UNIVERSITY OF NIŠ
FACULTY OF PHILOSOPHY
Department of Mathematics

FILOMAT

9:1 (1995)

УНИВЕРЗИТЕТ У НИШУ - ФИЛОЗОФСКИ ФАКУЛТЕТ -

NIŠ

FILOMAT

a continuation of
ZBORNIK RADOVA FILOZOFSKOG FAKULTETA U NIŠU
SERIJA MATEMATIKA

This publication was supported by the Faculty of Philosophy, University of Niš

Published by the Faculty of Philosophy, University of Niš

Printed by "Prosveta", Niš

UNIVERZITET U NIŠU
FILOZOFSKI FAKULTET

FILOMAT

9:1 (1995)

NIŠ

FILOMAT (Niš) 9:1 (1995), 1–8

A NUMERICAL PROCEDURE FOR COEFFICIENTS IN GENERALIZED GAUSS-TURÁN QUADRATURES

Gradimir V. Milovanović* and Miodrag M. Spalević

ABSTRACT. *A numerical procedure for the coefficients in the generalized Gauss-Turán quadrature formulas is presented. The corresponding nodes as the zeros of s-orthogonal polynomials can be determined by a stable algorithm given in* [10]. *A numerical example is included.*

1. Introduction

We consider the generalized Gauss-Turán quadrature formula (see [17])

$$\int_{\mathbb{R}} f(t)\,d\lambda(t) = \sum_{i=0}^{2s}\sum_{\nu=1}^{n} A_{i,\nu} f^{(i)}(\tau_\nu) + R_n(f), \tag{1.1}$$

where $d\lambda(t)$ is a nonnegative measure on the real line $\mathbb{R}$, with compact or infinite support, for which all moments

$$\mu_k = \int_{\mathbb{R}} t^k d\lambda(t), \quad k = 0, 1, \ \ldots\ ,$$

exist and are finite, and $\mu_0 > 0$. The formula (1.1) is exact for all polynomials of degree at most $2(s+1)n - 1$, i.e.,

$$R_n(f) = 0 \qquad \text{for} \quad f \in \mathcal{P}_{2(s+1)n-1}.$$

The knots τ_ν $(\nu = 1, \ldots, n)$ in (1.1) are the zeros of the monic polynomial $\pi_n^s(t)$, which minimizes the following integral

$$\int_{\mathbb{R}} \pi_n(t)^{2s+2}\, d\lambda(t),$$

Received 11.11.1993; Revised February 15.02.1994
1991 *Mathematics Subject Classification*: 03F65, 13A99
* Supported by Grant 0401A of RFNS through Math. Inst. SANU

where $\pi_n(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$. This polynomial π_n^s is known as s-orthogonal (or s-self associated) polynomial with respect to the measure $d\lambda(t)$ (for some details see [2–7], [11–13]). For $s = 0$, we have the standard case of orthogonal polynomials, and (1.1) then becomes well-known Gauss-Christoffel formula.

In [10] one of us gave a stable method for numerical constructing s-orthogonal polynomials and obtaining the nodes of the generalized Gauss-Turán quadrature formula (1.1). It was an iterative method with quadratic convergence based on a discretized Stieltjes procedure and the Newton-Kantorovič method.

In this paper, in Section 2, we give a numerical procedure for finding the coefficients $A_{i,\nu}$ in (1.1). An alternative method was given by Stroud and Stancu [16] (see also [15]). A numerical example is given in Section 3.

2. The Coefficients in the Generalized Gauss-Turán Quadrature

Let $\tau_\nu = \tau_\nu(s,n)$, $\nu = 1,\ldots,n$, be the zeros of the s-orthogonal polynomial $\pi_n(t)\,(\equiv p_n^s(t))$. If we define ω_ν by

$$\omega_\nu(t) = \left(\frac{\pi_n(t)}{t-\tau_\nu}\right)^{2s+1}, \qquad \nu = 1,\ldots,n,$$

then the coefficients $A_{i,\nu}$ in the generalized Gauss-Turán quadrature (1.1) can be expressed in the form (see [15])

$$A_{i,\nu} = \frac{1}{i!(2s-i)!}\left[D^{2s-i}\frac{1}{\omega_\nu(t)}\int_{\mathbb{R}}\frac{\pi_n(x)^{2s+1} - \pi_n(t)^{2s+1}}{x-t}\,d\lambda(x)\right]_{t=\tau_\nu},$$

where D is the standard differentiation operator. Especially, for $i = 2s$, we have

$$A_{2s,\nu} = \frac{1}{(2s)!(\pi_n{}'(\tau_\nu))^{2s+1}}\int_{\mathbb{R}}\frac{\pi_n(x)^{2s+1}}{t-\tau_\nu}\,d\lambda(x),$$

i.e.,

$$A_{2s,\nu} = \frac{B_\nu{}^{(s)}}{(2s)!(\pi_n'(\tau_\nu))^{2s}}, \qquad \nu = 1,\ldots,n,$$

where $B_\nu{}^{(s)}$ are the Christoffel numbers of the following Gaussian quadrature (with respect to the measure $d\mu(t) = \pi_n^{2s}(t)d\lambda(t)$),

$$\int_{\mathbb{R}} g(t)\,d\mu(t) = \sum_{\nu=1}^{n} B_\nu^{(s)} g(\tau_\nu) + R_n(g), \qquad R_n(\mathcal{P}_{2n-1}) = 0.$$

Since $B_\nu^{(s)} > 0$, we conclude that $A_{2s,\nu} > 0$. The expressions for the other coefficients ($i < 2s$) become very complicated. For the numerical calculation we can use a tringular system of linear equations obtained from the formula (1.1) by replacing f with the Newton polynomials: $1, t-\tau_1, \dots, (t-\tau_1)^{2s+1}$, $(t-\tau_1)^{2s+1}(t-\tau_2), \dots, (t-\tau_1)^{2s+1}(t-\tau_2)^{2s+1}\cdots(t-\tau_n)^{2s}$.

Here, we give a method for the numerical calculation of coefficients of the generalized Gauss-Turán quadrature formula (1.1), starting from the Hermite interpolation problem

$$H_m^{(i)}(\tau_\nu) = f^{(i)}(\tau_\nu),$$

where $\nu = 1, \dots, n$; $i = 0, 1, \dots, \alpha_\nu - 1$, $\alpha_1 + \cdots + \alpha_n = m + 1$.

Taking $\alpha_i = 2s+1$, $i = 1, \dots, n$ and integrating $f(t) - H_m(t) = r(f;t)$, we obtain

$$\int_{\mathbb{R}} f(t)d\lambda(t) = \sum_{i=0}^{2s}\sum_{\nu=1}^{n} f^{(i)}(\tau_\nu)\int_{\mathbb{R}} l_{\nu,i}(t)d\lambda(t) + R_n(f), \tag{2.1}$$

where

$$l_{\nu,i}(t) = \frac{1}{i!}\sum_{k=0}^{2s-i}\frac{1}{k!}\left[\frac{(t-\tau_\nu)^{2s+1}}{\Omega(t)}\right]_{t=\tau_\nu}^{(k)}\frac{\Omega(t)}{(t-\tau_\nu)^{2s-i-k+1}},$$

$$\Omega(t) = \left[(t-\tau_1)(t-\tau_2)\cdots(t-\tau_n)\right]^{2s+1},$$

and $R_n(f) = \int_{\mathbb{R}} r(f;t)d\lambda(t)$ is the corresponding remainder term.

Hence, (2.1) becomes the generalized Gauss-Turán quadrature formula (1.1), where $A_{i,\nu}$ are the Cotes numbers of higher order and given by

$$A_{i,\nu} = \int_{\mathbb{R}} l_{\nu,i}(t)d\lambda(t),$$

i.e.,

$$A_{i,\nu} = \frac{1}{i!}\sum_{k=0}^{2s-i}\frac{1}{k!}\left[\frac{(t-\tau_\nu)^{2s+1}}{\Omega(t)}\right]_{t=\tau_\nu}^{(k)}\int_{\mathbb{R}}\Omega_{\nu,i+k}(t)d\lambda(t),$$

where $i = 0, 1, \dots, 2s$; $\nu = 1, \dots, n$ and

$$\Omega_{\nu,j}(t) = \frac{\Omega(t)}{(t-\tau_\nu)^{2s-j+1}} = (t-\tau_\nu)^j\prod_{j\neq\nu}(t-\tau_j)^{2s+1}. \tag{2.2}$$

For $i+k \le 2s$, we can see that $\Omega_{\nu,i+k}(t)$ is a polynomial of degree at most

$$(n-1)(2s+1)+2s=(2s+1)n-1 \le 2(s+1)n-1=2N-1,$$

where $N=(s+1)n$.

Hence, the problem of determining of the coefficients of the Gauss-Turán quadrature formula (1.1) is reduced to determination of integrals in (2.2). All of the above integrals in can be found exactly, except for rounding errors, by using a Gauss-Christoffel quadrature formula with respect to the measure $d\lambda(t)$,

$$\int_{\mathbb{R}} g(t)\,d\lambda(t)=\sum_{k=1}^{N} A_k^{(N)} g(\tau_k^{(N)})+R_N(g), \tag{2.3}$$

taking $N=(s+1)n$ knots. This formula is exact for all polynomials of degree at most $2N-1=2(s+1)n-1$.

In order to calculate the derivatives

$$\left[\frac{(t-\tau_\nu)^{2s+1}}{\Omega(t)}\right]^{(k)}_{t=\tau_\nu} \qquad (k=0,1,\dots,2s;\ \nu=1,\dots,n), \tag{2.4}$$

we need the following auxiliary result:

Lemma 2.1. *If* $g\in C^{(m)}(E)$, $m\in \mathbf{N}_0$, $E\subset \mathbb{R}$, *then*

$$(e^g)^{(0)}=e^g,\quad (e^g)^{(p)}=\sum_{l=1}^{p}\binom{p-1}{l-1} g^{(l)}(e^g)^{(p-l)},\quad p=1,\dots,m.$$

Proof. Since $(e^g)'=g'e^g$, applying the Leibnitz's formula for the derivative of the product of functions, we have

$$\begin{aligned}(e^g)^{(p)}=(g'e^g)^{(p-1)}&=\sum_{j=0}^{p-1}\binom{p-1}{j}(g')^{(j)}(e^g)^{(p-j-1)}\\ &=\sum_{l=1}^{p}\binom{p-1}{l-1} g^{(l)}(e^g)^{(p-l)},\end{aligned}$$

where $p=1,\dots,m$. □

Let $a = \tau_0 < \tau_1 < \cdots < \tau_n < \tau_{n+1} = b$, where $[a, b]$ is the smallest closed interval containing $\mathrm{supp}(d\lambda)$, or $a = -\infty$, $b = +\infty$. For $t \in (\tau_{\nu-1}, \tau_{\nu+1})$ we define u_ν by

$$u_\nu(t) = \prod_{i\neq\nu}(t-\tau_\nu)^{-(2s+1)} = (-1)^{n-\nu}\exp\Big[-(2s+1)\sum_{i\neq\nu}\log|t-\tau_i|\Big],$$

i.e., $u_\nu(t) = (-1)^{n-\nu}e^{h_\nu(t)}$, where

$$h_\nu(t) = -(2s+1)\sum_{i\neq\nu} g_i(t), \qquad g_i(t) = \log|t-\tau_i|.$$

Since $g_i^{(j)}(\tau_\nu) = (-1)^{j-1}(j-1)!(\tau_\nu-\tau_i)^{-j}$, $j \geq 1$, we have

$$h_\nu^{(j)}(\tau_\nu) = -(2s+1)(-1)^{j-1}(j-1)!\sum_{i\neq\nu}(\tau_\nu-\tau_i)^{-j}.$$

It is clear that the derivatives in (2.4) are exactly the derivatives of $u_\nu(t)$ in the point $t = \tau_\nu$. Thus, using Lemma 2.1, we can express they in terms of $h_\nu^{(j)}(\tau_\nu)$.

This numerical method for calculating the coefficients $A_{i,\nu}$ can be summarized in the following form:

Proposition 2.2. *Let τ_ν, $\nu = 1, \ldots, n$, be zeros of the s-orthogonal polynom $\pi_n^s(t)$, with respect to the measure $d\lambda(t)$ on $\mathbb{R}$. Then, coefficients of the generalized Gauss-Turán quadrature formula,*

$$\int_{\mathbb{R}} f(t)\,d\lambda(t) = \sum_{i=0}^{2s}\sum_{\nu=1}^{n} A_{i,\nu}f^{(i)}(\tau_\nu) + R(f),$$

can be expressed in the form

$$A_{i,\nu} = \frac{1}{i\,!}(-1)^{n-\nu}\sum_{k=0}^{2s-i}\frac{1}{k\,!}\big[e^{h_\nu(t)}\big]^{(k)}_{t=\tau_\nu}\sum_{j=1}^{N}A_j^{(N)}\Omega_{\nu,i+k}(\tau_j^{(N)}),$$

where $A_j^{(N)}$ and $\tau_j^{(N)}$ are weights and nodes of the Gauss-Christoffel quadrature formula (2.3) in $N = (s+1)n$ points, the polynomial $\Omega_{\nu,j}(t)$ is given by (2.2), and $\big[e^{h_\nu(t)}\big]^{(k)}_{t=\tau_\nu}$ is determined by Lemma 2.1.

To conclude this section we mention a particulary interesting case of the Chebyshev mesure $d\lambda(t) = (1-t^2)^{-1/2}dt$. In 1930, S. Bernstein [1] showed that the monic Chebyshev polynomial $\hat{T}_n(t) = T_n(t)/2^{n-1}$ minimizes all integrals of the form

$$\int_{-1}^{1} \frac{|\pi_n(t)|^{k+1}}{\sqrt{1-t^2}}\, dt, \qquad k \geq 0.$$

Thus, the Chebyshev-Turán formula

$$\int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}}\, dt = \sum_{i=0}^{2s} \sum_{\nu=1}^{n} A_{i,\nu} f^{(i)}(\tau_\nu) + R_n(f), \tag{2.5}$$

with $\tau_\nu = \cos \frac{(2\nu-1)\pi}{2n}$, $\nu = 1, \dots, n$ is exact for all polynomials of degree at most $2(s+1)n-1$. Turán has stated a problem of explicit determination of $A_{i,\nu}$ and its behavior as $n \to +\infty$ (see Problem XXVI in [18]). Some characterizations and solution for $s = 2$ were obtained by Micchelli and Rivlin [9], Riess [14], and Varma [19]. One simple answer to Turán question was given by Kis [9].

3. Numerical Example

In this section we give an example when is preferable to use a formula of Turán type instead of the standard Gaussian formula

$$\int_{\mathbb{R}} f(t)\, d\lambda(t) = \sum_{\nu=1}^{n} A_\nu f(t_\nu) + R_n(f), \tag{3.1}$$

for which $R_n(\mathcal{R}_{2n-1}) = 0$. All computations were done on the MICROVAX 3400 computer in Q-arithmetic (machine precision $\approx 1.93 \times 10^{-34}$).

Consider the following simple numerical example

$$I = \int_{-1}^{1} e^t \sqrt{1-t^2}\, dt = 1.775499689212180946878576537 2\dots\,.$$

Here we have $f(t) = e^t$ and $d\lambda(t) = \sqrt{1-t^2}\, dt$ on $[-1, 1]$ (the Chebyshev measure of the second kind). Notice that $f^{(i)}(t) = f(t)$ for every $i \geq 0$.

The Gaussian formula (3.1) and the corresponding Gauss-Turán formula (1.1) give

$$I \approx I_n^G = \sum_{\nu=1}^{n} A_\nu e^{t_\nu} \tag{3.2}$$

and

$$I \approx I^T_{n,s} = \sum_{\nu=1}^{n} C_\nu^{(s)} e^{\tau_\nu}, \tag{3.3}$$

respectively, where $C_\nu^{(s)} = \sum_{i=0}^{2s} A_{i,\nu}$.

Table 3.1 shows the relative errors $|(I^T_{n,s} - I)/I^T_{n,s}|$ for $n = 1(1)5$ and $s = 0(1)5$. (Numbers in parentheses indicate decimal exponents and m.p. is the machine precision.)

TABLE 3.1
Relative errors in quadrature sums $I^T_{n,s}$

n	$s=0$	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$
1	1.15(−1)	4.71(−3)	9.72(−5)	1.21(−6)	1.01(−8)	5.98(−11)
2	2.38(−3)	2.05(−7)	3.06(−12)	1.36(−17)	2.40(−23)	1.88(−29)
3	1.97(−5)	1.15(−12)	4.02(−21)	9.26(−31)	m.p.	m.p.
4	8.76(−8)	1.71(−18)	4.68(−31)	m.p.	m.p.	m.p.
5	2.43(−10)	9.40(−25)	m.p.	m.p.	m.p.	m.p.

For $s = 0$ the quadrature formula (3.3) reduces to (3.2), i.e., $I^T_{n,0} \equiv I^G_n$. Notice that Turán formula (3.3) with n nodes has the same degree of exactness as Gaussian formula with $(s+1)n$ nodes.

REFERENCES

[1] S. BERNSTEIN, *Sur les polinomes ortogonaux relatifs à un segment fini*, J. Math. Pures Appl. **9**(9) (1930), 127–177.

[2] H. ENGELS, *Numerical Quadrature and Cubature*, Academic Press, London, 1980.

[3] W. GAUTSCHI, *A survey of Gauss-Christoffel quadrature formulae*, In: E. B. Christoffel (P. L. Butzer and F. Fehér, eds.), Birkhäuser, Basel, 1981, pp. 72–147.

[4] W. GAUTSCHI, *On generating orthogonal polynomials*, SIAM J. Sci. Statist. Comput. **3** (1982), 289–317.

[5] A. GHIZZETTI and A. OSSICINI, *Su un nuovo tipo di sviluppo di una funzione in serie di polinomi*, Rend. Accad. Naz. Lincei **8** (43) (1967), 21–29.

[6] S. GUERRA, *Polinomi generati da successioni peso e teoremi di rappresentazione di una funzione in serie di tali polinomi*, Rend. Ist. Mat. Univ. Trieste **8** (1976), 172–194.

[7] S. GUERRA, *Su un determinate collegato ad un sistema di polinomi ortogonali*, Rend. Ist. Mat. Univ. Trieste **10** (1978), 66–79.

[8] O. KIS, *Remark on mechanical quadrature*, Acta Math. Acad. Sci. Hungar. **8** (1957), 473–476 (Russian).

[9] C. A. Micchelli and T. J. Rivlin, *Turán formulae highest precision quadrature rules for Chebyshev coefficients*, IBM J. Res. Develop. **16** (1972), 372–379.

[10] G. V. Milovanović, *Construction of s-orthogonal polynomials and Turán quadrature formulae*, In: Numerical Methods and Approximation Theory III (Niš, 1987) (G. V. Milovanović, ed.), Univ. Niš, Niš, 1988, pp. 311–328.

[11] A. Ossicini, *Construzione di formule di quadratura di tipo Gaussiano*, Ann. Mat. Pura Appl. (4) **72** (1966), 213–238.

[12] A. Ossicini and F. Rosati, *Funzioni caratteristiche nelle formule di quadratura gaussiane con nodi multipli*, Bull. Un. Mat. Ital. (4) **11** (1975), 224–237.

[13] A. Ossicini and F. Rosati, *Sulla convergenza dei funzionali ipergaussiani*, Rend. Mat. (6) bf11 (1978), 97–108.

[14] R. D. Riess, *Gauss-Turán quadratures of Chebyshev type and error formulae*, Computing **15** (1975), 173–179.

[15] D. D. Stancu, *Asupra unor formule generale de integrate numerica*, Acad. R. P. Romîne. Stud. Cerc. Mat. **9** (1958), 209–216.

[16] A. H. Stroud and D. D. Stancu, *Quadrature formulas with multiple Gaussian nodes*, J. SIAM Numer. Anal. Ser. B **2** (1965), 129–143.

[17] P. Turán, *On the theory of mechanical quadrature*, Acta Sci. Math. Szeged **12** (1950), 30–37.

[18] P. Turán, *On some open problems of approximation theory*, J. Approx. Theory **29** (1980), 23–85.

[19] A. K. Varma, *On optimal quadrature formulae*, Studia Sci. Math. Hungar. **19** (1984), 437–446.

University of Niš, Faculty of Electronic Engineering, Departement of Mathematics, P. O. Box 73, 18000 Niš, Yugoslavia.

University "Svetozar Marković", Faculty of Natural Sciences and Mathematics, Department of Mathematics, R. Domanovića 12, 34000 Kragujevac, Yugoslavia.

FILOMAT (Niš) 9:1 (1995), 9–20

PAWLEY MULTIPLE ANTISYMMETRY THREE-DIMENSIONAL SPACE GROUPS $G_3^{l,p'}$ I. SYMMORPHIC GROUPS

Slavik V. Jablan

ABSTRACT. *By use of antisymmetric characteristic method, Pawley multiple antisymmetry three-dimensional space groups* $G_3^{l,p'}$ $(p = 3, 4, 6)$, *are derived.*

Crystallographic (p')-symmetry three-dimensional space groups (or Pawley colored antisymmetry groups) $G_3^{p'}$ $(p = 3, 4, 6)$ are derived by A. F. Palistrant [1,2,3,4]. From 73 symmorphic space groups G_3 are derived 670 junior $G_3^{p'}$ (96 $G_3^{3'}$ + 266 $G_3^{4'}$ + 308 $G_3^{6'}$), from 54 hemisymmorphic G_3 are derived 562 junior $G_3^{p'}$ (75 $G_3^{3'}$ + 252 $G_3^{4'}$ + 235 $G_3^{6'}$), and from 103 asymmorphic G_3 are derived 980 junior $G_3^{p'}$ (138 $G_3^{3'}$ + 432 $G_3^{4'}$ + 410 $G_3^{6'}$); this means, the category $G_3^{p'}$ $(p = 3, 4, 6)$ consists of 2212 junior groups (309 $G_3^{3'}$ + 950 $G_3^{4'}$ + 953 $G_3^{6'}$).

By the use of the generalized antisymmetric characteristic method (AC-method) [5,6,7], we will derive all crystallographic $(p', 2^l)$-symmetry three-dimensional space groups $G_3^{l,p'}$ $(p = 3, 4, 6)$.

1. Some General Remarks on (p')- and $(p', 2^l)$-symmetry

Pawley (p')-symmetry is a particular case of the general P- symmetry with $P = D_{p(2p)}$, where $D_{p(2p)}$ is the regular dihedral permutation group, generated by the permutations $e_1 = (1...p)(2p...p+1)$ and $e_2 = (1\ p+1)(2\ p+2)...(p\ 2p)$, $(p \geq 2)$ satisfying the relations:

$$e_1^p = e_2^2 = (e_1 e_2)^2 = E.$$

For every p the group $D_{p(2p)}$ is irreducible.

Received 5.10.1994
1991 *Mathematics Subject Classification*: 20H15
Supported by Grant 0401A of RFNS through Math. Inst. SANU

By introducing l antiidentity transformations $e_3, ..., e_{l+2}$ [8,9] ($l \in N$) commuting between themeselves and with e_1, e_2, we have $(p', 2^l)$-symmetry, with the group $P = D_{p(2p)} \times C_2^l$.

In this work only junior groups of complete $(p', 2^l)$-symmetry will be considered. Every junior (p')-symmetry group $G^{p'}$ is derived from certain generating symmetry group G, as well as every junior $(p', 2^l)$-symmetry group $G^{l,p'}$ is derived from certain junior (p')-symmetry group [1,2,8].

Theorem 1. *a) A $(p', 2^l)$-symmetry group $G^{l,p'}$ is the junior $(p', 2^l)$-symmetry group if all relations given in the presentation of its generating symmetry group G remain satisfied after replacing the generators of the group G by the corresponding $(p', 2^l)$-symmetry group generators;*

b) a junior $(p', 2^l)$-symmetry group is called the M^m-type $(p', 2^l)$- symmetry group, if it is a M^m-type group regarded as a l- multiple antisymmetry group;

c) a junior M^m-type $(p', 2^l)$-symmetry group $G^{l,p'}$ is a group of the complete $(p', 2^l)$-symmetry, if for every i $(i = 1, ..., l+2)$ e_i-transformation can be obtained in the group $G^{l,p'}$ as an independent $(p', 2^l)$-symmetry transformation.

If only the condition c) it is not satisfied, such a group $G^{l,p'}$ is the uncomplete junior $(p', 2^l)$-symmetry group of the M^m-type.

Because the derivation of $(3', 2^l)$-symmetry groups coincides to the derivation of $(32, 2^l)$-symmetry groups [10], as the basis for the derivation of all crystallographic $(p', 2^l)$-symmetry groups ($p = 3, 4, 6$), (4')- and (6')-symmetry groups will be sufficient. The derivation will be realised by the use of generalized AC:

Definition 1. *Let all the products of (p')-symmetry generators of a group $G^{p'}$, within which every generator participates once at the most, be formed, and then subsets of transformations equivalent with regard to (p')-symmetry, be separated. The resulting system is called the antisymmetric characteristic of the group $G^{p'}$ and denoted by $AC(G^{p'})$ [5,6,7,10].*

Theorem 2. *Two $(p', 2^l)$-symmetry groups of the M^m-type derived from the same (p')-symmetry group for m fixed $(m = 1, ..., l)$ are equal iff they possess equal antisymmetric characteristics.*

The problem of differing between complete and uncomplete $(p', 2^l)$-symmetry junior M^m-type groups can be solved by the use of the homomorphism of the subgroup $C_p = \{e_1\}$ of the group $D_{p(2p)}$ to the group C_2 at $p = 0 (\mathrm{mod}\ 2)$:

$$e_1^{2k-1} \to e_1, \quad e_1^{2k} \to E \quad (1 \leq k \leq (p+1)/2) \quad [5,10].$$

2. Symmorphic $(p', 2^l)$-symmetry Three-dimensional Space Groups $G_3^{l,p'}$ $(p = 3, 4, 6)$

For denoting space symmetry groups the original Fedorov symbols [1,2,8], Zamorzaev notation and International symbols [11] are used, where p'-symmetry transformations e_1^q, e_2, e_1e_2 $(p = 3, 4, 6;\ q|p)$ and $e_1^{p/2}e_2$ $(p = 4, 6)$ are denoted by $(p/q$, '), ") and (2' respectively.

The application of the theoretical assumptions given above will be illustrated by example of complete $(p', 2^l)$-symmetry junior three-dimensional space groups of the M^m-type $(p = 3, 4, 6)$ derived in the family with the common generating symmetry group G =**7s** (P2/m), $\{a, b, c\}(2 : m)$ with the AC: $\{m, cm\}\{2, 2a, 2b, 2ab\}$ belonging to the AC- equivalency class VII [6, Tab.1]. At $p = 3$ we have two junior (3')-symmetry groups:

1) $\{a, b, c^{(3}\}(2 : m^{')})$,

2) $\{a^{(3}, b, c\}(2^{')} : m)$.

Because of the e_2-transformation $m^{')}$, the AC of the first group is of the form $\{e_2, e_2\}\{E, E, E, E\}$ and of the type $(2)(5)^1$, and the AC of the second is of the form $\{E, E\}\{e_2, e_2, e_2, e_2\}$ and of the same type $(3)(5)^1$. Hence, for the both of them $N_1 = 7$, $N_2 = 64$, $N_3 = 700$, $N_4 = 6720$ [6,10].

So, we have the following complete $(3', 2)$-symmetry groups:

$\{^*a, b, c^{(3}\}(2 : m^{')})$, $\{a, b,^* c^{(3}\}(2 : m^{')})$, $\{a, b, c^{(3}\}(^*2 : m^{')})$,

$\{^*a, b,^* c^{(3}\}(2 : m^{')})$, $\{^*a, b, c^{(3}\}(2 :^* m^{')})$, $\{a, b,^* c^{(3}\}(^*2 : m^{')})$,

$\{a, b, c^{(3}\}(^*2 :^* m^{')})$, $\{^*a^{(3}, b, c\}(2^{')} : m)$, $\{a^{(3}, b,^* c\}(2^{')} : m)$,

$\{a^{(3}, b, c\}(2^{')} :^* m)$, $\{^*a^{(3}, b,^* c\}(2^{')} : m)$, $\{^*a^{(3}, b, c\}(2^{')} :^* m)$,

$\{a^{(3}, b,^* c\}(^*2^{')} : m)$ and $\{a^{(3}, b, c\}(^*2^{')} :^* m)$,

where the antisymmetries are denoted by an asterisk.

At $p = 0(\text{mod } 2)$, the form and, consequently, the type of $AC(G^{p'})$ is obtained by the use of the homomorphism mentioned in Chapter 1. By treating in this way the six (4')-symmetry groups belonging to this family, we have the following results: the three of them, $\{a^{(4}, b, c\}(2^{')} : m)$, $\{a^{(4}, b, c\}(2^{')} : m^{(2)})$ and $\{a^{(4}, b, c^{(2}\}(2^{')} : m)$, possess the AC of the form $\{E, E\}\{e_2, e_2, e_1e_2, e_1e_2\}$ and of the type $(3)(\underline{9})$, where by $(\underline{9})$ is denoted the type of the term $\{e_2, e_2, e_1e_2, e_1e_2\}$ which contains e_2- and e_1e_2-transformations. These transformations are nonequivalent in the sense multiple antisymmetry, so according to the multiple antisymmetry the type of the term mentioned is (9). Contrariwise, they are equivalent in the sense of (p')-symmetry, so the type of this term is denoted by $(\underline{9})$. This is the reason why the derivation of multiple antisymmetry groups from the (p')- symmetry groups with such antisy mmetric cha racteristics cannot be simply reduced on the theory of multiple antisymmetry, this means, on the derivation of multiple antisymmetry groups of the M^{m+2}-type from the M^2-type groups, as it has been done in the case of $(p2, 2^l)$-symmetry groups. From the first group $\{a^{(4}, b, c\}(2') : m)$ we derive $N_1(\{a^{(4}, b, c\}(2') : m)) = 9$ junior complete (4',2)-symmetry groups of the type M^1:

$\{a^{(4}, b, c\}(2') :^* m)$ with the AC: $\{e_3, e_3\}\{e_2, e_2, e_1e_2, e_1e_2\}$ of the type $(3)(\underline{9})^3$;

$\{a^{(4}, b, c\}(^*2') :^* m)$ with the AC: $\{e_3, e_3\}\{e_2e_3, e_2e_3, e_1e_2e_3, e_1e_2e_3\}$ of the type $(3)(\underline{9})^3$;

$\{^*a^{(4}, b, c\}(2') :^* m)$ with the AC: $\{e_3, e_3\}\{e_2, e_2, e_1e_2e_3, e_1e_2e_3\}$ of the type $(3)(9)^3$;

$\{a^{(4}, b, ^*c\}(2') : m)$ with the AC: $\{E, e_3\}\{e_2, e_2, e_1e_2, e_1e_2\}$ of the type $(4)(\underline{9})^3$;

$\{a^{(4}, b, ^*c\}(^*2') : m)$ with the AC: $\{E, e_3\}\{e_2e_3, e_2e_3, e_1e_2e_3, e_1e_2e_3\}$ of the type $(4)(\underline{9})^3$;

$\{^*a^{(4}, b, ^*c\}(2') : m)$ with the AC:$\{E, e_3\}\{e_2, e_2, e_1e_2e_3, e_1e_2e_3\}$ of the type $(4)\ (9)^3$;

$\{a^{(4}, ^*b, c\}(2') : m)$ with the AC: $\{e_3, e_3\}\{e_2, e_1e_2, e_2e_3, e_1e_2e_3\}$ of the type $(3)(\underline{16})^3$;

$\{a^{(4}, ^*b, c\}(2') :^* m)$ with the AC: $\{e_3, e_3\}\{e_2, e_1e_2, e_2e_3, e_1e_2e_3\}$ of the type (3)
$(\underline{16})^3$;

$\{a^{(4}, ^*b, c\}(2') : m)$ with the AC: $\{E, e_3\}\{e_2, e_1e_2, e_1e_2e_3, e_2e_3\}$ of the type $(4)(\underline{16})^3$.

From the groups with the AC of the type $(3)(\underline{9})^3$ can be derived the 6 M^2-type groups: 2 of the type $(4)(\underline{9})^4$, 1 of the type $(4)(9)^4$, 2 of the type $(3)(\underline{16})^4$ and 1 of the type $(4)(\underline{16})^4$; from the group with the AC of the type $(3)(9)^3$ the 7 M^2-type groups: 4 of the type $(4)(9)^4$, 2 of the type $(3)(16)^4$ and 1 of the type $(4)(16)^4$; from the groups with the AC of the type $(4)(\underline{9})^3$ the 10 M^2-type groups: 4 of the type $(4)(\underline{9})^4$, 2 of the type $(4)(9)^4$ and 4 of the type $(4)(\underline{16})^4$; from the group with the AC of the type $(4)(9)^3$ the 12 M^2-type groups: 8 of the type $(4)(9)^4$ and 4 of the type $(4)(16)^4$; from the group with the AC of the type $(3)(\underline{16})^3$ the 12 M^2-type groups: 4 of the type $(3)(\underline{16})^4$, 2 of the type $(3)(16)^4$, 4 of the type $(4)(\underline{16})^4$ and 2 of the type $(4)(16)^4$; from the group with the AC of t he type $(4)(\underline{16})^3$ the 18 M^2-type groups: 12 of the type $(4)(\underline{16})^4$ and 6 of the type $(4)(16)^4$. Hence, $N_2(\{a^{(4}, b, c\}(2') : m)) = 93$. Because from the groups of the types $(4)(\underline{9})^4$ and $(4)(9)^4$ can be derived 4 M^3-type groups, from the groups of the type

$(3)(\underline{16})^4$ 6, from the groups of the type $(3)(16)^5$ 8, from the groups of the type $(4)(\underline{16})^4$ 12 and from the groups of the type $(3)(\underline{16})^4$ 16 M^3-type groups, $N_3(\{a^{(4}, b, c\}(2^{')} : m)) = 840$.

The remaining three (4')-symmetry groups $\{a, b, c^{(4}\}(2 : m^{'}))$, $\{a, b, c^{(4}\}$ $(2^{(2} : m^{'}))$ and $\{a^{(2}, b, c^{(4}\}(2 : m^{'}))$ possess the AC of the form $\{e_2, e_1e_2\}$ $\{E, E, E, E\}$ and of the type $(\underline{4})(5)^2$, where by $(\underline{4})$ is denoted the type of the term $\{e_2, e_1e_2\}$. In the case of (p')-symmetry groups with the AC in which the term $\{e_2, e_1e_2\}$ occurs once and only once, the series of the numbers $N_m^{p'}$ can be simply computed using the following theorem:

Theorem 3. *Let in the $AC(G^{p'})$ the term $\{e_2, e_1e_2\}$ occurs once and only once. If by N_m is denoted the number of the junior M^{m+2}-type multiple antisymmetry groups derived from the $AC(G^{p'})$ treated as the AC of a 2-multiple antisymmetry group, then $N_m(G^{p'}) = (2^m + 1)N_m/2^{m+1}$ $(m = 1, ..., l)$.*

Proof: Because the term $\{e_2, e_1e_2\}$ occurs once and only once in the $AC(G^{p'})$ it is independent from the other part of the AC. For $m = 1$ it is transformed into the four terms different in the sense of 3-multiple antisymmetry: $\{e_2, e_1e_2\}$, $\{e_2e_3, e_1e_2\}$, $\{e_2, e_1e_2e_3\}$, $\{e_2e_3, e_1e_2e_3\}$, resulting in the three terms different in the sense of $(p', 2)$-symmetry: $\{e_2, e_1e_2\}$, $\{e_2e_3, e_1e_2\} = \{e_2, e_1e_2e_3\}$, $\{e_2e_3, e_1e_2e_3\}$. Hence, $N_1(G^{p'}) = 3N_1/4$. Proceeding in the same way, for every m $(m = 2, ..., l)$ it is transformed into the 2^{m+1} terms different in the sense of $(m + 2)$-multiple antisymmetry, resulting in the $2^m + 1$ terms different in the sense of $(p', 2^l)$-symmetry, so $N_m(G^{p'}) = (2^m + 1)N_m/2^{m+1}$.

Treated as the AC of a 2-multiple antisymmetry group, the AC of the form $\{e_2, e_1e_2\}\{E, E, E, E\}$ and of the type $(4)(5)^2$ gives $N_1 = 8$, $N_2 = 64$, $N_3 = 448$, so for the (4')- symmetry group $G^{4'} = \{a, b, c^{(4}\}(2 : m^{'}))$ with the same AC of the type $(\underline{4})(5)^2$, $N_1(G^{4'}) = 6$, $N_2(G^{4'}) = 40$, $N_3(G^{4'}) = 252$. The same holds for the other two (4')-symmetry groups $\{a^{(4}, b, c\}(2^{(2} : m^{'}))$, $\{a^{(2}, b, c^{(4}\}(2 : m^{'}))$ with the identical AC. Hence, for the symmetry group **7s** (P2/m), $N_1^{4'}(\mathbf{7s}) = 45$, $N_2^{4'}(\mathbf{7s}) = 399$, $N_3^{4'}(\mathbf{7s}) = 3276$.

From the ten (6')-symmetry groups of the same family, the two of them, $\{a, b, c^{(3}\}$ $(2^{(2} : m^{'}))$ and $\{a^{(3}, b, c\}(2^{')} : m^{(2)})$ possess the AC of the type $(3)(5)^2$ giving $N_1^{6'} = 5$, $N_2^{6'} = 34$, $N_3^{6'} = 234$; the one of them, $\{a^{(2}, b, c^{(3}\}(2 : m^{'}))$ the AC of the type $(3)(9)^2$ giving $N_1^{6'} = 11$, $N_2^{6'} = 132$, $N_3^{6'} = 1344$; the one of them, $\{a^{(3}, b, c^{(2}\}(2^{')} : m)$ the AC of the type $(4)(5)^2$ giving $N_1^{6'} = 8$, $N_2^{6'} = 64$, $N_3^{6'} = 448$; the two of them, $\{a^{(6}, b, c\}(2^{')} : m)$ and $\{a^{(6}, b, c\}(2^{')} : m^{(2)})$ the AC of the type $(3)(\underline{9})^2$ giving $N_1^{6'} = 9$, $N_2^{6'} = 93$,

$N_3^{6'} = 840$; the three of them, $\{a, b, c^{(6}\}(2 : m^{'}))$, $\{a, b, c^{(6}\}(2^{(2} : m^{'}))$ and $\{a^{(2}, b, c^{(6}\}(2 : m^{'}))$ the AC of the type $(\underline{4})(9)^2$ giving $N_1^{6'} = 12$, $N_2^{6'} = 150$, $N_3^{6'} = 1512$; and the one of them, $\{a^{(6}, b, c^{(2}\}(2^{'}) : m)$ the AC of the type $(4)(\underline{9})^2$ giving $N_1^{6'} = 13$, $N_2^{6'} = 168$, $N_3^{6'} = 1680$. Hence, $N_1^{6'}(\mathbf{7s}) = 84$, $N_2^{6'}(\mathbf{7s}) = 848$, $N_3^{6'}(\mathbf{7s}) = 7616$.

The possible applications of the generalized colored symmetry groups are considered by V.A.Koptsik [12].

3. Partial Catalogue of Symmorphic $(p', 2^l)$-symmetry Three-dimensional Space Groups $G_3^{l,p'}$ $(p = 3, 4, 6)$

In the same manner, the partial catalogue of all complete $(p', 2^l)$-symmetry junior symmorphic three-dimensional space groups of the M^m-type $G_3^{l,p'}$ $(p = 3, 4, 6)$, is realized. According to the work [6], this partial catalogue gives the possibility for their complete cataloguation.

The complete results are given only for the first ten symmetry groups $1s - 10s$. The remaining tables of this partial catalogue can be ordered from the author.

$2s$ $(P1)$ $\{a, b, c\}(2)$, $AC : \{2, 2a, 2b, 2c, 2ab, 2ac, 2bc, 2abc\}$, II

1) $\{a^{(3}, b, c\}(2'))$, $(9)^1$, $N1 = 1$ $N2 = 1$ $N3 = 1$
2) $\{a^{(4}, b, c\}(2'))$, $(25)^2$, $N1 = 1$ $N2 = 1$
3) $\{a^{(6}, b, c\}(2'))$, $(25)^2$, $N1 = 1$ $N2 = 1$

$3s$ $(P2)$, $\{a, b, c\}(2)$, $AC : \{c\}\{2, 2a, 2b, 2ab\}$, III

1) $\{a^{(3}, b, c\}(2^{'}))$, $(2)(5)^1$, $N_1 = 4$ $N_2 = 16$ $N_3 = 56$
2) $\{a^{(4}, b, c\}(2^{'}))$, $(2)(9)^2$, $N_1 = 5$ $N_2 = 18$
3) $\{a^{(4}, b, c(2\}(2^{'}))$, $(2)(9)^2$, $N_1 = 5$ $N_2 = 18$
4) $\{a^{(6}, b, c\}(2^{'}))$, $(2)(9)^2$, $N_1 = 5$ $N_2 = 18$
5) $\{a^{(3}, b, c(2\}(2^{'}))$, $(2)(5)^2$, $N_1 = 2$ $N_2 = 4$
6) $\{a^{(6}, b, c(2\}(2^{'}))$, $(2)(9)^2$, $N_1 = 5$ $N_2 = 18$

$4s$ $(B2)$ $\{a, b, (a + c)/2\}(2)$, $AC : \{2, 2b\}\{2(a + c)/2, 2b(a + c)/2\}$, IV

1) $\{a, b^{(3}, (a + c)/2\}(2^{'}))$, $(3)(3)^1$, $N_1 = 3$ $N_2 = 6$
2) $\{a^{(2}, b, (a + c)^{'})/2)\}(2^{''}))$, $(3)(3)^2$, $N_1 = 1$
3) $\{a, b^{(4}, (a + c)/2\}(2^{'}))$, $(4)(4)^2$, $N_1 = 3$
4) $\{a^{(2}, b, (a + c)^{(4}/2\}(2^{'}))$, $(3)(3)^2$, $N_1 = 1$
5) $\{a, b^{(6}, (a + c)/2\}(2^{'}))$, $(4)(4)^2$, $N_1 = 3$
6) $\{a^{(3}, b, (a + c)^{(6}/2\}(2^{'}))$, $(3)(3)^2$, $N_1 = 1$

$5s$ (Pm) $\{a, b, c\}(m)$, $AC : \{a, b, ab\}\{m, mc\}$, V

1) $\{a,b,c^{(3}\}(m^{'})$, $(4)(3)^1$, $N_1=4$ $N_2=22$ $N_3=112$
2) $\{a,b,c^{(4}\}(m^{'})$, $(4)(4)^2$, $N_1=3$ $N_2=10$
3) $\{a^{(2},b,c^{(4}\}(m^{'})$, $(4)(4)^2$, $N_1=3$ $N_2=10$
4) $\{a,b,c^{(6}\}(m^{'})$, $(4)(4)^2$, $N_1=3$ $N_2=10$
5) $\{a^{(2},b,c^{(3}\}(m^{'})$, $(6)(3)^2$, $N_1=5$ $N_2=24$
6) $\{a^{(2},b,c^{(6}\}(m^{'})$, $(6)(4)^2$, $N_1=6$ $N_2=30$

$6s$ (Bm) $\{a,b,(a+c)/2\}(m)$, $AC:\{m\}\{(a+c)/2,b(a+c)/2\}$, VI

1) $\{a,b,(a+c)^{(3}/2\}(m^{'})$, $(2)(3)^1$, $N_1=4$ $N_2=12$
2) $\{a^{(2},b,(a+c)^{')}/2)\}(m^{"})$, $(2)(3)^2$, $N_1=2$
3) $\{a,b,(a+c)^{(4}/2\}(m^{'})$, $(2)(3)^2$, $N_1=2$
4) $\{a,b^{(2},(a+c)^{(4}/2\}(m^{'})$, $(2)(3)^2$, $N_1=2$
5) $\{a,b,(a+c)^{(6}/2\}(m^{'})$, $(2)(3)^2$, $N_1=2$
6) $\{a,b^{(2},(a+c)^{(3}/2\}(m'))$, $(2)(4)^2$, $N_1=4$

$7s$ $(P2/m)$ $\{a,b,c\}(2:m)$, $AC:\{m,cm\}\{2,2a,2b,2ab\}$, VII

1) $\{a,b,c^{(3}\}(2:m^{'})$, $N_4=6720$ $(3)(5)^1$, $N_1=7$ $N_2=64$ $N_3=700$
2) $\{a^{(3},b,c\}(2^{'}):m)$, $N_4=6720$ $(3)(5)^1$, $N_1=7$ $N_2=64$ $N_3=700$
3) $\{a,b,c^{(4}\}(2:m^{'})$, $(4)(5)^2$, $N_1=6$ $N_2=40$ $N_3=252$
4) $\{a,b,c^{(4}\}(2(2:m^{'})$, $(4)(5)^2$, $N_1=6$ $N_2=40$ $N_3=252$
5) $\{a^{(4},b,c\}(2^{'}):m)$, $(3)(9)^2$, $N_1=9$ $N_2=93$ $N_3=840$
6) $\{a^{(4},b,c\}(2^{'}):m^{(2})$, $(3)(9)^2$, $N_1=9$ $N_2=93$ $N_3=840$
7) $\{a^{(2},b,c^{(4}\}(2:m^{'})$, $(4)(5)^2$, $N_1=6$ $N_2=40$ $N_3=252$
8) $\{a^{(4},b,c^{(2}\}(2^{'}):m)$, $(3)(9)^2$, $N_1=9$ $N_2=93$ $N_3=840$
9) $\{a,b,c^{(3}\}(2^{(2}:m^{'})$, $(3)(5)^2$, $N_1=5$ $N_2=34$ $N_3=224$
10) $\{a^{(3},b,c\}(2^{'}):m^{(2})$, $(3)(5)^2$, $N_1=5$ $N_2=34$ $N_3=224$
11) $\{a,b,c^{(6}\}(2:m^{'})$, $(4)(5)^2$, $N_1=6$ $N_2=40$ $N_3=252$
12) $\{a,b,c^{(6}\}(2^{(2}:m^{'})$, $(4)(5)^2$, $N_1=6$ $N_2=40$ $N_3=252$
13) $\{a^{(6},b,c\}(2^{'}):m)$, $(3)(9)^2$, $N_1=9$ $N_2=93$ $N_3=840$
14) $\{a^{(6},b,c\}(2^{'}):m^{(2})$, $(3)(9)^2$, $N_1=9$ $N_2=93$ $N_3=840$
15) $\{a^{(3},b,c^{(2}\}(2^{'}):m)$, $(4)(5)^2$, $N_1=8$ $N_2=64$ $N_3=448$
16) $\{a^{(2},b,c^{(3}\}(2:m^{'})$, $(3)(9)^2$, $N_1=11$ $N_2=132$ $N_3=1344$
17) $\{a^{(2},b,c^{(6}\}(2:m^{'})$, $(4)(9)^2$, $N_1=12$ $N_2=150$ $N_3=1512$
18) $\{a^{(6},b,c^{(2}\}(2^{'}):m)$, $(4)(9)^2$, $N_1=13$ $N_2=168$ $N_3=1680$

$8s$ $(B2/m)$ $\{a,b,(a+c)/2\}(2:m)$, $AC:\{m\}\{2,2b\}\{(a+c)/2,b(a+c)/2\}$, $VIII$

1) $\{a,b,(a+c)^{(3}/2\}(2:m^{'})$, $(2)(3)(3)^1$, $N_1=8$ $N_2=60$ $N_3=336$
2) $\{a,b^{(3},(a+c)/2\}(2^{'}):m)$, $(2)(3)(3)^1$, $N_1=8$ $N_2=60$ $N_3=336$

3) $\{a^{(2}, b, (a+c)'/2)\}(2^{"}) : m^{"})$, $(2)(3)(3)^2$, $N_1 = 6$ $N_2 = 24$
4) $\{a, b, (a+c)^{(4}/2\}(2 : m^{'})$, $(2)(3)(3)^2$, $N_1 = 6$ $N_2 = 24$
5) $\{a, b, (a+c)^{(4}/2\}(2^{(2} : m^{'})$, $(2)(3)(3)^2$, $N_1 = 6$ $N_2 = 24$
6) $\{a, b^{(4}, (a+c)/2\}(2^{'}) : m)$, $(2)(4)(4)^2$, $N_1 = 9$ $N_2 = 60$
7) $\{a, b^{(4}, (a+c)/2\}(2^{'}) : m^{(2)})$, $(2)(4)(4)^2$, $N_1 = 9$ $N_2 = 60$
8) $\{a, b^{(2}, (a+c)^{(4}/2\}(2 : m^{'})$, $(2)(3)(3)^2$, $N_1 = 6$ $N_2 = 24$
9) $\{a^{(2}, b, (a+c)^{(4}/2\}(2^{'}) : m)$, $(2)(3)(3)^2$, $N_1 = 6$ $N_2 = 24$
10) $\{a^{(2}, b, (a+c)^{(4}/2\}(2^{'}) : m^{(2)})$, $(2)(3)(3)^2$, $N_1 = 6$ $N_2 = 24$
11) $\{a, b, (a+c)^{(3}/2\}(2^{(2} : m^{'})$, $(2)(3)(3)^2$, $N_1 = 6$ $N_2 = 24$
12) $\{a, b^{(3}, (a+c)/2\}(2^{'}) : m^{(2)})$, $(2)(3)(3)^2$, $N_1 = 6$ $N_2 = 24$
13) $\{a, b, (a+c)^{(6}/2\}(2 : m^{'})$, $(2)(3)(3)^2$, $N_1 = 6$ $N_2 = 24$
14) $\{a, b, (a+c)^{(6}/2\}(2^{(2} : m^{'})$, $(2)(3)(3)^2$, $N_1 = 6$ $N_2 = 24$
15) $\{a, b^{(2}, (a+c)^{(3}/2\}(2 : m^{'})$, $(2)(4)(4)^2$, $N_1 = 12$ $N_2 = 96$
16) $\{a, b^{(6}, (a+c)/2\}(2^{'}) : m)$, $(2)(4)(4)^2$, $N_1 = 9$ $N_2 = 60$
17) $\{a, b^{(6}, (a+c)/2\}(2^{'}) : m^{(2)})$, $(2)(4)(4)^2$, $N_1 = 9$ $N_2 = 60$
18) $\{a^{(3}, b, (a+c)^{(6}/2\}(2^{'}) : m)$, $(2)(3)(3)^2$, $N_1 = 6$ $N_2 = 24$
19) $\{a^{(3}, b, (a+c)^{(6}/2\}(2^{'}) : m^{(2)})$, $(2)(3)(3)^2$, $N_1 = 6$ $N_2 = 24$

9s $(P222)$ $a, b, c(2 : 2')$, $AC : \{\{c\}\{2, 2a, 2b, 2ab\}, \{b\}\{2', 2'a, 2'c, 2'ac\}, \{a\}$ $\{22', 22'b, 22'c, 22'bc\}\}\{\{2, 2', 22'\}, \{2a, 2'a, 22'\}, \{2', 2b, 22'b\}, \{2'a, 2ab, 22'b\},$ $\{2, 2'c, 22'c\}, \{2a, 2'ac, 22'c\}, \{2b, 2'c, 22'bc\}, \{2ab, 2'ac, 22'bc\}\}$, IX

1) $\{a, b, c(3\}(2 : 2'))$, $(4)(5, (5, 5))^1$, $N_1 = 8$ $N_2 = 96$ $N_3 = 1516$ $N_4 = 20160$
2) $\{a, b, c^{(4}\}(2 : 2^{'}))$, $(6)(5, (9, 9))^2$, $N_1 = 7$ $N_2 = 88$ $N_3 = 840$
3) $\{a, b, c^{(4}\}(2^{(2} : 2^{'}))$, $(6)(5, (9, 9))^2$, $N_1 = 7$ $N_2 = 88$ $N_3 = 840$
4) $\{a^{(2}, b, c^{(4}\}(2 : 2^{'}))$, $(6)(5, (9, 9))^2$, $N_1 = 7$ $N_2 = 88$ $N_3 = 840$
5) $\{a^{(2}, b^{(2}, c^{(4}\}(2 : 2^{'}))$, $(6)(5, (9, 9))^2$, $N_1 = 7$ $N_2 = 88$ $N_3 = 840$
6) $\{a, b, c^{(3}\}(2^{(2} : 2^{'}))$, $(4)(5, (5, 5))^2$, $N_1 = 7$ $N_2 = 88$ $N_3 = 840$
7) $\{a, b, c^{(6}\}(2 : 2^{'}))$, $(6)(5, (9, 9))^2$, $N_1 = 7$ $N_2 = 88$ $N_3 = 840$
8) $\{a, b, c^{(6}\}(2^{(2} : 2^{'}))$, $(6)(5, (9, 9))^2$, $N_1 = 7$ $N_2 = 88$ $N_3 = 840$
9) $\{a^{(2}, b, c^{(6}\}(2 : 2^{'}))$, $(6)(9, 9, 9)^2$, $N_1 = 20$ $N_2 = 384$ $N_3 = 5376$
10) $\{a^{(2}, b^{(2}, c^{(6}\}(2 : 2^{'}))$, $(4)(9, (9, 9))^2$, $N_1 = 9$ $N_2 = 156$ $N_3 = 1680$
11) $\{a^{(2}, b, c^{(3}\}(2 : 2^{'}))$, $(6)(5, 9, 9)^2$, $N_1 = 13$ $N_2 = 196$ $N_3 = 1680$
12) $\{a^{(2}, b^{(2}, c^{(3}\}(2 : 2^{'}))$, $(6)(9, (9, 9))^2$, $N_1 = 11$ $N_2 = 172$ $N_3 = 1680$

10s $(C222)$ $\{a, (a+b)/2, c\}(2 : 2')$, $AC : \{(a+b)/2\}\{(2', 22'), (2'c, 22'c)\}$, $VIII$

1) $\{a, (a+b)^{(3}/2, c\}(2^{'}) : 2^{'})$, $(2)(4, 4)^1$, $N_1 = 14$ $N_2 = 168$ $N_3 = 1344$
2) $\{a, (a+b)/2, c^{(3}\}(2 : 2^{'}))$, $(2)(3, 3)^1$, $N_1 = 8$ $N_2 = 60$ $N_3 = 336$
3) $\{a^{(2}, (a+b)'/2), c\}(2 : 2^{"}))$, $(2)(3, 3)^2$, $N_1 = 6$ $N_2 = 24$
4) $\{a^{(2}, (a+b)'/2), c^{(2}\}(2 : 2^{"}))$, $(2)(3, 3)^2$, $N_1 = 6$ $N_2 = 24$

5) $\{a, (a+b)/2, c^{(4}\}(2:2'))$, $(2)(3,3)^2$, $N_1 = 7$ $N_2 = 36$

6) $\{a, (a+b)/2, c^{(4}\}(2^{(2}:2'))$, $(2)(3,3)^2$, $N_1 = 7$ $N_2 = 36$

7) $\{a, (a+b)^{(2}/2, c^{(4}\}(2:2'))$, $(2)(3,3)^2$, $N_1 = 7$ $N_2 = 36$

8) $\{a, (a+b)^{(2}/2, c^{(4}\}(2^{(2}:2'))$, $(2)(3,3)^2$, $N_1 = 7$ $N_2 = 36$

9) $\{a, (a+b)^{(4}/2, c\}(2'):2'))$, $(2)(4,4)^2$, $N_1 = 8$ $N_2 = 56$

10) $\{a, (a+b)^{(4}/2, c\}(2'):2^{(2'})$, $(2)(4,4)^2$, $N_1 = 8$ $N_2 = 56$

11) $\{a, (a+b)^{(4}/2, c^{(2}\}(2'):2'))$, $(2)(4,4)^2$, $N_1 = 8$ $N_2 = 56$

12) $\{a, (a+b)/2, c^{(3}\}(2^{(2}:2'))$, $(2)(4,4)^2$, $N_1 = 7$ $N_2 = 36$

13) $\{a, (a+b)^{(3}/2, c\}(2'):2^{(2'})$, $(2)(4,4)^2$, $N_1 = 8$ $N_2 = 56$

14) $\{a, (a+b)/2, c^{(6}\}(2:2'))$, $(2)(3,3)^2$, $N_1 = 7$ $N_2 = 36$

15) $\{a, (a+b)/2, c^{(6}\}(2^{(2}:2'))$, $(2)(4,4)^2$, $N_1 = 7$ $N_2 = 36$

16) $\{a, (a+b)^{(2}/2, c^{(3}\}(2:2'))$, $(2)(3,3)^2$, $N_1 = 6$ $N_2 = 24$

17) $\{a, (a+b)^{(2}/2, c^{(3}\}(2^{(2}:2'))$, $(2)(4,4)^2$, $N_1 = 7$ $N_2 = 36$

18) $\{a, (a+b)^{(2}/2, c^{(6}\}(2:2'))$, $(2)(3,3)^2$, $N_1 = 7$ $N_2 = 36$

19) $\{a, (a+b)^{(2}/2, c^{(6}\}(2^{(2}:2'))$, $(2)(4,4)^2$, $N_1 = 7$ $N_2 = 36$

20) $\{a, (a+b)^{(6}/2, c\}(2'):2'))$, $(2)(4,4)^2$, $N_1 = 8$ $N_2 = 56$

21) $\{a, (a+b)^{(6}/2, c\}(2'):2^{(2'})$, $(2)(4,4)^2$, $N_1 = 8$ $N_2 = 56$

22) $\{a, (a+b)^{(3}/2, c^{(2}\}(2'):2'))$, $(2)(4,4)^2$, $N_1 = 8$ $N_2 = 56$

23) $\{a, (a+b)^{(6}/2, c^{(2}\}(2'):2'))$, $(2)(4,4)^2$, $N_1 = 8$ $N_2 = 56$

The complete results are given in Table 1.

Table 1.

	(3')	(4')	(6')								
2s	1	1	1	24s		8		48s	3	2	7
3s	1	2	3	25s		7		49s	1		1
4s	1	3	2	26s	1	4	3	50s	2		6
5s	1	2	3	27s	1	2	1	51s	3	1	3
6s	1	3	2	28s	1	6	5	52s	1	1	1
7s	2	6	10	29s	1	7	3	53s	2	2	5
8s	2	8	9	30s	1	13	5	54s	5	2	14
9s	1	4	7	31s	1	10	3	55s	3	2	7
10s	2	9	12	32s	1	10	5	56s	4	2	11
11s	1	3	3	33s	1	11	5	57s	2	2	5
12s	1	3	4	34s	1	7	3	58s	3	4	17
13s	1	6	11	35s	1	7	3	63s		1	
14s	1	6	7	36s	1	18	11	65s	1		1
15s	2	10	12	37s	1	16	7	66s	1		1
16s	1	4	3	40s	2		2	67s	1		
17s	1	4	7	41s	1		1	68s	1		1
18s	1	6	11	42s	1		1	69s	1	1	1
19s	2	15	22	43s	1	1	1	70s	1		

20s	1	6	5	44s	4	1	4	71s	1		2
21s	1	6	9	45s	6	1	6	72s	1	2	3
22s		2		46s	4	1	4	73s	1		1
23s		3		47s	2	2	5				

	$(3',2)$	$(4',2)$	$(6',2)$	$(3',2^2)$	$(4',2^2)$	$(6',2^2)$
2s	1	1	1	1	1	1
3s	4	10	12	16	36	40
4s	3	5	4	6		
5s	4	6	14	22	20	64
6s	4	6	6	12		
7s	14	45	84	128	399	848
8s	16	54	66	120	264	360
9s	8	28	81	96	352	1260
10s	22	64	88	228	360	520
11s	4	6	8	12		
12s	5	17	23	42	108	156
13s	16	90	184	300	1440	3216
14s	14	64	84	168	480	672
15s	20	84	108	192	528	720
16s	6	12	12	24		
17s	14	48	84	168	384	672
18s	16	90	274	450	2340	9636
19s	38	252	420	804	4392	8016
20s	10	44	48	96	256	336
21s	18	168	228	432	4032	5376
22s		8				
24s		80			576	
25s		28				
26s	3	10	7	6		
27s	1					
28s	7	42	34	54	234	192
29s	6	28	12	24		
30s	7	80	34	54	420	192
31s	6	40	12	24		
32s	8	69	34	60	372	168
33s	7	70	34	54	378	192
34s	6	28	12	24		
35s	4	22	8	12		
36s	16	294	202	300	5040	3720
37s	14	192	84	168	1536	672
40s	4					
41s	2					
42s	2					
43s	1					

44s	4					
45s	6					
46s	4					
47s	8	6	13	24		
48s	12	6	20	36		
49s	2					
50s	12		24	48		
51s	3					
52s	1					
53s	8	6	14	24		
54s	20	6	38	60		
55s	12	6	19	36		
56s	16	6	29	48		
57s	8	6	13	24		
58s	30	36	160	288	240	1104
65s	2					
66s	2					
68s	1					
69s	2					
71s	4					
72s	6	8	12	24		
73s	2					

	$(3',2^3)$	$(4',2^3)$	$(6',2^3)$	$(3',2^4)$	$(4',2^4)$	$(6',2^4)$	$(3',2^5)$
2s	1						
3s	56						
5s	112						
7s	1400	3276	7616	13440			
8s	672						
9s	1516	3360	13776	20160			
10s	1680						
12s	336						
13s	5712	18144	43008	80640			
14s	1344						
15s	2688						
17s	1344						
18s	17220	77112	364224	685440	2056320	10321920	19998720
19s	16464	49392	106848	241920			
20s	672						
21s	10080	64512	86016	161280			
28s	336						
30s	336						
32s	336						
33s	336						
36s	5712	57456	45024	80640			

37s 1344
58s 2016

For the complete $(p', 2^l)$-symmetry junior symmorphic three-dimensional space groups of the M^m-type the numbers $N_m^{p'}$ $(p = 3, 4, 6)$ are the following:

$$N_0^{p'} = 96G_3^{3'} + 266G_3^{4'} + 308G_3^{6'} = 670$$
$$N_1^{p'} = 496G_3^{1,3'} + 2171G_3^{1,4'} + 2644G_3^{1,6'} = 5311$$
$$N_2^{p'} = 4709G_3^{2,3'} + 24088G_3^{2,4'} + 38133G_3^{2,6'} = 66930$$
$$N_3^{p'} = 71713G_3^{3,3'} + 273252G_3^{3,4'} + 666512G_3^{3,6'} = 1011477$$
$$N_4^{p'} = 1283520G_3^{4,3'} + 2056320G_3^{4,4'} + 10321920G_3^{4,6'} = 13661760$$
$$N_5^{p'} = 19998720G_3^{5,3'} = 19998720$$

References

[1] Zamorzaev A.M., Gelyarskij E.I., Palistrant A.F., *Tsvetnaya simmetriya, eyo obobscheniya i prilozeniya*, Shtiintsa, Kishinev, 1978.
[2] Zamorzaev A.M., Karpova Yu.S., Lungu A.P., Palistrant A.F., *P-simmetriya i eyo dal'nejshee razvitie*, Shtiintsa, Kishinev, 1986.
[3] Palistrant A.F., *Prostranstvennye gruppy (p')-simmetrii (polievy) i ih primenenie k vyvodu pyatimernyh kristallograficheskih grupp simmetrii*, DAN SSSR 254, 5 (1980), 1126-1130.
[4] Palistrant A.F., *Gruppy tsvetnoi simmetrii, ih obobscheniya i prilozeniya*, Avtoref. dis. ... dr-a fiz.-mat. nauk, KGU, Kishinev, 1981.
[5] Jablan S.V., *Plane and Layer $(p', 2^l)$-symmetry Groups*, Mat. Vesnik 42 (1990), 87-95.
[6] Jablan S.V., *A New Method of Deriving and Cataloguing Simple and Multiple Antisymmetry G_3^l Space Groups*, Acta Cryst., A43 (1987), 326-337.
[7] Jablan S.V., *Algebra of Antisymmetric Characteristics*, Publ. Inst. Math. 47 (61), 1990, 39-55.
[8] Zamorzaev A.M., *Teoriya prostoj i kratnoj antisimmetrii*, Shtiintsa, Kishinev, 1976.
[9] Zamorzaev A.M., Palistrant A.F., *Antisymmetry, its Generalizations and Geometrical Applications*, Z.Kristall., 151 (1980), 231-248.
[10] Jablan S.V., *$(p2, 2^l)$-symmetry Space Groups $G_3^{l,p2}$*, Acta Crystall. A48 (1992) (to appear).
[11] Hahn Th. (Editor), *International Tables for Crystallography*, Vol. A, D. Reidel Publ. Co., Dordecht, 1983, 1987.
[12] Koptsik V.A., *Generalized Symmetry in Crystal Physics*, Comput. Math. Applic. 16, 5-8 (1988), 407-424.

Department of Mathematics, Phylosophical Faculty, 18000 Niš, Ćirila i Metodija 2, Yugoslavia
E-mail: eslavik@ubbg.etf.bg.ac.yu

FILOMAT (Niš) 9:1 (1995), 21–31

THE GENERAL CONCEPT OF CLEAVABILITY OF MAPPINGS

M. Bonanzinga

ABSTRACT. *The aim of the paper is to give some answers to the following general question: "If X and Y are topological spaces and $f : X \to Y$ is a continuous mapping cleavable over the class $\mathcal{P}$ of topological spaces, is it true that f is a $\mathcal{P}$-mapping? ". Answers are given for some classes of topological spaces.*

Introduction and preliminary. In 1985 Arhangel'skii ([1], [2]), introduced the notion of cleavability for topological spaces. Following a general idea ([22]) to investigate mappings instead of spaces, in this paper we want to introduce the notion of cleavability for mappings. So, the concept of $\mathcal{P}$-mapping ([14]) is a basic notion. Let $\mathcal{P}$ be a topological property; a continuous mapping is called a *$\mathcal{P}$-mapping* if it satisfies a property $G_{\mathcal{P}}$ depending on $\mathcal{P}$ and every continuous mapping on a $\mathcal{P}$-space has the property $G_{\mathcal{P}}$. We want to study the $\mathcal{P}$-mappings when the property $\mathcal{P}$ is the *cleavability over a class of topological spaces*; in this way we want to obtain a more general notion of cleavability of mappings over a class of spaces as a generalization of the notion of cleavability of a space over the same class of spaces.

In particular we are interested in answering the following question: "If $f : X \to Y$ is a continuous mapping *cleavable* over a class $\mathcal{P}$ of topological spaces, is it true that f is a $\mathcal{P}$-mapping?"In this paper we shall use the following notations: (X, τ) or simply X means a topological space; $\overline{A}$, A^o are the closure and the interior of A respectively, where A is a subset of X; if $\overline{A}^o = A$ ($\overline{A^o} = A$) we say that A is a regular open (regular closed) subset of X; $C(X, Y)$ is the set of all continuous mappings from X to Y, where Y is

Received 03.12.1994; Revised 02.05.1995

1991 *Mathematics Subject Classification.* 54A20, 54A25, 54C05, 54E18, 54E30.

Key words and phrases. cleavable space, cleavable mapping, $\mathcal{P}$-mapping..

This paper was presented to the XII National Conference on Topology at Perugia University, May 1994.

a topological space. For notations not explicitly mentioned here, the reader is referred to [6], [15] and [19].

Let $\mathcal{P}$ be a class of topological spaces and $\mathcal{M}$ a class of continuous mappings. We recall the following

Definition 1. **[1].** *A space X is $\mathcal{M}$-cleavable over $\mathcal{P}$ if for every $A \subset X$ there exist $Y \in \mathcal{P}$ and $f \in \mathcal{M}$, $f : X \to Y$, such that $A = f^{-1}f(A)$ (or equivalently $f(A) \cap f(X - A) = \varnothing$).*

If $\mathcal{M}$ is the class of all continuous mappings, we shall just say that X is *cleavable over* $\mathcal{P}$. If $\mathcal{M}$ is the class of all open, closed, perfect, quotient mappings, we shall say that X is respectively *open, closed, perfect, quotient cleavable over* $\mathcal{P}$.

Remark 1 Let f be a one-to-one continous mapping of a space X into a space $Y \in \mathcal{P}$. Then obviously X is cleavable over $\mathcal{P}$. Note, that in the definition of cleavability the mapping f depends on the subset A of X. Thus we might say that a space X is said to be *absolutely cleavable* over $\mathcal{P}$ if there exists a one-to-one continuous mapping of X into some space $Y \in \mathcal{P}$ ([5]). Then cleavability over $\mathcal{P}$ may be regarded as a generalization of continuous bijections (onto some $Y \in \mathcal{P}$).

Definition 2. **[6].** *A space X is $\mathcal{M}$-pointwise cleavable over $\mathcal{P}$ if for every point $x \in X$, there exist $Y \in \mathcal{P}$ and $f \in \mathcal{M}$, $f : X \to Y$, where such that $\{x\} = f^{-1}f(x)$.*

Definition 3. **[6].** *A space X is $\mathcal{M}$-double cleavable over $\mathcal{P}$ if for any subsets A and B of X, there exist $Y \in \mathcal{P}$ and $f \in \mathcal{M}$, $f : X \to Y$, such that $A = f^{-1}f(A)$ and $B = f^{-1}f(B)$.*

Remark 2 If X is absolutely cleavable over $\mathcal{P}$, then X is double cleavable over $\mathcal{P}$; if X is double cleavable over $\mathcal{P}$, then X is cleavable over $\mathcal{P}$; moreover, if a space X is cleavable over $\mathcal{P}$, then X is pointwise cleavable over $\mathcal{P}$.

Then we can give the following definitions for the cleavability of a mapping.

Definition 4. *A continuous mapping $f : X \to Y$ is $\mathcal{M}$-cleavable over $\mathcal{P}$ if for every $y \in Y$ and $A \subset f^{-1}(y)$ there exist $Z \in \mathcal{P}$ and $g \in \mathcal{M}$, $g : X \to Z$, such that $A = g^{-1}g(A)$.*

Remark 3 The previous definition is not trivial if f is onto.

If $\mathcal{M}$ is the class of all continuous mappings, we shall just say that f is *cleavable over* $\mathcal{P}$. If $\mathcal{M}$ is the class of all open, closed, perfect, quotient mappings, we shall say that f is respectively *open, closed, perfect, quotient cleavable over* $\mathcal{P}$.

Further f is said to be *absolutely cleavable* over $\mathcal{P}$ if the mapping g is one-to-one.

Definition 5. *A continuous mapping $f : X \to Y$ is $\mathcal{M}$-pointwise cleavable over $\mathcal{P}$ if for every $y \in Y$ and $\{x\} \subset f^{-1}(y)$, there exist $Z \in \mathcal{P}$ and $g \in \mathcal{M}$, $g : X \to Z$ such that $\{x\} = g^{-1}g(x)$.*

Remark 4 The previous definition is equivalent to the definition of pointwise cleavability of X over $\mathcal{P}$.

Definition 6. *A continuous mapping $f : X \to Y$ is $\mathcal{M}$-double cleavable over $\mathcal{P}$ if for every $y \in Y$ and for every subset A and B of $f^{-1}(y)$, there exist $Z \in \mathcal{P}$ and $g \in \mathcal{M}$, $g : X \to Z$ such that $A = g^{-1}g(A)$ and $B = g^{-1}g(B)$.*

Remark 5 If $f : X \to Y$ is absolutely cleavable over $\mathcal{P}$, then f is double cleavable over $\mathcal{P}$; if f is double cleavable over $\mathcal{P}$, then f is cleavable over $\mathcal{P}$; moreover, if f is cleavable over $\mathcal{P}$, then f is pointwise cleavable over $\mathcal{P}$.

We have

Proposition 1. *A space X is $\mathcal{M}$-cleavable ($\mathcal{M}$-pointwise cleavable, ...) over $\mathcal{P}$ iff every continuous mapping $f : X \to Y$ is $\mathcal{M}$-cleavable ($\mathcal{M}$-pontwise cleavable, ...) over $\mathcal{P}$.*

Proof. ($\Rightarrow$) Let $f : X \to Y$ be a continuous mapping, $y \in Y$ and $A \subset f^{-1}(y)$. As X is $\mathcal{M}$-cleavable over $\mathcal{P}$, then there exist $Z \in \mathcal{P}$ and $g \in \mathcal{M}$, $g : X \to Z$ such that $g^{-1}g(A) = A$; this proves that f is $\mathcal{M}$-cleavable over $\mathcal{P}$. ($\Leftarrow$) Now suppose that every continuous mapping with domain X is $\mathcal{M}$-cleavable over $\mathcal{P}$. Let $A \subset X$ and let $Y = (Y, \tau)$ be Sierpinski's 2-point space (i.e., $Y = \{0, 1\}$ and $\tau = \{\varnothing, Y, \{1\}\}$. Define $f : X \to Y$ by $f(\overline{A}) = \{0\}$, $f(X - \overline{A}) = \{1\}$; then f is continuous. Since $A \subset f^{-1}(0)$ and f is a $\mathcal{M}$-cleavable mapping, there exist $Z \in \mathcal{P}$ and $g \in \mathcal{M}$, $g : X \to Z$ such that $g^{-1}g(A) = A$. Thus X is $\mathcal{M}$-cleavable over $\mathcal{P}$. $\square$

So we have the following natural question

Question - A. *Does there exist a continuous mapping f that is $\mathcal{M}$-cleavable over $\mathcal{P}$ such that its domain X is not $\mathcal{M}$-cleavable over $\mathcal{P}$?*

We have the following

Proposition 2. *A space X is $\mathcal{M}$-pointwise cleavable over $\mathcal{P}$ iff every continuous one-to-one mapping $f : X \to Y$ is $\mathcal{M}$-cleavable over $\mathcal{P}$.*

Proof. ($\Rightarrow$) Let $f : X \to Y$ be a continuous one-to-one mapping. Then, for every $y \in Y$ the fiber $f^{-1}(y)$ is a single point of X. So, if X is $\mathcal{M}$-pointwise cleavable over $\mathcal{P}$ we have that f is $\mathcal{M}$-cleavable over $\mathcal{P}$. ($\Leftarrow$) Now suppose that every continuous one-to-one continuous mapping with domain X is $\mathcal{M}$-cleavable over $\mathcal{P}$. Let $x \in X$. By hypothesis, the identity mapping on X, id_X, is $\mathcal{M}$-cleavable over $\mathcal{P}$; since $\{x\} = id_X^{-1}(x)$, X is $\mathcal{M}$-pointwise cleavable over $\mathcal{P}$. □

Note that if a space X is $\mathcal{M}$-pointwise cleavable but not $\mathcal{M}$-cleavable over $\mathcal{P}$, then the identity mapping on X, id_X, is $\mathcal{M}$-cleavable over $\mathcal{P}$; this shows that the notion of cleavability of a mapping is more general than the notion of cleavability of a space, in fact there exist mappings $f : X \to Y$ $\mathcal{M}$-cleavable over $\mathcal{P}$ such that X is not $\mathcal{M}$-cleavable over $\mathcal{P}$. Then we have an affirmative answer to the question A as the following example show

Example 1. *If $\mathcal{P} = \{\mathbb{R}\}$, the circumference S^1 is not cleavable over $\mathcal{P}$ ([4]) while the mapping $id : S^1 \to S^1$ is cleavable over $\mathcal{P}$.* □

Now we have the following natural question

Question - B. *Does there exist a continuous mapping f that is $\mathcal{M}$-pointwise cleavable over $\mathcal{P}$ such that its domain is not $\mathcal{M}$-pointwise cleavable over $\mathcal{P}$?*

By the definitions, the answer to the previous question is the following: "A continuous mapping $f : X \to Y$ is $\mathcal{M}$-pointwise cleavable over $\mathcal{P}$ iff X is pointwise cleavable over $\mathcal{P}$".

Some particular forms of cleavability of mappings imply particular forms of cleavability of spaces, as show the following four results

Proposition 3. *A constant mapping $f : X \to Y$ is $\mathcal{M}$-cleavable ($\mathcal{M}$-pointwise cleavable, ...) over $\mathcal{P}$ iff X is $\mathcal{M}$-cleavable ($\mathcal{M}$-pointwise cleavable, ...) over $\mathcal{P}$.*

Proposition 4. *If $f : X \to Y$ is cleavable over $\mathcal{P}$, where $\mathcal{P}$ is a $card(Y)$-productive class of spaces, then X is cleavable over $\mathcal{P}$.*

Proof. Let $A \subset X$ and $y \in f(A)$. By hypothesis, there exist a space $Z_y \in \mathcal{P}$ and a continuous mapping $g_y : X \to Z_y$ such that $g_y^{-1}g_y(A \cap f^{-1}(y)) = A \cap f^{-1}(y)$. Let $Z = \prod_{y \in f(A)} Z_y$; then, by hypothesis, $Z \in \mathcal{P}$. Define a mapping $g : X \to Z$, by $g(x) = \{g_y(x)\}_{y \in f(A)}$, for all $x \in X$. We will show that $g^{-1}g(A \cap f^{-1}(y)) = A \cap f^{-1}(y)$. Only need to show that $g^{-1}g(A \cap f^{-1}(y)) \subseteq A \cap f^{-1}(y)$. Let $x \in g^{-1}g(A \cap f^{-1}(y)$; so, $g(x) \in g(A \cap f^{-1}(y))$. Then, there exists $a \in A \cap f^{-1}(y)$ such that $g(x) = g(a)$; in particular, $f(a) = y$. Then, for every $z \in f(A)$, we have that $g_z(x) = g_z(a)$. So $x = g_z^{-1}g_z(a)$, for all $z \in f(A)$, and then, by hypothesis, $x \in A \cap f^{-1}(y)$. Thus $g^{-1}g(A) = A$. □

Remark 6 In the case in which $\mathcal{P}$ is a $card(Y)$-productive class of spaces, the previous property gives a negative answer to the question A.

Definition 7. *If f is a mapping from the space X to a space Y, the cardinality of f is defined as the number*

$$card(f) = card(f(X)) \times Sup\{card(f^{-1}(y)) : y \in Y\}.$$

Proposition 5. *If $f : X \to Y$ is pointwise cleavable over $\mathcal{P}$, where $\mathcal{P}$ is a $card(f)$-productive class of spaces, then X is absolutely cleavable over $\mathcal{P}$.*

Proof. Let $y \in Y$ and $x \in f^{-1}(y)$; then there exist a space $Z_x \in \mathcal{P}$ and a continuous mapping $g_x : X \to Z_x$ such that $\{x\} = g_x^{-1}g_x(x)$. Let $Z_y = \prod_{x\in f^{-1}(y)} Z_x$; by hypothesis, $Z_y \in \mathcal{P}$. Define the mapping $g_y : X \to Z_y$, by $g_y(z) = \{g_x(z)\}_{x\in f^{-1}(y)}$ for all $z \in X$. The mapping g_y is continuous: recall that g_y is continuous iff $p_s g_y$ is continuous, for $s \in f^{-1}(y)$, where $p_s : \prod_{x\in f^{-1}(y)} Z_x \to Z_s$ is the s^{th} projection mapping; since $p_s g_s(t) = g_s(t)$ for all $t \in X$, we have that $p_s g_y$ is a continuous mapping. Further $g_y|f^{-1}(y) : f^{-1}(y) \to Z_y$ is one-to-one: let $s, t \in f^{-1}(y)$ such that $s \neq t$. By hypothesis, $g_s(t) \neq g_s(s)$; then $\{g_x(s)\}_{x\in f^{-1}(y)} \neq \{g_x(t)\}_{x\in f^{-1}(y)}$, or equivalentely, $g_y|f^{-1}(s) \neq g_y|f^{-1}(t)$. Let $Z = \prod_{y\in f(X)} Z_y$; by hypothesis, $Z \in \mathcal{P}$. Define the mapping $g : X \to Z$, by $g(z) = \{\{g_x(z)\}_{x\in f^{-1}(y)}\}_{y\in f(X)}$. The mapping g is continuous: let $p_t : \prod_{y\in f(X)} Z_y \to Z_t$ the t^{th} projection mapping (recall that $Z_t = \prod_{x\in f^{-1}(t)} Z_x$); since $p_t g(s) = g_t(s)$, for all $s \in X$ and we have proved that g_t is continuous for all $t \in Y$, we have that $p_t g$ is continuous for all $t \in f(X)$ and then g is continuous. Since $g_y|f^{-1}(y) : f^{-1}(y) \to Z_y$ is one-to-one, for all $y \in Y$, we have that g is one-to-one. Then X is absolutely cleavable over $\mathcal{P}$. $\square$

Remark 7 In the case in which $\mathcal{P}$ is a $card(f)$-productive class of spaces, the previous property gives a negative answer to the question A.

Proposition 6. *If $f : X \to Y$ is closed pointwise cleavable over $\mathcal{P}$, where $\mathcal{P}$ is a $card(f)$-productive class of spaces, then X can be embedded as subspace into some space of $\mathcal{P}$.*

Proof. The proof is similar to the proof of Proposition 5 noting that, by hypothesis, every continuous mapping $g_x : X \to Z_y$ is closed and then $g : X \to g(X)$ is a closed mapping. Now we prove this fact. Let $A \subset X$ be closed. We want to prove that $g(A) = \prod_{y\in f(X)} \prod_{x\in f^{-1}(y)} g_x(A) \cap g(X)$, where $\prod_{y\in f(X)} \prod_{x\in f^{-1}(y)} g_x(A)$ is a closed subset of Z. The inclusion $g(A) \subseteq$

$\prod_{y\in f(X)}\prod_{x\in f^{-1}(y)} g_x(A)\cap g(X)$ is obvious. Let $t\in\prod_{y\in f(X)}\prod_{x\in f^{-1}(y)} g_x(A)\cap g(X)$ and $s\in X$ such that $g(s)=t$. Then $g_y(s)=\{g_x(s)\}_{x\in f^{-1}(y)}\in\prod_{x\in f^{-1}(y)} g_x(A)$, for all $y\in f(X)$. Let $\overline{y}=f(s)$. Then $g_x(s)\in g_x(A)$, for all $x\in f^{-1}(\overline{y})$. Since $s\in f^{-1}(\overline{y})$, we have that $g_s(s)\in g_s(A)$; so, there exists $a\in A$ such that $g_s(s)=g_s(a)$. Then, by hypothesis, $s\in A$ and the proof is complete. □

Remark 9 If $\mathcal{P}$ is a $card(f)$-productive and hereditary class of spaces, the previous property is equivalent to say that if X is pointwise cleavable over $\mathcal{P}$, then X is closed absolutely cleavable over $\mathcal{P}$.

Remark 10 In the following we will use the terms *e-cleavable mapping* or *e-cleavable space* over $\mathcal{P}$ to indicate that cleavability, pointwise cleavability, double cleavability and absolute cleavability of a mapping or of a space over $\mathcal{P}$ are equivalent.

By Propositions 5 and 6 we have the following:

Theorem 1. *Let $f: X\to Y$ be a continuous mapping and let $\mathcal{P}$ be a $card(f)$-productive class of spaces. The following conditions are equivalent:*
(i) f is e-cleavable over $\mathcal{P}$;
(ii) X is e-cleavable over $\mathcal{P}$;

Theorem 2. *Let $f: X\to Y$ be a continuous mapping and let $\mathcal{P}$ be a $card(f)$-productive and hereditary class of spaces. The following conditions are equivalent:*
(i) f is closed e-cleavable over $\mathcal{P}$;
(ii) X is closed e-cleavable over $\mathcal{P}$.

1. Cleavability over T_0, T_1, T_2, functionally Hausdorff and Urysohn spaces.

Note that, by the previous results, in the case in which $\mathcal{P}$ is a productive class of spaces, we have that the classic problem on cleavability: "If X is (closed) e-cleavable over the class $\mathcal{P}$, is it true that X belongs to $\mathcal{P}$?", can be reformulated in the following way:"If $f: X\to Y$ is (closed) e-cleavable over $\mathcal{P}$, is it true that $X\in\mathcal{P}$?". Further, in the case in which the answer is affirmative, the mapping f is a $\mathcal{P}$-mapping.

Following [15], we give

Definition 1.1. *A class $\mathcal{P}$ of topological spaces is said to be expansive if the existence of a continuous bijection $f: Y\to X$ from a space Y onto a space $X\in\mathcal{P}$ implies $Y\in\mathcal{P}$.*

By Corollary 1.1 in [10] in the case in which $\mathcal{P}$ is a productive, hereditary and expansive class of spaces, we have that if $f : X \to Y$ is e-cleavable over $\mathcal{P}$, then $X \in \mathcal{P}$, and f is a $\mathcal{P}$-mappings. In particular, the previous result is true for the classes $\mathcal{P}$ of T_0, T_1, T_2, *functionally Hausdorff* or *Urysohn* spaces. Recall the definitions of $\mathcal{P}$-mapping in these cases.

Definition 1.2 [14].

- *$f \in C(X,Y)$ is T_0 if for every pair of distinct points $x, y \in X$ such that $f(x) = f(y)$, there exists some neighbourhood U of x which not contains y or some neighborhood V of y which not contains x;*
- *$f \in C(X,Y)$ is T_1 if for every pair of distinct points $x, y \in X$ such that $f(x) = f(y)$, there exist two neighbourhoods U and V of x and y respectively, such that U does not contains y and V does not contains x;*
- *$f \in C(X,Y)$ is T_2 if for every pair of distinct points $x, y \in X$ such that $f(x) = f(y)$, there exist two disjoint open neighbourhoods U and V of x and y respectively;*
- *$f \in C(X,Y)$ is Urysohn if for every pair of distinct points x, y such that $f(x) = f(y)$, there exist a neighbourhood W of $f(x)$ and two open subsets U, V of $f^{-1}(W)$ such that $x \in U$, $y \in V$ and $\overline{U} \cap \overline{V} = \varnothing$, where the closures are in $f^{-1}(W)$.*

Further we give the following:

Definition 1.3.

- *$f \in C(X,Y)$ is functionally Hausdorff if for every pair of distinct points $x, y \in X$ such that $f(x) = f(y)$, there exists a continuous mapping $g : X \to [0,1]$ such that $g(x) = 0$ and $g(y) = 1$;*

2. Cleavability over regular, completely regular, semiregular and almost regular spaces.

Now we consider the classes of *regular* and *completely regular spaces.*

Definition 2.1 [14].

- *$f \in C(X,Y)$ is regular if for every point $x \in X$ and every closed $C \subset X$ such that $x \notin C$ there exist an open neighbourhood W of $f(x)$ and two open subsets U, V of $f^{-1}(W)$ such that $x \in U$, $C \cap f^{-1}(W) \subset V$ and $U \cap V = \varnothing$.*

Further we give the following

Definition 2.2.

- *$f \in C(X,Y)$ is completely regular if for every point $x \in X$ and every closed $C \subset X$ such that $x \notin C$ and $f(x) \in f(C)$, there exists a continuous mapping $g : X \to [0,1]$ such that $g(x) = 0$ and $g(C) = \{1\}$.*

Note that we can not consider the previous remarks for the classes of regular and completely regular spaces because they are not expansive. However, e-cleavability of a mapping $f : X \to Y$ over the class $\mathcal{P}$ of regular or completely regular spaces does not imply that X belongs to $\mathcal{P}$, and, in particular, that f is a $\mathcal{P}$-mapping; in fact there exists the following

Example 2. *Let τ^* be a topology on $\mathbb{R}$ generated by adding to the natural topology τ on the real line the set of rational numbers. $(\mathbb{R}, \tau)$ is regular (completely regular) while $(\mathbb{R}, \tau^*)$ is not regular (completely regular). Since $id : (\mathbb{R}, \tau^*) \to (\mathbb{R}, \tau)$ is a continuous bijection, $(\mathbb{R}, \tau^*)$ is absolutely cleavable over the class $\mathcal{P}$ of regular (completely regular) spaces; so id is absolutely cleavable over $\mathcal{P}$. However id is not regular (completely regular); in fact if id would be regular (completely regular), then $(\mathbb{R}, \tau^*)$ would be regular (completely regular), a contadiction.* □

By Corollary 1.3 in [10] in the case in which $\mathcal{P}$ is a productive and hereditary class of spaces, we have that if $f : X \to Y$ is closed e-cleavable over $\mathcal{P}$, then $X \in \mathcal{P}$, and f is a $\mathcal{P}$-mapping. In particular, the previous result is true for the classes $\mathcal{P}$ of regular or completely regular spaces.

Now we consider the classes of *semiregular* ([23]) and *almost regular* ([24]) *spaces.*

Definition 2.3 [14].

- *$f \in C(X, Y)$ is semiregular if for every open $A \subset X$ and every point $x \in A$ there exist an open neighbourhood W of $f(x)$ and a regular open subset R of $f^{-1}(W)$ such that $x \in R \subset (A \cap f^{-1}(W))$.*
- *$f \in C(X, Y)$ is almost regular if for every point $x \in X$ and every regular closed $C \subset X$ such that $x \notin C$ and $f(x) \in f(C)$, there exist an open neighbourhood W of $f(x)$ and two disjoint open subsets U, V of $f^{-1}(W)$ such that $x \in U$, $C \subset V$.*

Since every space can be embedded as a closed subspace into a semiregular space ([16]), every space is e-cleavable over the class of semiregular spaces and then *every continuous mapping is e-cleavable* over that class of spaces. Note that the classes of semiregular and almost regular space are productive but not hereditary, so we can not consider the previous remarks for these classes of spaces. However, the closed e-cleavability of a mapping $f : X \to Y$ over the classes $\mathcal{P}$ of semiregular or almost regular spaces does not imply that $X \in \mathcal{P}$ and, in particular, that f is a $\mathcal{P}$-mapping. In fact for the class of semiregular spaces we can consider Example 2 noting that the mapping *id* is closed, while for the class of almost regular we have the following

Example 3. *Let τ^{**} be a topology on $\mathbb{R}$ generated by adding to the natural topology τ on the real line the sets $\mathbb{Q}_1$ and $\mathbb{Q}_2$ such that $\{\mathbb{Q}_1, \mathbb{Q}_2\}$ is a par-*

tition of $\mathbb{Q}$. *By Example 4 in [10], we have that* $(\mathbb{R}, \tau^{**})$ *is absolutely closed cleavable over the class* $\mathcal{P}$ *of almost regular spaces, but it does not belongs to* $\mathcal{P}$. *Then every constant mapping* f *on* $(\mathbb{R}, \tau^{**})$ *is absolutely closed cleavable over the class* $\mathcal{P}$ *but it is not almost regular; in fact if* f *would be almost regular then* $(\mathbb{R}, \tau^{**})$ *would be almost regular, a contradiction.* □

3. Cleavability over *H-closed* spaces.

Now we consider the class of *H-closed spaces* (see [23],[15]).

Definition 3.1 [12].

- *Let* X, Y, Z, W *be spaces and* $f : X \to Y$ *and* $g : Z \to W$ *be continuous mappings.* f *is said to be embedded in* g *if* $Y = W$, X *is a subspace of* Z *and the restriction* $g|X$ *is equal to* f.
- *A mapping* $f : X \to Y$ *is called H-closed if it is a Hausdorff mapping and for every embedding of* f *into a Hausdorff mapping* $g : Z \to Y$, X *is closed in* Z.

We will need the following known result

Proposition 3.1. *Every Hausdorff space can be embedded as a closed subspace into a H-closed space.*

Theorem 3.1. *Let* $\mathcal{H}$ *be the class of H-closed spaces and let* $f : X \to Y$ *be a contonuous mapping. The following conditions are equivalent*

(1) X *is e-cleavable over* $\mathcal{H}$;
(2) f *is e-cleavable over* $\mathcal{H}$;
(3) X *is Hausdorff;*
(4) X *is closed absolutely cleavable over* $\mathcal{H}$;
(5) X *is closed double cleavable over* $\mathcal{H}$;
(6) X *is closed cleavable over* $\mathcal{H}$;
(7) X *is closed pointwise cleavable over* $\mathcal{H}$;
(8) f *is closed absolutely cleavable over* $\mathcal{H}$;
(9) f *is closed double cleavable over* $\mathcal{H}$;
(10) f *is closed cleavable over* $\mathcal{H}$;
(11) f *is closed pointwise cleavable over* $\mathcal{H}$.

Proof. The equivalence (1)⇔(2) follows by Theorem 1 . Now we prove that (1)⇔(3). Let $\mathcal{P}$ the class of Hausdorff spaces and suppose that X is e-cleavable over $\mathcal{H}$. Since $\mathcal{H} \subset \mathcal{P}$, X is e-cleavable over $\mathcal{P}$; then, by Corollary 1.2 in [10], $X \in \mathcal{P}$. Now suppose that X is Hausdorff; then , by Proposition 3.1, X can be embedded as a closed subspace into a H-closed space, that is X is closed absolutely cleavable over $\mathcal{H}$ and then X is absolutely cleavable over $\mathcal{H}$. Now we prove the equicalences (3)-(6). By Proposition 3.1, (3)⇒(4);

the implications (4)$\Rightarrow$(5)$\Rightarrow$(6)$\Rightarrow$(7) are obvious. Further, (7) implies that X is pointwise cleavable over $\mathcal{H}$ and then, by the equivalence (1)$\Leftrightarrow$(3), X is Hausdorff. Now we prove the equivalences (7)-(11). We know that (7)$\Leftrightarrow$(4) and the implications (4)$\Rightarrow$(9)$\Rightarrow$(10)$\Rightarrow$(11) are obvious. But (11) implies that X is closed pointwise cleavable over $\mathcal{H}$, so the proof is complete. □

Note that the class $\mathcal{H}$ is productive but not hereditary, so we can not consider the previous remarks for that class of spaces. However, the closed e-cleavability of a mapping $f : X \to Y$ over the class $\mathcal{H}$ does not imply that $X \in \mathcal{H}$ and, in particular, that f is a $\mathcal{H}$-mapping. In fact there exists the following

Example 4. *Let X be an Hausdorff but not an H-closed space. Then X can be embedded as a closed subspace into a H-closed space, that is X is closed absolutely cleavable over $\mathcal{H}$. So by Proposition 3, every continuous and constant mapping f on X is closed absolutely cleavable over $\mathcal{H}$. However, f is not an H-closed mapping, because otherwise we would have that $X \in \mathcal{H}$, a contradiction.* □

4. Open questions.

Note that all the classes of spaces we have considered are productive.

Question - 1. *Do there exist not-productive classes $\mathcal{P}$ of spaces such that the cleavability of a mapping $f : X \to Y$ over $\mathcal{P}$ is equivalent to the cleavability of the space X over $\mathcal{P}$?.*

Note that a metrizable separable space need not be cleavable over $\mathcal{P} = \{\mathbb{R}\}$; so we have the following natural question:"Does there exist a space Y and a continuous mapping $f : X \to Y$ such that X is a metrizable separable space and f is cleavable over $\mathcal{P}$?". However it is known that every metrizable space X is pointwise cleavable over $\{\mathbb{R}\}$ or, equivalently, if X is a metrizable space, then the mapping id_X is cleavable over $\{\mathbb{R}\}$.

Question - 2. *What classic results about cleavability of spaces can be generalized to cleavability of mapping?*

References

[1] A.V.Arhangel'skii, *A general conception of splittability of topological spaces*, Abstracts Tirasp, Symp. (1985), Štiinca Kišinev, (1985), 8-10 (in Russian).

[2] A.V.Arhangel'skii, *Some new trends in the theory of continuous mappings*, In: Continuous functions or topological spaces, LGU, Riga (1986), 5-35 (in Russian).

[3] A.V.Arhangel'skii, *The general concept of cleavability of a topologiacal space*, Topology Appl. 44(1992), 27-36.

[4] A.V.Arhangel'skii, *Cleavability over reals*, Topology Appl. 48(1992), 163-178.

[5] A.V.Arhangel'skii, *A survey on cleavability*, to appear.

[6] A.V.Arhangel'skii, F.Cammaroto, *On different types of cleavability of topological spaces: pointwise, closed, open and pseudoopen*, J. Austral. Math. Soc (Ser.A) 57(1995), 1-17.

[7] A.V.Arhangel'skii, D.B.Shakhmatov, *On pointwise approximation of arbitrary functions by countable collections of continuous functions*, Journ. of Soviet Math, 50(2) (1990), 1497-1511. (Translated from: Trudy Sem. I.G.Petrovskogo 13(1988), 206-227).

[8] A.Bella, F.Cammaroto, Lj.Kočinac, *Remarks on splittability of topological spaces*, Q and A in General Topology 9(1991), 89-99.

[9] M.Bonanzinga, *On cleavability over relative spaces*, Q and A in General Topology 12(1994), 185-192.

[10] M.Bonanzinga, F.Cammaroto, *On cleavability over $T_{i,\rho}$ spaces*, submitted.

[11] F.Cammaroto, *On splittability of topological spaces*, Proc. VI Brasilian Topological Meeting (1990) (to apprear).

[12] F.Cammaroto, V.V.Fedorchuk, J.R.Porter, *On H-closed extension of Hausdorff mappings*, submitted.

[13] F.Cammaroto, Lj.Kočinac, *Developable spaces and cleavability*, Rendiconti di Matem., Serie VII, 1(1995), 1-17.

[14] F.Cammaroto, G.Nordo, *On Urysohn, almost regular and semiregular functions*, Filomat 8(1994).

[15] R.Engelking, *General Topology*, PWN, Warszawa, 1977.

[16] M.Katětov, *On H-closed exstensions of topological spaces*, Časopis Pěst. Mat. Fys. 72(1947), 17-32.

[17] Lj.Kočinac, *Metrizability and cardinal invariants using splittability* , Comt. Rend. Acad. Bulg. Sci. 43(1990), 9-12.

[18] Lj.Kočinac, *Cleavability and divisibility of topological spaces*, Atti Accademia Pelor. "dei Pericolanti" 70(1992), 1-16.

[19] Lj.Kočinac, F.Cammaroto, A.Bella, *Some results on splittability of topological spaces*, Atti Accademia Pelor. "dei Pericolanti" 68(1990), 41-60.

[20] G.Marchisa, *Su una generalizzazione degli assiomi di separazione T_2, T_3, T_4, T_5*, Rend. Sem. Mat. Univ. e Politec. Torino (1974-75), 33(1976), 103-106.

[21] B.A.Pasynkov, *The dimension and geometry of mappings*, Doul. Akad. Nauk SSSR 221(1975), 543-546 = Soviet Math. Dokl., 16(1975), 384-388.

[22] B.A.Pasynkov, *On extension onto mappings of some notions and statements about space*, (in Russian) in Mappings and functors, IZD. Moskow Univ., Moskow W (1984), 72-102.

[23] J.R.Porter, R.G.Woods, *Extensions and absolutes of Hausdorff spaces*, Springer-Verlag, 1988.

[24] M.K.Singal, S.P.Arya, *On almost regular spaces*, Glasnik Mat. 4 (24) (1969), 89-99.

Dipartimento di Matematica - Universita' di Messina - Contrada Papardo - Salita Sperone 98168 Messina - Italy.

FILOMAT (Niš) 9:1 (1995), 33–38

A SUFFICIENT UNIVALENCE CONDITION

N. N. Pascu, M. Obradović* and D. Răducanu

ABSTRACT. *This paper is concerned with a sufficient univalence condition for analytic functions in the unit disc. This condition generalizes some well-known univalence criteria.*

1. Introduction and preliminaries

Let U_r denote the disc $\{z \in \mathbb{C} : |z| < r\}$, $r \in (0,1]$ and let A denote the class of functions f which are analytic in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ and $f(0) = f'(0) - 1 = 0$.

Definition 1. *Let $f, g : U \mapsto \mathbb{C}$ be analytic function in U. The function f is subordinate to the function g ($f \prec g$) if there is an analytic function φ in U, which satisfies the conditions $\varphi(0) = 0$, $|\varphi(z)| < 1$, $z \in U$, and $f = g \circ \varphi$.*

Definition 2. *The function $L : U \times I \mapsto \mathbb{C}$, $I = [0, \infty)$ is a Loewner chain if the function $L(z,t)$ is analytic and univalent in U for all $t \in I$ and $L(z,s) \prec L(z,t)$ for all $0 \le s < t$.*

Definition 3 [5]. *The function $F : U_r \times \mathbb{C} \mapsto \mathbb{C}$, $F = F(u,v)$ satisfies the Pommerenke's conditions in U_r if:*

i) the function $L(z,t) = F(e^{-t}z, e^t z)$ is analytic in U_r, for all $t \in I$, locally absolutely continuous in I, locally uniform with respect to U_r.

ii) the function $G(e^{-t}z, e^t z)$, where $G(u,v) = \dfrac{u}{v}\dfrac{\partial F}{\partial u} \Big/ \dfrac{\partial F}{\partial v}$ is analytic in U_r for all $t \in I$ and has an analytic extension in $\overline{U} = \{z \in \mathbb{C} : |z| \le 1\}$ for all $t > 0$ and in U for $t = 0$. The analytic extension of the function G is denoted by $H = H(e^{-t}z, e^t z)$ and is called the associate function of F.

iii) $\dfrac{\partial F}{\partial v}(0,0) \neq 0$ and $\dfrac{\partial F}{\partial u}(0,0) \Big/ \dfrac{\partial F}{\partial v}(0,0) \notin (-\infty, -1]$.

Received 31.01.1995
1991 *Mathematics Subject Classification*: 30C55
* Supported by Grant 0401A of RFNS through Math. Inst. SANU

iv) *the family of functions* $\left\{F(e^{-t}z, e^t z)\Big/\left[e^{-t}\dfrac{\partial F}{\partial u}(0,0)+e^t\dfrac{\partial F}{\partial v}(0,0)\right]\right\}_{t\in I}$ *forms a normal family in* U_r.

We shall need the following theorem to prove our results:

Theorem 1 [5]. *Let* $F : U_r \times \mathbb{C} \mapsto \mathbb{C}$, $F = F(u,v)$ *be a function which satisfies Pommerenke's conditions in* U_r *and let* H *be the associate function of* F. *If*

$$\text{(1)}\qquad \begin{aligned} |H(z,z)| &< 1, \quad z \in U \text{ and} \\ |H(z,1/\bar{z})| &\le 1, \quad z \in U \setminus \{0\}, \end{aligned}$$

then the function $F(e^{-t}z, e^t z)$ *for all* $t \in I$, *has an analytic and univalent extension in* U.

2. Main results

Theorem 2. *Let* $f, h \in A$ *and* $\alpha \in \mathbb{C}$. *If* $\operatorname{Re}\alpha > 1/2$ *and*

$$\text{(2)}\qquad \begin{aligned} &\left|\frac{1-\alpha}{\alpha}\left[|z|^2 + \frac{1-|z|^2}{2}\frac{zh''(z)}{h'(z)}\right]\left[|z|^2 + \frac{1-|z|^2}{2}\frac{zh''(z)}{h'(z)}\right.\right. \\ &\left. + (1-|z|^2)\frac{zf'(z)}{f(z)}\right] + |z|^2(1-|z|^2)\left[\frac{zh''(z)}{h'(z)} + \frac{zf''(z)}{f'(z)}\right] \\ &\left. + \frac{(1-|z|^2)^2}{2}\frac{zh''(z)}{h'(z)}\frac{zf''(z)}{f'(z)} - \frac{(1-|z|^2)^2}{2}z^2 S_h(z)\right| \le |z|^2 \end{aligned}$$

for all $z \in U$, *then* f *is an univalent function in* U.

Remark: We denote by $S_h(z)$ the Schwarz's derivative of the function h, i.e.

$$S_h(z) = \left[\frac{h''(z)}{h'(z)}\right]' - \frac{1}{2}\left[\frac{h''(z)}{h'(z)}\right]^2.$$

Proof: Let $F : U \times \mathbb{C} \mapsto \mathbb{C}$ be the function

$$\text{(3)}\qquad F(u,v) = [f(u)]^{1-\alpha}\left[f(u) + (v-u)f'(u)\left(1 + \frac{v-u}{2}\frac{h''(u)}{h'(u)}\right)^{-1}\right]^{\alpha},$$

$(u,v) \in U \times \mathbb{C}$, and let $L : U \times I \mapsto \mathbb{C}$ be the function

$$\text{(4)}\qquad \begin{aligned} L(z,t) = F(e^{-t}z, e^t z) &= f(e^{-t}z)\left[1 + (e^{2t}-1)\frac{e^{-t}zf'(e^{-t}z)}{f(e^{-t}z)}\right. \\ &\left.\left(1 + \frac{e^{2t}-1}{2}\frac{e^{-t}zh''(e^{-t}z)}{h'(e^{-t}z)}\right)^{-1}\right]^{\alpha}, \ (z,t) \in U \times I. \end{aligned}$$

Since, there is $r' \in (0,1)$ such that $f(z) \neq 0$ for all $z \in U_{r'} \setminus \{0\}$, the function $f_1(z,t) = \dfrac{e^{-t} z f'(e^{-t}z)}{f(e^{-t}z)} = 1 + \cdots$ is analytic in $U_{r'}$.

The function $f_2(z,t) = \dfrac{e^{-t} z h''(e^{-t}z)}{h'(e^{-t}z)} = b_1 e^{-t} z + \cdots$ is analytic in U.

Then, there exists $r \in (0, r')$ such that the function $f_3(z,t) = 1 + \dfrac{(e^{2t}-1) f_1(z,t)}{1 + 2^{-1}(e^{2t}-1) f_2(z,t)} = e^{2t} + \cdots$ is analytic in U_r and $f_3(z,t) \neq 0$ for all $z \in U_r$ and $t \in I$. Hence, for the function $f_4(z,t) = \left[f_3(z,t)\right]^{\alpha} = e^{2\alpha t} + \cdots$, we can choose an analytic branch in U_r and the function $L(z,t) = f(e^{-t}z) f_4(z,t) = e^{(2\alpha-1)t} z + \cdots$ is analytic in U_r.

Using (4) we obtain

$$\frac{\partial L(z,t)}{\partial t} = -e^{-t} z \frac{\partial F}{\partial u}(e^{-t}z, e^{t}z) + e^{t} z \frac{\partial F}{\partial v}(e^{-t}z, e^{t}z)$$

and we observe that $\left|\dfrac{\partial L(z,t)}{\partial t}\right|$ is bounded on $[0,T]$, for any $T > 0$ fixed and for all $z \in U_r$. Therefore, the function $L(z,t) = F(e^{-t}z, e^{t}z)$ is locally absolutely continuous in I, locally uniform with respect to U_r.

Since

$$a_1(t) = e^{-t} \frac{\partial F}{\partial u}(0,0) + e^{t} \frac{\partial F}{\partial v}(0,0) = e^{(2\alpha-1)t}$$

we obtain $a_1(t) \neq 0$ and $\lim\limits_{t\to\infty} |a_1(t)| = \lim\limits_{t\to\infty} e^{t\,\mathrm{Re}(2\alpha-1)} = \infty$.

It is easy to prove that there exists $k > 0$ such that $|F(e^{-t}z, e^{t}z)/a_1(t)| \leq k$ for all $z \in U_r$ and $t \in I$. Hence $\{F(e^{-t}z, e^{t}z)/a_1(t)\}_{t\in I}$ is a normal family in U_r.

Using (3) we obtain

$$\begin{aligned} G(u,v) = \frac{u}{v}\frac{\partial F}{\partial u} \Big/ \frac{\partial F}{\partial v} &= \frac{1-\alpha}{\alpha}\frac{u}{v}\left[1 + \frac{v-u}{2}\frac{h''(u)}{h'(u)}\right]\left[1 + \frac{v-u}{2}\frac{h''(u)}{h'(u)}\right. \\ &\left. + (v-u)\frac{f'(u)}{f(u)}\right] + \frac{u}{v}(v-u)\left[\frac{h''(u)}{h'(u)} + \frac{f''(u)}{f'(u)}\right] \\ &+ \frac{u}{v}\frac{(v-u)^2}{2}\left[\frac{f''(u)}{f'(u)}\frac{h''(u)}{h'(u)} - S_h(u)\right]. \end{aligned}$$

It results that the function $G(e^{-t}z, e^{t}z)$ has an analytic extension

$H(e^{-t}z, e^t z)$, where

$$H(e^{-t}z,e^tz)=\frac{1-\alpha}{2}e^{-2t}\left[1+\frac{e^{2t}-1}{2}\frac{e^{-t}zh''(e^{-t}z)}{h'(e^{-t}z)}\right]\cdot$$
$$\left[1+\frac{e^{2t}-1}{2}\frac{e^{-t}zh''(e^{-t}z)}{h'(e^{-t}z)}+(e^{2t}-1)\frac{e^{-t}zf'(e^{-t}z)}{f(e^{-t}z)}\right]$$
$$+e^{-2t}(e^{2t}-1)\left[\frac{e^{-t}zh''(e^{-t}z)}{h'(e^{-t}z)}+\frac{e^{-t}zf''(e^{-t}z)}{f'(e^{-t})}\right]$$
$$+e^{-2t}\frac{(e^t-e^{-t})^2}{2}z^2\left[\frac{h''(e^{-t}z)}{h'(e^{-t}z)}\frac{f''(e^{-t}z)}{f'(e^{-t}z)}-S_h(e^{-t}z)\right].$$

We have:
$|H(z,z)|=|\frac{1-\alpha}{\alpha}|<1$ for all $z\in U$ and $\alpha\in\mathbb{C}$ with $\operatorname{Re}\alpha>1/2$ and

$$|H(z,1/\bar z)|=\left|\frac{1-\alpha}{\alpha}\frac{1}{|z|^2}\left[|z|^2+\frac{1-|z|^2}{2}\frac{zh''(z)}{h'(z)}\right]\left[|z|^2\frac{1-|z|^2}{2}\frac{zh''(z)}{h'(z)}\right.\right.$$
$$\left.+(1-|z|^2)\frac{zf'(z)}{f(z)}\right]+(1-|z|^2)\left[\frac{zh''(z)}{h'(z)}+\frac{zf''(z)}{f'(z)}\right]$$
$$\left.+\frac{z}{\bar z}\frac{(1-|z|^2)^2}{2}\left[\frac{h''(z)}{h'(z)}\frac{f''(z)}{f'(z)}-S_h(z)\right]\right|\le 1,$$

for all $z\in U\setminus\{0\}$.

Therefore we can conclude, using Theorem 1, that the function $F(e^{-t}z, e^t z)$, $t\in I$ has an analytic and univalent extension $F_1(e^{-t}z,e^tz)$ in U for all $t\in I$. In particular, the function $f(z)=F_1(z,z)$, $z\in U$, is an univalent function in U.

3. Remarks

1. For $\alpha=1$ we obtain the following sufficient univalence condition:

Corollary 1. *If $f,h\in A$ and*

$$\left||z|^2(1-|z|^2)\left[\frac{zh''(z)}{h'(z)}+\frac{zf''(z)}{f'(z)}\right]+\frac{(1-|z|^2)^2}{2}\frac{zh''(z)}{h'(z)}\frac{zf''(z)}{f'(z)}\right.$$
$$\left.-\frac{(1-|z|^2)^2}{2}z^2S_h(z)\right|\le|z|^2, \tag{5}$$

for all $z\in U$, then the function f is univalent in U.

2. For $\alpha\to\infty$ we obtain an another sufficient univalence condition:

Corollary 2. *If* $f, h \in A$ *and*

$$\begin{aligned}&\Big|\frac{(1-|z|^2)^2}{2}\frac{zh''(z)}{h'(z)}\frac{zf''(z)}{f'(z)}-\frac{(1-|z|^2)^2}{2}z^2S_h(z)\\ (6)\quad &-\left[|z|^2+\frac{1-|z|^2}{2}\frac{zh''(z)}{h'(z)}\right]\left[|z|^2+\frac{1-|z|^2}{2}\frac{zh''(z)}{h'(z)}+(1-|z|^2)\frac{zf'(z)}{f(z)}\right]\\ &+|z|^2(1-|z|^2)\left[\frac{zh''(z)}{h'(z)}+\frac{zf''(z)}{f'(z)}\right]\Big|\le |z|^2,\end{aligned}$$

for all $z \in U$, *then the function* f *is univalent in* U.

3. If $h' = 1/f'$ we obtain the following univalence criterion which generalize the generalize the criterion due to Nehari:

Theorem 3 [7]. *Let* $f \in A$ *and* $\alpha \in \mathbb{C}$. *If* $\operatorname{Re}\alpha > 1/2$ *and*

(7)

$$\begin{aligned}&\Big|\frac{1-\alpha}{\alpha}\left[|z|^2-\frac{1-|z|^2}{2}\frac{zf''(z)}{f'(z)}\right]\left[|z|^2+(1-|z|^2)\left(\frac{zf'(z)}{f(z)}-\frac{1}{2}\frac{zf''(z)}{f'(z)}\right)\right]\\ &+z^2\frac{(1-|z|^2)^2}{2}S_f(z)\Big|\le |z|^2,\quad z\in U,\end{aligned}$$

then f *is a univalent function in* U.

$\alpha = 1$ in Theorem 3 gives us the Nehari's sufficient univalence condition:

Theorem 4 [4]. *If* $f \in A$ *and*

$$|S_f(z)| \le 2(1-|z|^2)^{-2},\quad z\in U,$$

then f *is univalent in* U.

4. If $h' = g'\cdot(f')^{-1}$ we obtain an univalence condition which generalize the criterion due to Epstein:

Theorem 5 [8]. *Let* $f, g \in A$ *and* $\alpha \in \mathbb{C}$. *If* $\operatorname{Re}\alpha > 1/2$ *and*

$$\begin{aligned}&\Big|\frac{1-\alpha}{\alpha}\Big[|z|^2-\frac{1-|z|^2}{2}\left(\frac{zf''(z)}{f'(z)}-\frac{zg''(z)}{g'(z)}\right)\\ (8)\quad &+(1-|z|^2)\frac{zf'(z)}{f(z)}\Big]\left[|z|^2-\frac{1-|z|^2}{2}\left(\frac{zf''(z)}{f'(z)}+\frac{zg''(z)}{g'(z)}\right)\right]\\ &+|z|^2(1-|z|^2)\frac{zg''(z)}{g'(z)}+z^2\frac{(1-|z|^2)^2}{2}\left[S_f(z)-S_g(z)\right]\Big|\le|z|^2,\ z\in U,\end{aligned}$$

then f *is a univalent function in* U.

$\alpha = 1$ in Theorem 4 gives us the Epstein's sufficient univalence condition:

Theorem 6 [2]. *If $f, g \in A$ and*

$$\left|\frac{(1-|z|^2)^2}{2}\left[S_f(z)-S_g(z)\right]+(1-|z|^2)\bar{z}\frac{g''(z)}{g'(z)}\right|\le 1,\ z\in U, \tag{9}$$

then f is a univalent function in U.

5. If $h(z)=z$ we obtain the following univalence criterion which generalize the criterion due to Becker:

Theorem 7 [3]. *Let $f \in A$ and $\alpha \in \mathbb{C}$. If $\operatorname{Re}\alpha > 1/2$ and*

$$\left|\frac{1-\alpha}{\alpha}\left[|z|^2+(1-|z|^2)\frac{zf'(z)}{f(z)}\right]+(1-|z|^2)\frac{zf''(z)}{f'(z)}\right|\le 1,\ z\in U, \tag{10}$$

then the function f is univalent in U.

$\alpha = 1$ in Theorem 7 gives us the Becker's sufficient univalence condition:

Theorem 8 [1]. *If $f \in A$ and*

$$(1-|z|^2)\left|\frac{zf''(z)}{f'(z)}\right|\le 1,\quad z\in U, \tag{11}$$

then the function f is univalent in U.

References

[1] J. Becker, *Über die Lösungsstruktur einer differentialgleichung in der Konformen Abbildung*, J. Reine Angew. Math., 285(1976), 66–74.

[2] C.L. Epstein, *Univalence criteria and surfaces in hyperbolic space*, J. Reine Angew. Math., (380)(1987), 196–214.

[3] S. Moldoveanu, *On an univalence condition*, Seminar of geometric function theory, Brasov, Preprint, nr. 3, 1993.

[4] Z. Nehari, The Schwarzian derivative and schlicht functions, Bull. Amer. Math. Soc., 55(1949), 545–551.

[5] N.N. Pascu, *The method of Loewner's chains for constructing univalence criteria*, (to appear)

[6] Ch. Pommerenke, *Univalent functions*, Vandenhoeck and Ruprecht, Götingen, 1975.

[7] D. Raducanu, *On an univalent criterion*, (to appear).

[8] D. Raducanu, *On some univalence conditions*, (to appear).

N. N. Pascu, D. Raducanu: Department of Mathematics, University "Transilvania", Brasov, 2200, Romania

M. Obradović: Department of Mathematics, Faculty of Technology and Metallurgy, 4 Karnegieva Street, 11 000 Belgrade, Yugoslavia
E-mail: obrad@elab.tmf.bg.ac.yu

FILOMAT (Niš) 9:1 (1995), 39–55

ON APPROXIMATION BY ANGLE FOR 2π PERIODIC FUNCTIONS

Miloš Tomić

ABSTRACT. *Approximations by angle from singular integrals of functions belonging to the space L_p, $1 \le p \le \infty$ are estimated using best aaproximations by angle from the trigonometric polynomials. The applications to Riesz's singular integrals are given.*

1. Introduction

It is well known that integrable 2π periodic functions can be obtained by different means of summation of their Fourier series. Approximations by sums of Fourier series can be compared with the best approximations as in paper [3] and [4]. In the paper [3] for function of one variable several inequalities are established by which the approximations are compared depending on whether $p > 1$ or $p = 1$. Those inequalities allow to compare classes of functions which are defined by approximations. Those are classes of Nikolski and saturation classes.

In the paper [4] we proved inequality concerning the approximation by angle for $1 < p < \infty$. The aim in this paper is to prove the inequality concerning also the approximation by angle but which concerns the space L_1 (the case $p = 1$).

To realize this aim we use one theorem of Timan of [3] (Theorem 1, inequality (3.11)) and one equality of [4] which in this paper we give as Lemma 2.

The difference between the quoted result of Timan and the results of this paper is following:

1) We generalise the result of Timan so that we consider an n-dimensional case of approximation by angle.

2) We give a theorem in a form which is more suitable for application in order to compare Nikolski's classes with saturation classes.

Received 02.03.1995; Revised 28.06.1995

1991 *Mathematics Subject Classification.* 42A10.

2. Auxiliary results

We say that $f \in L_p([o,2\pi]^n)$ if $f = f(x_1,\dots,x_n)$ is measurable on Δ_n and is a 2π periodic function with respect to every variable $x_1,\dots,x_n$ for which $\|f\| < \infty$, where

$$\|f\| = \|f\|_p = \left(\int_{\Delta_n} |f(x_1,\dots,x_n)|^p\right)^{1/p}, \quad 1 \le p < \infty,$$

$$\|f\|_\infty = \sup \mathrm{vrai}\,|f(x)|,$$

$$\Delta_n = \{x = (x_1,\dots,x_n),\, 0 \le x_i \le 2\pi,\, i = 1,\dots,n\}.$$

We will use the set of all sets of indices $i_1,\dots,i_m$ such that $1 \le i_j \le n$, $1 \le j \le m \le n$.

Let $T_{l_{i_j}}(x_1,\dots,x_n) \in L_p$ be a trigonometrical polynomial of order l_{i_j} with respect to variable x_{i_j} but with respect to all other variables $T_{l_{i_j}}$ is a arbitrary function.

The best approximation by m-dimensional angle for the function f to variables $x_{i_1},\dots,x_{i_m}$ is the quantity (see [2]):

$$Y_{l_{i_1}\dots l_{i_m}}(f)_p = \inf_T \Big\| f - \sum_{j=1}^{m} T_{l_{i_j}} \Big\|_p, \; l_{i_j} = 0,1,2,\dots \tag{2.1}$$

Let $\mathcal{X}^j_{l_j}(t)$, $j = 1,\dots,n$, $l_j = 1,2,\dots$ be the kernels such that $\mathcal{X}(-t) = \mathcal{X}(t)$, and

$$\int_0^{2\pi} \mathcal{X}(t)dt = 2\pi, \quad \int_0^{2\pi} |\mathcal{X}(t)|dt \le M, \quad \lim_{l_j\to\infty} \int_{0<\delta\le|t|} |\mathcal{X}_{l_j}(t)| = 0, \tag{2.2}$$

where the constant M does not depend on l_j.

A Fourier series of the kernel $\mathcal{X}^j_{l_j}(t)$ can be stated in the form

$$\mathcal{X}^j_{l_j}(t) \equiv 1 + \sum_{k_j=1}^{\infty} \gamma^j_{l_j}(k_j)\cos k_j t, \quad (j = 1,\dots,n). \tag{2.3}$$

For the function $f \in L_p$ by these kernels we can define singular integrals

$$I^j_{l_j} f = \frac{1}{2\pi}\int_0^{2\pi} f(x_1,\dots,x_j - t_j,\dots,x_n)\mathcal{X}^j_{l_j}(t_j)dt_j, \tag{2.4}$$

$$I_{l_i l_j} f = I_{l_i}^i I_{l_j}^j f, \ldots, I_{l_1,\ldots,l_n} f = I_{l_1}^1 \ldots I_{l_n}^n f.$$

By these singular integrals we can determine all m-dimensional angles ($1 \le i_j \le n$, $1 \le j \le m \le n$);

$$A_{l_{i_1},\ldots,l_{i_m}} f = I_{l_{i_1}}^{i_1} f + \cdots + I_{l_{i_m}}^{i_m} f - I_{l_{i_1} l_{i_2}} f - \cdots - I_{l_{i_{m-1}} l_{i_m}} f + \cdots + (-1)^{m-1} I_{l_{i_1} \ldots l_{i_m}} f. \tag{2.5}$$

Without loss of generality, in order to simplify the exposition, we will give a proof for the case $n = 2$, i.e. for a function of two variables. In that case we have three angles

$$A_{l_1} f = I_{l_1}^1 f, \quad A_{l_2} f = I_{l_2}^2 f, \quad A_{l_1 l_2} f = I_{l_1}^1 f + I_{l_2}^2 f - I_{l_1 l_2} f.$$

two one-dimensional angles and one two-dimensional angle.

For a function $f(x_1, x_2) \in L_p$ we will use singular integrals

$$S_{l_1} f = S_{l_1 \infty} f = \frac{1}{\pi} \int_0^{2\pi} f(x_1 - t_1, x_2) D_{l_1}(t_1) dt_1$$

$$S_{l_2} f = S_{\infty l_2} f = \frac{1}{\pi} \int_0^{2\pi} f(x_1, x_2 - t_2) D_{t_2}(t_2) dt_2, \quad S_{l_1 l_2} f = S_{l_1}(S_{l_2} f).$$

where $D_l(t) = \dfrac{\sin(l + 1/2)t}{2 \sin t/2}$ is the Dirichlet's kernel.

In order to prove our main result, we need the singular integrals of de la Vallee-Poussin (see [2]) $V_{l_1} f = V_{l_1 \infty} f$, $V_{l_2} f = V_{\infty l_2} f$, $V_{l_1 l_2} f = V_{l_1}(V_{l_2} f)$, $W_{l_1 l_2} f = V_{l_1} f + V_{l_2} f - V_{l_1 l_2} f$, $l_j = 0, 1, 2, \ldots$.

The functions $V_{l_j} f$, $j = 1, 2$, are trigonometrical polynomials of degree $2l_j - 1$ with respect to x_j and satisfies $\|V_{l_j} f\| \le B\|f\|$, $1 \le p \le \infty$, where B is an absolute constant.

Lemma 1 ([2], lemma 3). *Let $f(x_1, x_2) \in L_p$, $1 \le p \le \infty$. Then*

$$\left\|f - W_{l_1 l_2} f\right\|_p \le C Y_{l_1 l_2}(f)_p, \quad \left\|f - V_{l_j} f\right\|_p \le C Y_{l_j}(f)_p, \quad l_j = 0, 1, 2, \ldots \tag{2.6}$$

where C is an absolute constant.

The most important tool in the proof will be the following lemma:

Lemma 2. *For a $f \in L_p$, $1 \le p \le \infty$ and $l_j, s_j = 1, 2, \ldots$ the equalities*

$$f - A_{l_1 l_2} f = \sum_{i=1}^{9} B_j \tag{2.7}$$

hold, where

$$\begin{aligned} &B_1 = f - W_{2^{s_1}2^{s_2}} f, \quad B_2 = -I^1_{l_1} B_1, \quad B_3 = -I^2_{l_2} B_1, \quad B_4 = I_{l_1 l_2} B_1, \\ &B_5 = V_{2^{s_1}}\big(f - I^1_{l_1} f - V_{2^{s_2}} f + I^1_{l_1} V_{2^{s_2}} f\big), \quad B_6 = -I^2_{l_2} B_5, \\ &B_7 = V_{2^{s_2}}\big(f - I^2_{l_2} f - V_{2^{s_1}} f + I^2_{l_2} V_{2^{s_1}} f\big), \quad B_8 = -I^1_{l_1} B_7, \\ &B_9 = V_{2^{s_1}2^{s_2}}\big(f - A_{l_2 l_2} f\big). \end{aligned} \tag{2.8}$$

Proof.. The equality in the lemma is obtained by using the theorem of Fubini. □

We note that similar equalities were established in the paper [4] in which de la Vallee-Poussin sums are replaced by Dirichlet's sums.

Now we will use the function $F^j_l(m,\theta)$ which is defined in [3]:

$$\begin{aligned} F^j_l(m,\theta) &= \frac{1-\gamma^j_l(m)}{2} + \sum_{k=1}^{m-1} [1-\gamma^j_l(m-k)] \cos k\theta \\ &= \frac{1-\gamma^j_l(m)}{2} + \sum_{k=1}^{m-1} [1-\gamma^j_l(k)] \cos(m-k)\theta, \end{aligned} \tag{2.9}$$

for $m = 2, 3, \ldots$ and

$$F^j_l(1,\theta) = \frac{1-\gamma^j_l(1)}{2}, \quad F^j_l(0,\theta) = 1-\gamma^j_l(1), \quad j = 1, 2, \ldots, n.$$

Lemma 3. *If*

$$T_m(t) = \sum_{\nu=0}^{m} \alpha_\nu \cos \nu t + \beta_\nu \sin \nu t$$

is a trigonometrical polynomial of order m in one variable t, and if the function F_l is defined by (2.9), then the following equalities hold

$$\sum_{k=1}^{m} [1-\gamma_l(k)](\alpha_k \cos kx + \beta_k \sin kx) = \frac{2}{\pi} \int_0^{2\pi} F_l(m,\theta) T_m(x+\theta) \cos m\theta \, d\theta \tag{2.10}$$

$$\int_0^{2\pi} F_l(\nu,\theta) T_m(t+\theta) \cos \nu\theta \, d\theta = \int_0^{2\pi} F_l(m,\theta) T_m(t+\theta) \cos m\theta \, d\theta, \quad \nu > m. \tag{2.11}$$

Proof. Equality (2.10) is proved in [3] (the equality (3.11)). Equality (2.11) can be proved in the same way. □

3. The main result

For every kernel $\mathcal{X}^j_{l_j}(t)$, $j=1,\dots,n$ we can identify the quantities $\phi = \phi_j(l_j)>0$, $\psi=\psi^j_{l_j}(k_j)$, $K=K_j(\psi^j,l_j,k_j)$, using equalities

$$(3.1)\qquad 1-\gamma^j_{l_j}(k_j)=\phi_j(l_j)\psi^j_{l_j}(k_j),\quad k_j,l_j=1,2,\dots,$$

$$(3.2)\qquad K=K_j(\psi^j,l_j,k_j)=\frac{1}{2}+\sum_{\nu_j=1}^{2^{k_j-2}}\frac{\psi^j_{l_j}(2^{k_j}-1-\nu_j)}{\psi^j_{l_j}(2^{k_j}-1)}\cos\nu_j\theta_j\,.$$

For a fixed number l_j we choose the number s_j such that $2^{s_j}\le l_j\le 2^{s_j+1}$. We will say that the quantities ϕ,ψ,K satisfy conditions $(\alpha),(\beta),(\gamma),(\delta)$ if

$$(\alpha)\qquad \left|\psi^j_{l_j}(k'_j)\right|\le C_1\left|\psi^j_{l_j}(k''_j)\right|,\quad 0\le k'_j\le k''_j\le 2^s$$

$$\left|\psi^j_{l_j}(1)\right|\le C_2,\quad \left(\psi^j_{l_j}(1)=\psi^j_{l_j}(0)\right),$$

$$(\beta)\qquad \left|\psi^j_{l_j}(2k_j)\right|\le C_3\left|\psi^j_{l_j}(k_j)\right|,\quad 2k_j\le 2^{s_j}$$

$$(\gamma)\qquad 0<C_4\le\phi_j(l_j)\left|\psi^j_{l_j}(2^{s_j})\right|,$$

$$(\delta)\qquad \left\|K_j(\psi^j,l_j,k_j)\right\|_1\le C_5,$$

where the constants $C_1,\dots,C_5$ don't depend on k_j and l_j.

We will use symbol the [ ] such that $[2^{k-1}]=2^{k-1}$ for $k\ge 1$ and $[2^{0-1}]=0$.

By $a<<b$, $a>0$, $b>0$, we will denote the inequality $a\le Cb$, where C is some positive constant.

The following theorem gives the estimation of the approximation $\|f-A_{l_{i_1}\dots l_{i_m}}f\|$ by the best approximation by angle.

Theorem 1. *Let the quantities ϕ,ψ,K satisfy the conditions (α), (β), (γ), (δ) and let $f\in L_p$, $1\le p\le\infty$. Then for all natural numbers i_j and m such that $1\le i_j\le n$, $1\le j\le m\le n$ the following inequalities hold*

$$(3.3)\qquad \left\|f-A_{l_{i_1}\dots l_{i_m}}f\right\|_p\le C\prod_{j=1}^m\phi_{i_j}(l_{i_j})\Bigg[\sum_{k_{i_1}=0}^{l_{i_1}}\cdots\sum_{k_{i_m}=0}^{l_{i_m}}\prod_{j=1}^m\frac{\|\psi^{i_j}_{l_{i_j}}(k_{i_j})\|}{k_{i_j}+1}Y_{k_{i_1}\dots k_{i_m}}(f)_p\Bigg]$$

with constant C independent on f and $l_j=1,2,\dots$.

To prove this theorem we need

Theorem 2. *Let the singular integrals $\mathcal{X}_{l_j}^j(t)$, $j = 1,2$ satisfy the condition (2.2) and let the function $f(x_1, x_2) \in L_p([0,2\pi]^2)$, $1 \le p \le \infty$. Then for approximation by angle the following inequalities hold*

$$\|f - A_{l_j} f\|_p \le C_1 \Big[Y_{2^{s_j}}(f)_p + \sum_{k_j=0}^{s_j} Y_{[2^{k_j}-1]}(f)_p \int_0^{2\pi} |F_{l_j}^j(2^{k_j}-1, \theta_j)| d\theta_j \Big], \tag{3.4}$$

$$\begin{aligned} \|f - A_{l_1 l_2} f\|_p \le C_2 \Big[& Y_{2^{s_1} 2^{s_2}}(f)_p + \\ & + \sum_{k_1=0}^{s_1} Y_{[2^{k_1}-1] 2^{s_2}}(f)_p \int_0^{2\pi} |F_{l_1}^1(2^{k_1}-1, \theta_1)| d\theta_1 \Big] + \\ & + C_2 \Big[\sum_{k_2=0}^{s_2} Y_{2^{s_1}[2^{k_1}-1]}(f)_p \int_0^{2\pi} |F_{l_2}^2(2^{k_2}-1, \theta_2)| d\theta_2 \Big] + \\ & + C_2 \Big[\sum_{k_1=0}^{s_1} \sum_{k_1=0}^{s_2} Y_{[2^{k_1}-1][2^{k_2}-1]}(f)_p \prod_{j=1}^{2} \int_0^{2\pi} |F_{l_j}^j(2^{k_j}-1, \theta_j)| d\theta_j \Big], \end{aligned} \tag{3.5}$$

where the constants C_1, C_2 do not depend on f and $s_j, l_j = 1, 2, \ldots$ $(j = 1,2)$.

Proof of Theorem 2. We have

$$\|f - A_{l_j} f\| = \|f - I_{l_j}^j f\| = \|f - V_{2^{s_j}} f + V_{2^{s_j}} f - I_{l_j}^j V_{2^{s_j}} f + I_{l_j}^j V_{2^{s_j}} f - I_{l_j}^j f\|$$

and therefore

$$\|f - A_{l_j} f\|_p \le C_3 Y_{2^{s_j}}(f)_p + \|V_{2^{s_j}} f - I_{l_j}^j V_{2^{s_j}} f\|_p \tag{3.6}$$

where the constant C_3 does not depend on f, l_j, s_j and the numbers l_j and s_j are arbitrary.

We consider

$$\begin{aligned} V_{2^{s_j}} f - I_{l_j}^j V_{2^{s_j}} f = G_{l_j}(f, 2^{s_j}, x) = & \sum_{k=0}^{2^{s_j}-1} \delta_k^{(2^{s_j})} A_k \\ - \sum_{k=0}^{2^{s_j+1}-1} \gamma_{l_j}^j(k) \delta_k^{(2^{s_j})} A_k = & \sum_{k=1}^{2^{s_j+1}-1} [1 - \gamma_{l_j}^j(k)] \delta_k^{(2^{s_j})} A_k \end{aligned} \tag{3.7}$$

where A_k is the term of the Fourier series of f as a function with respect to the variable x_j and δ_j is a factor of a product which is determined by the sum of Vallee-Poussin.

For G we will use the expression

$$G_{l_j}(f, 2^{s_j}, x) = \sum_{k=1}^{s_j} \left[G_{l_j}(f, 2^k, x) - G_{l_j}(f, 2^{k-1}, x) \right] + G_{l_j}(f, 1, x). \tag{3.8}$$

In view of (3.7) the function $G_{l_j}(f, 2^k, x)$ is a trigonometrical polynomial with respect to x_j. Therefore, using Lemma 3, (2.10), we have

$$\begin{aligned} &G_{l_j}(f, 2^{k_1}, x) = \\ &= \frac{2}{\pi} \int_0^{2\pi} F^1_{l_1}(2^{k_1+1} - 1, \theta_1) V_{2^{k_1}} f(x_1 + \theta_1, x_2) \cos(2^{k_1+1} - 1)\theta_1 d\theta_1. \end{aligned} \tag{3.9}$$

$$\begin{aligned} &G_{l_j}(f, 2^{k_1-1}, x) = \\ &= \frac{2}{\pi} \int_0^{2\pi} F^1_{l_1}(2^{k_1} - 1, \theta_1) V_{2^{k_1-1}} f(x_1 + \theta_1, x_2) \cos(2^{k_1} - 1)\theta_1 d\theta_1. \end{aligned} \tag{3.10}$$

and similar equalities with respect to the variable x_2.

It follows by Lemma 3, (2.11), that holds

$$\begin{aligned} &G_{l_j}(f, 2^{k_1}, x) \\ &= \frac{2}{\pi} \int_0^{2\pi} F^1_{l_1}(2^{k_1} - 1, \theta_1) V_{2^{k_1}} f(x_1 + \theta_1, x_2) \cos(2^{k_1} - 1)\theta_1 d\theta_1. \end{aligned} \tag{3.11}$$

The equalities (3.10) and (3.11) give

$$\begin{aligned} &G_{l_j}(f, 2^{k_1}, x) - G_{l_1}(f, 2^{k_1-1}, x) = \frac{2}{\pi} \int_0^{2\pi} F^1_{l_1}(2^{k_1} - 1, \theta_1) \\ &\left[V_{2^{k_1}} f(x_1 + \theta_1, x_2) - V_{2^{k_1-1}} f(x_1 + \theta_1, x_2) \right] \cos(2^{k_1} - 1)\theta_1 d\theta_1. \end{aligned} \tag{3.12}$$

Since by Lemma 1

$$\left\| V_{2^{k_1}} f - V_{2^{k_1-1}} f \right\| \le \left\| V_{2^{k_1}} f - f \right\| + \left\| f - V_{2^{k_1-1}} f \right\| << 2Y_{[2^{k_1}-1]}(f) \tag{3.13}$$

we conclude from (3.12) that

$$\begin{aligned} &\left\| G_{l_j}(f, 2^{k_1}, x) - G_{l_1}(f, 2^{k_1-1}, x) \right\| \\ &<< Y_{[2^{k_1}-1]}(f) \int_0^{2\pi} \left| F^1_{l_1}(2^{k_1} - 1, \theta_1) \right| d\theta_1 \end{aligned} \tag{3.14}$$

holds.

For the quantity $G_{l_j}(f,1,x)$ we have

$$\begin{aligned} G_{l_j}(f,1,x) &= V_1 f - I^1_{l_1} V_1 f = S_1 f - I^1_{l_1} S_1 f \\ &= A_1 f - \gamma^1_{l_1}(1) A_1 f = \left[1 - \gamma^1_{l_1}(1)\right] A_1 f \\ &= \left[1 - \gamma^1_{l_1}(1)\right] [S_1 f - S_0 f] = \left[1 - \gamma^1_{l_1}(1)\right] [V_1 f - V_0 f] \end{aligned}$$

because $V_1 f = S_1 f$, $V_0 f = S_0 f$. Therefore

$$\|G_{l_1}(f,1,x)\| \ll Y_0(f) \left|1 - \gamma^1_{l_1}(1)\right|$$

hence, using the definition for $F^1_{l_1}(0,\theta)$ we obtain

$$(3.15) \qquad \|G_{l_1}(f,1,x)\|_p \ll Y_0(f)_p \int_0^{2\pi} \left|F^1_{l_1}(0,\theta_1)\right| d\theta_1.$$

Now, in view of (3.7), (3.8), (3.14) and (3.15) , we obtain

$$(3.16) \quad \left\|V_{2^{s_1}} f - I^1_{l_1} V_{2^{s_1}} f\right\| \ll \sum_{k_1=0}^{s_1} Y_{[2^{k_1}-1]}(f) \int_0^{2\pi} \left|F^1_{l_1}\left(2^{k_1} - 1, \theta_1\right)\right| d\theta_1.$$

From (3.6) using (3.16) it follows the inequality (3.4) for $j = 1$. In the same way we establish the inequality (3.16) for $j = 2$. Thus, the inequality (3.4) is proved.

To establish the inequality (3.5) concerning the approximation by two-dimensional angle we use Lemma 2.

It is clear that

$$(3.17) \qquad \|B_j\| \ll Y_{2^{s_1}2^{s_2}}(f)_p, \quad j = 1,2,3,4,$$

holds.

To estimate the quantity B_5 we will write

$$(3.18) \qquad B_5 = V_{2^{s_1}} \Phi - I^1_{l_1} V_{2^{s_1}} \Phi$$

where

$$\Phi = f - V_{2^{s_2}} f.$$

We consider the function B_5 as a function of the variable x_1 and apply the metod by which we estimated the expression $G_{l_1}(f, 2^{s_1}, x)$. So we derive

$$B_5 = B_5\left(2^{s_1}\right) = \sum_{k_1=1}^{s_1}\left[B_5\left(2^{k_1}\right) - B_5\left(2^{k_1-1}\right)\right] + B_5(1), \tag{3.8'}$$

$$\begin{aligned} &B_5\left(2^{k_1}\right) - B_5\left(2^{k_1-1}\right) = \\ &= \frac{2}{\pi}\int_0^{2\pi} F_{l_1}^1\left(2^{k_1}-1,\theta_1\right)\left[V_{2^{k_1}}\Phi\left(x_1+\theta_1,x_2\right) - \right. \\ &\left. - V_{2^{k_1-1}}\Phi\left(x_1+\theta_1,x_2\right)\right]\cos\left(2^{k_1}-1\right)\theta_1 d\theta_1, \end{aligned} \tag{3.12'}$$

$$\left\|V_{2^{k_1}}\Phi - V_{2^{k_1}}\Phi\right\| \leq \left\|V_{2^{k_1}}\Phi - \Phi\right\| + \left\|\Phi - V_{2^{k_1-1}}\Phi\right\|.$$

Since

$$V_{2^{k_1}}\Phi - \Phi = -\left[f - \left(V_{2^{k_1}}f + V_{2^{s_2}}f - V_{2^{k_1}2^{s_2}}f\right)\right]$$

we obtain

$$\left\|V_{2^{k_1}}\Phi - \Phi\right\|_p \ll Y_{2^{k_1}2^{s_2}}(f)_p,$$
$$\left\|\Phi - V_{2^{k_1-1}}\Phi\right\|_p \ll Y_{2^{k_1-1}2^{s_2}}(f)_p.$$

Thus

$$\left\|V_{2^{k_1}}\Phi - V_{2^{k_1-1}}\Phi\right\|_p \ll Y_{2^{k_1-1}2^{s_2}}(f)_p. \tag{3.19}$$

For $B_5(1)$ we have

$$B_5(1) = V_1\Phi - I_{l_1}^1\Phi = \left[1-\gamma_{l_1}^1(1)\right]A_1\Phi = \left[1-\gamma_{l_1}^1(1)\right]\left[V_1\Phi - V_0\Phi\right].$$

Since

$$\begin{aligned} &\|V_1\Phi - V_0\Phi\| \leq \|V_1\Phi - \Phi\| + \|\Phi - V_0\Phi\|, \\ &\|V_1\Phi - \Phi\| = \left\|f - \left(V_1 f + V_{2^{s_2}}f - V_{12^{s_2}}f\right)\right\| \ll Y_{12^{s_2}}(f), \\ &\|\Phi - V_0\Phi\| = \left\|f - \left(V_0 f + V_{2^{s_2}}f - V_{02^{s_2}}f\right)\right\| \ll Y_{02^{s_2}}(f), \\ &\left|1-\gamma_{l_1}^1(1)\right| = \left|F_{l_1}^1\left(0,\theta_1\right)\right| \end{aligned}$$

we derive

$$\left\|B_5(1)\right\| \ll Y_{02^{s_2}}(f)\int_0^{2\pi}\left|F_{l_1}^1\left(0,\theta_1\right)\right| d\theta_1. \tag{3.20}$$

In view of (3.18) , (3.8') , (3.12') , (3.19) , (3.20) it follows that

$$\|B_5\| \ll \sum_{k_1=0}^{s_1} Y_{[2^{k_1-1}]2^{s_2}}(f) \int_0^{2\pi} \left|F_{l_1}^1\left(2^{k_1}-1,\theta_1\right)\right| d\theta_1. \tag{3.21}$$

It is clear that

$$\|B_6\| \ll \|B_5\|. \tag{3.22}$$

In the same way we obtain

$$\|B_7\| \ll \sum_{k_2=0}^{s_2} Y_{2^{s_1}[2^{k_2-1}]}(f) \int_0^{2\pi} \left|F_{l_2}^2\left(2^{k_2}-1,\theta_2\right)\right| d\theta_2 \tag{3.23}$$

$$\|B_8\| \ll \|B_7\|. \tag{3.24}$$

To estimate B_9 we use the equality

$$B_9 = V_{2^{s_1}}P - V_{2^{s_1}}I_{l_1}^1 P, \quad P = V_{2^{s_2}}f - I_{l_2}^2 V_{2^{s_2}}f.$$

In the same way as we obtained the expression for B_5 we derive

$$\begin{aligned} B_9 &= \sum_{k_1=0}^{s_1}\left[B_9\left(P,2^{k_1}\right) - B_9\left(P,\left[2^{k_1-1}\right]\right)\right] \\ &= \sum_{k_1=0}^{s_1}\frac{2}{\pi}\int_0^{2\pi}\left\{V_{2^{k_1}}P\left(x_1+\theta_1,x_2\right) - V_{[2^{k_1-1}]}P\left(x_1+\theta_1,x_2\right)\right\}\cdot \\ &\cdot F_{l_1}^1\left(2^{k_1}-1,\theta_1\right)\cos\left(2^{k_1}-1\right)\theta_1 d\theta_1. \end{aligned} \tag{3.25}$$

We consider the function P as a function with respect to x_2 and obtain

$$\begin{aligned} &P(x_1+\theta_1,x_2) = V_{2^{s_2}}f(x_1+\theta_1,x_2) - V_{2^{s_2}}I_{l_2}^2 f(x_1+\theta_1,x_2) = \\ &\sum_{k_2=0}^{s_2}\left\{B_7\left(f,2^{k_2}\right) - B_7\left(f,\left[2^{k_2-1}\right]\right)\right\} = \\ &\sum_{k_2=0}^{s_2}\frac{2}{\pi}\int_0^{2\pi}\left\{V_{2^{k_2}}f\left(x_1+\theta_1,x_2+\theta_2\right) - \right. \\ &\left. V_{[2^{k_2-1}]}f\left(x_1+\theta_1,x_2+\theta_2\right)\right\}\cdot F_{l_2}^2\left(2^{k_2}-1,\theta_2\right)\cos\left(2^{k_2}-1\right)\theta_2 d\theta_2. \end{aligned} \tag{3.26}$$

Using (3.25) and (3.26) we get

$$(3.27)\qquad \begin{aligned} B_9 = \sum_{k_1=0}^{s_1}\sum_{k_2=0}^{s_2}\frac{4}{\pi}\int_0^{2\pi}\int_0^{2\pi}&\{V_{2^{k_1}2^{k_2}}f - V_{2^{k_1}[2^{k_2-1}]}-\\ &- V_{[2^{k_1-1}]2^{k_2}}f + V_{[2^{k_1-1}][2^{k_2-1}]}f\}\cdot\\ &\cdot\prod_{j=1}^{2}F_{l_j}^{j}\left(2^{k_j}-1,\theta_j\right)\cos\left(2^{k_j}-1\right)\theta_j d\theta_j. \end{aligned}$$

Since

$$(3.28)\qquad \begin{aligned} Q &= V_{2^{k_1}2^{k_2}}f - V_{2^{k_1}[2^{k_2-1}]}f - V_{[2^{k_1-1}]2^{k_2}}f + V_{[2^{k_1-1}][2^{k_2-1}]}f\\ &= W_{2^{k_1}2^{k_2}}f - W_{2^{k_1}[2^{k_2-1}]}f - W_{[2^{k_1-1}]2^{k_2}}f + W_{[2^{k_1-1}][2^{k_2-1}]}f \end{aligned}$$

we obtain

$$(3.29)\qquad \|Q\| \ll Y_{[2^{k_1-1}][2^{k_2-1}]}(f).$$

From (3.27) , in view of (3.28) and (3.29) it follows that

$$(3.30)\qquad \|B_9\| \ll \sum_{k_1=0}^{s_1}\sum_{k_2=0}^{s_2}Y_{[2^{k_1-1}][2^{k_2-1}]}(f)\prod_{j=1}^{2}\int_0^{2\pi}\left|F_{l_j}^{j}\left(2^{k_j}-1,\theta_j\right)\right|d\theta_j.$$

Finally, using Lemma 2 and the inequalities (3.17), (3.2), (3.22), (3.23), (3.24), (3.30) we obtain the inequality (3.5). The proof of Theorem 2 is complete .

Proof of Theorem 1. First we establish the inequality (3.3) for $m = 1$, $n = 2$. We will use the inequality (3.4), (Theorem 2), and the conditions of Theorem 1.

In view of (3.1) and (3.2) we derive

$$(3.31)\qquad F_{l_1}^{1}\left(2^{k_1}-1,\theta_1\right) = \phi_1(l_1)\psi_{l_1}^{1}\left(2^{k_l}-1\right)K_1\left(\psi^1,l_1,k_1\right),$$

hence, using the condition (δ), it follows that

$$(3.32)\qquad \left\|F_{l_1}^{1}\left(2^{k_1}-1,\theta_1\right)\right\| \ll \phi_1(l_1)\left|\psi_{l_1}^{1}\left(2^{k_1}-1\right)\right|.$$

From (3.4) by (3.32) we obtain

$$\|f - A_{l_1} f\| \ll Y_{2^{s_1}}(f) + \phi_1(l_1) \sum_{k_1=0}^{s_1} \left|\psi_{l_1}^1 (2^{k_1} - 1)\right| Y_{[2^{k_1-1}]}(f). \tag{3.33}$$

Now, from (3.33) using the condition (γ) it follows that

$$\begin{aligned} \|f - A_{l_1} f\| \ll \phi_1(l_1)\{ \left|\psi_{l_1}^1 (2^{s_1})\right| Y_{2^{s_1}}(f) \\ + \sum_{k_1=0}^{s_1} \left|\psi_{l_1}^1 (2^{k_1-1})\right| Y_{[2^{k_1-1}]}(f)\}. \end{aligned} \tag{3.34}$$

hence by the conditions (α) and (β) we derive

$$\|f - A_{l_1} f\| \ll \phi_1(l_1) \sum_{k_1=0}^{s_1+1} \left|\psi_{l_1}^1 ([2^{k_1-1}])\right| Y_{[2^{k_1-1}]}(f). \tag{3.35}$$

We conclude, using the conditions (β), (α) that (see [4]):

$$\sum_{k=1}^{s} \left|\psi_l (2^k)\right| Y_{2^k} \ll \sum_{\nu=2}^{2^s} \frac{|\psi_l(\nu)|}{\nu} Y_\nu. \tag{3.36}$$

Finally, from (3.35) by (3.36) we obtain the inequality (3.3) for the case $n = 2$, $m = 1$ (with respect to the variable x_1).

In the same way using the inequality (3.5) of Theorem 2 and the conditions of Theorem 1 (see the proof of the corresponding theorem in [4]), we obtain the inequality (3.3) for $m = n = 2$.

The proof of Theorem 1 is complete .

4. Applications

The obtained result (Theorem 1) we apply to Riesz's singular integrals.

Riesz's singular integral is given by the kernel (see [1])

$$\chi_l^{(\lambda)}(t) = 1 + \sum_{j=1}^{l} \left(1 - \frac{\lambda_j}{\lambda_{l+1}}\right) \cos jt \tag{4.1}$$

where the sequence λ_l, $l = 1, 2, \dots$ satisfies following conditions: (i) $0 < \lambda_l < \lambda_{l+1}$, (ii) $\lambda_l \to \infty$, $l \to \infty$, (iii) $\Delta_2 \lambda < 0$ or $\Delta_2 \lambda \geq 0$, (iv) $\lambda_{2l} = O(\lambda_l)$ if $\Delta_2 \lambda_l \geq 0$.

For this singular integral (this method of summation of Fourier series of functions) we have

$$\gamma_l^{(\lambda)}(0) = 1,\ \gamma_l^{(\lambda)}(j) = 1 - \frac{\lambda_j}{\lambda_{l+1}},\ j = 1, \ldots, l,$$
$$\gamma_l^{(\lambda)}(j) = 0,\ j = l+1, l+2, \ldots.$$

The quantities ϕ, ψ, K are

$$\psi_l^{(\lambda)}(j) = \lambda_j,\ j = 1, 2, \ldots, l$$
$$\psi_l^{(\lambda)}(j) = \lambda_{l+1},\ j \geq l+1,\ \phi^{(\lambda)}(l) = \frac{1}{\lambda_{l+1}},\ l = 1, 2, \ldots \tag{4.3}$$
$$K = K\left(\psi^{(\lambda)}, k, \theta\right) = \frac{1}{2} + \sum_{\nu=1}^{2^k-2} \frac{\lambda_{2^k-1-\nu}}{\lambda_{2^k-1}} \cos \nu\theta.$$

We will prove that the quantities ϕ, ψ, K satisfy the conditions (α), (β), (γ), (δ) of Theorem 1.

Since (i) and (4.3) the condition (α) holds for ψ.

To prove that the condition (β) is satisfied we use the inequality

$$\lambda_{k+1} - \lambda_k \leq C\frac{\lambda_k}{k}, \quad (C = \text{constant}),$$

which is proved in the paper [1] of Aljancic (if $\Delta_2\lambda \geq 0$ the condition (β) is obviously satisfied, the condition (iv)). From this inequality it follows that

$$\frac{\lambda_{k+1}}{\lambda_k} \leq 1 + C\frac{1}{k}.$$

Putting $k, k+1, \ldots, 2k-1$ in this inequality, by multiplication we derive

$$\frac{\lambda_{2k}}{\lambda_k} \leq \left(1 + \frac{C}{k}\right)^k,$$

and then $\lambda_{2k} = O(\lambda_k)$.

The condition (γ) is equivalent to the condition

$$\frac{\lambda_l}{\lambda_{2^s}} \leq C,\ 2^s \leq l < 2^{s+1}.$$

We have

$$\frac{\lambda_l}{\lambda_{2^s}} \leq \frac{\lambda_{2^{s+1}}}{\lambda_{2^s}} \leq C,$$

that means that condition (γ) is satisfied.

Now we will prove that the function K satisfies the condition (δ).

By applying Abel's identity we have

$$K\left(\psi^{(\lambda)}, k, \theta\right) - \frac{1}{\lambda_{2^k-1}} \sum_{\nu=0}^{2^k-3} \Delta\lambda_{2^k-1-\nu} D_\nu(\theta) + \frac{\lambda_1}{\lambda_{2^k-1}} D_{2^k-2}(\theta) \tag{4.4}$$

where D is Dirichlet's kernel.

To estimate the free term in (4.4) we introduce the new condition

$$\frac{l}{\lambda_l} \leq C \quad (C = \text{constant}), \tag{4.5}$$

independent on l.

In view of (4.5) we have

$$K\left(\psi^{(\lambda)}, k, \theta\right) = \frac{1}{\lambda_{2^k-1}} \sum_{\nu=0}^{2^k-3} \Delta\lambda_{2^k-1-\nu} D_\nu(\theta) + O(1). \tag{4.6}$$

If we apply Abel's identity again, from (4.6) we obtain

$$\begin{aligned} K\left(\psi^{(\lambda)}, k, \theta\right) &= \frac{1}{\lambda_{2^k-1}} \sum_{\nu=0}^{2^k-4} \Delta_2\lambda_{2^k-1-\nu} \sum_{j=0}^{\nu} D_j(\theta) \\ &+ \frac{1}{\lambda_{2^k-1}} \Delta\lambda_2 \sum_{j=0}^{2^k-3} D_j(\theta) + O(1). \end{aligned} \tag{4.7}$$

Since

$$\sum_{j=0}^{\nu} D_j(\theta) = (\nu+1)F_\nu(\theta)$$

where F_ν is Fejer's kernel , it follows that

$$\begin{aligned} K\left(\psi^{(\lambda)}, k, \theta\right) &= \frac{1}{\lambda_{2^k-1}} \sum_{\nu=0}^{2^k-4} (\nu+1)F_\nu(\theta)\Delta_2\lambda_{2^k-1-\nu} + \\ &+ \frac{1}{\lambda_{2^k-1}} \Delta\lambda_2 \left(2^k-2\right) F_k(\theta) + O(1). \end{aligned} \tag{4.8}$$

The equality (4.8) implies

$$\left\|K\left(\psi^{(\lambda)},k,\theta\right)\right\|_1 \ll \frac{1}{\lambda_{2^k-1}}\sum_{\nu=0}^{2^k-4}(\nu+1)\left|\Delta_2\lambda_{2^k-1-\nu}\right| \cdot \frac{|\Delta\lambda_2|}{\lambda_{2^k-1}}\left(2^k-2\right)+O(1). \tag{4.9}$$

Using (4.5) from (4.9) we obtain

$$\left\|K\left(\psi^{(\lambda)},k,\theta\right)\right\|_1 \ll \frac{1}{\lambda_{2^k-1}}\sum_{\nu=0}^{2^k-4}(\nu+1)\left|\Delta_2\lambda_{2^k-1-\nu}\right|+O(1) \tag{4.10}$$

where $\Delta\lambda_\nu=\lambda_\nu-\lambda_{\nu+1}$, $\Delta_2\lambda=\Delta(\Delta\lambda)$.

Since

$$\sum_{\nu=0}^{j}(\nu+1)\Delta_2\mu_\nu=2\mu_0-\mu_1+(\mu_{j+2}-\mu_{j+1})(j+1)-\mu_{j+1} \tag{4.11}$$

then, putting $\mu_\nu=\lambda_{2^k-1-\nu}$, $j=2^k-4$ we derive

$$\sum_{\nu=0}^{2^k-4}(\nu+1)\left|\Delta_2\lambda_{2^k-1-\nu}\right|=2\lambda_{2^k-1}-\lambda_{2^k-2}+(\lambda_1-\lambda_2)\left(2^k-3\right)-\lambda_2. \tag{4.12}$$

Finally, from (4.10) in view of (4.12) we obtain that $\|K\|_1\le C$. This means that the function K of Riesz's singular integrals satisfies the condition (δ).

In this way we prove the following

Theorem 3. *Let the sequences $\lambda_l^{(k)}$, $k=1,\ldots,n$, $l=1,2,\ldots$, satisfy the conditions (i) - (iv) and (v) $l=O(\lambda_l)$, $l\to\infty$. Let $A^{(\lambda)}_{l_{i_1},\ldots,l_{i_m}}f$ be m-dimensional angles which are obtained from singular integrals which are associated with the given sequences .*

Then, for $f\in L_p$, $1\le p\le\infty$, and all natural numbers i_j and m such that $1\le i_j\le n$, $1\le j\le m\le n$, the following inequalities hold

$$\left\|f-A^{(\lambda)}_{l_{i_1},\ldots,l_{i_m}}f\right\|\le C\left[\prod_{j=1}^{m}\lambda^{(i_j)}_{l_{i_j}}\right]^{-1}\sum_{k_{i_1}=0}^{l_{i_1}}\cdots \sum_{k_{i_m}=0}^{l_{i_m}}\prod_{j=1}^{m}\frac{\lambda^{(i_j)}_{k_{i_j}}}{k_{i_j}+1}Y_{k_{i_1}\ldots k_{i_m}}(f) \tag{4.13}$$

where the constant C does not depend on f and $l_j = 1, 2, \ldots$.

Particularly we consider the sequences

$$\lambda_j = \lambda_j^{(r,s)} = \lambda_j(r,s) = j^r \log^s(j+2),\ j = 1, 2, \ldots \tag{4.14}$$

where the real numbers r and s satisfy $r \geq 0$, $s \geq 0$.

Conditions (i) - (iv) of Theorem 3 are satisfied. Condition (iii) is satisfied because the function $\lambda(x) = x^r \log^s x$ has derivative $\lambda''(x) \geq 0$ for $r \geq 1$, $s \geq 0$, $x \geq b$ where b is the base of the logarithm.

Thus, we can apply Theorem 3 and obtain

Theorem 4. *Let $A^{(r,s)}_{l_{i_1} \ldots l_{i_m}} f$ be an m-dimensional angle from singular integrals which are determined by the sequences $\lambda_j(r,s) = j^r \log^s(j+2)$, $j = 1, 2, \ldots$ for $r \geq 1$, $s \geq 0$. Then for $f \in L_p([0, 2\pi]^n)$, $1 \leq p \leq \infty$, the following inequalities hold*

$$\begin{aligned} \left\| f - A^{(r,s)}_{l_{i_1} \ldots l_{i_m}} f \right\| &\leq C \prod_{j=1}^{m} l_{i_j}^{-r_{i_j}} \log^{-s_{i_j}} \left(l_{i_j} + 2\right) \cdot \\ &\cdot \sum_{k_{i_1}=0}^{l_{i_1}} \cdots \sum_{k_{i_m}=0}^{l_{i_m}} \prod_{j=1}^{m} \left(k_{i_j} + 1\right)^{r_{i_j} - 1} \log^{s_{i_j}} \left(k_{i_j} + 2\right) Y_{k_{i_1} \ldots k_{i_m}}(f) \end{aligned} \tag{4.15}$$

where the constant C does not depend on f and $l_j = 1, 2, \ldots$, $1 \leq i_j \leq n$, $1 \leq j \leq m \leq n$.

Puting $s_j = 0$, $j = 1, \ldots n$ we obtain from (4.15) the following inequalities

$$\begin{aligned} \left\| f - A^{(r)}_{l_{i_1} \ldots l_{i_m}} f \right\|_p &\leq C \prod_{j=1}^{m} l_{i_j}^{-r_{i_j}} \cdot \sum_{k_{i_1}=0}^{l_{i_1}} \cdots \\ &\sum_{k_{i_m}=0}^{l_{i_m}} \prod_{i=1}^{m} \left(k_{i_j} + 1\right)^{r_{i_j} - 1} Y_{k_{i_1} \ldots k_{i_m}}(f) \end{aligned} \tag{4.16}$$

where $f \in L_p([0, 2\pi]^n)$, $1 \leq p \leq \infty$, $r_j \geq 1$, $1 \leq i_j \leq n$, $1 \leq j \leq m \leq n$.

For $n = 1$ we have the case of a function of one variable. Then $Y = E$ and from (4.16) we obtain

$$\left\| f - A_l^{(r)} f \right\|_p \leq C \frac{1}{l^r} \sum_{k=0}^{l} (k+1)^{r-1} E_k(f)_p \tag{4.17}$$

where $1 \le p \le \infty$, $r \ge 1$, $l = 1, 2, \dots$ and

$$A_l^{(r)} f = \frac{a_0}{2} + \sum_{k=0}^{l} \left[1 - \frac{k^r}{(l+1)^r}\right] (a_k \cos kx + b_k \sin kx)\,,$$

a_k, b_k are the Fourier coeficients of the function f.

The theorem proved above make it possible to compare the classes of functions which are defined by the approximations. We will show that comparing the following classes.

Let the numbers $r_1, \dots r_n$, $r_j \ge 1$, $j = 1, \dots, n$, be given. We identify the classes

$$S_p^r H = \left\{ f \in L_p : Y_{l_{i_1} \dots l_{i_m}}(f)_p = O\left(\prod_{j=1}^{m} l_{i_j}^{-r_{i_j}}\right), \right.$$

$$\left. l_j = 1, 2, \dots,\ 1 \le i_j \le n,\ 1 \le j \le m \le n \right\}$$

$$V_p^r R = \left\{ f \in L_p : \left\| f - A^{(r)}_{l_{i_1} \dots l_{i_m}} f \right\|_p = O\left(\prod_{j=1}^{m} l_{i_j}^{-r_{i_j}}\right), \right.$$

$$\left. l_j = 1, 2, \dots,\ 1 \le i_j \le n,\ 1 \le j \le m \le n \right\}$$

where $A^{(r)}$ are angles which are determined by the sequances $\lambda_j(r_k) = j^{r_k}$, $k = 1, \dots, n$, $j = 1, 2, \dots$.

Then in view of the inequalities (4.16) we conclude that

$$S_p^{r+\varepsilon} H \subset V_p^r R \subset S_p^r H, \quad 1 \le p \le \infty,$$

where $r + \varepsilon$ is determined by numbers $r_j + \varepsilon_j$, $r_j \ge 1$, $\varepsilon_j > 0$, $j = 1, \dots, n$.

The classes $S_p^r H$ are the classes of Nikolski which are defined by the mixed dominated modulus of smoothness (see [2]).

References

[1] Aljancic S., *Classe de saturation du procede de moyennes typiques de Riesz*, Publ. de l'inst. math. t XIII (1959), 113-122.

[2] Potapov M. K., *O priblizenii uglom,*, russian, Conference o Constructive Theory of Functions, Academiai Kiado, Budapest, (1969), 371-399.

[3] Timan M. F., *Nailucsee priblizenie funkcij i linejnye metody summirovanija rjadov Fur'e*, russian, IANSSSR, ser. mat. 29 (1965), 587-604.

[4] Tomic M., *Priblizenie 2π periodiceskih funkcij iz singularnyh integralov*, russian, Publ.de l'inst. math. n. s., 26(40) (1979), 297-305.

Faculty of Mechanical Engineering, Department of Mathematics, University of Priština, 38 000 Priština, Yugoslavia

FILOMAT (Niš) 9:1 (1995), 57–62

POWER SEMIGROUPS THAT ARE ARCHIMEDEAN

Stojan Bogdanović and Miroslav Ćirić

ABSTRACT. *Power semigroups of various semigroups were studied by a number of authors. Here we give structural characterizations for semigroups whose power semigroups are Archimedean and we generalize some results from* [1], [8], [10] *and* [11].

Throughout this paper, $\mathbf{Z}^+$ will denote the set of all positive integers. For an element a of a semigroup S, $\langle a \rangle$ wil denote the *cyclic subsemigroup* of S generated by a. For a semigroup S, let $\mathbf{P}(S) = \{A \mid \varnothing \neq A \subseteq S\}$. If the multiplication on $\mathbf{P}(S)$ is defined by $AB = \{ab \mid a \in A,\ b \in B\}$, then $\mathbf{P}(S)$ is a semigroup which will be called the *power semigroup* of S, [11].

A semigroup S is *intra-π-regular* if for each $a \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n \in Sa^{2n}S$. A semigroup S is *left π-regular* if for each $a \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n \in Sa^{n+1}$, and it is *left regular* if for any $a \in S$, $a \in Sa^2$. *Right π-regular* and *right regular* semigroups are defined dually.

A semigroup S is *Archimedean* if for any $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n \in SbS$. A semigroup S is *left Archimedean* (*weakly left Archimedean*) if for any $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n \in Sb$ ($a^n \in Sba$), [4]. *Right Archimedean* and *weakly right Archimedean* semigroups are defined dually. A semigroup S is *t-Archimedean* (*weakly t-Archimedean*) if it is both left and right Archimedean (weakly left and weakly right Archimedean). A semigroup S is *power joined* if for any $a, b \in S$ there exists $m, n \in \mathbf{Z}^+$ such that $a^m = b^n$. A semigroup S is *left completely Archimedean* if it is Archimedean and left π-regular. *Right completely Archimedean* semigroups are defined dually. A semigroup S is *completely Archimedean* if it is both left and right completely Archimedean. A semigroup S is *left completely simple* if it is simple and left regular. *Right completely simple* semigroups

Received 12.03.1995
1991 *Mathematics Subject Classification.* Primary 20 M 10.
Supported by Grant 0401B of RFNS through Math. Inst. SANU

are defined dually. A semigroup S is *completely simple* if it is both left and right completely simple.

Further, $S = S^0$ will means that S is a semigroup with zero 0. A semigroup $S = S^0$ is a *nil-semigroup* if for any $a \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n = 0$. For $n \in \mathbf{Z}^+$, a semigroup $S = S^0$ is *n-nilpotent* if $S^n = \{0\}$, and $S = S^0$ is *nilpotent* if it is n-nilpotent, for some $n \in \mathbf{Z}^+$. An ideal extension S of a semigroup T will be called a *nil-extension* (*nilpotent extension, n-nilpotent extension*) if S/T is a nil-semigroup (nilpotent semigroup, n-nilpotent semigroup).

Let T be a subsemigroup of a semigroup S. A mapping φ of S onto T will be called a *right retraction* if $a\varphi = a$, for each $a \in S$, and $(ab)\varphi = a(b\varphi)$, for all $a, b \in S$. *Left retractions* are defined dually. A mapping φ of S onto T is a *retraction* if it is a homomorphism and $a\varphi = a$, for each $a \in T$. If T is an ideal of S, then φ is a retraction of S onto T if and only if it is both left and right retraction of S onto T. An ideal extension S of a semigroup T is a (*left, right*) *retractive* extension of T if there exists a (*left, right*) *retraction* of S onto T. A (left, right) retractive extension by an n-nilpotent semigroup will be called a (*left, right*) *n-inflation*, 2-inflations will be called simply *inflations*, and (left, right) retractive extensions by nilpotent semigroups will be called (*left, right*) *inflationary* extensions.

A semigroup S is a *singular band* if it is either a left zero band or a right zero band.

For undefined notions and notations we refer to [2], [3] and [7].

Theorem 1. *The following conditions on a semigroup S are equivalent:*

(i) $\mathbf{P}(S)$ *is Archimedean;*
(ii) $\mathbf{P}(S)$ *is a nil-extension of a simple semigroup;*
(iii) $\mathbf{P}(S)$ *is Archimedean with an idempotent.*

Proof. (i) $\Rightarrow$ (iii). Asumme $a \in S$. For $\{a\}$, $\langle a\rangle \in \mathbf{P}(S)$ there exists B, $C \in \mathbf{P}(S)$ and $n \in \mathbf{Z}^+$ such that $\{a\}^n = B\langle a\rangle C$, so for $b \in B$, $c \in C$ and $a^{2n} \in \langle a\rangle$ we have

$$a^n = ba^{2n}c \in Sa^{2n}S.$$

Therefore, S is intra-π-regular semigroup. Since S is also Archimedean, then by Theorem VI 1.1 [2], S is a nil-extension of a simple semigroup K. Thus, $\mathbf{P}(S)$ is an Archimedean semigroup with an idempotent K.

(iii) $\Rightarrow$ (ii). This follows by Theorem 3.2 [6].

(ii) $\Rightarrow$ (i). This follows by Theorem VI 1.1 [2]. □

Corollary 1. *If $\mathbf{P}(S)$ is Archimedean, then S is a nilpotent extension of a simple semigroup.*

Proof. By the proof of (i) $\Rightarrow$ (iii) in Theorem 1, S is a nil-extension of a simple semigroup K. Since $\mathbf{P}(S)$ is Archimedean, there exists $n \in \mathbf{Z}^+$, $A,\ B \in \mathbf{P}(S)$ such that $S^n = AKB$, whence $S^n = AKB \subseteq K = K^n \subseteq S^n$. Therefore, $S^n = K$, so S is a nilpotent extension of a simple semigroup. □

Theorem 2. *The following conditions on a semigroup S are equivalent:*

(i) $\mathbf{P}(S)$ *is left completely Archimedean;*
(ii) $\mathbf{P}(S)$ *is completely Archimedean;*
(iii) $\mathbf{P}(S)$ *is a nil-extension of a rectangular band;*
(iv) S *is a nilpotent extension of a rectangular band.*

Proof. (i) $\Rightarrow$ (ii). By Theorem 1, $\mathbf{P}(S)$ has an idempotent, so by Corollary 4 [4], $\mathbf{P}(S)$ is completely Archimedean.

(ii) $\Rightarrow$ (iv). Let $a \in S$. By Theorem 1, $S^n = K$ is a simple semigroup, for some $n \in \mathbf{Z}^+$. Also, by Theorem VI 2.2.1 [2], there exists $m \in \mathbf{Z}^+$, $C \in \mathbf{P}(S)$ such that $\{a\}^m = \{a\}^m \langle a\rangle C\{a\}^m$. Now, for any $c \in C$ we have

$$a^m = a^m aca^m = a^m a^2 ca^m = aa^m aca^m = aa^m = a^{m+1},$$

and by this it follows that K is a rectangular band.

(iv) $\Rightarrow$ (iii). Let $S^n = K$ be a rectangular band, for some $n \in \mathbf{Z}^+$. By Lemma 4 [8], $\mathbf{P}(K)$ is an ideal of $\mathbf{P}(S)$, and by Theorem 4 [10], $\mathbf{P}(K)$ is an inflation of a rectangular band T. Since $T^2 = T$, T is an ideal of $\mathbf{P}(K)$ and $\mathbf{P}(K)$ is an ideal of $\mathbf{P}(S)$, then T is an ideal of $\mathbf{P}(S)$. Also, for $A \in \mathbf{P}(S)$, $A^n \subseteq S^n = K$, so $A^n \in \mathbf{P}(K)$, whence $A^{2n} \in T$. Thus, $\mathbf{P}(S)$ is a nil-extension of a rectangular band T.

(iii) $\Rightarrow$ (i). This follows immediately. □

Corollary 1. *The following conditions on a semigroup S are equivalent:*

(i) $\mathbf{P}(S)$ *is an inflation of a rectangular band;*
(ii) S *is an inflation of a rectangular band;*
(iii) $(\forall x, y, z \in S)\ xz = xyz$.

Proof. (ii) $\Leftrightarrow$ (iii). This follows by Corollary 3.5 [5].

(iii) $\Rightarrow$ (i). For $A, B, C \in \mathbf{P}(S)$, by (iii) we obtain that $AC = ABC$, so by (ii) $\Leftrightarrow$ (iii) we obtain (i).

(i) $\Rightarrow$ (ii). This follows immediately. □

Theorem 3. *The following conditions on a semigroup S are equivalent:*

(i) $\mathbf{P}(S)$ *is weakly left Archimedean;*
(ii) $\mathbf{P}(S)$ *is a right zero band of nil-extensions of left zero bands;*
(iii) S *is a right inflationary extension of a rectangular band.*

Proof. (i) ⇒ (ii). By Theorem 1, $\mathbf{P}(S)$ has an idempotent, so by Theorem 7 [4] we obtain (ii).

(ii) ⇒ (i). This follows immediately.

(i) ⇒ (iii). By Theorem 2, S is a nilpotent extension of a rectangular band K. On the other hand, it is not hard to check that S is weakly left Archimedean, so by Theorem 7 [4], S is a right retractive nil-extension of a rectangular band T. Clearly, $K = T$, so (iii) holds.

(iii) ⇒ (i). Let S be a right inflationary extension of a rectangular band K and let φ be a right retraction of S onto K. By the proof of Theorem 2, $\mathbf{P}(S)$ is a nil-extension of $\mathbf{P}(K)$ and $\mathbf{P}(K)$ is an inflation of a rectangular band T. Further, T is a right zero band Y of left zero bands T_α, $\alpha \in Y$, so $\mathbf{P}(K)$ is a right zero band Y of semigroups P_α, $\alpha \in Y$, where for each $\alpha \in Y$, P_α is an inflation of T_α. Assume $A, B \in \mathbf{P}(S)$. Then $A^n, B^n \in T$, for some $n \in \mathbf{Z}^+$, and $A^n \in T_\alpha$, $B^n \in T_\beta$, for some $\alpha, \beta \in Y$. Now, $A\varphi \in \mathbf{P}(K)$, i.e. $A\varphi \in P_\gamma$, for some $\gamma \in Y$, so

$$A^n = A^{n+1} = A^{n+1}\varphi = (A^n A)\varphi = A^n(A\varphi) \in P_\alpha P_\gamma \subseteq P_\gamma,$$

and by $A^n \in T_\alpha$ we obtain $\gamma = \alpha$, i.e. $A\varphi \in P_\alpha$, whence

$$B^n A = (B^n A)\varphi = B^n(A\varphi) \in T_\beta P_\alpha \subseteq T \cap P_\alpha = T_\alpha.$$

Therefore, $A^n, B^n A \in T$, whence $A^n = A^n B^n A$, since T_α is a left zero band. Hence, $\mathbf{P}(S)$ is weakly left Archimedean. □

Corollary 3. *The following conditions on a semigroup S are equivalent:*

(i) $\mathbf{P}(S)$ *is weakly t-Archimedean;*

(ii) $\mathbf{P}(S)$ *is a matrix of nil-semigroups;*

(iii) S *is an inflationary extension of a rectangular band.*

Proof. This follows by Theorems 1 and 3 and Corollary 5 [4]. □

Theorem 4. *The following conditions on a semigroup S are equivalent:*

(i) $\mathbf{P}(S)$ *is left Archimedean;*

(ii) $\mathbf{P}(S)$ *is a nil-extension of a left zero band;*

(iii) S *is a nilpotent extension of a left zero band.*

Proof. (i) ⇒ (ii). By Theorem 1, $\mathbf{P}(S)$ has an idempotent, so by Theorem VI 3.2.1 [2], $\mathbf{P}(S)$ is a nil-extension of a left group. On the other hand, by Theorem 2, $\mathbf{P}(S)$ is a nil-extension of a rectangular band, and so $\mathbf{P}(S)$ is a nil-extension of a left zero band.

(ii) ⇒ (iii). Let $\mathbf{P}(S)$ be a nil-extension of a left zero band T. By Theorem 2, S is an n-nilpotent extension of a rectangular band K, for some $n \in \mathbf{Z}^+$.

For $a, b \in K$, $\{a\}, \{b\} \in T$, whence $\{a\} \cdot \{b\} = \{a\}$, i.e. $ab = a$. Thus, K is a left zero band.

(iii) $\Rightarrow$ (ii). Let S be an n-nilpotent extension of a left zero band K, for some $n \in \mathbf{Z}^+$. By Theorem 2, $\mathbf{P}(S)$ is a nil-extension of a rectangular band T. Let $A, B \in T$. Then $A = A^n \subseteq S^n = K$ and also $B \subseteq K$, whence $AB = A$. Therefore, T is a left zero band.

(ii) $\Rightarrow$ (i). This follows immediately. □

Corollary 4. *The following conditions on a semigroup S are equivalent:*

(i) $\mathbf{P}(S)$ *is left completely simple;*
(ii) $\mathbf{P}(S)$ *is completely simple;*
(iii) $\mathbf{P}(S)$ *is a rectangular band;*
(iv) $\mathbf{P}(S)$ *is a singular band;*
(v) S *is a singular band.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). This follows by Theorem 2.

(iii) $\Rightarrow$ (v). By (iii), each subset of S is its subsemigroup, so by the well-known result of L. Rédei [9], S is an ordinal sum of singular bands (for the definition of an ordinal sum see [7]). By Theorem 2, S is semilattice indecomposable, whence S is a singular band.

(v) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (i). This follows immediately. □

Corollary 5. *The following conditions on a semigroup S are equivalent:*

(i) $\mathbf{P}(S)$ *is t-Archimedean;*
(ii) $\mathbf{P}(S)$ *is power joined;*
(iii) $\mathbf{P}(S)$ *is a nil-extension of a group;*
(iv) $\mathbf{P}(S)$ *is a nil-semigroup;*
(v) $\mathbf{P}(S)$ *is nilpotent;*
(vi) S *is nilpotent.*

Proof. The equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) was proved by S. Bogdanović [1], and in the commutative case, (i) $\Leftrightarrow$ (vi) was proved by M.S. Putcha [8]. □

References

[1] S. Bogdanović, *A note on power semigroups*, Math. Japon. **6** (1983), no. 28, 725–727.

[2] S. Bogdanović, *Semigroups with a system of subsemigroups*, Inst. of Math., Novi Sad, 1985.

[3] S. Bogdanović and M. Ćirić, *Semigroups*, Prosveta, Niš, 1993. (in Serbian)

[4] S. Bogdanović and M. Ćirić, *Semilattices of left completely Archimedean semigroups*, Filomat (Niš) (to appear).

[5] S. Bogdanović and S. Milić, *Inflations of semigroups*, Publ. Inst. Math. **41 (55)** (1987), 63–73.
[6] J. L. Chrislock, *On medial semigroups*, J. Algebra **12** (1969), 1–9.
[7] M. Petrich, *Lectures in semigroups*, Akad.-Verlag, Berlin, 1977.
[8] M. S. Putcha, *On the maximal semilattice decomposition of the power semigroup of a semigroup*, Semigroup Forum **15** (1978), 263–267.
[9] L. Rédei, *Algebra* I, Pergamon Press, Oxford, 1967, p. 81.
[10] T. Tamura, *Power semigroups of rectangular groups*, Math. Japon. **29** (1984), 671–678.
[11] T. Tamura and J. Shafer, *Power semigroups*, Math. Japon. **12** (1967), 25–32.

University of Niš, Faculty of Economics, Trg JNA 11, 18000 Niš, Yugoslavia
E-mail address: mciric@archimed.filfak.ni.ac.yu

University of Niš, Faculty of Philosophy, Ćirila i Metodija 2, 18000 Niš, Yugoslavia
E-mail address: root@eknux.eknfak.ni.ac.yu

FILOMAT (Niš) 9:1 (1995), 63–67

SOME CARDINAL FUNCTIONS ON URYSOHN SPACES

Ljubiša D. Kočinac

ABSTRACT. *We give some results on the cardinality of Urysohn H-closed topological spaces involving a new cardinal function denoted by* $sqL_\theta(X)$.

1. Introduction

In [7], the following cardinal function was introduced. For a space X, $sqL(X)$ is the smallest infinite cardinal τ such that there exists a subset A in X of cardinality $\leq 2^\tau$ satisfying: for every family $\mathcal{U}$ of open subsets of X there exist a subfamily $\mathcal{V}$ of $\mathcal{U}$ and a subset B of A such that $|\mathcal{V}| \leq \tau$, $|B| \leq \tau$ and $\cup\mathcal{U} \subset \overline{B} \cup (\cup\mathcal{V})$. In [10], this cardinal function was studied in some details. In a similar way we define here another cardinal function, denoted by $sqL_\theta(X)$, and prove some results on the cardinality of Urysohn spaces involving this function. These results improve some results from [6] and [10].

2. Notation and terminology. Definitions

Notations and terminology in this paper are standard as in [2], [4], [5]. Unless otherwise indicated, all spaces are assumed to be at least T_1 and infinite. α, β, γ, δ are ordinal numbers, while τ, λ denote infinite cardinals; τ^+ is the successor cardinal of τ. As usual, cardinals are assumed to be initial ordinals. If S is a set, then $[S]^{\leq\tau}$ denote the collection of all subsets of X having cardinality $\leq \tau$.

We recall some definitions that we need.

2.1. A space X is *Urysohn* if for every two distinct points x and y in X there are open sets U and V such that $x \in U$, $y \in V$ and $\overline{U} \cap \overline{V} = \emptyset$.

2.2. If X is a space and A a subset of X, then we put

Received 20.04.1995

1991 *Mathematics Subject Classification*: 54A25

Supported by Grant 0401A of RFNS through Math. Inst. SANU

$$Cl_\theta A = \{x \in X : \overline{U} \cap A \neq \emptyset \text{ for every neighbourhood } U \text{ of } x\}.$$

The set $Cl_\theta A$ is called the ***θ-closure*** of A. A is ***θ-closed*** if $Cl_\theta A = A$.

2.3. A Hausdorff space X is called ***H-closed*** if every open cover $\mathcal{U}$ of X has a finite subcollection $\mathcal{V}$ whose union is dense in X.

2.4. ([1]) The ***θ-bitightness*** of a space X, denoted by $bt_\theta(X)$, is the smallest cardinal τ such that for each non-θ-closed set $A \subset X$ there exist a point $x \in X \setminus A$ and a collection $\mathcal{S} \in [[A]^{\leq\tau}]^{\leq\tau}$ such that $\{x\} = \cap\{Cl_\theta S : S \in \mathcal{S}\}$.

2.5. ([6]) Call a subset A of a space X ***θ-dense*** in X if $Cl_\theta A = X$, i.e. if for every open set $U \subset X$, $\overline{U} \cap A \neq \emptyset$. The ***θ-density*** of X is

$$d_\theta(X) = \omega \cdot \min\{|A| : A \text{ is a } \theta\text{-dense subset of } X\}.$$

Clearly, for every space X, $d_\theta(X) \leq d(X)$. There are spaces X for which $d_\theta(X) < d(X)$ holds.

2.6. ([9]) The ***θ-spread*** $s_\theta(X)$ of a space X is the supremum of the cardinalities of subsets D of X such that for every $x \in D$ there exists a neighbourhood U of x with $\overline{U} \cap D = \{x\}$. The inequality $s_\theta(X) < s(X)$ is possible.

2.7. A Hausdorff space X is said to be of *closed pseudocharacter* τ, denoted by $\psi_c(X) = \tau$, if τ is the smallest cardinal such that for each point $x \in X$ there exists a family $\{U_\alpha : \alpha \in \tau\}$ of neighbourhoods of x with $\{x\} = \cap\{\overline{U}_\alpha : \alpha \in \tau\}$.

3. Results

In [9], the following lemma is proved.

Lemma 3.1. *Let X be a topological space and $s_\theta(X) = \tau$. If $\mathcal{U}$ is a family of open subsets of X, then there exist $A \in [\cup\mathcal{U}]^{\leq\tau}$ and $\mathcal{V} \in [\mathcal{U}]^{\leq\tau}$ such that $\cup\mathcal{U} \subset Cl_\theta A \cup \cup\{\overline{V} : V \in \mathcal{V}\}$.* □

After this lemma and the definition of $sqL(X)$ it is reasonable to introduce:

Definition 3.2. *Let X be a space. Then $sqL_\theta(X)$ is defined to be the smallest cardinal τ such that there exists a subset A in X of cardinality $\leq 2^\tau$ satisfying: for every family $\mathcal{U}$ of open subsets of X there exist $\mathcal{V} \in [\mathcal{U}]^{\leq\tau}$ and $B \in [A]^{\leq\tau}$ such that $\cup\mathcal{U} \subset Cl_\theta B \cup (\cup\overline{\mathcal{V}})$.* □

Fact 1. $sqL_\theta(X) \leq sqL(X) \leq d(X)$.
Fact 2. $sqL_\theta(X) \leq d_\theta(X)$.

We shall also need the following lemma which is a version of the fundamental result on spread due to Shapirovskii (see [8;T.3]).

Lemma. 3.3 ([6;Prop. 3.3]). *Let X be a Urysohn space with $hs_\theta(X) \leq \tau$. Then there is a subset A of X such that $|A| \leq 2^\tau$ and $\cup\{Cl_\theta B : B \in [A]^{\leq\tau}\} = X$.*

Proposition 3.4. *For every Urysohn H-closed space X, we have*

$$sqL_\theta(X) \leq hs_\theta(X).$$

Proof. Let $hs_\theta(X) = \tau$. By Lemma 3.3 there exists a set $A \subset X$ with $|A| \leq 2^\tau$ such that $X = \cup\{Cl_\theta(B) : B \in [A]^{\leq\tau}\}$. Let us show that A witnesses $sqL_\theta(X) \leq \tau$. Take a collection $\mathcal{U}$ of open subsets of X. By Lemma 3.1 there exist $\mathcal{V} \in [\mathcal{U}]^{\leq\tau}$ and $M \in [\cup\mathcal{U}]^{\leq\tau}$ such that $\cup\mathcal{U} \subset Cl_\theta M \cup (\cup\overline{\mathcal{V}})$. For every $p \in M$ there exists some $S_p \in [A]^{\leq\tau}$ with $p \in Cl_\theta S_p$. Put $S = \cup\{S_p : p \in M\}$. Then $S \in [A]^{\leq\tau}$ and $M \subset \cup\{Cl_\theta S_p : p \in M\} \subset Cl_\theta(\cup\{S_p : p \in M\}) = Cl_\theta S$. As the θ-closure operator is idempotent in Urysohn H-closed spaces we have $Cl_\theta M \subset Cl_\theta(Cl_\theta S) = Cl_\theta S$. Hence, $\cup\mathcal{U} \subset Cl_\theta S \cup (\cup\overline{\mathcal{V}})$ and the proposition is proved. □

Example. Let X be the Niemytzki plane T equipped with the topology $\mathcal{T} = \{U \setminus C : U$ is open in T and $C \subset T$ is countable$\}$. Then $hs_\theta(X) = s(T) = 2^\omega$ and $sqL_\theta(X) = sqL(T) = \omega$. □

Theorem 3.5. *For every Urysohn H-closed space X, we have*

$$\psi_c(X) \leq 2^{sqL_\theta(X)}.$$

Proof. Let $sqL_\theta(X) = \tau$ and let $A \subset X$ be a set witnessing this fact. Fix a point $x \in X$. Since X is Urysohn, for every $y \in X \setminus \{x\}$ there are neighbourhoods U_y of x and V_y of y with $\overline{U}_y \cap \overline{V}_y = \emptyset$. Applying the definition of $sqL_\theta(X)$ to the family $\mathcal{V} = \{V_y : y \in X \setminus \{x\}\}$ (and A) one can find sets $Y = \{y_\alpha : \alpha \in \tau\} \in [X \setminus \{x\}]^{\leq\tau}$ and $B \in [A]^{\leq\tau}$ such that

$$X \setminus \{x\} \subset Cl_\theta B \cup (\cup\{\overline{V}_{y_\alpha} : \alpha \in \tau\}).$$

Put $\mathcal{U}_x = \{X \setminus Cl_\theta C : C \subset B, x \notin Cl_\theta C\} \cup \{U_{y_\alpha} : \alpha \in \tau\}$. Then $|\mathcal{U}_x| \leq 2^\tau$ so that we need to check $\{x\} = \cap\{\overline{U} : U \in \mathcal{U}_x\}$.

Let $p \in X \setminus \{x\} \subset Cl_\theta B \cup (\cup\{\overline{V}_{Y_\alpha} : \alpha \in \tau\})$. Consider two possibilities:

(i) $p \in Cl_\theta B$. Take neighbourhoods U_p of p and V_p of x such that $\overline{U}_p \cap \overline{V}_p = \emptyset$. It is easy to see that $p \in Cl_\theta(B \cap \overline{V}_p) \subset Cl_\theta\overline{V}_p = \overline{V}_p$ (in Urysohn H-closed spaces it holds $Cl_\theta\overline{G} = \overline{G}$ for each open set G). Therefore, $C = B \cap \overline{V}_p$ provides a subset of B with $\overline{U}_p \cap Cl_\theta C = \emptyset$, hence $\overline{U}_p \subset X \setminus Cl_\theta C = \emptyset$ and thus $\overline{U}_p \subset \overline{X \setminus Cl_\theta C}$ which gives $\{x\} = \cap\{\overline{U} : U \in \mathcal{U}_x\}$.

(ii) $p \in \cup\{\overline{V}_{y_\alpha} : \alpha \in \tau\}$. Then $x \in \cap\{\overline{U}_{y_\alpha} : \alpha \in \tau\}$, but $p \notin \cap\{\overline{U}_{y_\alpha} : \alpha \in \tau\}$. □

The following theorem is an improvement of Lemma 3.3.

Theorem 3.6. *Let X be a Urysohn H-closed space with $sqL_\theta(X) \leq \tau$. Then there is a subset A of X such that $|A| \leq 2^\tau$ and $\cup\{Cl_\theta B : B \in [A]^{\leq\tau}\} = X$.*

Proof. Let S be a set in X witnessing $sqL_\theta(X) \leq \tau$. According to Theorem 3.5, for every $x \in X$ one can choose a collection $\mathcal{U}_x$ of neighbourhoods of x such that $|\mathcal{U}_x| \leq 2^\tau$ and $\cap\{\overline{U} : U \in \mathcal{U}_x\} = \{x\}$. By transfinite induction we shall construct a sequence $\{M_\alpha : \alpha < \tau^+\}$ of subsets of X and a sequence $\{\mathcal{U}_\alpha : \alpha < \tau^+\}$ of families of open subsets of X satisfying the following conditions:

(a) $|M_\alpha| \leq 2^\tau$, $\alpha < \tau^+$;

(b) $\mathcal{U}_\alpha = \cup\{\mathcal{U}_x : x \in \cup\{M_\beta : \beta < \alpha\}\}$ (so $|\mathcal{U}_\alpha| \leq 2^\tau$), $\alpha < \tau^+$;

(c) If $T \in [S]^{\leq\tau}$, $\mathcal{V} \in [U_\alpha]^{\leq\tau}$ and $Cl\theta T \cup \cup\overline{\mathcal{V}} \neq X$, then $M_\alpha \setminus (Cl_\theta T \cup \cup\overline{\mathcal{V}}) \neq \emptyset$.

Suppose we have already defined all M_β and $\mathcal{U}_\beta$ for $\beta < \alpha$. Let us define M_α and $\mathcal{U}_\alpha$. For every $T \in [S]^{\leq\tau}$ and every $\mathcal{V} \in [\mathcal{U}_\beta]^{\leq\tau}$ choose a point $x(T, \mathcal{V}) \in X \setminus (Cl_\theta T \cup \cup\overline{\mathcal{V}})$ whenever the last set is not empty (otherwise the construction has been finished). Let

$$M_\alpha = \{x(T, \mathcal{V}) : T \in [S]^{\leq\tau} \text{ and } \mathcal{V} \in [U_\beta]^{\leq\tau}\}$$
$$\mathcal{U}_\alpha = \cup\{\mathcal{U}_x : x \in \cup\{M_\beta : \beta < \alpha\}\}.$$

It is easy to check that M_α and $\mathcal{U}_\alpha$ satisfy (a), (b) and (c). Put $M = \cup\{M_\alpha : \alpha < \tau^+\}$, $A = M \cup S$ and prove that A is the set we are looking for. First of all $|A| \leq 2^\tau$. Let $x \in X$. If $x \in A$ there is nothing to prove. Let $x \in X \setminus A$. Then $x \notin M$ so that for every $y \in M$ one can find a neighbourhood $V_y \in \mathcal{U}_y$ of y such that $x \notin \overline{V}_y$. So, $x \notin \cup\{\overline{V}_y : y \in M\}$. By the properties of S one can choose $B \in [S]^{\leq\tau}$ and $\{y_\gamma : \gamma \in \tau\} \in [M]^{\leq\tau}$ such that $M \subset \cup\{V_y : y \in M\} \subset Cl_\theta B \cup (\cup\{\overline{V}_{y_\gamma} : \gamma \in \tau\})$. Let us prove $x \in Cl_\theta B$. Suppose not. Then $Cl_\theta B \cup (\cup\{\overline{V}_{y_\gamma} : \gamma \in \tau\}) \neq X$. Since τ^+ is regular, there exists some $\delta < \tau^+$ such that $\{y_\gamma : \gamma \in \tau\} \subset M_\delta$. Then $\{V_{y_\gamma} : \gamma \in \tau\} \in [\mathcal{U}_\delta]^{\leq\tau}$. By (c), $M_{\delta+1} \setminus (Cl_\theta B \cup (\cup\{\overline{V}_{y_\gamma} : \gamma \in \tau\})) \neq \emptyset$. But this contradicts the fact $Cl_\theta B \cup (\cup\{\overline{V}_{y_\gamma} : \gamma \in \tau\}) \supset M \supset M_{\delta+1}$. The theorem is proved. □

The next two theorems improve Theorems 3.4 and 3.5, respectively, from [6]. The first of them is an immediate corollary of the previous theorem.

Theorem 3.7. *For every Urysohn H-closed space X we have*

$$d_\theta(X) \leq 2^{sqL_\theta(X)}. \quad \square$$

Theorem 3.8. *For every Urysohn H-closed space X we have*

$$|X| \leq 2^{sqL_\theta(X)bt_\theta(X)}.$$

Proof. Theorem 2.3 in [6] states that for every Urysohn space X, $|X| \leq [d_\theta(X)]^{bt_\theta(X)}$. Using now Theorem 3.7 we have $|X| \leq (d_\theta(X))^{bt_\theta(X)} \leq \left(2^{sqL_\theta(X)}\right)^{bt_\theta(X)} = 2^{sqL_\theta(X)bt_\theta(X)}$. □

The famous theorem of Hajnal-Juh'asz says: if X is a Hausdorff space, then $|X| \leq 2^{2^{s(X)}}$ [3], [4], [5]. In [9], it was shown that for a Urysohn space X this inequality can be improved to $|X| \leq 2^{2^{s_\theta(X)}}$. Our next result is an improvement of the last estimation for Urysohn H-closed spaces.

Theorem 3.9. *For every Urysohn H-closed space X we have*

$$|X| \leq 2^{2^{sqL_\theta(X)}}.$$

Proof. By Theorem 2.6 in [6], $|X| \leq 2^{d_\theta(X)\psi_c(X)}$ so that, by Theorems 3.5 and 3.7, one obtains $|X| \leq 2^{d_\theta(X)\psi_c(X)} \leq 2^{2^{sqL_\theta(X)}} \cdot 2^{sqL_\theta(X)} = 2^{2^{sqL_\theta(X)}}$. □

References

[1] F. Cammaroto, Lj. Kočinac, *On θ-tightness*, Facta Universitatis (Niš), Ser. Math. Inform. 8(1993), 77-85.

[2] R. Engelking, *General Topology*, PWN, Warszawa, 1977.

[3] A. Hajnal, I. Juhász, *Discrete subspaces of topological spaces*, Indag. Math. 29(1967), 343-356.

[4] R. Hodel, *Cardinal functions I*, In: K. Kunen, J.E. Vaughan (eds.), Handbook of Set-theoretic Topology, North-Holland, Amsterdam, 1984, 1-61.

[5] I. Juhász, *Cardinal functions in topology - ten years later*, Mathematical Centre Tracts 123, Amsterdam, 1980.

[6] Lj. Kočinac, *On the cardinality of Urysohn spaces*, Q & A in General Topology, 13(1995), 211–216.

[7] X.S. Liu, *Two topological cardinal functions*, Acta Math. Sinica 29 (1987), 494-497 (In Chinese).

[8] B. Shapirovskii, *On discrete subspaces of topological spaces. Weight, tightness and the Souslin number*, Doklady AN SSSR 202(1972), 779-782 (In Russian).

[9] J. Schröder, *Urysohn cellularity and Urysohn spread*, Math. Japonica 38(1993), 1129-1133.

[10] S.H. Sun, K.G. Choo, *Some new cardinal inequalities involving a cardinal function less than the spread and the density*, Comment. Math. Univ. Carolinae 31(1990), 395-401.

29. novembra 132, 37230 Aleksandrovac, Yugoslavia

FILOMAT (Niš) 9:1 (1995), 69–72

A NOTE ON CERTAIN CLASSES OF UNIVALENT FUNCTIONS

Ivan Jovanović and Milutin Obradović

ABSTRACT. *We give some results on $f'(z)$, $f(z)/z$ and $zf'(z)/f(z)$ for certain classes of univalent functions in the unit disc $|z|<1$.*

1. Introduction and preliminaries

Let A denote the class of functions f analytic in the unit disc $U=\{z: |z|<1\}$ with $f(0)=f'(0)-1=0$.

Ozaki [4] proved that if $f\in A$ and

$$\text{(1)}\qquad \operatorname{Re}\left\{1+z\frac{f''(z)}{f'(z)}\right\}<\frac{3}{2},\ z\in U,$$

then f is univalent in U. Later, Umezawa [7] showed that if $f\in A$ satisfies the condition (1), then f is univalent and convex in one direction. Sakaguchi [5] proved that if $f\in A$ satisfies (1), then $|\arg f'(z)|<\pi/2$, $z\in U$, i.e. f is close-to-convex function. Finally, R. Singh and S. Singh [6] proved that the same class is the subclass of starlike functions in U.

In his paper [3] Nunokawa considered the class of functions $f\in A$ such that

$$\text{(2)}\qquad \operatorname{Re}\left\{1+z\frac{f''(z)}{f'(z)}\right\}<1+\frac{\alpha}{2},\ z\in U,$$

for some $0<\alpha\le 1$.

He proved that for such class

$$\left|\arg\frac{zf'(z)}{f(z)}\right|<\frac{\pi}{2}\alpha,\ z\in U.$$

It is evident that for $\alpha=1$ in (2) we have class defined by (1) and the classes defined by (2) are the subclasses of the class defined by (1).

Received 25.04.1995
1991 *Mathematics Subject Classification*: 30C45
Supported by Grant 0401A of RFNS through Math. Inst. SANU

In this note we consider the values $zf'(z)/f(z)$, $f'(z)$ and $f(z)/z$ for the classes defined by (2).

We need the following definition and lemmas.

Let f and g be analytic in the unit disc U. We say that f is subordinate to g, written $f \prec g$, or $f(z) \prec g(z)$, if there exists an analytic function ω in U satisfying $\omega(0) = 0$, $|\omega(z)| < 1$, $z \in U$, and $f(z) = g(\omega(z))$. In patrticular, if g is univalent in U, then f is subordinate to g if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Lemma A ([2]). *Let ω be nonconstant and analytic in U with $\omega(0) = 0$. If $|\omega|$ attains its maximum value on the circle $|z| = r < 1$ at z_0, we have $z_0\omega'(z_0) = k\omega(z_0)$, $k \geq 1$.*

Lemma B ([1]). *Let g be a convex function in U and let γ be a complex number with $\mathrm{Re}\{\gamma\} > 0$. If f is analytic in U and $f \prec g$, then*

$$z^{-\gamma}\int_0^z f(w)w^{\gamma-1}dw \prec z^{-\gamma}\int_0^z g(w)w^{\gamma-1}dw.$$

We noté the that we can find more details of the classes of the functions we mentioned above in any standard book on univalent functions

2. Results and consequences

We start with the following

Theorem 1. *Let f satisfy the condition (2). Then*

$$z\frac{f'(z)}{f(z)} \prec (1-z)\Big(1 - \frac{1}{\alpha+1}z\Big)^{-1}. \tag{3}$$

Proof. Let's put $a = 1/(\alpha + 1)$ and

$$z\frac{f'(z)}{f(z)} = \frac{1-\omega(z)}{1-a\omega(z)}. \tag{4}$$

Evidently $\omega(0) = 0$. We want to prove that $|\omega(z)| < 1$, $z \in U$. From (4), after taking logarithmical differentiation, we get

$$1 + z\frac{f''(z)}{f'(z)} = \frac{1-\omega(z)}{1-a\omega(z)} - \frac{z\omega'(z)}{1-\omega(z)} + \frac{az\omega'(z)}{1-a\omega(z)}. \tag{5}$$

If it is not $|\omega(z)| < 1$, then by Lemma A, there exists a z_0, $|z_0| < 1$. such that $z_0\omega'(z_0) = k\omega(z_0)$ and $|\omega(z_0)| = 1$, $k \geq 1$. If we put $\omega(z_0) = e^{i\theta}$, then for such z_0 from (5) we have

$$\operatorname{Re}\left\{1 + z\frac{f''(z)}{f'(z)}\bigg|_{z=z_0}\right\} = \frac{1+a}{2a} + \frac{(1-a^2)(2a-1)}{2a(1-2a\cos\theta+a^2)}$$
$$+ (k-1)\frac{1-a^2}{2(1-2a\cos\theta+a^2)} \geq \frac{1+a}{2a}$$
$$+ \frac{(1-a^2)(2a-1)}{2a(1+a)^2} = 1 + \frac{\alpha}{2} + \frac{\alpha(1-\alpha)}{2(2+\alpha)} \geq 1 + \frac{\alpha}{2},$$

which is a contradiction to (2). Therefore, $|\omega(z)| < 1$, $z \in U$, and from (4) we finally get the relation (3).

We note that the function on the right side of (3) is univalent and maps the unit disc U onto the disc with the diameter end points 0 and $2(\alpha+1)/(\alpha+2)$.

Remark 1. *For $\alpha = 1$ in Theorem 1 and the previously cited result of Nunokawa we have that the image of U under $zf'(z)/f(z)$, where $f \in A$ satisfies (2), lies in the intersection of the angle $\{w : |\arg\omega| < \alpha\pi/2\}$ and the disc which is the image of U under the function $w = (1-z)/(1-z/(\alpha+1))$.*

Also we have

Theorem 2. *Let $f \in A$ satisfy the condition* (2). *Then*

(6) a) $f'(z) \prec (1+z)^\alpha$;

b) $\dfrac{f(z)}{z} \prec \dfrac{(1+z)^{\alpha+1}-1}{(\alpha+1)z}$.

Proof. a) From the condition (2) we conclude that f' has no zero in U. Let's put

(7) $$(f'(z))^{1/\alpha} = 1 + \omega(z)$$

(where we take the principal value). Evidently $\omega(0) = 0$. We want to prove that $|\omega(z)| < 1$, $z \in U$. From (7) after some transformations, we have

(8) $$1 + z\frac{f''(z)}{f'(z)} = 1 + \alpha\frac{z\omega'(z)}{1+\omega(z)}.$$

If it is not $|\omega(z)| < 1$, $z \in U$, then by Lemma A there exists a z_0, $|z_0| < 1$, such that $z_0\omega'(z_0) = k\omega(z_0)$ and $|\omega(z_0)| = 1$, $k \geq 1$. If we put $\omega(z_0) = e^{i\theta}$, from (8), we get

$$\operatorname{Re}\left\{1 + z\frac{f''(z)}{f'(z)}\bigg|_{z=z_0}\right\} = 1 + \frac{\alpha k}{2} \geq 1 + \frac{\alpha}{2},$$

which is a contradiction to (2). Therefore, $|\omega(z)| < 1$, $z \in U$, and from (7) we conclude that the relation (6) is true.

b) Since the function $(1+z)^\alpha$, $0 < \alpha \le 1$, is convex, then the result follows directly from the result of Lemma B, for $\gamma = 1$. $\square$

From Theorem 2, for $\alpha = 1$, we easily obtain

Corollary 1. *If $f \in A$ satisfies (1), then*

a) $\mathrm{Re}\{f'(z)\} > 0,\ z \in U,$

(which is the earlier result given in [5]);

b) *f is bounded in U and $|f(z)| < 3|z|/2$, $z \in U$.*

References

[1] D.J. Hallenbeck and S. Ruscheweyh, *Subordination by convex functions*, Proc. Amer. Math. Soc., 52(1975), 191–195.

[2] I.S. Jack, *Functions starlike and convex of order α*, J. London Math. Soc., (2)3(1971), 469–474.

[3] M. Nunokawa, *On certain starlike functions*, preprint.

[4] S. Ozaki, *On the theory of multivalent functions, II*, Sci. Rep. Tokyo Bunrika Daigaku, 4(1941), 45–87.

[5] K. Sakaguchi, *A property of convex functions and an application to criteria for univalence*, Bull. Nara Univ. Educ., 22(2) (1973), 1–5.

[6] R.Singh, S.Singh, *Some sufficient conditions for univalence and starlikeness*, Colloq. Math., 47(1982), 309–314.

[7] T. Umezava, *Analytic functions convex in one direction*, J. Math. Soc. Japan, 4(1952), 194–202.

Department of Mathematics, Faculty of Philosophy, Ćirila i Metodija 2, 18 000 Niš, Yugoslavia.
E-mail: ivan@archimed.filfak.ni.ac.yu

Department of Mathematics, Faculty of Technology and Metallurgy, 4 Karnegieva Street, 11 000 Belgrade, Yugoslavia.
E-mail: obrad@elab.tmf.bg.ac.yu

FILOMAT (Niš) 9:1 (1995), 73–75

SOME THEOREMS ABOUT PRIMARY COIDEALS

Daniel A. Romano

ABSTRACT. *In this short note we give some theorems about primary coideals of commutative ring with an apartness.*

Throughout this paper R will denote a commutative ring with an apartness in sense of the book [1] and paper [2]. A subset S of R is a *coideal* ([2]) of R iff $\varnothing \# S$, $-a \in S \;\Rightarrow\; a \in S$, $a+b \in S \;\Rightarrow\; a \in S \vee b \in S$, $ab \in S \;\Rightarrow\; a \in S \wedge b \in S$. The coideal S of R is a strongly extensional subset of R and $S \neq \varnothing \;\Rightarrow\; 1 \in S$ holds. The coideal S of R is a *prime coideal* iff $a \in S \wedge b \in S \;\Rightarrow\; ab \in S$. If S is a coideal of R, then the set $c(S) = \{b \in R : (\forall n \in \mathbf{N})(b^n \in S\}$ is a coideal of R under S, called *coradical* of S. The coideal Q of R is a *primary coideal* of R iff $c(Q) \subset Q$. If Q is a primary coideal of R, the the coradical $c(Q)$ of Q is a prime coideal. In this case, we say that the primary coideal Q belonging to the prime coideal $c(Q)$. Let S be a coideal of R and let X be a subset of R. Then the set $[S : X] = \{b \in R : (\exists x \in X)(bx \in S\}$ is a coideal of R called *quotient coideal* of Q by the subset X. It is clear that $[S : X] \subset S$ and $X \cap S = \varnothing \;\Rightarrow\; [S : X] = \varnothing$.

First, we shall give a description of irreducibility of a primary coideal Q as the union of the coradical $c(Q)$ and of one coideal S under Q.

Theorem 1. *Let Q be a primary coideal of R.*

(1) If $c(Q) \subsetneq Q$, then it does not exist a coideal S of R under Q such that $Q = c(Q) \cup S$.

(2) If $Q = c(Q) \cup S$, where S is a coideal of R under Q such that $S \subsetneq Q$, then $Q = c(Q)$.

Proof. (1) There exists an elemenat b in Q such that $b \# c(Q)$. Suppose that S is a coideal of R under Q such that $Q = c(Q) \cup S$. Let z be an arbitrary

Received 03.05.1995
1991 *Mathematics Subject Classification.* 03F65, 13A99.

element of Q. Then $z \in c(Q)$ or $z \in S$. If $z \in c(Q)$, then $zb \in c(Q) \subset Q = c(Q) \cup S$, and

$$z \in [c(Q) \cup S : b] = [c(Q) : b] \cup [S : b] = \varnothing \cup [S : b] \subset S.$$

Therefore $Q \subset S$. It is a contradiction.

(2) Suppose that there is a coideal S of R under Q such that $S \subsetneqq Q$ and $Q = c(Q) \cup S$. Then there exists an element b in Q such that $b \# S$. Thus $b \in Q = c(Q) \cup S$ and $b \neq S$ implies that $b \in c(Q)$. Therefore

$$\begin{aligned} z \in Q \ &\Rightarrow\ zb \in Q\, c(Q) \subset Q = c(Q) \cup S \\ &\Rightarrow\ z \in [c(Q) \cup S : b] = [c(Q) : b] \cup [S : b] = [c(Q) : b] \subset c(Q). \quad \square \end{aligned}$$

Let S be a subset of R. We say that S is a *stable* subset of R iff $(\forall x \in R)(\neg\neg(x \in S) \ \Rightarrow\ x \in S)$. If S is a stable coideal of R and if X is a multiplicative subset of R, then the set $\langle S : X \rangle = \{b \in R : bX \subset S\}$ is a coideal of R. $\langle S : X \rangle \subset S$. In the next theorem we shall give a construction of a primary coideal $\langle Q : c(Q) \rangle$, where Q is a stable coideal of R such that $c(Q)$ is a prime coideal of R.

Theorem 2. *Let Q be a stable coideal of R such that $c(Q)$ is a prime coideal of R. Then the set $\langle Q : c(Q) \rangle$ is a primary coideal of R belonging to $c(Q)$.*

Proof. Let Q be a stable coideal of R such that $c(Q)$ is a prime coideal of R. Then the set $\langle Q : c(Q) \rangle$ is a coideal of R and it holds

$$\begin{aligned} c(Q) \subset Q \ &\Rightarrow\ c(Q) = \langle c(Q) : c(Q) \rangle \subset \langle Q : c(Q) \rangle \subset Q \\ &\Rightarrow\ c(Q) = c(c(Q)) \subset c(\langle Q : c(Q) \rangle) \subset c(Q) \end{aligned}$$

Further, we have

$$\begin{aligned} & a \in \langle Q : c(Q) \rangle \ \ \&\ \ b \in c(Q) = \langle c(Q) : c(Q) \rangle \Longleftrightarrow \\ & a\, c(Q) \subset Q \ \ \&\ \ b\, c(Q) \subset c(Q) \ \Rightarrow \\ & ab\, c(Q) \subset a\, c(Q) \subset Q \ \Rightarrow \\ & ab \in \langle Q : c(Q) \rangle . \quad \square \end{aligned}$$

Let $\mathcal{F} = (P_j)_{j \in J}$ be a family of coideals of a ring R. We say that a coideal P of R is an *isolated coideal* from the family $\mathcal{F}$ iff $(\exists p \in P)(p \# \cup P_j)$. Let S be a stable coideal of R. If $c(S)$ is the union of prime coideals under S $c(S) = \cup_{j \in J} P_j$, where J is a discrete set, and if P_i is an isolated coideal from the family $\mathcal{F} = (P_j)_{j \in J \setminus \{i\}}$, then the set $Q_i = \langle S : P_i \rangle$ is a primary coideal of R belonging to P_i. Therefore, the stable coideal S of R such that $c(S) = \cup_{j \in J} P_j$ contains an union of primary coideals $Q_i = \langle S : P_i \rangle$ where P_i are isolated prime coideals of R under S.

Theorem 3. *Let S be a stable coideal of R such that $c(S) = \cup_{j\in J} P_j$, where the P_j 's are prime coideals of R under S, and the set J is discrete. If P_i is an isolated prime coideal from the family $(P_j)_{j\in J\setminus\{i\}}$, then the coideal $Q_i = \langle S : P_i\rangle$ is a primary coideal of R belonging to P_i.*

Proof. Without difficultes one can verify that the set Q_i is a coideal of R under S. On the other hand, we have $c(Q_i) = c(\langle S : P_i\rangle) = \langle c(S) : P_i\rangle = \langle \cup_{j\in J} P_j : P_i\rangle$. Suppose that b is an arbitrary element of $c(Q_i)$. Then $b\,P_i \subset \cup_{j\in J} P_j$. As P_i is an isolated prime coideal from the family $(P_j)_{j\in J\setminus\{i\}}$, there exists an element p_i in P_i such that $p_i \# \cup \{P_j : j \in J \;\;\&\;\; j \neq i\}$. Now, we have

$$\begin{aligned} b \in \big\langle \cup_{j\in J} P_j : P_i \big\rangle &\iff bP_i \subset \cup_{j\in J} P_j = \cup_{j\neq i} P_j \cup P_i \\ &\iff (\forall p \in P_i)(bp \cup_{j\neq i} P_j \cup P_i) \\ &\implies bp_i \cup_{j\neq i} P_j \cup P_i \\ &\iff b \in \big[\cup_{j\neq i} P_j \cup P_i : p_i\big] = \big[\cup_{j\neq i} P_j : p_i\big] \cup \big[P_i : p_i\big] \\ &\iff b \in \varnothing \cup P_i \\ &\iff b \in P_i. \end{aligned}$$

Therefore $c(Q_i) = P_i$. Thus

$$\begin{aligned} & a \in Q_i = \langle S : P_i\rangle \;\;\&\;\; b \in P_i = \langle P_i : p_i\rangle \iff \\ & aP_i \subset S \;\;\&\;\; bP_i \subset P_i \implies \\ & abP_i \subset aP_i \subset S \implies \\ & ab \in \langle S : P_i\rangle = Q_i. \quad \square \end{aligned}$$

References

[1] R. Mines, F. Richman and W. Ruitenburg, *A Course of Constructive Algebra*, Springer, New York, 1988

[2] D. A. Romano, *Rings and fields: a constructive view*, Z. Math. Logik Grundl. Math., 34(1)(1988), 25–40

Department of Mathematics, Faculty of Philosophy, University of Banja Luka, Bana Lazarevića 1, Banja Luka

FILOMAT (Niš) 9:1 (1995), 77–82

ON $\mathcal{M}$-HARMONIC SPACE $\mathcal{D}_p^s$

Miroljub Jevtić

ABSTRACT. *We show that the $\mathcal{M}$-harmonic Dirichlet space $\mathcal{D}_p^s$ is equal to the weighted Bergman space $\mathcal{A}_p^s$ for $0 < p < 1$ and $s > n$.*

1. Introduction

In [6, chapter 10] author considered the relationship between the weighted Bergman spaces $\mathcal{A}_p^s$ of $\mathcal{M}$-harmonic functions in the open unit ball B in $\mathbb{C}^n$ and the Dirichlet spaces $\mathcal{D}_p^s$. He showed that if $s > n$ and $1 \le p < \infty$, then $\mathcal{A}_p^s = \mathcal{D}_p^s$. In this note we show that also $\mathcal{A}_p^s = \mathcal{D}_p^s$ in the case $s > n$, $0 < p < 1$.

Let B be the open unit ball in $\mathbb{C}^n$ and $S = \partial B$ the unit sphere in $\mathbb{C}^n$. We denote by ν the normalized Lebesgue measure on B and by σ the rotation invariant probability mesure on S.

Let $\tilde{\Delta}$ be the invariant Laplacian on B. That is, $\tilde{\Delta} f(z) = \Delta(f \circ \varphi_z)(0)$, $f \in C^2(B)$, where Δ is the ordinary Laplacian and φ_z the standard automorphism of B, $\varphi_z \in \mathrm{Aut}(B)$, taking 0 to z (see [5]). The C^2-functions f that are anihilated by $\tilde{\Delta}$ are called $\mathcal{M}$-harmonic ($f \in \mathcal{M}$).

Definition 1.1. *For $0 < p < \infty$, and $s \in \mathbb{R}$, the weighted Bergman space $\mathcal{A}_p^s$ is defined as the space of $\mathcal{M}$-harmonic functions f on B for which*

$$\|f\|_{\mathcal{A}_p^s} = \left[\int_B (1-|z|^2)^s |f(z)|^p d\lambda(z)\right]^{1/p} < \infty.$$

Here, $d\lambda(z) = (1-|z|^2)^{-n-1} d\nu(z)$ is the measure on B that is invariant under the group $\mathrm{Aut}(B)$.

For $f \in C^1(B)$, $\nabla f = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_{2n}})$, $z_k = x_{2k-1} + i x_{2k}$, $k = 1, 2, \dots, n$, denotes the real gradient of f and let $\tilde{\nabla} f(z) = \nabla(f \circ \varphi_z)(0)$, $z \in B$, be the invariant real gradient of f.

Received 24.06.1995
1991 *Mathematics Subject Classification*: 32A37
Supported by Grant 0401A of RFNS through Math. Inst. SANU

Definition 1.2. *For $0 < p < \infty$, and $s \in \mathbb{R}$, the $\mathcal{M}$-harmonic Dirichlet space $\mathcal{D}_p^s$ is defined as the space of $\mathcal{M}$-harmonic functions f on B for which*

$$\int_B |\widetilde{\nabla} f(z)|^p (1-|z|^2)^s d\lambda(z) < \infty.$$

For $f \in \mathcal{D}_p^s$, set

$$\| |f| \|_{p,s} = |f(0)| + \left(\int_B |\widetilde{\nabla} f(z)|^p (1-|z|^2)^s d\lambda(z) \right)^{1/p}.$$

For the proof of our main result the following Theorem will be needed.

Theorem 1.3 ([4]). *Let $0 < p < \infty$, $s > n - p/2$ and $f \in \mathcal{M}$. Then following statements are equivalent:*

(*i*) $f \in \mathcal{D}_p^s$,

(*ii*) $\displaystyle\int_B |\nabla f(z)|^p (1-|z|^2)^{s+p} d\lambda(z) < \infty$,

(*iii*) $\displaystyle\int_B (1-|z|^2)^{s+p} \big(|Rf(z)| + |R\overline{f}(z)| \big)^p d\lambda(z) < \infty$.

As usual, $Rf(z) = \displaystyle\sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}$ is the radial derivative of f.

Theorem 1.4. *Let h be $\mathcal{M}$-harmonic on B.*

(*i*) *For all p, $0 < p < \infty$, and $s \in \mathbb{R}$, there exists a constant C, independent of h, such that*

$$\int_B (1-|z|^2)^s |\widetilde{\nabla} h(z)|^p d\lambda(z) \le C \int_B (1-|z|^2)^s |h(z)|^p d\lambda(z).$$

(*ii*) *For all p, $0 < p < \infty$, and $s > n$, there exists a positive constant C, independent of h, such that*

(1.1)
$$\int_B (1-|z|^2)^s |h(z)|^p d\lambda(z) \le C \left(|h(0)|^p + \int_B (1-|z|^2)^{s+p} |\nabla h(z)|^p d\lambda(z) \right).$$

Item (i) was proved in [6], Theorem 10.10. If $1 \le p < \infty$, then the second part follows from Theorem 1.3 and Theorem 10.10 [6]. So it remains to show that (1.1) holds for $0 < p < 1$. The proof will be given in section 2.

Corollary 1.5. *For all* p, $0 < p < \infty$, *and* $s > n$, *we have* $\mathcal{A}_p^s = \mathcal{D}_p^s$.

Next, we consider the relationship between the $\mathcal{M}$-harmonic Hardy space $\mathcal{H}^p$ and the spaces $\mathcal{D}_p^n$. For $0 < p < \infty$, $\mathcal{H}^p$ denotes the set of $\mathcal{M}$-harmonic functions f on B for which

$$\|f\|_p^p = \int_S [M_\alpha f(\xi)]^p d\sigma(\xi) < \infty, \text{ for some (any) } \alpha > 1.$$

Here $M_\alpha f(\xi) = \sup_{z\in D_\alpha(\xi)} |f(z)|$, $\xi \in S$, where $D_\alpha(\xi) = \{z \in B : |1 - \langle z,\xi\rangle| < \frac{\alpha}{2}(1-|z|^2)\}$, $\alpha > 1$, denotes the Koranyi admissible approach regions.

By Theorem 6.18 ([6]) for $1 < p < \infty$, $f \in \mathcal{H}^p$ if and only if

$$\int_B (1-|z|^2)^n |f(z)|^{p-2} |\widetilde{\nabla} f(z)|^2 d\lambda(z) < \infty.$$

Thus when $p = 2$, $\mathcal{H}^2 = \mathcal{D}_2^n$.

For all p, $2 \le p < \infty$, $\mathcal{H}^p \subset \mathcal{D}_p^n$, with $\|\,|f|\,\|_{p,n} \le C_{n,p}\|f\|_p$, for all $f \in \mathcal{H}^p$, where $C_{n,p}$ is a constant depending only on n and p (see [3], [6]).

For all p, $0 < p \le 2$, $\mathcal{D}_p^n \subset \mathcal{H}^p$.

For $\alpha > 1$, $\xi \in S$, let

$$S_\alpha f(\xi) = \left(\int_{D_\alpha(\xi)} |\widetilde{\nabla} f(z)|^2 d\lambda(z)\right)^{1/2}$$

denote the area integral of f. In [1] it is shown that if $f \in \mathcal{M}$ then $f \in \mathcal{H}^p$, $0 < p < \infty$, if and only if $S_\alpha f \in L^p(\sigma)$. From this and the inequality

$$\int_S [S_\alpha f(\xi)]^p d\sigma(\xi) \le C \int_B (1-|w|^2)^n |\widetilde{\nabla} f(w)|^p d\lambda(w),$$

where $f \in \mathcal{M}$ and $0 < p \le 2$ (see [6]), it follows that $\mathcal{D}_p^n \subset \mathcal{H}^p$, $0 < p \le 2$. We note that this inclusion was proved in [6] for $1 < p \le 2$.

In this note we follow the custom of using the letter C to stand for a positive constant which changes its value from one appearance to another while remaining independent of the important variables.

2. Proof of (1.1), case $0 < p < 1$

If $0 < r < 1$, we set $E_r(z) = \{w \in B : |\varphi_z(w)| < r\} = \varphi_z(rB)$. $E_r(z)$ is an ellipsoid and its volume is given by $\nu(E_r(z)) = \dfrac{r^{2n}(1-|z|^2)^{n+1}}{(1-r|z|)^{n+1}}$ (see [5], p.30).

For the proof of (1.1), $0 < p < 1$, the following lemmas will be needed.

Lemma 2.1. *If $s > 1$, then*

$$\int_0^1 \frac{dt}{|1 - t\langle z, w\rangle|^s} \le \frac{C}{|1 - \langle z, w\rangle|^{s-1}}, \quad z, w \in B.$$

Lemma 2.2 ([4]). *Let $0 < r < 1$ and $0 < p < \infty$. There is a constant $C > 0$ such that if $f \in \mathcal{M}$ then*

$$\left(\frac{|\nabla f(w)|}{|1 - \langle z, w\rangle|}\right)^p \le C \int_{E_r(w)} \left(\frac{|\nabla f(\xi)|}{|1 - \langle z, \xi\rangle|}\right)^p d\lambda(\xi), \; z, w \in B.$$

Lemma 2.3 ([2]). *For $1 < p < r < \infty$, $0 < q < \infty$ and a measurable $f \in L^{p,q-1}$ $(\|f\|^p_{p,q-1} = \int_B |f(z)|^p (1 - |z|^2)^{q-1} d\nu(z) < \infty)$ we have*

$$\left(\int_B \left(\int_B \frac{|f(w)|(1 - |w|^2)^{q-1}}{|1 - \langle z, w\rangle|^{n+q}} d\nu(w)\right)^r (1 - |z|^2)^{r(\frac{n+q}{p} - \frac{n}{r}) - 1} d\nu(z)\right)^{1/r}$$
$$\le C\|f\|_{p,q-1}.$$

Lemma 2.4 ([5], p.17). *If $\alpha > 0$, then*

$$\int_S \frac{d\sigma(\xi)}{|1 - \langle \xi, z\rangle|^{n+\alpha}} = O\left(\frac{1}{(1 - |z|)^\alpha}\right), \quad z \in B.$$

Lemma 2.5. *For $0 < s < t$ we have*

$$\int_0^1 \frac{(1 - r)^{s-1} dr}{(1 - r\rho)^t} \le C(1 - \rho)^{s-t}, \quad 0 \le \rho < 1.$$

Assume now that $s > n$, $0 < p < 1$ and $\int_B (1-|z|^2)^{s+p} |\nabla h(z)|^p d\lambda(z) < \infty$. Since $|\nabla h(z)|$ has $\mathcal{M}$-subharmonic behavior, i.e. $|\nabla h(w)| \le C \int_{E_r(w)} |\nabla h(z)| d\lambda(z)$, $w \in B$, for some $0 < r < 1$, we have for any $a > 0$

$$|h(z)|^p \le C\left(|h(0)|^p + \left(\int_0^1 \int_{E_r(tz)} |\nabla h(w)| d\lambda(w) dt\right)^p\right)$$
$$\le C\left(|h(0)|^p + \left(\int_0^1 \int_B \frac{|\nabla h(w)|(1 - |w|^2)^a}{|1 - t\langle z, w\rangle|^{n+a+1}} d\nu(w)\, dt\right)^p\right)$$
$$= C\left(|h(0)|^p + \left(\int_B |\nabla h(w)|(1 - |w|^2)^a d\nu(w) \int_0^1 \frac{dt}{|1 - t\langle z, w\rangle|^{n+a+1}}\right)^p\right)$$
$$\le C\left(|h(0)|^p + \left(\int_B \frac{|\nabla h(w)|(1 - |w|^2)^a}{|1 - \langle z, w\rangle|^{n+a}} d\nu(w)\right)^p\right),$$

by Lemma 2.1.

Applying Lemma 2.3 to the function $F(w) = \big(|\nabla h(w)|\,|1-\langle z,w\rangle|^{-n-a}\big)^{p/2}$, $w\in B$ ($z\in B$-fixed) and replacing p,r,q by $2, 2/p, p(a+n+1)-n$ respectively and using Lemma 2.2 we find that

$$\begin{aligned}
&\int_B \frac{|\nabla h(w)|(1-|w|^2)^a}{|1-\langle z,w\rangle|^{n+a}}\,d\nu(w)\\
&\le C\int_B\bigg(\int_{E_r(w)}\frac{F(\xi)(1-|\xi|^2)^{p(a+n+1)-n-1}d\nu(\xi)}{|1-\langle w,\xi\rangle|^{p(a+n+1)}}\bigg)^{2/p}(1-|w|^2)^a d\nu(w)\\
&\le C\bigg(\int_B\bigg(\int_B\frac{F(\xi)(1-|\xi|^2)^{p(a+n+1)-n-1}}{|1-\langle w,\xi\rangle|^{p(a+n+1)}}d\nu(\xi)\bigg)^{2/p}(1-|w|^2)^a d\nu(w)\\
&\le C\bigg(\int_B\frac{|\nabla h(w)|^p(1-|w|^2)^{p(a+n+1)-n-1}}{|1-\langle z,w\rangle|^{p(n+a)}}d\nu(w)\bigg)^{1/p},
\end{aligned}$$

we may assume that $a > \frac{s}{p} - n$.
Thus, by using Fubini's theorem, Lemma 2.4 and Lemma 2.5 we obtain

$$\begin{aligned}
&\int_B(1-|z|^2)^s|h(z)|^p d\lambda(z)\le C\bigg[|h(0)|^p+\int_B(1-|z|^2)^{s-n-1}d\nu(z)\times\\
&\int_B\frac{|\nabla h(w)|^p(1-|w|^2)^{p(a+n+1)-n-1}}{|1-\langle z,w\rangle|^{p(n+a)}}d\nu(w)\bigg]=C\bigg[|h(0)|^p\\
&+\int_B|\nabla h(w)|^p(1-|w|^2)^{p(a+n+1)-n-1}d\nu(w)\int_B\frac{(1-|z|^2)^{s-n-1}d\nu(z)}{|1-\langle z,w\rangle|^{p(a+n)}}\bigg]\\
&\le C\bigg[|h(0)|^p+\int_B|\nabla h(w)|^p(1-|w|^2)^{s+p-n-1}d\nu(w)\bigg].
\end{aligned}$$

This finishes the proof of Theorem 1.4.

References

[1] M. Arsenović, M. Jevtić, *Area integral characterizations of $\mathcal{M}$-harmonic Hardy spaces on the unit ball*, Rocky Mountain J. (to appear)

[2] F. Beatorus, J. Burbea, *Holomorphic Sobolev spaces on the ball*, Dissertationes Math. **270** (1989), 1–57.

[3] M. Jevtić, *Embedding derivatives of $\mathcal{M}$-harmonic Hardy spaces into Lebesgue spaces*, Publications de l'Institut Mathmatique, **52(66)** (1992), 43–46.

[4] M. Jevtić, *On $\mathcal{M}$-harmonic space $\mathcal{B}_p^s$*, Publications de l'Institut Mathmatique Belgrade, (to appear).

[5] W. Rudin, *"Function theory in the unit ball of C^n"*, Springer-Verlag, New York, 1980.

[6] M. Stoll, *Invariant potential theory in the unit ball of C^n*, Cambridge University Press, Cambridge, 1994.

Matematički fakultet, Studentski trg 16, 11 000 Beograd, Serbia, Yugoslavia

FILOMAT (Niš) 9:1 (1995), 83–91

ITERATION METHOD FOR THE EQUATIONS OF I. N. VECUA TYPE OF HIGHER ORDER WITH ANALYTICAL COEFFICIENTS

Miloje Rajović

ABSTRACT. *We use two methods for the integration of Vecua equations: 1. the method of the areolar series, 2. the method of the iterations.*

We solve the following I. N. Vecua equation

$$\frac{\partial W}{\partial \overline{z}} = AW + B\overline{W} + F, \tag{1}$$

where $A(z,\overline{z})$, $B(z,\overline{z})$ and $F(z,\overline{z})$ are given analytical coefficients of two variables. We use two methods:

1° By the method of areolar series (used by B. Ilievski [1]).

$$A = \sum_{i,j=0}^{\infty} a_{ij} z^i \overline{z}^j, \quad B = \sum_{i,j=0}^{\infty} b_{ij} z^i \overline{z}^j, \quad F = \sum_{i,j=0}^{\infty} f_{ij} z^i \overline{z}^j, \tag{2}$$

$$W = \sum_{i,j=0}^{\infty} c_{ij} z^i \overline{z}^j, \tag{3}$$

we have a solution in the form of series of coefficients and of the integration element $\Phi(z)$ - an arbitrary analytic function in the role of integration "constant" :

Received 20.07.1995
1991 *Mathematics Subject Classification*: 35A20

$$
\begin{aligned}
W(z,\overline{z}) = \Phi &+ \int A\Phi d\overline{z} + \int A d\overline{z} + \int A d\overline{z} \int A\Phi d\overline{z} \\
&+ \int A d\overline{z} \int A d\overline{z} \int A\Phi d\overline{z} + \int B\overline{\Phi} dz + \int B d\overline{z} \int \overline{B\Phi} d\overline{z} \\
&+ \int B d\overline{z} \int \overline{B} dz \int \overline{B}\Phi dz + \cdots + \int A d\overline{z} \int B\overline{\Phi} d\overline{z} + \int B d\overline{z} \int \overline{A\Phi} dz \\
&+ \int A d\overline{z} \int A dz \int B\overline{\Phi} d\overline{z} + \int B d\overline{z} \int \overline{B} dz \int \overline{A\Phi} d\overline{z} + \cdots \\
&+ \int F d\overline{z} + \int A d\overline{z} \int F d\overline{z} + \int B d\overline{z} \int \overline{F} dz + \int A d\overline{z} \int A d\overline{z} \int F d\overline{z} \\
&+ \int B d\overline{z} \int \overline{B} dz \int F d\overline{z} + \cdots
\end{aligned} \tag{4}
$$

The series (4) is convergent in a finite closed domain G of the complex plane $z = x + iy$ and the coefficients are analytic functions of z and $\overline{z}$.

2° By the method of the Vecua integral equation (see [2,3]). With applications in the theory of iteration, we get the solution in the form

$$
\begin{aligned}
W(z,\overline{z}) &= \frac{1}{\pi} \iint_G \frac{A(y)W(y) + B(y)\overline{W}(y)}{\zeta - z} d\xi d\eta \\
&\quad + \frac{1}{\pi} \iint_G \frac{F(\zeta) d\xi d\eta}{\zeta - z} + \psi(z)
\end{aligned} \tag{5}
$$

or for $F = 0$,

$$
W = \Phi(z) e^{\omega(z)}, \tag{6}
$$

where Φ is an analytic C function of z, and

$$
\omega(z) = \frac{1}{\pi} \iint_G \left[A(\zeta) + B(\zeta) \frac{\overline{W(\zeta)}}{W(\zeta)} \right] \frac{d\xi d\eta}{\zeta - z}, \quad (\zeta = \xi + i\eta \in G). \tag{7}
$$

In what follows we consider the problem to apply a similar procedure for the Vecua equations with conjugations of higher order.

We start with the equation

$$
\frac{\partial^2 \omega}{\partial \overline{z}^2} + A(z) W = 0
$$

which is an ordinary areolar equation of the second order and is analogous to an ordinary differential equation

$$
y'' + \lambda y = 0
$$

with constant coefficients.

The equation

$$\frac{\partial^2 \omega}{\partial \overline{z}^2} + A(z, \overline{z})W = 0$$

is a complex analogy of the real equations of Hill, Lame and Mathie. Both of them are not Vecua equations of higher order.

The first equation which could be named a Vecua equation of higher order is

$$\frac{\partial^2 \omega}{\partial \overline{z}^2} + A(z, \overline{z})\overline{W} = 0 \tag{8}$$

and that contains the conjugation of a unknown function. Since

$$W = \rho e^{i\phi}, \qquad \overline{W} = We^{-2i\phi} = We^{-2i\,\mathrm{arctg}((W-\overline{W})/(i(W+\overline{W}))}$$

we have that equation (8) is not linear but trancendental, because the operation (the rotation of an argument while the modul remains the same) is such operation. Because of this, here we have an essential difference between areolar equations wich are almost completly analogie to ordinary differential equations and Vecua type equations, wich are specific in some sense.

Denote by $\hat{\int}$ the inverse operator of $\frac{\partial}{\partial \overline{z}}$. Then we have:

$$\frac{\partial}{\partial \overline{z}}\Big(\frac{\partial W}{\partial \overline{z}}\Big) = -A\overline{W}$$

$$\frac{\partial \omega}{\partial \overline{z}} = -\hat{\int} A\overline{W} = -\int A\overline{W}\,d\overline{z} + \Phi_2(z).$$

If we define the operator

$$T^1 W = \Phi_2(z) - \int A\overline{W}\,d\overline{z}, \tag{9}$$

then it is easy to prove that this is a contraction operator for every analytic coefficient $A(z, \overline{z})$.

Next, we have

$$\begin{aligned} W &= \hat{\int}\Big[-\hat{\int} A\overline{W}\Big] = \hat{\int}\Big[\Phi_2 - \int A\overline{W}\,d\overline{z}\Big] \\ &= \Phi_1(z) + \int \Phi_2(z)\,d\overline{z} - \int\Big[\Big[\int A\overline{W}\,d\overline{z}\Big]d\overline{z} \end{aligned} \tag{10}$$

It easy to check:

Theorem. *The operator*

$$T^2W = \Phi_1 + \Phi_2\overline{z} - \iint A(z,\overline{z})\overline{W}d\overline{z}^2 \tag{11}$$

is a contraction operator for every analytical choise $\Phi_1(z), \Phi_2(z), A(z.\overline{z})$. If we substitute W by (10) in (11) we can define the sequence $T^3W, \dots, T^nW$ and than prove that T^nW is a contraction operator.

Since, by the Cauchy theorem, the analytic equation has an analytic solution, the right side in T^nW is always continuous so the iteration method is valid for (8). Puting $W_1 = T^2W$ we have

$$\begin{aligned} W_2 &= \Phi_1 + \Phi_2\overline{z} - \iint A\left[\overline{\Phi_1 + \Phi_2\overline{z} - \iint A\overline{W}d\overline{z}^2}\right]d\overline{z}^2 \\ &= \Phi_1 + \Phi_2\overline{z} - \iint A\overline{\Phi}_1 d\overline{z} - \iint A\overline{\Phi}_2 z d\overline{z}^2 + \iint Ad\overline{z}^2 \iint \overline{A}W dz^2. \end{aligned}$$

The last member has a role a remainder. Its estimation is of the order $|A|^2|W||z|^5/5!$.
Next, we have

$$\begin{aligned} W_3 &= \Phi_1 + \Phi_2\overline{z} - \iint A\overline{\Phi}_1 d\overline{z} - \iint A\Phi_2 z d\overline{z}^2 \\ &+ \iint Ad\overline{z}^2 \iint \overline{A}\Phi_1 dz^2 + \iint Ad\overline{z}^2 \iint \overline{A}\Phi_2 dz^2 \\ &- \iint Ad\overline{z}^2 \iint \overline{A}dz^2 \iint A\overline{\Phi}_1 d\overline{z} - \iint Ad\overline{z}^2 \iint \overline{A}dz^2 \iint A\overline{\Phi}_2 d\overline{z}^2 + R_3, \end{aligned}$$

where the remainder R_3 is given by

$$R_3 = \iint Ad\overline{z}^2 \iint \overline{A}dz^2 \iint Ad\overline{z}^2 \iint \overline{A}W d\overline{z}^2.$$

In the next step we give th 4^{th} approximation

(12)

$$
\begin{aligned}
W_4 \simeq W(z,\overline{z}) &= \Phi_1(z) + \Phi_2(z)\overline{z} - \iint A\overline{\Phi}_1 d\overline{z}^2 - \iint A\overline{\Phi}_2 z d\overline{z}^2 \\
&+ \iint A d\overline{z}^2 \iint \overline{A}\Phi_1 dz + \iint A d\overline{z}^2 \iint \overline{A}\Phi_2 \overline{z} dz^2 \\
&- \iint A dz^2 \iint \overline{A} d\overline{z}^2 \iint A\overline{\Phi}_1 d\overline{z}^2 - \iint A d\overline{z}^2 \iint \overline{A} dz^2 \iint A\overline{\Phi}_2 z\overline{z} dz^2 \\
&+ \iint A d\overline{z}^2 \iint \overline{A} dz^2 \iint A d\overline{z}^2 \iint \overline{A}\Phi_1 dz^2 \\
&+ \iint A d\overline{z}^2 \iint \overline{A} dz^2 \iint A d\overline{z}^2 \iint \overline{A}\Phi_2 \overline{z} dz^2 \\
&- \iint A d\overline{z}^2 \iint \overline{A} dz^2 \iint A d\overline{z}^2 \iint \overline{A} dz^2 \iint A\overline{\Phi}_1 d\overline{z}^2 \\
&- \iint A dz^2 \iint \overline{A} dz^2 \iint A d\overline{z}^2 \iint \overline{A} dz^2 \iint A\overline{\Phi}_2 z d\overline{z}^2 \\
&+ \iint A d\overline{z}^2 \iint \overline{A} dz^2 \iint A d\overline{z}^2 \iint \overline{A} dz^2 \iint A d\overline{z}^2 \iint \overline{A}\Phi_1 d\overline{z}^2 \\
&+ \iint A d\overline{z}^2 \iint \overline{A} dz^2 \iint A d\overline{z}^2 \iint \overline{A} dz^2 \iint A d\overline{z}^2 \iint \overline{A}\Phi_2 \overline{z} dz^2 \\
&- \iint A d\overline{z}^2 \iint \overline{A} dz^2 \iint A d\overline{z}^2 \iint \overline{A} dz^2 \iint A d\overline{z}^2 \iint \overline{A} d\overline{z}^2 \iint \overline{A}\Phi_1 d\overline{z}^2 \\
&- \iint A d\overline{z}^2 \iint \overline{A} dz^2 \iint A d\overline{z}^2 \iint \overline{A} dz^2 \iint A d\overline{z}^2 \iint \overline{A} dz^2 \iint A\Phi_2 z dz^2 + R_4,
\end{aligned}
$$

where the remainder R_4 is

$$
\begin{aligned}
R_4 = &\iint A d\overline{z}^2 \iint \overline{A} dz^2 \iint A d\overline{z}^2 \iint \overline{A} d\overline{z}^2 \iint A d\overline{z}^2 \iint \overline{A} dz^2 \iint A d\overline{z}^2 \\
&\iint \overline{A} W(z,\overline{z}) dz^2
\end{aligned}
$$

and can be easily estimated. We may be satisfied by the second, third and if it is necessary by the fourth approximation.

Consequences. Definitions of new special functions. Conjugate exponent of the Vecua equation

$$\frac{\partial W}{\partial \overline{z}} + 1 \cdot \overline{W} = 0$$

defines a new exponent ${}^{\overline{z}}e$, as it was showed by Vecua. The Polozij [4] has tried to introduce p-exponential function denoted by ${}^{p}e^{z}$. A similar exponent can be defined by the equation

$$\frac{\partial W}{\partial \overline{z}} + A(z)\overline{W} = 0,$$

where $A(z)$ is an analytic function of z. We will denote this exponent by $\overset{\overline{z}}{_A}e^{z}$, the equation

$$\frac{\partial W}{\partial \overline{z}} + A(z,\overline{z})\overline{W} = 0$$

defines some other kinds of functions.

Conjugate areolar cosinus and sinus. If in (8) we take $A(z,\overline{z}) = 1$, then the equation

$$\frac{\partial^2 W}{\partial \overline{z}^2} + \overline{W} = 0$$

in a natural way defines $\sin\overline{z}$ and $\cos\overline{z}$. For $\Phi_1 \equiv 1$, $\Phi_2 \equiv 0$ we have by the definition:

$$\cos_A \overline{z} \equiv 1 - \frac{\overline{z}^2}{2!} + \frac{\overline{z}^2}{2!}\frac{z^2}{2!} - \cdots$$

Similarly, for $\Phi_1 \equiv 0$, $\Phi_2 \equiv 1$ we have

$$\sin_A \overline{z} \equiv z - z\frac{\overline{z}^2}{2!} + \cdots$$

Conjugate hyperbolic functions. If $A(z,\overline{z}) = 1$, $\Phi_1 \equiv 1$, $\Phi_2 \equiv 0$, we have

$$\mathrm{ch}_A \overline{z} \equiv 1 + \frac{\overline{z}^2}{2!} + \frac{\overline{z}^2}{2!}\frac{z^2}{2!} + \cdots$$

and for $A(z,\overline{z}) = 1$, $\Phi_1 \equiv 0$, $\Phi_2 \equiv 1$, we have

$$\mathrm{sh}_A \overline{z} \equiv \overline{z} + z\frac{z^3}{3!} + \frac{\overline{z}^2}{2!}\frac{\overline{z}^3}{3!} + \cdots$$

This suggests the following definition (loke the Euler - formula)

$$\overset{\overline{z}}{_A}e = \mathrm{ch}_A \overline{z} - \mathrm{sh}_A \overline{z}.$$

Conjugate Bessel's cylindrical and other functions. If we take $A(z,\overline{z}) = \overline{z}$ then by using the equation

$$\frac{\partial^2 W}{\partial \overline{z}^2} + \overline{z}\overline{W} = 0$$

and its solution (12) we can introduce variations of Bessel's functions. We also can make this if we choose $A(z,\overline{z}) \equiv z$, where now A has a role a constant with respect to operators $\frac{\partial}{\partial \overline{z}}$ and $\int d\overline{z}$: $\frac{\partial^2 W}{\partial \overline{z}^2} + z\overline{W} = 0$.
In this way we will have Bessel's functions and the corresponding functions of different classes and of different categories of transcendentality.
The same we can do for the general "constant" coefficient $A(z)$:

$$\frac{\partial^2 W}{\partial \overline{z}^2} + A(z)\overline{W} = 0$$

and hence we can have a new complex trigonometry.

If this way, the equation

$$\frac{\partial^2 W}{\partial \overline{z}^2} + B(\overline{z})A(z)\overline{W} = 0$$

can be regarded as a generatrise of some conjugate functions of Hille, Lame and Mathie of two complex variables z and $\overline{z}$.

So we can extend the spaces of elementary transcendental functions in the complex plane.

The Vecua equation of the second order with analytic coefficients. Consider the equation

$$\frac{\partial^2 W}{\partial \overline{z}^2} = A(z,\overline{z})W + B(z,\overline{z})\overline{W} + F(z,\overline{z}) \tag{13}$$

with analytic coefficients

$$A = \sum_{i,j=0}^{\infty} a_{ij} z^i \overline{z}^j, \quad B = \sum_{i,j=0}^{\infty} b_{ij} z^i \overline{z}^j, \quad F = \sum_{i,j=0}^{\infty} f_{ij} z^i \overline{z}^j.$$

By the method of areolar series

$$W(z,\overline{z}) = \sum_{i,j=0}^{\infty} c_{ij} z^i \overline{z}^j$$

or by the method of analytic change (in fact, by iterations) one get approximations of an analytic solution.

From

$$\frac{\partial W}{\partial \overline{z}} = \hat{\int} (AW + B\overline{W} + F)$$

one obtains the first integral

$$\frac{\partial W}{\partial \overline{z}} = \int (AW + B\overline{W} + F) d\overline{z}^2 + \Phi_1(z),$$

and also the second integral

$$W = \iint (AW + B\overline{W} + F) d\overline{z}^2 + \Phi_1(z)\overline{z} + \Phi_2(z);$$

using now iteration method we have a solution

$$\begin{aligned}
W &= \Phi_2 + \iint A\Phi_2 d\overline{z}^2 + \iint B\overline{\Phi}_2 d\overline{z}^2 \\
&+ \iint A d\overline{z}^2 \iint A\Phi_2 d\overline{z}^2 + \iint A d\overline{z}^2 + \iint B\overline{\Phi}_2 d\overline{z}^2 \\
&+ \iint B d\overline{z}^2 \iint \overline{A\Phi}_2 dz^2 + \iint B d\overline{z}^2 \iint \overline{B}\Phi_2 dz^2 \\
&+ \Phi_1 \overline{z} + \iint \overline{z} A\Phi_1 d\overline{z}^2 + \iint z B\overline{\Phi}_1 d\overline{z}^2 \\
&+ \iint A d\overline{z}^2 \iint \overline{z} A\Phi_1 d\overline{z}^2 + \iint A d\overline{z}^2 \iint z B\Phi_1 d\overline{z}^2 \qquad (14)\\
&+ \iint B d\overline{z}^2 \iint z\overline{A\Phi}_1 dz^2 + \iint B d\overline{z}^2 \iint \overline{B}\Phi_1 \overline{z} dz^2 \\
&+ \iint F d\overline{z}^2 \iint A d\overline{z}^2 \iint F d\overline{z}^2 + \iint B d\overline{z}^2 \iint \overline{F} dz^2 \\
&+ \iint A d\overline{z}^2 \iint A d\overline{z}^2 \iint F d\overline{z}^2 + \iint A d\overline{z}^2 \iint B d\overline{z}^2 \iint \overline{F} dz^2 \\
&+ \iint B d\overline{z}^2 \iint \overline{A} dz^2 \iint \overline{F} dz^2 + \iint B d\overline{z}^2 \iint \overline{B} dz^2 + \iint F d\overline{z}^2 + R_2,
\end{aligned}$$

where the remainder R_2 has a similar form as R_4 for the solution (12).

Possibilities. Without diffculties this method can be extended to any linear (pseudolinear) equation of Vecua type of the order n with the conjugation of function

$$F\left(z, \overline{z}, W(z,\overline{z}), \overline{W(z,\overline{z})}, \frac{\partial W}{\partial \overline{z}}, \frac{\partial^2 W}{\partial \overline{z}^2}, \ldots, \frac{\partial^n W}{\partial \overline{z}^n}\right) = 0$$

which has anallytic coefficients.

The consequences are general solutions in the form of series of coefficients of systems of real partial equations.

REFERENCES

[1] B. Ilievski, *Certaines solutions analytiques d'une classe des equations Vecua*, Bull. Mathematique de la SMI de la Republique Macedonie, Skopie **14** (1990), 79–86.
[2] I. N. Vecua, *Obopsćenie analitičeskie funkcii*, Moskva, 1959.
[3] ______, *Sistemi differenncialnih uravnenii eliptičeskogo tipa i graničnie zadači e primenenie v terii oboloček*, Mat. Sbornik 31 **73** (1959).
[4] G. N. Polozij, *p-analitičeskie i (p,q) analitičeskie funkcii*, Kiev, 1969.

Faculty of Mechanical Engineering, 36000 Kraljevo, Yugoslavia

FILOMAT (Niš) 9:1 (1995), 93–99

INEQUALITIES FOR COEFFICIENTS OF ALGEBRAIC POLYNOMIALS

I. Ž. Milovanović, L. Z. Rančić and R. Ž. Đorđević

ABSTRACT. *Let $\mathcal{P}_n$ be the class of algebraic polynomials $P(x) = \sum_{\nu=0}^{n} a_\nu x^\nu$ of degree at most n and $\|P\|_{d\sigma} = (\int_{\mathbb{R}} |P(x)|^2 d\sigma(x))^{1/2}$, where $d\sigma(x)$ is a nonnegative measure on $\mathbb{R}$. We consider the best constant in the inequality $|a_\nu| \le C_{n,\nu}(d\sigma)\|P\|_{d\sigma}$, when $P \in \mathcal{P}_n$ and such that $P(\xi_k) = 0$ $(k = 1, 2, \ldots, m)$. The cases $C_{n,n}(d\sigma)$ and $C_{n,n-1}(d\sigma)$ were studied by Milovanović and Guessab [2] and for an arbitrary ν by Milovanović and Rančić [5], where they gave explicit expressions for some classical measures. In this paper we determine the best constants $C_{n,\nu}$ for the generalized Gegenbauer measure on $(-1, 1)$ and for the generalized Hermite measure on $(-\infty, +\infty)$.*

1. Introduction

Let $\mathcal{P}_n$ be the class of algebraic polynimials $P(x) = \sum_{\nu=0}^{n} a_\nu x^\nu$ of degree of most n. Denote by

$$\|P\|_{d\sigma} = \sqrt{(P, P)} = \Big(\int_{\mathbb{R}} |P(x)|^2 d\sigma(x)\Big)^{1/2} \tag{1.1}$$

the norm of a polynomial $P \in \mathcal{P}_n$, where $d\sigma(x)$ is a given nonnegative measure on the real line $\mathbb{R}$, with compact or infinite support, for which all moments $\mu_k = \int_{\mathbb{R}} x^k \, d\sigma(x)$, $k = 0, 1, \ldots$, exist and are finite, and $\mu_0 > 0$. We will consider the problem of determining the best possible constants $C_{n,\nu}(d\sigma)$ such that the following inequalities

$$|a_\nu| \le C_{n,\nu}(d\sigma) \, \|P\|_{d\sigma} \qquad (0 \le \nu \le n), \tag{1.2}$$

are valid.

Polynomials in (1.2) belong to the restrictive class of polynomials

$$\mathcal{P}_n(\xi_1, \xi_2, \ldots, \xi_n) = \{P \in \mathcal{P}_n \mid P(\xi_k) = 0, \ \xi_k \in \mathbb{C}, \ k = 1, 2, \ldots, m\}.$$

Received 22.07.1995
1991 *Mathematics Subject Classification*: Primary 26C05, 26D05, 33C45, 91A44
Supported by Grant 0401A of RFNS through Math. Inst. SANU

Let

$$\prod_{i=1}^{m}(x-\xi_i) \equiv x^m - s_1 x^{m-1} + \cdots + (-1)^{m-1} s_{m-1} x + (-1)^m s_m,$$

where s_k denotes the elementary symmetric functions of $\xi_1, \xi_2, \ldots, \xi_m$, i.e.,

$$s_k = \sum \xi_1 \xi_2 \cdots \xi_k \qquad (k = 1, 2, \ldots, m). \tag{1.3}$$

For $k = 0$ we have $s_0 = 1$ and $s_k = 0$ for $k > m$.

For the measure $d\sigma$ there exists an unique set of orthonormal polynomials $\pi_n(\cdot) = \pi_n(\cdot\,; d\sigma)$, $n = 0, 1, \ldots$, defined by

$$\pi_n(x) = b_n^{(n)}(d\sigma)x^n + b_{n-1}^{(n)}(d\sigma)x^{n-1} + \cdots + b_0^{(n)}(d\sigma) \qquad (b_n^{(n)}(d\sigma) > 0)$$

and

$$(\pi_n, \pi_m) = \delta_{nm} \qquad (n, m \geq 0),$$

where

$$(f, g) = \int_{\mathbb{R}} f(x)\overline{g(x)}\, d\sigma(x) \qquad (f, g \in L^2(\mathbb{R})).$$

Denote by $d\hat{\sigma}(t)$ the weight

$$d\hat{\sigma}(x) = \prod_{k=1}^{m} |x - \xi_k|^2\, d\sigma(x). \tag{1.4}$$

The problem (1.2) was considered in [2] by Milovanović and Guessab (see also [4]). They have proved the following result:

Theorem A. *If $P \in \mathcal{P}_n(\xi_1, \xi_2, \ldots, \xi_m)$ and $\hat{b}_\nu^{(n)} = b_\nu^{(n)}(d\hat{\sigma})$, then inequalities*

$$|a_n| \leq \hat{b}_{n-m}^{(n-m)} \, \|P\|_{d\sigma} \tag{1.5}$$

and

$$|a_{n-1}| \leq \left(\left(\hat{b}_{n-m-1}^{(n-m)} - s_1\, \hat{b}_{n-m}^{(n-m)} \right)^2 + \left(\hat{b}_{n-m-1}^{(n-m-1)} \right)^2 \right)^{1/2} \|P\|_{d\sigma} \tag{1.6}$$

hold.

Inequality in (1.5) *and* (1.6) *are attained if and only if* $P(x)$ *is a constant multiple of*

$$\hat{\pi}_{n-m}(x\,;\,d\hat{\sigma})\prod_{k=1}^{m}|x-\xi_k|$$

and

$$\left(\left(\hat{b}_{n-m-1}^{(n-m)}-s_1\,\hat{b}_{n-m}^{(n-m)}\right)\hat{\pi}_{n-m}(x)+\hat{b}_{n-m-1}^{(n-m-1)}\,\hat{\pi}_{n-m-1}(x)\right)\prod_{k=1}^{m}|x-\xi_k|,$$

respectively.

Milovanović and Rančić in [5] considered the corresponding problem for arbitrary k and proved the following inequality

$$|a_{n-k}|\le\left(\sum_{j=0}^{k}\left(\sum_{i=j}^{k}(-1)^{k-i}s_{k-i}\,\hat{b}_{n-m-i}^{(n-m-j)}\right)^2\right)^{1/2}\|P\|_{d\sigma},\tag{1.7}$$

with extremal polynomial

$$\left(\sum_{j=0}^{k}\hat{\pi}_{n-m-j}(x)\sum_{i=j}^{k}(-1)^{k-i}s_{k-i}\,\hat{b}_{n-m-i}^{(n-m-j)}\right)\prod_{k=1}^{m}(x-\xi_k).$$

In the above mentioned papers the authors have determined explicit constants $C_{n,\nu}(d\sigma)$ for some weights corresponding to the classical orthogonal polynomials. In this paper we are going to determine explicit constants $C_{n,\nu}(d\sigma)$ for weights that correspond to the generalized Gegenbauer and the generalized Hermite polynomials. This is significant because of the growing importance of these polynomials in many applications, particulary in numerical approximation (see for example [3]).

2. The Generalized Gegenbauer Case

At first we observe the generalized Gegenbauer case

$$d\sigma(x)=|x|^{\mu}(1-x^2)^{\alpha}dx\qquad(\mu,\alpha>-1).$$

Let $\beta=(\mu-1)/2$ and let $\{W_n^{(\alpha,\beta)}(x)\}$ be a sequence of the generalized Gegenbauer monic polynomials orthogonal with respect to the measure

$d\sigma(x)$ on $(-1,1)$ (which was introduced by Lascenov in [1]). For such polynomials we have the following recurrence relation

$$W_{n+1}^{(\alpha,\beta)}(x) = xW_n^{(\alpha,\beta)} - \lambda_n W_{n-1}^{(\alpha,\beta)}(x), \qquad n = 0,1,\dots, \tag{2.1}$$

with $W_{-1}^{(\alpha,\beta)}(x) = 0$ and $W_0^{(\alpha,\beta)}(x) = 1$, where

$$\lambda_{2n} = \frac{n(n+\alpha)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}$$

and

$$\lambda_{2n+1} = \frac{(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)},$$

for $n = 0,1,\dots,$ except when $\alpha+\beta=-1$; then $\lambda_1 = (\beta+1)(\alpha+\beta+2)$.

Using the norm of $W_n^{(\alpha,\beta)}(x)$,

$$\|W_{2n}^{(\alpha,\beta)}\|^2 = \frac{n!}{(n+\alpha+\beta+1)_n} B(n+\alpha+1, n+\alpha+1),$$

$$\|W_{2n+1}^{(\alpha,\beta)}\|^2 = \frac{n!}{(n+\alpha+\beta+2)_n} B(n+\alpha+1, n+\beta+2),$$

where B is the beta function, we can obtain the loading coefficients

$$b_n^{(n;\,\alpha,\beta)} = b_n^{(n)}(d\sigma)$$

in the corresponding orthonormal polynomials $\hat{W}_n^{(\alpha,\beta)}(x)$

$$b_n^{(n;\,\alpha,\beta)} = \begin{cases} \left(\dfrac{(k+\alpha+\beta+1)_k}{k!B(k+\alpha+1,k+\beta+1)}\right)^{1/2} & (n=2k), \\ \left(\dfrac{(k+\alpha+\beta+2)_k}{k!B(k+\alpha+1,k+\beta+2)}\right)^{1/2} & (n=2k+1). \end{cases}$$

Using Theorem A and (1.7) we obtain:

Theorem 2.1. *Under restriction* $P^{(i)}(0) = 0$ $(i = 0,1,\dots,k-1)$ *and* $P^{(i)}(-1) = P^{(i)}(1) = 0$ $(i = 0,1,\dots,s-1)$, *where* $s = (m-k)/2 \in \mathbb{N}$ *for* $l = 0$ *and* $l = 1$, *we have that*

$$|a_{n-l}| \le b_{n-l-m}^{(n-l-m;\, m-k+\alpha,\, \beta+k)} \|P\|_{d\sigma}.$$

The equality is attained if and only if

$$P(x) = Ax^k(x^2-1)^s\,\hat{W}_{n-l-m}^{(m-k+\alpha,\,\beta+k)}(x) \qquad (A = \text{const}).$$

Proof. Since restrictions on polynomials are given only in the points $\xi_1 = \xi_2 = \cdots = \xi_k = 0$, $\xi_{k+1} = \xi_{k+2} = \cdots = \xi_s = -1$ and $\xi_{s+1} = \xi_{s+2} = \cdots = \xi_m = 1$, the new measure $d\hat{\sigma}(x)$ is again the generalized Gegenbauer measure

$$d\hat{\sigma}(x) = |x|^{2k+\mu}(1-x^2)^{m-k+\alpha}dx \qquad (\mu+2k,\ m-k+\alpha > -1).$$

Since the weight function is even, then according to (2.1) it is not difficult to prove that $b_\nu^{(n;\,\alpha,\beta)} = 0$ when $n-\nu = 2r+1$ ($r \in \mathbb{N}$, $\nu = 0,1,\dots,n$). The required result can be directly obtained from Theorem A and (1.7), where, in our case, $s_1 = 0$. □

3. The Generalized Hermite Case

Consider now the generalized Hermite measure $d\sigma(x) = |x|^{2k}e^{-x^2}dx$ ($k > -1/2$) on $(-\infty, +\infty)$. With $H_n^{(k)}(x)$ we denote the generalized Hermite monic polynomial. For such polynomials the following differential equation is satisfied

$$xy'' + 2(k-x^2)y' + (2nx - \varepsilon x^{-1})y = 0, \tag{3.1}$$

where $\varepsilon = 0$, for n is even, and $\varepsilon = 2k$, for n is odd.

Using (3.1) we can obtain:

1° If n is even then

$$H_n^{(k)}(x) = (-1)^{\frac{n}{2}}\Gamma\Big(k+\frac{n}{2}+\frac{1}{2}\Big)\sum_{\nu=0}^{n/2}(-1)^\nu\binom{n/2}{\nu}\frac{x^{2\nu}}{\Gamma\Big(k+\nu+\dfrac{1}{2}\Big)}$$

and

$$\|H_n^{(k)}\|^2 = \Big(\frac{n}{2}\Big)!\,\Gamma\Big(k+\frac{n}{2}+\frac{1}{2}\Big).$$

2° If n is odd, then

$$H_n^{(k)}(x) = (-1)^{\frac{n-1}{2}}\Gamma\Big(k+\frac{n-1}{2}+\frac{3}{2}\Big)\sum_{\nu=0}^{(n-1)/2}(-1)^\nu\binom{\frac{n-1}{2}}{\nu}\frac{x^{2\nu+1}}{\Gamma\Big(k+\nu+\dfrac{3}{2}\Big)}$$

and

$$\|H_n^{(k)}\|^2 = \Big(\frac{n-1}{2}\Big)!\,\Gamma\Big(k + \frac{n-1}{2} + \frac{3}{2}\Big).$$

The coefficients $\hat{b}_\nu^{(n)}(d\sigma)$ $(\nu = 0, 1, \dots, n)$ in the coresponding orthonormal polynomials $\hat{H}_n^{(k)}(x)$ are given by:

1° If n is even then

$$\hat{b}_\nu^{(n)}(d\sigma) = (-1)^{n/2+\nu}\, \frac{\binom{n/2}{\nu/2}\sqrt{\Gamma\Big(k + \frac{n}{2} + \frac{1}{2}\Big)}}{\Gamma\Big(k + \frac{\nu}{2} + \frac{1}{2}\Big)\sqrt{\Big(\frac{n}{2}\Big)!}}$$

if ν is even, and $\hat{b}_\nu^{(n)}(d\sigma) = 0$ if ν is odd.

2° If n is odd then

$$\hat{b}_\nu^{(n)}(d\sigma) = (-1)^{(n-1)/2+\nu}\, \frac{\binom{(n-1)/2}{(\nu-1)/2}\sqrt{\Gamma\Big(k + \frac{n}{2} + 1\Big)}}{\Gamma\Big(k + \frac{\nu}{2} + 1\Big)\sqrt{\Big(\frac{n-1}{2}\Big)!}}$$

if ν is odd, and $\hat{b}_\nu^{(n)}(d\sigma) = 0$ if ν is even.

Similarly, as in the previous section, we can prove the following theorem:

Theorem 3.1. *Let $P^{(i)}(0) = 0$ $(i = 0, 1, \dots, m-1)$ and let the measure $d\hat{\sigma}(x) = x^{2m} d\sigma(x)$ and the norm $\|P\|_{d\sigma}$ be given by* (1.1). *Then*

$$|a_{n-l}| \le \sqrt{A_{n,l}}\, \|P\|_{d\sigma} \qquad (l = 0, 1, \dots, n), \tag{3.2}$$

where $A_{n,l} = 0$ for $n - l - m < 0$, and otherwise

$$A_{n,l} = \frac{1}{K_1} \sum_{j=0}^{[l/2]} \binom{k + \frac{n-l+m}{2} - \frac{1}{2} + j}{j} \binom{\frac{n-l-m}{2} + j}{j} \qquad (n - l - m \text{ is even}),$$

where $K_1 = \Big(\frac{n-l-m}{2}\Big)!\,\Gamma\Big(k + \frac{n-l+m}{2} + \frac{1}{2}\Big)$, *and*

$$A_{n,l} = \frac{1}{K_2} \sum_{j=0}^{[l/2]} \binom{k + \frac{n-l+m}{2} + j}{j} \binom{\frac{n-l-m-1}{2} + j}{j} \qquad (n - l - m \text{ is odd}),$$

where $K_2 = \left(\frac{n-l-m-1}{2}\right)!\,\Gamma\left(k+\frac{n-l+m}{2}+1\right)$.

The inequality (3.2) *reduces to an equality if and only if*

$$P(x) = Ax^m \sum_{j=0}^{[l/2]} \hat{b}_{n-l-m}^{(n-l-m+2j)} \hat{H}_{n-l-m+2j}^{(k+m)} \qquad (A = \text{const}).$$

References

[1] R. V. Lascenov, *On a class of orthogonal polynomials*, Učen. Zap. Leningrad. Gos. Ped. Inst. **89**(1953), 191–206 (Russian).

[2] G. V. Milovanović and A. Guessab, *An estimate for Coefficients of Polynomials in L^2 norm*, Proc. Amer. Math. Soc. **120**(1994), 165–171.

[3] G. V. Milovanović and M. A. Kovačević, *Least squares approximation with constraint: Generalized Gegenbauer case*, Facta Univ. Ser. Math. Inform. 1(1986), 73–81.

[4] G. V. Milovanović, D. S. Mitrinović and Th. M. Rassias, *Topics in Polynomials: Extremal Problems, Inequalities, Zeros*, World Scientific, Singapore-New Jersey-London-Hong Kong, 1994.

[5] G. V. Milovanović and L. Z. Rančić, *An estimate for coefficients of polynomials in L^2 norm. II*, (to appear).

University of Niš, Faculty of Electronic Engineering, Departement of Mathematics, P. O. Box 73, 18000 Niš, Yugoslavia

FILOMAT (Niš) 9:1 (1995), 101–111

ON PROPERTIES OF SOME NONCLASSICAL ORTHOGONAL POLYNOMIALS

G. V. Milovanović, P. M. Rajković and Z. M. Marjanović

ABSTRACT. *In this paper we consider some sequences of nonclassical orthogonal polynomials which were studied in the papers* [3], [4] *and* [6]. *We find some new relations which they satisfy and discuss their zeros.*

1. Polynomials of the Laguerre type

We consider the generalized Laguerre functional

$$\boldsymbol{I}^{(s)}(P) = \int_0^{+\infty} P(x)x^s e^{-x}\,dx, \quad s \in \mathbb{N}_0, \quad \boldsymbol{I}^{(0)} = \boldsymbol{I},$$

and the monic generalized Laguerre polynomials $\{\widehat{L}_n^{(s)}(x)\}$, which satisfy the following three-term recurrence relation

$$\begin{aligned}(1.1)\qquad &\widehat{L}_{n+1}^{(s)}(x) = (x - 2n - s - 1)\widehat{L}_n^{(s)}(x) - n(n+s)\widehat{L}_{n-1}^{(s)}(x),\\ &\widehat{L}_{-1}^{(s)}(x) = 0, \quad \widehat{L}_0^{(s)}(x) = 1.\end{aligned}$$

These polynomials can be expressed in the form

$$(1.2)\qquad \widehat{L}_n^{(s)}(x) = \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k}(k+s+1)_{n-k}x^k,$$

where

$$(m)_s = m(m+1)\cdots(m+s-1), \qquad (m)_0 = 1.$$

We introduce the functional $\boldsymbol{\Delta}_a$ by

$$\boldsymbol{\Delta}_a(f) = f(a),$$

Received 22.07.1995
1991 *Mathematics Subject Classification*: 33C45
Supported by Grant 0401A of RFNS through Math. Inst. SANU

and define the functional I_1 by

$$\boldsymbol{I}_1 = \boldsymbol{I} + c\boldsymbol{\Delta}_0, \quad c \in \mathbb{R}.$$

By $\{L_n(x;c)\}$ we denote the corresponding sequence of orthogonal polynomials of the *Laguerre type*. Such polynomials were expressed as a linear combination of $L_n^{(0)}(x)$ and $xL_n^{(2)}(x)$ (see [6]). In the same paper, it was proved that the n-th polynomial of this sequence can be expressed in the form

$$L_n(x;c) = (-1)^n n! \sum_{k=0}^{n} (-1)^k \frac{1 + k(1 + c(n+1))}{(k+1)!} \binom{n}{k} x^k. \tag{1.3}$$

In this section we prove some properties of these polynomials.

Denoting the monic polynomials of the Laguerre type by $\widehat{L}_n(x;c)$, we yield

$$L_n(x;c) = (1 + nc)\widehat{L}_n(x;c), \quad c \neq -\frac{1}{n}, \quad n \in \mathbb{N}. \tag{1.3'}$$

From (1.3) we see that

$$\deg L_n(x;-1/n) = n - 1 \qquad \text{and} \qquad L_n(x;-1/n) = -nL_{n-1}(x;-1/n).$$

Hence, for each $n \in \mathbb{N}$, the sequences $\{\widehat{L}_\nu(x;-1/n)\}_{\nu=0}^{\infty}$ are *quasi-orthogonal* of the order one with respect to $\boldsymbol{I}_1$, with $c = -1/n$, and the next reccurence relations are not valid for $L_n(x;-1/n)$ and its neighbours.

Theorem 1.1. *The polynomial $\widehat{L}_n(x;c)$ can be expressed in the form*

$$\widehat{L}_n(x;c) = \widehat{L}_n^{(1)}(x) + \lambda_n \widehat{L}_{n-1}^{(1)}(x), \tag{1.4}$$

where

$$\lambda_n = n\,\frac{1 + (n+1)c}{1 + nc},$$

i.e., $\{\widehat{L}_n(x;c)\}$ is quasi-orthogonal of the order one with respect to the functional $\boldsymbol{I}^{(1)}$.

Proof. Suppose that λ_n exists such that (1.4) is true. Using (1.2) and (1.3)–(1.3′), for the coefficient of the term x^k we yield

$$\frac{1 + k(1 + c(n+1))}{1 + nc} \cdot \frac{n}{n-k} = \frac{n(n+1)}{n-k} - \lambda_n.$$

After some computation we find λ_n which does not depend on k. □

Using the previous relations we can prove the three-term recurrence relation for polynomials $\widehat{L}_n(x;c)$:

Theorem 1.2. *The sequence* $\{\widehat{L}_n(x;c)\}$ *satisfies the three-term recurrence relation*

$$\widehat{L}_{n+1}(x;c) = (x-\alpha_n)\widehat{L}_n(x;c) - \beta_n \widehat{L}_{n-1}(x;c),$$

where

$$\alpha_n = \frac{2n+1+4n(n+1)c+n(n+1)(2n+1)c^2}{(1+nc)(1+(n+1)c))},$$
$$\beta_n = n^2\,\frac{(1+(n-1)c)(1+(n+1)c)}{(1+nc)^2}.$$

The norm of $\widehat{L}_n(x;c)$ is given by

$$\|\widehat{L}_n(x;c)\|^2 = \beta_0\beta_1\cdots\beta_n = (n!)^2\,\frac{1+(n+1)c}{1+nc}\,.$$

Theorem 1.3. *The zeros of* $\widehat{L}_n(x;c)$ *are real, simple and positive, except for* $c<-1/n$ *when one of them is negative. Denoting these zeros by*

$$\zeta_{n1} < \zeta_{n2} < \cdots < \zeta_{nn},$$

we have, for $c < -2/(n+1)$, *that the lowest zero* ζ_{n1} *satisfies*

$$-\frac{\zeta_{nn}}{n-1} < \zeta_{n1} < 0 \quad \wedge \quad |\zeta_{n1}| < |\zeta_{n2}|. \tag{1.5}$$

Proof. Let ζ_{nj}, $j=k,\ldots,n$, be positive zeros of odd multiplicity. Defining $q(x) = \prod_{j=k}^{n}(x-\zeta_{nj})$, we see that the polynomial $xq(x)\widehat{L}_n(x;c)$ does not change sign for $x>0$. Hence

$$I_1(xq(x)\widehat{L}_n(x;c)) = \int_0^{+\infty} xq(x)\widehat{L}_n(x;c)e^{-x}\,dx \neq 0.$$

Since $\{\widehat{L}_n(x;c)\}$ is quasi-orthogonal of the order one, with respect to the functional $I^{(1)}$, it follows $\deg q(x) \geq n-1$.

From (1.3) we have that $\widehat{L}_n(0;c) = (-1)^n n!/(1+nc)$. Hence

$$\operatorname{sign}\widehat{L}_n(0;c) = \begin{cases} (-1)^n, & \text{for } c>-1/n,\\ (-1)^{n+1}, & \text{for } c<-1/n.\end{cases}$$

Since $\widehat{L}_n(x;c) = \prod_{i=1}^{n}(x-\zeta_{ni})$ and $\widehat{L}_n(0;c) = (-1)^n \prod_{i=1}^{n} \zeta_{ni}$, we conclude that all zeros are positive for $c > -1/n$, and only one of them is negative for $c < -1/n$. Also, differentiating (1.3) with respect to x, we find

$$\widehat{L}_n'(0;c) = (-1)^{n+1}\frac{n!n}{2}\cdot\frac{2+c(n+1)}{1+nc}.$$

Thus, for $c < -2/(n=1)$, it is $\widehat{L}_n'(0;c)/\widehat{L}_n(0;c) > 0$. Because of

$$\frac{\widehat{L}_n'(x;c)}{\widehat{L}_n(x;c)} = \sum_{i=1}^{n}\frac{1}{x-\zeta_{ni}},$$

we have $\sum_{i=1}^{n}(-1/\zeta_{ni}) > 0$. Then, from $-1/\zeta_{n1} > \sum_{i=2}^{n}(1/\zeta_{ni})$, we yield

$$\frac{1}{-\zeta_{n1}} > \frac{n-1}{\zeta_{nn}} \qquad \text{and} \qquad \frac{1}{-\zeta_{n1}} > \frac{1}{\zeta_{n2}},$$

from which follows (1.5). $\square$

EXAMPLE 1.1. The polynomial $\widehat{L}_3(x;1) = x^3 - \frac{11}{4}x^2 + \frac{27}{2}x - \frac{3}{2}$, has positive zeros: $x_1 \approx 0.119747$, $x_2 \approx 2.06541$, $x_3 \approx 6.06483$, but the polynomial $\widehat{L}_3(x;-2) = x^3 - \frac{39}{5}x^2 + \frac{54}{5}x + \frac{6}{5}$, has one negative zero: $x_1 \approx -0.103301$, $x_2 \approx 1.95187$, $x_3 \approx 5.95143$.

Theorem 1.4. *The zeros of $\widehat{L}_n^{(1)}(x)$ and $\widehat{L}_{n-1}^{(1)}(x)$ interlace the zeros of the polynomial $\widehat{L}_n(x;c)$.*

Proof. Let $x_{m,i}^{(1)}$, $i = 1,\ldots,m$, be the zeros of $\widehat{L}_m^{(1)}(x)$. Then from (1.4), we have $\widehat{L}_n(x_{n,i}^{(1)};c) = \lambda_n\widehat{L}_{n-1}^{(1)}(x_{n,i}^{(1)})$ and $\widehat{L}_n(x_{n,i+1}^{(1)};c) = \lambda_n\widehat{L}_{n-1}^{(1)}(x_{n,i+1}^{(1)})$. Since between $x_{n,i}^{(1)}$ and $x_{n,i+1}^{(1)}$ there exists an unique zero of $\widehat{L}_{n-1}^{(1)}(x)$, we conclude that $\widehat{L}_{n-1}^{(1)}(x_{n,i}^{(1)})$ and $\widehat{L}_{n-1}^{(1)}(x_{n,i+1}^{(1)})$ must have the opposite signs. Hence, a zero of $\widehat{L}_n(x;c)$ exists in the interval $(x_{n,i}^{(1)}, x_{n,i+1}^{(1)})$ for $i = 1,\ldots,n-1$.

In the same way, we can conclude that $\widehat{L}_n(x;c)$ has a zero in the interval $(x_{n-1,i}^{(1)}, x_{n-1,i+1}^{(1)})$ $(i = 1,\ldots,n-2)$. $\square$

Remark. In [8] it was proved that $\widehat{L}_n(x)$ and $\widehat{L}_{n-1}(x)$ interlace the zeros of $\widehat{L}_n(x;c)$.

2. Polynomials of the Jacobi type

In this section we consider the functional

$$\boldsymbol{I}^{(\beta,\alpha)}(P)=\int_0^1 x^\beta(1-x)^\alpha P(x)\,dx,\qquad \beta,\alpha>-1,$$

and the monic Jacobi polynomials $\{\widehat{Q}_n^{(\beta,\alpha)}(x)\}$ orthogonal with respect to the functional $\boldsymbol{I}^{(\beta,\alpha)}$. Such polynomials can be expressed by the sum

$$\widehat{Q}_n^{(\beta,\alpha)}(x)=\frac{(-1)^n n!}{(n+\alpha+\beta+1)_n}\sum_{k=0}^{n}\frac{(-1)^k}{k!(n-k)!}(n+\alpha+\beta+1)_k\frac{(\beta+1)_n}{(\beta+1)_k}x^k,$$

and they satisfy the three-term recurrence relation

$$\widehat{Q}_{n+1}^{(\beta,\alpha)}(x)=(x-a_n)\widehat{Q}_n^{(\beta,\alpha)}(x)-b_n\widehat{Q}_{n-1}^{(\beta,\alpha)}(x),\ \ \widehat{Q}_{-1}^{(\beta,\alpha)}(x)=0,\ \ \widehat{Q}_0^{(\beta,\alpha)}(x)=1,$$

where

$$a_n=\gamma_{n+1}-\gamma_n,\qquad \gamma_n=\frac{n(n+\beta)}{2n+\alpha+\beta},$$
$$b_n=\left\{\frac{(n-1)(\beta+n-1)}{2(2n+\alpha+\beta-1)}-a_n\right\}\gamma_n-\frac{n(\beta+n)}{2(2n+\alpha+\beta+1)}\gamma_{n+1}.$$

Let $\boldsymbol{I}_2=\boldsymbol{I}^{(0,\alpha)}+c\boldsymbol{\Delta}_0$, $c\in\mathbb{R}$, be a new functional and let $\{P_n^{(0,\alpha)}(x;c\,|\,0)\}$ be the corresponding monic orthogonal polynomials of the *Jacobi type*. In [6] it was proved that the n-th polynomial can be expressed by

$$P_n^{(0,\alpha)}(x;c\,|\,0)=(-1)^n n!\sum_{k=0}^{n}(-1)^k\frac{1+k(1+c(n+1)(n+\alpha))}{(k+1)!(n+\alpha+k+1)_{n-k}}\binom{n}{k}x^k.$$

Denoting the monic polynomials of the Jacobi type by $\{\widehat{P}_n^{(0,\alpha)}(x;c\,|\,0)\}$, we have

$$P_n^{(0,\alpha)}(x;c\,|\,0)=(1+n(n+\alpha)c)\widehat{P}_n^{(0,\alpha)}(x;c\,|\,0),\quad c\neq-\frac{1}{n(n+\alpha)},\quad n\in\mathbb{N}.$$

Hence it is $\deg P_n^{(0,\alpha)}(x;-\frac{1}{n(n+\alpha)}\,|\,0)=n-1$ and the next reccurence relations are not valid for that polynomial and its neighbours.

Like in Section 1 we can prove the following results:

Theorem 2.1. *The polynomial* $\widehat{P}_n^{(0,\alpha)}(x;c\,|\,0)$ *can be expressed in the form*

$$\widehat{P}_n^{(0,\alpha)}(x;c\,|\,0)=\widehat{Q}_n^{(1,\alpha)}(x)+\lambda_n\widehat{Q}_{n-1}^{(1,\alpha)}(x),$$

where

$$\lambda_n=\frac{n(n+\alpha)}{(2n+\alpha)(2n+\alpha+1)}\cdot\frac{1+(n+1)(n+\alpha+1)c}{1+n(n+\alpha)c}.$$

Theorem 2.2. *The sequence* $\{\widehat{P}_n^{(0,\alpha)}(x;c\,|\,0)\}$ *satisfies the three-term recurrence relation*

$$\widehat{P}_{n+1}^{(0,\alpha)}(x;c\,|\,0)=(x-\alpha_n)\widehat{P}_n^{(0,\alpha)}(x;c\,|\,0)-\beta_n\widehat{P}_{n-1}^{(0,\alpha)}(x;c\,|\,0),$$

where

$$\alpha_n=a_n+\lambda_{n+1}-\lambda_n,\quad \beta_n=b_{n-1}\frac{\lambda_n}{\lambda_{n-1}}.$$

Theorem 2.3. *All zeros of* $\widehat{P}_n^{(0,\alpha)}(x;c\,|\,0)$ *are real, simple and positive, with exception one of them for* $c<-1/(n(n+\alpha))$. *Furthermore, for* $c<-2/((n+1)(n+\alpha))$ *the inequalities* (1.5) *hold.*

EXAMPLE 2.1. The polynomial $\widehat{P}_3^{(0,0)}(x;1\,|\,0)=x^3-\frac{27}{20}x^2+\frac{21}{50}x-\frac{1}{200}$, has all zeros in $(0,1)$: $x_1\approx 0.0123940$, $x_2\approx 0.459337$, $x_3\approx 0.878268$, but the polynomial $\widehat{P}_3^{(0,0)}(x;-2\,|\,0)=x^3-\frac{45}{34}x^2+\frac{33}{85}x+\frac{1}{340}$, has one negative zero: $x_1\approx -0.103301$, $x_2\approx 1.95187$, $x_3\approx 5.95143$.

Theorem 2.4. *The zeros of* $\widehat{Q}_n^{(1,\alpha)}(x)$ *and* $\widehat{Q}_{n-1}^{(1,\alpha)}(x)$ *interlace the zeros ofthe polynomial* $\widehat{P}_n^{(0,\alpha)}(x;c\,|\,0)$.

Remark. It was proved the interlacing property for $\widehat{Q}_n^{(0,\alpha)}(x)$ and $\widehat{Q}_{n-1}^{(0,\alpha)}(x)$ with respect to $\widehat{P}_n^{(0,\alpha)}(x;c\,|\,0)$ (see [8]).

3. Polynomials of the Legendre type

The sequence of Legendre polynomials $\{P_n(x)\}$ is orthogonal with respect to the functional

$$\boldsymbol{L}(P)=\int_{-1}^{1}P(x)\,dx.$$

This sequence satisfies the three-term recurrence relation

$$(n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x),$$

with initial values $P_{-1}(x) = 0$ and $P_0(x) = 1$. The polynomial $P_n(x)$ can be expressed in the form

$$P_n(x) = \frac{1}{2^n} \sum_{i=0}^{[n/2]} (-1)^i \binom{n}{i} \binom{2n-2i}{n} x^{n-2i}.$$

Let

$$\boldsymbol{I}_3 = \boldsymbol{L} + c(\boldsymbol{\Delta}_{-1} + \boldsymbol{\Delta}_1), \qquad c \in \mathbb{R},$$

be a new functional, and $\{P_n(x; c\,|-1, 1)\}$ the corresponding orthogonal polynomials of the *Legendre type*. Then the moments $\tilde{\mu}_n = I_3(x^n)$ are given by

$$\tilde{\mu}_n = \begin{cases} 0 & \text{for odd } n, \\ \dfrac{2}{n+1} + 2c & \text{for even } n. \end{cases}$$

The polynomial $P_n(x; c\,|-1, 1)$ can be expressed in a determinant form,

$$P_n(x; c\,|-1, 1) = \det \begin{vmatrix} \tilde{\mu}_0 & \tilde{\mu}_1 & \cdots & \tilde{\mu}_{n-1} \\ \tilde{\mu}_1 & \tilde{\mu}_2 & & \tilde{\mu}_n \\ \vdots & & & \\ \tilde{\mu}_{n-1} & \tilde{\mu}_n & & \tilde{\mu}_{2n-1} \\ 1 & x & & x^n \end{vmatrix}.$$

Using a method as in Gautschi and Milovanović [5], we determine

$$\tilde{\Delta}_n = \det[\tilde{\mu}_{i+j}]_{i,j=0,\ldots,n-1}.$$

It is known that for determinants

$$H_n = \det\left[\tfrac{1}{2i+2j-3}\right]_{i,j=1}^{n}, \quad H_0 = 1,$$

holds

$$H_n = \frac{(2n-2)!!^2}{(2n-1)^2 \cdots (4n-5)^2 (4n-3)} H_{n-1}, \quad n = 3, 4, \ldots .$$

Introducing

$$H_n(c) = \left[\tfrac{1}{2i+2j-3} + c\right]_{i,j=1}^{n},$$

we obtain

$$H_n(c) = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ -c & 1 & \frac{1}{3} & & \frac{1}{2n-1} \\ -c & \frac{1}{3} & \frac{1}{5} & & \frac{1}{2n+1} \\ \vdots & & & & \\ -c & \frac{1}{2n-1} & \frac{1}{2n+1} & & \frac{1}{4n-3} \end{bmatrix}.$$

Lemma 3.1. *We have*

$$H_n(c) = \left(1 + c\binom{2n}{2}\right) H_n.$$

Proof. For $H_n(c)$ we have

$$H_n(c) = H_n + cD_n,$$

where

$$D_n = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ -1 & 1 & \frac{1}{3} & & \frac{1}{2n-1} \\ -1 & \frac{1}{3} & \frac{1}{5} & & \frac{1}{2n+1} \\ \vdots & & & & \\ -1 & \frac{1}{2n-1} & \frac{1}{2n+1} & & \frac{1}{4n-3} \end{bmatrix}.$$

We can prove that $D_n = \binom{2n}{2}H_n$, $n \in \mathbf{N}$. Subtracting the last row in D_n from all others, except the first one, we obtain

$$D_n = \frac{(2n-2)!!}{(2n-1)\cdots(4n-3)} \begin{bmatrix} 0 & 2n-1 & 2n+1 & \cdots & 4n-3 \\ 0 & 1 & \frac{1}{3} & & \frac{1}{2n-1} \\ 0 & \frac{1}{3} & \frac{1}{5} & & \frac{1}{2n+1} \\ \vdots & & & & \\ -1 & 1 & 1 & & 1 \end{bmatrix}.$$

Also, subtracting the last column from all others, except the first one, we obtain

$$D_n = \frac{(2n-2)!!^2}{(2n-1)^2\cdots(4n-5)^2(4n-3)}\{(4n-3)H_{n-1} + D_{n-1}\},$$

from which, by induction, we finish the proof. □

Lemma 3.2. *If*

$$H_n'(c) = \det\left[\tfrac{1}{2i+2j-1} + c\right]_{i,j=1}^{n}, \qquad H_n' = H_n'(0),$$

then

$$H_n'(c) = \left[1 + c\binom{2n+1}{2}\right] H_n', \quad n = 1, 2, \dots .$$

Thus,

$$\tilde{\Delta}_n = \Delta_n \left[1 + \binom{n}{2}c\right] \left[1 + \binom{n+1}{2}c\right],$$

where

$$\Delta_n = \det[L(x^{i+j})]_{i,j=0,\dots,n-1}.$$

Theorem 3.3. *The polynomials $P_n(x;c\,|-1,1)$ satisfy the following three-term recurrence relation*

$$(n+1)\left(1+\tbinom{n}{2}c\right)P_{n+1}(x;c\,|-1,1)=(2n+1)\left(1+\tbinom{n+1}{2}c\right)xP_n(x;c\,|-1,1)$$
$$-\,n\left(1+\tbinom{n+2}{2}c\right)P_{n-1}(x;c\,|-1,1).$$

Proof. By $\{\widehat{P}_n(x;c\,|-1,1)\}$ we denote the sequence of monic polynomials of the Legendre type. According to the property $(xf,g)=(f,xg)$ of the inner product defined by the functional $\boldsymbol{I}_3$: $(f,g)=\boldsymbol{I}_3(fg)$, we conclude (cf. [2] and [9]) that this sequence satisfies a three-term reccurence relation of the form

$$\widehat{P}_{n+1}(x;c\,|-1,1)=x\widehat{P}_n(x;c|-1,1)-\beta_n\widehat{P}_{n-1}(x;c\,|-1,1),$$

where (see [3])

$$\beta_n=\frac{\tilde{\Delta}_{n-1}\tilde{\Delta}_{n+1}}{\tilde{\Delta}_n^2}.$$

Knowing a relation for the monic Legendre polynomials and the determinant Δ_n, we yield

$$\widehat{P}_{n+1}(x;c\,|-1,1)=x\widehat{P}_n(x;c\,|-1,1)$$
$$-\frac{n^2}{4n^2-1}\frac{(1+\binom{n-1}{2}c)(1+\binom{n+2}{2}c)}{(1+\binom{n}{2}c)(1+\binom{n+1}{2}c)}\widehat{P}_{n-1}(x;c\,|-1,1).$$

Putting

$$P_n(x;c\,|-1,1)=\frac{(2n)!}{2^n n!^2}\widehat{P}_n(x;c|-1,1),$$

we finish the proof. □

By induction and using the three-term recurrence relation, we obtain:

Theorem 3.4. *The polynomial $P_n(x;c\,|-1,1)$ can be expressed in the form*

$$P_n(x;c\,|-1,1)=\frac{1}{2^n}\sum_{k=0}^{[n/2]}(-1)^k\binom{n}{k}\binom{2n-2k}{n}\left\{1+\left(\binom{n}{2}+2k\right)c\right\}x^{n-2k},$$

where $n=2,3,\ldots$.

Remark. This formula was derived in [6] with a mistake.

The last formula shows that the sequence $\{P_n(x;c\,|-1,1)\}$ has a degenerated property in the sence that is

$$\widehat{P}_n\left(x;-\tbinom{n}{2}^{-1}|-1,1\right)=\widehat{P}_{n-2}\left(x;-\tbinom{n}{2}^{-1}|-1,1\right).$$

Theorem 3.5. *The polynomials $P_n(x;c\,|-1,1)$ are quasi-orthogonal of the second order with respect to the functional*

$$J^{(1,1)}(P)=\int_{-1}^{1} P(x)(1-x^2)\,dx.$$

The polynomial $P_n(x;c\,|-1,1)$ has at least $n-2$ different zeros with odd multiplicity in $(-1,1)$.

Proof. Because of the orthogonality, we have for any polynomial $p(x)$ of degree k $(k\le n-3)$ that

$$I_3(P_n(x;c\,|-1,1)p(x)(1-x^2))=0,$$

i.e.,

$$\int_{-1}^{1} P_n(x;c\,|-1,1)p(x)(1-x^2)\,dx=0.$$

So, we yield quasi-orthogonality of the order 2. Finally, let $x_{n1},x_{n2},\ldots,x_{nk}$ be all distinct zeros of $P_n(x;c\,|-1,1)$ with odd multiplicity and which are in $(-1,1)$. If we introduce the node polynomial $p(x)=(x-x_{n1})(x-x_{n2})\cdots(x-x_{nk})$, then the polynomial $P_n(x;c\,|-1,1)p(x)(1-x^2)$ does not change sign in $(-1,1)$. Therefore,

$$\int_{-1}^{1} P_n(x;c\,|-1,1)p(x)(1-x^2)\,dx\neq 0.$$

Because of that, we conclude that $\deg p(x)\ge n-2$. □

EXAMPLE 3.2. The polynomial

$$P_3(x;c\,|-1,1)=5(1+3c)x^3-3(1+5c)x,$$

has the zeros

$$x_1=0,\quad x_{2,3}=\pm\sqrt{\frac{3(1+5c)}{5(1+3c)}}.$$

All zeros of $P_3(x;c\,|-1,1)$ are in $(-1,1)$ if $c>-1/3$. For $c<-1/3$, two of them are out of $(-1,1)$.

Remark. According to Favard's theorem (see [2]) there are exist the weight distributions $w_i(x)$, $i=1,2,3$, corresponding to the previous functionals J_i:

$$J_i(P)=\int_{-\infty}^{+\infty} P(x)\,dw_i(x),\qquad i=1,2,3.$$

These distributions are (see [3])

$$w_1(x)=e^{-x}+c\delta(x),\;\; w_2(x)=(1-x)^{\alpha}+c\delta(x),\;\; w_3(x)=1+c\big(\delta(x-1)+\delta(x+1)\big),$$

where $\delta(x)$ is the Dirac's delta function.

References

[1] G. Szego, *Orthogonal Polynomials*, 4th ed., Amer. Math. Soc. Colloq. Publ. Vol. 23, Providence, R.I., 1975.

[2] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.

[3] L. L. Littlejohn and S. D. Shore, *Nonclassical orthogonal polynomials as solutions to second order differential equations*, Canad. Math. Bull. **25** (1982), 291–295.

[4] A. B. Mingarelli and A. M. Krall, *Legendre type polynomials under an indefinite inner product*, SIAM J. Math. Anal. **14** (1983), 399–402.

[5] W. Gautschi and G. V. Milovanović, *Polynomials orthogonal on the semicircle*, J. Approx. Theory **46** (1986), 230–250.

[6] A. L. Kuzmina, *On some nonclassical orthogonal polynomials*, Izv. Vyssh. Uchebn. Zaved. Mat. **1987**, 47–52. (Russian)

[7] M. Alfaro, F. Marcellan, M. L. Rezola and A. Ronveaux, *On orthogonal polynomials of Sobolev type: Algebraic properties and zeros*, SIAM J. Math. Anal. **23** (1992), 737–757.

[8] T. E. Perez and M. A. Pinar, *Global properties of zeros for Sobolev-type orthogonal polynomials*, J. Comput. Appl. Math. **49** (1993), 225–232.

[9] W. D. Evans, L. L. Littlejohn, F. Marcellan, C. Markett and A. Ronveaux, *On recurrence relations for Sobolev orthogonal polynomials*, SIAM J. Math. Anal. **26** (1995), 446–467.

Grdimir V. Milovanović and Zvezdan M. Marjanović: Faculty of Electronic Engineering, Department of Mathematics, University of Niš, 18000 Niš, Yugoslavia

Predrag M. Rajković: Faculty of Mechanical Engineering, Department of Mathematics, University of Niš, 18000 Niš, Yugoslavia

FILOMAT (Niš) 9:1 (1995), 113–114

WORKSHOP ON TOPOLOGY
(FIFTH NIŠ-SOFIAN TOPOLOGICAL SEMINAR)
Guyletchitza, June 27 - 30, 1995

The fifth Niš-Sofian topological seminar was held in the Scientific Center Guyletchitza, on the mountain Rila, near Sofia on June 27 to June 30, 1995. Besides topologists from Niš and Sofia two topologists from Italy and two from Russia were also participants of the seminar. The following papers were presented:

June 27, 1995

Chairman: Gino Tironi

15:00 - 15:30 D. Doitchinov: On uniform frames
15:30 - 16:00 G. Dimov , D. Vakarelov: S-systems and duality theorems
16:00 - 16:30 V. Gutev: On set-valued selections for lsc mappings

Chairman: Doitchin Doitchinov

17:00 - 17:30 I. Gotchev , H. Mintchev: On sequential properties of Noetherian topological spaces
17:30 - 18:00 S. Popvassilev: On some topologies on $\mathbb{R}^n$
18:00 - 18:30 D. Karaivanov: The commutativity between subspaces and hyperspaces

June 28, 1995

Chairman: Stoyan Nedev

9:30 - 10:00 A. Gryzlov: On some theorems from the partition calculus and the theory of cardinal invariants
10:00 - 10:30 P. Semenov: Countable sets of non-convex valued non lsc mappings
10:30 - 11:00 M. Stanojević: On hyperspaces with the locally finite topology

Chairman: Ljubiša Kočinac

11:30 - 12:00 D. Milovančević: U-systems and U-spaces
12:00 - 12:30 D. Vakarelov: Equivalence of some separation principles
12:30 - 13:00 D. Stavrova: A generic approach for restricting cardinality of topological and more general spaces

Chairman: Momir Stanojević

15:30 - 16:00 A. Bella: On the number of compact, H-closed and H-sets in Hausdorff spaces
16:00 - 16:30 M. Žižović: Some topologies on lattices

Chairman: Pavel Semenov

17:00 - 17:30 R. Krstić: Fuzzy topological separation axioms
17:30 - 18:00 S. Vučić: Ordered fuzzy topological spaces and separation axioms

June 29, 1995

Chairman: Doitchin Doitchinov

9:30 - 10-00 S. Nedev: On some selection theorems
10:00 - 10:30 P. Semenov: Some results on set-valued selections

Chairman: Anatolij Gryzlov

11:00 - 11:30 Lj. Kočinac: Around a G_δ-diagonal
11:30 - 12:00 G. Tironi: The product of pseudoradial spaces

FILOMAT

FILOMAT is a continuation of *Zbornik radova Filozofskog fakulteta u Nišu, Serija Matematika (vol. 1(1987)- vol. 6(1992)* and is published yearly. It publishes original papers in all fields of pure and applied mathematics.

Instructions for authors

All manuscripts (the original and a copy) must be written in English. The extent of the papers is limited to ten pages in length; manuscripts over 10 pages are accepted only exceptionally. Manuscripts should not require many language corrections.

FILOMAT is produced using $\mathcal{AMS}$-TEX. Authors are encouraged to prepare their manuscripts using $\mathcal{AMS}$-TEX. Only a hard copy should be submitted for assessment, but if the paper is accepted the author will be asked to send the text on an IBM PC compatible diskette.

The author(s) should write their names, addresses and the title of the paper on a separate sheet. All manuscripts should start with a short Abstract and include the footnote 1991 Mathematics Subject Classification on the first page. Definitions, theorems, lemmas, remarks, proofs etc. should be written using only one of two alternative styles: **Definition** 2.1. or 3.2. **Theorem** consistently throughout the paper.

Figures are included in the text and must be numbered (by arabic numbers) and mentioned in the text. Equations (or formulas) must be numbered (for future references) in parentheses ( ) at the left margin.

References should be listed alphabetically in the following form:

[3] E.Hewitt and K.A.Ross, *Abstract Harmonic Analysis*, Vol. I, Springer-Verlag, Berlin, 1963.

[11] Đ.Kurepa, *On regressive functions*, Z. Math. Logik 4 (1958), 148-156.

[15] P.Petrović, *Neka svojstva ...*, Doctoral dissertation, University of Belgrade, 1980.

A total of 30 reprints of each paper will be available free of charge; additional reprints can be ordered.

All correspondence concerning both manuscripts and exchange should be addressed to:

Filozofski fakultet (FILOMAT), ul. Ćirila i Metodija 2, 18000 Niš, Yugoslavia.

The subscription price is 20 dollars USA per volume, post free.

For subscriptions write to the same address.

CONTENTS

ISSN 0354–5180