}_0^t$ such that for each $i < \omega$ there is an $n_i \in a$ such that

$$f_a(n_i) - h(n_i) > i.$$

Choose $b \in \mathcal{I}_0^t$ such that $a \subseteq^* b$. Then, since n_i's must converge to ∞, we can find an $i > 2k$ such that n_i is also an element of b. Then, by $\{a,b\} \in K_1$, we have that $\|f_a - f_b\| \leq \Delta(a,b) < k$, so

$$2k < f_a(n_i) - h(n_i) \leq |f_a(n_i) - f_b(n_i)| + |f_b(n_i) - h(n_i)|,$$

and so we must have that

$$|f_b(n_i) - h(n_i)| > k.$$

By the definition of $h(n_i)$, there is a $c \in \mathcal{I}_0^s$ such that $h(n_i) = f_c(n_i)$. (The set from the definition of $h(n_i)$ is not empty as, for example, a belongs to it). It follows that

$$\|f_b - f_c\| > k.$$

On the other hand, $\{b,c\} \in K_1$, so we also have that

$$\|f_b - f_c\| \leq \Delta(b,c) < k,$$

a contradiction. □

Now we turn to another application of OCA in the field of Boolean algebras.

THEOREM 10.7 (OCA) Every uncountable Boolean algebra contains an uncountable subset of pairwise incomparable elements.

REMARK We shall construct a set of incomparable, not incompatible elements. Sometimes a former is called an antichain in a Boolean algebra.

LEMMA 10.4 Let X and Y be two uncountable linearly ordered sets and consider $X \times Y$ ordered coordinatewise: $\langle x,y\rangle \leq \langle x',y'\rangle$ iff $x \leq x'$ and $y \leq y'$. If $X \times Y$ is the union of countably many chains, then both X and Y must have uncountable families of disjoint nontrivial intervals.

PROOF Suppose that $X \times Y = \bigcup_{n<\omega} C_n$. Let $f: X \to Y$ be such that $F = \text{range}(f)$ is uncountable and $f^{-1}(y)$ is uncountable for every $y \in F$. Using the simple counting arguments find $n \in \omega$ and an uncountable $F' \subseteq F$ such that for all $y \in F'$ the set $(f^{-1}(y) \times \{y\}) \cap C_n$ is uncountable. Consider the function $f' = f \cap C_n$. For every $y \in F'$ choose x_y^0 and x_y^1 from $f'^{-1}(y)$ such that $x_y^0 < x_y^1$ and the interval (x_y^0, x_y^1) is nontrivial. It is enough to prove that for all $y \neq z \in F'$ the intervals (x_y^0, x_y^1) and (x_z^0, x_z^1) are disjoint. Suppose the contrary, that $x_y^0 < x_z^1 < x_y^1$. Remember that C_n is a chain, so we have that $\langle x_y^0, y\rangle$, $\langle x_z^1, z\rangle$ and $\langle x_y^1, y\rangle$ are comparable; thus $y < z < y$—a contradiction. □

It follows that if X and Y are two uncountable sets of reals then $X \times Y$ is not the union of $\aleph_0$ chains.

If B is a Boolean algebra, then B^+ is the set of all its nonzero elements. For an element a of B the symbol $B \restriction a$ denotes the set of all $x \in B$ such that $x \leq a$. We want to embed B into $\mathcal{P}(\omega)$; hence we need the following Lemma.

LEMMA 10.5 Let B be a Boolean algebra with no uncountable antichain; then there is a countable $D \subseteq B^+$ such that for every $a \in B^+$ there is a $b \in D$ such that $b \subseteq a$—or in other words, there is a countable dense set in B^+.

PROOF Suppose that for any two disjoint elements a and b of B at least one of algebras $B \upharpoonright a$ and $B \upharpoonright b$ has a countable dense set; then we may inductively construct a maximal antichain $D_0 \subseteq B$ such that for any $d \in D_0$ the algebra $B \upharpoonright d$ has a countable dense set. By the assumption, D_0 must be countable. For every d in D_0 choose a countable dense $A_d \subseteq (B \upharpoonright d)^+$ and set $D = \bigcup_{d \in D_0} A_d$. Then A is the required countable dense set in B^+.

Now suppose that there are disjoint a_0 and b_0 such that neither $B \upharpoonright a_0$ nor $B \upharpoonright b_0$ has a countable dense set. Then we can inductively choose sequences a_α $(\alpha < \omega_1)$ and b_α $(\alpha < \omega_1)$ of elements of $B \upharpoonright a_0$ and $B \upharpoonright b_0$ respectively, such that:

(1) $a_\alpha \not\subseteq a_\beta$ for $\alpha > \beta$, and
(2) $b_\alpha \not\subseteq b_\beta$ for $\alpha > \beta$.

Then it is easily checked that

$$\{a_\alpha \cup (b_0 \setminus b_\alpha) : \alpha < \omega_1\}$$

is an uncountable set of pairwise incomparable elements. □

PROOF (Theorem 10.7) Suppose that B has no uncountable antichain. By Lemma 10.4, we have a countable dense set in B and thus we may suppose that B is an uncountable subalgebra of $\mathcal{P}(\omega)$. Consider the partition $[B]^2 = K_0 \cup K_1$ defined by:

$$\{x, y\} \in K_0 \quad \text{iff} \quad x \not\subseteq y \quad \text{and} \quad y \not\subseteq x$$

With the topology on B induced from 2^ω this is an open partition. Let us check the alternatives given by OCA:

1^0 An uncountable 0-homogeneous set is an uncountable antichain, and it concludes the proof.

2^0 Suppose that $B = \bigcup_{n \in \omega} B_n$, and every B_n is 1-homogeneous—that is, a chain. In particular, one of the B_n's is uncountable. Notice that the subalgebra of B generated by B_n is isomorphic to an *interval algebra* $\text{Int}(L)$ for some uncountable linearly ordered set L. ($\text{Int}(L)$ is a Boolean algebra generated by the family of intervals $I_a = \{x : x \leq a\}$ for all $a \in L$). Therefore, we may now assume that $B = \text{Int}(L)$ for some L of this kind.

If possible, divide L into two uncountable components L_0 and L_1 such that $L_0 < L_1$. (Notice that if such L_0 and L_1 cannot be found, then it is easy to find an uncountable family of pairwise disjoint intervals of L). Let L_1^* be the converse of L_1 and consider the product ordering on $L_0 \times L_1^*$, as in Lemma 10.3: $\langle x_0, y_0 \rangle \leq \langle x_1, y_1 \rangle$ iff $x_0 \leq y_0$ and $x_1 \leq y_1$. Every element of this poset corresponds to an element of $\text{Int}(L)$ in such a way that two elements of $L_0 \times L_1^*$ are comparable iff the corresponding intervals of $\text{Int}(L)$ are comparable. By the assumption, $L_0 \times L_1^*$ is representable as the union of countably many chains. By Lemma 10.3, L_0 contains an uncountable family of disjoint intervals, so we are done. □

Recall that Ramsey's Theorem implies that every infinite sequence of reals has an infinite monotonic subsequence. The following two facts show that OCA gives us much stronger versions of this phenomena.

THEOREM 10.8 (OCA) Let f be a real function on the uncountable set of reals E; then there exists an uncountable $F \subseteq E$ such that $f \upharpoonright F$ is monotone. If the function f is one-to-one, the restriction $f \upharpoonright F$ can be chosen to be, moreover, a homeomorphism.

PROOF Define a partition $[E]^2 = K_0 \cup K_1$ by

$$\{x, y\} \in K_0 \quad \text{iff} \quad f(x) < f(y) \quad \text{and} \quad x < y$$

We identify $x \in E$ with $\langle x, f(x) \rangle$. In the topology induced from $\mathbb{R}^2$ this is an open partition. We have either an uncountable 1-homogeneous or an uncountable 0-homogeneous subset F of E, and $f \upharpoonright F$ is monotone in both cases. □

THEOREM 10.9 (OCA) Let X and Y be two uncountable sets of reals. There is an uncountable $X_0 \subseteq X$ and strictly increasing $f : X_0 \to Y$.

PROOF Look at the partition $X \times Y = K_0 \cup K_1$ defined by

$$\{\langle x, y \rangle, \langle x', y' \rangle\} \in K_0 \quad \text{iff} \quad x < x' \to y < y'$$

The partition is evidently open in the product topology, so we can consider the two alternatives:

1^0 There is an uncountable 0-homogeneous set U. Set $X_0 = \{x \in X : \exists y \langle x, y \rangle \in U\}$, and for every $x \in X_0$ choose for $f(x)$ some $y \in Y$ such that $\langle x, y \rangle \in U$; then f is strictly increasing.

2^0 The set $X \times Y$ is a union of countably many 1-homogeneous sets. This defines a partition of the set $X \times Y^*$ into countably many chains. By Lemma 10.3 the set X contains an uncountable family of disjoint intervals, and this is impossible. □

11. A BASE FOR THE CLASS OF COMETRIZABLE SPACES

In this section we give an application of OCA to obtain a partial solution of an open problem in topology—a problem of finding a base for the class of regular spaces. For more details on this and some related problems consult §3 of [Todorčević 1989].

DEFINITION 11.1 *Let $\mathfrak{T}$ be a class of regular (T_3) topological spaces. If $\mathcal{B}$ is a subclass of spaces in $\mathfrak{T}$ such that for any space X in $\mathfrak{T}$ there is a space $Y \in \mathcal{B}$ that is topologically embeddable in X, we say that $\mathcal{B}$ is a base for $\mathfrak{T}$.*

DEFINITION 11.2 *In a topological space X a set $\mathcal{N}$ of subsets of X is a network iff for every $x \in X$ and every open B containing x there is some $C \in \mathcal{N}$ such that $x \in C \subseteq B$.*

FACT Every T_1-topological space that is a continuous image of a separable metric space M has a countable network, and conversely. (For the proof see Appendix E.0.1).

DEFINITION 11.3 *The Sorgenfry line (or the arrow-space) is a topological space that we get from $\mathbb{R}$ by refining its topology by the intervals $[a, b)$ for all $a, b \in \mathbb{R}$.*

EXAMPLE 11.1 Consider a T_3 space X. Define $T_X \subseteq X \times \mathrm{Open}(X)$ to be the set of all pairs $\langle x, B\rangle$ such that $x \in B$ ($\mathrm{Open}(X)$ is the family of all open subsets of X; similarly, $\mathrm{Closed}(X)$ is the family of all closed subsets of X) and consider the following partition $[T_X]^2 = K_0 \cup K_1$:

$$\{\langle x, B\rangle, \langle y, C\rangle\} \in K_0 \quad \text{iff} \quad x \notin C \quad \text{or} \quad y \notin B.$$

It is easy to see that X has a countable network iff T_X is the union of countably many 1-homogeneous sets. So it is natural to consider the Ramsey-type statement saying that either T_X is a countable union of 1-homogeneous sets or there is an uncountable 0-homogeneous set. Here is a topological translation of the two alternatives:

1^0 Suppose that we have a partition $T_X = \bigcup_{n\in\omega} T_n$ such that every T_n is 1-homogeneous. Set $C_n = \{y : \exists C\, \langle y, C\rangle \in T_n\}$ and $\mathcal{N} = \{C_n\}_{n\in\omega}$. Then every pair $\langle x, B\rangle$ from $X \times \mathrm{Open}(X)$ is in some C_n; hence $x \in C_n \subseteq B$ and thus $\mathcal{N}$ forms a countable network of X.

2^0 Suppose that there is an uncountable 0-homogeneous set $U \subseteq T_X$. Then we consider the set

$$\{y : \exists B\,\langle y, B\rangle \in U\}.$$

Then Y is an uncountable *weakly separated* subspace of X, i.e. for every y in Y we can associate a neighborhood U_y such that for every $x \neq y$ in Y either $x \notin U_y$ or $y \notin U_x$. Note that the arrow-space itself is weakly separated as the neighborhoods $[x, \infty)$ $(x \in \mathbb{R})$ witness.

All this leads to the following hypothesis about a possible basis for the class of all regular spaces.

HYPOTHESIS For every regular space X one of the following conditions holds:

(1) X has a countable network,
(2) X contains an uncountable subspace of the arrow-space,
(3) There is an uncountable discrete subspace in X.

The solution of this Hypothesis requires a rather strong Ramsey–type result. Using OCA we shall now show that the Hypothesis holds for a wide class of topological spaces described in the following definition, which covers most of the known examples of regular spaces.

DEFINITION 11.4 *A set of real numbers X with the topology τ forms a cometrizable topological space iff there is a family $\mathcal{F}$ of closed subsets of $\mathbb{R}$ such that for every $x \in X$, for every set U open in X there is an $F \in \mathcal{F}$ such that $x \in \operatorname{int}_\tau(F) \subseteq F \subseteq U$, where $\operatorname{int}_\tau(F)$ denotes the interior of F with respect to the topology τ.*

To apply OCA to partitions like the one in Example 11.1 we need a topology on the set $P(X)$, or some subset thereof.

DEFINITION 11.5 *An exponential space, $\exp(X)$, is defined on the set Closed(X) with the exponential (or Vietoris) topology given by its subbase, that consists of all the sets of the following two kinds:*

(1) $\{F : F \cap U \neq \emptyset\}$, *for some fixed open U, and*
(2) $\{F : F \subseteq U\}$, *for some fixed open U.*

FACT If X is compact metric then $\exp(X)$ is also compact metric (see Appendix, Theorem E.0.3). □

THEOREM 11.1 (OCA) Let X be a T_3 cometrizable space. Then one of the following conditions holds:

(1) X is a continuous image of a separable metric space,
(2) X contains an uncountable subspace of the arrow-space,
(3) X contains an uncountable discrete subspace.

PROOF In what follows it is convenient to take $\mathbb{R}$ to be equal to the Cantor cube $\{0,1\}^\omega$ with the (usual) lexicographical ordering. Let X, τ and $\mathcal{F}$ be as in the definition of a cometrizable space. Define a set $T \subseteq X \times \mathcal{F}$ by $T = \{\langle x, F\rangle : x \in \operatorname{int}_\tau(F)\}$, and two partitions of $[T]^2$: the first, $[T]^2 = K_0^r \cup K_1^r$, by:

$$\{\langle x, F\rangle, \langle y, G\rangle\} \in K_0^r \quad \text{iff} \quad x < y \ \text{ and } \ y \notin F$$

and the second $[T]^2 = K_0^l \cup K_1^l$ by:

$$\{\langle x, F\rangle, \langle y, G\rangle\} \in K_0^l \quad \text{iff} \quad x < y \ \text{ and } \ x \notin G$$

Notions like 0^r homogeneous, 1^r-homogeneous, etc. have the obvious meanings. It is easy to see that in the product topology of $\mathbb{R} \times \exp(\mathbb{R})$ both partitions are open. Now examine the alternatives given by the OCA:

1^0 There is a partition $X = \bigcup_{n\in\omega} T_n$, where each T_n is both 1^r- and 1^l- homogeneous; then we have a countable network

$$\mathcal{N} = \{\{x \in X : \exists F\,\langle x, F\rangle \in T_n\} : n < \omega\}$$

as in Example 11.1.

2^0 There is an uncountable 0^l- (or 0^r-) homogeneous set U. By the symmetry we may suppose that U is 0^l-homogeneous. Set $Y = \{y : \exists F_y^l\,\langle y, F_y^l\rangle \in U\}$, and consider the restriction of the second partition on the set

$$T_Y = \{\langle y, F\rangle : y \in Y,\ F \in \mathcal{F}\}.$$

Again applying OCA we get the alternatives:

2.1^0 There is an uncountable 0^r-homogeneous set $V \subseteq U$. Consider the set

$$Z = \{z \in Y : \exists F_z^r\,\langle z, F_z^r\rangle \in V\}.$$

For any $z \in Z$ there is an open (in τ) set $O_z = \operatorname{int}_\tau F_z^r \cap \operatorname{int}_\tau F_z^l$ such that $O_z \cap Z = \{z\}$; hence Z is discrete.

2.2^0 $T_Y = \bigcup_{n\in\omega} T_n$, and every T_n is 1-homogeneous. Set

$$C_n = \{x \in Y : \exists F\,\langle x, F\rangle \in T_n\} \qquad \text{for } n \in \omega.$$

For every $x \in Y$ and for every $U \in \tau$, if $x \in U$ then there exists $F \in \mathcal{F}$ such that $x \in \operatorname{int}_\tau(F) \subseteq F \subseteq U$ and for some n we have that $\langle x, F\rangle \in T_n$.

To prove that Y is a right-facing arrow-space, we need to find an open C_x for every x in Y such that $x \in C_x \subseteq [x, +\infty)$. Having this goal in mind, we first note the following fact which immediately follows from the definitions.

CLAIM 1 Suppose that $x \in Y \cap U$ for some $U \in \tau$. Choose $F \in \mathcal{F}$ such that $x \in \operatorname{int}_\tau(F) \subseteq F \subseteq U$ and let n be such that $\langle x, F\rangle \in T_n$. Then

$$x \in (C_n \cap [x, +\infty)) \subseteq F \subseteq U. \qquad \square$$

If the sets C_n are clopen relative to Y then every $x \in Y$ has an open neighborhood $C_n \cap [x, +\infty)$, and so Y is homeomorphic to an uncountable subspace of a arrow-space. While this may not be true in general, the next fact shows that we can always find an uncountable subset of Y having this property.

CLAIM 2 There is an uncountable $Z \subseteq Y$ such that for every n the set $Z \cap C_n$ is clopen in the topology induced from $\mathbb{R}$.

PROOF We may suppose that C_n's are closed by taking their closures (for K_1^r is closed). Let σ be a topology induced on Y from $\mathbb{R}$ and refined by extending the topology in such a way that every C_n is clopen. This topology is zero-dimensional with countable weight, so by Theorem 10.7 there is an uncountable $Z \subseteq Y$ such that σ and τ coincide on Z, and we are done. □

By the two last Claims we have that $\langle Z, \tau\rangle$ is homeomorphic to an uncountable subspace of the arrow-space. □

12. THE PROPER FORCING AXIOM, PFA

Let $\mathcal{P}$ be a poset, and let $\dot{Q}$ be a $\mathcal{P}$-name for a poset. Define a poset $\mathcal{P} * \dot{Q} = \{\langle p, \dot{q}\rangle : p \Vdash \dot{q} \in \dot{Q}\}$ ordered by

$$\langle p_0, \dot{q}_0\rangle \leq \langle p_1, \dot{q}_1\rangle \qquad \text{iff} \qquad p_0 \leq p_1 \ \&\ p_0 \Vdash \dot{q}_0 \leq \dot{q}_1.$$

A natural poset for collapsing the continuum, Coll$\,(2^{\aleph_0})$, is the poset of all countable partial functions from ω_1 to $\mathbb{R}$ with the ordering given by $p \leq q$ iff $p \supseteq q$. Forcing with Coll$\,(2^{\aleph_0})$ collapses the continuum to $\aleph_1$. A poset $\mathcal{P}$ is *σ-closed* iff for every countable decreasing sequence $p_0 \geq p_1 \geq \ldots, \geq p_i \geq \ldots$ there exists an element q of $\mathcal{P}$ such that $p_i \geq q$ for all $i \in \omega$. It is known that forcing with a σ-closed poset *preserves countable sets of ordinals*—this means that the family of countable sets of ordinals in the generic model is exactly the same to the family of countable sets of ordinals in the ground model (exercise). The poset Coll$\,(2^{\aleph_0})$ is evidently σ-closed. Now we give a forcing axiom stronger than OCA (see for example [Baumgartner], [Shelah 1977], [Todorčević 1989]).

DEFINITION 12.1 *The Proper Forcing Axiom (PFA): If a poset $\mathcal{P}$ is proper and $\{\mathcal{D}_\alpha\}_{\alpha<\omega_1}$ is a family of open dense subsets of $\mathcal{P}$, then there exists a filter $G \subseteq \mathcal{P}$ such that $G \cap D_\alpha \neq \emptyset$ for all $\alpha < \omega_1$.*

Notice that we have not defined a *proper* partial ordering, for all we shall need in this section is the fact that

If $\dot{Q}$ is a Coll$\,(2^{\aleph_0})$-name for a ccc poset, then Coll$\,(2^{\aleph_0}) * \dot{Q}$ is proper.

Thus, PFA is a considerable strengthening of MA(ω_1) because every ccc poset is proper. It is frequently used to obtain some results that do not follow from MA(ω_1) itself. It turns out that PFA is very strong indeed, in particular, it has a large cardinal strength. The large cardinal strength of PFA, a definite sign of its power, sometimes adds a bit of a confusion (frequently misplaced) among those mathematicians interested in exact independence results. This is one of the reasons why we introduce weaker forcing axioms (like OCA), which are free of large cardinals.

THEOREM 12.1 For every two sets of reals of power of the continuum, X and Y, there is a function $f\colon X \xrightarrow{1-1} Y$ such that the poset $\mathcal{P} = \mathcal{P}_f$ of all finite increasing subfunctions of f has the following properties:

(1) for every disjoint $\mathcal{K} \subseteq \mathcal{P}$ of power continuum there are $p \neq q \in \mathcal{K}$ such that $p \cup q$ is in $\mathcal{P}$, and

(2) let $\mathcal{P}^{<\omega} = \{p \in \mathcal{P}^\omega : p_i \neq \emptyset$ for finitely many $i \in \omega\}$. Then for every $\mathcal{K} \subseteq \mathcal{P}^{<\omega}$ of cardinality $\mathfrak{c}$ there are $p \neq q \in \mathcal{K}$ such that $p_i \cup q_i \in \mathcal{P}$ for every $i \in \omega$.

PROOF Enumerate X as $\{x_\alpha\}_{\alpha<\mathfrak{c}}$. We shall use the fact that for any function $f: \mathbb{R}^n \to \mathbb{R}$ the set of all points $x \in \mathbb{R}^n$ such that f is continuous in x is G_δ. Let $\{g_\alpha\}_{\alpha<\mathfrak{c}}$ be the set of all continuous functions g such that

(3) $\mathrm{dom}(g)$ is a G_δ subset of $\mathbb{R}^n$ for some $n \in \omega$, and

(4) $\mathrm{range}(g)$ is a subset of $\mathbb{R}$.

We define f inductively so that

$$f(x_\alpha) \in Y \setminus \{f(x_\xi) : \xi < \alpha\}, \qquad \text{and} \qquad f(x_\alpha) \notin \{g_\xi(p^\frown x_\alpha) : \xi < \alpha\},$$

where p ranges over the set of all finite sequences of elements of $\{x_\xi : \xi \leq \alpha\} \cup \{f(x_\xi) : \xi < \alpha\}$ of appropriate length. Fix a family $\mathcal{K}$ as in (1). We may suppose that for some $n \in \omega$ every s in $\mathcal{K}$ is of length n. So

$$\mathcal{K} = \{s^\xi\}_{\xi<\mathfrak{c}}, \qquad \text{and} \qquad s^\xi = \{\langle x_i^\xi, f(x_i^\xi)\rangle\}_{i<n}$$

The proof of (1) now proceeds by induction on n. Define a partial function $\bar{g}: \mathbb{R}^{2n-1} \to \mathbb{R}$ in the following way: For any $s^\xi \in \mathcal{K}$, set

$$\bar{s}^\xi = \langle x_0^\xi, f(x_0^\xi), \ldots, f(x_{n-2}^\xi), x_{n-1}^\xi\rangle \quad \text{and} \quad g(\bar{s}^\xi) = f(x_{n-1}^\xi).$$

By the constructions of f and $\bar{g}$, any continuous partial function h from $\mathbb{R}^{2n-1}$ to $\mathbb{R}$ coincides with $\bar{g}$ in less than $\mathfrak{c}$ points. Define $\mathcal{K}_0$ as the set of all $s \in \mathcal{K}$ such that $\bar{g}$ is not continuous in the point $\bar{s}$ (notice that $|\mathcal{K}_0| = \mathfrak{c}$). We may assume that

(1) There is a set of rational intervals $I_0, I_1, \ldots, I_{2n-1}$ that *separates* each s^ξ— that is, $x_i^\xi \in I_j$ iff $i = j$ for every pair $i, j < n$,

(2) For every s^ξ in $\mathcal{K}_0$ there is a sequence s^{ξ_i} of elements of $\mathcal{K}$ such that

$$\lim_{i\to\infty} s^{\xi_i} = s^\xi \quad \text{and} \quad \lim_{i\to\infty} \bar{g}(s^{\xi_i}) \neq \bar{g}(s^\xi)$$

By symmetry, we may assume that $\lim_{i\to\infty} \bar{g}(s^{\xi_i}) < \bar{g}(s^\xi)$ for every ξ, and moreover that

(3) $\lim_{i\to\infty} \bar{g}(s^{\xi_i}) < d < \bar{g}(s^\xi)$ for some rational number d and every $s^\xi \in \mathcal{K}_0$.

By the induction hypothesis there are different s^ξ and s^η in $\mathcal{K}_0$ such that the first $n-1$ pairs in s^ξ together with the first $n-1$ pairs in s^η form an increasing function. Assume that $x_n^\xi < x_n^\eta$ and fix intervals $J_0, J_1, \ldots, J_{2n-1}$ so that $x_i^\xi \notin J_i$ and $x_i^\eta \in J_i$ for all $i = 0, 1, \ldots 2n-1$. By the construction, every element of $\mathcal{K}_0$ is an accumulation point of $\mathcal{K}$, so we can find $s^\zeta \in \mathcal{K}$ such that $x_i^\zeta \in J_i$ for all $i < 2n-1$ and $\bar{g}(s^\zeta) > d$. Evidently $s^\xi \cup s^\zeta$ is increasing. The proof of (2) is essentially the same but a little bit longer, so we shall skip it. □

THEOREM 12.2 (PFA) Let X and Y be sets of reals of power $\aleph_1$. There is a function $f: X \xrightarrow{1-1} Y$ such that f is the union of countably many partial strictly increasing functions, $f = \bigcup_{i\in\omega} f_i$.

PROOF First apply Theorem 12.1 in the forcing extension of Coll $(2^{\aleph_0})$ and get a name $\dot{f}$ for a 1-1 function from $\dot{X}$ into $\dot{Y}$ such that the corresponding poset $\mathcal{P}_{\dot{f}}^{<\omega}$ is ccc. Now apply PFA to the composition Coll $(2^{\aleph_0}) * \mathcal{P}_{\dot{f}}^{<\omega}$ and the following family of $\aleph_1$ dense sets:

$$D_x = \{(p, \dot{q}) \in \text{Coll}\,(2^{\aleph_0}) * \mathcal{P}_{\dot{f}}^{<\omega} : x \in \mathrm{dom}(q_i) \text{ for some } i < \omega\}, \quad (x \in X). \qquad \square$$

DEFINITION 12.2 *A set of reals X is $\aleph_1$-dense iff for all $x < y$ in X the set $(x, y) \cap X$ is of size $\aleph_1$.*

THEOREM 12.3 (PFA) Every two $\aleph_1$-dense sets of reals are isomorphic. (See [Baumgartner 1973]).

PROOF Let X and Y be a given two $\aleph_1$-dense sets of reals. Without loss of generality, we may assume that X and Y are actually dense in $\mathbb{R}$. Using Theorem 12.2, for every pair of rational intervals I and J we fix two bijections

$$f_{I,J}: X \cap I \to Y \cap J, \qquad f_{I,J} = \bigcup_{i\in\omega} f_{I,J}^i$$
$$g_{I,J}: Y \cap J \to X \cap I, \qquad g_{I,J} = \bigcup_{i\in\omega} g_{I,J}^i$$

in such a way that all $f_{I,J}^i$ and $g_{I,J}^i$ are strictly increasing. Let $\mathcal{P}$ be a set of all finite increasing functions p from X to Y with the property:

(0) For every $x \in \mathrm{dom}(p)$ there are i_x, I_x and J_x such that $p(x) = f_{I_x,J_x}^{i_x}(x)$ or $x = g_{I_x,J_x}^{i_x}(p(x))$.

CLAIM $\mathcal{P}$ is ccc.

PROOF Fix an uncountable family $\{p_\xi : \xi < \omega_1\} \subseteq \mathcal{P}$. We may suppose without loss of generality that

(1) All p_ξ's are of the same cardinality n and $p_\xi = \{(x_\xi^j, p(x_\xi^j)) : j \leq n-1\}$,

(2) There are i^j, I^j and J^j for every $j \leq n-1$ such that: for every x_ξ^j we have $i_{x_\xi^j} = i^j$, $I_{x_\xi^j} = I^j$ and $J_{x_\xi^j} = J^j$,

(3) $I^j < I^{j+1}$ and $J^j < J^{j+1}$ for all $j < n-1$ (for every p_ξ is increasing), and

(4) For every $j < n$, either $f_{I^j,J^j}(x_\xi^j) = p_\xi(x_\xi^j)$ for all $\xi < \omega_1$ or $g_{I^j,J^j}(p(x_\xi^j)) = x_\xi^j$ for all $\xi < \omega_1$.

Now it is easily seen that in such a subfamily the union of any two p_ξ's is in $\mathcal{P}$. □

REMARK Notice that in (1)-(4) we have only used the counting arguments; hence $\mathcal{P}$ is not only ccc, but it is in fact σ-centered (a poset is *σ-centered* iff it is a countable union of centered sets).

Now notice that the sets $\mathcal{D}_x = \{p \in \mathcal{P} : x \in \mathrm{dom}(p)\}$ (for $x \in X$) and $\mathcal{E}_y = \{p \in \mathcal{P} : y \in \mathrm{range}(p)\}$ (for $y \in Y$) are dense in $\mathcal{P}$, so by PFA there is a filter $G \subseteq \mathcal{P}$ intersecting them. Let $f = \bigcup G$. Then f is an isomorphism between X and Y. □

REMARK The analogue theorem is not true for $\mathfrak{c}$-dense sets of reals; thus the conclusion of Theorem 12.3 contradicts CH. The construction of nonisomorphic $\mathfrak{c}$-dense sets of reals is a classical result proved by using a diagonalization similar to the construction of g in Theorem 12.1 (with $X = Y$ and $n = 1$).

Proof of the following Theorem goes along similar lines as the proof of Theorem 12.1 (see [Todorčević 1989, Theorem 4.4]).

THEOREM 12.4 For $X \subseteq \mathbb{R}$ let $[X]^2 = K_0 \cup K_1$ be an open partition such that X is not decomposable into less than $\mathfrak{c}$ 1-homogeneous subsets; then there exists an $Y \subseteq X$ of the cardinality $\mathfrak{c}$ such that a set $\mathcal{P} = \{p \in [Y]^{<\omega} : p \text{ is 0-homogeneous}\}$ has the following property: *If $\mathcal{K}$ is a set of disjoint elements of $\mathcal{P}$ of size of the continuum, then there are $p, q \in \mathcal{K}$ such that $p \cup q \in \mathcal{P}$.* □

COROLLARY PFA implies OCA for second countable spaces.

PROOF Let X be a set of reals and let $[X]^2 = K_0 \cup K_1$ be the given open partition such that X is not the union of countably many 1-homogeneous sets. Notice that the forcing with Coll$(2^{\aleph_0})$ does not introduce any new closed sets. [It does not introduce new reals because it is σ-closed, and closed sets are coded by reals—see §1 or Appendix C.] The closure of a 1-homogeneous set is also 1-homogeneous, so X is not the union of countably many 1-homogeneous sets even in the forcing extension by Coll$(2^{\aleph_0})$. The forcing with Coll$(2^{\aleph_0})$ makes the continuum equal to ω_1, so by Theorem 12.4 there is a Coll$(2^{\aleph_0})$-name $\dot{Y}$ for an uncountable subset of X such that the poset $\dot{Q}$ of all finite 0-homogeneous subsets of Y is ccc. Forcing internally by Coll$(2^{\aleph_0}) * \dot{Q}$ gives us an uncountable 0-homogeneous subset of X. □

It is interesting that the following Theorem has found its application in solving an automatic continuity problem of Banach algebras (see [Dales]).

THEOREM 12.5 (PFA)

a. $\langle \{0,1\}^{\omega_1}, <_{\mathrm{Lex}} \rangle$ is not embeddable in $\langle \omega^\omega, <^* \rangle$.
b. $\mathcal{P}(\omega_1)$ is not embeddable into $\mathcal{P}(\omega)/\mathrm{Fin}$.

PROOF The proofs of a. and b. are similar; hence we shall prove only a. Suppose the contrary, that there is an embedding $\varphi : \{0,1\}^{\omega_1} \to \langle \omega^\omega, <^* \rangle$. Let f be a function $f : \omega_1 \to \{0,1\}$ such that $f^{-1}(0)$ and $f^{-1}(1)$ are uncountable. Consider the following sequences of elements of $\langle \omega^\omega, <^* \rangle$ (here 0 and 1 denote constant functions in $\{0,1\}^{\omega_1}$):

$$a_f^\xi = \varphi(f \restriction \alpha_\xi \cup 0 \restriction [\alpha_\xi, \omega_1)), \qquad \alpha_\xi = \min\{\beta \geq \xi : f(\beta) = 1\}$$
$$b_f^\xi = \varphi(f \restriction \beta_\xi \cup 1 \restriction [\beta_\xi, \omega_1)), \qquad \beta_\xi = \min\{\beta \geq \xi : f(\beta) = 0\}.$$

Now $a = \{a_f^\xi\}_{\xi<\omega_1}$ ($b = \{b_f^\xi\}_{\xi<\omega_1}$) forms an increasing (decreasing) sequence and $a_f^\xi <^* \phi(f) <^* b_f^\xi$ for all $\xi < \omega_1$. By making finite changes of a_f^ξ's assume that $a_f^\xi(n) \leq b_f^\xi(n)$ for all n and ξ. We have constructed $2^{\aleph_1}$ such pairs, while there are $2^{\aleph_0}$ elements of ω^ω to fill these pre-gaps. Note also that this association can be done in any Coll$(2^{\aleph_0})$-generic model $V[G]$, where we have that $2^{\aleph_1} > 2^{\aleph_0} = \aleph_1$. Therefore, we can find a name $\dot{f}$ for a function such that the pair $\langle a_{\dot{f}}, b_{\dot{f}} \rangle$ is a gap in $V[G]$. We want to use PFA to find the interpretation of the name $\dot{f}$ in the ground model.

THEOREM 12.6 Let $\{a_\xi, b_\xi : \xi < \omega_1\} \subseteq \omega^\omega$ be a (ω_1, ω_1^*)-gap in $\langle \omega^\omega, <^* \rangle$. Let

$$Q = \{q \in [\omega_1]^{<\omega} : (\forall \xi < \eta \in q)(\exists n \in \omega) a_\xi(n) > b_\eta(n)\},$$

and let Q be ordered by: $p \leq q$ iff $p \supseteq q$. Then Q is ccc.

PROOF Let $\mathcal{K} = \{q_\xi : \xi < \omega_1\}$ be a given uncountable subset of Q. We may suppose that all the q_ξ's are disjoint and that $q_\xi < q_\eta$ for all $\xi < \eta$. Now set $\alpha_\xi = \min q_\xi$ and $\beta_\xi = \max q_\xi$. For each $q_\xi \in \mathcal{K}$ there is $m_\xi \in \omega$ such that

(1) $a_\alpha(n) \leq b_\beta(n)$ for all $n \geq m_\xi$ and $\alpha, \beta \in q_\xi$.

Refining $\mathcal{K}$ we may assume that there exists $m < \omega$ such that for all $\xi < \omega_1$:

(2) $m_\xi = m$

The pair $\{a_{\alpha_\xi}, b_{\beta_\xi}\}_{\xi<\omega_1}$ also forms a (ω_1, ω_1^*)-gap; hence there exist $\eta < \xi < \omega_1$ and $n > m$ such that $a_{\alpha_\xi}(n) > b_{\beta_\eta}(n)$. Then $q_\xi \cup q_\eta$ is 0-homogeneous. □

REMARK Theorem 12.6 holds also with ω_1 replaced by any regular uncountable cardinal κ. For such gap the corresponding poset Q is κ-cc and the proof remains unchanged.

Returning to the proof of Theorem 12.5, associate $\dot{Q}$ to the gap $\langle a_{\dot{f}}, b_{\dot{f}} \rangle$ defined above and force internally by Coll$(2^{\aleph_0}) * \dot{Q}$. In this way, we get an $f \in \{0,1\}^{\omega_1}$ such that the pre-gap $\{a_f^\xi, b_f^\xi : \xi < \omega_1\}$ contains a cofinal special gap (see Lemma 9.1 and Corollary 9.1). This means that $\phi(f)$ can't be defined, and this is in contradiction with the assumption that ϕ is an embedding. □

LEMMA 12.1 ($\mathfrak{m} > \omega_1$) The structure $\langle \{0,1\}^{\omega_1}, <_{\mathrm{Lex}} \rangle$ is embedded into the structure $\langle \omega^\omega/\mathcal{U}, \leq_\mathcal{U} \rangle$ for every nonprincipal ultrafilter $\mathcal{U}$ on ω.

PROOF Recall that the elements of $\langle \omega^\omega/\mathcal{U}, <_\mathcal{U} \rangle$ are the equivalence classes $[a]_\mathcal{U} = \{b \in \omega^\omega : \{n \in \omega : a(n) = b(n)\} \in \mathcal{U}\}$, and the ordering $\leq_\mathcal{U}$ is defined by $[a]_\mathcal{U} \leq_\mathcal{U} [b]_\mathcal{U}$ iff $\{n \in \omega : a(n) \leq b(n)\} \in \mathcal{U}$.

Consider sequences a_α and b_α ($\alpha < \omega_1$) in ω^ω such that a_α is $<^*$-increasing, b_α is $<^*$-decreasing and for all $n \in \omega$ and all $\alpha < \omega_1$ holds $a_\alpha(n) \leq b_\alpha(n)$. We shall prove that in $\omega^\omega/\mathcal{U}$ this set (more precisely, $\langle [a_\alpha]_\mathcal{U}, [b_\alpha]_\mathcal{U} \rangle$) is not a gap. Define a partition of $[\omega_1]^3$ into K_0 and K_1 by $\{\alpha, \beta, \gamma\} \in K_0$ iff for every $n \in \omega$ at least two intervals from the set

$$\{[a_\alpha(n), b_\alpha(n)], [a_\beta(n), b_\beta(n)], [a_\gamma(n), b_\gamma(n)]\}$$

have nonempty intersection.

CLAIM This partition is ccc.

PROOF Fix a family $\{F_\xi : \xi < \omega_1\}$ of finite 0-homogeneous sets. We may suppose that F_ξ's are of the same size, k. So set $F_\xi = \langle a_\xi^i, b_\xi^i \rangle_{i<k}$ We may refine this family to get an integer m such that

(1) For all $n > m$, all $\xi < \omega_1$ and all $\alpha < \beta \in F_\xi$ we have that $a_\xi(n) < a_\eta(n) < b_\eta(n) < b_\xi(n)$, and
(2) All F_ξ's coincide below m—that is, there are s_i and t_i for all $i < k$ such that $a_\xi^i \restriction m = s_i$ and $b_\xi^i \restriction m = t_i$.

After this refining is done, the union of any two F_ξ's is 0-homogeneous. □

Use $\mathfrak{m} > \omega_1$ to find an uncountable 0-homogeneous set X. Notice that X is cofinal in ω_1; thus, it is enough to fill the pre-gap $(a_\alpha, b_\alpha)_{\alpha\in X}$. Consider a family of intervals $\mathcal{I}_n = \{[a_\alpha(n), b_\alpha(n)] : \alpha \in X\}$. Choose an interval $[a_\alpha(n), b_\alpha(n)]$ that is antilexicographically minimal in $\mathcal{I}_n$ and denote it by I_n ($\max I_n$ is the smallest among $\max I$ for $I \in \mathcal{I}_n$ and $\min I_n \le \min I$ for all I in $\mathcal{I}_n$ with $\max I = \max I_n$). Now we partition $\mathcal{I}_n$ into two sets:

$$\mathcal{I}_n^0 = \{I \in \mathcal{I}_n : I \cap I_n \neq \emptyset\} \quad \text{and} \quad \mathcal{I}_n^1 = \{I \in \mathcal{I}_n : I \cap I_n = \emptyset\}.$$

The set $\bigcap(\mathcal{I}_n^0)$ is not empty (it contains at least $b_\alpha(n)$). Also, any two intervals in $\mathcal{I}_n^1$ have a nonempty intersection, for X is homogeneous. By Helly's Theorem about convex sets, $\mathcal{I}_n^1$ has the finite intersection property, and so by compactness it also has a nonempty intersection. Thus we may define functions c and d from ω^ω by (setting $\min\emptyset = 0$):

$$c(n) = \min\bigcap\mathcal{I}_n^0 \quad \text{and} \quad d(n) = \min\bigcap\mathcal{I}_n^1.$$

Fix $\alpha \in X$. Then for every n we have either $a_\alpha(n) \le c(n) \le b_\alpha(n)$ or $a_\alpha(n) \le d(n) \le b_\alpha(n)$, and $\mathcal{U}$ decides for one of these possibilities—namely, there is a set $A \in \mathcal{U}$ such that for all $n \in A$ the same possibility holds. In other words, we have that either $[a_\alpha]_\mathcal{U} \le_\mathcal{U} [c]_\mathcal{U} \le_\mathcal{U} [b_\mathcal{U}]_\mathcal{U}$ or $[a_\alpha]_\mathcal{U} \le_\mathcal{U} [d]_\mathcal{U} \le_\mathcal{U} [b_\mathcal{U}]_\mathcal{U}$. So, there is an uncountable set $Y \subseteq X$ such that one of these two alternatives, say the first, holds for every α in Y. It follows that in this case, $[c]_\mathcal{U}$ splits the gap in $\omega^\omega/\mathcal{U}$.

Our next step is to embed the structure $\langle 2^{<\omega_1}, <_{\mathrm{Lex}}\rangle$ into $\langle\omega^\omega/\mathrm{Fin}, <^*\rangle$, preserving order. This is easily accomplished by induction. Then for every $f \in 2^{\omega_1}$ we have associated a pre-gap $\langle a_{f|\alpha}, b_{f|\alpha}\rangle_{\alpha<\omega_1}$. Denote by c_f the element of $\omega^\omega/\mathcal{U}$ that fills this pre-gap in $\omega^\omega/\mathcal{U}$ and check that $f \mapsto c_f$ is the required embedding. □

It is interesting that Theorem 12.5 and Lemma 12.1 have the following application in the seemingly quite distant theory of Banach algebras.

COROLLARY (PFA) Let X be a compact topological space and let $C(X)$ denote the algebra of all continuous real functions with domain X. Then for every commutative Banach algebra A, every homomorphism $H: C(X) \to A$ is continuous.

PROOF Suppose that there is such a discontinuous homomorphism. It is known (see [Dales] and [Dales–Woodin]) that the existence of a discontinuous homomorphism H from $C(X)$ into A implies the existence of an ultrafilter $\mathcal{U}$ on ω such that the ultrapower $\langle\omega^\omega/\mathcal{U}, \le_\mathcal{U}\rangle$ embeds into $\langle\omega^\omega, <^*\rangle$. By Lemma 12.1, it follows that $\langle\{0,1\}^{\omega_1}, <_{\mathrm{Lex}}\rangle$ also embeds into $\langle\omega^\omega, <^*\rangle$. But this would contradict Theorem 12.5. □

REMARK It is known that the existence of a discontinuous homomorphism follows from CH, so the use of the additional axiom PFA above is in some sense necessary. The CH-construction of a discontinuous homomorphism between commutative Banach algebras is a quite deep result of Dales and Esterle. The consistency of the statement that every homomorphism from $C(X)$ into a commutative Banach algebra must be continuous was first proved by Solovay. We refer the reader to [Dales] and [Dales–Woodin] for a survey of this interesting subject.

13. SOME CARDINAL INVARIANTS OF THE CONTINUUM

In this section we will deal with the following five cardinal invariants associated in some way with the set of natural numbers.

$\mathfrak{m} = \min\{\theta :$ there is a ccc poset $\mathcal{P}$ with a family $\{\mathcal{D}_\xi\}_{\xi<\theta}$ of dense sets such that there is no $\{\mathcal{D}_\xi : \xi < \theta\}$-generic filter$\}$, i.e. the least cardinal κ such that MA(κ) fails (see also §10).

$\mathfrak{b} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega$ and $\mathcal{F}$ is $<^*$-unbounded$\}$ (see also §9).

$\mathfrak{d} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega$ and $(\forall g \in \omega^\omega)(\exists f \in \mathcal{F})\ g <^* f\}$, i.e. the minimal cardinality of a *dominating subset of* ω^ω.

$\mathfrak{u} = \min\{|\mathcal{B}| : \mathcal{B} \subseteq [\omega]^\omega$ and $\mathcal{B}$ generates an ultrafilter$\}$.

To describe the remaining cardinal $\mathfrak{g}$ we need some additional definitions.

DEFINITION 13.1 *A family $\mathcal{D} \subseteq [\omega]^\omega$ is groupwise dense iff*

(1) *$\mathcal{D}$ is closed under $\subseteq^*$ (in symbols, $(\forall X \in \mathcal{D})(\forall Y \in [\omega]^\omega)(Y \subseteq^* X \to Y \in \mathcal{D})$), and*

(2) *For every infinite disjoint family Π of nonempty intervals of ω there is an infinite subfamily $\Pi' \subseteq \Pi$ such that $\bigcup\Pi' = X$ for some X in $\mathcal{D}$.*

Every groupwise dense set $\mathcal{D}$ is, evidently, dense in a poset $\langle[\omega]^\omega, \subseteq^*\rangle$. An example of a dense, but not groupwise dense family is the set of all $A \subseteq \omega$ which contain only finitely many pairs of consecutive numbers. An equivalent reformulation of (2) is the following:

(2') If $f: \omega \to \omega$ is a one-to-one function, then set $\{A \in [\omega]^\omega : f^{-1}(A) \in \mathcal{D}\}$ is dense in poset $\langle[\omega]^\omega, \subseteq\rangle$.

Also, in (2) we may replace "infinite disjoint family of nonempty intervals" with "a partition of ω into finite intervals" or "a partition of an infinite set $X \subseteq \omega$ into finite sets".

EXAMPLE 13.1 An example of a groupwise dense set is any ideal in $[\omega]^\omega$ of a *cellularity* $n < \omega$, i.e. the ideal $\mathcal{I}$ such that for every partition $\omega = \bigcup_{i<n} X_i$ there is a set $X_i \in \mathcal{I}$. To see why every ideal of a finite cellularity satisfies (2), suppose that $\omega = \bigcup_{j\in\omega} A_j$ is a disjoint partition of ω into finite sets, and let $X_i = \bigcup_{j\in\omega} A_{jn+i}$

for all $i < n$. Then sets $\langle X_i : i < n\rangle$ form a disjoint partition of ω into n sets, and there is an $X_i \in \mathcal{I}$.

Now we are ready to define our fifth cardinal invariant.

$\mathfrak{g} = \min\{\kappa :$ some family of κ groupwise dense subsets of $[\omega]^\omega$ has the empty intersection$\}$.

All cardinals defined in this section are between ω_1 and $\mathfrak{c}$; hence it is consistent that they are all equal (to $\mathfrak{c}$). Various equalities (or inequalities) involving these cardinals may be considered as weakenings of CH (or strengthenings of $\neg$CH). For example, in §7 we have seen that the assumption $\mathfrak{m} > \omega_1$ is a quite powerful forcing axiom which decides many problems about the continuum for which $\neg$CH is just too weak to decide. In this section we consider the assumption $\mathfrak{u} < \mathfrak{g}$ which can be considered as another powerful forcing axiom giving us a fine structure and connections to several areas of mathematics. Here is a list of some of the simpler relations between these numbers which can be proved with no much difficulties.

FACT $\mathfrak{m} \le \mathfrak{b} \le \mathfrak{u}$, $\mathfrak{b} \le \mathrm{cf}(\mathfrak{d})$ and $\mathfrak{g} \le \mathfrak{d}$.

PROOF (Sketches) ($\mathfrak{m} \le \mathfrak{b}$) Let $\kappa < \mathfrak{m}$ and $\{f_\xi : \xi < \kappa\}$ is a subset of ω^ω. Define a poset $\mathcal{P} = \{\langle s, F\rangle : s$ is a function mapping some $n \in \omega$ to ω, and $F \in [\kappa]^{<\omega}\}$. An ordering is defined by: $\langle p, F\rangle \le \langle q, G\rangle$ iff $p \supseteq q$ and $(\forall \xi \in G)(\forall i \in \operatorname{dom} p \setminus \operatorname{dom} q) p(i) > f_\xi(i)$. This poset is ccc, and every $\mathcal{D}_\xi = \{\langle p, F\rangle \in \mathcal{P} : \xi \in F\}$ is dense.

($\mathfrak{b} \le \mathfrak{u}$) To every increasing $f \in \omega^\omega$ we associate S_f, an infinite subset of ω, by $S_f = \bigcup_{i<\omega}[f^{2n}(0), f^{2n+1}(0))$. If $X \subseteq \omega$ is infinite, let $f_X \in \omega^\omega$ be the enumeration of X, the function mapping i to the i-th element (in the increasing order) of X. Prove that if f is strictly increasing and eventually dominates f_X, then $S_f \not\supseteq X$ and $S_f \cap X$ is infinite.

($\mathfrak{b} \le \mathrm{cf}(\mathfrak{d})$) Let $\mathcal{F} = \{f_\xi : \xi < \mathfrak{d}\}$ be a dominating family in ω^ω such that $f_\eta \not\le^* f_\xi$ for all $\xi < \eta < \mathfrak{d}$. Let $\{\xi_\alpha : \alpha < \mathrm{cf}(\mathfrak{d})\}$ be a cofinal subset of $\mathfrak{d}$; then $\{f_{\xi_\alpha} : \alpha < \mathrm{cf}(\mathfrak{d})\}$ is unbounded in $\mathcal{F}$, and hence in ω^ω.

($\mathfrak{g} \le \mathfrak{d}$) To every $f \in \omega^\omega$ associate a groupwise dense set

$$\mathcal{D}_f = \{X \in [\omega]^\omega : f_X \text{ is not dominated by } f\}.$$

□

More interesting and difficult is the following theorem, most of which will be proved here after some preliminary results:

THEOREM 13.1 If $\mathfrak{u} < \mathfrak{g}$ then $\mathfrak{b} = \mathfrak{u} < \mathfrak{g} = \mathfrak{d} = \mathfrak{c}$.

Hence, $\mathfrak{u} < \mathfrak{g}$ appears to be quite influential even to the properties of the continuum which do not play any direct role in the defining of the numbers $\mathfrak{u}$ and $\mathfrak{g}$. This suggests the following question:

PROBLEM 13.1 Does $\mathfrak{u} < \mathfrak{g}$ imply $\mathfrak{u} = \omega_1$?

DEFINITION 13.2 *For a function $f : \omega \to \omega$ and an ultrafilter $\mathcal{F}$ on ω define $f(\mathcal{F})$ to be the filter $\{A : f^{-1}(A) \in \mathcal{F}\}$.*

We assume that all the filters we deal with are nontrivial—that is, they are proper and contain the filter of cofinite sets (Frechet filter).

THEOREM 13.2 The inequality $\mathfrak{u} < \mathfrak{g}$ implies the following *Filter Dichotomy*, FD: For every nontrivial filter $\mathcal{F}$ on ω there is a finite-to-one function $f : \omega \to \omega$ such that $f(\mathcal{F})$ is either a filter of cofinite sets or an ultrafilter.

PROOF Let $\mathcal{B}$ be a base of an ultrafilter $\mathcal{U}$ on ω of size $\mathfrak{u}$. For $B \in \mathcal{B}$ define a set $\mathcal{D}_B$ as the set of all $X \subseteq \omega$ such that there is an $A \in \mathcal{F}$ such that every interval with both ends lying in X intersects B only if it intersects A; in symbols

$$\mathcal{D}_B = \{X \in [\omega]^\omega : (\exists A \in \mathcal{F})(\forall x < y \in X)[x, y) \cap A \ne \emptyset \to [x, y) \cap B \ne \emptyset\}.$$

CLAIM If $f(\mathcal{F})$ is not contained in a Frechet filter for any finite-to-one f then the set $\mathcal{D}_B$ is groupwise dense in $[\omega]^\omega$.

PROOF We shall check (1) and (2) from Definition 13.1.

(1) Trivial.

(2) Let Π be a partition of ω into intervals. By making it coarser suppose that $B \cap I \ne \emptyset$ for all $I \in \Pi$. Enumerate Π as $\{I_i\}_{i\in\omega}$, and suppose that for every $A \in \mathcal{F}$ the set $\{i \in \omega : A \cap I_i = \emptyset\}$ is finite. For $f : \omega \to \omega$ defined by $f''I_i = \{i\}$ we have that f is finite-to-one and that $f(\mathcal{F})$ consists of cofinite sets, a contradiction. Hence there is an $A \in \mathcal{F}$ such that the set $\Pi' = \{I_i \in \Pi : A \cap I_i = \emptyset\}$ is infinite. The set $X = \bigcup \Pi'$ is in $\mathcal{D}_B$.

To check this, notice that for any pair $x < y \in X$ either $[x, y)$ is the subset of $\bigcup \Pi'$, thus $[x, y) \cap A$ is empty, or $[x, y)$ contains a whole interval from $\Pi \setminus \Pi'$, and thus has a nonempty intersection with B. This proves the Claim. □

A family $\{\mathcal{D}_B : B \in \mathcal{B}\}$ of groupwise dense sets is of size $\mathfrak{u}$ (and $\mathfrak{u} < \mathfrak{g}$); thus it has a nonempty intersection. So let Y be a fixed element of this intersection; enumerate Y increasingly as $\{y_i\}_{i\in\omega}$ and define a partition $\Pi_Y = \{[y_i, y_{i+1})\}_{i\in\omega}$ of ω into finite intervals. Define a function $f : \omega \to \omega$ by $f''J_i = \{i\}$.

CLAIM $f(\mathcal{U}) \subseteq f(\mathcal{F})$.

PROOF For any B in $\mathcal{B}$ there is an $A \in \mathcal{F}$ such that for all $x < y \in Y$:

$$A \cap [x, y) \ne \emptyset \to B \cap [x, y) \ne \emptyset$$

thus we have $i \in f''A \to i \in f''B$, and so $f''A \subseteq f''B$. On the other hand, the set $\{f''B : B \in \mathcal{B}\}$ generates $f(\mathcal{U})$, the set $\{f''A : A \in \mathcal{F}\}$ is a subset of $f(\mathcal{F})$, and so we are done. □

It remains to say that $f(\mathcal{U})$ is an ultrafilter and $f(\mathcal{F})$ is a proper filter; hence $f(\mathcal{F})$ is an ultrafilter, too. □

We shall need the following result in the proof of Theorem 13.1. A family $\mathcal{F}$ is *almost disjoint* iff all $x \ne y \in \mathcal{F}$, the intersection $x \cap y$ is finite.

LEMMA 13.1 There is an almost disjoint family $\mathcal{A} \subseteq [\omega]^\omega$ of size continuum.

PROOF It suffices to construct such family of subsets of some countable set, e.g. the set $[\omega]^{<\omega}$. To every $X \subseteq \omega$ we associate $A_X = \{X \cap n : n \in \omega\}$. Notice that $A_X \subseteq [\omega]^{<\omega}$ is infinite for an infinite X, and that for different X and Y the set $A_X \cap A_Y$ is finite. Hence $\{A_X : X \in [\omega]^\omega\}$ is the desired family. □

PROOF (of Theorem 13.1) ($\mathfrak{b} = \mathfrak{u}$) Choose an $<^*$-increasing $<^*$-unbounded sequence $(f_\xi)_{\xi<\mathfrak{b}}$ of increasing functions in ω^ω. Since $\mathfrak{b} \leq \mathfrak{u} < \mathfrak{g} < \mathfrak{d}$ this is not a dominating family and there is a $g \in \omega^\omega$ not eventually dominated by any f_ξ; without loss of generality we may assume that g is increasing. Let $\mathcal{F}$ be the filter on ω generated by sets

$$A_\xi = \{n : g(n) > f_\xi(n)\} \qquad (\xi < \mathfrak{b})$$

and all cofinite sets. It is easy to prove that this filter contains only infinite sets (use the fact that the set $\{n : f_\xi(n) < f_\eta(n)\}$ is cofinite for all $\xi < \eta < \mathfrak{b}$). By Theorem 13.2 there is a finite-to-one function $f: \omega \to \omega$ such that $f(\mathcal{F})$ is either an ultrafilter or the Frechet filter.

CLAIM The filter $f(\mathcal{F})$ is not equal to the Frechet filter; hence it is an ultrafilter.

PROOF Let $k_0 < k_1 < \ldots$ be the increasing enumeration of the range of f. Define $h: \omega \to \omega$ by $h(n) = g(\max(f^{-1}(k_{i+1})))$, where $k_i = f(n)$ for $n \in \omega$. If $f(\mathcal{F})$ were the Frechet filter then the image of every A_ξ would be cofinite; this means that for every ξ there is a $N_\xi \in \omega$ such that for all $k_i > N_\xi$ there is an $n = n(\xi, i) \in f^{-1}(k_i)$ satisfying $g(n) > f_\xi(n)$. But then for all $n > \max(f^{-1}\{k_i : k_i \leq N_\xi\})$ we would have that:

$$\begin{aligned} h(n) = g(\max(f^{-1}(k_{i+1}))) &\geq g(n(\xi, i+1)) \\ &> f_\xi(n(\xi, i+1)) = k_{i+1} > k_i = f_\xi(n), \end{aligned}$$

where $k_i = f_\xi(n)$. This means that h eventually dominates every f_ξ ($\xi < \mathfrak{b}$); a contradiction. □ (Claim)

The ultrafilter $f(\mathcal{F})$ is generated by $\mathfrak{b}$ sets $\{f''A_\xi : \xi < \mathfrak{b}\}$; hence we have $\mathfrak{u} \leq \mathfrak{b}$, and therefore $\mathfrak{u} = \mathfrak{b}$.

($\mathfrak{d} = \mathfrak{c}$) Fix a dominating family $\mathcal{D} \subseteq \omega^\omega$; without loss of generality we may suppose that it consists of strictly increasing functions. For a given $f \in \mathcal{D}$ let $0 = k_0^f < k_1^f < \ldots$ be a sequence of natural numbers such that $f(k_i^f) < k_{i+1}^f$ for all $i \in \omega$. Let

$$A_f = \bigcup_{i<\omega} [k_{4i}^f, k_{4i+1}^f] \quad \text{and} \quad B_f = \bigcup_{i<\omega} [k_{4i+2}^f, k_{4i+3}^f].$$

The point of this construction is that no interval of the form $[n, f(n)]$ intersects both A_f and B_f. Let $\mathcal{U}_f$ and $\mathcal{V}_f$ be arbitrary ultrafilters containing sets A_f and B_f respectively. Set $\mathcal{F} = \bigcap_{f\in\mathcal{D}}(\mathcal{U}_f \cap \mathcal{V}_f)$; it contains all cofinite sets. By FD there is a finite-to-one $g: \omega \to \omega$ such that $g(\mathcal{F})$ is either an ultrafilter or the Frechet filter.

CLAIM 1 $g(\mathcal{F})$ is not an ultrafilter.

PROOF Suppose that it is. Notice that in this case we have $g(\mathcal{U}_f) = g(\mathcal{V}_f) = g(\mathcal{F})$ for all $f \in \mathcal{D}$; therefore $g(A_f), g(B_f) \in g(\mathcal{F})$, and the set $g(A_f) \cap g(B_f)$ is infinite for all $f \in \mathcal{D}$. We shall show that this is impossible. The function $n \mapsto \max(g^{-1}(g(n)))$ is dominated by some $f \in \mathcal{D}$, i.e. for some $N_0(f) \in \omega$ we have

$$(\forall i > N_0(f))\, g^{-1}(i) \subseteq [n, f(n)), \quad \text{where } n = \min(g^{-1}(i)).$$

CLAIM 2 For every $\ell \in \omega$ such that $\min(g^{-1}(\ell)) > N_0(f)$ the set $g^{-1}(\ell)$ intersects at most two intervals $[k_i^f, k_{i+1}^f]$, which moreover must be consecutive.

PROOF Let $n = \min(g^{-1}(\ell)) \in [k_{i-1}^f, k_i^f]$ for some $i \in \omega$; then

$$\max(g^{-1}(g(n))) < f(n) \leq f(k_i^f) < k_{i+1}^f;$$

hence $g^{-1}(\ell) \subseteq [k_{i-1}, k_{i+1})$. □

From Claim 2 and definitions of A_f and B_f it follows that $g(A_f) \cap g(B_f) \subseteq N_0(f)$, and this proves Claim 1. □

Now we know that $g(\mathcal{F})$ is the Frechet filter; hence for every infinite $C \subseteq \omega$ there is an ultrafilter $\mathcal{W}_C$ from our family $\{\mathcal{U}_f, \mathcal{V}_f : f \in \mathcal{D}\}$ containing the preimage $f^{-1}(C)$ (if C is cofinite, this is clear; if it is not and there is no such $\mathcal{W}_C$, then the set $\omega \setminus C$ is in $f(\mathcal{F})$). Notice also that if the set $C \cap D$ is finite, so is the set $f^{-1}(C) \cap f^{-1}(D)$; hence the ultrafilters $\mathcal{W}_C$ and $\mathcal{W}_D$ are different. By Lemma 13.1, it follows that $\mathfrak{d} = \mathfrak{c}$.

The proof of $\mathfrak{g} = \mathfrak{d}$ uses similar ideas, although it is slightly more involved and it depends on Theorem 13.6 below which will not be proved here (see [Blass 1990] and [Laflamme]). □

DEFINITION 13.3 *A filter $\mathcal{F}$ on ω has cellularity n iff for every partition of ω into n sets, $\omega = \bigcup_{i<n} X_i$ there is an $i < n$ such that $\omega \setminus X_i \in \mathcal{F}$.*

Notice that ultrafilters are filters with cellularity 2, an intersection of two ultrafilters has cellularity 3, and so on.

THEOREM 13.3 (FD) For every pair of ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on ω there is a finite-to-one function $f: \omega \to \omega$ such that $f(\mathcal{U}) = f(\mathcal{V})$. The conclusion of this Theorem is called *Near Coherence of Filters*, shortly NCF.

PROOF Let $\mathcal{F}$ be a filter $\mathcal{U} \cap \mathcal{V}$. By Theorem 13.2 there is a finite-to-one function $f: \omega \to \omega$ such that $f(\mathcal{F})$ is the Frechet filter or an ultrafilter. For $\epsilon \in \{0, 1, 2\}$ define a set $X(\epsilon) = \bigcup_{i\in\omega} f^{-1}(3i+\epsilon)$. The filter $\mathcal{F}$ is of cellularity 2 and so there is ϵ, say 1, such that $X(\epsilon)$ is in $\mathcal{F}$. Thus we have that the non-cofinite set $\{3i+1\}_{i\in\omega}$ is in $f(\mathcal{F})$; hence $f(\mathcal{F})$ is not the Frechet filter, and so it is an ultrafilter. But $f(\mathcal{U}) \supseteq f(\mathcal{F})$ and $f(\mathcal{V}) \supseteq f(\mathcal{F})$, which means that we must have that $f(\mathcal{U}) = f(\mathcal{V}) = f(\mathcal{F})$. □

The following Theorem illustrates the influence of NCF on some relatively distant fields of mathematics; for its proof and other applications of NCF see [Blass 1987]. We need some additional definitions.

DEFINITION 13.4 *A topological space X is a continuum iff it is connected, compact and Hausdorff. A continuum is decomposable if it is a sum of two proper subcontinuums. It is known that $\beta\mathbb{R}^+$ is indecomposable. In an indecomposable continuum X a relation $\sim$ defined by: "$x \sim y$ iff there is a proper subcontinuum $Y \subseteq X$ such that x and y are in Y" is an equivalence relation. Equivalence classes for $\sim$ are called composants.*

DEFINITION 13.5 *Let H be an infinite-dimensional, separable, complex Hilbert space. Consider the algebra $\mathcal{L}(H)$ of all bounded linear operators on H (i.e. all linear $L: H \to H$ such that $L(x) \leq M\|x\|$ for some $M \in \mathbb{R}^+$ and all $x \in H$). An operator L on H is compact iff the image of the unit ball $U = \{x : \|x\| = 1\}$ is a compact set. The set of all compact operators on H forms a (two-sided) ideal of $\mathcal{L}(H)$, and it is denoted by $\mathcal{K}(H)$.*

THEOREM 13.4 The following statements are equivalent:

(1) For an infinite-dimensional, separable Hilbert space H over $\mathbb{C}$, the ideal $\mathcal{K}(H)$ is not the sum of two proper subideals.
(2) $\beta\mathbb{R}^+ \setminus \mathbb{R}^+$ has only one composant.
(3) NCF. □

Theorem 13.4 shows us that the results involving cardinal invariants can (and should) sometimes be formulated without mentioning any cardinals at all. Here is another example of such a phenomenon which is based on the proof that the additivity of Lebesgue measure is less than the additivity of a Baire category (see e.g. [Bekkali], [Fremlin 1985], ...)

THEOREM 13.5 There is an operator $T: \mathcal{N} \to \mathcal{M}$ such that

(1) T is monotone—$T(A) \subseteq T(B)$ for all $A \subseteq B$ in $\mathcal{N}$, and
(2) For every meager set M there is a null set N such that $T(N) \supseteq M$.

(Here, $\mathcal{N}$ is the ideal of subsets of $\mathbb{R}$ of Lebesgue measure zero and $\mathcal{M}$ is the ideal of first category subsets of $\mathbb{R}$). □

We also have the following strengthening of Theorem 13.2 proved recently by Laflamme (see [Laflamme] and [Blass 1990]) which gives another reformulation of $\mathfrak{u} < \mathfrak{g}$ not involving the cardinal numbers:

THEOREM 13.6 The inequality $\mathfrak{u} < \mathfrak{g}$ is equivalent to the following statement: For every nonempty family $\mathcal{A}$ of infinite subsets of ω there is a finite-to-one $f: \omega \to \omega$ such that

(1) $f''\mathcal{A}$ is dense in $[\omega]^\omega$, or
(2) $f''\mathcal{A}$ generates an ultrafilter, or
(3) $f''\mathcal{A}$ contains only cofinite sets. □

For more of cardinal characteristics of the continuum see [Blass 1987, 1990], [Fremlin 1985], or the references therein.

14. SOME OPEN PROBLEMS

Here we give a list of open mathematical problems which may or may not be related to forcing, but in many cases the forcing method might provide at least a better understanding of the problem in question.

The partition symbol $\kappa \overset{ccc}{\to} (\lambda)^m_n$ asserts the same thing as $\kappa \to (\lambda)^m_n$, but restricted to ccc partitions. It is known that $\mathrm{MA}(\omega_1)$ is equivalent to the assumption that every Suslin partition of $[\omega_1]^{<\omega}$ has an uncountable homogeneous set.

PROBLEM 14.1 Is $\mathrm{MA}(\omega_1)$ equivalent to the partition relation $\omega_1 \overset{ccc}{\to} (\omega_1)^2_2$?

The partition relation in the following problem is a relation for ordered sets; thus $\alpha \to (\beta, \gamma)^n$ means "For every ordered set of type α and every $f: [\alpha]^n \to 2$ there is either a 0-homogeneous set for f of type β or a 1-homogeneous set for f of type γ."

PROBLEM 14.2 Is $\omega_1 \to (\alpha, n)^3$ true for every countable ordinal α and every positive integer n?

REMARK This would be a considerable strengthening of the partition relation $\omega_1 \to (\alpha)^2_n$ (Baumgartner–Hajnal). The best result to date about this problem is given in [Milner–Prikry].

PROBLEM 14.3 Prove the following dense-set version of HL for infinitely many trees:

For every sequence $\langle T_i \rangle_{i<\omega}$ of finitely branching trees of height ω and for every partition $\bigotimes_{i<\omega} T_i = G_0 \cup G_1$ there exists $\epsilon \in \{0,1\}$ and $t_i \in T_i$ $(i < \omega)$ such that for every $n < \omega$ there is an n-dense sequence $\langle A_i : i < \omega \rangle$ for $\langle T_i[t_i] : i < \omega \rangle$ such that $A_0 \times A_1 \times \cdots \subseteq G_\epsilon$.

REMARK Notice that it is not required that $\langle t_i : i < \omega \rangle \in \bigotimes_{i<\omega} T_i$; this would be a too strong requirement. For more information about this problem see [Laver].

PROBLEM 14.4 Prove that for every partition of $[\mathbb{R}]^2$ into three parts $c: [\mathbb{R}]^2 \to 3$ there is an $X \subseteq \mathbb{R}$ homeomorphic to $\mathbb{Q}$ such that $|c''[X]^2| \leq 2$.

REMARK In the forcing extension $V^{C_{\beth_\omega}}$ the answer is positive. The solution of the problem might involve an extension of the HL Theorem.

PROBLEM 14.5 Prove that for every sequence $\{B_i\}_{i\in\omega}$ of Borel linearly ordered sets (i.e. sets of reals that are Borel and linearly ordered by a Borel relation) there are $i < j \in \omega$ such that $B_i \leq B_j$. In the other words, prove that the quasi-ordering of Borel linearly ordered sets ordered by the relation "is embeddable in" is well-quasi-ordered.

REMARK This is true for countable linearly ordered sets (Laver).

DEFINITION 14.1 *A submeasure on a Boolean algebra B is a function $\mu: B \to \mathbb{R}^+$ such that*

(1) $\mu(a) = 0$ *iff* $a = 0$, *and*
(2) $\mu(a \cup b) \leq \mu(a) + \mu(b)$ *for all a and b such that* $a \cap b = 0$.

A submeasure μ is exhaustive iff for every $\epsilon > 0$ there is no infinite disjoint sequence $\{a_i\}_{i\in\omega}$ such that $\mu(a_i) \geq \epsilon$ for every $i \in \omega$. A submeasure is uniformly exhaustive iff for every $\epsilon > 0$ there is an $n_\epsilon \in \omega$ such that there is no disjoint sequence $\{a_i\}_{i<n_\epsilon}$ of length n_ϵ such that $\mu(a_i) \geq \epsilon$ for every $i < n_\epsilon$.

PROBLEM 14.6 (The Control Measure Problem) Is every exhaustive submeasure on an infinite Boolean algebra uniformly exhaustive?

REMARK For more details on this problem and some of its variants see [Fremlin 1989].

PROBLEM 14.7 Let $\mathcal{P}$ be a poset such that $\mathcal{P} = \bigcup_{i<\omega} \mathcal{P}_i$ and $\mathcal{P}_i$ does not contain an infinite antichain for any $i \in \omega$. Is there a partition $\mathcal{P} = \bigcup_{i\in\omega} \mathcal{P}_i^+$ such that for every i there is an $n_i \in \omega$ such that $\mathcal{P}_i^+$ does not contain an antichain of cardinality n_i?

REMARK This is the only yet unsolved problem from the list given by [Horn–Tarski] in 1948.

PROBLEM 14.8 Is $P(\omega)/\,\mathrm{Fin}$ isomorphic to $P(\omega_1)/\,\mathrm{Fin}$?

REMARK The negative answer is trivially a consequence of CH, but are there some real (i.e. ZFC) combinatorial reasons making the distinction between these two algebras?

PROBLEM 14.9 Prove that for every metric space M there is a decomposition $M = M_0 \cup M_1$ such that every $P \subseteq M$ homeomorphic to the Cantor set intersects both M_0 and M_1.

REMARK This is true for $M = \mathbb{R}$ (Bernstein) and also for any M assuming $V = L$ (Weiss).

PROBLEM 14.10 (The Invariant Subspace Problem) Does every linear operator T in a separable Hilbert space have a nontrivial closed invariant subspace?

REMARK This does not hold for Banach spaces in general (Euflo, Read). On the other hand, if T is *polynomially compact* (there exists a polynomial $P(x)$ such that $P(T)$ is compact), then the answer is positive. This is proved using the methods of Non-standard Analysis (Bernstein–Robinson Theorem).

APPENDIX: A. AXIOMATIC SET THEORY

In this Appendix we shall describe a mathematical theory called the *Zermelo-Fraenkel Set Theory*, shortly ZF, and its extension ZFC, i.e. ZF plus the Axiom of Choice added. The material presented here is intended to be a logical background for the preceding chapters. Every of the cited textbooks in set theory gives more details if the reader wishes them. We assume some basic knowledge of Mathematical Logic (see [Mendelson]). By x, y, z, t, u, v we denote variables, while a, b, c denote constants. Capitals $X, Y, \ldots$ stand for proper classes, $\vec{u}$ is the n-tuple of variables, and $\mathrm{Len}(\vec{u})$ is its length. The same notation applies to n-tuples of constants. If $\vec{u} = \langle u_0, \ldots, u_{n-1}\rangle$ is an n-tuple, then $\vec{u}^\frown a$ is the $(n+1)$-tuple $\langle u_0, \ldots, u_{n-1}, a\rangle$.

A.1 ZFC The language of this theory has only one non-logical symbol, "$\in$", and the formula $x \in y$ is interpreted as "x is a member of y".

AXIOM 1 *Extensionality*: $\forall xy(\forall z(z \in x \leftrightarrow z \in y) \to x = y)$.

This axiom says that each set is entirely determined by its elements—two sets with the same elements are equal. From now on we shall not formulate axioms in a formal language, leaving this as an exercise for the reader.

AXIOM 2 *Pairing Axiom*: For every two sets x and y there exists a set $\{x, y\}$ whose elements are only x and y.

For every x and y we define the *ordered pair* $\langle x, y\rangle$ to be the set $\{x, \{x, y\}\}$. This set exists by Pairing, and by the Extensionality $\langle x, y\rangle = \langle z, t\rangle$ iff $x = z$ and $y = t$. We may then define the notion of an n-tuple recursively using the notion of ordered pair, but we shall later take a different approach. Until then we use the term "n-tuple" in an usual, intuitive way.

AXIOM 3 *Union Axiom*: The set $\bigcup x = \bigcup\{y : y \in x\} = \{z : (\exists y \in x) z \in y\}$ exists for every set x.

For every x and y define $x \cup y$ to be $\bigcup\{x, y\}$. Also, we define the relation $\subseteq$ by: $x \subseteq y$ iff $\forall z(z \in x \to z \in y)$. We shall freely use this symbol in our formulas having in mind that for every formula containing "$\subseteq$" there is an equivalent formula of the language of ZF. For example, the formula $(y \subseteq x \rightarrow y \in z)$ is equivalent to $(\forall t(t \in y \rightarrow t \in x) \rightarrow y \in z)$. We can define other relations by formulas in the language $\{\in\}$. E.g. the relation "x is a set" is defined by $x = x$. This relation is unary; thus it defines a subcollection of our universe. A collection of all x satisfying certain formula $\phi(x)$ need not be a set (by Russel's Paradox), and if this is the case then we say that it is a *proper class*. When we say that "X is a class", this means that X is a proper class or a set. The relation "x is a set" is an example of a proper class—it coincides with the universe. If X is a proper class, then we shall write $X(x)$ instead of $x \in X$, because "$\in$" denotes a relation between sets. Another important class is defined by:

"x is a set of ordered pairs, and (for all u, v, w) if $\langle u, v\rangle \in x$ and $\langle u, w\rangle \in x$ then $v = w$".

This class is the class of all functions. If f is a function and x is a subset of dom(f), then the *restriction of f to x* ($f \restriction x$) is the set $\{\langle u, v\rangle : \langle u, v\rangle \in f \;\&\; u \in x\}$. The reader may try to write down the defining formulas for "x is a function and $y = \text{dom}(x)$ (or $y = \text{range}(x)$)" and "x is an equivalence (partial ordering) relation".

AXIOM 4 *Power Set Axiom*: For every set x there exists a set $\mathcal{P}(x)$ the elements of which are all y such that $y \subseteq x$.

If $\phi(x_1, x_2, \ldots, x_n)$ is the formula with n variables x_i $(1 \le i \le n)$ we shall write $\phi(\vec{x})$. If the formula $\phi(x, y, \vec{z})$ and the n-tuple $\vec{a}$ are such that

$$\forall xyt((\phi(x, y, \vec{a}) \;\&\; \phi(x, t, \vec{a})) \rightarrow y = t),$$

then we say that $\phi(x, y, \vec{a})$ is *function-defining*. We implicitly assume that every function is a set, but the collection of all (x, y) such that $\phi(x, y, \vec{a})$ may be a proper class; hence we shall use the term *functional relation defined by $\phi(x, y, \vec{a})$*.

If f is a function, then by $f''x$ we denote the image of x, i.e. the set $\{f(u) : u \in x\}$.

AXIOM 5 *Replacement Scheme*: For every set x and every functional relation f, the set $f''x$ exists.

(Notice that x need not be contained in the domain of f). The following Corollary is often called the *Comprehension Scheme*:

COROLLARY For every set x, formula $\phi(u, \vec{v})$, and a Len$(\vec{v})$-tuple $\vec{a}$, the set $\{u \in x : \phi(u, \vec{a})\}$ exists.

PROOF Let $f(u) = u$ iff $\phi(u, \vec{a})$, undefined otherwise. □

So Replacement Scheme asserts the existence of a definable subset of a given set. Now we may define $x \cap y = \{u \in x : u \in y\}$, and $\bigcap x = \{u \in \bigcup x : (\forall t \in x) u \in t\}$. We also define the *empty set*, $\emptyset$, as $\{u \in x : u \neq u\}$. By last Corollary and Extensionality, this definition is valid if there exists some set x, but axioms mentioned so far still do not guarantee even the existence of the empty set; but this follows from the usual axioms for the First-order Predicate Calculus.

DEFINITION A.1.1 *For all x and y the direct product $x \times y$ of x and y is the set $\{\langle u, v\rangle : u \in x, v \in y\}$.*

To see that $x \times y$ exists for all x and y notice that this set is a definable subset of $\mathcal{P}(\mathcal{P}(x \cup y))$. Similarly, the set of all functions mapping x to y (denoted by ${}^x y$) exists as a definable subset of $\mathcal{P}(\mathcal{P}(\mathcal{P}(x \cup y)))$. If I and A are sets and f is a 1-1 function mapping I to A, then the set $\{x \in A : \exists i\, f(i) = x\}$ is denoted by $\langle x_i : i \in I\rangle$ and is called the *family of sets indexed by I*. The reader can now easily prove the existence of sets $\bigcup_{i \in I} x_i$ and $\bigcap_{i \in I} x_i$.

A.2 ORDINALS

DEFINITION A.2.1 *A binary relation ρ on the given set x is the partial ordering iff it is reflexive (i.e. $a\rho a$ for every $a \in x$), antisymmetric (i.e. $a\rho b$ and $b\rho a$ implies $a = b$ for all $a, b \in x$), and transitive (i.e. $a\rho b$ and $b\rho c$ implies $a\rho c$ for all $a, b, c \in x$). If ρ satisfies these conditions, then we say that the structure $\langle x, \rho\rangle$ is a partially ordered set, or a poset. Two elements a and b of a poset are comparable iff either $a\rho b$ or $b\rho a$. A partial ordering is total (or linear) iff every two elements of x are comparable.*

DEFINITION A.2.2 *If ρ is a partial ordering on a set A, then we say that the structure $\langle A, \rho\rangle$ is well-founded iff every nonempty subset B of A has a minimal element with respect to ρ—an element $x \in B$ with the property that there is no $y \in B \setminus \{x\}$ such that $y\rho x$. If ρ is a linear ordering, then we say that A is well-ordered by the relation ρ.*

DEFINITION A.2.3 *A set M is transitive iff every element of M is also a subset of M; or in other words, M is closed under the unary operation $x \mapsto \bigcup x$.*

DEFINITION A.2.4 *A set is an ordinal iff it is transitive and well-ordered by the relation $\in$. We always assume that the letters at the beginning of the Greek alphabet (like $\alpha, \beta, \gamma \ldots$) denote ordinals. Let Ord denote the class of all ordinals.*

LEMMA A.2.1 Every initial segment of an ordinal is an ordinal. Every proper initial segment of Ord is an ordinal. □

THEOREM A.2.1 The class Ord is well-ordered by the relation "$\in$". □

We shall write $\alpha < \beta$ iff $\alpha \in \beta$. For every α the set $S(\alpha) = \alpha \cup \{\alpha\}$ is an ordinal, and it is the least ordinal greater than α. Ordinals that are equal to $S(\alpha)$ for some α are said to be *successor* ordinals. An ordinal that is not a successor ordinal is a *limit* ordinal. Notice that Ord is not a set, otherwise it would be an ordinal—the greatest one. But every ordinal has a successor, etc.

DEFINITION A.2.5 *We define the natural numbers as ordinals: 0 is $\emptyset$, and $n+1$ is $S(n)$ for every n. Hence, all natural numbers (except 0) are successor ordinals.*

DEFINITION A.2.6 *If the relation ρ defined on the class C is such that for every $x \in C$ the class "of all y in C such that $y\rho x$" is a set, then we say that the relation ρ is set-like on C.*

LEMMA A.2.2

(1) If f is an *ordermorphism* (isomorphism with respect to the relation $<$) between α and β, then f is the identity map; thus $\alpha = \beta$.

(2) If the set x is well-ordered by some relation ρ, then there is an ordermorphism between x and an ordinal, unique by (1). We say that this ordinal is the *order type* of x (compare with the definition of the order type in §6).
(3) If the relation ρ is a set-like well-ordering on the proper class C, then C is ordermorphic to Ord.

Proof (1) Consider the least ξ such that $f(\xi) \neq \xi$. (2) Let y be the set of all initial segments of x that are ordermorphic to some ordinal. y is itself an initial segment of x, and for any $u, v \in y$ the corresponding ordermorphisms f_u and f_v coincide on the set $u \cap v$. (Otherwise, fix the least z such that $f_u(z) \neq f_v(z) \ldots$) Hence, the set $y' = \bigcup y \cup \{\min_\rho(x \setminus \bigcup y)\}$ is isomorphic to an ordinal (simply extend $\bigcup_{u \in y} f_u$). This is the contradiction because $y' \not\subseteq y$. (3) is proved like (2). □

Theorem A.2.2 (Definitions by Recursion on the Ordinals) Let M be a class of functions f such that dom(f) is an ordinal and let H be a function with domain M. There is an unique function F defined on Ord such that $F(\beta) = H(F \restriction \beta)$ for all β.

Proof See Theorem A.3.3 below. □

Similarly, we have the following *schema of proof by induction on the ordinals:*

$$\forall \beta[(\forall \alpha < \beta)\phi(\alpha) \to \phi(\beta)] \to \forall \gamma \phi(\gamma).$$

Actually there is nothing special about ordinals used in these two schemas. One can have also *definition by ρ-recursion* or the *proof by ρ-induction* for every well-founded relation ρ. The most frequently used ρ is the $\in$-relation restricted to some set x (or unrestricted). Thus, for example, the *schema of proof by $\in$-induction* has the form

$$\forall u[(\forall v \in u)\phi(v) \to \phi(u)] \to \forall w \phi(w).$$

Of course, we have to assume in this case that $\in$ is indeed a well-founded relation which is not automatic. One needs a special axiom (Axiom of Foundation, Axiom 8 in our A.4) to postulate this. We shall later need the following version of the definition by recursion, and in order to state it we need a new definition.

Definition A.2.7 *An increasing family of sets $\langle W_\alpha \rangle$ is continuous iff $W_\alpha = \bigcup_{\beta<\alpha} W_\beta$ for all limit α. For such family define a class $W = \bigcup_{\mathrm{Ord}(\beta)} W_\beta$, and for every $x \in W$ let $\rho(x)$ be the least α such that $x \in W_\alpha$.*

Fact (With the notation from Definition A.2.7) If W is a proper class and $x \subseteq W$ is a set, then there is α such that $x \subseteq W_\alpha$. □

Theorem A.2.3 Let W_α and ρ be as in the Definition A.2.7. Let $\mathcal{F}$ be some family of functions such that $(\forall f \in \mathcal{F})(\exists \alpha)\,\mathrm{dom}(f) = W_\alpha$. Then for every functional relation H with the domain $W \times \mathcal{F}$ there is the unique functional relation F with the domain W such that $F(x) = H(x, F \restriction \{y \in W : \rho(y) < \rho(x)\})$. □

Proof Similar to the proof of Theorem A.2.2, but this time we fix $x \in W$ such that ρx is minimal and x is not in the domain of any partial function approximating F. □

Axiom 6 *Axiom of Choice* (or AC): For every family $\mathcal{F}$ of nonempty sets there is a function $f : \mathcal{F} \to \bigcup \mathcal{F}$ such that $f(F) \in F$ for every $F \in \mathcal{F}$.

The function f whose existence is asserted by this axiom is called the *Choice Function for the family $\mathcal{F}$.*

Theorem A.2.4 (Zermelo; compare with the more general Theorem A.3.3) Every set can be well-ordered; more precisely, on every set x there is an ordering $<$ such that $\langle x, < \rangle$ is isomorphic to some $\langle \alpha, \in \rangle$.

Proof Fix a set x and a choice function f for the family $\mathcal{P}(x)$. Define a functional relation $H : \mathrm{Ord} \to x$ by $H(\alpha) = f(x \setminus \bigcup\{H(\beta) : \beta < \alpha\})$. By Theorem A.2.2, such H exists. Let α be the least ordinal such that $\bigcup\{H(\beta) : \beta < \alpha\} = x$. If such ordinal does not exist, then H is a bijection between Ord and x, hence (by Replacement), Ord would be a set. Then $H \restriction \alpha$ is a bijection between α and x. □

Definition A.2.8 *For two ordinals α and β we define the ordinal $\alpha + \beta$ (the ordinal $\alpha \cdot \beta$) as the order type of the set $\{0\} \times \alpha \cup \{1\} \times \beta$ (set $\beta \times \alpha$) ordered lexicographically.*

Notice that e.g. $2 \cdot \omega = \omega$, but $\omega \cdot 2 = \omega + \omega \neq \omega$. Also notice that $S(\alpha) = \alpha + 1$. Definition A.2.8 applies also to other order types.

A.3 Cardinals

Definition A.3.1 *Two sets x and y are equipotent (denoted by $x \sim y$) iff there is a bijective mapping $f : x \to y$. An ordinal is a cardinal iff it is not equipotent to any smaller ordinal. The linear ordering on Card is inherited from Ord. We usually denote cardinals by letters κ, λ, θ.*

As a corollary to the Theorem A.2.4 every set x is equipotent to a (unique) cardinal. This is the *cardinality* of the set x, denoted by $|x|$. We also say that x is of *power* κ or of *size* κ. Without the AC, the situation is much more complicated (see [Kuratowski–Mostowski]).

Theorem A.3.1 For every two sets x and y the following are equivalent:

(1) There is a 1-1 mapping from x to y.
(2) There is an onto mapping from y to x.
(3) $|x| \leq |y|$.

Proof Easy, using AC. □

Corollary (Cantor–Bernstein) If there is a 1-1 mapping from X to Y and a 1-1 mapping from Y to X then $X \sim Y$. □

Exercise Prove the Cantor–Bernstein theorem without using AC.

Theorem A.3.2 (Cantor) For any set x there is no bijection between x and $\mathcal{P}(x)$.

Proof Fix a function $f : x \to \mathcal{P}(x)$. Define $y = \{z \in x : z \notin f(z)\}$ and argue by contradiction to prove that there is no $z \in x$ such that $f(z) = y$; consequently, f is not onto. □

In fact, there is another proof of Cantor's theorem which uses another principle and which at the same time gives Zermelo's theorem. To state this proof, for a set x let $\mathcal{W}(x)$ be the set of all well-orderable subsets of x, i.e. subsets y of x for which there is a relation $\rho \subseteq y^2$ such that (y, ρ) is well-ordered.

THEOREM A.3.3 For every function $F: \mathcal{W}(x) \to x$ there exist $y \subseteq z$ in $\mathcal{W}(x)$ such that $F(y) = F(z) \in z \setminus y$.

PROOF Suppose that such y and z in $\mathcal{W}(x)$ cannot be found and define $G: \mathrm{Ord} \to x$ recursively by $G(\alpha) = F(G''\alpha)$. This is well-defined because the $G''\alpha$ is clearly an element of $\mathcal{W}(x)$. The fact that there are no $y \subseteq z$ such that $F(y) = F(z) \in z \setminus y$ is used to inductively show that G must be a 1-1 map, a contradiction. □

DEFINITION A.3.2 *For two cardinals κ and λ we define cardinals $\kappa + \lambda$, $\kappa \cdot \lambda$ and κ^λ:*

(1) $\kappa + \lambda = |\kappa \times \{0\} \cup \lambda \times \{1\}|$,
(2) $\kappa \cdot \lambda = |\kappa \times \lambda|$, *and*
(3) $\kappa^\lambda = |\{f : f: \lambda \to \kappa\}|$.
(4) $k^{<\lambda} = \sup_{\theta<\lambda} k^\theta$.

All natural numbers are cardinals, and the addition, multiplication and exponentiation defined in Definition A.3.2 coincide with the ordinary arithmetic for natural numbers. Functions $+$ and $\cdot$ are increasing, but not strictly increasing (see Theorem A.3.4).

AXIOM 7 *Axiom of Infinity*: There is a set x such that $\emptyset \in x$ and for every $y \in x$ also $S(y) \in x$.

It is easy to see that an ordinal satisfies this Axiom if and only if it is a limit ordinal. So the least ordinal satisfying this Axiom is the least ordinal containing all natural numbers, and we denote it by ω. The cardinal $|\omega|$ is denoted by $\aleph_0$ and $\aleph_\alpha$ is the αth cardinal greater than $\aleph_0$. The order type of $\aleph_\alpha$ is denoted by ω_α. Thus, $\aleph_\alpha$ and ω_α denote the same object, but they will be used in order to distinguish whether we are referring to a cardinal or to an ordinal. The least cardinal greater than the cardinal κ is denoted κ^+, and such cardinals are called *successor* cardinals. Non-successor cardinals are *limit*. Thus $\aleph_\alpha$ is limit iff α is limit. (Note that there is always a cardinal greater than κ. This follows from Cantor's Theorem using AC, but it can be proved directly without using AC. For example, let κ^+ be the set of all ordinals α for which there is a 1-1 function $f: \alpha \to \kappa$).

DEFINITION A.3.3 *The set x is infinite iff there is a 1-1 function $f: \omega \to x$. Otherwise, it is finite. The set is countable iff its cardinality is $\aleph_0$.*

DEFINITION A.3.4 *For $n \in \omega$ an ordered n-tuple is the function f such that* $\mathrm{dom}(f) = n$. *Instead of "an n-tuple $\vec{a}$ of elements of the set x" we shortly say "an n-tuple $\vec{a}$ in x."*

By Theorem A.3.3, the relation $2^\kappa \geq \kappa^+$ is true for every cardinal κ. The *Continuum Hypothesis (or CH)* is the statement that $2^{\aleph_0} = \aleph_0$. The *Generalized Continuum Hypothesis* (GCH) is the statement that $2^\kappa = \kappa^+$ for all cardinals κ.

There is another hierarchy of cardinal numbers worth mentioning, $\beth_\alpha$, defined in the following way:

$$\beth_\alpha = \begin{cases} \aleph_0, & \text{if } \alpha = 0 \\ 2^{\beth_\beta}, & \text{if } \alpha = \beta + 1, \\ \sup_{\beta<\alpha} \beth_\beta, & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

Hence CH is the statement that $\aleph_1 = \beth_1$, while GCH is the statement that "$\aleph_\alpha = \beth_\alpha$ for all ordinals α".

THEOREM A.3.4 $\kappa^n = \kappa$ for every infinite cardinal κ and a positive integer n. In particular, the set of all finite sequences of ordinals from κ has cardinality κ, i.e. $\kappa^{<\aleph_0} = \kappa$.

On the other hand, $\kappa^{\aleph_0}$ need not be equal to κ; e.g. if $\kappa = \aleph_0$. But there exist arbitrarily large cardinals κ such that $\kappa^{\aleph_0} = \kappa$, e.g. $\kappa = \beth_\alpha$ whenever $\mathrm{cf}\,\alpha > \omega$.

PROOF Notice that it is enough to prove the statement for $n = 2$. Let $<_2$ be an ordering of Ord × Ord defined by: $\langle\alpha, \beta\rangle <_2 \langle\xi, \eta\rangle$ iff (see Fig. 19)

(1) $\max(\alpha, \beta) <_2 \max(\xi, \eta)$, or
(2) $\max(\alpha, \beta) = \max(\xi, \eta)$ and $\langle\alpha, \beta\rangle <_{Lex} \langle\eta, \xi\rangle$.

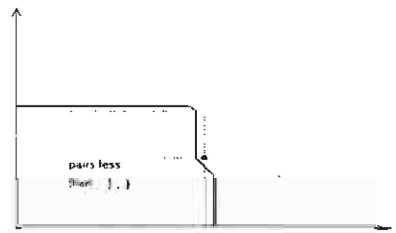

Fig. 19

This relation is a set-like well-ordering on Ord^2, so there is an isomorphism

$$I: (\mathrm{Ord}^2, <_2) \to (\mathrm{Ord}, <).$$

Notice that $|\kappa^2| = \kappa$ follows from $I''(\kappa \times \kappa) \subseteq \kappa$. So we shall prove the latter statement by induction on κ. For $\kappa = \omega$, the restriction $I \restriction \omega^2$ can be explicitly defined as follows:

$$I(m, n) = \binom{m + n + 1}{2} + m.$$

So, suppose that $\kappa > \omega$ and that $I''\lambda^2 \subseteq \lambda$ for all cardinals $\lambda < \kappa$. If κ is a limit cardinal then $I''\kappa^2 = \bigcup\{I''\lambda^2 : \mathrm{Card}(\lambda)\ \&\ \lambda < \kappa\}$ which gives the conclusion. So assume that $\kappa = \lambda^+$ for some λ. Notice that the diagonal Δ of κ^2 is unbounded in $(\kappa^2, <^2)$, so it suffices to show that $I''\Delta \subseteq \kappa$. But if $I(\langle\gamma, \gamma\rangle) \geq \kappa$ for some $\gamma < \kappa$, the set γ^2 would have the cardinality κ, contradicting the fact that $|\gamma^2| = |\lambda^2|$ and that the Theorem holds for λ. □

COROLLARY For all infinite κ and λ:

(1) Operations $+$ and $\cdot$ coincide for infinite cardinals, $\kappa+\lambda=\kappa\cdot\lambda=\max(\kappa,\lambda)$.
(2) For any x of power κ there are exactly κ distinct finite sequences of elements of x.
(3) The union of at most κ sets each of cardinality at most κ has the cardinality at most κ.
(4) $2^\kappa=\kappa^\kappa$. (Because $2^\kappa \le \kappa^\kappa \le (2^\kappa)^\kappa = 2^{\kappa\cdot\kappa} = 2^\kappa$). □

DEFINITION A.3.5 *The cofinality of an ordinal α, in symbols $\mathrm{cf}(\alpha)$, is the least ordinal β such that there is a cofinal mapping $f:\beta\to\alpha$, i.e. a mapping with the property that for every $\xi<\alpha$ there exists $\eta<\beta$ such that $\xi\le f(\eta)$.*

LEMMA A.3.1 (See [Kunen, Lemma 10.31]) There is a strictly increasing cofinal mapping $g:\mathrm{cf}(\alpha)\to\alpha$. □

LEMMA A.3.2 (See [Kunen, Lemma 10.32]) If there is a cofinal strictly increasing map $f:\alpha\to\beta$, then $\mathrm{cf}(\alpha)=\mathrm{cf}(\beta)$. So for every β we have $\mathrm{cf}(\mathrm{cf}(\beta))=\mathrm{cf}(\beta)$. □

We say that a limit ordinal α is *regular* iff $\mathrm{cf}(\alpha)=\alpha$.

LEMMA A.3.3 Every regular ordinal is a cardinal. □

Cardinals that are not regular are said to be *singular*.

LEMMA A.3.4 Every successor cardinal is regular.

PROOF Suppose that $f:\alpha\to\kappa^+$ is cofinal for some $\alpha<\kappa^+$. Then for every $\xi<\alpha$ the ordinal $f(\xi)$ is of cardinality at most κ, and there is $|\alpha|\le\kappa$ such ordinals. So $\kappa^+=\left|\bigcup_{\xi<\alpha}f(\xi)\right|\le\kappa$, a contradiction. □

All limit cardinals one can imagine are singular—take $\aleph_\omega$ or $\aleph_{\omega_1}$ for example. But there seems to be nothing which prevents the existence of limit regular cardinals greater than ω, so-called *weakly inaccessible* cardinals. For a limit ordinal α we have that $\mathrm{cf}(\aleph_\alpha)=\mathrm{cf}(\alpha)$, thus a cardinal $\aleph_\alpha$ is weakly inaccessible iff $\aleph_\alpha=\alpha=\mathrm{cf}(\alpha)$. A weakly inaccessible cardinal κ is *strongly inaccessible* iff it is also greater than 2^λ for all $\lambda<\kappa$. If GCH is true then these two notions coincide. When we say "inaccessible" we mean "strongly inaccessible".

A.4 THE AXIOM OF FOUNDATION

AXIOM 8 *Axiom of Foundation*: For every set x there is a $y\in x$ such that $y\cap x=\emptyset$.

It follows that $x\cap\{x\}=\emptyset$ for every set x, and, therefore, $x\notin x$ for every set x. In a similar way we prove that there is no infinite $\in$-decreasing sequence $x_0\ni x_1\ni\ldots$ (consider the set $\{x_i : i\in\omega\}$), so Foundation states that the relation $\in$ is well-founded. Therefore we can define an ordinal as a transitive set that is linearly ordered by $\in$.

Now we define *The Cumulative Hierarchy*, V, by recursion on Ord in the following way (see Fig. 20):

$$V_0=\emptyset \qquad V_{\alpha+1}=\mathcal{P}(V_\alpha) \qquad V_\alpha=\bigcup_{\beta<\alpha}V_\beta, \quad \text{if } \alpha \text{ is a limit ordinal.}$$

$$\text{and finally,}\quad V=\bigcup_{\mathrm{Ord}(\alpha)}V_\alpha$$

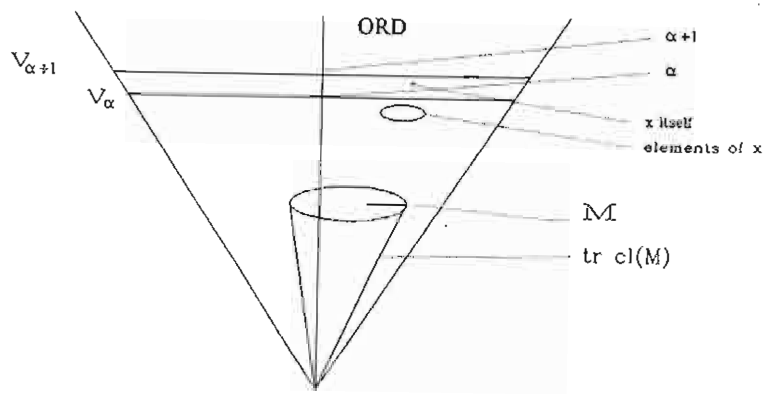

Fig. 20

For a set x in V, the *rank* of x (denoted $\mathrm{rank}(x)$) is the least ordinal α such that $x\in V_\alpha$. Rank is always a successor ordinal. A set x is in V iff all of its elements are in V, and its rank is $\sup\{\mathrm{rank}(y): y\in x\}+1$. So each V_α is transitive.

LEMMA A.4.1 Every ordinal α is in V, and its rank is $\alpha+1$.

PROOF ($\mathrm{rank}(\alpha)\le\alpha+1$) Fix the least ordinal $\alpha\notin V_{\alpha+1}$. Every $\beta<\alpha$ is in $V_{\beta+1}\subseteq V_\alpha$, so $\alpha\in V_{\alpha+1}$. ($\mathrm{rank}(\alpha)\ge\alpha+1$) Fix the first $\alpha\in V_\alpha$. We have $\alpha\in\bigcup_{\beta<\alpha}V_\beta$, thus for some $\beta<\alpha$ we have $\alpha\subseteq V_\beta$, and hence $\beta\in V_\beta$. □

For a given set M define its *transitive closure* $\mathrm{tr\,cl}(M)$ as the least transitive set containing M (see Fig. 20). It can be explicitly defined by

$$\mathrm{tr\,cl}(M)=\bigcup\left\{\bigcup^n M : n\in\omega\right\}, \qquad \text{where} \quad \bigcup^n M=\underbrace{\bigcup\cdots\bigcup}_{n} M$$

THEOREM A.4.1 Axiom of Foundation is equivalent to the statement $(\forall x)x\in V$, i.e. that $(\forall x)(\exists\alpha)x\in V_\alpha$. □

Thus, the Axiom of Foundation tells us that our universe is restricted to V. Note that almost every mathematical object can be defined inside V, and that a major part of mathematics is placed already in $V_{\omega+\omega}$.

DEFINITION A.4.1 *For a set x we say that it is extensional iff for all $y \neq z \in x$ the sets $y \cap x$ and $z \cap x$ are distinct.*

THEOREM A.4.2 (MOSTOWSKI'S COLLAPSING THEOREM) For every extensional set x there is a unique transitive set M and a unique isomorphism

$$\pi : \langle x, \in \rangle \to \langle M, \in \rangle.$$

PROOF The isomorphism π is defined by an $\in$-recursion as follows:

$$\pi(y) = \{\pi(z) : z \in y\} \quad (y \in x).$$

Let $M = \{\pi(y) : y \in x\}$. Then M is transitive and $\pi : \langle x, \in \rangle \to \langle M, \in \rangle$ is an isomorphism. Uniqueness is proved using the $\in$-induction. □

REMARK Note that if $y \in x$ is a transitive set then $\pi(y) = y$, i.e. the fixed points of the collapsing isomorphism are precisely the transitive sets.

One often needs also the following version of Theorem A.4.2 which is proved similarly.

THEOREM A.4.3 Let U be a set and let E be a binary relation on U such that

(1) $\langle U, E \rangle$ is well-founded,
(2) $\forall x, y \in U (x \neq y \to (\exists z \in U)(z E x \leftrightarrow \neg z E y))$.

Then there is a unique transitive set M and a unique isomorphism $\pi : \langle U, E \rangle \to \langle M, \in \rangle$. □

APPENDIX: B. THE MODEL THEORY OF V

B.1 RELATIVIZATION Let X be a class, $\phi(\bar{x})$ a formula of the language $\{\in\}$, and let $\bar{a}$ be an n-tuple in X. We say that $\phi(\bar{a})$ is the *formula with parameters from X*. We define the *relativization*, $\phi^X(\bar{a})$, of the formula $\phi(\bar{a})$ to X inductively in the following way (we shall not write down the parameters in formulas to save space):

(1) If ϕ is *atomic* (i.e. ϕ is $x \in y$ or $x = y$ for some variables or constants x and y), then ϕ^X is ϕ.
(2) If ϕ is $\neg\psi$ for some formula ψ then ϕ^X is $\neg\psi^X$.
(3) If ϕ is ψ_1 & ψ_2 for some formulas ψ_1 and ψ_2 then ϕ^X is ψ_1^X & ψ_2^X.
(4) If ϕ is $(\exists x)\psi$ for some variable x and formula ψ, then ϕ^X is $(\exists x)(X(x)$ & $\psi^X)$ (here $X(x)$ is the defining formula of the class X).

(Recall that all logical connectives can be expressed by using only $\neg$ and &). So, we get the formula ϕ^X by restricting all quantifiers occurring in ϕ to X. It is easy to see that the set X is extensional iff (Extensionality Axiom)^X is true. More generally,

FACT Formula ϕ^X is true (in the universe) iff the formula ϕ is true in X. □

If the formula ϕ^X is true then we say that X is a *model* for ϕ, or that X is *correct* for ϕ. We denote this fact by $x \models \phi$ ("x *models* ϕ").

THEOREM B.1.1 All axioms of ZFC are true in V.

PROOF We give two sample proofs.

Extensionality. If $x, y \in V$, then all elements of x and y are in V. Thus if $x \cap V = y \cap V$ then (by the Extensionality) $x = y$.

Union. If $x \in V_{\alpha+1}$ then (by transitivity) $\bigcup x \subseteq V_\alpha$, hence $\bigcup x \in V_{\alpha+1}$.

Etc. □

Thereupon V is a model for ZFC, but it is a proper class and we would like to prove that there is a model for ZFC that is a set. This happens to be impossible (by Gödel's Incompleteness Theorem; of course, if we assume that ZFC is consistent), but we will produce a good enough approximation.

DEFINITION B.1.1 *For every cardinal κ by H_κ we denote the set of all sets x such that $|\operatorname{tr}\operatorname{cl}(x)| < \kappa$. The sets in $H_{\aleph_0}$ ($= V_\omega$) are said to be hereditarily finite, while the sets in $H_{\aleph_1}$ are hereditarily countable.*

THEOREM B.1.2 For any cardinal κ:

(1) $H_\kappa \subseteq V_\kappa$.
(2) H_κ is transitive.
(3) If $y \subseteq x \in H_\kappa$, then $y \in H_\kappa$.
(4) If κ is regular then $x \in H_\kappa$ iff $x \subseteq H_\kappa$ and $|x| < \kappa$.
(5) If κ is regular and uncountable then H_κ is a model for ZFC$-$P (i.e. theory ZFC minus the Power Set Axiom).

PROOF (1) For $x \in H_\kappa$ we have $\operatorname{rank}(x) = \{\operatorname{rank}(u) + 1 : u \in x\}$, and this set is an ordinal $\leq |x| < \kappa$.

(2), (3), (4) Easy.

(5) Extensionality follows from (2), Union from (2) and (3), Replacement from (2), (3), the regularity of κ, and the fact that the union of $< \kappa$ sets of cardinality $< \kappa$ is of cardinality $< \kappa$. Infinity follows from $\omega < \kappa$ and the fact that $(x = \omega)^{H_\kappa}$ iff $x = \omega$; for the proof of the latter fact see subsection B.2. □

The major part of mathematics is placed in $H_{\mathfrak{c}^+}$. Usually H_κ is much smaller than V_κ; for example, notice that already $V_{\omega+2}$ is not a subset of $H_{\mathfrak{c}}$.

THEOREM B.1.3 For a regular $\kappa > \omega$, $H_\kappa = V_\kappa$ iff κ is strongly inaccessible.

PROOF If κ is not strongly inaccessible fix a $\lambda < \kappa$ such that $2^\lambda \geq \kappa$; then $\mathcal{P}(\lambda) \in V_{\lambda+2} \setminus H_\kappa \subseteq V_\kappa \setminus H_\kappa$. Otherwise for any $\lambda < \kappa$ we have that $|V_\lambda| < \kappa$, hence $V_\lambda \in H_\kappa$, and any $x \in V_\kappa$ is in V_λ for some $\lambda < \kappa$. □

THEOREM B.1.4 If κ is strongly inaccessible, then V_κ is a model for ZFC.

PROOF By Theorem B.1.2 (5) and Theorem B.1.3, we only have to prove that V_κ satisfies the Power Set Axiom. But V_ξ satisfies it for any limit ξ: if $x \in V_\alpha$ then all subsets of x are in $V_{\alpha+1}$, and $\mathcal{P}(x)$ is in $V_{\alpha+2}$. □

For a given theory T by Con(T) we denote the statement "T is consistent". If a theory is rich enough (like ZFC or even Peano Arithmetic) this statement can be expressed as a statement in its own language, and by Gödel's Incompleteness Theorem it is not a consequence of T, unless T is inconsistent. So, we have the following

COROLLARY The consistency of ZFC does not imply the consistency of ZFC + "There is an inaccessible cardinal". □

The existence of an inaccessible cardinal is one of the weakest axioms of the long list of large cardinal axioms. A statement ϕ is a *large cardinal axiom*, or more precisely, an *axiom with a large cardinal strength*, if Con(ZFC) does not imply Con(ZFC$+\phi$). Such statements usually state the existence of a cardinal with some property, but there are certain large cardinal axioms that do not mention such properties; one of them is PFA. In fact, there is a number of classical problems with a large cardinal hidden in them. There are even natural propositions involving only (hereditarily) finite sets and having large cardinal strength.

B.2 REFLECTION In this subsection we shall use the word "induction" in its usual meaning—induction on the set of natural numbers.

DEFINITION B.2.1 *If X is a class and $\phi(\bar{x})$ is a formula then we say that ϕ is absolute for X iff for any valuation $\bar{a} \in X^n$ we have that $\phi(\bar{a})$ is true iff $\phi^X(\bar{a})$ is true. A formula is absolute iff it is absolute for any X. If X is a class and Y is its subclass, then $\phi(\bar{x})$ is absolute for X and Y iff for any valuation $\bar{a} \in Y^n$ we have $\phi^Y(\bar{a})$ iff $\phi^X(\bar{a})$.*

LEMMA B.2.1 Every formula without quantifiers is absolute. If ϕ and ψ are absolute, then $\phi \mathbin{\&} \psi$ and $\neg\phi$ are absolute. □

A formula is in the *prenex normal form* (PNF) iff it is $(Q_1 x_1)\ldots(Q_n x_n)\phi(\bar{x})$, where each Q_i is either $\forall$ or $\exists$, and ϕ is without quantifiers. A basic result of the predicate calculus says that every formula is equivalent to some formula in PNF, so we shall restrict our attention to PNF-formulas.

LEMMA B.2.2 Let Y be a subclass of X, and let $\phi(v, \bar{x})$ be absolute for X and Y. The formula $\exists v\phi(v, \bar{x})$ is absolute for X and Y iff for every Len($\bar{x}$)-tuple $\bar{a}$ in Y such that $\exists v\phi(v, \bar{a})$ is true in X exists $b \in Y$ such that $\phi(b, \bar{a})$ is true in Y.

PROOF We have two cases (fix a Len($\bar{x}$)-tuple $\bar{a}$ in Y):

(1) If $\exists v\phi(v, \bar{a})$ is true in X, then there is a $b \in Y$ such that $\phi^Y(b, \bar{a})$ is true, hence $\phi^X(b, \bar{a})$ is true.
(2) $\exists v\phi(v, \bar{a})$ is false in X. Suppose that it is true in Y; then there is a $b \in Y$ such that $\phi^Y(b, \bar{a})$ is true. But $\phi(v, \bar{x})$ is absolute for X and Y, and $\phi^X(b, \bar{a})$ is true; a contradiction. □

LEMMA B.2.3 Let $\langle x_i : i \in \omega\rangle$ be the increasing family of sets. If a formula ψ and all its subformulas are absolute for each x_i, then ψ is absolute for $x = \bigcup_{i\in\omega} x_i$.

PROOF We may assume that ψ is $(Q_1 v_1)\ldots(Q_n v_n)\phi(\bar{u}, \bar{v})$ (where Len($\bar{v}$) $= n$) for some formula ϕ without quantifiers. The proof goes by induction on the length of the string of quantifiers. Firstly we substitute every occurrence of $\forall v_i$ with $\neg\exists v_i\neg$, so (by Lemma B.2.1) we need to check the induction step only for $\exists v_i$. Formula $\phi(\bar{u}, \bar{v})$ is, being without quantifiers, absolute. Suppose that $(Q_{k+1} v_{k+1})\ldots(Q_n v_n)\phi(\bar{u}, \bar{v})$ is absolute for x, and denote it by $\phi_k(\bar{u}, v_1, \ldots, v_k)$. Proof that this formula is absolute for x is just like that of Lemma B.2.2, where in the analogue of the case (1) we use the fact that the valuation for ϕ_k in x lies in some x_i. □

THEOREM B.2.1 (REFLECTION THEOREM) For any formula $\phi(\bar{x})$, increasing and continuous family $\langle W_\alpha\rangle$, and any ordinal β there is a $\xi > \beta$ such that $\phi(\bar{x})$ is absolute for W_ξ and W.

PROOF We suppose that $\phi(\bar{x})$ is in the PNF, and proceed by induction on the length of the string of quantifiers, like in the Lemma B.2.3. We have to check only the case when $\phi_{k+1}(\bar{y})$ is $\exists v\psi_k(v, \bar{y})$. By the inductive hypothesis there is a $\beta_0 > \beta$ such that W_{β_0} is absolute for ψ_k and all its subformulas. We define a functional relation g: Ord $\to$ Ord in the following way:

For every formula $\exists x\phi(x, \bar{y})$ define $f_\phi: V^{\operatorname{Len}(\bar{y})} \to V$ by:

$$f_\phi(\bar{a}) = \begin{cases} \alpha, & (\exists u \in W_\alpha)\phi(u, \bar{a}) \mathbin{\&} (\forall\beta < \alpha)(\forall v \in W_\beta)\neg\phi(v, \bar{a}) \\ 0, & \neg(\exists v)\phi(v, \bar{a}). \end{cases}$$

Notice that $f''_{\psi_k}[W_\delta]^{\mathrm{Len}(\bar{y})}$ is a set for any δ, and let $g(\delta)$ be the least γ such that W_γ includes this set. Define a sequence of ordinals β_i ($i \in \omega$) as follows:

(1) β_0 is the least ordinal $> \alpha$ such that ψ_k is absolute for W_{β_0},
(2) $\beta_{2k+1} = g(\beta_{2k})$, and
(3) β_{2k} is the least ordinal greater than β_{2k-1} such that ψ_k and all its subformulas are absolute for $W_{\beta_{2k}}$.

Let ξ_k be the supremum of this sequence. By Lemma B.2.3, W_{ξ_k} is absolute for ψ_k, while by Lemma B.2.2 it is absolute for $\exists v \psi_k(v, \bar{y})$, and this ends the proof. □

Concerning possible applications of Theorem B.2.1, notice that the family $\langle V_\xi \rangle$ is continuous. Although the family $\langle H_{\aleph_\xi} \rangle$ is not continuous, we still have Theorem B.2.2 (5). Recall that by the Axiom of Foundation every set lies in V, so we have

COROLLARY For any formula $\phi(\bar{x})$ and any β there are $\alpha, \kappa > \beta$ such that $\phi(\bar{x})$ is absolute for V_α and for H_κ. □

If we have any finite subtheory of ZF or ZFC, we can put it into one formula and apply Theorem B.2.1. By the Compactness Theorem of Predicate Calculus, every theorem of ZFC is a consequence of a finite list of axioms; therefore we can always construct a model satisfying all relevant statements, i.e. a *large enough part of ZFC*.

THEOREM B.2.2 Any finite subtheory of ZFC has a model. □

Also, again by Gödel's Incompleteness Theorem,

THEOREM B.2.3 ZFC is not finitely axiomatizable. □

We now have a model for any finite part of ZFC, and we may also construct this model so that it contains any object of interest to us—all we need is to start with the large enough V_α (or H_κ). This model is even transitive, but we wanted our models also to be countable (for a good enough reason see e.g. Corollary to Theorem 2.1), and V_α is uncountable for any $\alpha > \omega$. But once we have a model we may use some Model Theory to shrink it.

DEFINITION B.2.2 *If $M \subseteq N$ and all formulas are absolute for M and N, then we say that M is an elementary submodel of N.*

The following result asserts the existence of a small elementary submodel of H_θ containing some prescribed set. Beginning with some H_θ and its well-ordering, $<_w$ (whose existence is guaranteed by Theorem A.2.4), let us define a function $f_\phi : [H_\theta]^{\mathrm{Len}(\bar{a})} \to H_\theta$ for every $\phi(v, \bar{a})$ by

$$f_\phi(\bar{a}) = \begin{cases} \text{"The } <_w\text{-least } b \in H_\theta \text{ such that } H_\theta \models \phi(b, \bar{a})\text{"}, & \text{if such } b \text{ exists,} \\ 0, & \text{otherwise.} \end{cases}$$

This function is called a *Skolem function* for the formula ϕ. For $x \in H_\theta$, define $F: H_\theta \to H_\theta$ by

$$F(w) = \bigcup \{ f''_\phi[w]^{\mathrm{Len}(\bar{v})} : \phi(u, \bar{v}) \text{ is a formula of the language of ZFC} \}.$$

(The formal definition of this function requires encoding formulas by elements of V_ω, and we are completely omitting this technical part). Notice that, by putting $u = v$ for $\phi(u, v)$, we have $F(w) \supset w$. The closure of a set x under the function F is called *Skolem hull of the set* x, and denoted by $\mathrm{Hull}_{H_\theta}(x)$. Notice that we can define Skolem functions on any well-ordered set, hence we can speak about e.g. $\mathrm{Hull}_{V_\alpha}(x)$. The following Lemma needed in Chapter 4 shows that we can build $\mathrm{Hull}_{H_\theta}(y)$ in a single step.

LEMMA B.2.4 $\mathrm{Hull}_{H_\theta}(y)$ is equal to the set of all $x \in H_\theta$ such that x is the unique element of H_θ such that $H_\theta \models \phi(x, \bar{a})$ for some formula ϕ and some sequence of parameters $\bar{a}$ in y. (Hence $|\mathrm{Hull}_{H_\theta}(y)| = |y| + \aleph_0$).

PROOF It is enough to prove that for every $x \in \mathrm{Hull}_{H_\theta}(y)$ there are ϕ and $\bar{a}$ as required, and this is easily done by induction on i. □

Notice that in Theorem B.2.1 there is no bound on the size of a model W_ξ. The following Theorem gives us a small transitive model.

THEOREM B.2.4 For every transitive set y and every finite fragment of ZFC there is a model M of this fragment containing y such that $|M| \leq |y| + \aleph_0$; moreover, M may be chosen to be transitive.

PROOF By the Reflection, take V_α containing y and satisfying the required fragment of ZFC. Take $M = \mathrm{Hull}_{V_\alpha}(y)$. Take transitive collapse of M for the desired transitive model. □

B.3 MORE ABSOLUTENESS

Certain formulas are not absolute for all models, but are absolute for all transitive models. By the Mostowski's Collapsing Theorem, this is all we need.

DEFINITION B.3.1 *A formula ψ is Δ_0 iff it is*

(1) *without quantifiers,*
(2) *$(\exists x \in y)\phi$, for some Δ_0 formula ϕ, or*
(3) *ϕ & ψ or $\neg\phi$ for some Δ_0 formulas ϕ and ψ.*

LEMMA B.3.1 All Δ_0 formulas are absolute for transitive models.

PROOF By induction on the length of a formula. The only nontrivial part, when ϕ is $(\exists x \in y)\psi$ and ψ is absolute, follows directly from the Lemma B.3.2 and transitivity. □

COROLLARY The following properties are equivalent to Δ_0 formulas, thus absolute for transitive models:

"x is transitive", as	$(\forall y \in x)(\forall z \in y)(z \in x)$
$\mathrm{Ord}(x)$, as	$(\forall y, z \in x)(y \in z \vee z \in y \vee z = y)$ & "x is transitive"
"x is a limit ordinal", as	$\mathrm{Ord}(x)$ & $(\forall y \in x)(y \cup \{y\} \in x)$
"$x = \omega$", as	"x is the least limit ordinal" □

The same applies to the properties "x is an ordered pair", "x is an n-tuple", "f is a function", "function f is 1-1", $x \in \omega^\omega$, etc. If X is a class, then $X^M = \{x : X^M(x)\}$. A class is *absolute* for M iff $X^M = X \cap M$, i.e. iff $X(x)$ is absolute for M. This definition applies also to sets; for example, the symbol $(\omega_1)^M$ denotes the set that is ("the least uncountable ordinal")^M.

COROLLARY If M is a transitive model for ZFC, then

(1) $\omega^M = \omega$,
(2) $\mathbb{R}^M = M \cap \mathbb{R}$. □

DEFINITION B.3.2 *A formula is Σ_1 (Π_1) iff it is equivalent to $\exists x \phi$ ($\forall x \phi$) for some Δ_0 formula ϕ. A formula is Δ_1 iff it is both Σ_1 and Π_1.*

LEMMA B.3.2 Δ_1 formulas are absolute for transitive models.

PROOF One part is proved in (2) of the proof of Lemma B.3.2; the other is dual. □

If we suppose that M is a model for ZFC then we have absoluteness of formulas equivalent to Δ_1 formulas in ZFC. Of course, this equivalence is always a consequence of some finite fragment of ZFC, and it is enough to take a model of this fragment.

COROLLARY "x is infinite" is absolute for transitive models.

PROOF This is equivalent to "$\exists f(f:\omega \xrightarrow{1-1} x)$" and to $\forall\langle f,n\rangle(f:x \to n) \to$"$f$ is not onto". (The reader should write down the latter formula correctly to see why it is given by this irregular quantification). □

On the other hand, the formula "x is uncountable" is not absolute—for example, for a countable transitive model M the ordinal $(\omega_1)^M$ is always countable.

THEOREM B.3.1 Let M be a countable transitive model for a large enough part of ZFC. If $x \in M$, then $([x]^{<\omega})^M = [x]^{<\omega}$.

PROOF Firstly, $y \in [x]^{<\omega}$ is equivalent to the Δ_0 formula $(\exists n \in \omega)(y: n \to x)$. So it remains to prove that every n-tuple in x is in M. Suppose the contrary, find $y \in [x]^{<\omega} \setminus ([x]^{<\omega})^M$ of minimal length; then $y = z^\frown u$ for some $z \in [x]^{<\omega}$ and $u \in x$, but this set exists by ZFC; hence $y \in M$. □

COROLLARY For any countable transitive model M for a large enough fragment of ZFC (remember that elements of $\mathbb{Q}$ are finite sequences of natural numbers):

(1) $C_\omega \in M$ and $C_\omega{}^M = C_\omega$, (2) $\mathbb{Q} \in M$ and $\mathbb{Q}^M = \mathbb{Q}$. □

By Theorem B.3.1, if M is a transitive model of ZFC and $x \in M$, then all finite subsets of x are in M. For infinite subsets this is not true; all we can say is that $(\mathcal{P}(x))^M = \mathcal{P}(x) \cap M$.

THEOREM B.3.2 Relation "R well orders x" is absolute for transitive models of ZFC.

PROOF This is equivalent to these two formulas:

$$(\forall y)(y \subseteq x \to (\exists z \in y)(\forall t \in y)(z \mathrel{R} t)) \qquad \text{(Definition A.2.4)} \quad \text{and}$$

$$(\exists\langle f,\alpha\rangle)((\mathrm{Ord}(\alpha)\ \&\ f:\alpha \to x\ \&\ f \text{ is an isomorphism}) \quad \text{(Lemma A.2.2, (2))},$$

and the result follows by Lemma B.3.2. □

APPENDIX: C. DESCRIPTIVE SET THEORY

Descriptive set theory is a study of "definable" sets of reals, or more generally, of subsets of any given complete separable metric space (or "Polish space", as it is frequently called). In most of the results the restriction to one of the standard spaces $\mathbb{R}$, $[0,1]$, $\{0,1\}^\omega$, or ω^ω(or their finite products) is not a loss of generality.

C.1 BOREL AND ANALYTIC SETS

The elements of any of standard spaces are called "reals". The *Borel sets* of reals are the elements of the smallest σ-algebra which contains the open sets. *Analytic sets* of reals are the projections of Borel sets, i.e. sets of the form

$$\{x : (\exists y \in \mathbb{R})\langle x,y\rangle \in B\}$$

for some Borel set $B \subseteq \mathbb{R}^2$. Their complements are the *coanalytic sets* of reals. Clearly, every Borel set is both analytic and coanalytic. A rather deep result (due to M. Suslin) says that the converse of this is true, i.e. if a set of reals is at the same time analytic and coanalytic, then it must be Borel. It is also known that there are sets of reals which are analytic but not Borel (and therefore not coanalytic). The algebra of Borel sets of a space X is generated in ω_1 many steps as follows: Let Σ^0_0 be a basis of the topology of X. Σ^0_1 is the family of open sets and Π^0_1 is the family of closed sets. For $2 \le \alpha < \omega_1$, Σ^0_α is the family of all subsets of X of the form

$$A = \bigcup_{n<\omega} A_n$$

where A_n's belong to $\bigcup_{\xi<\alpha} \Pi^0_\xi$, while Π^0_α is the collection of complements of sets from Σ^0_α. Thus, Σ^0_2 are the F_σ-subsets of X and Π^0_2 are the G_δ-subsets of X. It follows that

$$\bigcup_{\alpha<\omega_1} \Sigma^0_\alpha = \bigcup_{\alpha<\omega_1} \Pi^0_\alpha$$

is the σ-algebra of Borel subsets of X.

A function $f: X \to Y$ is *Borel* (or *Borel measurable*) if $f^{-1}(U)$ is a Borel subset of X for every open set $U \subseteq Y$.

THEOREM C.1.1 If $f: X \to \mathbb{R}$ is Borel measurable, then its graph

$$G_f = \{(x, f(x)) : x \in X\}$$

is a Borel subset of $X \times \mathbb{R}$.

PROOF Let I_n ($n < \omega$) be an enumeration of all open intervals of $\mathbb{R}$ with rational endpoints. Then

$$G_f = \bigcap_{n<\omega} [(X \times I_n) \cup (f^{-1}(\mathbb{R} \setminus I_n) \times \mathbb{R})]. \qquad \square$$

REMARK The converse of this result is true when X is a Polish space.

THEOREM C.1.2 If $X \subseteq \mathbb{R}$, then the following statements are equivalent:

(1) X is analytic.
(2) X is a continuous image of ω^ω.
(3) There is a family X_σ ($\sigma \in \{0,1\}^{<\omega}$) of closed (or Borel) sets such that $X = \bigcup_{f \in \{0,1\}^\omega} \bigcap_{n \in \omega} X_{f \restriction n}$. $\square$

(In (3) we say that X is a result of the *A-operation* applied to a sequence X_s ($s \in \omega^{<\omega}$)).

C.2 BAIRE SPACES AND THE BAIRE PROPERTY A space X is *Baire* if the intersection of any countable sequence of dense open subsets of X is dense in X.

THEOREM C.2.1 (BAIRE CATEGORY THEOREM) Every complete metric space (and every compact Hausdorff space) is Baire. $\square$

A subset $N \subseteq X$ is *nowhere dense* if for every nonempty open $U \subseteq X$ there is a nonempty open $G \subseteq U$ such that $G \cap N = \emptyset$. Countable unions of nowhere dense sets are called *meager* subsets of X. Their complements are the *comeager* subsets of X. A set $A \subseteq X$ has the *Baire property* if there is an open set $U \subseteq X$ such that $A \Delta U$ is meager. The family of sets with the Baire property is a σ-algebra of subsets of X which contains all Borel sets. A function $f: X \to Y$ is *Baire* if $f^{-1}(U)$ has the Baire property for every open set $U \subseteq Y$. Clearly, every Borel map is Baire.

THEOREM C.2.2 If f is a Baire map from X into a second countable space Y then f is continuous on a comeager subset of X.

PROOF Let U_n ($n < \omega$) be a basis of Y and for each n fix an open G_n such that $G_n \Delta f^{-1}(U_n)$ is meager. Then f is continuous on

$$X \setminus \bigcup_{n<\omega} (G_n \Delta f^{-1}(U_n)). \qquad \square$$

C.3 MEASURE THEORY OF THE STANDARD SPACES The measure of $\mathbb{R}$ and $[0,1]$ is the usual *Lebesgue measure* λ. The measure of the Cantor cube $\{0,1\}^\omega$ is the *product measure* (or the *Haar measure*) generated by its values on the basic open sets $[s] = \{f \in \{0,1\}^\omega : s \subseteq f\}$ ($s \in \{0,1\}^{<\omega}$):

$$\mu([s]) = 1/2^{|s|}.$$

The measure of the Baire space ω^ω is also the product of the *counting measure* ν on ω where $\nu(\{n\}) = 1/2^{n+1}$ (i.e. $\nu(A) = \sum_{n \in A} 1/2^{n+1}$ for $A \subseteq \omega$). Thus, for every $s \in \omega^{<\omega}$,

$$\nu([s]) = 1/2^{|s|+s(0)+\cdots+s(|s|-1)},$$

where as before $[s] = \{f \in \omega^\omega : s \subseteq f\}$.

C.4 THE STANDARD INJECTION I OF ω^ω INTO $\{0,1\}^\omega$ is defined by:

$$I(f) = 0^{f(0)}1\,0^{f(1)}1\,0^{f(2)}1\ldots,$$

where 0^m denotes the m-tuple of 0's.

Then I is a homeomorphism between the Baire space and the set of all non-eventually 0 elements of the Cantor cube. The countable sets of points of the Cantor cube are clearly unimportant in any translation of the Baire category arguments between the spaces ω^ω and $\{0,1\}^\omega$. Also, a subset $A \subseteq \omega$ is ν-measurable iff $I''A$ is μ-measurable in which case we have $\mu(I''A) = \nu(A)$.

C.5 THE STANDARD SURJECTION J OF $\{0,1\}^\omega$ ONTO $[0,1]$ is defined by:

$$J(f) = \sum_{n=0}^{\infty} \frac{f(n)}{2^{n+1}}.$$

This map is also 1-1 except on the countable set of all eventually constant members of the Cantor cube which clearly do not make difference in any arguments involving Baire category or measure theory of these two spaces. For example, note that for every $s \in \{0,1\}^{<\omega}$, $J''[s]$ is an interval of $[0,1]$ of the same measure. More generally, a subset $A \subseteq \{0,1\}^\omega$ is Haar measurable iff $J''A$ is Lebesgue measurable in which case $\lambda(J''A) = \mu(A)$. This explains why in many arguments, especially those involving measurability of sets or the additivity of the Lebesgue measure, one may choose any of the standard spaces in the role of the set of reals in order to avoid unessential technical difficulties.

C.6 BOREL SETS IN MODELS OF ZFC If M is a countable transitive model then any set of reals from M is countable and therefore F_σ. But it is a consequence of ZFC that not every set of reals is Borel, so there are sets of reals in M that are not (as seen from M) Borel. So we need a subtler means to pinpoint Borel sets in M. In §2 we had already defined codes for some simpler Borel sets (up to G_δ and F_σ), now we shall use the same idea to code all Borel sets, using reals (i.e. elements of ω^ω) as codes.

DEFINITION C.6.1 *We enumerate all open intervals with rational endpoints in a canonical way as I_i for $i \in \omega$. For $c \in \omega^\omega$ and $i \in \omega$ define $c_i \in \omega^\omega$ by $c_i(n) = c(2^i(2n+1)-1)$. A Borel set A_c for $c \in \omega^\omega$ is defined recursively as follows:*

(1) *if $c(0) > 1$ then $A_c = \bigcup_{i>0} I_{c(i)}$,*
(2) *if $c(0) = 1$ then $A_c = \bigcup_{i\in\omega} A_{c_i}$, provided A_{c_i} has been defined for every $i < \omega$,*
(3) *if $c = 0^\frown d$ then $A_c = \mathbb{R} \setminus A_d$, provided A_d has been defined.*

The set $\mathrm{BC} \subseteq \omega^\omega$ of codes for Borel sets is the set of all reals c for which the set A_c has been defined.

So a code $c \in \mathrm{BC}$ gives us a not only a Borel set A_c, but also a method to construct it. Notice that a code for a fixed Borel set is not unique.

FACT A Borel set is open iff it has a code c such that $c(0) > 1$, while it is closed iff it has a code c such that $c(0) = 0$ and $c(1) > 1$. The former codes are called Σ_1, while the latter are called Π_1. □

LEMMA C.6.1 Relations "=" and "⊆", and functions "∪", "∩", and "\", when naturally defined on BC, are absolute. □

LEMMA C.6.2 If $\langle b^i : i \in \omega\rangle$ is the sequence of elements of BC that is in M, then $\bigcup_{i\in\omega} A_{b^i}$ is absolute for M.

PROOF Define c such that $c(0) = 1$ and $c_i = b^i$ for all $i \in \omega$; then c is in BC, therefore $(A_c = \bigcup_{i\in\omega} A_{b^i})^M$, and by the absoluteness of "=" the assumption follows. □

THEOREM C.6.1 The property "A_c is null" is absolute for transitive models of ZFC.

PROOF If c is Σ_1, then $\mu(A_c) = \sum_{i=1}^{\infty} \mu\left(I_{c(i)} \setminus \bigcup_{j=1}^{i-1} I_{c(j)}\right)$, and this formula is absolute. Similarly, the measure of sets with Π_1-codes is absolute. If A_c is null in M then (for every $\epsilon > 0)^M$ there is a $c_\epsilon \in (\Sigma_1)^M$ such that $(\mu(A_{c_\epsilon}) < \epsilon)^M$ and $(A_c \subseteq A_{c_\epsilon})^M$. So $A_c \subseteq A_{c_\epsilon}$, and by the absoluteness of used predicates, A_c is null. If A_c is not null in M, then there is a $d \in (\Pi_1)^M$ such that $(\mu(A_d) > 0)^M$ and $(A_d \subseteq A_c)^M$, and this is absolute. □

One similarly proves (see also §2):

THEOREM C.6.2 The properties "A_c is nowhere dense" and "A_c is meager" are absolute for transitive models of ZFC. □

A canonical coding for analytic sets is obtained using subtrees of $\omega^{<\omega} \times \omega^{<\omega}$ as codes (see §5). There is also a coding of Σ^1_2 sets of reals (i.e. the projections of coanalytic sets), using subtrees of $\omega^{<\omega} \times \omega_1^{<\omega}$. Although they are too large to fit into a countable transitive model, there is a number of applications of such tree representations of Σ^1_2 sets of reals.

APPENDIX: D. COMBINATORIAL SET THEORY

Let X be a set and θ a cardinal. Then

$$[X]^\theta = \{Y \subseteq X : |Y| = \theta\}$$
$$[X]^{<\theta} = \{Y \subseteq X : |Y| < \theta\}.$$

Thus $[X]^{<\aleph_0}$ is the collection of all finite subsets of X and $[X]^2$ is the set of all unordered pairs $\{x, y\}$ of elements of X such that $x \neq y$.

D.1 ARROW NOTATION For cardinals (or ordinals) α and β_i ($i < k$) and an integer $r > 0$, the formula

$$\alpha \to (\beta_i)^r_{i<k}$$

denotes the statement: "For every partition $f : [\alpha]^r \to k$ there exist $i < k$ and $B \subseteq \alpha$ of cardinality (or order type) β_i such that $f''[B]^r = \{i\}$".

THEOREM D.1.1 (RAMSEY'S THEOREM) $\omega \to (\omega)^r_k$ for $1 \leq r, k < \omega$.

PROOF See [Erdös–Hajnal–Maté–Rado III.10.2]. □

THEOREM D.1.2 (FINITE VERSION OF RAMSEY'S THEOREM) For every m, r and k in ω there exists n in ω such that $n \to (m)^r_k$.

PROOF Follows from Ramsey's Theorem using compactness. □

D.2 STATIONARY SETS Let $\lambda > 0$ be a limit ordinal. A subset $C \subseteq \lambda$ is *closed* if $\sup A \in C$ for every nonempty $A \subseteq C$ with supremum $< \lambda$. A *closed and unbounded* set in λ is a closed set which is cofinal in λ. A subset $S \subseteq \lambda$ is *stationary* in λ if it meets every closed and unbounded subset of λ.

LEMMA D.2.1 (THE PRESSING-DOWN LEMMA) If S is a stationary subset of ω_1 and if $f : S \to \omega_1$ is such that $f(\xi) < \xi$ for every ξ in S, then f is constant on a stationary subset of S.

PROOF See [Kunen, II.6.15]. □

THEOREM D.2.1 $\omega_1 \to (\omega_1, \omega+1)^2$.

PROOF Let $p: [\omega_1]^2 \to \{0,1\}$ be a given partition. For every limit ordinal $\delta < \omega_1$, let A_δ be a maximal subset of δ such that $p''[A_\delta \cup \{\delta\}]^2 = \{1\}$. If for some δ the set A_δ is infinite, then we are done. So assume that A_δ is finite for every δ and let $f(\delta) = \max A_\delta$. By the Pressing-Down Lemma choose a stationary set S such that $f''S = \{\xi\}$ for some $\xi < \omega_1$. Since there exist only countably many finite subsets of $\xi+1$ there exists an uncountable $T \subseteq S$ and $A \subseteq \xi+1$ such that $A_\delta = A$ for all $\delta \in T$. Now it is easily checked that $p''[T]^2 = \{0\}$. □

THEOREM D.2.2 $2^{\aleph_0} \not\to (\aleph_1, \aleph_1)^2$.

PROOF Let $<_w$ be a well-ordering of $\mathbb{R}$ and define a partition $[\mathbb{R}]^2 = K_0 \cup K_1$ by

$$\{x,y\} \in K_0 \quad \text{iff} \quad x < y \text{ is equivalent to } x <_w y$$ □

THEOREM D.2.3 $\beth_{r-1} \to (\aleph_0)^r_{\aleph_0}$ for every $1 \le r < \omega$.

PROOF See [Erdös–Hajnal–Maté–Rado]. □

LEMMA D.2.2 (Δ-SYSTEM LEMMA) For every uncountable family $\mathcal{F}$ of infinite sets there exist a set D and an uncountable $\mathcal{F}' \subseteq \mathcal{F}$ such that $E \cap F = D$ for every two distinct elements E and F of $\mathcal{F}'$. The family $\mathcal{F}'$ is called a *Δ-system* with root D.

PROOF See [Kunen, II.1.5], or Lemma 1.2 of this text. □

THEOREM D.2.4 Let n be a positive integer and let F_i $(i < \omega)$ be a family of n-element sets. Then there is an infinite $I \subseteq \omega$ such that F_i $(i \in I)$ forms a Δ-system.

PROOF Clearly, we may assume $F_i \subseteq \omega$ for every i. For $k < n$ and $i < \omega$, let $F_i(k)$ denote kth element of F_i in its increasing enumeration. Shrinking the family we may assume that for every $k < n$ the sequence of integers $F_i(k)$ $(i < \omega)$ is either constant or increasing. [To see this apply Ramsey's Theorem to $p_k: [\omega_1]^2 \to \{0,1\}$ defined by $p_k(\{i,j\}) = 0$ iff $i < j \leftrightarrow F_i(k) < F_j(k)$.] Define now $p: [\omega]^2 \to n \times n$ by letting $p(\{i,j\})$ be the lexicographically minimal pair (k,ℓ) $(k \ne \ell)$ with the property $F_i(k) = F_j(\ell)$ if there is any; otherwise let $p(\{i,j\}) = (0,0)$. Apply now Ramsey's Theorem and show that the constant value must be $(0,0)$. □

REMARK The proof of the Δ-system Lemma can also be given using the above proof of Theorem D.2.4. Namely, we may assume that $\mathcal{F}$ is of the form F_α $(\alpha < \omega_1)$ and that for some integer $n > 0$, $|F_\alpha| = n$ for all α. Apply now the proof of Theorem D.2.4 and notice that the relation $\omega_1 \to (\omega_1, 3, \ldots, 3)^2$ is all that is needed. Notice that this relation itself is an immediate corollary of Theorem D.2.1 and Ramsey's Theorem.

The idea of using a Ramsey-type result in proving a Δ-system Lemma is quite flexible indeed. For example, a similar proof gives the following Δ-system Lemma for Borel families F_x $(x \in \mathbb{R})$ of finite sets of reals, i.e. families for which $x \mapsto F_x$ is a Borel map from $\mathbb{R}$ into its exponential space.

THEOREM D.2.5 For every Borel family F_x $(x \in \mathbb{R})$ of finite sets of reals there is a perfect set of reals P such that F_x $(x \in P)$ forms a Δ-system.

PROOF To prove this result, first find a perfect $P_0 \subseteq \mathbb{R}$ and an integer k such that $x \mapsto F_x$ is continuous on P_0 and $|F_x| = k$ for all x in P_0. Now apply the proof of Theorem D.2.5 where Ramsey's theorem is replaced by either OCA* or its consequence, the case $n = 2$ of Theorem 6.11. □

D.3 TREES A *tree* is a partially ordered set such that $\{s \in T : s < t\}$ is well-ordered for every t in T. The order type of $\{s \in T : s < t\}$ is the *height* of the node t of T, denoted by $\mathrm{ht}(t)$. The *height of* T, $\mathrm{ht}(T)$, is equal to the ordinal

$$\{\mathrm{ht}(t) : t \in T\}.$$

The *levels* of T are the sets of the form

$$T_\alpha = \{t \in T : \mathrm{ht}(t) = \alpha\}.$$

A *branch* of T is its maximal chain.

A *Suslin tree* is an uncountable tree T with no uncountable chain or antichain. The following fact shows that in many instances of checking that a tree T is Suslin it suffices to show that T has no uncountable antichains.

LEMMA D.3.1 Let T be an uncountable tree such that every node t of T has two incomparable successors t_0 and t_1. Then T is Suslin iff T has no uncountable antichains.

PROOF Let b be a branch of T and for each $t \in b$ pick $s_t \in \{t_0, t_1\}$ such that $s_t \notin b$. Then $\{s_t : t \in b\}$ is an antichain of T. □

A tree T is *Aronszajn* if T has the height equal to ω_1 and every level of T is countable, but T has no uncountable chains. A tree T is *special* if it is the union of countably many antichains. Clearly, every Suslin tree is also Aronszajn, but not conversely. The following results (of Aronszajn and Kurepa) explain the connection between these notions.

LEMMA D.3.2 A tree T is special iff the set $T_{\lim}$ of all limit nodes of T is special.

PROOF To prove the nontrivial direction, fix some antichain decomposition $a: T_{\lim} \to \omega$. For every t in T let s_t denote the maximal limit node of T bellow t, and let n_t be the number of elements in the interval $[s_t, t)$. Define $b: T \to \omega \times \omega$ by $b(t) = \langle a(s_t), n_t \rangle$. Then b is an antichain decomposition of T. □

THEOREM D.3.1 A tree T is special iff there is a strictly increasing map from T into the rationals.

PROOF We prove only the nontrivial direction. Let $a: T \to \omega$ be such that $a^{-1}(n)$ is an antichain for each n, and define $f: T \to \{0,1\}^\omega$ by: $f(t)(n) = 1$ iff $a(t) \ge n$ and there is an $s \le t$ such that $a(s) = n$. Then f is strictly increasing and its values are the "diadic rationals" of the Cantor cube (i.e. its elements that are eventually zero). □

THEOREM D.3.2 There is a special Aronszajn tree.

PROOF Let $\mathbb{Q}_0$ be the set of all finite sequences of natural numbers with the following ordering:

$$s <_0 t \quad \text{iff} \quad s \supsetneq t \quad \text{or} \quad s(n) < t(n) \quad \text{for } n = \min\{i : s(i) \neq t(i)\}.$$

This is another natural copy of the rationals. (Its order type is equal to $\eta + 1$ as the empty sequence is its top element). For every limit countable ordinal α fix a subset C_α of α of order type ω with supremum α in such a way that every element of C_α is a successor ordinal. Let $C_\emptyset = \emptyset$ and $C_{\alpha+1} = \{\alpha\}$. Define $\rho_0 : [\omega_1]^2 \to \mathbb{Q}_0$ recursively by:

$$\rho_0(\alpha, \beta) = \langle \operatorname{tp}(C_\beta \cap \alpha)\rangle^\frown \rho_0(\alpha, \min(C_\beta \setminus \alpha)),$$

with the convention that $\rho_0(\alpha, \alpha)$ is the empty sequence for every α. Show that for every β the naturally defined function $\rho_0(\cdot, \beta)$, mapping β into $\mathbb{Q}_0$ is strictly increasing and that for every α the set of all restrictions of these functions to α is at most countable. Thus,

$$T(\rho_0) = \{\rho_0(\cdot, \beta) \restriction \alpha : \alpha \leq \beta < \omega_1\}$$

is an Aronszajn tree. To see that it is special, note that the function which maps $\rho_0(\cdot, \beta) \restriction \alpha$ to $\rho_0(\alpha, \beta)$ is a strictly increasing map from the set of limit nodes of $T(\rho_0)$ into $\mathbb{Q}_0$. □

APPENDIX: E. TOPOLOGY

A topological space X is T_0 iff for every pair of its points, one of the points has a neighborhood which does not contain the another. A topological space X is T_1 iff for every pair of its points, each one has a neighborhood which does not contain the another. A topological space X is Hausdorff (or T_2) iff for every pair of points x, y of X has a pair of disjoint open neighborhoods. Space X is regular (or T_3) iff for every closed $F \subseteq X$ and every point x of X that is not in F there are disjoint open neighborhoods of x and F.

The *weight* of a topological space X, denoted by $w(X)$, is the minimal cardinality of a base of X. A π*-base* of X is a collection $\mathcal{B}$ of nonempty open subsets of X such that every nonempty open subset of X includes a member of $\mathcal{B}$. The π*-weight*, $\pi w(X)$, is the minimal cardinality of a π-base of X. A typical example of a space X for which weight and π-weight are different is the *arrow-space* (or *Sorgenfry line*) on $\mathbb{R}$ generated by the half-open intervals $[x, y)$ of $\mathbb{R}$, and denoted $(\mathbb{R}, \to)$.

EXERCISE Use the Exercise of Section 6.A to show that the topology of the arrow space cannot be induced by any linear ordering of $\mathbb{R}$, i.e. that $(\mathbb{R}, \to)$ is not a linearly ordered topological space.

PROPOSITION E.0.1 $w((\mathbb{R}, \to)) = 2^{\aleph_0}$ and $\pi w((\mathbb{R}, \to)) = \aleph_0$. □

An interesting property of the arrow-space is that it is *separated*. More precisely, we say that a subset Y of a topological space X is *separated* (or, *weakly separated*, as they are frequently called) if for every $y \in Y$ we can assign a neighborhood U_y such that for every $x \neq y$ in Y either $x \notin U_y$ or $y \notin U_x$. Thus, the assignment

$$x \mapsto [x, \infty) \quad (x \in \mathbb{R})$$

is a separation of the arrow-space. Notice that every *discrete subspace* Y of X is separated since if U_y $(y \in Y)$ is a sequence of open sets witnessing the discreteness of Y (i.e. $U_y \cap Y = \{y\}$ for all $y \in Y$), then it also separates Y. The arrow space is an example of a space which is separated but it has no uncountable discrete subspace. A family $\mathcal{N}$ of subsets of X is a *network* if for every $x \in X$ and every open set U containing x there is an $N \in \mathcal{N}$ such that

$$x \in N \subseteq U.$$

THEOREM E.0.1 A T_0-space X has a countable network iff it is a continuous image of a separable metric space M.

PROOF ($\Leftarrow$) Suppose $f: M \to X$ is continuous and onto, where M is a separable metric space. Let B_i $(i < \omega)$ be a basis of M and set

$$N_i = f''B_i \quad (i < \omega).$$

Then N_i $(i < \omega)$ is a network of X. ($\Rightarrow$) Assume that X has a countable network $\mathcal{N}$ and let τ be the topology on X generated by

$$\{N, X \setminus N : N \in \mathcal{N}\}$$

as subbasis. Then X is second countable (and therefore separable metric) and the identity map is continuous. □

REMARK Note that if X has a countable network then the metric space M of Theorem E.0.1 can, in fact, be a subspace of the Cantor cube, i.e. a set of reals. In the other words, every T_0 space X with countable network is a continuous image of a set of reals. This explains why OCA for second countable spaces (or, in fact, for spaces with a countable network) is equivalent to OCA for the sets of reals (see §10).

The *tightness* of a space X, denoted by $t(X)$, is the minimal cardinal θ such that whenever x of X is in the closure of some $A \subseteq X$ then there is a subset B of A of size at most θ such that x is in the closure of B. If the tightness of X is countable, then X is called a *countably tight* space. The following result of Shapirovskiĭ gives an unexpected connection between two quite diverse notions.

THEOREM E.0.2 Every compact Hausdorff space without uncountable discrete subspaces is countably tight.

PROOF Suppose that x is in the closure of A, but not in the closure of any countable subset of A. Clearly we may increase A and assume that the closure of every countable subset of A is in fact included in A, since this bigger set has the same property as A with respect to x. Recursively choose x_ξ $(\xi < \omega_1)$ in A and neighborhoods U_ξ of x_ξ as follows: If for some $\eta < \omega_1$, $\{x_\xi : \xi < \eta\} \subseteq A$ has been determined then (since it can't accumulate to x) we can choose a neighborhood U_η of x such that (remember that our space is, being compact and Hausdorff, also regular)

$$\overline{U_\eta} \cap \overline{\{x_\xi : \xi < \eta\}} = \emptyset.$$

Note that A is a *countably compact* (i.e. every countable open covering has a finite subcovering) subspace of X, so we can choose x_η from the intersection

$$A \cap \bigcap_{\xi \leq \eta} \overline{U_\xi}.$$

This gives us x_ξ $(\xi < \omega_1)$ which is a discrete subspace of X, a contradiction. In fact, x_ξ $(\xi < \omega_1)$ is more than discrete—it is a *free sequence* in X, i.e. for every $\eta < \omega_1$ the closures of the two sets

$$\{x_\xi : \xi < \eta\} \quad \text{and} \quad \{x_\xi : \eta \leq \xi < \omega_1\}$$

are disjoint. □

Let X be a Hausdorff space and let $\exp(X)$ denote the set of all nonempty closed subsets of X. The *Vietoris* (or *exponential*) *topology* of $\exp(X)$ is the topology generated by the sets of the form

$$\{F \in \exp(X) : F \subseteq U\} \quad (U \text{ is open in } X),$$
$$\{F \in \exp(X) : F \cap V \neq \emptyset\} \quad (V \text{ is open in } X).$$

THEOREM E.0.3 If X is compact metric, then $\exp(X)$ is also compact metric.

PROOF Since $x \mapsto \{x\}$ is an isomorphical embedding of X into a closed subspace of $\exp(X)$, one direction is trivial. To see the other direction, we define the *Hausdorff metric* ρ on $\exp(X)$ using the metric d of X as follows (here $d(x, F) = \inf_{y \in F} d(x, y)$):

$$\rho(E, F) = \max\{\sup_{x \in E} d(x, F), \sup_{y \in F} d(y, E)\}$$

We leave to the reader to check that this metric indeed generates the Vietoris topology on $\exp(X)$. To see that $\exp(X)$ is compact, let F_i $(i < \omega)$ be a given sequence of elements of this space. For every $n < \omega$, let $\mathcal{F}_n$ be a finite cover of X by closed balls of diameter at most $1/(n+1)$. Choose recursively a decreasing sequence $A_0 \supseteq A_1 \supseteq \ldots$ of infinite subsets of ω and $\mathcal{G}_n \subseteq \mathcal{F}_n$ such that for all $i \in A_n$,

$$\mathcal{G}_n = \{F \in \mathcal{F}_n : F \cap F_i \neq \emptyset\}.$$

Let $i_n = \min(A_n \setminus n)$ for $n < \omega$. Then F_{i_n} $(n < \omega)$ converges to

$$F = \bigcap_{n<\omega} \bigcup \mathcal{G}_n. \qquad □$$

Point x is a *complete accumulation point* of a set A iff cardinalities of sets $U \cap A$ and A are same for every open neighborhood U of x.

PROPOSITION E.0.2 A space X is compact if and only if every infinite $A \subseteq X$ has a complete accumulation point.

PROOF We prove only the direct implication, leaving the converse to the interested reader. Suppose that A has no complete accumulation point. Then for every x in X we can find an open set U_x containing x such that $|U_x \cap A| < |A|$. Since $\{U_x\}_{x \in X}$ is an open cover of X, there exist finitely many elements $x_0, \ldots, x_n$ in X such that $X = \bigcup_{i=0}^{n} X_{x_i}$. Then

$$A = \bigcup_{i=0}^{n} (U_{x_i} \cap A)$$

So the infinite set A is covered by finitely many sets $U_{x_i} \cap A$ $(0 \leq i \leq n)$ of smaller sizes than A, a contradiction. □

REMARK If the space X is metric then the set A of Proposition E.0.2 can be assumed (without loss of generality) to be countably infinite, i.e. a metric space X is compact iff every sequence $\{x_n\}_{n=0}^{\infty}$ of elements of X has an accumulation point. Notice that we have used this characterization in the proof of Theorem E.0.3.

16. REFERENCES

James E. Baumgartner, *All $\aleph_1$-dense sets of reals can be isomorphic*, Fundamenta Mathematicae 79 (1973), 100–106.

———, *Applications of Proper Forcing Axiom*, Handbook of Set-Theoretic Topology (Kenneth Kunen and Jerry E. Vaughan, eds.), North-Holland, 1984, pp. 913–959.

Mohamed Bekkali, *Topics in Set Theory*, vol. 1476, Springer-Verlag Lecture Notes in Mathematics, 1991.

Andreas Blass, *A Partition Theorem for Perfect Sets*, Proc. AMS 82 (1981), 271–277.

———, *Near coherence of filters II: Applications to Operator Ideals, the Stone-Čech Remainder of the Half Line, Order Ideals of Sequences, and Slenderness of Groups*, Trans. AMS 300 (1987), 557–581.

———, *Groupwise Density and Related Cardinals*, Archive for Mathematical Logic (1990), 1–11.

H.G. Dales, *Automatic continuity: a survey*, Bull. London Math. Soc. 10 (1978), 129–183.

H.G. Dales and W. Hugh Woodin, *An Introduction to Independence for Analysts*, vol. 115, Cambridge University Press, London Math. Soc. Lecture Note Series, 1987.

Dennis Devlin, *Some Partition Theorems and Ultrafilters on ω*, Thesis (1979), Dartmouth.

Robert P. Dilworth, *A Decomposition Theorem for Partially Ordered Sets*, Annals of Math. 51 (1950), 161–166.

Ryszard Engelking, *General Topology*, PWN, Warszawa, 1977.

Paul Erdős, András Hajnal, Atilla Máté and Richard Rado, *Combinatorial Set Theory—Partition Relations for Cardinals*, North-Holland, 1984.

David Fremlin, *Cichon's Diagram*, Séminaire d'initiation à l'analyse (G, Choquet et al., eds.), vol. 61, Publications Mathématiques de l'Université Pierre et Marie Currie, Paris, 1985.

———, *Measure Algebras*, Handbook of Boolean Algebras (J. Donald Monk and Robert Bonnet, eds.), North-Holland, 1989.

James D. Halpern and Hans Laüchli, *A Partition Theorem*, Trans. Amer. Math. Soc. **124** (1966), 360–367.

James D. Halpern and Azriel Lévy, *The Boolean Prime Ideal Theorem does not Imply the Axiom of Choice*, Proc. Symp. Pure Math., vol. 13, Amer. Math. Soc., Providence, Rhode Island, 1971, pp. 83–134.

Leo A. Harrington, Alexander S. Kechris and Alain Louveau, *A Glimm–Effros Dichotomy for Borel Equivalence Relations*, preprint.

Leo A. Harrington, David Marker and Saharon Shelah, *Borel Orderings*, Trans. AMS **310** (1988), 293–302.

Felix Hausdorff, *Summen von $\aleph_1$ Mengen*, Fundamenta Mathematicae **26** (1936), 241–255.

Aarno Hohti, *On Relative ω-cardinality and Locally Fine Coreflection of Products*, Topology Proc. **13** (1988), 93–106.

Alfred Horn and Alfred Tarski, *Measures in Boolean Algebras*, Trans. AMS **64** (1948), 467–497.

Thomas Jech, *Set Theory*, Academic Press, 1978.

Alexander S. Kechris, *Lecture Notes on Borel Equivalence Relations.*

Kenneth Kunen, *Set Theory–An Introduction to Independence Proofs*, North–Holland, 1980.

Kazimierz Kuratowski and Andrzej Mostowski, *Set Theory (With an Introduction to Descriptive Set Theory)*, North–Holland, 1976.

Đuro Kurepa, *The Cartesian Multiplication and the Cellularity Number*, Publ. Inst. Math. (N.S.)(Beograd) **2(16)** (1962), 121–139.

Claude Laflamme, *Equivalence of Families of Functions on the Natural Numbers*, Trans. AMS **330** (1992), 307–319.

Richard Laver, *Products of Infinitely Many Perfect Trees*, J. London Math. Soc. **29**, (2) (1984), 385–396.

Donald A. Martin and Robert M. Solovay, *Internal Cohen Extensions*, Ann. Math. Logic **2** (1970), 143–178.

Adrian R.D. Mathias, *Happy Families*, Ann. Math. Logic **12** (1977), 59–111.

Elliott Mendelson, *Introduction to Mathematical Logic*, Van Nostrand, 1964.

Eric C. Milner and Karel Prikry, *A Partition Relation for Triples Using a Model of Todorčević*, Discrete Mathematics **95** (1991), 183–191.

John C. Oxtoby, *Measure and Category*, Springer–Verlag, 1970.

Jeff Paris, *Combinatorial Statements Independent of Arithmetic*, Mathematics of Ramsey Theory (Jaroslav Nešetřil and Vojtěch Rödl, eds.), Springer–Verlag, 1990, pp. 232–245.

Janusz Pawlikowski, *Small Subset of the Plane Which Almost Contains Almost All Borel Functions*, Pacific J. Math. **144** (1990), 155–160.

Fritz Rothberger, *Sur les familles indénombrables de suites de nombres naturels et les problèmes concernant la propriété C*, Pro. Cambridge Phil. Soc. **37** (1941), 109–126.

Boris Shapirovskiĭ, *Cardinal Invariants in Compact Spaces*, Transactions of the American Math. Soc. (2) **134** (1987), 93–118.

Laurent Schwartz, *Radon Measures on Arbitrary Topological Spaces and Cylindrical Measures*, Oxford University Press, London, 1973.

Saharon Shelah, *A Two-cardinal Theorem and a Combinatorial Theorem*, Proc. Amer. Math. Soc. **62** (1977), 134–136.

______, *Proper Forcing*, vol. 940, Springer–Verlag Lecture Notes in Mathematics, 1977.

Waclaw Sierpiński, *Sur une problème concertant les sous-ensembles croissants de continu*, Fund. Math. **3** (1922), 65–71.

Stevo Todorčević, *Remarks on Cellularity in Products*, Compositio Math. **57** (1986), 357–394.

______, *Partition Problems in Topology*, AMS Providence, Rhode Island, 1989.

______, *Free Sequences*, Topology and its Applications **35** (1990), 235–238.

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