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History of Mathematical and Mechanical Sciences
Book 3

Slavik Jablan

Geometry in the Pre-scientific Period

Ornament today

Belgrade
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Geometry in the Pre-scientific Period

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Recenzent: dr Neda Bokan, docent Matematičkog fakulteta u Beogradu

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GEOMETRY IN THE PRE-SCIENTIFIC PERIOD

Throughout history there were always links between geometry and the art of painting. These links become especially evident when in the study of ornamental art we apply the theory of symmetry. Therefore, ornamental art is called by H.Weyl [38] the "oldest aspect of higher mathematics given in an implicit form" and by A.Speiser the "prehistory of group theory".

The idea to study ornaments of different cultures from the point of view of the theory of symmetry, given by G.Pólya [28] and A.Speiser [34], and supported by the intensive development of the theory of symmetry in the 20th century, caused the appearance of a whole series of works dedicated mostly to the ornamental art of ancient civilizations, to the cultures which contributed the most to the development of ornamental art (Egyptian, Arab, Moorish,...) [2,14,16,17,26] or to the ornamental art of primitive peoples [9,10]. Only in more recent works, research has turned to the very roots, the origins of ornamental art: to the ornamental art of the Paleolithic and Neolithic [22]. The extensions of the classical theory of symmetry— the antisymmetry and colored symmetry made possible the more profound analysis of the "black-white" [19,20] and colored ornamental motifs in the ornamental art of the Neolithic and ancient civilizations.

This work gives the results of a symmetry analysis of Paleolithic and Neolithic ornamental art. It is dedicated to the search for "ornamental archetypes"— the universal basis of complete ornamental art. The development of ornamental art started together with the beginnings of mankind and represents one of the oldest records of human attempts to note, understand and express regularity— the underlying basis of every scientific knowledge.

The final conclusion of this work is that most of the ornamental motifs which have been discussed from the standpoint of the theory of symmetry are of a much earlier date than we can expect. This places the beginning of ornamental art, the oldest aspect of geometric cognition, back to several thousands years before the ancient civilizations, i.e. in the Paleolithic and Neolithic.

Since ornamental art is mostly limited to the two-dimensional plane presentation of ornamental motifs, the subject of this art from the point of view of the theory of symmetry is given by the plane symmetry groups: the symmetry groups of rosettes, friezes and ornaments. The discrete groups of symmetry of rosettes consists of two infinite classes of symmetry groups: cyclic groups **Cn** generated by the rotation of order *n* ($n \in N$) and dihedral groups **Dn** generated by two reflections, the reflection lines of which cross in the invariant point, center of rotation of order *n*. Seven discrete groups of symmetry of friezes can be denoted by symbols: p11, p1g, p12, pm1, p1m, pmg and pmm where the symbol p denotes a translation, g a glide reflection, m a reflection and n (n=1,2) a rotation of order *n* (*n*=1,2). All this symbols are treated in the coordinate sense, so that on the first position is the translation symbol p, the elements of symmetry on the second position are perpendicular to the translation axis, and the elements of symmetry on the third position are parallel with the translation axis (Figure 1).

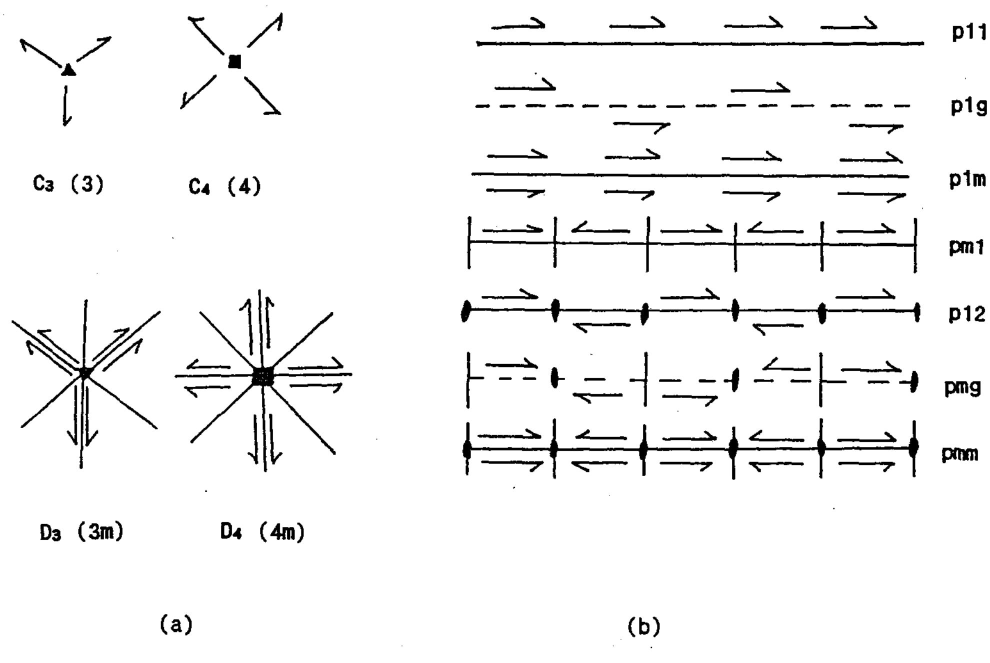


Figure 1. Tables of the graphic symbols of symmetry elements of (a) rosettes; (b) seven discrete symmetry groups of the friezes.

Analogously, with the symbols of symmetry groups of ornaments the symbol p denotes a two-dimensional translation subgroup, while the symbols m, g, n (n=2,3,4,6) have respectively the same meanings as in the case of symmetry groups of friezes

(Figure 2). When we talk about the continuous groups of symmetry of friezes, the presence of a continuous translation is denoted by a subscript 0, while with the antisymmetry groups, antigenerators are denoted by '. Antisymmetry groups are denoted also by the group/subgroup symbols *G/H* [31].

By the term "pre-scientific period" we understand the Paleolithic and Neolithic epochs, covering the period from the end of the Quaternary epoch (around 12000-10000 B.C.) till the end of the IV millennium B.C., when we have signs of the first alphabet.

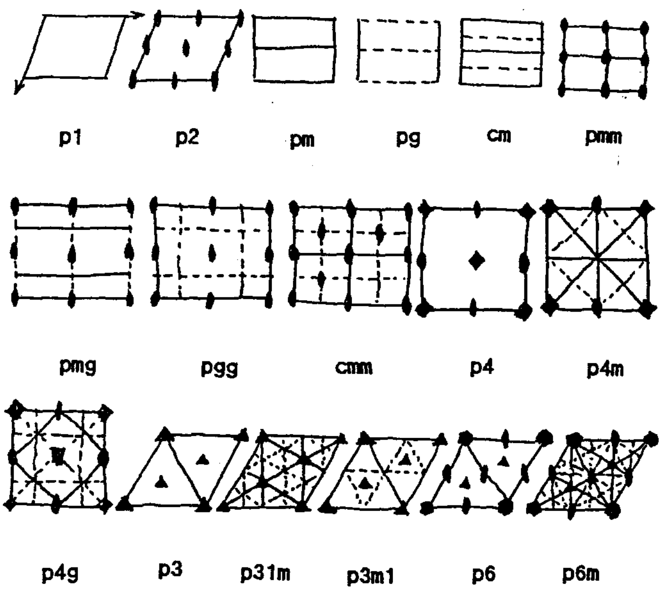
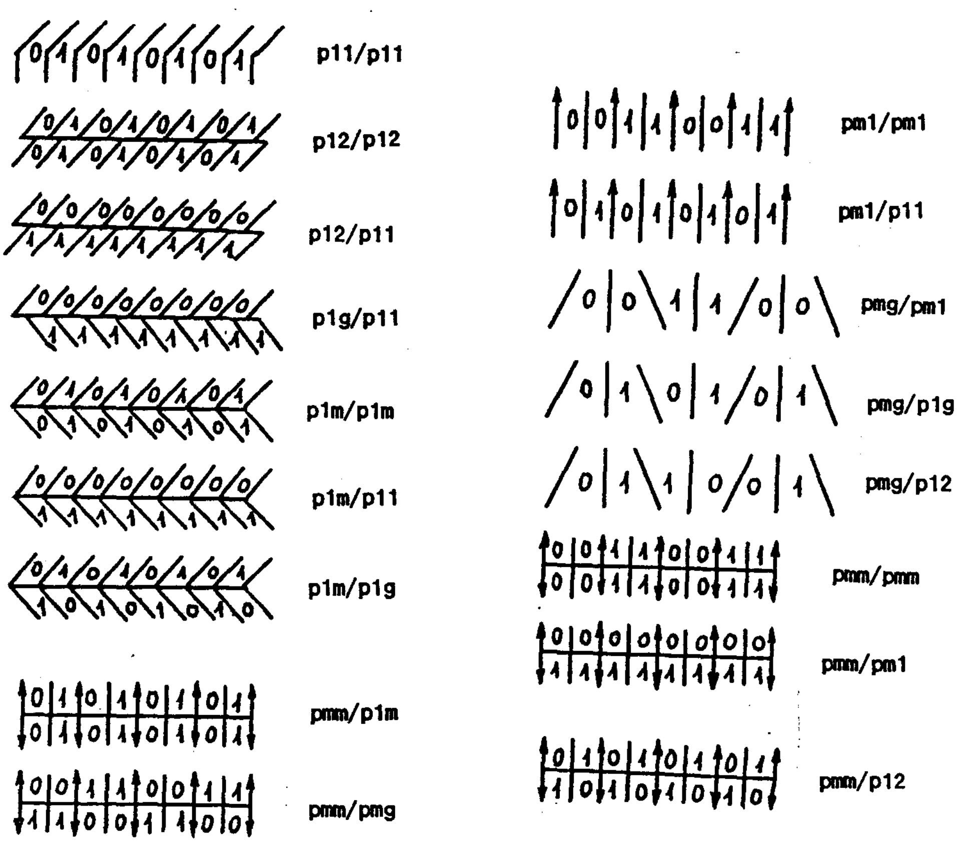


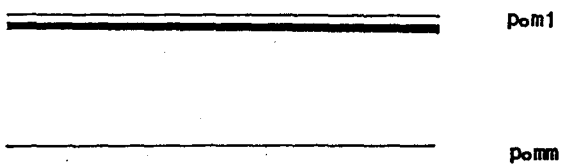
Figure 2. Table of the graphic symbols of symmetry elements of the seventeen discrete groups of symmetry of ornaments.

In the absence of written sources, the study of geometry of the prehistoric period is based on the analyses of artifacts, which offer information on geometric knowledge in an implicit form. Among the artifacts mentioned we distinguish: ornamental motifs realized in the form of bone engravings, carvings and drawings on stone from the Paleolithic and Neolithic, ornamental motifs in ceramics obtained in the Neolithic phase by engraving, pressing, drawing or coloring, as well as architectural objects and constructions from the Neolithic period, so called megalithic monuments.

The simplest ornamental form are the rosettes, figures which correspond to the symmetry groups C_n and D_n, denoted in



(a)



(b)

Figure 3. (a) Seventeen "black-white" antisymmetry groups of friezes; (b) visually presentable continuous groups of symmetry of friezes **p∞m1** and **p∞mm**.

Shubnikov's notation by **n** and **nm** respectively.

The continuous symmetry group of rosettes D_∞ (∞m) corresponds to the maximal symmetric rosette - the circle. Due to the maximal visual and constructional simplicity and maximal symmetry, the circle represents a primary geometric shape, a geometric archetype. Within ornamental art it appears in the Paleolithic, as an independent rosette or in combination with some concentric rosette of a lower degree of symmetry, usually circumscribed or inscribed in a circle. Since the group D_∞ (∞m) contains all the other groups of symmetry of rosettes as subgroups, rosettes of a lower degree of symmetry are often derived by the desymmetrization of a circle. Owing to its visual-geometric properties: completeness, compactness, boundedness and uniformity of its structural segments, the circle may serve as a universal symbol of completeness and perfection. At the very beginnings of ornamental art the circle becomes the symbol of the Sun, remaining that throughout history (Figure 4).

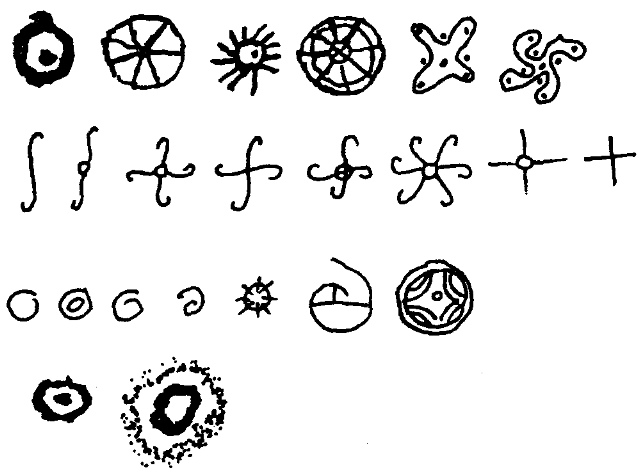


Figure 4. Variations of the Sun symbol in the ornamental art of the Paleolithic and Neolithic.

The continuous symmetry group of rosettes C_∞ (∞), the continuous group of rotation, a physical interpretation of which could be a circle uniformly rotating around the center, is visually interpretable exclusively by the use of textures [32]: by applying the same asymmetric figures statistically distributed in accordance with the desired symmetry C_∞ (∞).

The spiral, one of the oldest dynamic visual symbols which in the visual sense suggests the rotational motion around the invariant point could be accepted as an adequate symbolic interpretation of this continuous symmetry group. In ornamental art, the spiral appeared already in the Paleolithic as an independent ornamental motif or in the form of a double spiral, a motif with symmetry group C_2 (2) generated by two-fold rotation (Figure 5).

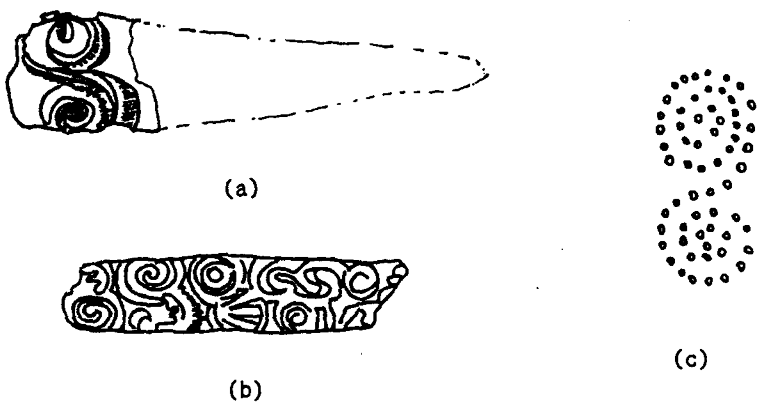


Figure 5. Spiral ornamental motifs (Paleolithic, the Magdalenian, about 10000 B.C.) from: (a) Arudy; (b) Isturiz; (c) Maljta (USSR).

Among the elementary geometric forms we have the line segment, usually placed in accordance with the basic natural directions, vertical and horizontal line. To a line segment corresponds the symmetry group D_2 (2m), generated by two reflections: one with the reflection line perpendicular and the other with the reflection line collinear to the line segment. However, from the point of view of visual perception, due to the action of the visual and gravitational dominant – the vertical line, we experience the symmetry of the line segment as the D_1 (m). In this case, the horizontal reflection is neglected. The combination of the vertical and horizontal line segment results in the cross form with symmetry group D_1 (m), D_2 (2m) or D_4 (4m). The rosettes with symmetry D_2 (2m) and D_4 (4m) possess another fundamental property: the existence of perpendicular, vertical and horizontal reflection lines. The form of cross with symmetry group D_4 (4m) is often subjectively, visually perceived as symmetry D_2 (2m), neglecting the presence of four-fold rotation.

Static rosettes with symmetry group D_1 (m) or D_2 (2m) are linked to the plane symmetry of man, its vertical attitude and perpendicularity to the base. Besides the rational mirror symmetry, which originated from motifs in nature, we have in the ornamental art the different aspects of symbolic symmetry D_1 (m): the duplicated figures, two-headed animals, etc. These examples result mostly from the often use of the vertical mirror symmetry as a visual dominant (Figure 6).

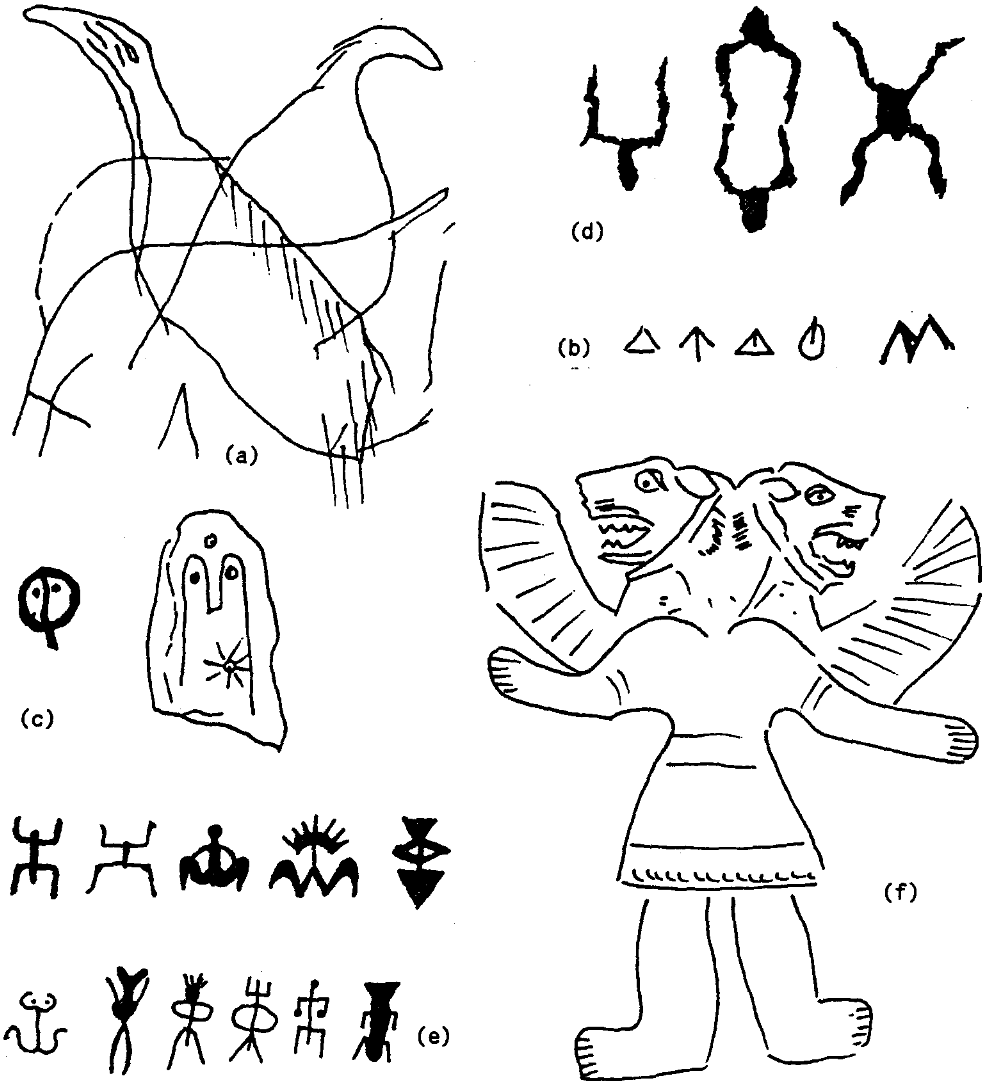


Figure 6. Examples of rosettes with symmetry group D_1 (m) and D_2 (2m) in the ornamental art of the Paleolithic and Neolithic: (a) D_1 (m), Paleolithic, El Pendo; (b) D_1 (m), Paleolithic of France and Spain; (c) stylizations of a face motif, D_1 (m); (d) derivation of the rosettes with symmetry group D_2 (2m) by the superposition of rosettes with symmetry group D_1 (m), Paleolithic of France; (e) stylizations of a human figure, D_1 (m), Paleolithic and Neolithic of Italy and Spain; (f) the two-headed winged lion, Tell Hallaf, about 5000 B.C.

In Paleolithic ornamental art we have also the rosettes with the symmetry group D_n (nm): D_3 (3m), D_4 (4m) and D_6 (6m), as well as the corresponding regular polygons: equilateral triangle, square and regular hexagon (Figure 7). Although with rosettes the principle of crystallographic restriction (n=1,2,3,4,6) is not respected, prevailing are the rosettes with the symmetry group D_n (nm) for these values of *n*. In a later stage, the Neolithic (Figure 8), we have also the rosettes with the symmetry group D_5 (5m) with the use of regular pentagon and pentagramme. The first appearance of pentagramme is dated by H.S.M.Coxeter [7, pp. 8] in the VII century B.C. The visual characteristics of rosettes with symmetry group D_n (nm) are stability, stationariness and absence of enantiomorphism. Enantiomorphism, the existence of a "right" and "left" modification of the same figure, appears with all figures possessing a symmetry group which does not contain indirect symmetry transformations.

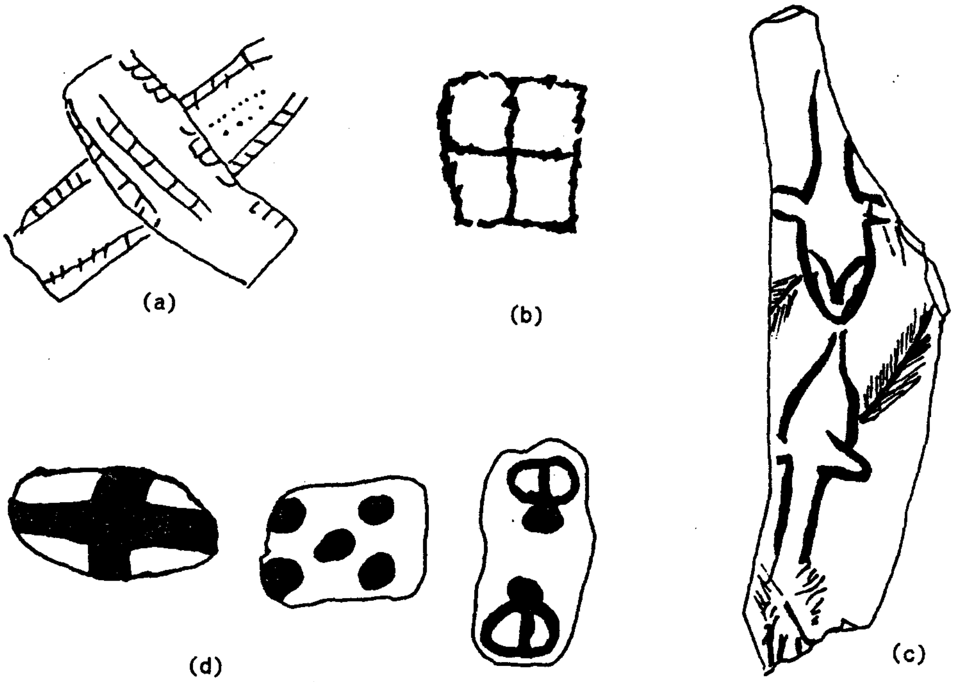


Figure 7. Examples of rosettes with symmetry group C_n (n) and D_n (nm) in the ornamental art of the Paleolithic: (a) Castilo, D_2 (m); (b) Paleolithic of France, D_4 (4m); (c) Laugerie Basse, C_2 (2); (d) Maz d' azil, D_2 (2m) and D_4 (4m).

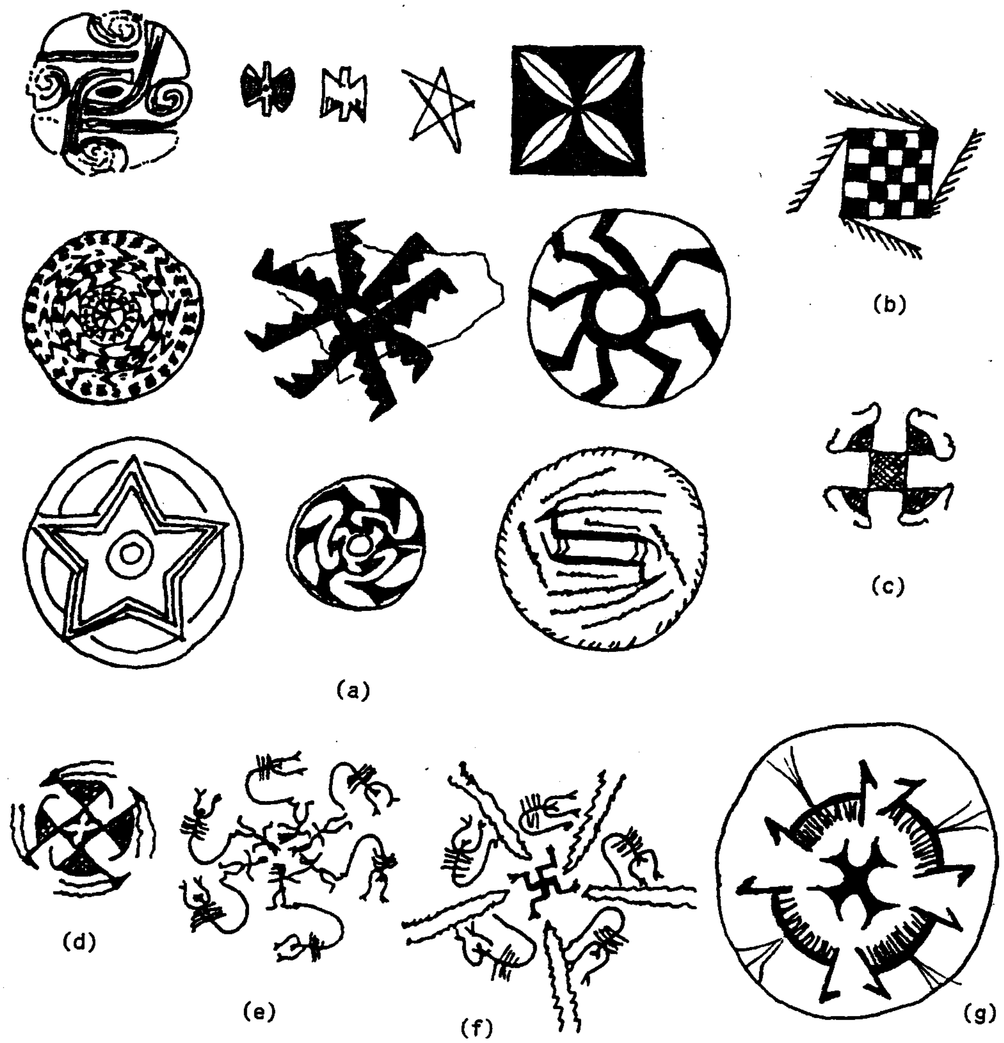


Figure 8. Examples of rosettes with symmetry group C_n (n) and D_n (nm) in the ornamental art of the Neolithic: (a) Susa, Hacilar, Catal Hüjük, Tell Hallaf, Eridu culture, about 6000-4500 B.C. (7500-5000 B.C.?); (b) Susa, C_6 (6), about 5500-5000 B.C.; (c) Samara, D_4 (4m); (d) Samara, C_4 (4); (e) Samara, C_6 (6); (f) Samara, the superposition of concentric rosettes with symmetry groups C_5 (5) and C_4 (4); (g) Susa, D_4 (4m).

In contradistinction to the static rosettes with symmetry group D_n (nm), the rosettes with symmetry group C_n (n) (e.g. triquetra with symmetry group C_3 (3), swastika with symmetry group C_4 (4)) are visually dynamic rosettes with the possibility of construction of enantiomorphic modifications, which suggests the impression of rotational motion (Figure 7,8,9,10).

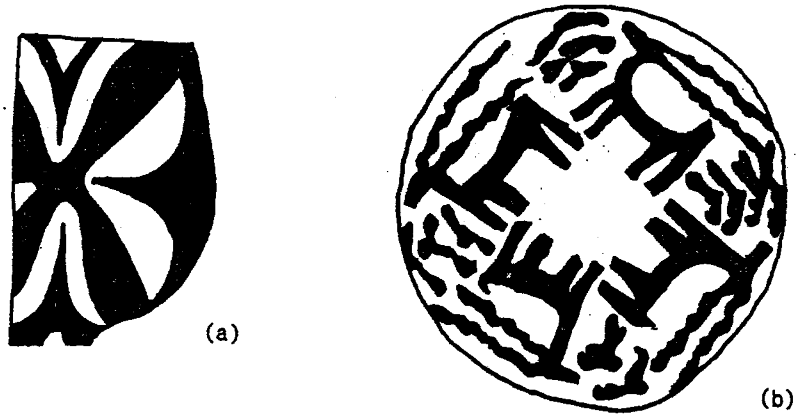


Figure 9. Examples of rosettes with symmetry group C_n (n) and D_n (nm) in Neolithic ceramics: (a) Aznabegovo-Vrshnik, Yugoslavia, D_4 (4m), about 5000 B.C.; (b) Samara, C_4 (4), about 5500-5000 B.C.

In the next stage of the development of ornamental art, i.e. in the Neolithic, after understanding the symmetry regularities on which the symmetry of rosettes is based and solving their elementary geometric constructions, the diversity of rosettes increases. This is followed by the application of plant and zoomorphic motifs and by varying the form of the fundamental region. Also, the superpositions of concentric rosettes which bring in a desymmetrization – a reduction to a lower degree of symmetry – are very common (Figure 8f, 10).

In the Neolithic, with two-colored ceramics, we have the antisymmetry "black-white" rosettes (Figure 11). In this case, the antisymmetry can be treated either as the mode of desymmetrization for obtaining the subgroups of index 2 of a given symmetry group or as an independent form of symmetry. In the table of antisymmetry groups every group is denoted by the group/subgroup symbol *G/H* [31] and followed by a system of (anti)generators. The factor-group *G/H* is isomorphic to a cyclic group of order 2, the group of color change "black"-"white".

Table of antisymmetry groups of rosettes:

D_{2n}/D_n (2nm/nm)	(2n)'m
D_n/C_n (nm/n)	nm'
C_{2n}/C_n (2n/n)	(2n)'

Figure 10. Neolithic rosette, C_4, Middle East, around 6000 B.C.

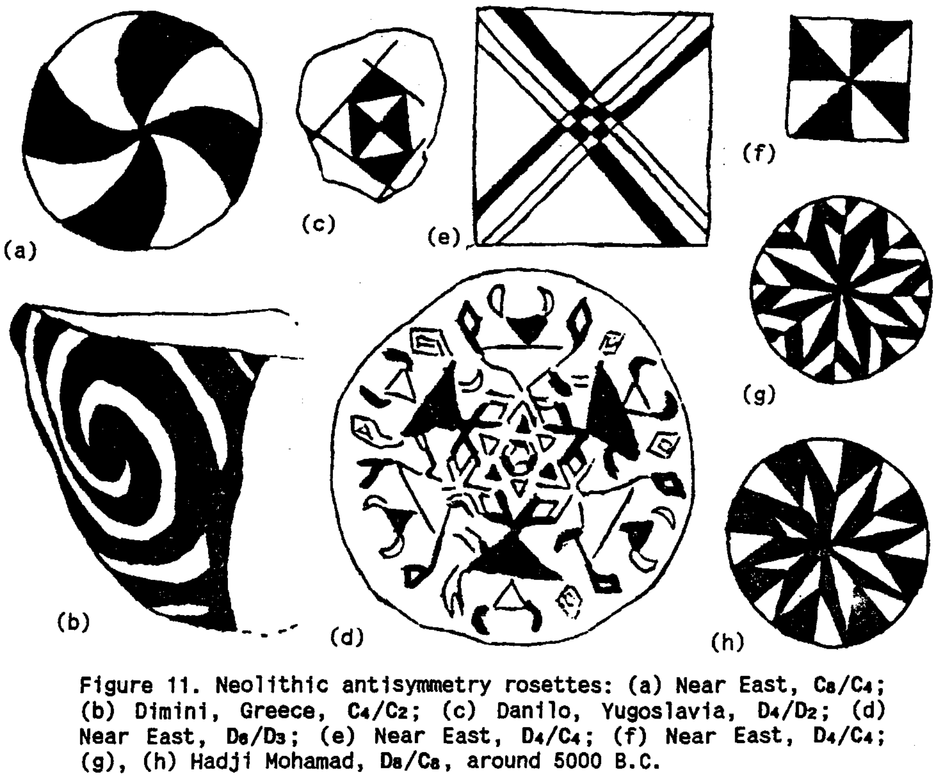


Figure 11. Neolithic antisymmetry rosettes: (a) Near East, C_8/C_4; (b) Dimini, Greece, C_4/C_2; (c) Danilo, Yugoslavia, D_4/D_2; (d) Near East, D_6/D_3; (e) Near East, D_4/C_4; (f) Near East, D_4/C_4; (g), (h) Hadji Mohamad, D_8/C_8, around 5000 B.C.

In the case of antisymmetry groups, there is a possibility for interpreting the color change "black"-"white" as the alternating change of some physical or geometric bivalent property. In ornamental art color change mentioned introduces a space component, a suggestion of relations "in front"-"behind", "up"-"down", "above"-"below". From the artistic point of view, it introduces the contrast between repeating congruent figures and specific equivalence of the "figure" and "background" thus expressing in a symbolical sense the dynamic conflict and duality.

In ornamental art the use of color in the sense of regular coloring, that means antisymmetry and colored symmetry, was opened a large unexplored field. Hence, in the history of ornamental art, we can consider the Neolithic as its peak, a period in which after the basic technical and constructional problems were solved, new possibilities for artistic research, imagination, variety of motifs and decorativeness, were opened.

In the late Paleolithic (Magdalenian, about 12000-10000 B.C.) we find the oldest examples of the symmetry groups of friezes, plane symmetry groups without invariant points and with invariant line. We have the examples of all the seven symmetry groups of friezes: **p11**, **p1g**, **p12**, **pm1**, **p1m**, **pmg**, **pmm** as well as two visually presentable continuous symmetry groups of friezes **p_0m1** and **$p_0$mm**.

Friezes are usually obtained by applying the rosettal method of construction: translational multiplication of an initial motif – a rosette, the symmetry of which directly conditions the symmetry of the frieze obtained. The other origin of the friezes are models found in nature which, by itself, possess the symmetry of a frieze (Figure 12).

The way friezes are derived after models found in nature, is illustrated by examples: a herd of deer reduced to the frieze with symmetry group **p11**, the motif of cult-dance rendering the frieze with the symmetry group **pm1**. Friezes with symmetry group **p12** and **pmg** can be considered as stylized waves. Models in nature with symmetry group **p1g** and **p1m** are found with the distribution of leaves of certain plants; they have served as the pretext for the construction of corresponding friezes in ornamental art. The importance of the plane symmetry in nature and the number of rosettes with the symmetry group D_1 (**m**) and D_2 (**2m**) caused the appearance and frequent occurrence of friezes with symmetry group **pmm**. These friezes can be derived by a translational multiplication of rosette with the symmetry group D_2 (**2m**), where the translation axis is parallel with one reflection line of the rosette. The symmetry group of friezes **pmm** is the maximal discrete group of symmetry of the friezes, generated by reflections. All the other symmetry groups of friezes are subgroups of the group **pmm**. Hence, the group **pmm** can serve for derivation of all the symmetry groups of friezes by desymmetrization. Examples of all the seven discrete symmetry groups are found in Paleolithic ornamental art (Figure 13).

Besides friezes with a concrete meaning, which are based on material models found in nature, the appearance of certain

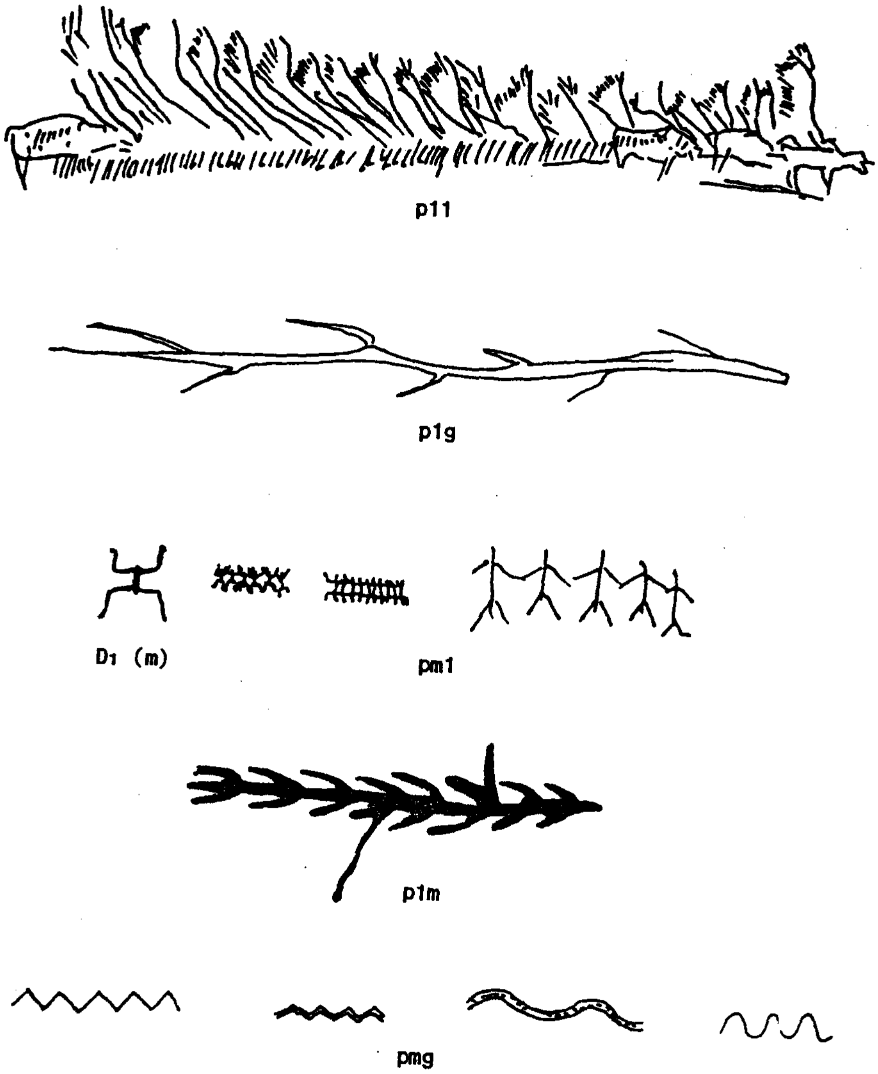


Figure 12. The origin of friezes with symmetry groups p11, p1g, pm1, p1m and pmg from models in nature (Paleolithic, Magdalenian).

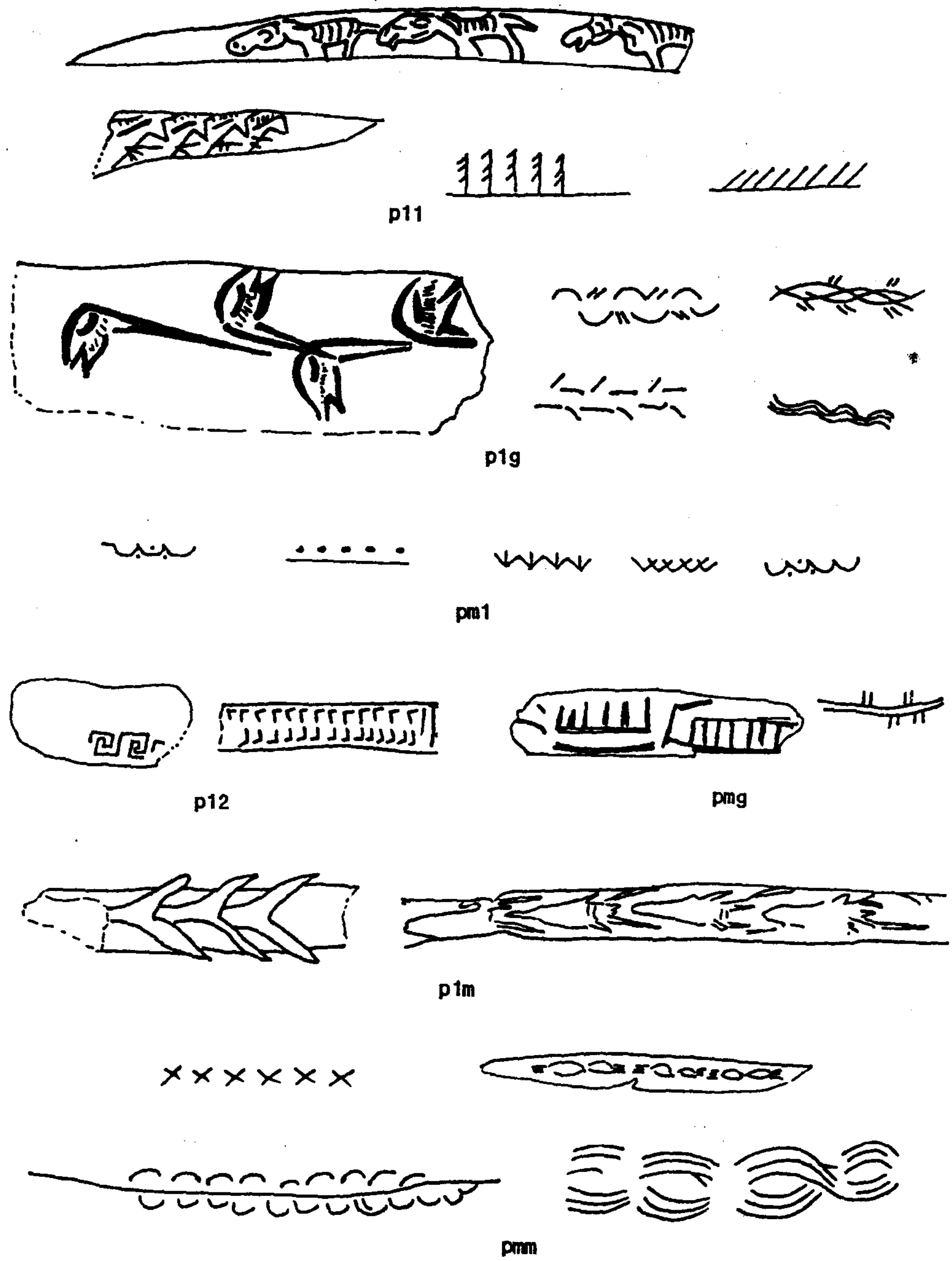


Figure 13. Examples of all the seven symmetry groups of friezes in the ornamental art of the Paleolithic.

friezes is caused also by the periodic change of many natural phenomena (the change of day and night, seasons, the tides, phases of the Moon...) so that friezes represent, at the same time, the oldest attempts to register the periodical change of natural phenomena, i.e. the first forms of calendars. These friezes can also be understood as ways of registering quantities, serving as tally boards, and thus indicating the beginnings of counting and notifying the results of counting, i.e. the appearance of the set of natural numbers.

Thanks to their symbolic meaning, certain "geometric" friezes became a means of visual communication. This is proved by the preserved names of friezes in the ornamental art of primitive peoples. This communication role of friezes, established in the Paleolithic, was partly preserved in the Neolithic. With the development of other communication forms, friezes lost their primary symbolic function which was partly or completely replaced by their decorative function. The beginning of this process can be registered already in Neolithic ornamental art (Figure 14).

The polarity, non-polarity and bipolarity of the translation axis of the friezes, the presence or absence of enantiomorphism, the presence or absence of the indirect symmetries within the frieze symmetry group [22], represent some of the relevant geometric properties which, in the same time, define the visual characteristics of the friezes, thus conditioning also the spectrum of symbolic meanings which friezes with the certain symmetry groups possess.

With regard to the frequency of occurrence, besides friezes originating directly from models found in nature, in the ornamental art of the pre-scientific period, friezes which satisfy the criterion of visual entropy [22]: maximal visual and constructional simplicity and maximal symmetry, are dominant.

The oldest examples of antisymmetry friezes, so called "black-white" friezes, date back to the Neolithic epoch, in which we have the examples of most of the 17 antisymmetry groups of friezes. Further investigations should show whether or not in this period there existed examples of all the 17 antisymmetry groups of friezes. With regard to the frequency of occurrence, the most numerous are "black-white" friezes derived from the symmetry groups of most frequent friezes by the use of antisymmetry desymmetrization method (Figure 14).

The frequency of occurrence of antisymmetry friezes depends also on the antisymmetry properties. Therefore, more frequent are antisymmetry friezes with oppositely colored adjacent fundamental regions. A domination of the "geometric" antisymmetry friezes over antisymmetry friezes inspired by models found in nature is also evident, due to the absence of antisymmetry in nature among models from plant and animal life. In contradistinction to this, many natural alternating phenomena followed by bivalent changes (e.g. the change of day and night, etc.) are represented by antisymmetry friezes, already in Neolithic ornamental art.

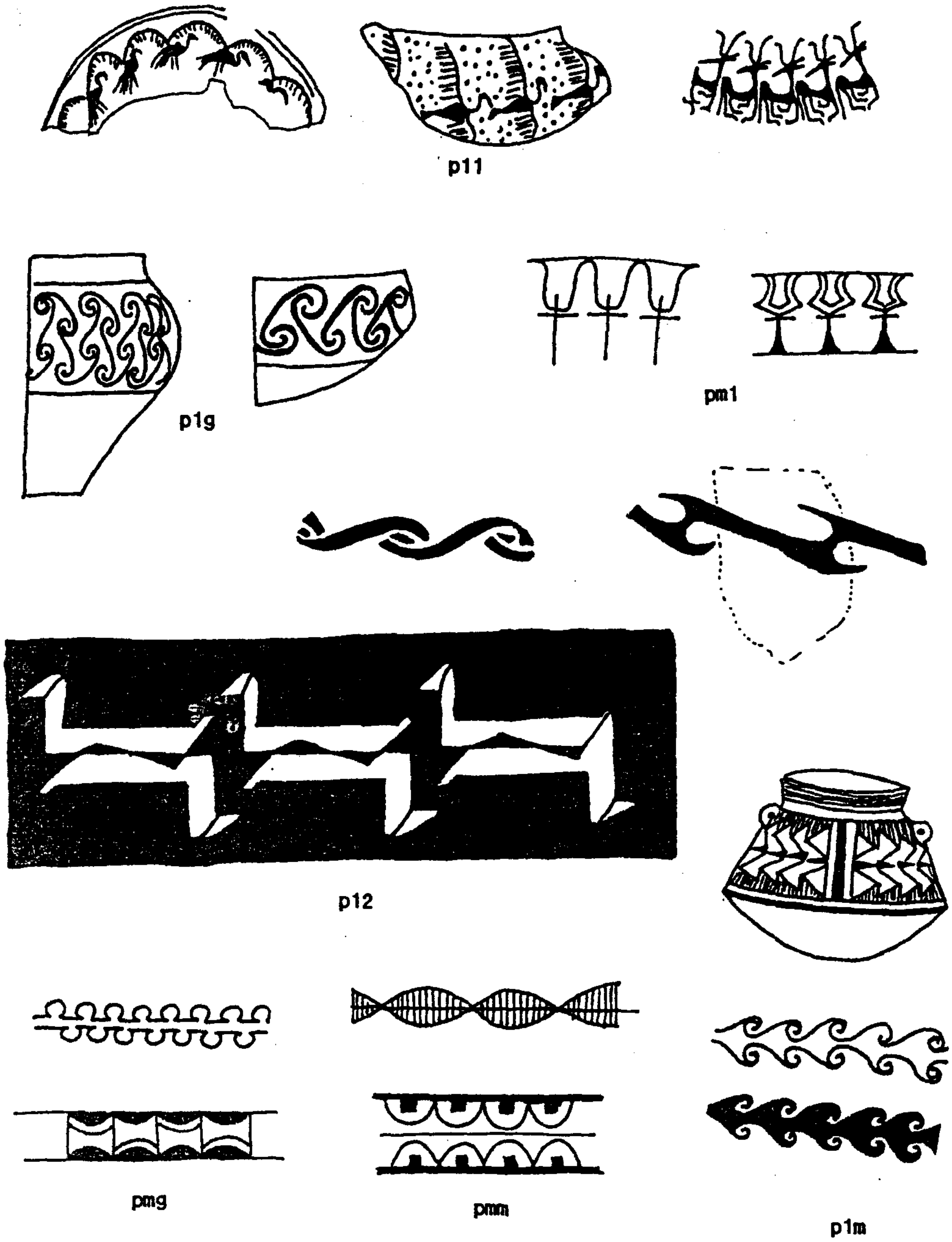


Figure 14. Examples of all the seven symmetry groups of friezes in Neolithic ornamental art.

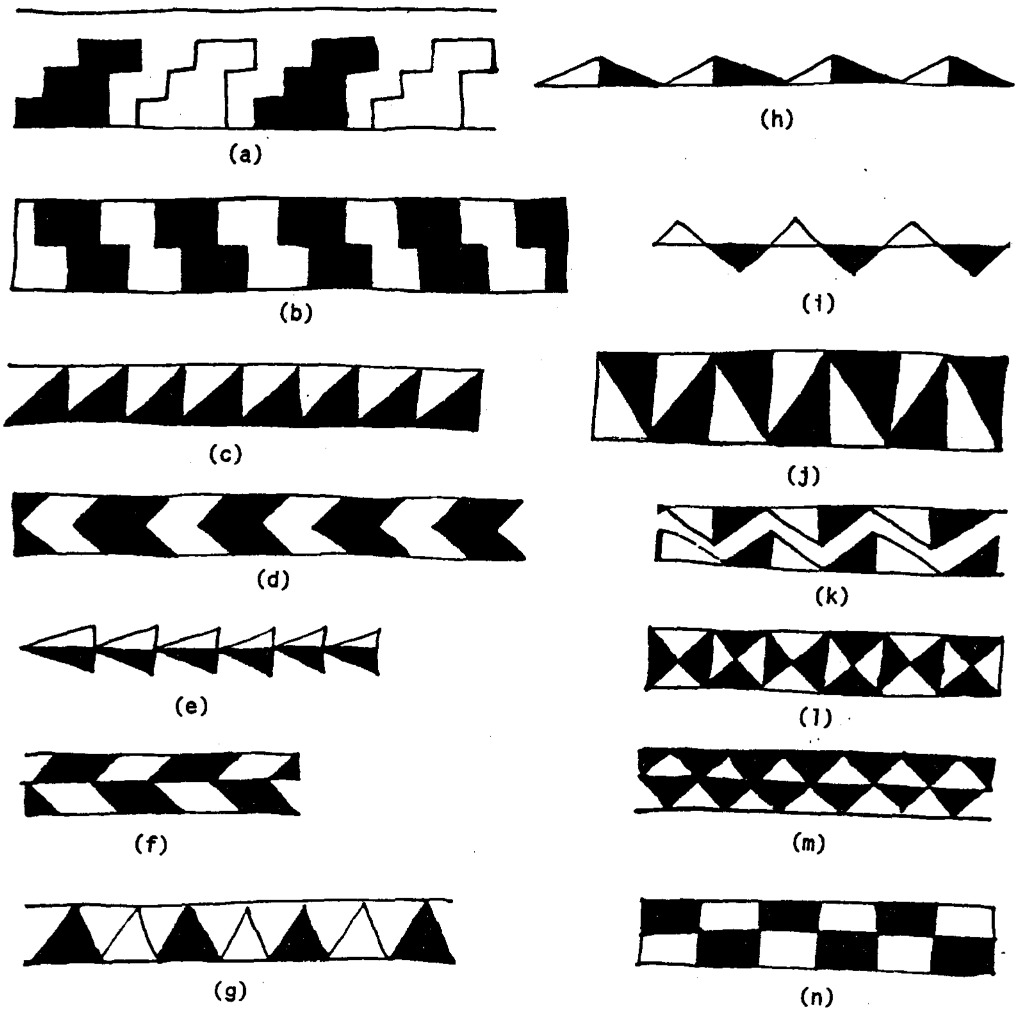


Figure 15. Examples of 14 antisymmetry groups of friezes in Neolithic ornamental art: (a) Greece, p11/p11, about 3000 B.C.; (b) Greece, p12/p12; (c) Near East, p12/p11, about 5000 B.C.; (d) Near East, p1m/p1m, about 5000 B.C.; (e) Near East, p1m/p11; (f) Anadolia, p1m/p1g, around 5000 B.C.; (g) Near East, pm1/pm1; (h) Near East, pm1/p11; (i) Greece, pmg/pm1; (j) Near East, pmg/p1g, about 5000 B.C.; (k) Anadolia, pmg/p12; (l) Tell el Hallaf, pmm/pmm, about 4900-4500 B.C.; (m) Hacilar, pmm/pm1, about 5500-5200 B.C.; (n) Near East, pmm/pmg.

There is a possibility of treating the antiidentity transformation of order 2 (color change "black"-"white") as the way to represent space symmetry structures, the bands (three-dimensional symmetry groups with invariant plane and included line, and without invariant points), in the plane, whereby the 31 antisymmetry groups of the friezes (7 generating + 7 senior + 17 junior antisymmetry groups) correspond to the 31 groups of symmetry of bands. From the artistic point of view, that gives a possibility to suggest space in a flat drawing plane. Besides this possibility, there are also many different geometric or non-geometric interpretations of the antiidentity transformation. In prehistoric ornamental art a primary symbolic function of "black-white" friezes, is evident.

In the theory of symmetry and ornamental art, the most interesting field of study is the 17 groups of symmetry of ornaments, two-dimensional symmetry groups without invariant lines and points. The common characteristic of ornaments is the presence of discrete two-dimensional translation subgroup, generated by two independent translations. How difficult it is to discover and construct the examples of all the 17 symmetry groups of ornaments is shown by the fact that many cultures with a very rich ornamental art do not possess within their early ornamental art the examples of all these groups [16,17]. The same is proved by the fact that in the mathematical studies of symmetry, the complete list of the 17 groups of symmetry of ornaments are found only in 1890, in the works of E.S.Fedorov, although this problem attracted also many other important mathematicians (for example, C.Jordan, L.Sohncke).

This is the reason why it is rather surprising that already in the ornamental art of the Paleolithic we can find examples of the nine symmetry groups of ornaments: p1, p2, pm, pmm, pmg, cm, cmm, p4m and p6m [22]. In the Neolithic phase we have the appearance of the other five symmetry groups of ornaments: pg, pgg, p4, p4g and p6, while examples of symmetry groups of ornaments p3, p3m1 and p31m can be found in the early ornamental art of ancient civilizations, and probably also in the late Neolithic.

According to the stated presence of the corresponding symmetry groups p4m and p6m in the Paleolithic, all the three regular tessellations: {4,4} with symmetry group p4m, {6,3} and {3,6} with symmetry group p6m , are known. Besides the regular square, hexagonal and triangular lattices, in Paleolithic ornamental art we find the remaining two Bravais lattices: the lattice of parallelograms with symmetry group p2 and the rhombic lattice with symmetry group cmm.

In the ornamental art of the Paleolithic and Neolithic, with regard to the construction methods used in obtaining the ornaments we distinguish four construction methods: multiplication of the friezes, multiplication of the rosettes, the method of Bravais lattices and the desymmetrization method. The first construction method is based on the translational repetition of a certain frieze by means of a discrete translation, non-parallel to the frieze axis. Because of the simplicity of this construction, and because of the existence of examples of all the seven discrete symmetry groups of friezes, this method was probably often used for the construction of ornaments. In the Paleolithic, it is probably used for the construction of ornaments with symmetry group **p1**, **p2**, **pm**, (**pg**)*), **pmg** and **pmm**. The similar rosette method of construction is based on the multiplication of a rosette by two independent discrete translations. The symmetry of the ornament obtained is completely defined by the properties of these translations and by the symmetry group of the rosette. The appearance of the Bravais lattices in the Paleolithic and Neolithic ornamental art originates from the models in nature (e.g. honeycomb, different net structures). An other cause is a very high degree of visual and constructional simplicity of the Bravais lattices. The most frequent Bravais lattices, regular tessellations {4,4}, {6,3} and {3,6}, to which correspond the maximal symmetry groups of ornaments **p4m** and **p6m** generated by reflections, have often served as the basis for the application of the desymmetrization method. The importance of this construction method increases especially with the appearance of (two) colored ceramics in the Neolithic, i.e. with the beginning of antisymmetry and colored symmetry ornaments. All these construction methods probably were used in the ornamental art of the pre-scientific period.

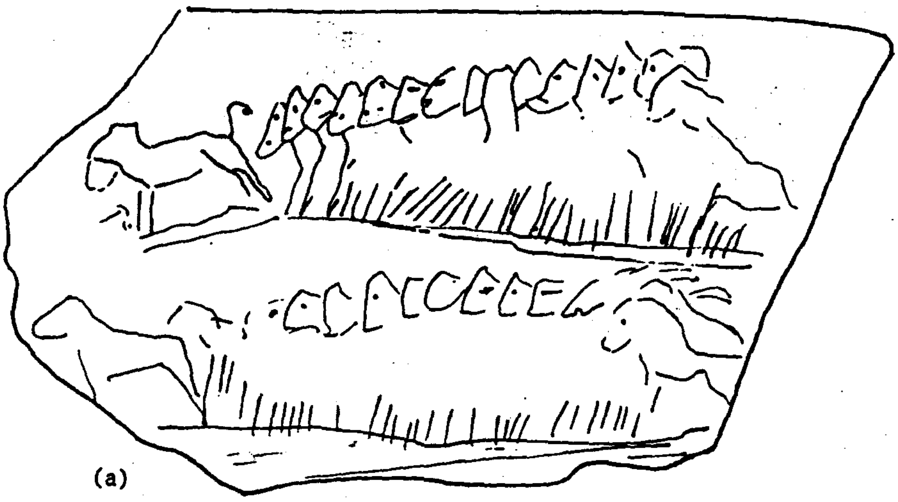
Since they point out to the very roots of ornamental art, ornaments from the Paleolithic, realized as bone engravings or stone carvings and drawings, deserve special attention.

Ornaments with the symmetry group **p1** are based on the multiplication of the frieze with the symmetry group **p11** by discrete translation, or on multiplication of an asymmetric figure by two discrete translations. Because of a low symmetry, they occur relatively seldom, and most often appear with stylized asymmetric models found in nature (Figure 16).

Ornaments with the symmetry group **p2** appear in the most elementary form: as a lattice of parallelograms. A high point of Paleolithic ornamental art are ornaments with application of the meander motif or double spiral, the rosette with symmetry group C_2 (2), which originates most probably on the territory of the USSR. This motifs will be, later on, often used in the ornamental art of almost all Neolithic cultures, mostly as a variation of the motif of waves. Because the forms with symmetry group **p2** are very rare in nature, ornaments with symmetry group **p2** are almost completely limited to geometric motifs or to symbolic stylized motifs (Figure 17).

Ornaments with the symmetry group **pm**, due to the presence of the reflections, belong to the class of static ornaments. Besides the geometric motifs, there is a frequent use of models with plane symmetry which are found in nature (Figure 18).

*) Although according to [22] no examples of ornaments with symmetry group **pg** have been found in Paleolithic, there are grounds to believe that they do appear in Paleolithic ornamental art, as there are examples of the group of symmetry of the friezes **p1g**.



(a)

(b)

Figure 16. Examples of ornaments with the symmetry group p1 in Paleolithic ornamental art: (a) Chaffaud; (b) bone engravings, Europe.

Regarding the frequency of occurrence and variety in Paleolithic and Neolithic ornamental art, ornaments with symmetry group **pmg** and **pmm** are prevailing. Both of these ornaments can be obtained by the frieze method of construction, by translational multiplication of the friezes **pmg** and **pmm** respectively. The ornament with symmetry group **pmg** appears in in its primary form almost always within the geometric ornaments as a stylization of the wave motif. The symmetry group **pmg** offers the possibility for different variations, expressing in the visual sense a specific balance between the static visual component caused by the presence of reflections and dynamic component resulting from the presence of the glide reflection, which suggests the alternating motion (Figure 19).

The static ornament **pmm** generated by reflections is realized in its earliest form as the rectangular lattice, by the multiplication of the frieze with the symmetry group **pmm** by means of a translation perpendicular to the frieze axis or by the rosette method of construction, i.e. by multiplication of the

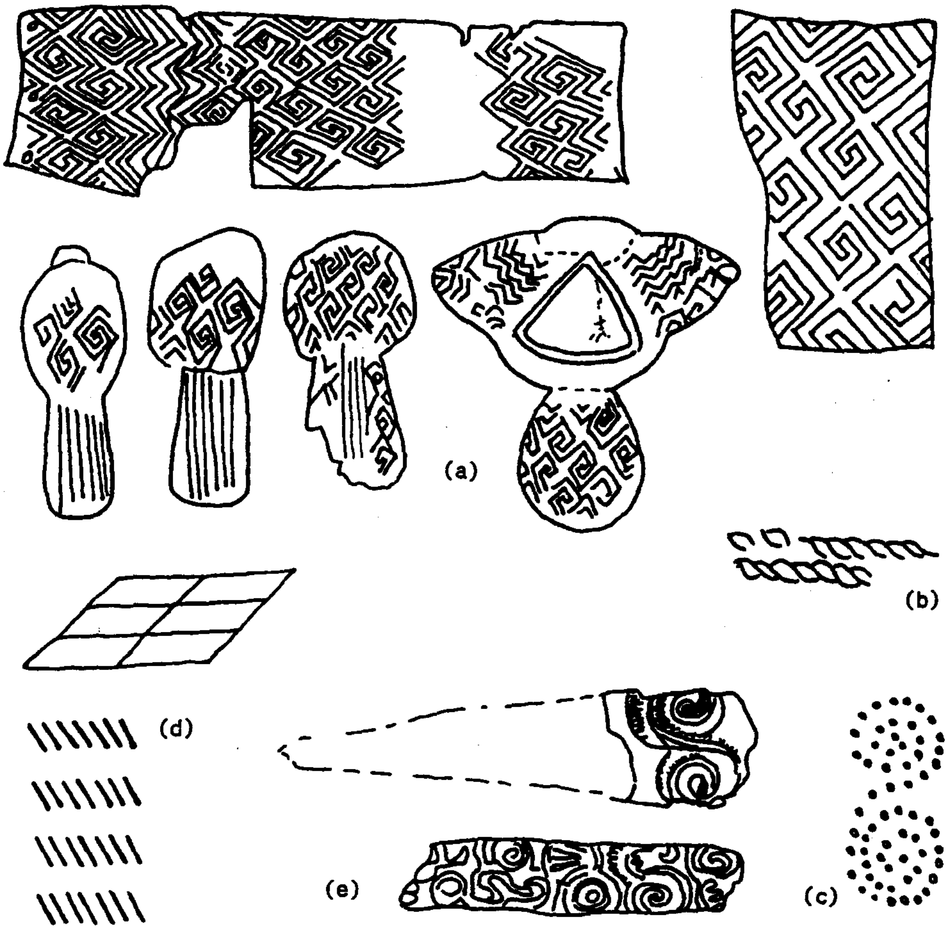


Figure 17. Examples of ornaments with the symmetry group p2 in Paleolithic ornamental art: (a) Mezin, USSR, about 12000-10000 B.C.; (b) Western Europe; (c) the motif of double spiral, Maljta, USSR; (d) the lattice of parallelograms, bone engraving from Western Europe; (e) Arudy, Isturiz.

with the symmetry group D_2 (2m) by means of two translations perpendicular to the corresponding reflection lines of the rosette (Figure 20).

Ornaments with the symmetry group cmm appear in the Paleolithic in the form of the rhombic lattice. These ornaments can be constructed from the ornament with symmetry group pmm by centering, i.e. by the procedure in which the gaps between the

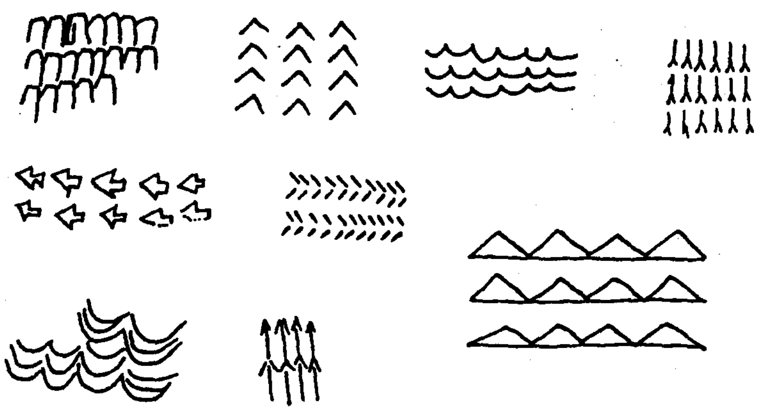


Figure 18. Examples of ornaments with the symmetry group pm in Paleolithic ornamental art (Ardales, Gorge d'enfer, Romanelli cave).

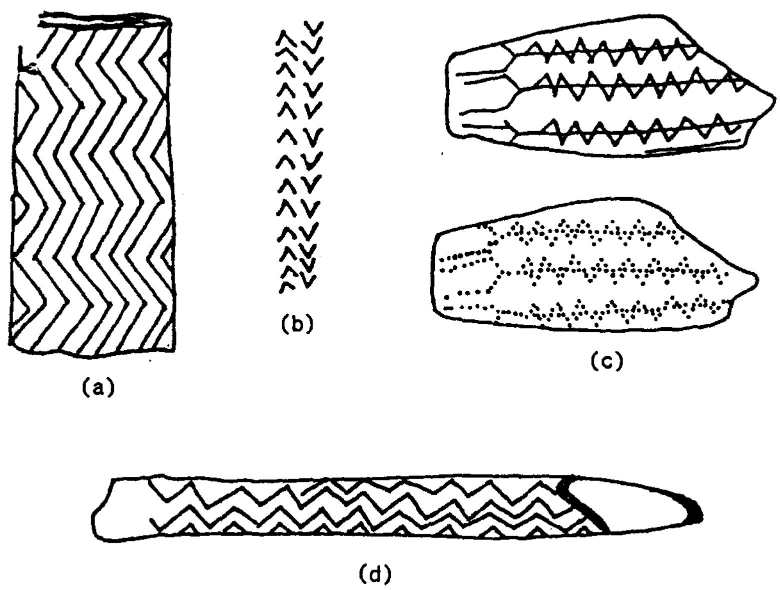


Figure 19. Examples of ornaments with the symmetry group pmg in Paleolithic ornamental art: (a) Mezin, USSR; (b) Western Europe; (c) Pernak, Estonia; (d) Shtetin.

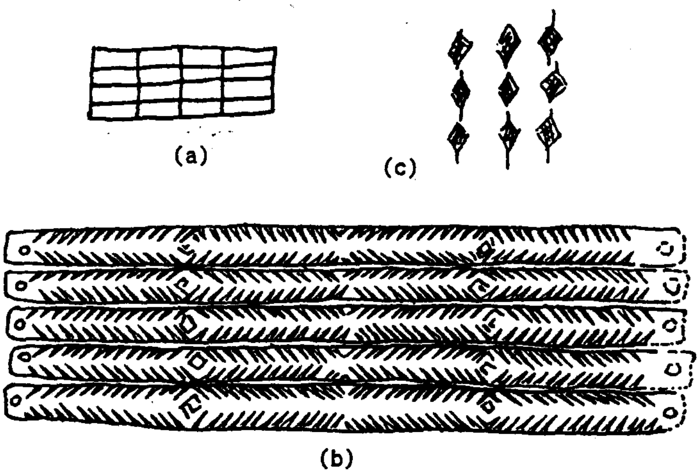


Figure 20. Examples of ornaments with the symmetry group **pmm** in Paleolithic ornamental art: (a) the Bravais lattice of parallelograms, Lasco cave; (b) Mezin, USSR, about 12000 B.C.; (c) Laugerie Haute.

rosettes D_2 (2m) forming this ornament are filled with the same rosettes (Figure 21).

The ornaments with the symmetry group cm are obtained from the ornaments with the symmetry group pm by the same procedure – by centering (Figure 22).

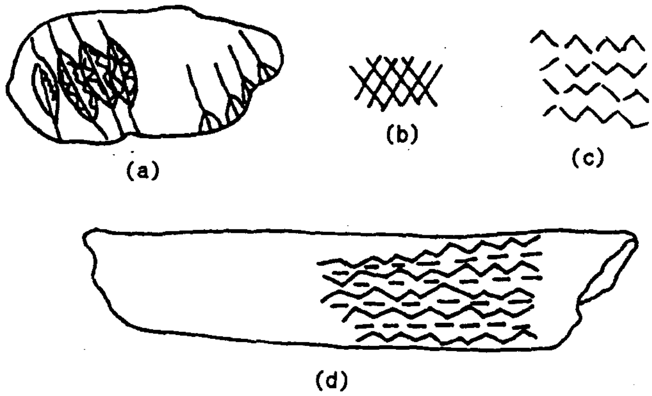
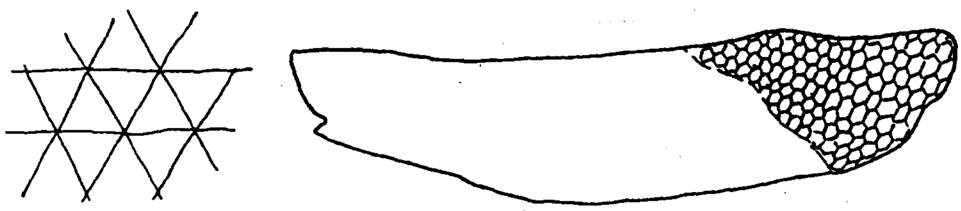


Figure 21. Examples of ornaments with the symmetry group **cmm** in Paleolithic ornamental art: (a) Polesini cave; (b) Laugerie Haute; (c) Pindel; (d) Vogelherd.

Figure 22. Example of the ornament with the symmetry group **cm** in Paleolithic ornamental art.

The symmetry groups of ornaments **p4m** and **p6m** correspond to the regular tessellations {4,4}, {6,3} and {3,6}. The regular tessellation consisting of regular hexagons, three of which are incident with each vertex of tessellation, most probably originates from its model in nature: the honeycomb (Figure 23). The regular tessellations {3,6} (Figure 23) and {4,4} (Figure 24) are from the same period, the Paleolithic.



(a) (b)

Figure 23. Examples of the regular tessellations with the symmetry group **p6m**: (a) {3,6}; (b) {6,3}, Yeliseevichi, USSR.

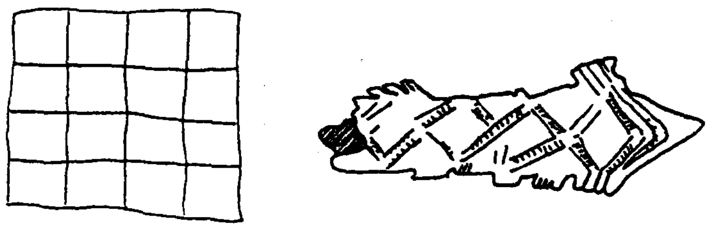


Figure 24. Example of the regular tessellation {4,4} and the ornament with the symmetry group **p4m** in Paleolithic ornamental art.

The principle of visual entropy: maximal visual and constructional simplicity and maximal symmetry is a common, universal characteristic of all Paleolithic ornaments. Hence, among Paleolithic ornaments five of the nine existing symmetry groups of ornaments correspond to the Bravais lattices, seven of the nine groups contain reflections and belong to a class of static ornaments. The almost complete absence of the dynamic elements of symmetry: polar translations, polar rotations and glide reflections, is evident.

In the Neolithic period we have the appearance of almost all the remaining symmetry groups of ornaments. A special place in Neolithic ornamental art have the antisymmetry, "black-white" ornaments. The majority of the 46 antisymmetry groups appear in Neolithic ornamental art, in particular in the ornamental art of the Near and Middle East (Tal el Hallaf, Hacilar, Catal Hüjük,...). If we treat antisymmetry ornaments with the antisymmetry group p6m/p3m1 as the classical-symmetry ornaments obtained by the method of antisymmetry desymmetrization, we can add to the list of symmetry groups of ornaments appearing in the Neolithic, also the symmetry group p3m1 (Figure 25).

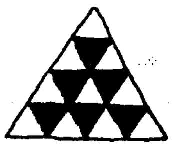


Figure 25. Example of the antisymmetry ornament with antisymmetry group p6m/p3m1 in Neolithic ornamental art. From the point of view of the classical theory of symmetry this ornament can be treated as an example of appearance of the symmetry group p3m1 in Neolithic ornamental art.

Neolithic ornamental art is one of the richest sources of different ornaments in all the history of ornamental art. The examples of 14 symmetry groups of ornaments (Figure 26) and 23 antisymmetry groups of ornaments (Figure 27) found in Neolithic ornamental art are the most complete testimony about the artistic creativity of Neolithic peoples.

Ornaments with the symmetry group p3, p3m1 and p31m represent quite a problem with regard to their construction. In classical-symmetry ornamental art they first appear in the ornamental art of ancient civilizations, or maybe earlier, in late Neolithic ornamental art (Figure 28).

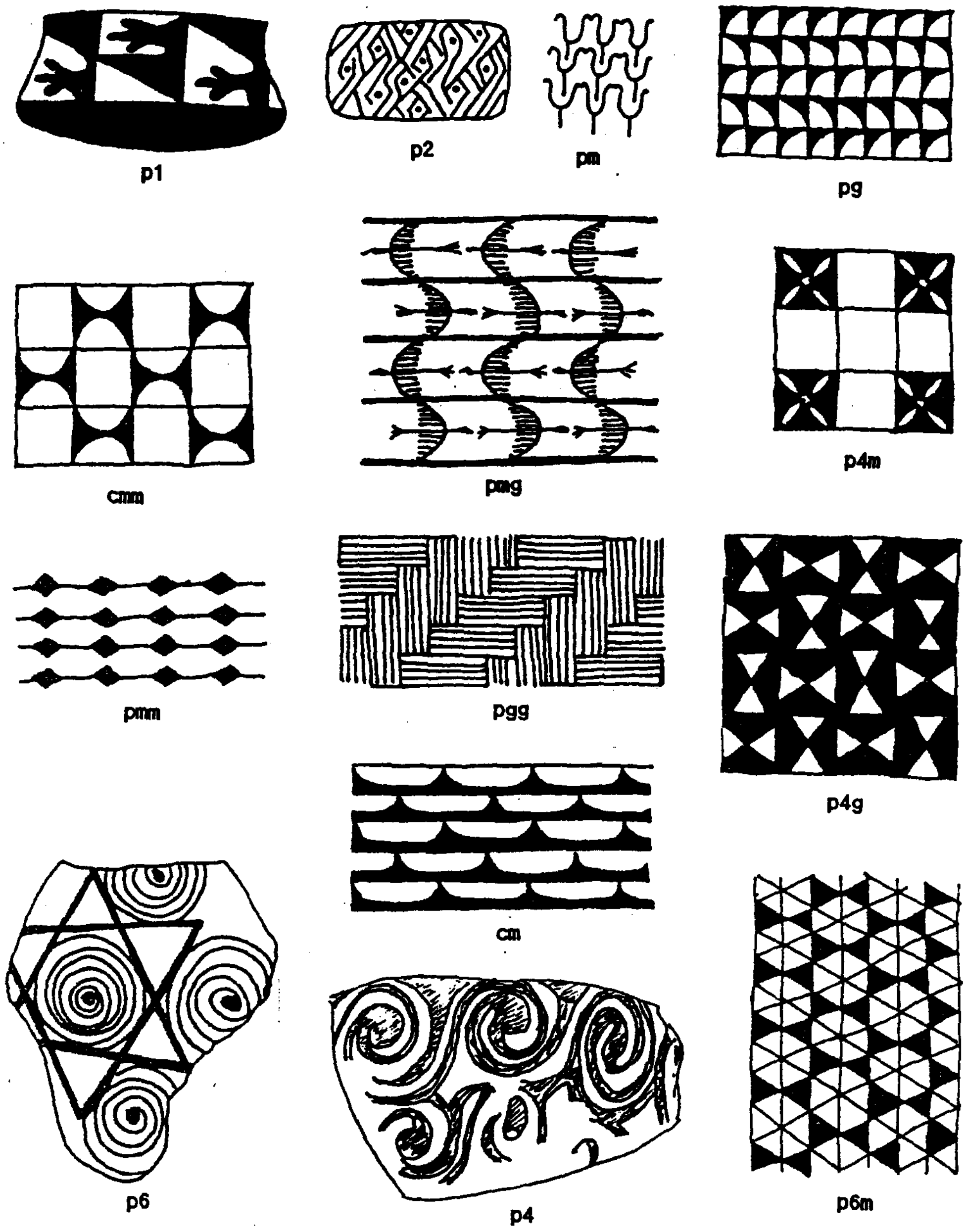
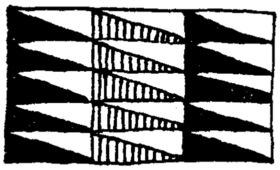
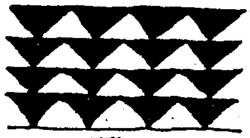
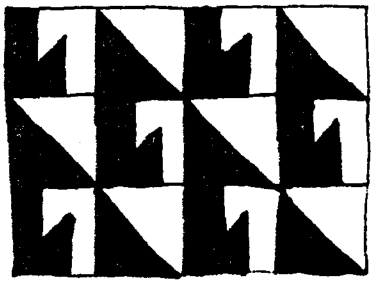


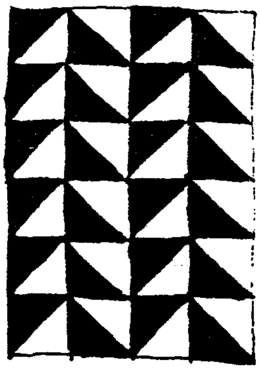
Figure 25. The examples of 14 symmetry groups of ornaments in Neolithic ornamental art.

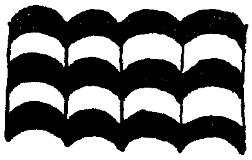

p1/p1

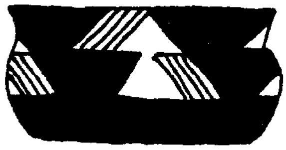

p2/p2

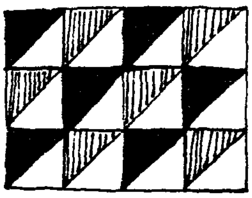

pmg/pm

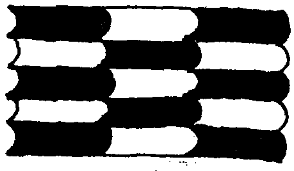

p2/p1

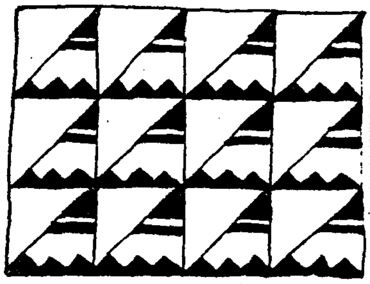

pmg/pg

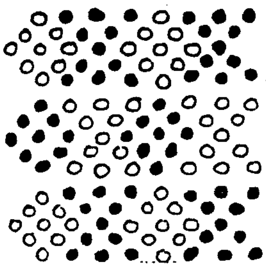

pm/pm1

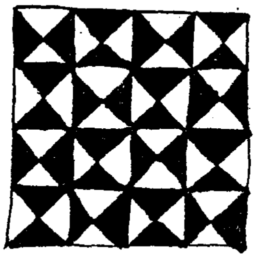

pg/pg

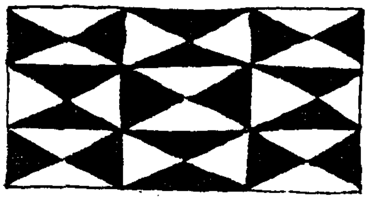

cm/pm

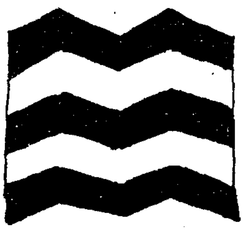

pm/cm

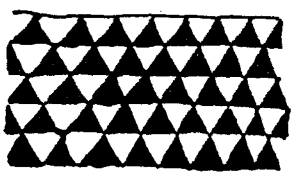

cm/p1

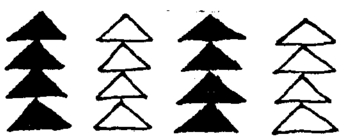

pmm/pmm


p4m/p4g


pmm/cmm


pmg/pmg


p6m/p3m1


pm/p1m

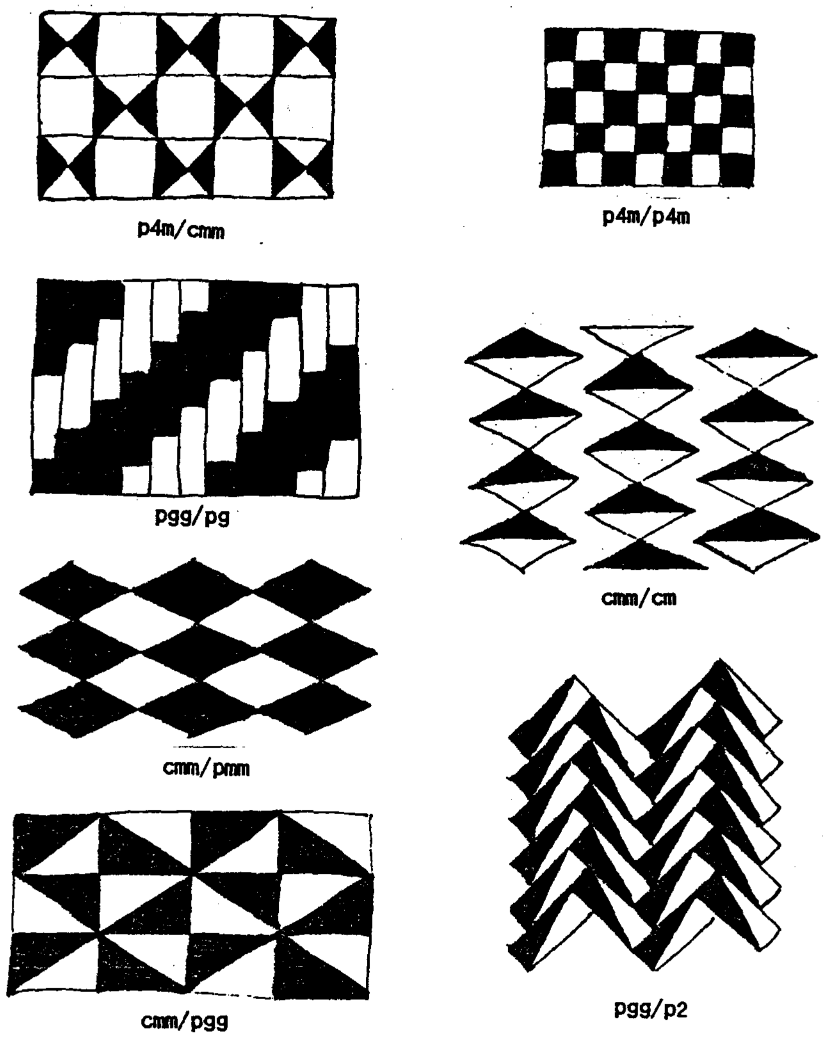


Figure 26. Examples of 23 antisymmetry groups of ornaments in Neolithic ornamental art.

The information on the geometric constructions, technical means and instruments used in the pre-scientific period cannot be gathered from the study of ornamental art, because the relatively small dimensions of ornaments did not require any use of instruments. Implicit data are reached by the study of megalithic monuments, stone constructions found mainly in Western Europe, the oldest examples of which date from the IV millennium B.C. Having in mind the independent development of Neolithic cultures in Western Europe and their later appearance with respect to the culturally superior East, even the study of somewhat younger monuments (about 3000 B.C.) can serve as a record of the original geometric-astronomic knowledge. A synthesis of the results of the archaeological studies of megalithic monuments is given by J.E.Woods [25].

Poles and ropes were the first geometric instruments used in this period for the construction of circles, spirals and ovoids. These constructions were realized with a very high degree of precision. The deviations occurred because the ropes of animal or plant origin, stretched by pulling. Therefore, the circles constructed in Stonehenge has a constructional deviation of less than 0,4% while the circle in Brodgar has a deviation of 1%.

The construction of the ellipse was discovered, most probably, by accident during the construction of a circle, when a rope got stuck against some obstacle. The shape obtained was interesting enough, and motivated the architects of megalithic monuments to investigate further the construction of ellipse.

The equidistant, Archimedes spiral, was constructed in the Neolithic by winding a rope around the pole. The ovoids were the result of combination of the circle and ellipse construction.

For the construction of the right angle the Pythagorean triangle (3,4,5) and the approximate Pythagorean triangles (8,9,12), (11,13,17), (12,35,37), are used. They are very often found in the basic length elements for the construction of the ellipse. The application of approximate Pythagorean triangles shows that the theorem of Pythagoras was not known. Geometric knowledge was of an empiric character, based on noticing in practice the triangles suitable for the construction of the right angle and remembering their dimensions.

By studying the metric characteristics of megalithic monuments we come to the hypothesis of the existence of a metric standard, the length unit: the megalithic yard, which is also indicated by the dimensions of megalithic monuments in Carnac (Bretagne), Avebury (England) and Brodgar (Orkney Islands).

The megalithic constructions, circles of stones, "temples" and "platforms" most probably had a function of solar and lunar observatories. It was noticed that in the "temple" in New Grange, in the day of the winter solstice, sun beams shine through the opening on the roof part, and illuminate the room. Platforms had a similar function. In the case of platforms as the reference points for astronomic observations outstanding points on the horizon of the landscape (e.g. tops of mountains, mountain passages) were used. Such use of natural reference points considerably increased the precision of the results of astronomic observations.

New data on the geometric knowledge of the pre-scientific period can be obtained from more detailed recent studies of the Neolithic monuments.

Very interesting and insufficiently explored fields related to the geometry of the pre-scientific period are still the following: dating of the appearance of all the plane symmetry structures and corresponding classical-symmetry, antisymmetry and color-symmetry groups, the registering of the most significant archaeological exavation sites from the point of view of the theory of symmetry and ornamental art, the links between the ornamental art of different cultures, the links between the friezes, natural numbers and calendars, etc. All these and many other similar questions relevant to the history of mathematics of the pre-scientific period should become a common field of research for mathematicians, archaeologists and specialists of different sciences.

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ORNAMENT TODAY

The origins of ornamental art date back to the Paleolithic (Magdalenian, about 10000 B.C.) and represent one of the first human attempts to perceive, understand and express symmetry, regularity and harmony. Already at this stage, we have examples of the symmetry groups of rosettes, all the seven discrete symmetry groups of friezes and most of the 17 discrete symmetry groups of ornaments (Figure 1).

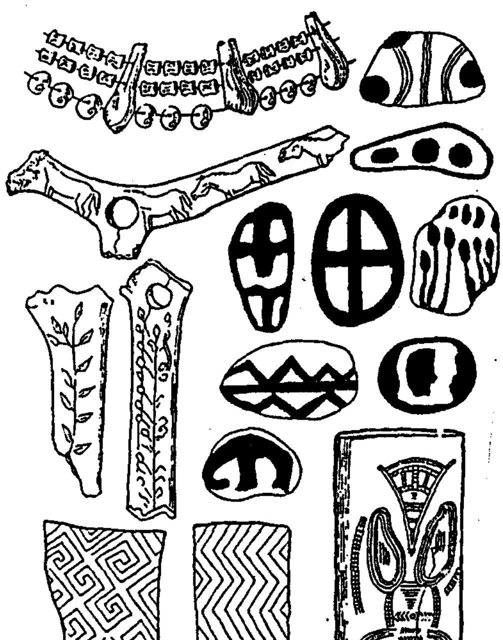


Figure 1. Examples of rosettes, friezes and ornaments in Paleolithic ornamental art.

In this period as well as in the Neolithic and in the art of the ancient civilizations (Egypt, China, Japan...) ornamental art is flourishing, so examples of all plane symmetry groups of isometries are being realized.

In the oldest phases of ornamental art, the symbolic role of ornaments is prevailing, so that ornaments became an aspect of communication. By history, symbolic meanings gradually disappear, retreating before other means of communication. After geometric rules on which ornaments are based have been understood and construction problems solved, ornamental art offers possibilities for a playing, artistic imagination and decorativeness (Figure 2, 3).

Figure 2. Ornaments of primitive peoples, New Guinea.

Ornamental art was of great importance in several pre-Renaissance cultures (e.g. Egyptian, Arab, Moorish...). The recent Western art brought the division of the fine arts into the "fine" and "decorative" arts, with ornamental art playing a role of a second-rate decorative art. Thus, unfortunately, the title "Ornament Today" is associated with everyday decorative products such as wallpapers, decorative fabrics, rather than with fine art creations. However, the 20th century art itself and its connections with natural sciences, mathematics, crystallography and the theory of symmetry, have opened new possibilities for a development of ornamental art, which are beyond reach of empirical studies.

After the studies of the space symmetry structures and corresponding symmetry groups, the 32 crystal classes, the 14 Bravais lattices and the 230 space symmetry groups (Fedorov groups) the interest of mathematicians and crystallographers turned in the thirties, towards so-called "small crystallographic groups" and "sub-periodic groups" – plane symmetry groups of rosettes, friezes and ornaments. Besides the rosette symmetry

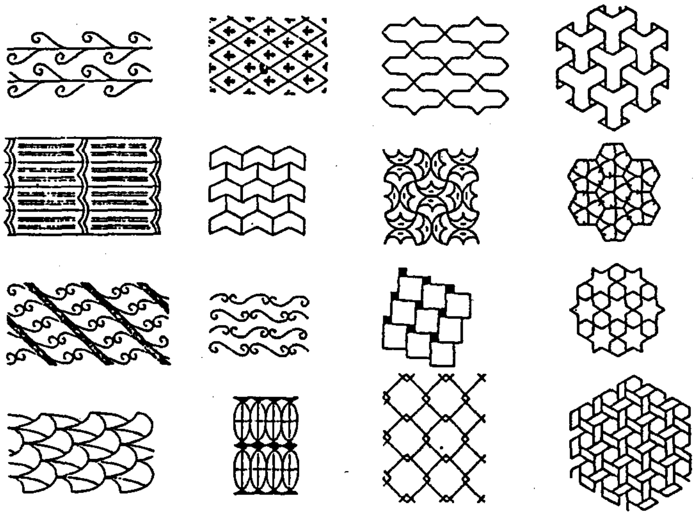


Figure 3. A choice of ornaments of different cultures, illustrating unlimited possibilities of variety of ornamental motifs.

groups C_n and D_n, which were discovered earlier, the mathematicians like G.Pólya, P.Niggli A.Speiser [37] and others prove the existence and completeness of the list of the seven discrete symmetry groups of friezes and the 17 discrete symmetry groups of ornaments as well as the list of the corresponding semicontinuous and continuous symmetry groups. The research then began based on the new possibilities arising from these discoveries (Figure 4).

Looking for examples, visual illustrations of corresponding symmetry groups, mathematicians make use of the rich ornamental heritage of ancient civilizations, mainly Egyptian, Arab and Moorish ornaments. The idea of studying ornaments belonging to different cultures from the point of view of the theory of symmetry distinctly differs from the previously accepted descriptive method of ornament classification. The new idea of classifying ornaments, announced in the work of A.Speiser, is quickly accepted and applied in the works of E.Müller, H.Weyl [39], A.V.Shubnikov and V.A.Koptsik [34], D.K.Washburn [38], D.W.Crowe [7,8] and others. Contrary to the former descriptive methods of classification, which divided ornaments according to motifs into the "geometric", "plant" or "animal", ornaments are now classified in accordance with corresponding symmetry groups. One of the most interesting investigations referred to the question of the appearance of all the 17 symmetry groups of

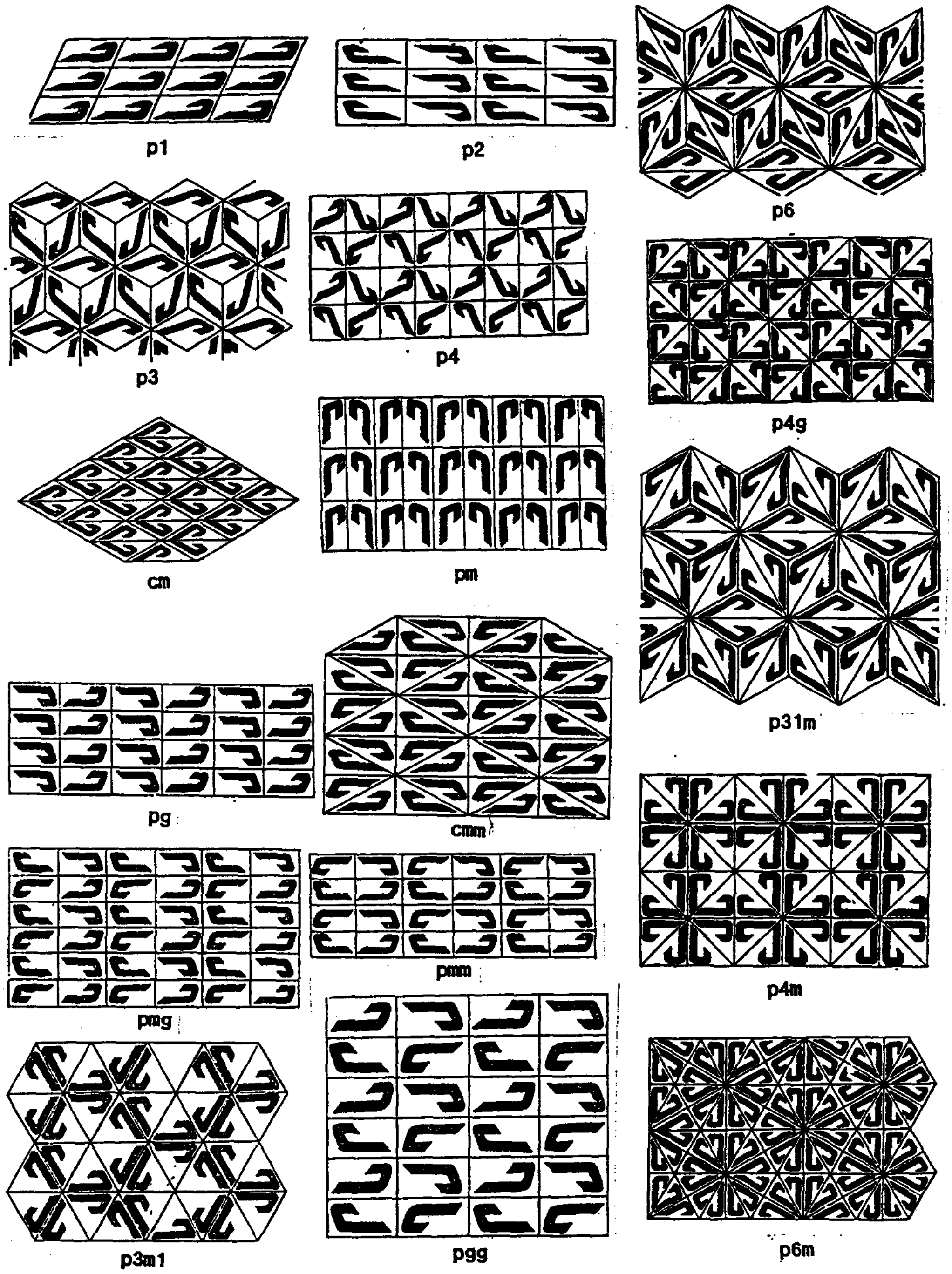


Figure 4. The 17 symmetry groups of ornaments.

ornaments in the ornamental art of Egypt [39,16]. A more detailed studies of the ornamental art of individual cultures, based mainly on archaeological sources from Africa are given in the works of D.W.Crowe [7,8] (Figure 5) and D.K.Washburn [38], who discusses the ornamental art of American Indians (Figure 6).

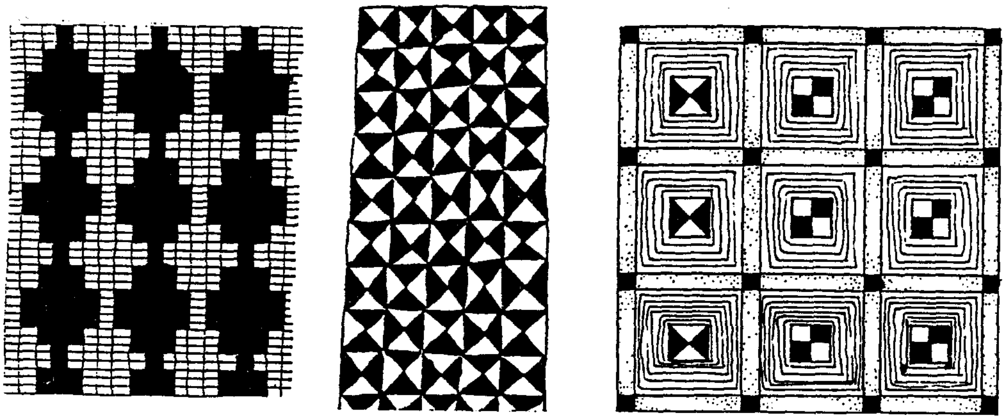


Figure 5. Ornamental motifs from Africa, Bakuba art.

In the search for the origins of ornamental art, in the monograph [24] it is stated the existence of examples of symmetry groups of rosettes, all the seven discrete groups of symmetry of friezes and the 11 of the 17 symmetry groups of ornaments in Paleolithic ornamental art. In that way, the empiric geometric heritage preserved in ornamental art, becomes accessible for a detailed analysis and classification making use of the theory of symmetry, offering at the same time the authentic information on the origins of mathematical thinking in the pre-historic period, and thus opening a whole new chapter for the study of the mathematical knowledge history.

The analysis of ancient ornaments, in particular that from Paleolithic and Neolithic ornamental art, brings up the hypothesis that the visual-geometric properties of ornaments define the time of the first appearance and the frequency of occurrence of certain symmetry groups in ornamental art. In accordance with the contemporary studies of visual perception from the standpoint of the gestalt-psychology [1], it is possible to formulate the principle of visual entropy – principle of the maximal constructional and visual simplicity and maximal symmetry. In its wider sense, this principle can also be applied to the studies of aestethic grounds on which different works of art are based and for a creation of more exact aesthetic criteria and analytic approaches to fine art works.

After the appearance of non-figurative art (abstraction,

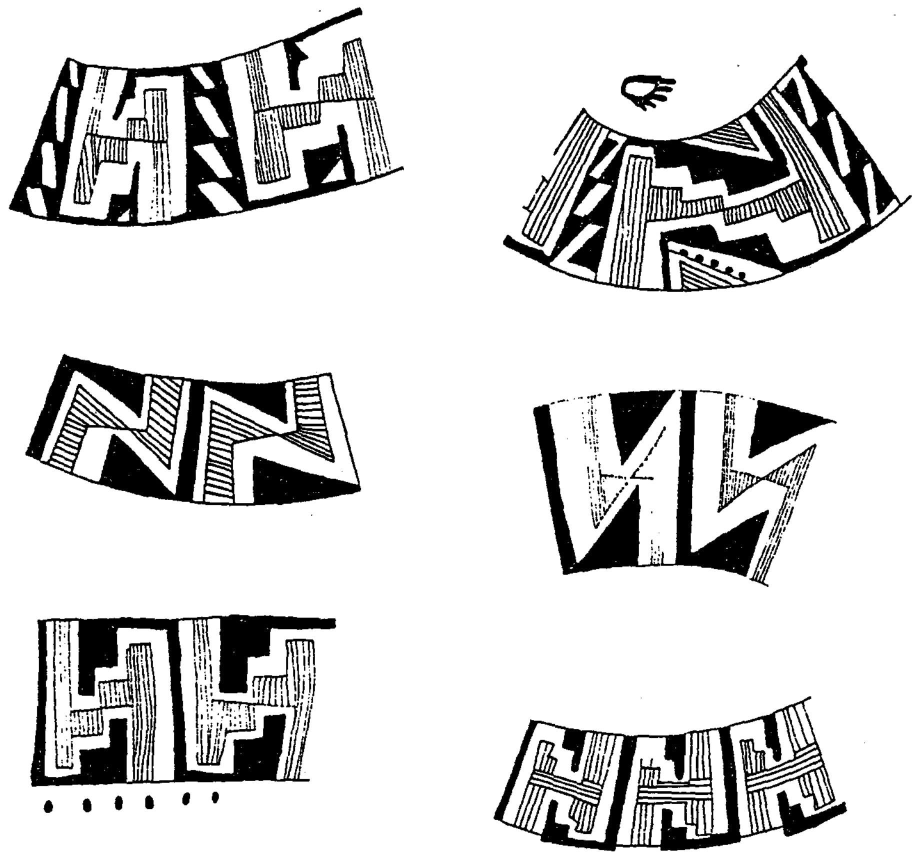


Figure 6. Ornamental motifs in the ornamental art of American Indians, Upper Gila Area.

especially geometric abstraction) of the 20th century, the classical descriptive language and criteria of the fine art aesthetics, containing numerous elements which do not refer to visual characteristics (e.g. subject, motif, degree of reality...), become insufficient or completely inadequate for contemporary aesthetic analyses. When observing the blank, white square by K.Malevich, and when trying to find out what makes it a unique, unrepeatable work of art, we have to apply visual criteria, closely connected to the laws of the theory of symmetry in its widest sense, that means, in the sense of "well organized forms" [1].

When searching for such criteria and looking for universal laws of the "well organization", harmony and accord, the simplest symmetry forms to which isometric symmetry groups correspond, can be used. Isometric ornamental plane figures: rosettes, friezes and ornaments, represent only the most elementary form of regular, symmetry plane structures. Other visual components such as a color, relation "figure"-"back-ground", convexity, concavity, topological equivalence, etc., can be discussed by extending the classical theory of symmetry to the antisymmetry, colored symmetry, curvilinear symmetry, similarity symmetry, conformal symmetry, non-Euclidean symmetry...

The first of these extensions - antisymmetry, i.e. two-colored "black-white" symmetry, includes besides geometric symmetry transformations an involutional, often non-geometric bivalent change (e.g. color change "black"-"white") commuting with all symmetry transformations (Figure 7).

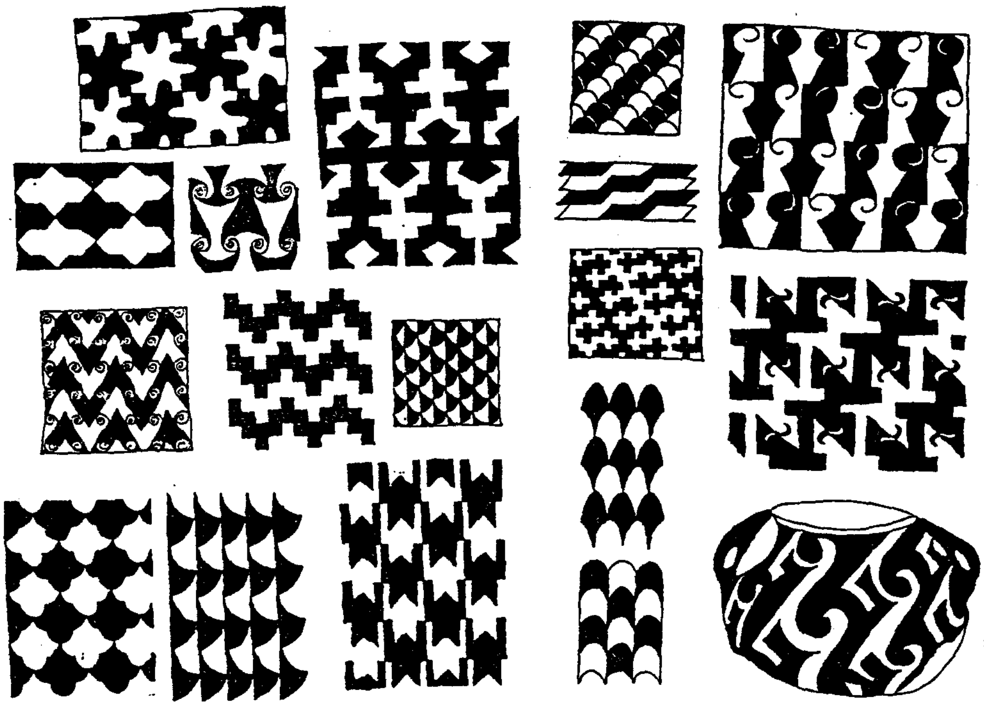


Figure 7. Examples of antisymmetry ornaments.

Antisymmetry ornaments from Neolithic ornamental art, are the result of artist's wish to enrich the ornament, express the mutually opposite alternating features – forms of duality in nature, to create the contrast and suggestion of space. They became a subject of mathematical studies in the thirties, and are discussed in the works of H.Heesch, A.V.Shubnikov [34,35], N.V.Belov [35], A.M.Zamorzaev [41,42] and others. As a starting point for introducing the antisymmetry, Weber "black-white" diagrams of bands are used. The color change "black"-"white" served as a possibility for the dimensional transition, i.e. for the interpretation of the three-dimensional space in a plane. An analogue approach results in the 46 antisymmetry groups of ornaments which together with the 17 generating and the 17 senior antisymmetry groups correspond to the 80 symmetry groups of layers. Applied to the n-dimensional symmetry groups, antisymmetry becomes a tool for derivations and analyses of sub-periodic symmetry groups of the (n+1)-dimensional space.

From the artistic point of view, multi-dimensional symmetry structures and their plane interpretations open a large unexplored field (Figure 8).

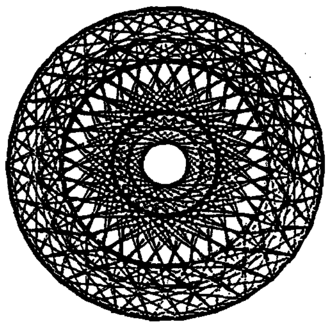


Figure 8.The projection of the four-dimensional polytope {3,3,5}.

Further generalizations of the theory of symmetry lead over the multiple antisymmetry to the colored symmetry, discussed in the works of N.V.Belov [35], A.Loeb [29], A.M.Zamorzaev, A.F.Palistrant and E.I.Galyarskii [42], M.Senechal, T.W.Wieting [40] and others. The colored symmetry is a polyvalent symmetry, which besides symmetry transformations includes a "color change", i.e. the change of any polyvalent feature (e.g. some physical property) commuting with symmetries.

The antisymmetry and colored symmetry offer at the same time a possibility to study and visualize various symmetry structures together with their physical properties. In that way, besides their symmetry – regular geometric organization of structures (e.g. crystal structures) it is possible to analyze different physical characteristics, whose change can be interpreted by color permutations. In spite of high scopes of the theory of colored symmetry, many problems originating from the ornamental art studies (e.g. such color-symmetry stuctures, where quantities of different colors are used in the given ratio $p:q$, or $p:q:r$), still remain unsolved [19].

The next series of problems refers to geometric and visual characteristics of ornaments considering the form of the fundamental region, elementary asymmetric figure by multiplication of which, using symmetry transformations of the given group, a plane isohedral tiling can be achieved (Figure 9).

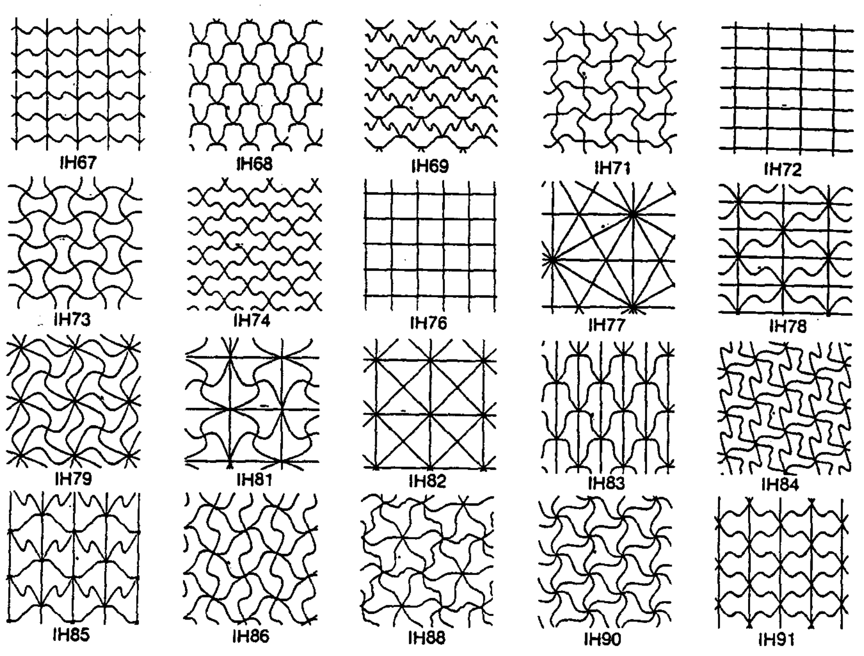


Figure 9. Examples of isohedral tilings.

Tilings, ideal mosaical coverings of a plane without gaps and overlaps, used in ornamental art of different cultures (Egyptian, Arab, Moorish...) (Figure 10, 11) have find its mathematical interpretation in the tiling theory. Of a special interest is the problem of the periodic monohedral tilings – isohedral tilings discussed in the works of H.Heesch [20], B.Grünbaum and G.C.Shephard [17,18], as well as similar problems related to the uniform tilings (Figure 12), tilings with star polygons (Figure 13), isogonal tilings (Figure 14), n-hedral tilings and colored tilings (Figure 15).

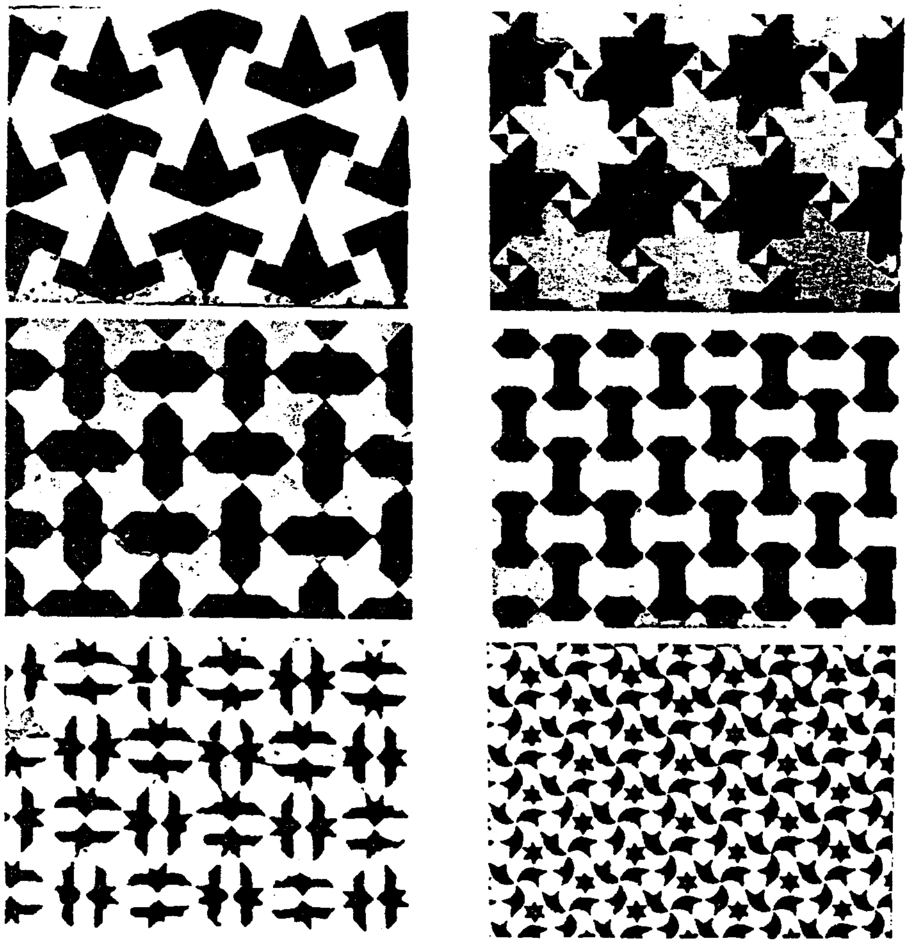


Figure 10. Mosaics from Alhambra.

Although a considerable progress has been made in this field, there are still numerous unsolved (and maybe unsolvable) problems. They are, e.g. to define the universal criterion offering an answer to the question whether the given figure can be a protile for constructions of monohedral tilings. The similar question is: how many isohedral tilings with different symmetry groups generate the chosen protile (Figure 16).

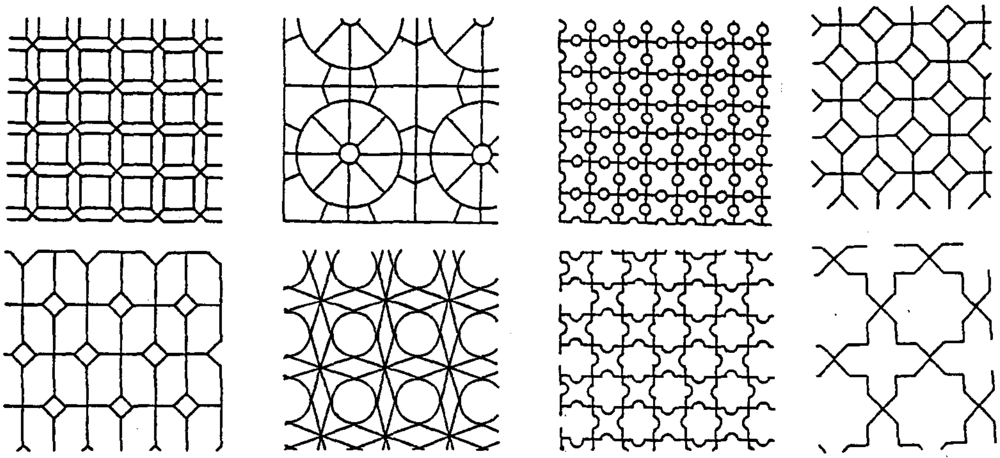


Figure 11. Floor ornamental tilings, Portugal, the XV century.

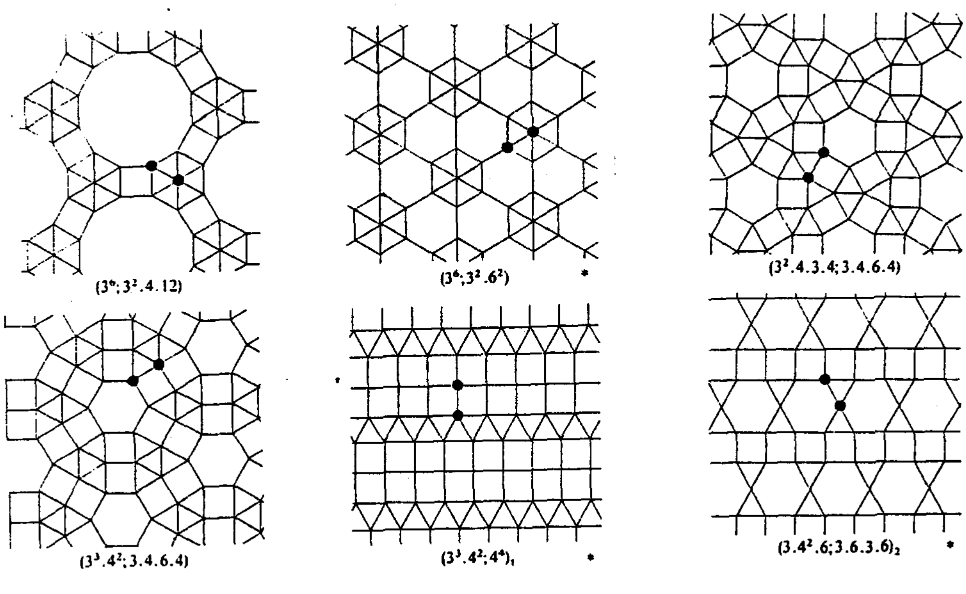


Figure 12. The six of the 20 types of 2-uniform tilings.

A separate chapter of the tiling theory represent the aperiodic tilings. In the case of aperiodic tilings the principle of crystallographic restriction (n=1,2,3,4,6) does not hold. The importance of aperiodic tilings becomes evident when quasicrystal structures as well as organic structures, like the DNA-structure called by E.Schrödinger "the aperiodic crystal", are discussed (Figure 17).

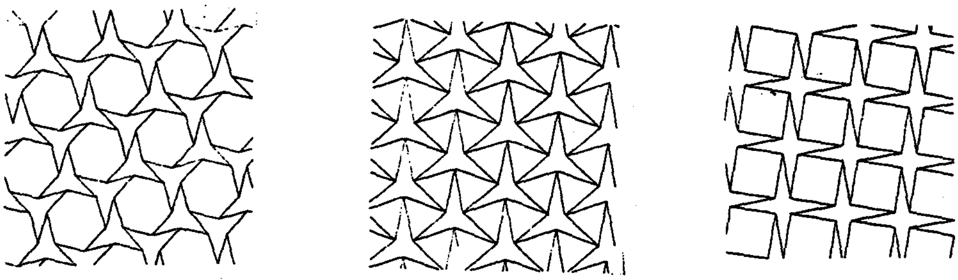


Figure 13. Uniform tilings by regular and star polygons.

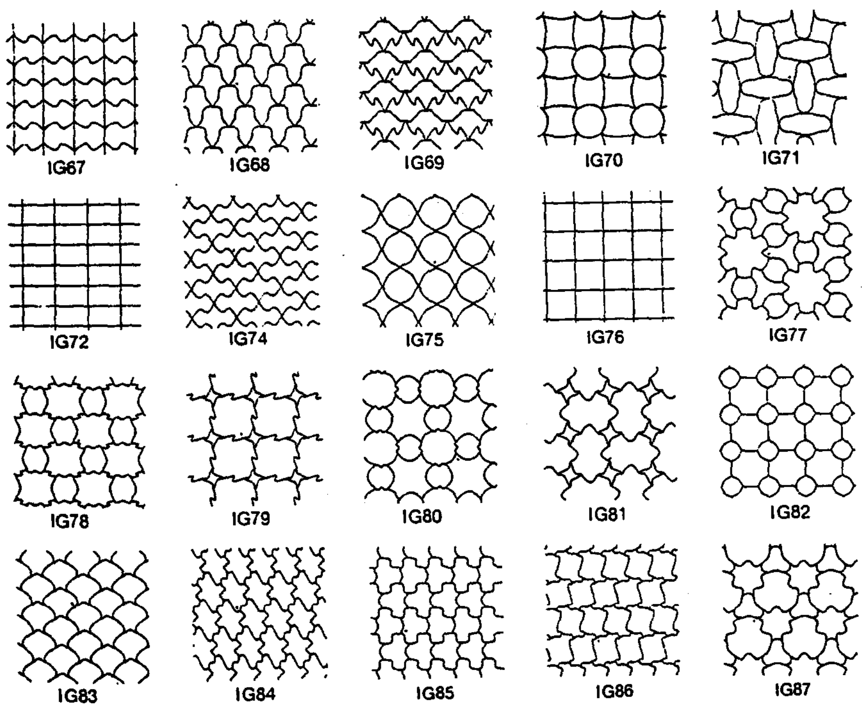


Figure 14. Isogonal tilings.

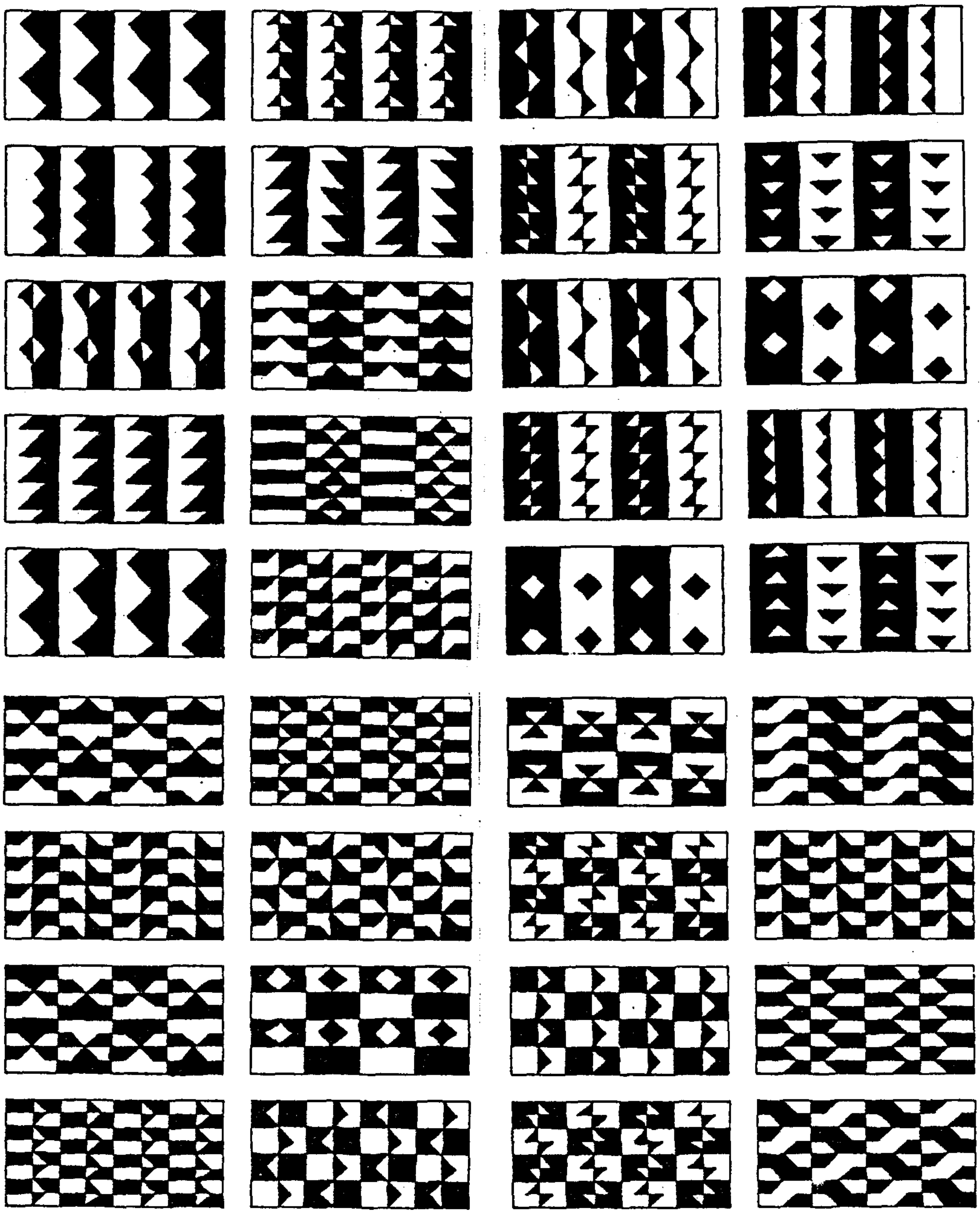


Figure 15. Different monohedral and 2-hedral "black-white" tilings obtained by the use of the multiple antisymmetry.

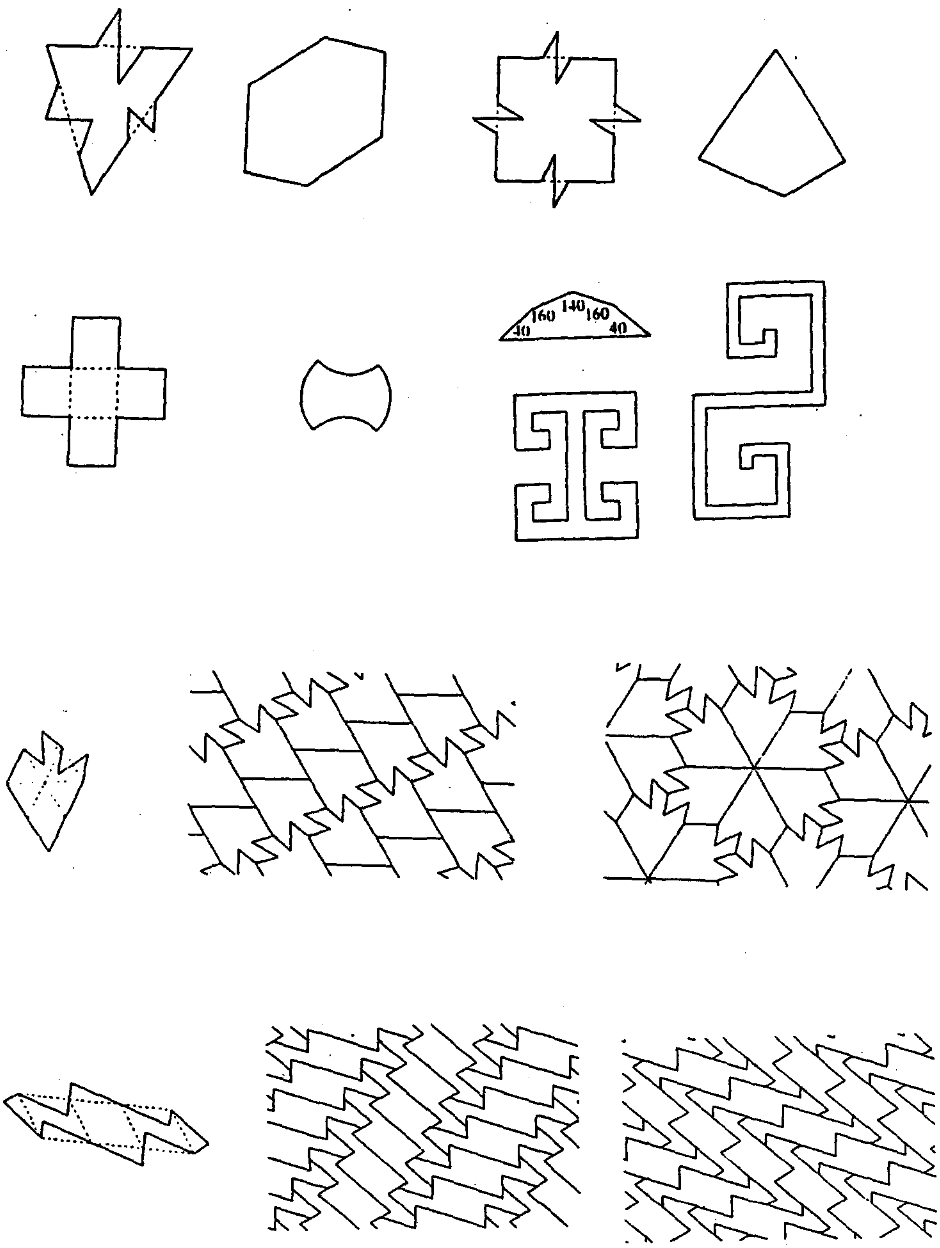


Figure 16. A survey of protiles which can be used for constructions of monohedral tilings. Some of them generate two or more different tilings.

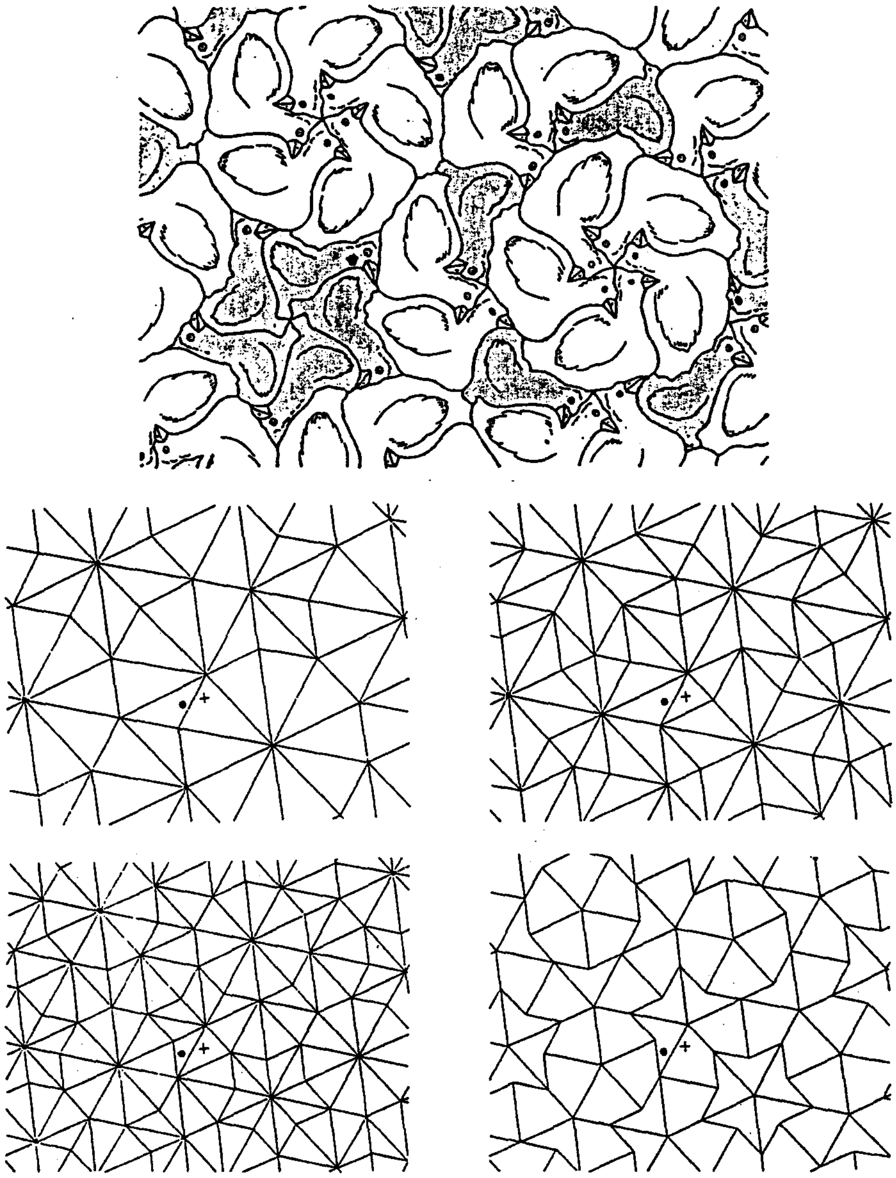


Figure 17. "Penrose chickens" and the schemes corresponding to Penrose aperiodic tilings.

Besides the structures mentioned, relatively uninvestigated are interlaced symmetry structures which find their place in ornamental art and the theory of symmetry (Figure 18).

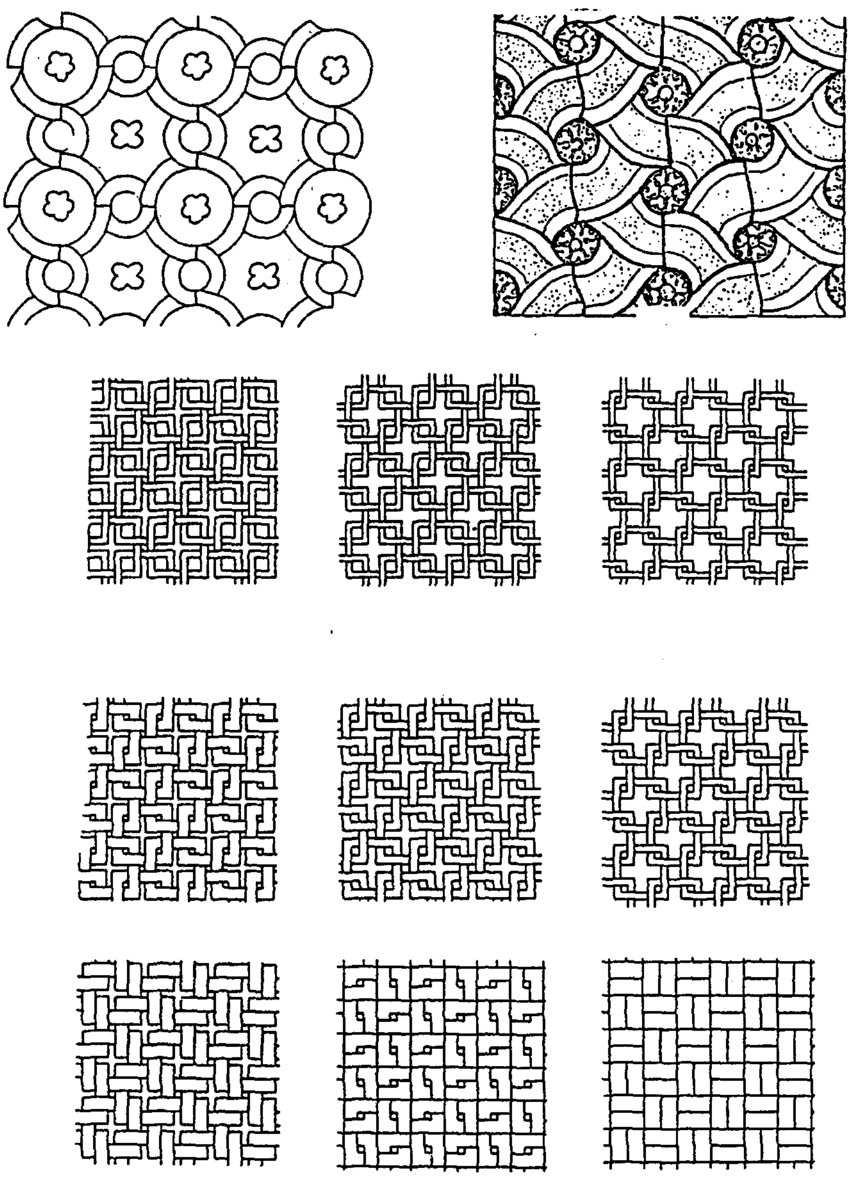


Figure 18. Different interlaced ornaments.

An advantage of the theory of symmetry as a scientific discipline is a possibility to formulate most of problems, assumptions and theorems in a simple language, enabling specialists of different scientific profiles, even the amateurs, to take part in their solving.

Similarity symmetry groups are discussed in the works of H.Weyl [39], A.V.Shubnikov [34], A.M.Zamorzaev, E.I.Galyarskii and A.F.Palistrant [42] and others. After the inspiring book of H.Weyl [39] in which appearances of the similarity symmetry in nature (the symmetry of Nautilus shell, sunflower Heliantus maximus, pineapple) are analyzed, the similarity symmetry groups are studied by different authors. Many examples of these groups are present in the ornamental art of different cultures. The appearance of all the plane similarity symmetry groups belonging to the five infinite classes C_nK, C_nL, C_nM, D_nK and D_nL, can be traced in the history of ornamental art [24]. Probably the most interesting in the visual sense are the similarity symmetry groups C_nL and D_nL, connected with the use of the logarithmic, equiangular spiral – invariant line of the group of linear transformations. The logarithmic spiral, geometric properties of which J.Bernoulli has described by the words "Eadem mutata resurgo", occurs in ornamental art from the ancient times, in particular in Greek-Roman ornamental art (Figure 19).

The further generalization – conformal symmetry groups, due to the absence of models in nature, indicates a different approach, which is characteristic for the 20th century science: a path from an abstract theory towards corresponding visual interpretations, understood as a model of this abstract theory. Such an approach, occurring in geometry at the beginning of non-Euclidean geometries, results in the theory of conformal symmetry groups and in visual models of all finite and infinite conformal symmetry groups. Since these groups are isomorphic with the symmetry groups of tablets and, comprising the similarity symmetry groups, with the symmetry groups of rods – three-dimensional line symmetry groups, there is, besides by antisymmetry, another possibility for interpretation of three-dimensional symmetry structures in a plane (Figure 20).

All symmetry groups afore mentioned can be extended by the antisymmetry and colored symmetry.

The non-Euclidean crystallography represents a special field of the theory of symmetry. The symmetry groups of the hyperbolic plane can be interpreted within Poincaré or Klein model of the hyperbolic plane. In these models, circle inversions have a role of line reflections in the hyperbolic plane [9,10]. In contradistinction from the 17 symmetry groups of ornaments of the Euclidean plane, in the case of hyperbolic ornaments there is an infinite number of symmetry groups.

The 20th century fine arts are characterized by abstraction. The geometric abstraction first appeared in the works of cubists and was developed further on in the works of many artists, in particular by the Russian constructivists, and suprematists (V.Suetin, K.Malevich and others). Parallel to this goes the development of geometric ornamental art. The problems of construction of different kinds of ornaments (antisymmetry, colored symmetry, similarity symmetry, conformal symmetry, non-

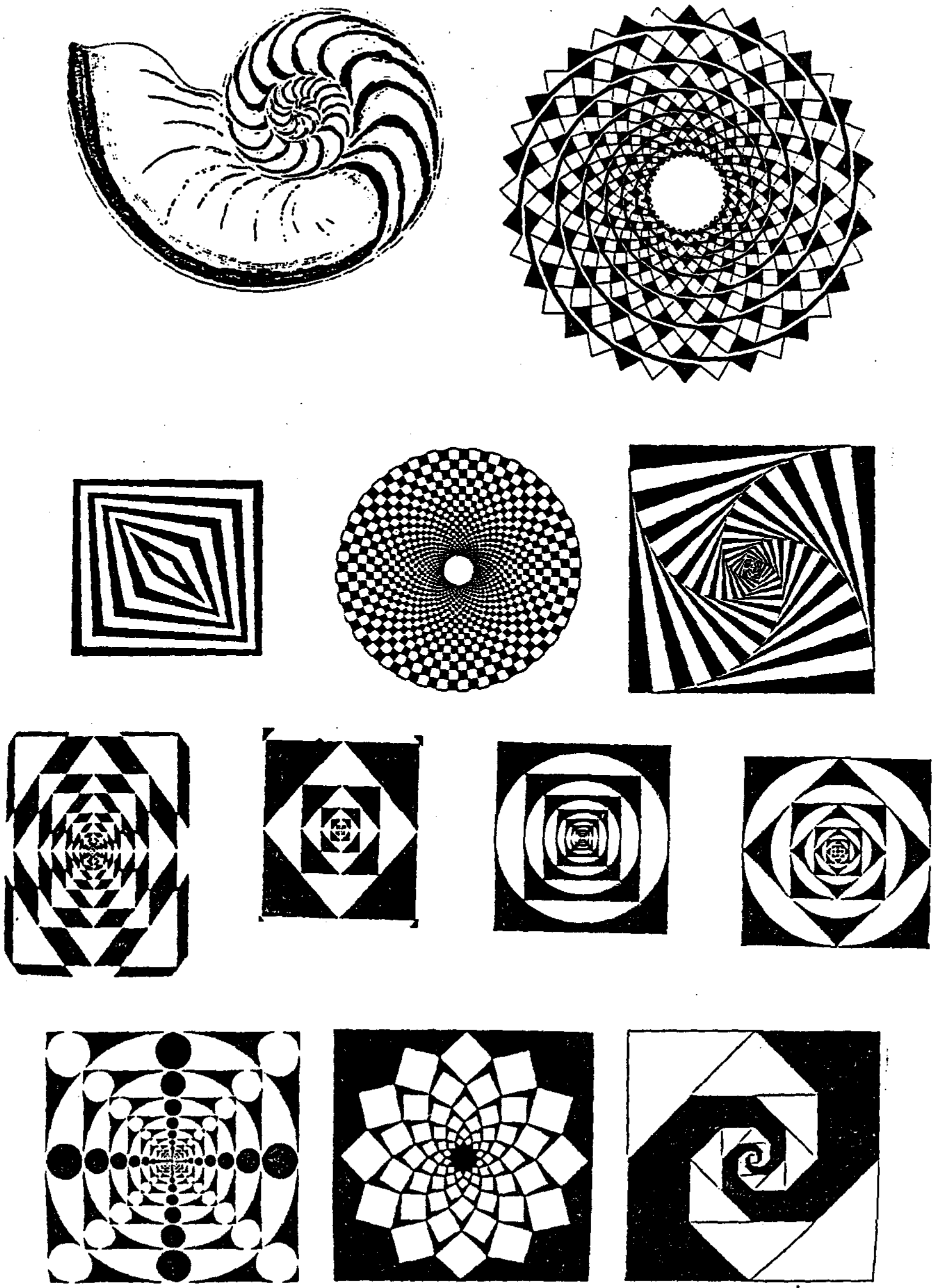
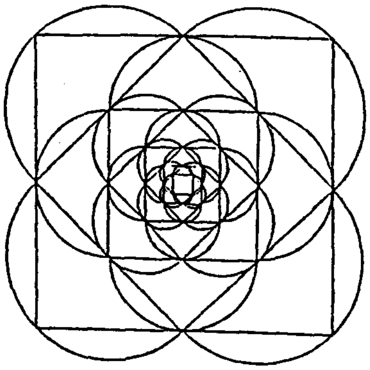
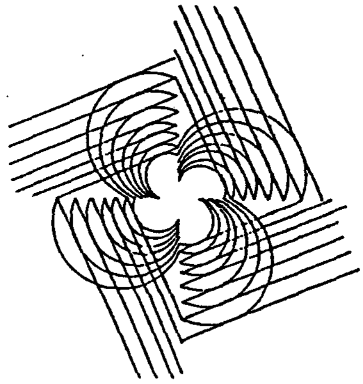


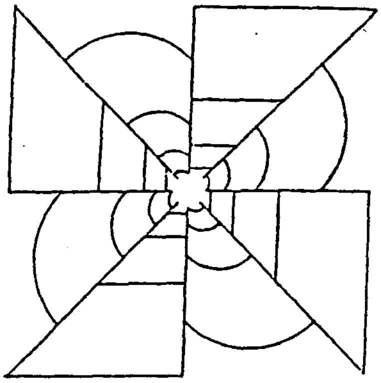
Figure 19. The similarity symmetry in nature (Nautilus shell) and in ornamental art.



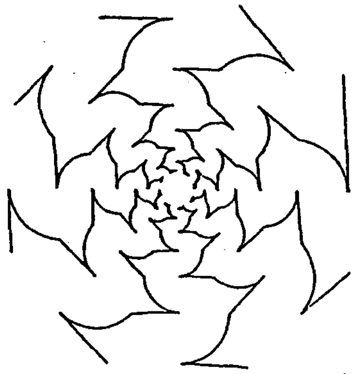
$L_8D_4R_I$



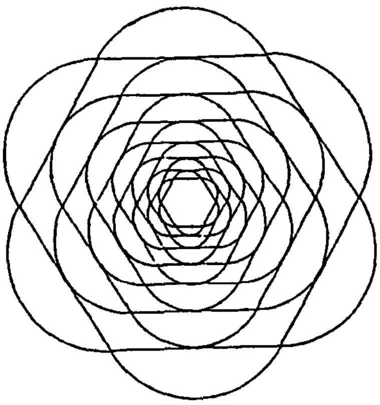
KC_4R_I



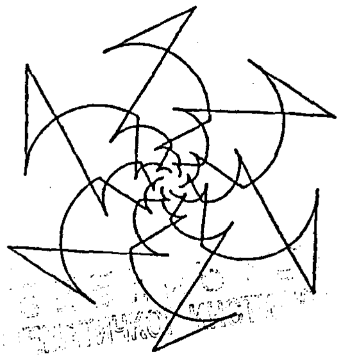
$K4_I$



MC_8R_I



KD_6R_I



$L_{12}C_6R_I$

Figure 20. Examples of the conformal symmetry rosettes.

Euclidean symmetry ornaments) has led artists to the study of principles of the theory of symmetry. That strengthened the links between the science and art and some borders between them have disappeared. So, for example, we have a complete derivation of the 46 antisymmetry groups of ornaments and their first interpretation by antisymmetry mosaics in the works of H.J.Woods, published in the Journal of the Textile Institute of Manchester in 1936 [8] (Figure 21).

Significant results have been achieved in the field of visual interpretations of symmetry groups, variations of the form of the fundamental region, visual modellings of the symmetry groups of a sphere, antisymmetry and colored symmetry groups, by H.Hinterreiter [31], a constructivist painter. The research of H.Hinterreiter on symmetry structures and their visual effect is based on the works of the German chemist and philosopher W.Ostvald. Besides by isometric transformations, many of the graphic works of H.Hinterreiter are realized by the use of symmetry structures subjected to the action of affine and projective transformations (Figure 22).

The highlights of the ornamental art of the 20th century came from the Dutch graphic artist M.C.Escher. His first attempts were to realize plane monohedral ornaments, based on the experiences acquired from Moorish ornaments. In his long creative work M.C.Escher solved a series of difficult geometric and artistic problems. Most of them belong to the field of the theory of symmetry: constructions of monohedral, 2-hedral or n-hedral tilings using as their protiles figures derived from models found in nature, combined with the use of the antisymmetry, colored symmetry, similarity symmetry, conformal symmetry, non-Euclidean symmetry, topological symmetry... (Figure 23,24,25,26,27).

Although the contacts that M.C.Escher had with one of the most significant contemporary mathematicians, H.S.M.Coxeter, and its certain knowledge of literature from the field of the theory of symmetry cannot be neglected, still remains the unavoidable fact that in many of his works M.C.Escher anticipated certain main problems of the theory of symmetry and its generalizations, having their almost visionary premonition. Some colored symmetry groups, for example, had appeared in his works before they were derived by mathematicians and crystallographers.

How difficult are the problems M.C.Escher had to face when constructing monohedral tilings with zoomorphic protiles can be seen if we try unaided to construct one of these tilings with the protile based on a model found in nature, even if we are familiar with the theory of tilings, which was still not completed when M.C.Escher created his ornaments.

In the contemporary art, from the point of view of the theory of symmetry and psychology of visual perception, the most interesting is the "op-art" ("optical art"), which reached its top in seventies in the works of V.Vasarely, J.R.Soto, W.Fangor, the group "Abstraction-Creation", B.Riley, J.Albers, F.Morellét and others.

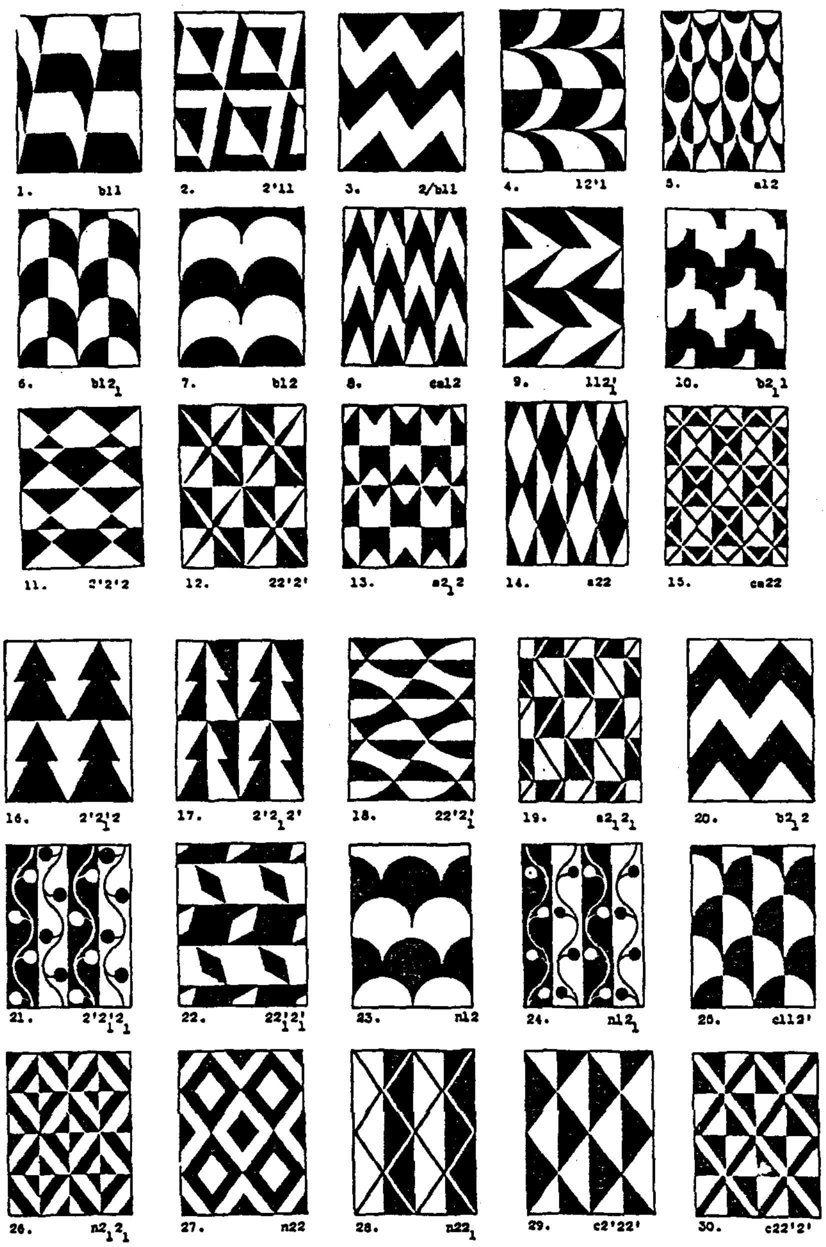


Figure 21. Antisymmetry ornaments by H.J.Woods.

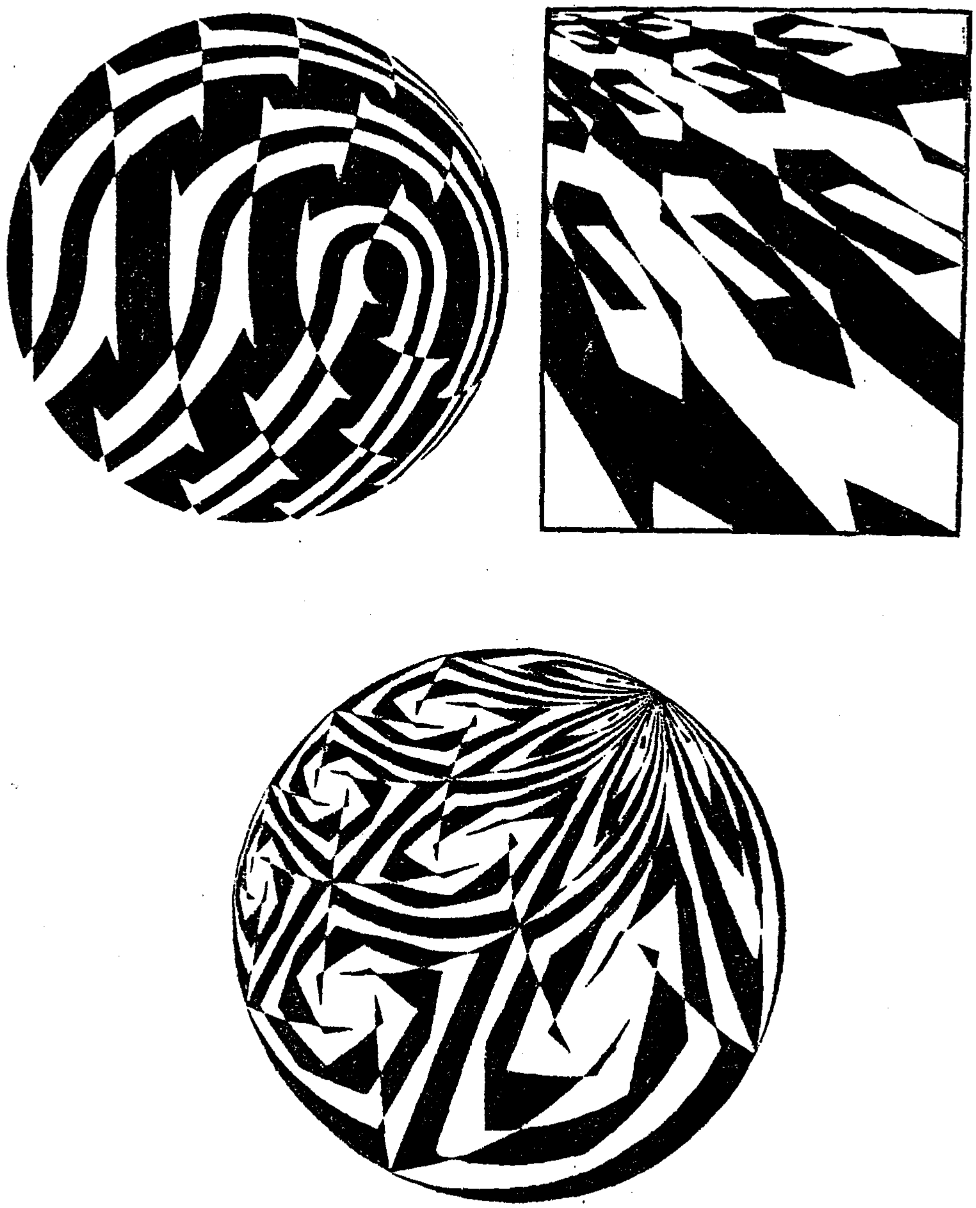


Figure 22. Graphic works by H.Hinterreiter.

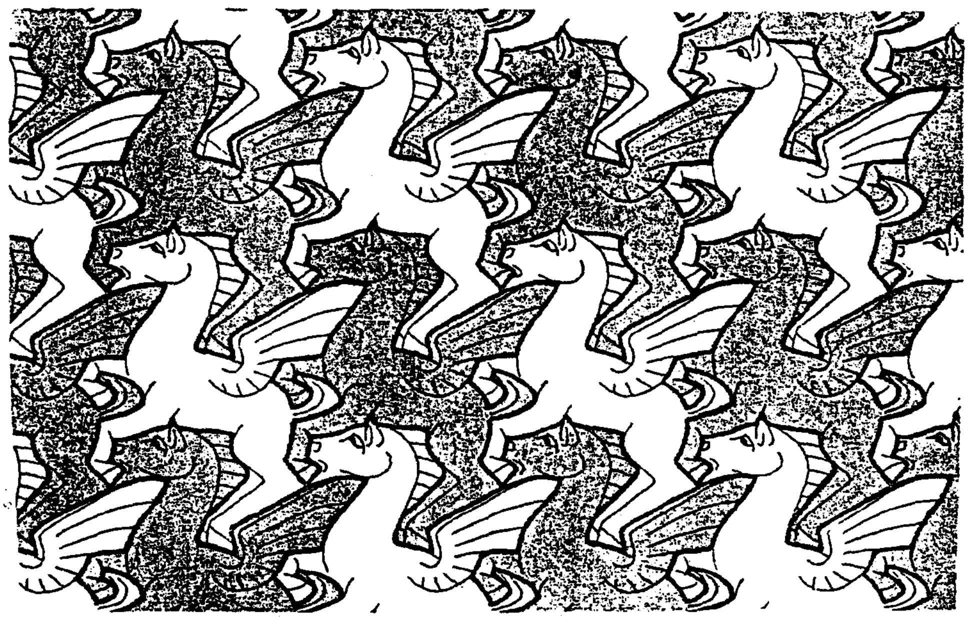


Figure 23. Antisymmetry ornaments by M.C.Escher with the antisymmetry groups pg/p1 and p1/p1.

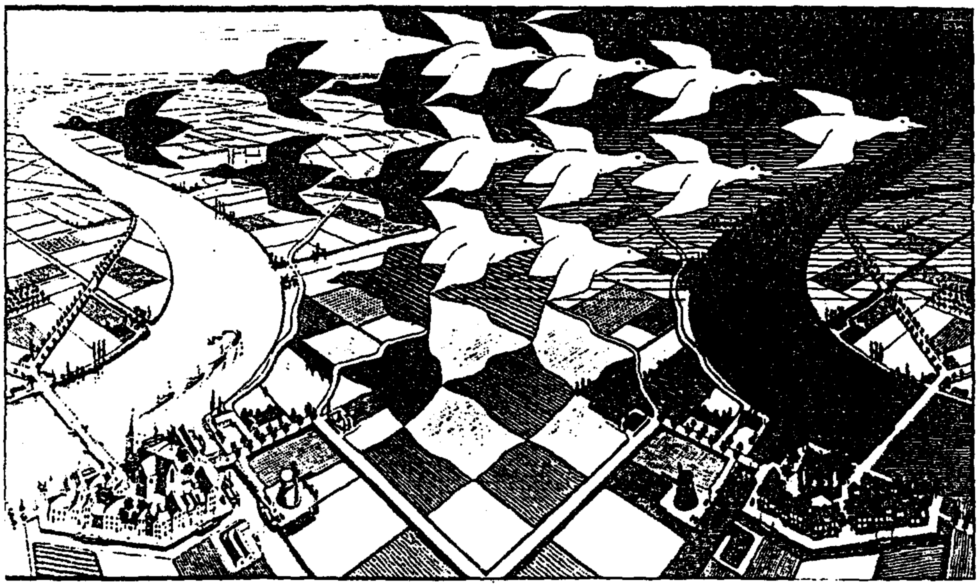


Figure 24. The transformation of the antisymmetry ornament with the antisymmetry group cmm onto the antisymmetry ornament with the antisymmetry group pg/p1.

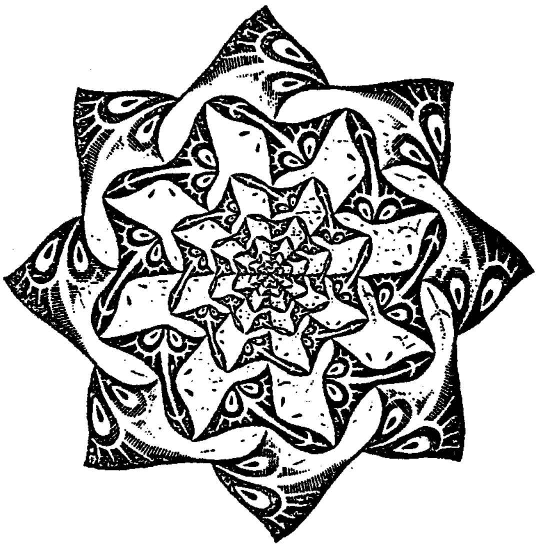
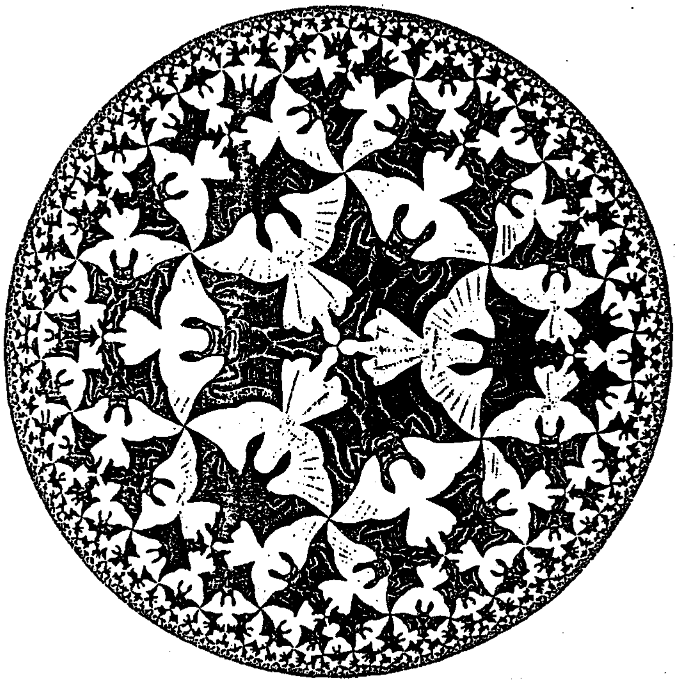


Figure 25. A conformal antisymmetry ornament by M.C.Escher.



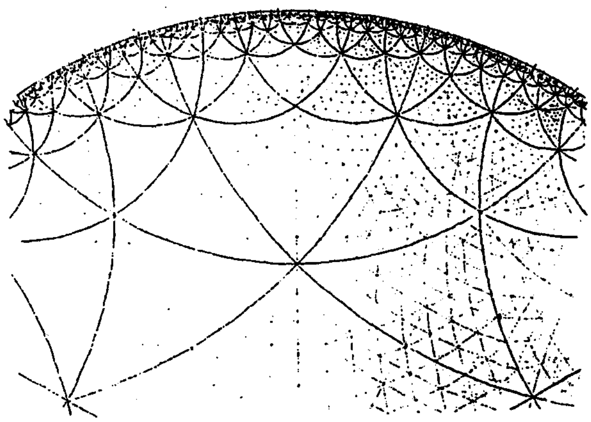


Figure 26. "Circle Limit IV", the hyperbolic plane ornament by M.C.Escher with the symmetry group [3+,8] and the geometric scheme which served as a basis for its construction.

Figure 27. A hyperbolic plane ornament by M.C.Escher, obtained using the Poincaré half-plane model of the Lobachevsky plane.

Being the synthesis of ideas of the geometric abstraction, constructivism, suprematism and the principles of the psychology of visual perception, the op-art makes experiments with visual structures producing a programmed visual effect to the observer. The planning of the visual effect and the adequate choice of the visual parameters is based on the knowledge of the physiologic-psychological laws of visual perception. Since the symmetry in its widest sense is one of the most important visual parameters, the op-art works are often symmetrical (Figure 28,29,30). Besides the already discussed discrete groups of symmetry and generalized symmetry (antisymmetry, colored symmetry...) an important role in the op-art is given to superposed net-structures resulting in a "moire effect" [34] and textures [35] (Figure 30b).

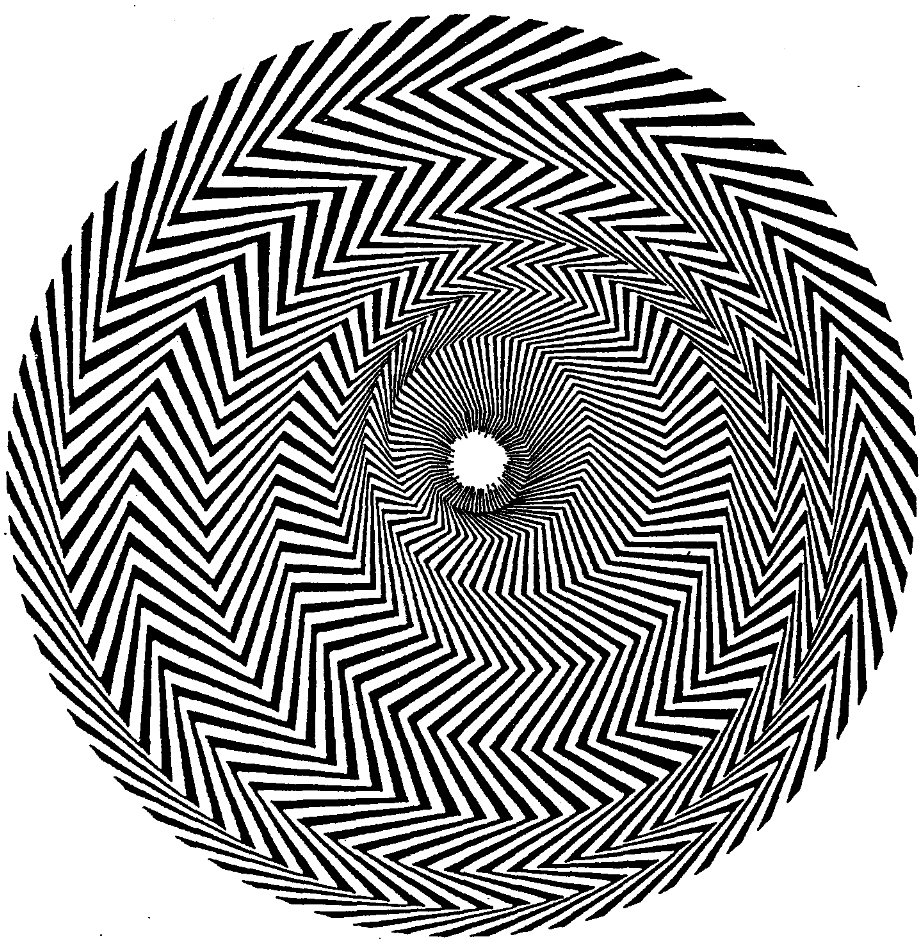


Figure 28. "Blaze I" by B.Riley.

From the standpoint of the theory of symmetry, textures make possible visual interpretations of all continuous symmetry groups which are not visually presentable (e.g. C_∞ – the symmetry group of a rotating circle). Since they are realized by the statistically uniform distribution of an asymmetric figure in accordance with the desired symmetry, in the physical terms they can be understood as results of different accidental dynamic processes, having a distinguished place in the modern physics and in all natural sciences. Some of such op-art motifs are realized

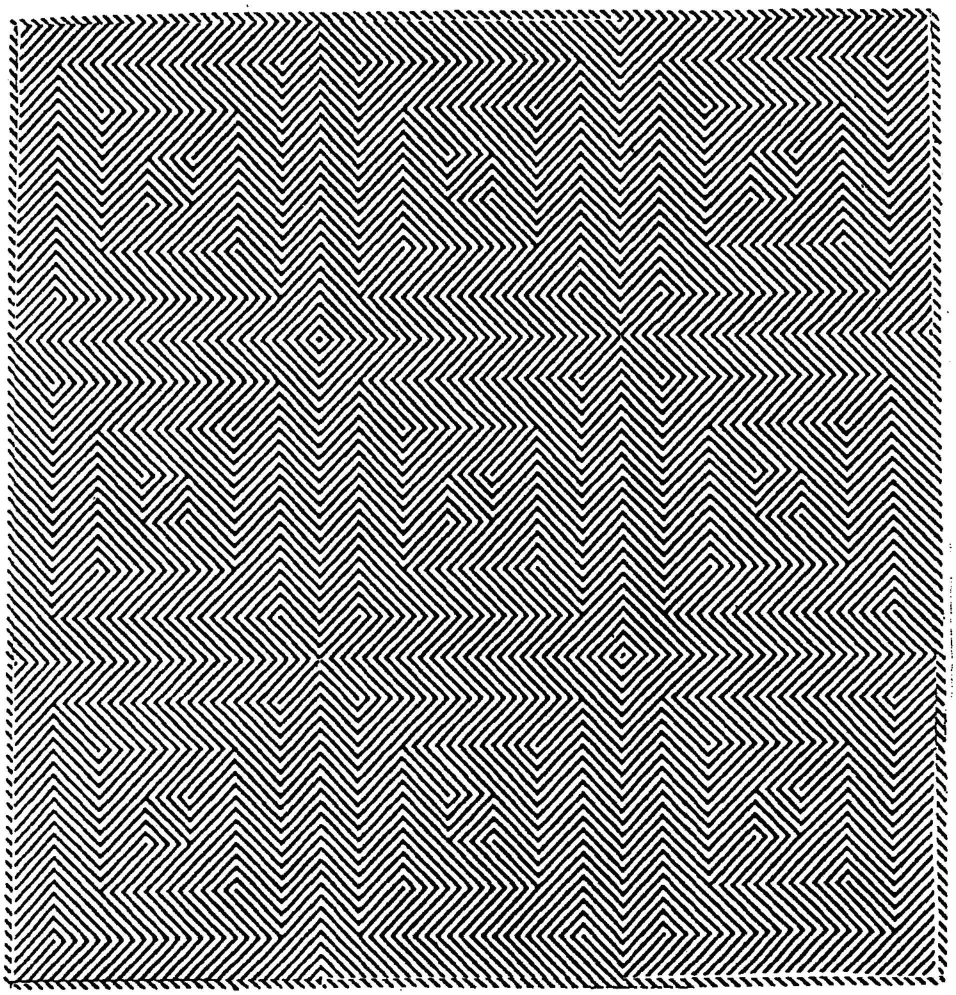


Figure 29. "Square of Three" by R.Neal.

by the use of a random number generator, so that they represent first applications of computers in art (Figure 31). In the last few years we can follow the rising of the new art discipline – computer art, which already deserved an independent study [33].

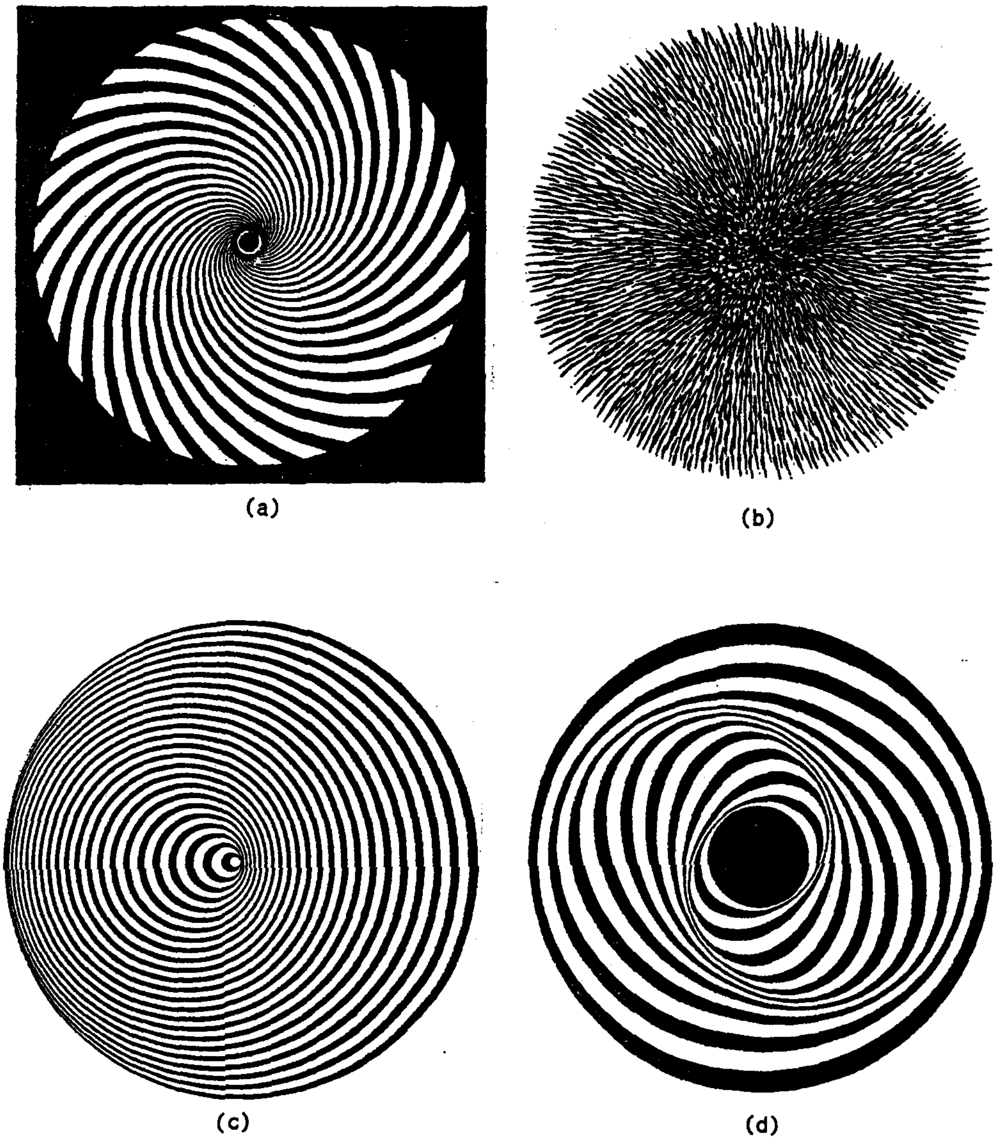


Figure 30. Different rosettal motifs in op-art: (a) continuous conformal antisymmetry rosette with the antisymmetry group $L_1C_{60}Z_I/L_1C_{30}Z_I$; (b) graphic by M.Shutej; (c) graphic by M.Apollonio; (d) graphic by F.Celentano.

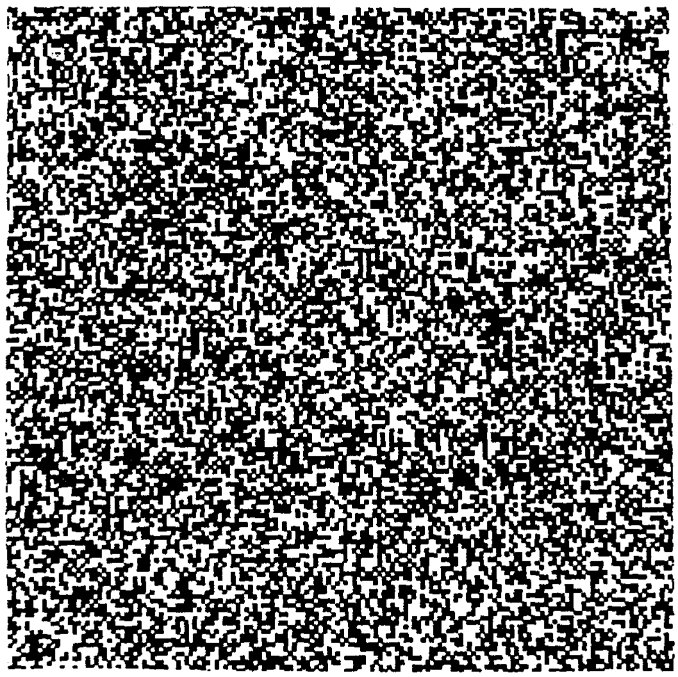


Figure 31. "Aleatoric Distribution" by F.Morellét.

The theory of symmetry is now established as a scientific discipline which does not offer almost any empirical research. In contrast, ornamental art based on the intuitive-empirical approach, centuries ago anticipated some knowledge of the theory of symmetry. Today, the roles are changed, and the level of the theory of symmetry mostly exceeds the mathematical range of ornamental art. Therefore, it is necessary to estimate the true place and importance of ornamental art at the very moment. The fact that the number and character of different possible symmetry structures is determined and fully defined by symmetry laws (e.g. the existence of exactly the 17 symmetry groups of ornaments, the 46 antisymmetry groups of ornaments, the 93 types of isohedral plane tilings, etc.) does not mean that the construction of ornaments is only a mechanical process deprived of any creativity. That only means that an artist need not to solve the technical construction problems anymore. Knowing the results of the theory of symmetry and its generalizations, construction rules and geometric possibilities for choosing visually relevant characteristics of ornaments (e.g. the form of the fundamental region, different possibilities for regular colorings...) he can direct all his creative potentials towards aesthetic aspects of ornaments and achievement of the desired visual effect.

On the other hand, mathematics and all other natural sciences can expect from ornamental art the most evident visual way of interpreting and modelling different symmetry structures and the corresponding symmetry groups. It is for sure that ornamental art, as a centuries old, inexhaustible source, nourished by the inspiration of generations of artists, can offer to the theory of symmetry a large number of relevant questions seeking an answer.

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SLAVIK V. JABLAN (born 1952) graduated in mathematics from the University of Belgrade (1971), where he also gained his M.A. degree (1981) and Ph.D. degree (1984) with the thesis "Theory of Simple and Multiple Antisymmetry in E^2 and $E^2\backslash\{O\}$". Participated in the Advanced Scientific Program in Kishinev (USSR) in 1985 and 1988. He published more than 30 papers on the theory of symmetry, antisymmetry and colored symmetry. Fields of interest are also the history of ornamental art and its analysis from the point of view of the theory of symmetry as well as ornamental design. His design "Two-Colored Ornamental Tilings" got the award – honourable mention at the International Competition of Industrial Design and New Technology CEVISAMA-'87 (Valencia, Spain), and "Colored Ornamental Tilings" is presented at the II Quadrienale Internazionale "La Ceramica nell' Arredo Urbano" (Faenza, Italy, 1989). Currently he is professor of mathematics at the University of Niš (Yugoslavia) and the member of the Mathematical Institute.

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