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Recenzenti: Gradimir Milovanović, Lav Ivanović

Primljeno za štampu odlukom Naučnog veća Matematičkog instituta od 24. 12. 1992.

Tehnički urednik: Dragan Blagojević

Tekst obradio u TEX-u: autor

Štampa: Grafički atelje "Galeb", Zemun, Kej oslobođenja 73

Štampanje završeno septembra 1993.

Klasifikacija američkog matematičkog društva
(AMS Mathematics Subject Classification 1990): 65–02, 65 N xx

Univerzalna decimalna klasifikacija: 517.956

CIP - Каталогизација у публикацији
Народна библиотека Србије, Београд

517.962.8

JOVANOVIĆ, Boško S.
[Finite Difference Method]
The Finite Difference Method for Boundary-Value Problems with Weak Solutions / Boško S. Jovanović. - Beograd : Matematički institut, 1993 (Beograd : Grafički atelje "Galeb"). - 91 str. ; 24 cm. - (Posebna izdanja / Matematički institut ; knj. 16)

Tiraž 420. - Bibliografija: str. 83-91.
ISBN 86-80593-15-X

517.956
a) Метод коначних разлика b) Диференцијалне једначине, парцијалне
17841932

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Preface

Many problems in science and engineering can be described by partial differential equations (PDE). In most cases, PDE's can not be solved exactly, so various approximative methods can be used instead. One of the most important and most widely used methods is the finite difference method.

Recently, there has been a growing interest in problems with weak solutions, reduced to PDE's with un–smooth or discontinuous coefficients. Consequently, there is a need for the generation of convergent difference schemes for such problems. The most important aims for this purpose are:

1) determination of the minimum smoothness of the input data allowing the convergence of the scheme;

2) determination of the relation between the rate of convergence and the smoothness of the input data.

This work is intended for the examination of those matters.

In order to demonstrate the main difficulties which are encountered in solving the above mentioned problems, let us consider as a model example the Dirichlet boundary–value problem for the Poisson equation in the unit square $\Omega = (0,\, 1)^2$. We approximate the problem by a "cross"–difference scheme on a regular mesh ω with the step h in the domain Ω. Then, the error $z = u - v$, where u is the solution of the original problem, and v is the solution of the difference scheme, satisfies the relation

$$(1) \qquad -\Delta_h z = \psi \equiv \Delta u - \Delta_h u \,.$$

By estimating the right–hand–side using the Taylor's expansion, we readily obtain the following convergence rate estimate,

$$(2) \qquad \|u - v\|_{W_2^2(\omega)} \le C\, h^2\, \|u\|_{C^4(\bar{\Omega})} \,.$$

By using the appropriate transformations, the right–hand–side of (1) can be expressed as a sum of several integrals of the fourth–order partial derivatives of the solution $u(x)$, which yields the following estimate

$$(3) \qquad \|u - v\|_{W_2^2(\omega)} \le C\, h^2\, \|u\|_{W_2^4(\Omega)} \,.$$

The estimate (3) is "finer" than the estimate (2), in the sense that it permits the convergence for even less smooth solutions. Expression (3) can be derived easier and more elegantly, by using the Bramble–Hilbert lemma [7], [8]. Thus, this lemma takes the role of the Taylor's formula for the functions from the Sobolev spaces.

Further investigations yield the estimates of the following form

$$(4) \qquad \|u-v\|_{W_2^k(\omega)} \le C\, h^{s-k}\, \|u\|_{W_2^s(\Omega)}\,, \qquad s \ge k\,.$$

Estimates of type (4) are said to be consistent with the smoothness of the solution of the boundary value problem (see Lazarov, Makarov & Samarskiĭ [67]). Note that, if a weaker norm is used for the estimate of the convergence rate, less smoothness is consequently required for the solution. Estimates of the type (4) are similar to those of the finite elements method

$$(5) \qquad \|u-v\|_{W_2^k(\Omega)} \le C\, h^{s-k}\, \|u\|_{W_2^s(\Omega)}\,, \qquad s \ge k\,,$$

which shows a relation between the finite differences and finite elements methods.

Let us show here some of the difficulties arising in the derivation of the type (4) estimates, and in the generation of the difference schemes satisfying those estimates.

— For $s \le 3$ (or $s \le n/2 + 2$ in the n–dimensional case) the right–hand–side of the original equation becomes discontinuous. Consequently, the difference schemes with averaged right–hand–sides must be used.

— For $k \ne 2$ (i.e. $k = 0,\, 1$), "good" a priori estimates $\|z\|_{W_2^k(\omega)}$ are required. In particular, such "good" estimates satisfy schemes with "divergent" right–hand–sides.

— The transition to a fractional s is based on a generalisation of the Bramble–Hilbert lemma on Sobolev spaces of fractional order (see Dupont & Scott [16]).

— The transition to a fractional k is based on multiplicative inequalities for discrete Sobolev–type norms.

— The transition to estimates in W_p^k–norms, for $p \ne 2$, requires a new technique for the derivation of a priori estimates, by the use of discrete Fourier multipliers (see Mokin [78]).

— The transition to equations with variable coefficients (primarily the linear elliptic equations) is the most troublesome. It is necessery to determine the widest possible classes of coefficients of the equations. (Such classes are sets of multipliers in Sobolev spaces, see Maz'ya & Shaposhnikova [77]). The error depends not only on the solution u, but also on the coefficients of the equation. Thus, instead of the standard Bramble–Hilbert lemma, its bilinear version is used. Instead of the Cauchy–Schwartz inequality, the Hölder's inequality is used. Finally, Sobolev spaces W_q^λ and $W_{2q/(q-2)}^\mu$ with "conjugated" lower indices are used consistently, as well as the imbedding theorems for Sobolev spaces.

— The transition to the parabolic case is based on the generalisation of the Bramble–Hilbert lemma to the anisotropic Sobolev–type spaces.

This work is based on the author's papers published in the past several years, and it is closely related to the results of Ivanović, Süli, Lazarov, Samarskiĭ, Makarov, Weinelt, Gavrilyuk, Voĭtsekhovskiĭ, and others. The theory of convergence of difference schemes is presented systematically for elliptic, parabolic and hyperbolic PDE's with variable coefficients. The only existing monograph in this field (Samarskiĭ, Lazarov & Makarov [85]) deals with elliptic equations, mostly with constant coefficients.

Although most of the material presented in this work has been previously published as my own results, I wish to emphasize the valuable contributions of my colleagues Lav Ivanović and Endre Süli. Namely, we jointly started the study in this field, posted and solved several problems together, and published a number of papers together. Those results were the base for my ensuing work. I strongly believe that our collaboration will continue and be more extensive in the future.

Belgrade, January 1992 Boško S. Jovanović

I Introductory Topics

In this chapter we shall introduce some basic terminology and results obtained in the theories of distributions, function spaces and differential equations, which are used in this work.

1. Preliminaries and Denotations

First, we introduce terms and denotations which shall be used.

Set

$\mathbf{N}$ = the set of natural numbers,
$\mathbf{N}_0 = \mathbf{N} \cup \{0\}$, and
$\mathbf{R}$ = the set of real numbers.

For $s \in \mathbf{R}$ let $[s]$ be the largest integer $\leq s$, and $[s]^-$ — the largest integer $< s$.

We shall represent the elements of the set $\mathbf{R}^n$ as vector–columns, and denote

$$x = (x_1, x_2, \dots, x_n)^T = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} .$$

We shall denote by $r_1, r_2, \dots, r_n$ the unit vectors of coordinate axes in $\mathbf{R}^n$. The elements of the set $\mathbf{N}_0^n$ will be called multi–indices, and will be denoted by

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)^T .$$

We shall also adopt the following notations

$$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n , \quad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} ,$$
$$|x| = \left(x_1^2 + x_2^2 + \dots + x_n^2\right)^{1/2} \quad \text{and} \quad dx = dx_1\, dx_2 \cdots dx_n .$$

An open and connected set $\Omega \subset \mathbf{R}^n$ will be called a domain. We assume that domains which are used are bounded and convex, unless it is stated otherwise. The boundary of Ω is the set $\Gamma = \partial\Omega = \overline{\Omega} \setminus \Omega$. The domain Ω' is called a subdomain of the domain Ω, and denoted by $\Omega' \Subset \Omega$, if $\overline{\Omega'} \subset \Omega$.

In the ensuing we shall use functions of the form $f : \Omega \to \mathbf{R}$, $\Omega \subset \mathbf{R}^n$. The value of the function f at the point $x \in \Omega$ will be denoted by $f(x)$ or $f(x_1, x_2, \dots, x_n)$.

The support of the function $f(x)$, denoted by $\operatorname{supp} f$, is the closure of the set of points such that $f(x) \neq 0$. If $\operatorname{supp} f$ is a bounded set, the function f is said to be finite.

The partial derivatives will be denoted by

$$D_i f = \frac{\partial f}{\partial x_i}, \qquad D^\alpha f = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n} f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}.$$

We also introduce the finite differences

$$\Delta_a f(x) = f(x+a) - f(x), \qquad a \in \mathbf{R}^n,$$
$$\Delta_{i,h} f(x) = \Delta_{h r_i} f(x), \qquad h \in \mathbf{R},$$
$$f_{x_i}(x) = f_{\bar{x}_i}(x + h r_i) = f_{\mathring{x}_i}(x + 0.5 h r_i) = \big(\Delta_{i,h} f(x)\big)/h.$$

We shall use the following function spaces:

$C^m(\Omega)$ — the space of functions which are continuous in Ω together with all their partial derivatives of the order $\leq m$ $(m \in \mathbf{N}_0 \cup \{\infty\})$.

$C(\Omega) = C^0(\Omega)$.

$C_0^m(\Omega)$ — the subspace of $C^m(\Omega)$ consisting of functions with compact support in Ω.

$C^m(\overline{\Omega})$ — the space of functions which are continuous on $\overline{\Omega}$, together with all their partial derivatives of order $\leq m$, with the norm

$$\|f\|_{C^m(\overline{\Omega})} = \max_{|\alpha| \leq m} \max_{x \in \overline{\Omega}} |D^\alpha f(x)|.$$

$L_p(\Omega)$ — Lebesgue space of functions which are measurable in Ω, and which satisfy the condition

$$\|f\|_{L_p(\Omega)} = \Big(\int_\Omega |f(x)|^p \, dx\Big)^{1/p} < \infty, \qquad 1 \leq p < \infty,$$

or

$$\|f\|_{L_\infty(\Omega)} = \operatorname*{sup\,ess}_{x \in \Omega} |f(x)| < \infty.$$

$L_{p,\,loc}(\Omega)$ — the space of locally integrable functions,

$$f(x) \in L_{p,\,loc}(\Omega) \quad \text{if} \quad f(x) \in L_p(\Omega') \quad \text{for any bounded subdomain} \quad \Omega' \Subset \Omega.$$

$C^m(\overline{\Omega})$ and $L_p(\Omega)$ are Banach spaces.

The hypersurface $S \subset \mathbf{R}^n$, whose dimensionality is $n-1$ is said to be of the class C^m, and denoted by $S \in C^m$, if in the neighbourhood of any point $x_0 \in S$, it can be represented by an equation of the form

$$\varphi_{x_0}(x) = 0,$$

where $\varphi_{x_0} \in C^m$. The hypersurface S is said to be continuous in the Lipschitz-sense if it can be divided into a finite number of partitions S_j, each of them being represented by an equation of the form

$$x_{i_j} = \psi_j(x_1, \ldots, x_{i_j-1}, x_{i_j+1}, \ldots, x_n),$$

where ψ_j is continuous in the Lipschitz-sense. A domain Ω is said to be Lipschitzian if its boundary is continuous in the Lipschitz-sense.

Finally, C and C_i denote positive generic constants which may have different values in various expressions.

2. Distributions: Definitions and Basic Properties

In the set of functions $C_0^\infty(\mathbf{R}^n)$ we define the convergence in the following manner.

DEFINITION 1. *The sequence of functions $\varphi_j \in C_0^\infty(\mathbf{R}^n)$ is said to converge to $\varphi \in C_0^\infty(\mathbf{R}^n)$ if the following conditions are satisfied*

1. There exists a compact set $K \subset \mathbf{R}^n$ such that $\operatorname{supp}\varphi_j \subseteq K$ *for every j;*

2. For every multi-index α, the sequence $D^\alpha\varphi_j$ converges uniformly to $D^\alpha\varphi$ on K when $j \to \infty$.

The set $C_0^\infty(\mathbf{R}^n)$ equipped with this topology will be called the set of basic functions and denoted by $\mathcal{D} = \mathcal{D}(\mathbf{R}^n)$. For a domain Ω in $\mathbf{R}^n$, $\mathcal{D}(\Omega) \subset \mathcal{D}(\mathbf{R}^n)$ denotes the set of basic functions with supports in Ω.

The linear bounded functionals on the set $\mathcal{D}(\Omega)$ are called distributions (see Schwartz [86], Rudin [83]), and the set of distributions will be denoted by $\mathcal{D}'(\Omega)$. The value of the distribution $f \in \mathcal{D}'(\Omega)$ on the basic function $\varphi \in \mathcal{D}(\Omega)$ will be denoted by

$$\langle f, \varphi\rangle = \langle f, \varphi\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)}.$$

DEFINITION 2. *The sequence of distributions $f_j \in \mathcal{D}'(\Omega)$ converges to $f \in \mathcal{D}'(\Omega)$ if*

$$\langle f_j, \varphi\rangle \to \langle f, \varphi\rangle, \quad j \to \infty, \quad \forall\varphi \in \mathcal{D}(\Omega).$$

DEFINITION 3. *Distribution $f \in \mathcal{D}'(\mathbf{R}^n)$ is said to vanish in the domain Ω, which can be written in the form*

$$f = 0 \quad in \ \Omega \qquad or \qquad f(x) = 0\,, \quad x \in \Omega$$

if

$$\langle f, \varphi \rangle = 0\,, \quad \forall \varphi \in \mathcal{D}(\Omega)\,.$$

Two distributions $f,\, g \in \mathcal{D}'(\mathbf{R}^n)$ are said to be equal in the domain Ω, i.e.

$$f(x) = g(x)\,, \quad x \in \Omega\,,$$

if

$$f(x) - g(x) = 0\,, \quad x \in \Omega\,.$$

If $f(x) \in L_{1,loc}(\mathbf{R}^n)$ then

$$\varphi(x) \mapsto \int_{\mathbf{R}^n} f(x)\, \varphi(x)\, dx$$

defines a bounded linear functonal on $\mathcal{D}(\mathbf{R}^n)$. In other words, each locally integrable function induces a distribution. Such distributions are called regular. In the following every regular distribution will be identified by the corresponding locally integrable function, i.e.

$$\langle f, \varphi \rangle = \int_{\mathbf{R}^n} f(x)\, \varphi(x)\, dx\,.$$

The distributions which are not regular are called singular. Such is, for exaple, the Dirack's distribution

$$\langle \delta, \varphi \rangle = \varphi(0)\,, \quad \forall \varphi \in \mathcal{D}(\mathbf{R}^n)\,.$$

If A is a regular real matrix of the n–th order, and b is a fixed vector in $\mathbf{R}^n$, than we can define the linear change of variables for a distribution $f \in \mathcal{D}'(\mathbf{R}^n)$ in the following manner

$$\big\langle f(A\,y + b),\, \varphi(y) \big\rangle = \Big\langle f(x),\, \frac{\varphi(A^{-1}(x-b))}{|\det A|} \Big\rangle\,, \quad \forall \varphi \in \mathcal{D}(\mathbf{R}^n)\,.$$

The multiplication of a distribution $f \in \mathcal{D}'(\Omega)$ by a smooth function $a \in C^\infty(\Omega)$ is defined by

$$\langle a\, f, \varphi \rangle = \langle f, a\, \varphi \rangle\,, \quad \forall \varphi \in \mathcal{D}(\Omega)\,.$$

The derivation of a distribution $f \in \mathcal{D}'(\Omega)$ is defined by

$$\langle D^\alpha f, \varphi \rangle = (-1)^{|\alpha|} \langle f, D^\alpha \varphi \rangle\,, \quad \forall \varphi \in \mathcal{D}(\Omega)\,.$$

Note that every distribution is infinitely differentiable.

3. Sobolev Spaces

Let Ω be a domain in $\mathbf{R}^n$, $k \in \mathbf{N}_0$ and $1 \le p \le \infty$. The Sobolev space $W_p^k(\Omega)$ is defined in the following manner (see Sobolev [89], Adams [1])

$$W_p^k(\Omega) = \{f \in L_p(\Omega) : D^\alpha f \in L_p(\Omega),\ |\alpha| \le k\},$$

where derivatives are taken in the distribution sense. In particular, for $k = 0$, set

$$W_p^0(\Omega) = L_p(\Omega).$$

The norm in $W_p^k(\Omega)$ is defined by

$$\|f\|_{W_p^k(\Omega)} = \Big(\sum_{i=1}^{k} |f|^p_{W_p^i(\Omega)}\Big)^{1/p}, \quad 1 \le p < \infty,$$

where,

$$|f|_{W_p^i(\Omega)} = \Big(\sum_{|\alpha|=i} \|D^\alpha f\|^p_{L_p(\Omega)}\Big)^{1/p},$$

or

$$\|f\|_{W_\infty^k(\Omega)} = \max_{0\le i\le k} |f|_{W_\infty^i(\Omega)}, \qquad |f|_{W_\infty^i(\Omega)} = \max_{|\alpha|=i} \|D^\alpha f\|_{L_\infty(\Omega)}.$$

$W_p^k(\Omega)$ is a Banach space. In particular, $W_2^k(\Omega)$ is a Hilbert space with the inner product

$$(f, g)_{W_2^k(\Omega)} = \sum_{|\alpha|\le k} \int_\Omega D^\alpha f(x)\, D^\alpha g(x)\, dx.$$

For $0 < \sigma < 1$ set

$$|f|_{W_p^\sigma(\Omega)} = \Big(\int_\Omega\int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{n+\sigma p}}\, dx\, dy\Big)^{1/p}, \quad 1 \le p < \infty,$$

or

$$|f|_{W_\infty^\sigma(\Omega)} = \sup_{x\in\Omega} \operatorname{ess} \frac{|f(x) - f(y)|}{|x-y|^\sigma}.$$

The Sobolev space $W_p^s(\Omega)$ with a fractional positive index $s = [s] + \sigma$, $0 < \sigma < 1$, is defined as a set of functions $f \in W_p^{[s]}(\Omega)$ with the finite norm

$$\|f\|_{W_p^s(\Omega)} = \Big(\|f\|^p_{W_p^{[s]}(\Omega)} + |f|^p_{W_p^s(\Omega)}\Big)^{1/p}, \quad 1 \le p < \infty,$$

where,

$$|f|_{W_p^s(\Omega)} = \Big(\sum_{|\alpha|=[s]} |D^\alpha f|_{W_p^{s-[s]}(\Omega)}\Big)^{1/p},$$

with the corresponding change for $p = \infty$.

The closure of $\mathcal{D}(\Omega)$ in the norm $W_p^s(\Omega)$ is a subspace of $W_p^s(\Omega)$ which shall be denoted by $\overset{\circ}{W}{}_p^s(\Omega)$.

Let s be a positive number, $1 < p < \infty$ and $1/p + 1/p' = 1$. The space $W_p^{-s}(\Omega)$ is defined as the dual space of $\overset{\circ}{W}{}_{p'}^s(\Omega)$, $W_p^{-s}(\Omega) = \left(\overset{\circ}{W}{}_{p'}^s(\Omega)\right)'$. Since $\mathcal{D}(\Omega) \subset \overset{\circ}{W}{}_{p'}^s(\Omega)$, we have $W_p^{-s}(\Omega) \subset \mathcal{D}'(\Omega)$, i.e. the elements of the space $W_p^{-s}(\Omega)$ are distributions.

Distributions of $W_p^{-s}(\Omega)$ can be represented as derivatives of ordinary functions. First, let s be a positive integer. If $u(x) \in W_{p'}^s(\Omega)$ than $D^\alpha u \in L_{p'}(\Omega)$ for every $|\alpha| \le s$. According to the Riesz's theorem (see Aljančić [2], Yosida [112]) an arbitrary bounded linear functional on $W_{p'}^s(\Omega)$ can be represented in the following manner

$$\eta(u) = \sum_{|\alpha|\le s} \int_\Omega D^\alpha u(x)\,(-1)^{|\alpha|} f_\alpha(x)\,dx\,,$$

where $f_\alpha(x) \in L_p(\Omega)$.

Let now η be a distribution of $W_p^{-s}(\Omega)$, with the same representation as above. Then η is fully determined by its values on functions $\varphi \in \mathcal{D}(\Omega)$. Consequently,

$$\begin{aligned}\langle \eta, \varphi\rangle = \eta(\varphi) &= \sum_{|\alpha|\le s} \int_\Omega D^\alpha \varphi(x)\,(-1)^{|\alpha|} f_\alpha(x)\,dx \\ &= \sum_{|\alpha|\le s} (-1)^{|\alpha|} \langle f_\alpha(x), D^\alpha\varphi(x)\rangle = \Big\langle \sum_{|\alpha|\le s} D^\alpha f_\alpha(x), \varphi(x)\Big\rangle\,.\end{aligned}$$

We may conclude that the elements of $W_p^{-s}(\Omega)$ can be represented as

(1) $$\eta = f(x) = \sum_{|\alpha|\le s} D^\alpha f_\alpha(x)\,, \quad \text{where} \quad f_\alpha(x) \in L_p(\Omega)\,.$$

This representation is not unique (Lions & Magenes [70]).

Now, let s be a positive non–integer number: $s = [s] + \sigma$, $0 < \sigma < 1$. If $u(x) \in W_{p'}^s(\Omega)$, then $D^\alpha u \in L_{p'}(\Omega)$ for each $|\alpha| \le [s]$ and $\dfrac{D^\alpha u(x) - D^\alpha u(y)}{|x-y|^{n/p'+\sigma}} \in L_{p'}(\Omega \times \Omega)$ for each $|\alpha| = [s]$. An arbitrary bounded linear functional on $W_{p'}^s(\Omega)$ can be represented in the following form (see Wloka [111])

$$\begin{aligned}\eta(u) = &\sum_{|\alpha|\le [s]} \int_\Omega D^\alpha u(x)\,(-1)^{|\alpha|} f_\alpha(x)\,dx \\ &+ \sum_{|\alpha|=[s]} \int_\Omega\int_\Omega \frac{D^\alpha u(x) - D^\alpha u(y)}{|x-y|^{n/p'+\sigma}}\,(-1)^{|\alpha|} F_\alpha(x, y)\,dx\,dy\,,\end{aligned}$$

where $f_\alpha(x) \in L_p(\Omega)$, $|\alpha| \le [s]$, and $F_\alpha(x, y) \in L_p(\Omega \times \Omega)$, $|\alpha| = [s]$. Using the same argument, the elements of the space $W_p^{-s}(\Omega)$ can be represented in the form

$$\eta = f(x) = \sum_{|\alpha|\le[s]} D^\alpha f_\alpha(x) + \sum_{|\alpha|=[s]} D^\alpha \int_\Omega \frac{F_\alpha(x, y) - F_\alpha(y, x)}{|x-y|^{n/p'+\sigma}}\, dy, \tag{2}$$

where $f_\alpha(x) \in L_p(\Omega)$, $F_\alpha(x, y) \in L_p(\Omega \times \Omega)$, and the integral is taken as the principal value. This representation is not unique.

From the definition of the Sobolev spaces it follows that

$$\textit{If} \quad u \in W_p^s(\Omega), \quad \textit{then} \quad D_i u \in W_p^{s-1}(\Omega) \quad \textit{for} \quad s \ge 1. \tag{3}$$

From (1) and (2) the same result follows for $s \le 0$. If $p = 2$, and Ω is a Lipschitzian domain, then (3) holds for every real s: $-\infty < s < +\infty$ (Triebel [96]).

CONSEQUENCE. *If Ω is a Lipschitzian domain in $\mathbf{R}^n$, $s > 0$, $s \ne$ integer$+1/2$, $f_\alpha \in L_2(\Omega)$ and $g_\alpha \in W_2^{[s]+1-s}(\Omega)$ then the distribution*

$$f(x) = \sum_{|\alpha|\le[s]} D^\alpha f_\alpha(x) + \sum_{|\alpha|=[s]+1} D^\alpha g_\alpha(x) \tag{4}$$

belongs to the space $W_2^{-s}(\Omega)$.

Let Ω be a domain in $\mathbf{R}^n$ and $\mathcal{X}$ an arbitrary Hilbert space. Sobolev spaces $W_p^s(\Omega; \mathcal{X})$ of functions $f : \Omega \to \mathcal{X}$ are defininied analogously, substituting the absolute value by the norm of the space $\mathcal{X}$. For example,

$$\|f\|_{L_p(\Omega;\mathcal{X})} = \Big(\int_\Omega \|f(x)\|_{\mathcal{X}}^p\, dx\Big)^{1/p}, \quad 1 \le p < \infty,$$

$$|f|_{W_p^i(\Omega;\mathcal{X})} = \Big(\sum_{|\alpha|=i} \|D^\alpha f\|_{L_p(\Omega;\mathcal{X})}^p\Big)^{1/p}, \quad i = 0, 1, 2, \ldots$$

$$|f|_{W_p^\sigma(\Omega;\mathcal{X})} = \Big(\int_\Omega\int_\Omega \frac{\|f(x) - f(y)\|_{\mathcal{X}}^p}{|x-y|^{n+\sigma p}}\, dx\, dy\Big)^{1/p}, \quad 0 < \sigma < 1,$$

etc.

In the following we will frequently use the Steklov averaging operators, with the step h

$$T_i^+ f(x) = \int_0^1 f(x + h\, t\, r_i)\, dt = T_i^- f(x + h\, r_i) = T_i f(x + 0.5\, h\, r_i).$$

These operators map the partial derivatives onto differences

$$T_i^+\big(D_i f(x)\big) = D_i\big(T_i^+ f(x)\big) = f_{x_i}(x).$$

The averaging increases the smoothness of the functions; if $f \in W_p^s(\Omega)$ then $T_1 T_2 \cdots T_n f \in W_p^{s+1}(\Omega_{h/2})$, where $\Omega_{h/2}$ is the subdomain of Ω, which consists of points whose Euclidian distance from the boundary is exceeding $h/2$.

The following result is valid.

THEOREM 1. *Let $f \in W_p^s(\Omega)$, $s > 0$, and let the boundary of the domain Ω be sufficiently smooth ($\Gamma \in C^{[s]^- +1}$). Then, there exists an extension of the function f outside of the domain Ω which also belongs to the class W_p^s.*

CONSEQUENCE. *If $f \in \overset{\circ}{W}{}_p^s(\Omega)$ then its extension by a zero outside of the domain Ω belongs to the class W_p^s.*

The fundamental role in the theory of Sobolev spaces is played by the imbedding and traces theorems (see Sobolev [89], Adams [1], Triebel [96]).

THEOREM 2. *Let $f \in W_p^s(\Omega)$, $s > 1/p$, $s \neq$ integer $+1/p$, and let the boundary of the domain Ω be sufficiently smooth ($\Gamma \in C^{[s]^- +1}$). Then there exists a trace of the function f on the boundary Γ, which belongs to the space $W_p^{s-1/p}(\Gamma)$, and the following estimate*

$$\|f\|_{W_p^{s-1/p}(\Gamma)} \leq C\,\|f\|_{W_p^s(\Omega)}$$

holds.

THEOREM 3. *Let $f \in W_p^s(\Omega)$, $s > 0$, and let the boundary of the domain Ω be continuous in the Lipschitz–sense. Then the following imbedings hold*

a) if $s \cdot p < n$ then

$$W_p^s(\Omega) \subseteq L_q(\Omega), \qquad p \leq q \leq \frac{np}{n - sp},$$

b) if $s \cdot p = n$ then

$$W_p^s(\Omega) \subseteq L_q(\Omega), \qquad p \leq q < \infty,$$

c) if $s \cdot p > n$ then

$$W_p^s(\Omega) \subseteq C(\overline{\Omega}).$$

THEOREM 4. *Let $0 \leq t \leq s < \infty$, $1 < p \leq q < \infty$ and $s - n/p \geq t - n/q$. Then,*

$$W_p^s(\Omega) \subseteq W_q^t(\Omega).$$

In the following, we shall need the estimates of the norms in the near–boundary region $\Omega^h = \Omega \setminus \Omega_h$, of width h, in the domain Ω. The following result hold (Oganesyan & Rukhovets [79]).

THEOREM 5. *Let the boundary Γ of the domain Ω belongs to the class C^1. Then, for every function $f(x) \in W_2^s(\Omega)$, $0 \le s \le 1$, the following estimate holds*

$$\|f\|_{L_2(\Omega^h)} \le C\,F(h)\,\|f\|_{W_2^s(\Omega)},$$

where,

$$F(h) = \begin{cases} h^s, & 0 \le s < 1/2 \\ h^{1/2}\,|\ln h|, & s = 1/2 \\ h^{1/2}, & 1/2 < s \le 1 \end{cases}.$$

4. Anisotropic Spaces. Weighted Spaces.

Sometimes we encounter functions which have different smoothness in different variables. Spaces of such functions are called anisotropic. Here we define one class of anisotropic function spaces.

Let $\mathbf{R}_+$ be the set of non-negative real numbers. In this paragraph elements of $\mathbf{R}_+^n$ will be called multi-indices. For $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)^T \in \mathbf{R}_+^n$ define

$$[\alpha] = ([\alpha_1], [\alpha_2], \dots, [\alpha_n])^T \quad \text{and} \quad [\alpha]^- = ([\alpha_1]^-, [\alpha_2]^-, \dots, [\alpha_n]^-)^T.$$

Let Ω be a domain in $\mathbf{R}^n$ with a Lipschitz continuous boundary. For $\alpha \in \mathbf{R}_+^n$ and $1 \le p < \infty$ the seminorm $|f|_{\alpha,p}$ can be defined in the following manner

$$|f|_{\alpha,p}^p = \|f\|_{L_p(\Omega)}^p, \quad \text{for} \quad \alpha_1 = \alpha_2 = \dots = \alpha_n = 0,$$

$$|f|_{\alpha,p}^p = \int_\Omega \int_{\Omega_i(x)} \frac{|\Delta_{i,h_i} f(x)|^p}{|h_i|^{1+p\alpha_i}}\, dh_i\, dx, \quad \text{for} \quad 0 < \alpha_i < 1, \quad a_k = 0, \quad \forall k \ne i,$$

$$|f|_{\alpha,p}^p = \int_\Omega \iint_{\Omega_{ij}(x)} \frac{|\Delta_{i,h_i} \Delta_{j,h_j} f(x)|^p}{|h_i|^{1+p\alpha_i}\,|h_j|^{1+p\alpha_j}}\, dh_i\, dh_j\, dx,$$

$$\text{for} \quad 0 < \alpha_i, \alpha_j < 1, \quad a_k = 0, \quad \forall k \ne i, j,$$

....

$$|f|_{\alpha,p}^p = \int_\Omega \int \cdots \int_{\Omega_{1\dots n}(x)} \frac{|\Delta_{1,h_1} \cdots \Delta_{n,h_n} f(x)|^p}{|h_1|^{1+p\alpha_1} \cdots |h_n|^{1+p\alpha_n}}\, dh_1 \cdots dh_n\, dx,$$

$$\text{for} \quad 0 < \alpha_1, \alpha_2, \dots, \alpha_n < 1,$$

$$|f|_{\alpha,p}^p = |D^{[\alpha]} f|_{\alpha-[\alpha],p}^p, \quad \text{if, for some } k, \quad a_k \ge 1.$$

Here,

$$\Omega_i(x) \quad = \{h_i \,:\, x + h_i r_i \in \Omega\},$$

$$\Omega_{ij}(x) \quad = \{(h_i, h_j)^T \,:\, x + c_i h_i r_i + c_j h_j r_j \in \Omega \,;\; c_i, c_j = 0, 1\},$$

....

$$\Omega_{1\dots n}(x) = \Big\{(h_1, \dots, h_n)^T \,:\, x + \sum_{k=1}^n c_k h_k r_k \in \Omega \,;\; c_k = 0, 1\Big\}.$$

For $p = \infty$ the corresponding integrals are substituted by sup ess,

$$|f|_{\alpha,\infty} = \|f\|_{L_\infty(\Omega)}, \qquad \text{for} \quad \alpha_1 = \alpha_2 = \ldots = \alpha_n,$$

$$|f|_{\alpha,\infty} = \operatorname*{sup\,ess}_{x\in\Omega,\, h_i\in\Omega_i(x)} \frac{|\Delta_{i,h_i} f(x)|}{|h_i|^{\alpha_i}}, \qquad \text{for} \quad 0 < \alpha_i < 1, \quad a_k = 0, \quad \forall k \neq i,$$

etc.

The finite set of multi-indices $A \subset \mathbf{R}^n_+$ is called regular if $0 = (0, 0, \ldots, 0)^T \in A$ so that for any $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T \in A$, there exist real numbers $\beta_k \geq \alpha_k$ $(k = 1, 2, \ldots, n)$ such that $\beta_k r_k \in A$.

If A is a regular set of multi-indices we define the following norms

$$\|f\|_{W_p^A(\Omega)} = \Big(\sum_{\alpha\in A} |f|^p_{\alpha,p}\Big)^{1/p}, \qquad 1 \leq p < \infty,$$

$$\|f\|_{W_\infty^A(\Omega)} = \max_{\alpha\in A} |f|_{\alpha,\infty}.$$

The closure of $C^\infty(\overline{\Omega})$ in the norm $\|\cdot\|_{W_p^A(\Omega)}$ will be denoted by $W_p^A(\Omega)$ (see Dražić [13]).

EXAMPLE. *Let* $A = A_0 \cup A_1$, *where*

$$A_0 = \Big\{\alpha \in \mathbf{N}_0^n \,:\, \sum_{k=1}^n \frac{\alpha_k}{s_k} < 1\Big\}, \qquad A_1 = \bigcup_{i=1}^n A_{1i},$$

$$A_{1i} = \Big\{\alpha \in \mathbf{R}^n_+ \,:\, \alpha_k \in \mathbf{N}_0 \ \text{for} \ k \neq i;\ \sum_{k=1}^n \frac{\alpha_k}{s_k} = 1\Big\},$$

and $s_1, \ldots, s_n$ *are given positive real numbers. Then* $W_p^A(\Omega) = W_p^{(s_1,\ldots,s_n)}(\Omega)$ *is an anisotropic Sobolev space, the top seminorm of which can be defined by*

$$|f|_{W_p^{(s_1,\ldots,s_n)}(\Omega)} = \Big(\sum_{\alpha\in A_1} |f|^p_{\alpha,p}\Big)^{1/p}, \qquad 1 \leq p < \infty.$$

For $s_1 = s_2 = \ldots = s_n$, $W_p^A(\Omega)$ *reduces to an ordinary, isotropic Sobolev space* $W_p^s(\Omega)$. *(More precisely, the norm of* $W_p^A(\Omega)$ *is then equivalent to the standard norm of* $W_p^s(\Omega)$*).*

Let Ω be, as before, a domain of $\mathbf{R}^n$, $I = (0, T) \subset \mathbf{R}$ and $Q = \Omega \times I$. If s and r are non-negative real numbers, we can introduce the space $W_p^{s,r}(Q) = L_p(I;\, W_p^s(\Omega)) \cap W_p^r(I;\, L_p(\Omega))$, with the norm

$$\|f\|_{W_p^{s,r}(Q)} = \Big(\int_0^T \|f(t)\|^p_{W_p^s(\Omega)}\, dt + \|f\|^p_{W_p^r(I;\, L_p(\Omega))}\Big)^{1/p}, \qquad 1 \leq p < \infty,$$

and with a corresponding expression for $p = \infty$.

$W_p^{s,r}(Q)$ reduces to a space of the type $W_p^A(Q)$. For example, if $s \in \mathbb{N}_0$ then

$$A = \{(\alpha_1, \ldots, \alpha_n, 0)^T : \alpha_i \in \mathbb{N}_0, \ \alpha_1 + \ldots + \alpha_n \leq s\}$$
$$\cup \{(0, \ldots, 0, \beta)^T : \beta \in \mathbb{N}_0, \ \beta < r\} \cup \{(0, \ldots, 0, r)^T\}.$$

We shall use the space $W_2^{s,s/2}(Q) = L_2(I; W_2^s(\Omega)) \cap W_2^{s/2}(I; L_2(\Omega))$. Here we can define the top seminorm by

$$|f|_{W_2^{s,s/2}(Q)} = \left(\int_0^T |f(t)|^2_{W_2^s(\Omega)}\, dt + |f|^2_{W_2^{s/2}(I; L_2(\Omega))}\right)^{1/2}.$$

We shall also introduce the space $\widehat{W}_2^{s,s/2}(Q) = W_2^{(s,\ldots,s,s/2)}(Q)$. Obviously,

$$\widehat{W}_2^{s,s/2}(Q) \subset W_2^{s,s/2}(Q).$$

Anisotropic spaces also satisfy certain imbedding theorems. We shall later need those from Lions & Magenes [70].

THEOREM 1. *If* $f \in W_2^{s,r}(Q)$, $s \geq 0$, $r > 1/2$, *then for* $k < r - 1/2$ $(k \in \mathbb{N}_0)$ *there exists a trace* $\dfrac{\partial^k f(x,0)}{\partial t^k} \in W_2^q(\Omega)$, *where* $q = \dfrac{s}{r}\left(r - k - \dfrac{1}{2}\right)$.

THEOREM 2. *Let* $f \in W_2^{s,r}(Q)$, $s, r > 0$, *and let* $\alpha \in \mathbb{N}_0^n$ *and* $k \in \mathbb{N}_0$ *satisfy* $\dfrac{|\alpha|}{s} + \dfrac{k}{r} \leq 1$. *Then,* $D_x^\alpha D_t^k f \in W_2^{\mu,\nu}(Q)$, *where* $\dfrac{\mu}{s} = \dfrac{\nu}{r} = 1 - \left(\dfrac{|\alpha|}{s} + \dfrac{k}{r}\right)$, *and* D_x *and* D_t *are partial derivatives with respect to* $x = (x_1, x_2, \ldots, x_n)^T$ *and* t.

CONSEQUENCE. $W_2^{s,s/2}(Q) = \widehat{W}_2^{s,s/2}(Q)$, *with the equivalence of their norms.*

Finally, let us consider the weighted Sobolev spaces. Let Ω be a domain of $\mathbf{R}^n$ and let $\varrho(x) \in C^\infty(\overline{\Omega})$ be a non-negative function on $\overline{\Omega}$. For $s = 0, 1, 2, \ldots$ we shall introduce the following spaces

$$W_{p,\varrho}^s(\Omega) = \left\{f : \varrho^{|\alpha|} D^\alpha f \in L_p(\Omega),\ 0 \leq |\alpha| \leq s\right\},$$

with norms

$$\|f\|_{W_{p,\varrho}^s(\Omega)} = \Big(\sum_{|\alpha| \leq s} \|\varrho^{|\alpha|} D^\alpha f\|^p_{L_p(\Omega)}\Big)^{1/p}, \quad 1 \leq p < \infty,$$

and with corresponding expressions for $p = \infty$.

In particular, if $\varrho(x)$ is of the same order as the distance $d(x, \Gamma)$ from the point $x \in \Omega$ to the boundary Γ,

$$0 < C_1 \leq \frac{\varrho(x)}{d(x,\Gamma)} \leq C_2, \quad \forall x \in \Omega,$$

then set $\Xi^s(\Omega) = W^s_{2,\varrho}(\Omega)$. Obviously,

$$\Xi^0(\Omega) = L_2(\Omega), \quad W^s_2(\Omega) \subseteq \Xi^s(\Omega) \subseteq L_2(\Omega), \quad s \geq 0.$$

Using the theory of interpolation of function spaces (Bergh & Löfström [4]), spaces $W^s_{p,\varrho}(\Omega)$ can be defined for real $s \geq 0$. In particular,

$$\Xi^s(\Omega) = \{f \in \mathcal{D}'(\Omega) : \varrho^s f \in \overset{\circ}{W}{}^s_2(\Omega)\}, \quad s > 0, \quad s \neq \text{integer} + 1/2.$$

For $s < 0$ set

$$\Xi^s(\Omega) = \left(\Xi^{-s}(\Omega)\right)'.$$

Similarly, anisotropic weighted spaces in the domain $Q = \Omega \times I$ can be introduced by

$$\Xi^{s,s/2}(Q) = L_2(I; \Xi^s(\Omega)) \cap \Xi^{s/2}(I; L_2(\Omega)), \quad s \geq 0,$$
$$\Xi^{s,s/2}(Q) = (\Xi^{-s,-s/2}(Q))', \quad s < 0.$$

5. Besov Spaces

Let Ω be a Lipschitzian domain in $\mathbf{R}^n$, $0 < s < \infty$, $1 < p < \infty$ and $1 \leq q \leq \infty$. The Besov space $B^s_{p,q}(\Omega)$ consists of all functions $f(x)$ having a finite norm

$$\|f\|_{B^s_{p,q}(\Omega)} = \left(\|f\|^q_{L_p(\Omega)} + \sum_{|\alpha|\leq k} \int_{\mathbf{R}^n} |y|^{-n-(s-k)q} \|\Delta^l_y D^\alpha f\|^q_{L_p(\Omega_{y,l})} dy\right)^{1/q},$$

where k and l are arbitrary integers satisfying the conditions $0 \leq k < s$, $l > s-k$, and $\Omega_{y,l} = \bigcap_{j=0}^{l} \{x \in \mathbf{R}^n : x + jy \in \Omega\}$. (In particular, one can set $k = [s]^-$, $l = 2$). For $q = \infty$ one should substitute in the above expression $\left(\int |\cdot|^q |y|^{-n} dy\right)^{1/q}$ by $\sup \operatorname{ess} |\cdot|$.

The Besov space can be normed also in various equivalent ways. (see Besov, Il'in & Nikol'skiĭ [6], Triebel [96]).

For $q = p$ and for a non-integer s, Besov spaces reduce to Sobolev spaces

$$B^s_{p,p}(\Omega) = W^s_p(\Omega).$$

Besov spaces satisfy various imbedding theorems. For example, for $1 < p < \infty$, $1 \leq q_1 \leq q_2 \leq \infty$ and arbitrary $\varepsilon > 0$, the following imbeddings hold (see Triebel [96])

$$B^{s+\varepsilon}_{p,\infty}(\Omega) \subseteq B^s_{p,1}(\Omega) \subseteq B^s_{p,q_1}(\Omega) \subseteq B^s_{p,q_2}(\Omega) \subseteq B^s_{p,\infty}(\Omega) \subseteq B^{s-\varepsilon}_{p,1}(\Omega).$$

For $1 < p \leq r < \infty$, $1 \leq q \leq \infty$, $t \leq s$ and $s - n/p \geq t - n/r$

$$B^s_{p,q}(\Omega) \subseteq B^t_{r,q}(\Omega).$$

6. Bramble–Hilbert Lemma

The well known Bramble–Hilbert lemma (see Bramble & Hilbert [7, 8], Dupont & Scott [16]) has a fundamental role for the estimation of linear functionals in Sobolev spaces.

LEMMA 1. *Let Ω be a Lipschitzian domain in $\mathbf{R}^n$, s — a positive real number, and $\mathcal{P}_s$ set of polynomials (with n variables) of degree $< s$. Then there exists a constant $C = C(\Omega, s, p)$ such that*

$$\inf_{P \in \mathcal{P}_s} \|f - P\|_{W^s_p(\Omega)} \leq C\, |f|_{W^s_p(\Omega)}, \quad \forall f \in W^s_p(\Omega).$$

This lemma can be easily transferred into anisotropic spaces of Sobolev's type. Let $A \subset \mathbf{R}^n_+$ be a regular set of non-negative real multi-indices. We shall denote by $\kappa(A)$ the convex envelope of the set A in $\mathbf{R}^n$. Let $\partial_0\kappa(A)$ be a part of the boundary $\kappa(A)$ not depending on coordinate hyperplanes and $A_\partial = A \cap \overline{\partial_0\kappa(A)}$. Let B be a subset of A_∂, such that $B \cup \{0\}$ is a regular set of multi-indices, and $\nu(B) = \left\{\beta \in \mathbf{N}^n_0 \,:\, D^{[\alpha]^-} x^\beta \equiv 0,\ \forall \alpha \in B\right\}$. With $\mathcal{P}_B$ we denote the set of polynomials of the form

$$P(x) = \sum_{\alpha \in \nu(B)} p_\alpha x^\alpha.$$

The following results are valid (see Dražić [13], Jovanović [35]).

LEMMA 2. *Let Ω be a Lipschitzian domain in $\mathbf{R}^n$ and let the sets of multi-indices A and B satisfy the above defined conditions. Then there exists a constant $C = C(\Omega, A, B, p)$ such that*

$$\inf_{P \in \mathcal{P}_B} \|f - P\|_{W^A_p(\Omega)} \leq C \sum_{\alpha \in B} |f|_{\alpha,p}, \quad \forall f \in W^A_p(\Omega).$$

The following statements are direct consequences of Lemma 2.

LEMMA 3. *Let $\eta(f)$ be a bounded linear functional on $W^A_p(\Omega)$ which vanishes for $f(x) = x^\alpha$, $\alpha \in \nu(B)$. Then, there exists a constant $C = C(\Omega, A, B, p)$ such that for every $f \in W^A_p(\Omega)$, the inequality*

$$|\eta(f)| \leq C \sum_{\alpha \in B} |f|_{\alpha,p}$$

holds.

LEMMA 4. *Let A_k, B_k and Ω_k satisfy in $\mathbf{R}^{n_k}$ $(k = 1, 2, \ldots, m)$ the same conditions as A, B and Ω. Let $\eta(f_1, f_2, \ldots, f_m)$ be a bounded multi-linear functional on $W_{p_1}^{A_1}(\Omega_1) \times W_{p_2}^{A_2}(\Omega_2) \times \ldots \times W_{p_m}^{A_m}(\Omega_m)$, which vanishes if some of its variables have the form $f_k = x^\alpha$, $x \in \Omega_k$, $\alpha \in \nu(B_k)$. Then there exists a constant $C = C(\Omega_1, A_1, B_1, p_1, \Omega_2, A_2, B_2, p_2, \ldots, \Omega_m, A_m, B_m, p_m)$ such that for every $(f_1, f_2, \ldots, f_m) \in W_{p_1}^{A_1}(\Omega_1) \times W_{p_2}^{A_2}(\Omega_2) \times \ldots \times W_{p_m}^{A_m}(\Omega_m)$ the inequality*

$$|\eta(f_1, f_2, \ldots, f_m)| \le C \prod_{k=1}^{m} \sum_{\alpha \in B_k} |f_k|_{\alpha, p_k}$$

holds.

7. Multipliers in Sobolev Spaces

Let V and W be real function spaces contained in $\mathcal{D}'(\Omega)$. A function $a(x)$ defined on Ω is called a point multiplier, or simply a multiplier, from V to W if, for every $f \in V$, the product $a(x) \cdot f(x)$ belongs to W. The set of such multipliers is denoted by $M(V \to W)$. In particular, if $V = W$ we set $M(V) = M(V \to V)$.

We shall now examine the multipliers in Sobolev spaces, i.e. the sets of the form $M(W_p^t(\Omega) \to W_q^s(\Omega))$, with the natural limitation $t \ge s$. We shall also limit ourselves to the case $q = p$.

To begin, we shall consider the case of Sobolev spaces on $\mathbf{R}^n$. From the definition of multiplication of a distribution by a smooth function

$$\langle a \cdot f, \varphi \rangle_{\mathcal{D}' \times \mathcal{D}} = \langle f, a \cdot \varphi \rangle_{\mathcal{D}' \times \mathcal{D}}$$

for $a \in M(W_p^t(\mathbf{R}^n) \to W_p^s(\mathbf{R}^n))$ and $f \in W_{p'}^{-s}(\mathbf{R}^n)$, $1/p + 1/p' = 1$, we shall now define the product $a \cdot f \in W_{p'}^{-t}(\mathbf{R}^n)$ by

$$\langle a \cdot f, \varphi \rangle_{W_{p'}^{-t} \times W_p^t} = \langle f, a \cdot \varphi \rangle_{W_{p'}^{-s} \times W_p^s}, \qquad \forall \varphi \in W_p^t(\mathbf{R}^n).$$

This definition implies that

$$M(W_{p'}^{-s}(\mathbf{R}^n) \to W_{p'}^{-t}(\mathbf{R}^n)) = M(W_p^t(\mathbf{R}^n) \to W_p^s(\mathbf{R}^n)),$$

and, therefore, it is sufficient to examine the properties of the sets $M(W_p^t(\mathbf{R}^n) \to W_p^s(\mathbf{R}^n))$ and $M(W_p^t(\mathbf{R}^n) \to W_p^{-s}(\mathbf{R}^n))$ for $t \ge s \ge 0$.

We present here, without proofs, some basic results on multipliers in Sobolev spaces (see Maz'ya & Shaposhnikova [77]).

LEMMA 1. *If $a \in M(W_p^t(\mathbf{R}^n) \to W_p^s(\mathbf{R}^n))$, $t \ge s \ge 0$ then*
$a \in M(W_p^{t-s}(\mathbf{R}^n) \to L_p(\mathbf{R}^n))$,

$a \in M(W_p^{t-\sigma}(\mathbf{R}^n) \to W_p^{s-\sigma}(\mathbf{R}^n))$, $\quad 0 < \sigma < s$,

$D^\alpha a \in M(W_p^t(\mathbf{R}^n) \to W_p^{s-|\alpha|}(\mathbf{R}^n))$, $\quad |\alpha| \le s$,

$D^\alpha a \in M(W_p^{t-s+|\alpha|}(\mathbf{R}^n) \to L_p(\mathbf{R}^n))$, $\quad |\alpha| \le s$.

LEMMA 2. *For* $t \ge s \ge 0$, $M(W_p^t(\mathbf{R}^n) \to W_p^s(\mathbf{R}^n)) \subseteq W^s_{p,\,unif}$

$$= \Big\{ f \, : \, \sup_{z \in \mathbf{R}^n} \|\eta(x-z)\cdot f(x)\|_{W_p^s} < \infty, \ \forall \eta \in \mathcal{D}(\mathbf{R}^n), \ \eta \equiv 1 \ \ on \ \ B_1 \Big\},$$

where B_1 *is the unit ball with the center* 0. *If* $t \cdot p > n$, *then the inverse inclusion* $M(W_p^t(\mathbf{R}^n) \to W_p^s(\mathbf{R}^n)) \supseteq W^s_{p,\,unif}$ *also holds.*

LEMMA 3. *Let* $t \ge s \ge 0$. *If* $a \in M(W_p^t(\mathbf{R}^n) \to W_p^s(\mathbf{R}^n))$ *then* $a \in M(W_p^t(\mathbf{R}^{n+k}) \to W_p^s(\mathbf{R}^{n+k}))$. *Also* $a \in M(W_p^{t,t/2}(\mathbf{R}^n \times \mathbf{R}) \to W_p^{s,s/2}(\mathbf{R}^n \times \mathbf{R}))$.

LEMMA 4. *For* $s \ge 0$ *the inclusion* $M(W_p^s(\mathbf{R}^n)) \subseteq L_\infty(\mathbf{R}^n)$ *holds.*

LEMMA 5. *Suppose* $1 < p < \infty$, *and let s and t be non-negative integers such that* $t \ge s$. *If*

$$a = \sum_{|\alpha| \le t} D^\alpha a_\alpha$$

and $a_\alpha \in M(W_p^t(\mathbf{R}^n) \to W_p^{t-s}(\mathbf{R}^n)) \cap M(W_{p'}^s(\mathbf{R}^n) \to L_{p'}(\mathbf{R}^n))$, *where* $1/p + 1/p' = 1$, *then* $a \in M(W_p^t(\mathbf{R}^n) \to W_p^{-s}(\mathbf{R}^n))$.

LEMMA 6. *Let* $p > 1$, $t > s > 0$, *and assume that either* $q \in [n/t, \infty]$ *and* $t \cdot p < n$, *or* $q \in (p, \infty)$ *and* $t \cdot p = n$. *If* $a \in B^s_{q,p,\,unif}$

$$= \Big\{ f \, : \, \sup_{z \in \mathbf{R}^n} \|\eta(x-z)\cdot f(x)\|_{B^s_{q,p}} < \infty, \ \forall \eta \in \mathcal{D}(\mathbf{R}^n), \ \eta \equiv 1 \ \ on \ \ B_1 \Big\},$$

then $a \in M(W_p^t(\mathbf{R}^n) \to W_p^s(\mathbf{R}^n))$. *This result is also valid for* $t = s$, *provided that* $a \in L_\infty(\mathbf{R}^n)$.

LEMMA 7. *If* $a_\alpha \in M(W_p^{s-|\alpha|}(\mathbf{R}^n) \to W_p^{s-k}(\mathbf{R}^n))$, $s \ge k$, *for every multi-index* α, *then the differential operator*

$$\mathcal{L}u = \sum_{|\alpha| \le k} a_\alpha(x)\, D^\alpha u, \quad x \in \mathbf{R}^n \tag{1}$$

defines a continuous mapping from $W_p^s(\mathbf{R}^n)$ *to* $W_p^{s-k}(\mathbf{R}^n)$.

Analogous result is valid for $s < 0$. If $p = 2$, then this result holds for every s. Under some conditions we have the inverse result.

LEMMA 8. *Let operator* (1) *define a continuous mapping from* $W_p^s(\mathbf{R}^n)$ *to* $W_p^{s-k}(\mathbf{R}^n)$, *and* $p(s-k) > n$, $p > 1$. *Then* $a_\alpha \in M(W_p^{s-|\alpha|}(\mathbf{R}^n) \to W_p^{s-k}(\mathbf{R}^n))$ *for every multi-index* α.

All of these results can be extended to Sobolev spaces in a domain of $\mathbf{R}^n$. More precisely, if Ω is a Lipschitzian domain in $\mathbf{R}^n$ and the function a belongs to $M\big(W_p^t(\Omega) \to W_p^s(\Omega)\big)$, then a can be extended to a function $\tilde{a}$, defined on $\mathbf{R}^n$, such that $\tilde{a} \in M\big(W_p^t(\mathbf{R}^n) \to W_p^s(\mathbf{R}^n)\big)$. The converse is also true, i.e. the restriction of a multiplier $a \in M\big(W_p^t(\mathbf{R}^n) \to W_p^s(\mathbf{R}^n)\big)$ on Ω belongs to $M\big(W_p^t(\Omega) \to W_p^s(\Omega)\big)$.

For bounded domains, $W^s_{p,unif}$ and $B^s_{q,p,unif}$ can be replaced by standard Sobolev and Besov spaces. Using Lemmas 2, 4 and 6, imbedding theorems for Sobolev and Besov spaces and the representation of distributions from negative Sobolev spaces, we obtain the following results.

LEMMA 9. *Suppose that Ω is a bounded Lipschitzian domain in $\mathbf{R}^n$, $s > 0$ and $p > 1$. If $a \in W_q^t(\Omega)$, where*

$q = p, \quad t = s, \qquad$ *when* $\; s \cdot p > n$, *or*

$q \ge n/s, \quad t = s + \varepsilon \notin \mathbf{N}, \quad \varepsilon > 0, \qquad$ *when* $\; s \cdot p \le n$,

then $a \in M\big(W_p^s(\Omega)\big)$.

LEMMA 10. *Let Ω be a bounded Lipschitzian domain in $\mathbf{R}^n$, $s > 0$ and $p > 1$. If $a \in L_q(\Omega)$, where*

$q = p, \quad$ *when* $\; s \cdot p > n$,

$q > p, \quad$ *when* $\; s \cdot p = n$, *and*

$q \ge n/s, \quad$ *when* $\; s \cdot p < n$,

then $a \in M\big(W_p^s(\Omega) \to L_p(\Omega)\big)$.

LEMMA 11. *Let Ω be a bounded Lipschitzian domain in $\mathbf{R}^n$ and*

$$a(x) = a_0(x) + \sum_{i=1}^{n} D_i\, a_i(x).$$

If $a_0 \in M\big(W_2^t(\Omega) \to L_2(\Omega)\big)$ and $a_i \in M\big(W_2^t(\Omega) \to W_2^{1-s}(\Omega)\big) \cap M\big(W_2^{t-1}(\Omega) \to L_2(\Omega)\big)$, $i = 1, 2, \ldots, n$, where $0 < s \le 1 \le t < 2$ and $s \ne 1/2$, then $a \in M\big(W_2^t(\Omega) \to W_2^{-s}(\Omega)\big)$.

8. Boundary–Value Problems for Partial Differential Equations

As a model problem in the elliptic case, let us consider the Dirichlet boundary-value problem for a linear self-adjoint partial differential equation of the second order

$$(1) \qquad \mathcal{L}u \equiv -\sum_{i,j=1}^{n} D_i\left(a_{ij}\, D_j u\right) + a\, u = f, \qquad x \in \Omega$$
$$u = g, \qquad x \in \Gamma = \partial\Omega.$$

This problem has been extensively studied (see Ladyzhenskaya & Ural'tseva [60], Lions & Magenes [70], Ladyzhenskaya [59]), particularly for the case of solutions from spaces $W_2^s(\Omega)$. Here the equation and the boundary condition are taken in sense of distributions. Solutions of this type, which in the general case represent distributions, are called generalised, or weak solutions.

Let the following conditions be satisfied (A):

I. Ω is a bounded convex domain in $\mathbf{R}^n$, with a boundary $\Gamma \in C^\infty$;

II. $a_{ij},\, a \in C^\infty(\overline{\Omega})$, $a_{ij} = a_{ji}$;

III. The operator $\mathcal{L}$ is strongly elliptic, i.e.

$$\sum_{i,j=1}^{n} a_{ij}\, y_i\, y_j \geq c_0 \sum_{i=1}^{n} y_i^2 , \qquad c_0 > 0 , \qquad \forall\, x \in \overline{\Omega} , \qquad \forall\, y \in \mathbf{R}^n ,$$

$$a(x) \geq 0 , \qquad \forall\, x \in \overline{\Omega} .$$

The following statement is true (Lions & Magenes [70]):

THEOREM 1. *Set $g \in W_2^{s-1/2}(\Gamma)$ and let one of the following conditions be satisfied*

a) $f \in W_2^{s-2}(\Omega)$, *for* $s \geq 2$,

b) $f \in \Xi^{s-2}(\Omega)$, *for* $0 < s < 2$,

c) $f \in \Xi^{s-2}(\Omega)$, *for* $s \leq 0$, $s \neq$ *integer* $+ 1/2$.

Then the boundary-value problem (1) *has the unique solution* $u \in W_2^s(\Omega)$.

One may use weaker conditions than (A). For example, it is sufficient that the coefficients of the equation (1) belong to the corresponding multiplier spaces (see Theorems 7.7 and 7.8)

$$a_{ij} \in M\big(W_2^{s-1}(\Omega)\big) , \qquad a \in M\big(W_2^s(\Omega) \rightarrow W_2^{s-2}(\Omega)\big) .$$

If the boundary is smooth stepwise, the solution has singularities at the breaking points. However, these singularities can be avoided if the input data satisfy certain additional consistency conditions. For example, the solution of the homogeneous Dirichlet boundary-value problem for the Poisson equation in the square $\Omega = (0,\, 1)^2 \subset \mathbf{R}^2$ belongs to the Sobolev space $W_2^s(\Omega)$ (for $s > 3$) if the right-hand side satisfies the following conditions

$$f = 0 ,$$

$$D_1^2 f - D_2^2 f = 0 ,$$

$$\dots\dots\dots\dots\dots\dots$$

$$D_1^{2k} f - D_1^{2k-2}\, D_2^2 f + \ldots + (-1)^k\, D_2^{2k} f = 0 , \qquad k = \left[\frac{s-3}{2}\right]^- ,$$

at all four vertices of the square Ω (see Volkov [108]).

In the parabolic case, as a model problem we shall consider the initial–boundary–value problem

$$\begin{aligned} &\frac{\partial u}{\partial t} + \mathcal{L}u = f, && (x,t) \in Q = \Omega \times (0,T], \\ &u = 0, && (x,t) \in \Gamma \times [0,T], \\ &u(x,0) = u_0(x), && x \in \Omega. \end{aligned} \tag{2}$$

Let the following conditions be satisfied (B):

I. Ω *is a bounded convex domain in* $\mathbf{R}^n$, *with a boundary* $\Gamma \in C^\infty$;

II. $a_{ij},\, a \in C^\infty(\overline{Q}), \quad a_{ij} = a_{ji}$;

III. The operator $\mathcal{L}$ *is strongly elliptic in* $\overline{Q}$, *i.e.*

$$\sum_{i,j=1}^{n} a_{ij}\, y_i\, y_j \ge c_0 \sum_{i=1}^{n} y_i^2, \qquad c_0 > 0, \qquad \forall (x,t) \in \overline{Q}, \qquad \forall y \in \mathbf{R}^n,$$
$$a(x,t) \ge 0, \qquad \forall (x,t) \in \overline{Q}.$$

Note also the following consistency conditions

$$\begin{aligned} &\textit{There exists a function } v \in W_2^{s,s/2}(Q) \textit{ such that} \\ &v = 0, \qquad (x,t) \in \Gamma \times [0,T], \\ &v(x,0) = u_0(x), \qquad x \in \Omega, \\ &\frac{\partial^k}{\partial t^k}\left(\frac{\partial v}{\partial t} + \mathcal{L}v\right)\bigg|_{t=0} = \frac{\partial^k f(x,0)}{\partial t^k}, \qquad \textit{for } \; 0 \le k \le \left[\frac{s-3}{2}\right]^- . \end{aligned} \tag{3}$$

The solution of the initial–boundary–value problem (2) belongs to the space $W_2^{s,s/2}(Q)$ under following conditions (see Lions & Magenes [70], Ladyzhenskaya, Solonnikov & Ural'tseva [61])

a) For $s \ge 2,\; s,\, s/2 \ne$ *integer* $+ 1/2$,
if $f \in W_2^{s-2,s/2-1}(Q), \quad u_0 \in W_2^{s-1}(\Omega)$ *and the consistency conditions* (3) *are satisfied.*

b) For $1 \le s < 3/2$,
if $f \in \left(W_2^{2-s,1-s/2}(Q)\right)', \quad u_0 \in W_2^{s-1}(\Omega)$.

c) For $0 < s < 1$,
if $f \in \left(W_2^{2-s,1-s/2}(Q)\right)', \quad u_0 \in (W_2^{1-s}(\Omega))'$.

d) For $s \le 0,\; s \ne$ *integer* $+ 1/2$,
if $f \in \Xi^{s-2,s/2-1}(Q), \quad u_0 \in \Xi^{s-1}(\Omega)$.

This result can be extended to the interval $3/2 < s < 2$, provided that the input data satisfy certain special consistency conditions.

Some of the conditions (B) can be reduced, similarly to those in the elliptic case.

Finally, let us consider the following hyperbolic initial-boundary-value problem

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} + \mathcal{L}u = f, && (x; t) \in Q = \Omega \times (0, T], \\
&u = 0, && (x, t) \in \Gamma \times [0, T], \\
&u(x, 0) = 0, && \frac{\partial u(x, 0)}{\partial t} = 0, \qquad x \in \Omega .
\end{aligned} \tag{4}
$$

Let the conditions (B) be satisfied.

Contrary to the elliptic and parabolic cases, in the hyperbolic case there arises a certain inconsistency between the smoothness of input data and the solution. For example, the following results hold true (Lions & Magenes [70]),

a) If $f \in L_2(Q)$, *then* $u \in W_2^{1,1}(Q)$;

b) If $f \in W_2^{0,1}(Q)$, *then* $u \in W_2^{2,2}(Q)$;

c) If $f \in \widetilde{W}_2^{s-2,s-1}(Q) = \left\{ f \in W_2^{s-2,s-1}(Q) : \dfrac{\partial^j u(x,0)}{\partial t^j} = 0,\ 0 \le j < s - \frac{3}{2} \right\}$, $s \ge 3$, $s \ne$ *integer* $+ 1/2$, *then* $u \in W_2^{s,s}(Q)$.

Using the interpolation theory of function spaces (Bergh & Löfström [4]) and the transposition, the result can be extended to the real values $s < 3$.

II Eliptic Equations

In this chapter we provide estimates of the convergence rate in discrete W_2^r–norms for finite difference schemes approximating boundary–value problems for partial differential equations of the elliptic type. We consider equations with variable coefficients from Sobolev classes.

In Paragraph 1 we introduce some necessary terms from the theory of difference schemes. In Paragraph 2 we obtain estimate of the convergence rate in the discrete W_2^1–norm for the difference scheme approximating the Dirichlet boundary–value problem for the second–order eliptic equation with variable coefficients. In Paragraph 3 for the same problem we obtain estimates of convergence rate in other W_2^r–norms $(0 \leq r \leq 2)$. In Paragraph 4 for the equation with separated variables, the L_2–estimate, consistent with the smoothness of data is obtained. The Paragraph 5 is devoted to the fourth–order equations.

1. Meshes, Mesh–functions, Operators and Norms

One–dimensional case. Let $h > 0$. We denote by $\mathbf{R}_h$ the uniform mesh with the step h on the real axis: $\mathbf{R}_h = \{x = j\,h \,:\, j = 0, \pm 1, \pm 2, \ldots\}$. For each node $x \in \mathbf{R}_h$ we consider the neighbourhood $i(x) = (x - h/2,\, x + h/2)$. The mesh $\vartheta \subset \mathbf{R}_h$ is said to be connected if the set $\cup_{x\in\vartheta}\, i(x)$ is connected. If the mesh $\vartheta \subset \mathbf{R}_h$ is bounded we denote $x^- = x^-(\vartheta) = \min \vartheta - h$, $x^+ = x^+(\vartheta) = \max \vartheta + h$ and $\overline{\vartheta} = \vartheta \cup \{x^-, x^+\}$.

Let $H(\vartheta)$ be the set of functions defined on the mesh ϑ and $\overset{\circ}{H}(\vartheta)$ be the set of functions defined on $\overline{\vartheta}$ and equal to zero on $\overline{\vartheta} \setminus \vartheta$. Both of these spaces can be furnished with a inner product

$$(v,\, w)_\vartheta = (v,\, w)_{L_2(\vartheta)} = h \sum_{x\in\vartheta} v(x)\, w(x)\,,$$

and the corresponding norm

$$\|v\|_\vartheta = \|v\|_{L_2(\vartheta)} = \|v\|_{W_2^0(\vartheta)} = (v,\, v)_\vartheta^{1/2}.$$

We also define, in $H(\overline{\vartheta})$, the inner product

$$[v,\, w]_\vartheta = [v,\, w]_{L_2(\vartheta)} = \frac{h}{2}\, v(x^-)\, w(x^-) + \frac{h}{2}\, v(x^+)\, w(x^+) + (v,\, w)_\vartheta\,,$$

and the norm

$$\|[v]|_\vartheta = \|[v]|_{L_2(\vartheta)} = \|[v]|_{W_2^0(\vartheta)} = [v,\, v]_\vartheta^{1/2}.$$

In the following we shall set $h = 1/n$, $n \in \mathbf{N}$, and consider the standard mesh $\theta = \mathbf{R}_h \cap (0,\, 1)$ on the unit interval. In this case $x^-(\theta) = 0$, $x^+(\theta) = 1$. Let us also denote $\theta^- = \theta \cup \{0\}$, $\theta^+ = \theta \cup \{1\}$.

We introduce the finite difference operators in the usual way

$$v_x = (v^+ - v)/h\,, \qquad v_{\bar{x}} = (v - v^-)/h\,, \qquad v_{\overset{\circ}{x}} = (v_x + v_{\bar{x}})/2\,,$$

where $v^\pm = v^\pm(x) = v(x \pm h)$.

The following expressions for the "derivation" of the product of mesh-functions

$$(v\, w)_x = v_x\, w^+ + v\, w_x = v_x\, w + v^+\, w_x\,,$$
$$(v\, w)_{\bar{x}} = v_{\bar{x}}\, w^- + v\, w_{\bar{x}} = v_{\bar{x}}\, w + v^-\, w_{\bar{x}}\,,$$

and partial summation

$$(v_x,\, w)_{\theta^-} = -(v,\, w_{\bar{x}})_{\theta^+} + v(1)\, w(1) - v(0)\, w(0)$$

hold.

In $\overset{\circ}{H}(\theta)$ we define the operator

$$\Lambda v = \begin{cases} -v_{x\bar{x}}\,, & x \in \theta \\ 0\,, & x \in \overline{\theta} \setminus \theta \end{cases}.$$

The operator Λ is linear, self-adjoint and positive definite. The following equality is satisfied

$$(\Lambda v,\, v)_\theta = -(v_{x\bar{x}},\, x)_\theta = \|v_x\|_{\theta^-}^2\,.$$

The eigenvalues of the operator Λ are

(1) $$\lambda_k = \frac{4}{h^2} \sin^2 \frac{k\pi h}{2}\,, \qquad k = 1, 2, \ldots, n-1\,,$$

and they satisfy the following inequalities

(2) $$8 \le \lambda_k < \frac{4}{h^2}\,, \qquad k = 1, 2, \ldots, n-1\,.$$

The corresponding eigenfunctions are

$$v^k = \sin k\pi x\,, \qquad x \in \overline{\theta}\,, \qquad k = 1, 2, \dots, n-1\,,$$

and they satisfy the orthogonality conditions

$$(\sin k\pi x,\, \sin l\pi x)_\theta = \frac{1}{2}\,\delta_{kl} = \begin{cases} 1/2, & k = l \\ 0, & k \neq l \end{cases}, \qquad k, l = 1, 2, \dots, n-1\,,$$

thus representing the basis of the space $\overset{\circ}{H}(\theta)$.

A mesh-function $v \in \overset{\circ}{H}(\theta)$ can be represented in the form

(3) $$v = \sum_{k=1}^{n-1} b_k \sin k\pi x\,, \qquad x \in \overline{\theta}\,,$$

where $b_k = 2\,(v,\, \sin k\pi x)_\theta$. We can easilly obtain the following relations

(4) $$\|v\|_\theta^2 = \frac{1}{2} \sum_{k=1}^{n-1} b_k^2\,,$$

(5) $$\|v_x\|_{\theta^-}^2 = (\Lambda v,\, v)_\theta = \frac{1}{2} \sum_{k=1}^{n-1} \lambda_k\, b_k^2\,,$$

and

(6) $$\|v_{x\bar{x}}\|_\theta^2 = (\Lambda v,\, \Lambda v)_\theta = \frac{1}{2} \sum_{k=1}^{n-1} \lambda_k^2\, b_k^2\,.$$

From (2) and (4–6) it follows that

(7) $$\|v_{x\bar{x}}\|_\theta \geq 2\sqrt{2}\,\|v_x\|_{\theta^-} \geq 8\,\|v\|_\theta\,.$$

Let us define the discrete Sobolev-like seminorms and norms

$$|v|_{W_2^1(\theta)} = \|v_x\|_{\theta^-}\,, \qquad |v|_{W_2^2(\theta)} = \|v_{x\bar{x}}\|_\theta\,,$$
$$\|v\|_{W_2^k(\theta)}^2 = \|v\|_{W_2^{k-1}(\theta)}^2 + |v|_{W_2^k(\theta)}^2\,, \qquad k = 1, 2\,.$$

From the relations (7) it follows that the seminorms $|v|_{W_2^1(\theta)}$ and $|v|_{W_2^2(\theta)}$ are equivalent with respect to the norms $\|v\|_{W_2^1(\theta)}$ and $\|v\|_{W_2^2(\theta)}$ on $\overset{\circ}{H}(\theta)$.

Let us define the operator

$$\overline{\Lambda} v = \begin{cases} -\dfrac{2}{h}\, v_x\,, & x = 0 \\ -\,v_{x\bar{x}}\,, & x \in \theta \\ \dfrac{2}{h}\, v_{\bar{x}}\,, & x = 1 \end{cases}$$

on the space $H(\overline{\theta})$. (For evenly extended mesh-functions the values at the nodes $x = 0$ and $x = 1$ can also be represented as second differences). The operator $\overline{\Lambda}$ is self-adjoint with respect to the inner product $[v, w]_\theta$. The eigenvalues of $\overline{\Lambda}$ are also represented by (1), for $k = 0, 1, \ldots, n$. Since $\lambda_0 = 0$, the operator $\overline{\Lambda}$ is non-negative, but not positive definite. The corresponding eigenfunctions are

$$v^0 = 1, \qquad v^k = \cos k\pi x, \qquad k = 1, 2, \ldots, n.$$

They satisfy the conditions of orthogonality

$$[\cos k\pi x, \cos l\pi x]_\theta = \begin{cases} 1, & k = l = 0, n \\ 1/2, & k = l = 1, 2, \ldots, n-1 \\ 0, & k \neq l \end{cases},$$

and represent the basis of the space $H(\overline{\theta})$.

A function $v \in H(\overline{\theta})$ can be presented in the form

$$v = \frac{a_0}{2} + \sum_{k=1}^{n-1} a_k \cos k\pi x + \frac{a_n}{2} \cos n\pi x, \tag{8}$$

where $a_k = 2[v, \cos k\pi x]_\theta$, $k = 0, 1, \ldots, n$. For $v \in \overset{\circ}{H}(\theta)$ the representation (3) coincides with (8) at all nodes of the mesh $\overline{\theta}$, and for $v \in H(\overline{\theta}) \setminus \overset{\circ}{H}(\theta)$ — at all nodes of θ. A simple argument shows that

$$|[v]|_\theta^2 = \frac{a_0^2}{4} + \frac{1}{2}\sum_{k=1}^{n-1} a_k^2 + \frac{a_n^2}{4},$$

$$\|v_x\|_{\theta^-}^2 = [\overline{\Lambda} v, v]_\theta = \frac{1}{2}\sum_{k=1}^{n-1} \lambda_k a_k^2 + \frac{\lambda_n a_n^2}{4},$$

$$|[\overline{\Lambda} v]|_\theta^2 = \frac{1}{2}\sum_{k=1}^{n-1} \lambda_k^2 a_k^2 + \frac{\lambda_n^2 a_n^2}{4}.$$

Also define the following norms

$$|[v]|_{W_2^1(\theta)}^2 = |[v]|_\theta^2 + \|v_x\|_{\theta^-}^2,$$

$$|[v]|_{W_2^2(\theta)}^2 = |[v]|_\theta^2 + \|v_x\|_{\theta^-}^2 + |[\overline{\Lambda} v]|_\theta^2.$$

Finally, introduce the discrete seminorms and norms of the non-integer order

$$|v|_{W_2^r(\theta)}^2 = \begin{cases} h^2 \displaystyle\sum_{\substack{x, t \in \overline{\theta} \\ x \neq t}} \frac{[v(x) - v(t)]^2}{|x - t|^{1+2r}}, & 0 < r < 1 \\ h^2 \displaystyle\sum_{\substack{x, t \in \theta^- \\ x \neq t}} \frac{[v_x(x) - v_x(t)]^2}{|x - t|^{1+2(r-1)}}, & 1 < r < 2 \end{cases},$$

$$\|v\|^2_{W_2^r(\theta)} = \|v\|^2_{W_2^{[r]^-}(\theta)} + |v|^2_{W_2^r(\theta)}, \qquad 0 < r < 2,$$

$$\|[v]\|^2_{W_2^r(\theta)} = \|[v]\|^2_{W_2^{[r]^-}(\theta)} + |v|^2_{W_2^r(\theta)}, \qquad 0 < r < 2.$$

Similarly we can define the discrete Sobolev-like norms of higher order.

LEMMA 1. *In the mesh-function space $\overset{\circ}{H}(\theta)$ the multiplicative inequality*

$$\|v\|_{W_2^r(\theta)} \le C(r)\,\|v\|^{1-r}_{L_2(\theta)}\,\|v\|^r_{W_2^1(\theta)}, \qquad 0 < r < 1 \tag{9}$$

holds.

Proof. Let us expand the function $v \in \overset{\circ}{H}(\theta)$ in sine series (3) and define the norm

$$B_r(v) = \Big(\frac{1}{2}\sum_{k=1}^{n-1} k^{2r}\, b_k^2\Big)^{1/2}.$$

For $0 < r < 1$, using the inequality

$$\sin x \ge \frac{2}{\pi}x, \qquad 0 \le x \le \frac{\pi}{2},$$

and the Hölder's inequality, we obtain

$$B_r(v) \le \Big[\frac{1}{2}\sum_{k=1}^{n-1}\Big(\frac{\lambda_k}{4}\Big)^r b_k^2\Big]^{1/2} = 2^{-r}\Big[\frac{1}{2}\sum_{k=1}^{n-1}(\lambda_k\, b_k^2)^r\, b_k^{2(1-r)}\Big]^{1/2}$$
$$\le 2^{-r}\Big(\frac{1}{2}\sum_{k=1}^{n-1}\lambda_k\, b_k^2\Big)^{r/2}\Big(\frac{1}{2}\sum_{k=1}^{n-1} b_k^2\Big)^{(1-r)/2},$$

i.e.

$$B_r(v) \le 2^{-r}\,\|v\|^{1-r}_{L_2(\theta)}\,|v|^r_{W_2^1(\theta)}. \tag{10}$$

Using the expansion (3), the function $v \in \overset{\circ}{H}(\theta)$ becomes evenly extended outside of the mesh $\overline{\theta}$. Set $\Theta = (-1,\, 1) \cap \mathbf{R}_h$ and define the norm

$$N_r(v) = \Big\{h^2 \sum_{x\in\overline{\Theta}}{}' \sum_{\substack{t\in\overline{\Theta}\\ t\neq 0}}{}' \frac{[v(x) - v(x-t)]^2}{|t|^{1+2r}}\Big\}^{1/2}$$

where,

$$h\sum_{x\in\overline{\Theta}}{}' w(x) = \frac{h}{2}w(-1) + \frac{h}{2}w(1) + h\sum_{x\in\Theta} w(x) = [w,\, 1]_\Theta.$$

The extended function (also denoted by $v(x)$) is periodical, so, using the expansion (3), we obtain

$$N_r^2(v) = h^2 {\sum_{x\in\overline{\Theta}}}' {\sum_{\substack{t\in\overline{\Theta}\\ t\neq 0}}}' |t|^{-1-2r} v(x)\,[\,-v(x-t)+2\,v(x)-v(x+t)]$$

$$= h^2 {\sum_{x\in\overline{\Theta}}}' {\sum_{\substack{t\in\overline{\Theta}\\ t\neq 0}}}' |t|^{-1-2r} \sum_{l=1}^{n-1} b_l \sin l\pi x \sum_{k=1}^{n-1} b_k \cdot 4\sin^2\frac{k\pi t}{2}\cdot \sin k\pi x$$

$$= \sum_{l=1}^{n-1}\sum_{k=1}^{n-1} b_l b_k \cdot h {\sum_{x\in\overline{\Theta}}}' \sin l\pi x \cdot \sin k\pi x \cdot h {\sum_{\substack{t\in\overline{\Theta}\\ t\neq 0}}}' |t|^{-1-2r}\cdot 4\sin^2\frac{k\pi t}{2}$$

$$= \sum_{k=1}^{n-1} b_k^2 \cdot 8 \cdot h {\sum_{t\in\theta+}}'' t^{-1-2r}\sin^2\frac{k\pi t}{2},$$

where,

$$h {\sum_{t\in\theta+}}'' w(t) = h\sum_{t\in\theta} w(t) + \frac{h}{2}\,w(1) = (w,\,1)_\theta + \frac{h}{2}\,w(1).$$

Thus we obtain

$$N_r^2(v) = 16\,(\pi/2)^{2r}\cdot\frac{1}{2}\sum_{k=1}^{n-1} k^{2r} b_k^2\, C(k,\,r),$$

where,

$$C(k,\,r) = \frac{k\pi h}{2} {\sum_{t\in\theta+}}'' \left(\frac{k\pi t}{2}\right)^{-1-2r} \sin^2\frac{k\pi t}{2}.$$

$C(k,\,r)$ is the expansion of the integral

$$\int_0^{k\pi/2} x^{-1-2r}\sin^2 x\,dx$$

and can be estimated from both sides,

$$0 < \frac{1}{8}\left(\frac{2}{\pi}\right)^{2r} \le C(k,\,r) \le \pi^{2-2r}\left(1+\frac{1}{2-2r}\right)+\frac{1}{2r}\left(\frac{2}{\pi}\right)^{2r}.$$

Hence we conclude that the norms $N_r(v)$ and $B_r(v)$ are equivalent.

Using this, the inequality (10), equivalence of the seminorm $|v|_{W_2^1(\theta)}$ and the norm $\|v\|_{W_2^1(\theta)}$, and the obvious inequality

$$|v|_{W_2^r(\theta)} \le N_r(v),$$

one obtains the statement of the lemma. ∎

REMARK 1. The same result can be obtained from the cosine expansion (8) and the norm

$$A_r(v) = \Big(\frac{1}{2}\sum_{k=1}^{n-1} k^{2r}\, a_k^2 + \frac{n^{2r}\, a_n^2}{4}\Big)^{1/2}.$$

This norm is also equivalent to $N_r(v)$, assuming that $v(x)$ is evenly extended outside of $\overline{\theta}$.

REMARK 2. Similarly, for $v \in \overset{\circ}{H}(\theta)$ one can prove the following multiplicative inequality

$$\|v\|_{W_2^r(\theta)} \le C(r)\, \|v\|^{2-r}_{W_2^1(\theta)}\, \|v\|^{r-1}_{W_2^2(\theta)}, \qquad 1 < r < 2. \tag{11}$$

Multidimensional case. As a two-dimensional case is sufficiently representative, using a relatively simple notation, we shall consider the following case.

The direct product $\mathbf{R}_h^2 = \mathbf{R}_h \times \mathbf{R}_h$ represents a uniform square mesh in $\mathbf{R}^2$. For each node $x = (x_1, x_2) \in \mathbf{R}_h^2$ we associate the neighbourhood $e(x) = i(x_1) \times i(x_2) = (x_1 - h/2,\, x_1 + h/2) \times (x_2 - h/2,\, x_2 + h/2)$. The mesh $\varpi \subset \mathbf{R}_h^2$ is said to be connected if the set $\overline{\bigcup_{x\in\varpi} e(x)}$ is connected.

Let ϖ be a bounded connected mesh. In the set $H(\varpi)$ of the functions defined on ϖ we can define the inner product

$$(v, w)_\varpi = (v, w)_{L_2(\varpi)} = h^2 \sum_{x\in\varpi} v(x)\, w(x),$$

and the norm

$$\|v\|_\varpi = \|v\|_{L_2(\varpi)} = \|v\|_{W_2^0(\varpi)} = (v, v)_\varpi^{1/2}.$$

Finite differences are defined in the above described manner (see Paragraph 1.1)

$$v_{x_i} = (v^{+i} - v)/h, \qquad v_{\bar{x}_i} = (v - v^{-i})/h, \qquad v_{\overset{\circ}{x}} = (v_{x_i} + v_{\bar{x}_i})/2,$$

where $v^{\pm i} = v^{\pm i}(x) = v(x \pm h\, r_i)$.

In the ensuing we shall use the standard mesh with the step $h = 1/n$, in the unit square $\Omega = (0, 1)^2$. Let $\Gamma = \partial\Omega$ be the boundary of the domain Ω and $\Gamma_{ik} = \{x \in \Gamma : x_i = k,\ 0 < x_{3-i} < 1\}$, $i = 1, 2$, $k = 0, 1$. Set $\omega = \Omega \cap \mathbf{R}_h^2 = \theta \times \theta$, $\overline{\omega} = \overline{\Omega} \cap \mathbf{R}_h^2 = \overline{\theta} \times \overline{\theta}$, $\gamma = \Gamma \cap \mathbf{R}_h^2 = \overline{\omega} \setminus \omega$, $\gamma_{ik} = \Gamma_{ik} \cap \mathbf{R}_h^2$, $\overline{\gamma}_{ik} = \overline{\Gamma}_{ik} \cap \mathbf{R}_h^2$ and $\gamma_* = \gamma \setminus \big(\bigcup_{i,k} \gamma_{ik}\big)$. Also set $\omega_i = \omega \cup \gamma_{i0}$, $i = 1, 2$, and $\omega_{kl} = \omega \cup \gamma_{1k} \cup \gamma_{2l} \cup \{(k, l)\}$, $k, l = 0, 1$. Let $\overset{\circ}{H}(\omega)$ be the set of mesh-functions defined on $\overline{\omega}$, which are equal to zero on γ.

Define the following Sobolev-like discrete seminorms and norms

$$|v|^2_{W_2^1(\omega)} = \|v_{x_1}\|^2_{\omega_1} + \|v_{x_2}\|^2_{\omega_2},$$
$$|v|^2_{W_2^2(\omega)} = \|v_{x_1\bar{x}_1}\|^2_{\omega} + \|v_{x_1x_2}\|^2_{\omega_{00}} + \|v_{x_2\bar{x}_2}\|^2_{\omega},$$
$$\|v\|^2_{W_2^k(\omega)} = \|v\|^2_{W_2^{k-1}(\omega)} + |v|^2_{W_2^k(\omega)}, \qquad k = 1, 2.$$

Also introduce the discrete Laplacean on $\mathbf{R}^2_h$

$$\Delta_h v = v_{x_1\bar{x}_1} + v_{x_2\bar{x}_2}.$$

The operator $\mathring{\Delta}_h : \mathring{H}(\omega) \to \mathring{H}(\omega)$ defined by

$$\mathring{\Delta}_h v = \begin{cases} \Delta_h v, & x \in \omega \\ 0, & x \in \gamma \end{cases}$$

is self-adjoint and negative definite, with respect to the inner product $(v, w)_\omega$. For $v \in \mathring{H}(\omega)$ the following relations

$$-(\mathring{\Delta}_h v, v)_\omega = -(\Delta_h v, v)_\omega = |v|^2_{W_2^1(\omega)},$$
$$\|\Delta_h v\|^2_\omega = \|v_{x_1\bar{x}_1}\|^2_\omega + 2\|v_{x_1x_2}\|^2_{\omega_{00}} + \|v_{x_2\bar{x}_2}\|^2_\omega \ge |v|^2_{W_2^2(\omega)},$$
$$\|\Delta_h v\|^2_\omega \ge 16(-\Delta_h v, v)_\omega \ge 16^2\|v\|^2_\omega$$

hold. Hense,

$$|v|_{W_2^2(\omega)} \ge 2\sqrt{2}\,|v|_{W_2^1(\omega)} \ge 8\sqrt{2}\,\|v\|_\omega.$$

Consequently, in the space $\mathring{H}(\omega)$, the seminorms $|v|_{W_2^1(\omega)}$ and $|v|_{W_2^2(\omega)}$ are respectively equivalent to norms $\|v\|_{W_2^1(\omega)}$ and $\|v\|_{W_2^2(\omega)}$.

Define the discrete Sobolev-like norms of the fractional order

$$|v|^2_{W_2^r(\omega)} = \sum_{i=1}^2 h^3 \sum_{\substack{x_i, t_i \in \bar{\theta} \\ x_i \ne t_i}} \sum_{x_{3-i} \in \theta} \frac{\left[v(x) - v(t_i r_i + x_{3-i} r_{3-i})\right]^2}{|x_i - t_i|^{1+2r}}, \qquad 0 < r < 1,$$

$$|v|^2_{W_2^r(\omega)} = \sum_{i=1}^2 h^3 \sum_{\substack{x_i, t_i \in \theta^- \\ x_i \ne t_i}} \sum_{x_{3-i} \in \theta} \frac{\left[v_{x_i}(x) - v_{x_i}(t_i r_i + x_{3-i} r_{3-i})\right]^2}{|x_i - t_i|^{1+2(r-1)}}$$
$$+ \sum_{i=1}^2 h^3 \sum_{\substack{x_{3-i}, t_{3-i} \in \bar{\theta} \\ x_{3-i} \ne t_{3-i}}} \sum_{x_i \in \theta^-} \frac{\left[v_{x_i}(x) - v_{x_i}(x_i r_i + t_{3-i} r_{3-i})\right]^2}{|x_{3-i} - t_{3-i}|^{1+2(r-1)}}, \qquad 1 < r < 2,$$

$$\|v\|^2_{W_2^r(\omega)} = \|v\|^2_{W_2^{[r]^-}(\omega)} + |v|^2_{W_2^r(\omega)}, \qquad 0 < r < 2.$$

The multiplicative inequalities

$$(12)\qquad \|v\|_{W_2^r(\omega)} \le C(r)\,\|v\|_{L_2(\omega)}^{1-r}\,\|v\|_{W_2^1(\omega)}^{r}, \qquad 0<r<1,$$

$$(13)\qquad \|v\|_{W_2^r(\omega)} \le C(r)\,\|v\|_{W_2^1(\omega)}^{2-r}\,\|v\|_{W_2^2(\omega)}^{r-1}, \qquad 1<r<2,$$

are direct consequences of (9) and (11).

Finally, define the following inner products and norms

$$[v,\,w]_\omega = h^2\sum_{x\in\omega} v(x)\,w(x) + \frac{h^2}{2}\sum_{x\in\gamma\setminus\gamma_*} v(x)\,w(x) + \frac{h^2}{4}\sum_{x\in\gamma_*} v(x)\,w(x),$$

$$[v,\,w]_{i,\omega} = h^2\sum_{x\in\omega_i} v(x)\,w(x) + \frac{h^2}{2}\sum_{x\in\gamma\setminus(\gamma_{i0}\cup\overline{\gamma}_{i1})} v(x)\,w(x), \qquad i=1,\,2,$$

$$|[v]|_\omega = |[v]|_{L_2(\omega)} = |[v]|_{W_2^0(\omega)} = [v,\,v]_\omega^{1/2},$$

$$|[v]|_i = |[v]|_{i,\omega} = [v,\,v]_{i,\omega}^{1/2},$$

$$[v]_{W_2^1(\omega)}^2 = |[v_{x_1}]|_1^2 + |[v_{x_2}]|_2^2,$$

$$[v]_{W_2^2(\omega)}^2 = |[v_{x_1\bar x_1}]|_\omega^2 + \|v_{x_1x_2}\|_{\omega_{00}}^2 + |[v_{x_2\bar x_2}]|_\omega^2,$$

$$|[v]|_{W_2^k(\omega)}^2 = |[v]|_{W_2^{k-1}(\omega)}^2 + [v]_{W_2^k(\omega)}^2, \qquad k=1,\,2.$$

If the function $v \in \overset{\circ}{H}(\omega)$ is oddly extended outside $\overline{\omega}$, then

$$|[v]|_{W_2^k(\omega)} = \|v\|_{W_2^k(\omega)}, \qquad k=0,\,1,\,2.$$

Analogously, we can define discrete Sobolev–like norms of higher order.

2. Difference Scheme for Second–Order Equations. Convergence in the W_2^1–norm.

As a model problem let us consider the Dirichlet boundary–value problem for the second–order linear elliptic equation with variable coefficients in the square $\Omega = (0,\,1)^2$

$$(1)\qquad -\sum_{i,j=1}^{2} D_i\,(a_{ij}\,D_j u) + a\,u = f \quad \text{in} \quad \Omega, \qquad u=0 \quad \text{on} \quad \Gamma = \partial\Omega.$$

Assume that the generalised solution of the problem (1) belongs to the Sobolev space $W_2^s(\Omega)$, $s>0$, and $f(x) \in W_2^{s-2}(\Omega)$. Consequently, the coefficients of the equation (1) must belong to the corresponding spaces of multipliers (see Paragraph 1.7)

$$a_{ij} \in M\big(W_2^{s-1}(\Omega)\big)\,, \qquad a \in M\big(W_2^s(\Omega) \to W_2^{s-2}(\Omega)\big)\,.$$

According to Lemmas 7.9–7.11, the sufficient conditions are

$$a_{ij} \in W_2^{|s-1|}(\Omega)\,, \qquad a \in W_2^{|s-1|-1}(\Omega)\,, \qquad \text{for} \quad |s-1| > 1\,,$$

$$a_{ij} \in W_p^{|s-1|+\delta}(\Omega)\,, \qquad a = a_0 + \sum_{i=1}^{2} D_i a_i\,,$$

$$a_0 \in L_{2+\varepsilon}(\Omega)\,, \qquad a_i \in W_p^{|s-1|+\delta}(\Omega)\,,$$

where, $\varepsilon > 0$,

$\delta > 0\,, \quad p \ge 2/|s-1| \quad$ for $\quad 0 < |s-1| \le 1\,,$ and

$\delta = 0\,, \quad p = \infty \quad$ for $\quad s = 1\,.$

The consecutive estimates do not depend on δ, so one can set $\delta = 0$.

Also assume that the following conditions hold

$$a_{ij} = a_{ji}\,,$$

$$\sum_{i,j=1}^{2} a_{ij}\, y_i\, y_j \ge c_0 \sum_{i=1}^{2} y_i^2\,, \qquad c_0 > 0\,, \qquad \forall x \in \Omega\,, \qquad \forall y \in \mathbf{R}^n\,,$$

$$a(x) \ge 0 \quad \text{in the sense of distributions, i.e.}$$

$$\langle a \cdot \varphi, \varphi \rangle_{\mathcal{D}' \times \mathcal{D}} \ge 0\,, \qquad \forall \varphi \in \mathcal{D}(\Omega)\,,$$

as well as the possible consistency conditions at the vertices of the domain Ω (see Paragraph 1.8).

We approximate the problem (1) on the mesh $\bar{\omega}$ with the following finite difference scheme

$$\mathcal{L}_h v = T_1^2 T_2^2 f \quad \text{in} \quad \omega\,, \qquad v = 0 \quad \text{on} \quad \gamma\,, \tag{2}$$

where,

$$\mathcal{L}_h v = -0.5 \sum_{i,j=1}^{2} \left[(a_{ij}\, v_{\bar{x}_j})_{x_i} + (a_{ij}\, v_{x_j})_{\bar{x}_i} \right] + (T_1^2 T_2^2 a)\, v\,,$$

and where T_i are the Steklov smoothing operators with the step h. Note that (2) is a standard symmetric difference scheme (Samarskiĭ [84]) with both the right-hand-side and the lowest-order coefficient being averaged. For $s \le 3$, $a(x)$ and $f(x)$ may not be continuous, and, consequently, the difference scheme with non-averaged data would not be well defined.

Let u be the solution of the boundary-value problem (1) and v be the solution of the difference scheme (2). For $s > 1$, $u(x)$ is a continuous function and the error $z = u - v$ is defined at the nodes of the mesh $\bar{\omega}$. It is easy to see that the conditions

$$\mathcal{L}_h z = \sum_{i,j=1}^{2} \eta_{ij,\bar{x}_i} + \eta \quad \text{in} \quad \omega\,, \qquad z = 0 \quad \text{on} \quad \gamma\,, \tag{3}$$

are satisfied, where,

$$\begin{aligned}\eta_{ij} &= T_i^+T_{3-i}^2(a_{ij}\,D_j u) - 0.5\,(a_{ij}\,u_{x_j} + a_{ij}^{+i}\,u_{\bar{x}_j}^{+i})\,, \quad \text{and}\\ \eta &= (T_1^2T_2^2 a)\,u - T_1^2T_2^2(a\,u)\,.\end{aligned}$$

Using the energy method (Samarskiĭ [84]), it is easy to prove the following result.

LEMMA 1. *The finite difference scheme* (3) *is stable in the sense of the a priori estimate*

$$\|z\|_{W_2^1(\omega)} \le C\Big(\sum_{i,j=1}^{2}\|\eta_{ij}\|_{\omega_i} + \|\eta\|_{\omega}\Big)\,. \tag{4}$$

The problem of deriving the estimate of convergence rate for the finite difference scheme (2) is now reduced to the estimation of the right–hand–side terms in (4). First, we represent η_{ij} in the following manner (see Jovanović, Ivanović & Süli [54])

$$\begin{aligned}\eta_{ij} &= \eta_{ij1} + \eta_{ij2} + \eta_{ij3} + \eta_{ij4}\,, \quad \text{where,}\\ \eta_{ij1} &= T_i^+T_{3-i}^2(a_{ij}\,D_j u) - (T_i^+T_{3-i}^2 a_{ij})\,(T_i^+T_{3-i}^2\,D_j u)\,,\\ \eta_{ij2} &= \big[T_i^+T_{3-i}^2 a_{ij} - 0.5\,(a_{ij} + a_{ij}^{+i})\big]\,(T_i^+T_{3-i}^2 D_j u)\,,\\ \eta_{ij3} &= 0.5\,(a_{ij} + a_{ij}^{+i})\,\big[T_i^+T_{3-i}^2 D_j u - 0.5\,(u_{x_j} + u_{\bar{x}_j}^{+i})\big]\,, \quad \text{and}\\ \eta_{ij4} &= -\,0.25\,(a_{ij} - a_{ij}^{+i})\,(u_{x_j} - u_{\bar{x}_j}^{+i})\,.\end{aligned}$$

For $1 < s \le 2$ we set $\eta = \eta_0 + \eta_1 + \eta_2$, where,

$$\begin{aligned}\eta_0 &= (T_1^2T_2^2 a_0)\,u - T_1^2T_2^2(a_0\,u)\,, \quad \text{and}\\ \eta_i &= (T_1^2T_2^2 D_i\,a_i)\,u - T_1^2T_2^2(u\,D_i a_i)\,, \qquad i = 1,\,2\,.\end{aligned}$$

For $2 < s \le 3$ we set $\eta = \eta_3 + \eta_4$, where,

$$\begin{aligned}\eta_3 &= (T_1^2T_2^2 a)\,(u - T_1^2T_2^2 u)\,, \quad \text{and}\\ \eta_4 &= (T_1^2T_2^2 a)\,(T_1^2T_2^2 u) - T_1^2T_2^2(a\,u)\,.\end{aligned}$$

Introduce now the elementary rectangles $e_0 = e_0(x) = \{y : |y_j - x_j| < h\,,\ j = 1,\,2\}$ and $e_i = e_i(x) = \{y : x_i < y_i < x_i + h\,,\ |y_{3-i} - x_{3-i}| < h\}$, $i = 1, 2$. The linear transformation $y = x + h\,x^*$ maps the rectangles e_0, e_i onto standard rectangles $E_0 = \{x^* : |x_j^*| < 1\,,\ j = 1,\,2\}$ and, respectively, $E_i = \{x^* : 0 < x_i^* < 1\,,\ |x_{3-i}^*| < 1\}$. Set $a_{ij}^*(x^*) = a_{ij}(x + h\,x^*)$, $u^*(x^*) = u(x + h\,x^*)$ etc.

The value η_{ij1} at the node $x \in \omega_i$ can be represented as

$$\eta_{ij1}(x) = \frac{1}{h}\left\{ \iint_{E_i} \left(1 - |x^*_{3-i}|\right) a^*_{ij}(x^*)\, \frac{\partial u^*}{\partial x^*_j}\, dx^* \right.$$
$$\left. - \iint_{E_i} \left(1 - |x^*_{3-i}|\right) a^*_{ij}(x^*)\, dx^* \cdot \iint_{E_i} \left(1 - |x^*_{3-i}|\right) \frac{\partial u^*}{\partial x^*_j}\, dx^* \right\}.$$

One can readily conclude that $\eta_{ij1}(x)$ is a bounded bilinear functional of $(a^*_{ij},\, u^*) \in W^\lambda_q(E_i) \times W^\mu_{2q/(q-2)}(E_i)$, where $\lambda \geq 0$, $\mu \geq 1$ and $q > 2$. Moreover, $\eta_{ij1} = 0$, if a^*_{ij} is a constant, or u^* is a first-degree polynomial. Using Lemma 1.6.4 one obtains

$$|\eta_{ij1}(x)| \leq \frac{C}{h}\, |a^*_{ij}|_{W^\lambda_q(E_i)}\, |u^*|_{W^\mu_{2q/(q-2)}(E_i)}, \qquad 0 \leq \lambda \leq 1, \qquad 1 \leq \mu \leq 2.$$

Switching back to the original variables,

$$|a^*_{ij}|_{W^\lambda_q(E_i)} = h^{\lambda - 2/q}\, |a_{ij}|_{W^\lambda_q(e_i)}, \qquad \text{and}$$
$$|u^*|_{W^\mu_{2q/(q-2)}(E_i)} = h^{\mu-(q-2)/q}\, |u|_{W^\mu_{2q/(q-2)}(e_i)}.$$

Consequently,

$$|\eta_{ij1}(x)| \leq C\, h^{\lambda+\mu-1}\, |a_{ij}|_{W^\lambda_q(e_i)}\, |u|_{W^\mu_{2q/(q-2)}(e_i)}, \qquad 0 \leq \lambda \leq 1, \qquad 1 \leq \mu \leq 2.$$

Summating over the nodes of the mesh ω_i, and using Hölder's inequality, one obtains

(5) $$\|\eta_{ij1}\|_{\omega_i} \leq C\, h^{\lambda+\mu-1}\, |a_{ij}|_{W^\lambda_q(\Omega)}\, |u|_{W^\mu_{2q/(q-2)}(\Omega)}, \quad 0 \leq \lambda \leq 1, \quad 1 \leq \mu \leq 2.$$

Set $\lambda = s-1$, $\mu = 1$ and $q = p$. From the imbedding Theorem 1.3.4, $W^s_2 \subseteq W^1_{2p/(p-2)}$ for $1 < s \leq 2$. Therefore, from (5),

(6) $$\|\eta_{ij1}\|_{\omega_i} \leq C\, h^{s-1}\, \|a_{ij}\|_{W^{s-1}_p(\Omega)}\, \|u\|_{W^s_2(\Omega)}, \qquad 1 < s \leq 2.$$

Similar estimates hold for η_{ij2}, η_{ij4}, η_1 and η_2.

Let now $q > 2$ be a constant. The following imbeddings are satisfied

$$W^{\lambda+\mu-1}_2 \subseteq W^\lambda_q \quad \text{for} \quad \mu > 2 - 2/q \qquad \text{and} \qquad W^{\lambda+\mu}_2 \subseteq W^\mu_{2q/(q-2)} \quad \text{for} \quad \lambda > 2/q.$$

Setting $\lambda + \mu = s$ one obtains from (5),

(7) $$\|\eta_{ij1}\|_{\omega_i} \leq C\, h^{s-1}\, \|a_{ij}\|_{W^{s-1}_2(\Omega)}\, \|u\|_{W^s_2(\Omega)}, \qquad 2 < s \leq 3.$$

In the same manner one can estimates η_{ij4}.

For $s > 2$, $\eta_{ij2}(x)$ is a bounded bilinear functional of $(a_{ij},\, u) \in W^{s-1}_2(e_i) \times W^1_\infty(e_i)$ which vanishes if either a_{ij} is a first-degree polynomial or if u is a constant.

Using Lemma 1.6.4 and the imbedding $W_2^s \subseteq W_\infty^1$, one obtains for η_{ij2} an estimate of the form (7).

Similarly, $\eta_{ij3}(x)$ is a bounded bilinear functional of $(a_{ij},\, u) \in C(\bar{e}_i) \times W_2^s(e_i)$, $s > 1$, which vanishes if u is a second-degree polynomial. In the same manner, using imbeddings $W_p^{s-1} \subseteq C$ (for $1 < s \leq 2$) and $W_2^{s-1} \subseteq C$ (for $s > 2$), one obtains again the estimates of the forms (6) and (7), for η_{ij3}.

Let $2 < q < 2/(3-s)$. For $2 < s \leq 3$, $\eta_3(x)$ is a bounded bilinear functional of $(a,\, u) \in L_q(e_0) \times W_{2q/(q-2)}^{s-1}(e_0)$. Moreover, $\eta_3 = 0$ if u is a first-degree polynomial. Using the Bramble–Hilbert lemma 1.6.1 and the imbeddings $W_2^{s-2} \subseteq L_q$ and $W_2^s \subseteq W_{2q/(q-2)}^{s-1}$ one obtains the estimate

$$\|\eta_3\|_\omega \leq C\, h^{s-1}\, \|a\|_{W_2^{s-2}(\Omega)}\, \|u\|_{W_2^s(\Omega)}, \qquad 2 < s \leq 3. \tag{8}$$

For $2 < s \leq 3$, $\eta_4(x)$ is a bounded bilinear functional of $(a,\, u) \in W_2^{s-2}(e_0) \times W_\infty^1(e_0)$ which vanishes if either a or u are constant. Using the same formality and the imbedding $W_2^s \subseteq W_\infty^1$, one obtains for η_4 an estimate of the form (8).

Finally, set $2 < q < \min\{2+\varepsilon,\, 2/(2-s)\}$. For $1 < s \leq 2$, $\eta_0(x)$ is a bounded bilinear functional of $(a_0,\, u) \in L_q(e_0) \times W_{2q/(q-2)}^{s-1}(e_0)$, which vanishes if u is a constant. Using imbeddings $L_{2+\varepsilon} \subseteq L_q$ and $W_2^s \subseteq W_{2q/(q-2)}^{s-1}$, one obtains the following estimate

$$\|\eta_0\|_\omega \leq C\, h^{s-1}\, \|a_0\|_{L_{2+\varepsilon}(\Omega)}\, \|u\|_{W_2^s(\Omega)}, \qquad 1 < s \leq 2. \tag{9}$$

Combining (4) with (6)–(9) we obtain the following result:

THEOREM 1. *The finite difference scheme* (2) *converges in the norm* $W_2^1(\omega)$ *and the following estimates*

$$\|u - v\|_{W_2^1(\omega)} \leq C\, h^{s-1} \big(\max_{i,j} \|a_{ij}\|_{W_2^{s-1}(\Omega)} + \|a\|_{W_2^{s-2}(\Omega)}\big)\, \|u\|_{W_2^s(\Omega)} \quad \textit{for} \;\; 2 < s \leq 3, \tag{10}$$

and

$$\|u - v\|_{W_2^1(\omega)} \leq C\, h^{s-1} \big(\max_{i,j} \|a_{ij}\|_{W_p^{s-1}(\Omega)} + \max_i \|a_i\|_{W_p^{s-1}(\Omega)} + \|a_0\|_{L_{2+\varepsilon}(\Omega)}\big)\, \|u\|_{W_2^s(\Omega)}, \quad \textit{for} \;\; 1 < s \leq 2 \tag{11}$$

hold.

The obtained estimates of the convergence rate are consistent with the smoothness of data.

3. Convergence in other Discrete Norms

From estimate of the convergence rate (2.10) of the difference scheme (2.2), obtained in the previous paragraph, and the self-evident inequality

(1) $$|z|_{W_2^2(\omega)} \le \frac{\sqrt{6}}{h}\, |z|_{W_2^1(\omega)}$$

one immediatly obtains the estimate

(2) $$\|u - v\|_{W_2^2(\omega)} \le C\, h^{s-2} \left(\max_{i,j} \|a_{ij}\|_{W_2^{s-1}(\Omega)} + \|a\|_{W_2^{s-2}(\Omega)}\right) \|u\|_{W_2^s(\Omega)},$$

for $2 < s \le 3$.

In order to derive the analogous estimate for $3 < s \le 4$ one needs the difference analogue to the so called "second fundamental inequality" (Ladyzhenskaya & Ural'tseva [60], Ladyzhenskaya [59], D'yakonov [17])

(3) $$|z|_{W_2^2(\omega)} \le C\, \|\mathcal{L}_h z\|_{L_2(\omega)}.$$

Here,

$$C = C(a_{11},\, a_{12},\, a_{22},\, a) = C_0 \left(1 + \|T_1^2 T_2^2 a\|_{L_q(\omega)}\right) \left(1 + \max_{i,j} \|a_{ij}\|^{q/(q-2)}_{W_q^1(\omega)}\right)$$

where $2 < q \le \infty$; $\|\cdot\|_{L_q(\omega)}$ and $\|\cdot\|_{W_q^1(\omega)}$ are the discrete analogues of the corresponding Sobolev norms

$$\|v\|^q_{L_q(\varpi)} = h^2 \sum_{x\in\varpi} |v(x)|^q, \qquad \varpi \subset \mathbf{R}_h^2, \qquad \text{and}$$

$$\|v\|^q_{W_q^1(\omega)} = \|v\|^q_{L_q(\omega)} + \sum_{i=1}^{2} \|v_{x_i}\|^q_{L_q(\omega_i)}.$$

(As usual, the case $q = \infty$ is obtained by taking the proper limit). Using the Bramble–Hilbert lemma 1.6.3 one can easilly show that

$$\|a_{ij}\|_{W_q^1(\omega)} \le C_1\, \|a_{ij}\|_{W_q^1(\Omega)}, \quad \text{and}$$
$$\|T_1^2 T_2^2 a\|_{L_q(\omega)} \le C_2\, \|a\|_{L_q(\Omega)}.$$

So, one can set in (3),

$$C = C(a_{11},\, a_{12},\, a_{22},\, a) = C_3 \left(1 + \|a\|_{L_q(\Omega)}\right) \left(1 + \max_{i,j} \|a_{ij}\|^{q/(q-2)}_{W_q^1(\Omega)}\right), \qquad 2 < q \le \infty.$$

It is worth noting that in the terminology used by Ladyzhenskaya "the first fundamental inequality" (in the discrete case) corresponds to the following inequality

(4) $$c_0\, |z|^2_{W_2^1(\omega)} \le (\mathcal{L}_h z,\, z)_\omega$$

which can easily be derived by using partial summation.

One can now derive the desired convergence rate estimate in the norm $W_2^2(\omega)$ for the finite difference scheme (2.2). From (2.3) and (3) it follows that

$$\|z\|_{W_2^2(\omega)} \le C\Big(\sum_{i,j=1}^{2} \|\eta_{ij,x_i}\|_\omega + \|\eta\|_\omega\Big). \tag{5}$$

Estimating η_{ij,x_i} and η by the method described in the previous paragraph, one obtains again the estimate (2), for $2 < s \le 4$ (see also Berikelashvili [5]).

The convergence rate estimate in the norm $L_2(\omega)$ is also based on "the second fundamental inequality" (2). For the sake of simplicity, let us consider an equation with $a(x) = 0$

$$-\sum_{i,j=1}^{2} D_i\,(a_{ij}\,D_j u) = f \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \Gamma = \partial\Omega, \tag{6}$$

and the corresponding difference scheme

$$\begin{aligned} \mathcal{L}_h v \equiv -0.5 \sum_{i,j=1}^{2} \big[(a_{ij}\,v_{\bar{x}_j})_{x_i} + (a_{ij}\,v_{x_j})_{\bar{x}_i}\big] &= T_1^2 T_2^2 f \quad \text{in} \quad \omega, \\ v = 0 \quad \text{on} \quad \gamma. & \end{aligned} \tag{7}$$

The error $z = u - v$ satisfies the conditions

$$\mathcal{L}_h z = \sum_{i,j=1}^{2} \eta_{ij,x_i} \quad \text{in} \quad \omega, \qquad z = 0 \quad \text{on} \quad \gamma. \tag{8}$$

The right–hand–side can be rewritten in the form

$$\sum_{i,j=1}^{2} \eta_{ij,x_i} = \sum_{i=1}^{2} \Big(\tilde{\mathcal{L}}_{ii}\,\xi_{ii} + \mathcal{K}_i\,\chi_i + \sum_{j=1}^{2} \upsilon_{ij,x_i}\Big), \tag{9}$$

where,

$$\begin{aligned} \tilde{\mathcal{L}}_{ii}\,v &= -\big[(T_i^+ T_{3-i}^2\,a_{ii})\,v_{x_i}\big]_{\bar{x}_i}, \\ \mathcal{K}_i\,v &= \big[(T_i^+ T_{3-i}^2\,a_{i,3-i})\,v_{x_{3-i}}\big]_{\bar{x}_i}, \\ \chi_i &= \varrho_i - 0.5\,\big(\xi_{i,3-i} + \xi_{i,3-i}^{+i,-(3-i)}\big), \\ \xi_{ij} &= u - 0.5\,\big(T_{3-i}^- T_{3-j}^+ u + T_{3-i}^+ T_{3-j}^- u\big), \\ \varrho_i &= 0.25\,\big[(T_{3-i}^- T_i^+ u - T_{3-i}^+ T_i^- u) - (T_{3-i}^- T_i^+ u - T_{3-i}^+ T_i^- u)^{+i,-(3-i)}\big], \\ \upsilon_{ij} &= T_i^+ T_{3-i}^2 (a_{ij}\,D_j u) - (T_i^+ T_{3-i}^2\,a_{ij})\,(T_i^+ T_{3-i}^2 D_j u) \\ &\quad + 0.5\,\big[(T_i^+ T_{3-i}^2\,a_{ij})\,(u_{x_j} + u_{\bar{x}_j}^{+i}) - a_{ij}\,u_{x_j} - a_{ij}^{+i}\,u_{\bar{x}_j}^{+i}\big]. \end{aligned}$$

One can assume that the solution $u(x)$ is extended, preserving the class, in the domain $(-h_0,\, 1+h_0)^2$, $h_0 = const > 0$, $h < h_0$.

LEMMA 1. *If* $a_{ij} \in W_q^1(\Omega)$, $q > 2$, *then the finite difference scheme* (8) *satisfies the a priori estimate*

$$(10) \qquad \|z\|_\omega \le C \sum_{i=1}^{2} \Big(\|\xi_{ii}\|_\omega + \|\xi_{i,3-i}\|_{\overline{\omega}} + \|\varrho_i\|_{\omega_{i-1,2-i}} + \sum_{j=1}^{2} \|v_{ij}\|_{\omega_i} \Big) .$$

Proof. Introduce an auxiliary function w satisfying the conditions

$$\mathcal{L}_h w = z \quad \text{in} \quad \omega\,, \qquad w = 0 \quad \text{on} \quad \gamma\,.$$

From (8) and (9) follows that

$$\|z\|_\omega^2 = (z,\, \mathcal{L}_h w)_\omega = (\mathcal{L}_h z,\, w)_\omega$$
$$= \sum_{i=1}^{2} \Big[(\tilde{\mathcal{L}}_{ii}\, \xi_{ii},\, w)_\omega + (\mathcal{K}_i\, \chi_i,\, w)_\omega + \sum_{j=1}^{2} (v_{ij,x_i},\, w)_\omega \Big]$$
$$= \sum_{i=1}^{2} \Big[(\xi_{ii},\, \tilde{\mathcal{L}}_{ii}\, w)_\omega + (\chi_i,\, \mathcal{K}_i^* w)_{\omega_{i-1,2-i}} - \sum_{j=1}^{2} (v_{ij},\, w_{x_i})_{\omega_i} \Big]$$
$$\le \sum_{i=1}^{2} \Big(\|\xi_{ii}\|_\omega \, \|\tilde{\mathcal{L}}_{ii}\, w\|_\omega + \|\chi_i\|_{\omega_{i-1,2-i}} \, \|\mathcal{K}_i^* w\|_{\omega_{i-1,2-i}} + \sum_{j=1}^{2} \|v_{ij}\|_{\omega_i} \, \|w_{x_i}\|_{\omega_i} \Big) .$$

From "the second fundamental inequality" (3) it follows that

$$\|\tilde{\mathcal{L}}_{ii}\, w\|_\omega\,, \quad \|\mathcal{K}_i^* w\|_{\omega_{i-1,2-i}}\,, \quad \|w_{x_i}\|_{\omega_i} \;\le\; C\, \|\mathcal{L}_h w\|_\omega\,.$$

Hence, one immediatly obtains the inequality (10). ∎

The "second fundamental inequality" is valid for $a_{ij} \in W_q^1(\Omega)$, $q > 2$, so a "good" convergence rate estimate can be expected only for $s = 2$. In this case, $\xi_{ij}(x)$ and $\varrho_i(x)$ are bounded linear functionals on W_2^2, vanishing on the first-degree polynomials. Using the Bramble–Hilbert lemma 1.6.3 one obtains

$$(11) \qquad \|\xi_{ii}\|_\omega\,, \quad \|\xi_{i,3-i}\|_{\overline{\omega}}\,, \quad \|\varrho_i\|_{\omega_{i-1,2-i}} \;\le\; C\, h^2\, \|u\|_{W_2^2(\Omega)}\,.$$

The term v_{ij}, as in the previous case, can be split into three terms, and estimated using bilinear version of the Bramble–Hilberta lemma 1.6.4

$$(12) \qquad \|v_{ij}\|_{\omega_i} \le C\, h^2 \left(\|a_{ij}\|_{W_\infty^1(\Omega)}\, \|u\|_{W_2^2(\Omega)} + \|a_{ij}\|_{W_\infty^2(\Omega)}\, \|u\|_{W_2^1(\Omega)} \right) .$$

From (10 – 12) one obtains the following convergence rate estimate for the finite difference scheme (7)

$$(13) \qquad \|u - v\|_{L_2(\omega)} \le C\, h^2 \max_{i,j} \|a_{ij}\|_{W_\infty^2(\Omega)}\, \|u\|_{W_2^2(\Omega)}\,.$$

The estimate (13) is not consistent with the smoothness of data since it requires that the coefficients are twice differentiable, rather than only once. This is a consequence of the rough estimate of the term $v_{ij,\bar{x}_i}$ in (10). A better estimate can be obtained for the scheme with averaged coefficients

$$\tilde{\mathcal{L}}_h v = \sum_{i,j=1}^{2} \tilde{\mathcal{L}}_{ij} v = T_1^2 T_2^2 f \quad \text{in} \quad \omega\,, \qquad v = 0 \quad \text{on} \quad \gamma \tag{14}$$

where,

$$\tilde{\mathcal{L}}_{ij} v = -0.5\left[(T_i^+ T_{3-i}^2\, a_{ij})\,(v + v^{+i,-j})_{x_j}\right]_{\bar{x}_i}.$$

In this case, the error $z = u - v$ satisfies the conditions

$$\tilde{\mathcal{L}}_h z = \sum_{i=1}^{2}\left(\tilde{\mathcal{L}}_{ii}\,\xi_{ii} + \mathcal{K}_i\,\chi_i + \sum_{j=1}^{2} \eta_{ij1,\bar{x}_i}\right) \quad \text{in} \quad \omega\,, \qquad z = 0 \quad \text{on} \quad \gamma\,,$$

where ξ_{ii}, χ_i and η_{ij1} are defined above.

For $a_{ij} \in W_p^1(\Omega)$, $p > 2$,

$$\|z\|_\omega \le C \sum_{i=1}^{2}\left(\|\xi_{ii}\|_\omega + \|\xi_{i,3-i}\|_{\bar{\omega}} + \|\varrho_i\|_{\omega_{i-1,2-i}} + \sum_{j=1}^{2} \|\eta_{ij1}\|_{\omega_i}\right).$$

Using the previously derived estimates (11) and (2.5), one obtains

$$\|u - v\|_{L_2(\omega)} \le C\, h^2 \max_{i,j} \|a_{ij}\|_{W_\infty^1(\Omega)}\, \|u\|_{W_2^2(\Omega)}\,. \tag{15}$$

The estimate (15) is "almost consistent" with the data smoothness: here $a_{ij} \in W_\infty^1(\Omega)$ instead of $W_p^1(\Omega)$. If one allows the inconsistency between the smoothness of the solution and the coefficients, assuming that instead (see Paragraph 2),

$$u \in W_2^s(\Omega)\,, \qquad a_{ij} \in W_p^{|s-1|+\delta}(\Omega)\,,$$

the following conditions hold,

$$u \in W_2^s(\Omega)\,, \quad 1 < s \le 2\,; \qquad a_{ij} \in W_\infty^1(\Omega)\,,$$

then, instead of (15) one obtains

$$\|u - v\|_{L_2(\omega)} \le C\, h^s \max_{i,j} \|a_{ij}\|_{W_\infty^1(\Omega)}\, \|u\|_{W_2^s(\Omega)}\,, \qquad 1 < s \le 2\,.$$

This estimate is not consistent with the smoothness of data, except for $s = 2$, when it reduces to (15).

From the derived convergence rate estimates and the multiplicative inequalities (1.12) and (1.13) one can easily obtains the new estimates in the fractional order

Sobolev norms. For example, for the finite difference scheme (2.2) from (2.10), (2) and (1.13) one obtain

$$\|u-v\|_{W_2^r(\omega)} \leq C\,h^{s-r}\,\big(\max_{i,j}\|a_{ij}\|_{W_2^{s-1}(\Omega)} + \|a\|_{W_2^{s-2}(\Omega)}\big)\,\|u\|_{W_2^s(\Omega)}\,,$$
$$\text{for}\quad 1 \leq r \leq 2 < s \leq 3\,.$$

From (2.11), (1) and (1.13),

$$\|u-v\|_{W_2^r(\omega)} \leq C\,h^{s-r}\,\big(\max_{i,j}\|a_{ij}\|_{W_p^{s-1}(\Omega)} + \max_{i}\|a_i\|_{W_p^{s-1}(\Omega)}$$
$$+\|a_0\|_{L_{2+\epsilon}(\Omega)}\big)\,\|u\|_{W_2^s(\Omega)}\,,\quad \text{for}\quad 1 \leq r < s \leq 2\,.$$

Similarly, from (15), (1.12), (1.13), the self–evident inequality

$$|z|_{W_2^1(\omega)} \leq \frac{2\sqrt{2}}{h}\,\|z\|_{L_2(\omega)}$$

and (1) one obtains the following convergence rate estimate for the difference scheme (14)

$$\|u-v\|_{W_2^r(\omega)} \leq C\,h^{2-r}\,\max_{i,j}\|a_{ij}\|_{W_\infty^1(\Omega)}\,\|u\|_{W_2^2(\Omega)}\,,\qquad 0 \leq r \leq 2\,.$$

4. Convergence in $L_2(\omega)$: The Case of Separated Variables

In the previous paragraph we have seen that the derivation of the convergence rate estimates in $L_2(\omega)$–norm is met with significant difficulties. A satisfactory estimate is obtained only for $s=2$, while for $s<2$ only the estimates inconsistent with the data smoothness are obtained.

A more precise result can be obtained for the equation with separated variables

(1) $$-\sum_{i=1}^{2} D_i\,(a_i\,D_i u) = f \quad \text{in} \quad \Omega\,, \qquad u=0 \quad \text{on} \quad \Gamma = \partial\Omega\,,$$

where,

$$a_i = a_i(x_i)\,, \qquad i = 1,\,2$$

and

$$0 < c_0 \leq a_i \leq c_1\,, \qquad x_i \in (0,\,1)\,, \qquad i = 1,\,2\,, \qquad c_0,\,c_1 = \text{const}\,.$$

The following conditions ensure that a_i belong to the space of multipliers $M(W_2^{s-1}(\Omega))$ (see Paragraph 1.7)

$$a_i \in W_p^{|s-1|+\delta}(0,\,1)$$

where,

$$\begin{array}{lll} p = 2\,, & \delta = 0 \quad - & \text{for} \quad |s-1| > 0.5\,, \\ p = p(s) \geq 1/|s-1|\,, & \delta > 0 \quad - & \text{for} \quad 0 < |s-1| \leq 0.5\,, \quad \text{and} \\ p = \infty\,, & \delta = 0 \quad - & \text{for} \quad s = 1\,. \end{array}$$

Similarly to the results of Paragraph 2, the estimates derived below do not depend on δ in any way so one can set $\delta = 0$.

Introduce now the solving operators of one-dimensional exact difference schemes (see Lazarov, Makarov & Samarskiĭ [67])

$$S_i f(x) = \frac{1}{h} \int_{x_i - h}^{x_i + h} \kappa_i(t)\, f(x + (t - x_i)\, r_i)\, dt\,, \qquad i = 1, 2$$

where,

$$\kappa_i(t) = \begin{cases} \displaystyle\int_{x_i-h}^{t} \frac{d\tau}{a_i(\tau)} \Big/ \int_{x_i-h}^{x_i} \frac{d\tau}{a_i(\tau)}\,, & t \in (x_i - h,\, x_i) \\ \displaystyle\int_{t}^{x_i+h} \frac{d\tau}{a_i(\tau)} \Big/ \int_{x_i}^{x_i+h} \frac{d\tau}{a_i(\tau)}\,, & t \in (x_i,\, x_i + h) \end{cases}.$$

These operators satisfy the following conditions

$$S_i\,(D_i\,(a_i\,D_i u)) = (\hat{a}_i\, u_{x_i})_{\bar{x}_i}\,,$$

where,

$$\hat{a}_i(x_i) = \left(\frac{1}{h}\int_{x_i}^{x_i+h} \frac{d\tau}{a_i(\tau)}\right)^{-1}, \qquad i = 1, 2\,.$$

For $a_i(x_i) \equiv 1$,

$$S_i = T_i^2 = T_i^+ T_i^-\,.$$

We approximate the problem (1) by the following finite difference scheme

$$(2) \qquad -\sum_{i=1}^{2} b_{3-i}\,(\hat{a}_i\, v_{x_i})_{\bar{x}_i} = S_1 S_2 f \quad \text{in} \quad \omega\,, \qquad v = 0 \quad \text{on} \quad \gamma\,,$$

where $b_i = S_i(1)$.

Since the solution $u(x)$ of the problem (1) is not necessarily a continuous function, we define the error

$$z = \overline{u} - v\,, \qquad \text{where} \quad \overline{u} = \begin{cases} T_1 T_2 u\,, & 0 < s \leq 1 \\ u\,, & 1 < s \leq 2 \end{cases}.$$

The error thus defined satisfies the conditions

$$-\sum_{i=1}^{2} b_{3-i}\,(\hat{a}_i\, z_{x_i})_{\bar{x}_i} = \sum_{i=1}^{2} (\hat{a}_i\, \psi_{i,x_i})_{\bar{x}_i} \quad \text{in} \quad \omega\,, \qquad z = 0 \quad \text{on} \quad \gamma\,,$$

where $\psi_i = S_{3-i}(u) - b_{3-i}\,\overline{u}\,,\quad i = 1, 2\,.$

The following a priori estimate holds

$$\|z\|_\omega \le C\left(\|\psi_1\|_\omega + \|\psi_2\|_\omega\right). \tag{3}$$

The problem of deriving the convergence rate estimate for the finite difference scheme (2) is now reduced to the estimation of the right-hand-side terms in (3).

The value ψ_i at the node $x \in \omega$ is a bounded linear functional of $u \in W_2^s(e_0)\,,\ s > 0.5$. Moreover, $\psi_i = 0$ if $u(x)$ is a constant. Using the Bramble–Hilbert lemma one obtains

$$|\psi_i| \le C(h)\,|u|_{W_2^s(e_0)}\,, \qquad 0.5 < s \le 1\,,$$

where $C(h) = C\,h^{s-1}$. A summation over the mesh ω yields

$$\|\psi_i\|_\omega \le C\,h^s\,|u|_{W_2^s(\Omega)}\,, \qquad 0.5 < s \le 1\,. \tag{4}$$

The main difficulty in the derivation of estimates for $s > 1$ lies in the fact that ψ_{3-i} is a nonlinear functional of a_i. However, ψ_{3-i} may be conveniently decomposed so to allow a direct estimate of the nonlinear terms. Set

$$\begin{aligned}
\psi_{3-i} &= \psi_{3-i,1} + \psi_{3-i,2} + \psi_{3-i,3}\,, \quad \text{where}\\
\psi_{3-i,1} &= \int_0^1 \left[u(x + h\tau r_i) - 2u(x) + u(x - h\tau r_i)\right]\left(\int_{x_i-h}^{x_i-h\tau}\frac{d\sigma}{a_i(\sigma)}\right)\\
&\quad\times\left(\int_{x_i-h}^{x_i}\frac{d\sigma}{a_i(\sigma)}\right)^{-1} d\tau\,,\\
\psi_{3-i,2} &= \int_0^1 \left[u(x + h\tau r_i) - u(x)\right]\left(\int_{x_i}^{x_i+h\tau}\frac{d\sigma}{a_i(\sigma)}\int_{x_i-h}^{x_i}\frac{d\sigma}{a_i(\sigma)}\right)^{-1}\\
&\quad\times\left(\int_{x_i+h\tau}^{x_i+h}\frac{d\sigma}{a_i(\sigma)}\right)h^{-1}\int_{x_i-h}^{x_i}\int_{x_i}^{x_i+h}\frac{a_i(t)-a_i(t')}{a_i(t)\,a_i(t')}\,dt\,dt'\,d\tau\,,\\
\psi_{3-i,3} &= \int_0^1 \left[u(x + h\tau r_i) - u(x)\right]\left(\int_{x_i-h}^{x_i}\frac{d\sigma}{a_i(\sigma)}\right)^{-1}\\
&\quad\times h^{-1}(1-\tau)^{-1}\int_{x_i-h}^{x_i-h\tau}\int_{x_i+h\tau}^{x_i+h}\frac{a_i(t)-a_i(t')}{a_i(t)\,a_i(t')}\,dt\,dt'\,d\tau\,.
\end{aligned}$$

The value $\psi_{3-i,1}$ at the node $x \in \omega$ is a bounded linear functional of $u \in W_2^s(e_0)\,,\ s > 1$, which vanishes on the first-degree polynomials. Using the Bramble–Hilbert lemma one obtains

$$\|\psi_{3-i,1}\|_\omega \le C\,h^s\,|u|_{W_2^s(\Omega)}\,, \qquad 1 < s \le 2\,. \tag{5}$$

For $1.5 < s \le 2\,,\ \psi_{3-i,2}$ is a bounded linear functional of $u \in W_2^s(e_0)$, i.e.

$$|\psi_{3-i,2}| \le C\,h^{\lambda-0.5}\left(h^{-1}\|u\|_{L_2(e_0)} + |u|_{W_2^1(e_0)} + h^{s-1}\,|u|_{W_2^s(e_0)}\right)|a_i|_{W_2^\lambda(i_0)}\,, \qquad \lambda > 0\,,$$

where $i_0 = i_0(x_i) = (x_i - h,\, x_i + h)$. Moreover, $\psi_{3-i,2} = 0$ if $u(x)$ is a constant, so one can eliminate the term $h^{-1}\|u\|_{L_2(e_0)}$ at the right–hand–side. Using a summation one obtains

$$\|\psi_{3-i,2}\|_\omega \le C\, h^{\lambda+0.5} \left(\max_{x_i} |u|_{W_2^1(\Omega_{hi})} + h^{s-1}\, |u|_{W_2^s(\Omega)} \right),$$

where $\Omega_{hi} = \Omega_{hi}(x) = \{ y \in \mathbf{R}^2 \,:\, x_i - h < y_i < x_i + h,\ 0 < y_{3-i} < 1 \}$. Setting $\lambda = s - 1$ and using the inequality

$$|u|_{W_2^1(\Omega_{hi})} \le C\, h^{0.5}\, \|u\|_{W_2^s(\Omega)}, \qquad s > 1.5,$$

which is a consequence of the Theorem 1.3.5, one obtains

$$\|\psi_{3-i,2}\|_\omega \le C\, h^s\, \|a_i\|_{W_2^{s-1}(0,1)}\, \|u\|_{W_2^s(\Omega)}, \qquad 1.5 < s \le 2. \tag{6}$$

Similarly,

$$\|\psi_{3-i,2}\|_\omega \le C\, h^s\, \|a_i\|_{W_p^{s-1}(0,1)}\, \|u\|_{W_2^s(\Omega)}, \qquad 1 < s \le 1.5. \tag{7}$$

The same kind of estimates holds for $\psi_{3-i,3}$.

Combining (3) and (4)–(7) one obtains the following result

THEOREM 1. *The finite difference scheme* (2) *converges and the estimate*

$$\|u - v\|_{L_2(\omega)} \le C\, h^s\, \max_i \|a_i\|_{W_p^{|s-1|}(0,1)}\, \|u\|_{W_2^s(\Omega)}, \qquad 0.5 < s \le 2 \tag{8}$$

holds.

The obtained convergence rate estimate is consistent with data smoothness.

REMARK. For $0 < s \le 0.5$ the function $S_1 S_2 f$ can be discontinuous, and, consequently, the difference scheme (2) is not applicable. In this case, instead of (2), one should use a scheme with a more strongly averaged right–hand–side. In the case of equation with constant coefficients, such schemes were studied by Jovanović [34] and Ivanović, Jovanović & Süli [30].

5. Fourth–Order Equation

In this paragraph we shall consider some boundary–value problems for the fourth–order symmetric elliptic equation with variable coefficients

$$(1) \qquad \mathcal{L}u \equiv D_1^2 M_1(u) + 2\, D_1\, D_2\, M_3(u) + D_2^2\, M_2(u) = f(x)\,, \qquad x \in \Omega$$

where the following notation is used

$$M_1(u) = a_1\, D_1^2 u + a_0\, D_2^2 u\,, \qquad M_2(u) = a_0\, D_1^2 u + a_2\, D_2^2 u\,,$$
$$M_3(u) = a_3\, D_1\, D_2 u\,, \qquad \Omega = (0,\, 1)^2\,.$$

Let the following conditions be satisfied

$$(2) \qquad \begin{array}{c} a_i \ge c_0 > 0\,, \quad i = 1,\, 2,\, 3\,, \quad a_1\, a_2 - a_0^2 \ge c_1 > 0\,, \quad x \in \Omega\,, \\ u \in W_2^s(\Omega)\,, \quad f \in W_2^{s-4}(\Omega)\,, \quad 2 < s \le 4\,. \end{array}$$

Thus the coefficients a_i must belong to the space of multipliers $M\big(W_2^{s-2}(\Omega)\big)$. The following conditions are sufficient (see Lemma 1.7.9)

$$(3) \qquad a_i \in W_p^{s-2+\varepsilon}(\Omega)\,, \qquad i = 0,\, 1,\, 2,\, 3\,,$$

where,

$$\begin{array}{llll} p = 2\,, & \varepsilon = 0 & - & \text{for } \; 3 < s \le 4\,, \\ p > 2\,, & \varepsilon = 0 & - & \text{for } \; s = 3\,, \\ p \ge 2/(s-2)\,, & \varepsilon > 0 \text{ — arbitrary} & - & \text{for } \; 2 < s < 3\,. \end{array}$$

First consider the problem with boundary conditions of the second kind

$$(4) \qquad u = 0 \quad \text{on} \quad \Gamma\,; \qquad D_i^2 u = 0 \quad \text{on} \quad \Gamma_{i0} \cup \Gamma_{i1}\,, \qquad i = 1,\, 2\,.$$

We approximate the problem (1),(4) by the following finite difference scheme

$$(5) \qquad \mathcal{L}_h v \equiv m_1(v)_{x_1\bar{x}_1} + 2\, m_3(v)_{\bar{x}_1\bar{x}_2} + m_2(v)_{x_2\bar{x}_2} = T_1^2 T_2^2 f\,, \qquad x \in \omega\,,$$
$$(6) \qquad v = 0\,, \quad x \in \gamma\,; \quad v_{x_i\bar{x}_i} = 0\,, \quad x \in \gamma_{i0} \cup \gamma_{i1}\,, \quad i = 1,\, 2\,,$$

where,

$$m_1(v) = a_1\, v_{x_1\bar{x}_1} + a_0\, v_{x_2\bar{x}_2}\,, \qquad m_2(v) = a_0\, v_{x_1\bar{x}_1} + a_2\, v_{x_2\bar{x}_2}\,,$$
$$m_3(v) = \tilde{a}_3\, v_{x_1 x_2} \quad \text{and} \quad \tilde{a}_3(x) = a_3(x_1 + 0.5\, h,\, x_2 + 0.5\, h)\,.$$

Note that the discrete solutions are defined at the external nodes, belonging to the domain $[-h,\, 1+h]^2$. Consequently, we assume that the solution u, and the coefficients a_i are extended, preserving the class, to the domain $(-h_0,\, 1+h_0)^2$, where $h_0 = \text{const} > 0$ and $h < h_0$.

The error $z = u - v$ satisfies the conditions

$$\mathcal{L}_h z = \varphi_{1,x_1\bar{x}_1} + 2\varphi_{3,\bar{x}_1\bar{x}_2} + \varphi_{2,x_2\bar{x}_2}\,, \qquad x \in \omega\,, \tag{7}$$

$$z = 0\,, \quad x \in \gamma; \quad z_{x_i\bar{x}_i} = u_{x_i\bar{x}_i}\,, \quad x \in \gamma_{i0} \cup \gamma_{i1}\,, \quad i = 1, 2\,, \tag{8}$$

where,

$$\varphi_i = m_i(u) - T_{3-i}^2\, M_i(u)\,, \qquad i = 1, 2; \qquad \varphi_3 = m_3(u) - T_1^+ T_2^+ M_3(u)\,.$$

Note that (4), (6) and (8) yields

$$m_i(z) = \varphi_i\,, \qquad x \in \gamma_{i0} \cup \gamma_{i1}\,, \qquad i = 1, 2\,.$$

Multiplying equation (7) by z, and using the partial summation and the Cauchy–Schwartz's inequality, one readily obtains the following a priori estimate

$$\|z\|^2_{W_2^2(\omega)} \le C\left(\|\varphi_1\|^2_\omega + \|\varphi_2\|^2_\omega + \|\varphi_3\|^2_{\omega_{00}}\right). \tag{9}$$

THEOREM 1. *If the solution and the coefficients of the boundary-value problem* (1), (4) *satisfy conditions* (2) *and* (3), *then the finite difference scheme* (5), (6) *converges and the estimate*

$$\|u - v\|_{W_2^2(\omega)} \le C\, h^{s-2} \max_i \|a_i\|_{W_p^{s-2+\epsilon}(\Omega)}\, \|u\|_{W_2^s(\Omega)}\,, \qquad 2.5 < s \le 4 \tag{10}$$

holds.

Proof. In order to obtain the convergence rate estimate for the difference scheme (5–6), it is sufficient to estimate the terms in the sum on the right–hand–side of the inequality (9). First, we represent φ_1 in the following manner

$$\begin{aligned}
\varphi_1 \quad &= \sum_{j=1}^{8} \varphi_{1,j}\,, \quad \text{where,}\\
\varphi_{i,k} \quad &= a_{2-k}\left(u_{x_k\bar{x}_k} - T_1^2 T_2^2 D_k^2 u\right),\\
\varphi_{i,k+2} &= \left(a_{2-k} - T_1^2 T_2^2 a_{2-k}\right)\left(T_1^2 T_2^2 D_k^2 u\right),\\
\varphi_{i,k+4} &= \left(T_1^2 T_2^2 a_{2-k}\right)\left(T_1^2 T_2^2 D_k^2 u\right) - T_1^2 T_2^2\left(a_{2-k}\, D_k^2 u\right),\\
\varphi_{i,k+6} &= T_1^2 T_2^2\left(a_{2-k}\, D_k^2 u\right) - T_2^2\left(a_{2-k}\, D_k^2 u\right), \qquad k = 1, 2\,.
\end{aligned}$$

Analogously we can represent φ_2. Also, set

$$\begin{aligned}
\varphi_3 \quad &= \varphi_{3,1} + \varphi_{3,2}\,, \quad \text{where}\\
\varphi_{3,1} \quad &= \left(\tilde{a}_3 - T_1^+ T_2^+ a_3\right) u_{x_1 x_2}\,, \quad \text{and}\\
\varphi_{3,2} \quad &= \left(T_1^+ T_2^+ a_3\right) u_{x_1 x_2} - T_1^+ T_2^+\left(a_3\, D_1 D_2 u\right).
\end{aligned}$$

For $s \geq 2$, the value $\varphi_{1,1}$ at the node $x \in \omega$, is a bounded linear functional of $u \in W_2^s(e_0)$,

$$|\varphi_{1,1}| \leq C(h) \|a_1\|_{C(\bar{\Omega})} \|u\|_{W_2^s(e_0)} .$$

Moreover, $\varphi_{1,1} = 0$ if u is a third-degree polynomial. Using Lemma 1.6.3 one obtains

$$|\varphi_{1,1}| \leq C h^{s-3} \|a_1\|_{C(\bar{\Omega})} |u|_{W_2^s(e_0)}, \qquad 2 \leq s \leq 4 .$$

Hence, using imbedding (see Theorem 1.3.3) $W_p^{s-2+\varepsilon} \subseteq C$, $s > 2$, and summing over the mesh ω,

$$\|\varphi_{1,1}\|_\omega \leq C h^{s-2} \|a_1\|_{W_p^{s-2+\varepsilon}(\Omega)} |u|_{W_2^s(\Omega)}, \qquad 2 \leq s \leq 4 . \tag{11}$$

We can estimate $\varphi_{1,2}$ in the same manner.

The value of $\varphi_{1,3}(x)$, $x \in \omega$, is a bounded bilinear functional of $(a_1, u) \in W_p^\lambda(e_0) \times W_q^2(e_0)$, where $\lambda \cdot p > 2$, $q = \infty$ for $p = 2$ and $q = 2p/(p-2)$ for $p > 2$. Moreover, $\varphi_{1,3} = 0$ if either a_1 or u is a first-degree polynomial. Using Lemma 1.6.4,

$$|\varphi_{1,3}| \leq C h^{\lambda-1} \|a_1\|_{W_p^\lambda(e_0)} |u|_{W_q^2(e_0)}, \qquad 2/p \leq \lambda \leq 2 ,$$

and, consequently,

$$\|\varphi_{1,3}\|_\omega \leq C h^\lambda \|a_1\|_{W_p^\lambda(\Omega)} \|u\|_{W_q^2(\Omega)} .$$

Setting $\lambda = s - 2 + \varepsilon$ and using the imbeddings

$$W_2^s \subseteq W_\infty^2, \quad \text{for} \quad s > 3 \qquad \text{and} \qquad W_2^s \subseteq W_{2p/(p-2)}^2, \quad \text{for} \quad 2 < s \leq 3 ,$$

one obtains

$$\|\varphi_{1,3}\|_\omega \leq C h^{s-2} \|a_1\|_{W_p^{s-2+\varepsilon}(\Omega)} \|u\|_{W_2^s(\Omega)}, \qquad 2 < s \leq 4 . \tag{12}$$

In the same manner we can estimate $\varphi_{1,4}$ and $\varphi_{3,1}$.

For $\lambda \geq 0$, $\mu \geq 2$ and $q > 2$ the value of $\varphi_{1,5}(x)$, $x \in \omega$, is a bounded bilinear functional of $(a_1, u) \in W_q^\lambda(e_0) \times W_{2q/(q-2)}^\mu(e_0)$. Moreover, $\varphi_{1,5} = 0$ if either a_1 is a constant or if u is a second-degree polynomial. Using Lemma 1.6.4, one obtains

$$\|\varphi_{1,5}\|_\omega \leq C h^{\lambda+\mu-2} \|a_1\|_{W_q^\lambda(\Omega)} \|u\|_{W_{2q/(q-2)}^\mu(\Omega)},$$

where $0 \leq \lambda \leq 1$ and $2 \leq \mu \leq 3$. Set $\lambda + \mu = s$. If $\lambda + \mu > 3$ one can find a $q = q(\lambda, \mu)$ such that $\lambda \geq 2/q \geq 3 - \mu$. Then,

$$W_p^{s-2+\varepsilon} = W_2^{\lambda+\mu-2+\varepsilon} \subseteq W_q^\lambda \qquad \text{and} \qquad W_2^s = W_2^{\lambda+\mu} \subseteq W_{2q/(q-2)}^\mu .$$

Analogously, if $2 < \lambda + \mu \leq 3$ one can find a q such that $\lambda \geq 2/q \geq 2/p - (\mu - 2)$. In that case,

$$W_p^{s-2+\varepsilon} = W_p^{\lambda+\mu-2+\varepsilon} \subseteq W_q^\lambda \qquad \text{and} \qquad W_2^s = W_2^{\lambda+\mu} \subseteq W_{2q/(q-2)}^\mu .$$

From theese imbeddings it follows that

(13) $$\|\varphi_{1,5}\|_\omega \le C\, h^{s-2}\, \|a_1\|_{W_p^{s-2+\epsilon}(\Omega)}\, \|u\|_{W_2^s(\Omega)}\,, \qquad 2 < s \le 4\,.$$

In the same manner we can estimate $\varphi_{1,6}$ and $\varphi_{3,2}$.

For $\lambda > 0.5$, the value $\varphi_{1,7}(x)$ at the node $x \in \omega$ is a bounded linear functional of $a_1\, D_1^2 u \in W_2^\lambda(e_0)$, vanishing on the first-degree polynomials. Using Lemma 1.6.3 one obtains

$$\|\varphi_{1,7}\|_\omega \le C\, h^\lambda\, |a_1\, D_1^2 u|_{W_2^\lambda(\Omega)}\,, \qquad 0.5 < \lambda \le 2\,.$$

From the inequality

$$|a_1\, D_1^2 u|_{W_2^\lambda(\Omega)} \le C\, \|a_1\|_{W_p^{\lambda+\epsilon}(\Omega)}\, \|D_1^2 u\|_{W_2^\lambda(\Omega)}\,,$$

setting $\lambda = s - 2$, it follows that

(14) $$\|\varphi_{1,7}\|_\omega \le C\, h^{s-2}\, \|a_1\|_{W_p^{s-2+\epsilon}(\Omega)}\, \|u\|_{W_2^s(\Omega)}\,, \qquad 2.5 < s \le 4\,.$$

In the same manner one can estimate $\varphi_{1,8}$.

Finally, from (11–14) and (9) one can obtain the desired estimate (10). ∎

Remark 1. Similarly to the case studied in the preceeding paragraph, the finite difference scheme (5–6) is not defined for $2 < s \le 2.5$, since in that case the right-hand-side $T_1^2 T_2^2 f$ is not continuous. Consequently, a scheme with a more strongly averaged right-hand-side must be used (see Ivanović, Jovanović & Süli [31], for the case of equation with constant coefficients).

Consider now the problem with Dirichlet boundary conditions

(15) $$u = 0 \quad \text{on} \quad \Gamma; \qquad D_i u = 0 \quad \text{on} \quad \Gamma_{i0} \cup \Gamma_{i1}\,, \qquad i = 1, 2\,.$$

Same as before, we approximate the equation (1) by (5), while the boundary conditions (15) are approximated by

(16) $$v = 0\,, \quad x \in \gamma; \qquad v_{\overset{\circ}{x}_i} = 0\,, \quad x \in \gamma_{i0} \cup \gamma_{i1}\,, \quad i = 1, 2\,.$$

The error $z = u - v$ satisfies the equation (7) and the boundary conditions

(17) $$z = 0\,, \quad x \in \gamma; \qquad z_{\overset{\circ}{x}_i} = u_{\overset{\circ}{x}_i}\,, \quad x \in \gamma_{i0} \cup \gamma_{i1}\,, \quad i = 1, 2\,.$$

Using the notation

$$\zeta_i = \big(u_{\overset{\circ}{x}_i} - D_i u\big)/h\,,$$

the second boundary condition in (17) can be written as

$$z_{\overset{\circ}{x}_i} = h\,\zeta_i\,, \qquad x \in \gamma_{i0} \cup \gamma_{i1}\,, \qquad i = 1, 2\,.$$

The a priori estimate

$$\|z\|^2_{W_2^2(\omega)} \le C\Big(\|\varphi_1\|^2_{\omega_1\cup\gamma_{11}} + \|\varphi_2\|^2_{\omega_2\cup\gamma_{21}} + \|\varphi_3\|^2_{\omega_{00}} + \sum_{i=1}^{2} h^2 \sum_{x\in\gamma_{i0}\cup\gamma_{i1}} \zeta_i^2\Big) \tag{18}$$

holds. The first three terms in the sum on the right–hand–side can be estimated in the same manner as in the previous case. For $s > 2$, ζ_i is a bounded linear functional of $u \in W_2^s(e_0)$, which vanishes on the second–degree polynomials. Using Lemma 1.6.3 one obtains the estimate

$$\Big(h^2 \sum_{x\in\gamma_{i0}} \zeta_i^2\Big)^{1/2} \le C\, h^{s-2}\, |u|_{W_2^s(\Omega_{i0})}, \qquad 2 < s \le 3,$$

where $\Omega_{i0} = \Omega_{hi}(0) = \{x \,:\, -h < x_i < h,\ 0 < x_{3-i} < 1\}$. Using Theorem 1.3.5 one obtains

$$\Big(h^2 \sum_{x\in\gamma_{i0}} \zeta_i^2\Big)^{1/2} \le C\, h^{\min\{s-2,\,1.5\}}\, |\ln h|^{1-|\mathrm{sgn}(s-3.5)|}\, \|u\|_{W_2^s(\Omega)}, \qquad 2 < s \le 4. \tag{19}$$

By analogy, one can estimate ζ_i on γ_{i1}.

From (18) and (19), and the previously derived estimates for φ_1, φ_2 and φ_3, one obtains the following convergence rate estimate for the finite difference scheme (5), (16),

$$\|u - v\|_{W_2^2(\omega)} \le C\, h^{\min\{s-2,\,1.5\}}\, |\ln h|^{1-|\mathrm{sgn}(s-3.5)|} \times \max_i \|a_i\|_{W_p^{s-2+\epsilon}(\Omega)}\, \|u\|_{W_2^s(\Omega)}, \qquad 2.5 < s \le 4. \tag{20}$$

REMARK 2. For $s < 3.5$, the solution of the boundary–value problem (1), (15) can be evenly extended outside of the boundary, preserving the class W_2^s. In this case $\zeta_i = 0$ on $\gamma_{i0} \cup \gamma_{i1}$, and the estimate (20) is a direct consequence of (11–14) and (18).

Finally, consider the problem with natural boundary conditions

$$M_i(u) = 0, \qquad D_i M_i(u) + 2 D_{3-i} M_3(u) = 0, \qquad x \in \Gamma_{i0} \cup \Gamma_{i1}, \qquad i = 1, 2; \qquad M_3(u) = 0, \qquad x \in \gamma_* . \tag{21}$$

The solution to the problem (1), (21) is determined with an accuracy of up to an additive first–degree polynomial. In order to obtain a unique solution of the problem, introduce the values at the three vertices of Ω as follows

$$u(0,0) = c_{00}, \qquad u(0,1) = c_{01}, \qquad u(1,0) = c_{10}. \tag{22}$$

One approximates the conditions (21) and (22) by

$$(23)\qquad \begin{gathered} m_i(v) = 0\,, \qquad m_i(v)_{\mathring{x}_i} + \left[m_3(v) + m_3(v)^{-i}\right]_{\bar{x}_{3-i}} = 0\,, \\ x \in \overline{\gamma}_{i0} \cup \overline{\gamma}_{i1}\,, \qquad i = 1,\, 2\,, \\ m_3(v) + m_3(v)^{-1} + m_3(v)^{-2} + m_3(v)^{-1,\,-2} = 0\,, \quad x \in \gamma_*\,, \end{gathered}$$

and

$$(24)\qquad v(0,\,0) = c_{00}\,, \qquad v(0,\,1) = c_{01}\,, \qquad v(1,\,0) = c_{10}\,.$$

Note that in that manner a discrete solution is also defined at external nodes at distances $2h$ from Γ and, consequently, the difference scheme (1), (23), (24) contains lewer equations than unknowns (or nodes). The missing conditions can be obtained from the approximation of equation (1) at boundary nodes. Introduce the asymmetric averaging operators in the following manner

$$T_{i\pm}^2 = 2\int_0^1 (1-t)\, f(x \pm t\, h\, r_i)\, dt\,, \qquad i = 1,\, 2\,,$$

and set

$$(25)\qquad \mathcal{L}_h v = \begin{cases} T_{i+}^2 T_{3-i}^2 f\,, & x \in \gamma_{i0} \\ T_{i-}^2 T_{3-i}^2 f\,, & x \in \gamma_{i1} \\ T_{1+}^2 T_{2+}^2 f\,, & x = (0,\,0) \\ \text{and analogous expressions for} & x = (0,\,1)\,,\ (1,\,0)\,,\ (1,\,1) \end{cases}$$

The error $z = u - v$ satisfies the a priori estimate

$$(26)\qquad \|[z]\|_{W_2^2(\omega)}^2 \le C\left(\|[\varphi_1]\|_\omega^2 + \|[\varphi_2]\|_\omega^2 + \|\varphi_3\|_{\omega_{00}}^2 + \|[\phi_1]\|_\omega^2 + \|[\phi_2]\|_\omega^2\right),$$

where,

$$\phi_i = \begin{cases} T_{3-i}^2\, M_i(u) - T_{(3-i)+}^2\, M_i(u)\,, & x \in \overline{\gamma}_{i0} \\ T_{3-i}^2\, M_i(u) - T_{(3-i)-}^2\, M_i(u)\,, & x \in \overline{\gamma}_{i1} \\ 0\,, & \text{at other nodes} \end{cases}$$

φ_1, φ_2 and φ_3 can be estimated in the same manner as it was done in previous cases. ϕ_i is a bounded linear functional of $M_i(u) \in W_2^\lambda$, $\lambda > 0.5$, which vanishes on constants. Using Lemma 1.6.3 and Theorem 1.3.5 one obtains

$$(27)\qquad \begin{aligned} &\|[\phi_i]\|_\omega \le C\, h^{\min\{s-2,\,1.5\}}\, |\ln h|^{1-|\operatorname{sgn}(s-3.5)|} \\ &\times \max_j \|a_j\|_{W_p^{s-2+\epsilon}(\Omega)}\, \|u\|_{W_2^s(\Omega)}\,, \qquad 2.5 < s \le 4\,. \end{aligned}$$

From (26), (27) and the previous estimates for φ_i, $i = 1,\, 2,\, 3$, one obtains the following convergence rate estimate for the finite difference scheme (1), (23), (25)

$$\begin{aligned} &\|[u - v]\|_{W_2^2(\omega)} \le C\, h^{\min\{s-2,\,1.5\}}\, |\ln h|^{1-|\operatorname{sgn}(s-3.5)|} \\ &\times \max_i \|a_i\|_{W_p^{s-2+\epsilon}(\Omega)}\, \|u\|_{W_2^s(\Omega)}\,, \qquad 2.5 < s \le 4\,. \end{aligned}$$

6. The Problem History and Comments

The principal purpose of this chapter is to establish a method for the derivation of convergence rate estimates, which are consistent with the data smoothness, for the difference schemes approximating boundary value problems for elliptic partial differential equations. This procedure is based on the Bramble–Hilbert lemma and its generalizations (1.6.1–1.6.4).

According to the definition (Lazarov, Makarov & Samarskiĭ [67]), a convergence rate estimate of the form

$$(1) \qquad \|u-v\|_{W_2^r(\omega)} \le C\,h^{s-r}\,\|u\|_{W_2^s(\Omega)}\,, \qquad s>r$$

is said to be consistent with the smoothness of the solution to the boundary–value problem. Note that similar estimates, of the form

$$\|u-v\|_{W_2^r(\Omega)} \le C\,h^{s-r}\,\|u\|_{W_2^s(\Omega)}\,, \qquad s>r$$

are characteristic for the finite elemenats method (see Strang & Fix [90], Ciarlet [9], [10]). In the case of equations with variable coefficients, constant C depends on the norms of coefficients, and consequently one can obtain estimates of the form

$$\|u-v\|_{W_2^r(\omega)} \le C\,h^{s-r}\left(\max_{i,j}\|a_{ij}\|_{W_p^{s-1}(\Omega)}+\|a\|_{W_p^{s-2}(\Omega)}\right)\|u\|_{W_2^s(\Omega)}\,.$$

(compare to (2.10), (2.11), (3.2), (3.15), (4.8), (5.10) and (5.20)).

Estimates of the form (1) were, to the best of our knowledge, derived by Weinelt [109], for $r=1$ and $s=2,3$, in case of the Poisson equation. Later, estimates of the form (1) were obtained by Lazarov, Makarov, Samarskiĭ, Weinelt, Jovanović, Ivanović, Süli, Gavrilyuk, Voĭtsekhovskiĭ and others, using systematically the Bramble–Hilbert lemma.

For example, families of difference schemes with averaged right–hand–sides for Poisson and Helmholtz equation were introduced in Jovanović's [34] and Ivanović's, Jovanović's & Süli's [30], [51] papers, yielding the spans of estimates of the form (1), in case of non–integer values of s.

The procedure for the determination of the constant in the Bramble–Hilbert lemma, using the mappings of elementary rectangles onto the standard ones, was suggested by Lazarov [62].

In the papers of Lazarov [62], Lazarov & Makarov [66] and Makarov & Ryzhenko [74, 75], the convergence of the difference schemes was examined for the Poisson equation in cylindrical, polar and spherical coordinates, and estimates of the type (1) were obtained in the corresponding weighted Sobolev spaces.

A scheme with an enhanced accuracy for equations with constant coefficients was derived by Jovanović, Süli & Ivanović [55], and similar results were obtained later by Voĭtsekhovskiĭ & Novichenko [107].

Difference schemes for the biharmonic equation were considered by Lazarov [63], Gavrilyuk, Lazarov, Makarov & Pirnazarov [20], Ivanović, Jovanović & Süli [31, 32], and for the equations of the elasticity theory — by Makarov & Kalinin [71, 57].

Application of the exact difference schemes was considered by Lazarov, Makarov & Samarskiĭ [67].

Equations with variable coefficients were studied later. At first, the difference schemes for the Helmholtz equation with variable lowest coefficient were studied (Weinelt, Lazarov & Makarov [97, 68], Voĭtsekhovskiĭ, Makarov & Shabliĭ [106]), and after that also problems with variable coefficients of the highest derivatives (Godev & Lazarov [26], Jovanović, Ivanović & Süli [54], Jovanović [36, 42, 43]). Equations with the lowest coefficients belonging to the negative Sobolev classes were considered by Voitsehovskii, Makarov & Rybak [105], and Jovanović [43].

The fourth-order equations with variable coefficients were studied by Gavrilyuk, Prikazchikov & Khimich [22], and Jovanović [45]. Quasilinear equations in arbitrary domains, solved by a combination of finite difference and fictitious domains methods were studied by Voĭtsekhovskiĭ & Gavrilyuk [100], Voĭtsekhovskiĭ, Gavrilyuk & Makarov [101] and Jovanović [37, 38, 44].

The technique described above was also used for solution of the eigenvalue problems (Prikazchikov & Khimich [81]), variational inequalities (Voĭtsekhovskiĭ, Gavrilyuk & Sazhenyuk [102], Gavrilyuk & Sazhenyuk [23]) and in the investigation of the superconvergence effects (Marletta [76]).

Finally, let us also mention the papers in which the convergence rate was estimated in discrete W_p^k–norms, for $p \neq 2$ (Lazarov & Mokin [69], Lazarov [64], Godev & Lazarov [24], Drenska [14, 15], Süli, Jovanović & Ivanović [91, 92]). In this case, the determination of a priori estimates is technically more complex — the theory of discrete Fourier multipliers (Mokin [78]) is used, rather than the energy estimates. The convergence rate estimates are obtained from the above described technique, using the Bramble–Hilbert lemma.

The Paragraph 2 was written, in most part, following the Ref. [43] by Jovanović. A simpler problem, with coefficients $a_{ij} \in W_\infty^{s-1}$ and $a \in W_\infty^{s-2} \cap L_\infty$, was studied by Jovanović, Ivanović & Süli [54] and Jovanović [36]. Paragraphs 4 and 5 contain results previously published by Jovanović [46], and [45], respectively.

III Non–stationary Problems

In this chapter the convergence rate of finite difference schemes approximating initial–boundary–value problems for non–stationary equations is examined. In Paragraph 1 we consider the first initial–boundary–value problem for linear second-order parabolic partial differential equation with variable coefficients. The convergence of the corresponding finite difference schemes is proved in the discrete $W_2^{1,1/2}$–norm. The obtained convergence rate estimates are consistent with the smoothness of data. In Paragraph 2 we consider an analogous problem for the second–order hyperbolic equation. The convergence is proved in a mixed norm $\|\cdot\|_{2,\infty}^{(1)}$.

1. The Parabolic Problem

The formulation of the problem. As a model problem we consider the first initial–boundary–value problem for the second-order linear symmetric parabolic partial differential equation with variable coefficients in the domain $Q = \Omega \times (0, T] = (0, 1)^2 \times (0, T]$

$$
\begin{aligned}
&\frac{\partial u}{\partial t} + \mathcal{L}u = f\,, && (x, t) \in Q\,, \\
&u = 0\,, && (x, t) \in \Gamma \times [0, T] = \partial\Omega \times [0, T]\,, \\
&u(x, 0) = u_0(x)\,, && x \in \Omega\,,
\end{aligned} \tag{1}
$$

where,

$$\mathcal{L}u = -\sum_{i,j=1}^{2} D_i\,(a_{ij}\,D_j u) + a\,u\,.$$

Assume that the generalised solution to the problem (1) belongs to the anisotropic Sobolev space $W_2^{s,s/2}(Q)$, $1 < s \le 3$, $f(x, t)$ belongs to $W_2^{s-2,s/2-1}(Q)$ and the coefficients $a_{ij} = a_{ij}(x)$ and $a = a(x)$ satisfy the same conditions as in the elliptic case (see Paragraph 2.2)

$$
\begin{array}{lll}
a_{ij} \in W_2^{s-1}(\Omega), & a \in W_2^{s-2}(\Omega), & \text{for} \quad s > 2, \\
a_{ij} \in W_p^{s-1+\delta}(\Omega), & a = a_0 + \sum_{i=1}^{2} D_i a_i, & \\
a_0 \in L_{2+\varepsilon}(\Omega), & a_i \in W_p^{s-1+\delta}(\Omega), & \text{where,} \\
\varepsilon > 0, \quad \delta > 0, & p \geq 2/(s-1), & \text{for} \quad 1 < s \leq 2.
\end{array}
$$

These conditions provide that the coefficients belong to the corresponding multiplier spaces

$$
a_{ij} \in M\big(W_2^{s-1,(s-1)/2}(Q)\big), \qquad a \in M\big(W_2^{s,s/2}(Q) \to W_2^{s-2,(s-2)/2}(Q)\big).
$$

The consecutive convergence rate estimates do not depend on δ by no means, so one can set $\delta = 0$.

Also assume that the following conditions hold

$$
a_{ij} = a_{ji},
$$

$$
\sum_{i,j=1}^{2} a_{ij}\, y_i\, y_j \geq c_0 \sum_{i=1}^{2} y_i^2, \qquad c_0 > 0, \qquad \forall x \in \Omega, \qquad \forall y \in \mathbf{R}^n,
$$

$$
a(x) \geq 0 \quad \text{in the sense of distributions in } \Omega, \text{ i.e.}
$$

$$
\langle a \cdot \varphi, \varphi \rangle_{\mathcal{D}' \times \mathcal{D}} \geq 0, \qquad \forall \varphi \in \mathcal{D}(\Omega),
$$

as well as the possible consistency conditions for the input data at the edges of the domain Q, enabling the existence of the solution $u \in W_2^{s,s/2}(Q)$ (see Paragraph 1.8).

Finally, suppose that the solution $u(x,\, t)$ of (1) is extended on $Q_d = (-d,\, 1+d)^2 \times (-d,\, T]$, where $d > 0$, preserving the class.

The finite difference scheme. Let $m \in \mathbf{N}$, $\tau = T/m$ and θ_τ be an uniform mesh with the step τ on $(0,\, T)$. Set $\theta_\tau^- = \theta_\tau \cup \{0\}$, $\theta_\tau^+ = \theta_\tau \cup \{T\}$, $\overline{\theta}_\tau = \theta_\tau \cup \{0,\, T\}$, $Q_{h\tau} = \omega \times \theta_\tau$, $Q_{h\tau}^- = \omega \times \theta_\tau^-$, $Q_{h\tau}^+ = \omega \times \theta_\tau^+$ and $\overline{Q}_{h\tau} = \overline{\omega} \times \overline{\theta}_\tau$, where ω is the previously introduced uniform mesh with the step h in the domain Ω (see Paragraph 2.1). Assume that

$$
c_1\, h^2 \leq \tau \leq c_2\, h^2, \qquad c_1,\, c_2 = const > 0
$$

and $h,\, \tau < d$.

For the function v defined on $\overline{Q}_{h\tau}$ introduce, in addition to the finite differences v_{x_i} and $v_{\bar{x}_i}$, the differences relativing to the variable t,

$$v_t = (v^+ - v)/\tau = v_{\bar{t}}^+ ,$$

where $v^\pm(x, t) = v(x, t \pm \tau)$.

Finally, together with the Steklov operators T_i, T_i^+ and T_i^- for the averaging over x_i (with the step h), introduce the operators for the averaging over the variable t (with the step τ)

$$T_t^+ f(x, t) = \int_0^1 f(x, t + t'\tau)\, dt' = T_t^- f(x, t + \tau) = T_t f(x, t + \tau/2) .$$

Approximate the initial–boundary–value problem (1) on the mesh $\overline{Q}_{h\tau}$ by the following difference scheme

$$(2) \qquad \begin{aligned} &v_{\bar{t}} + \mathcal{L}_h v = T_1^2 T_2^2 T_t^- f && \text{in } Q_{h\tau}^+ ,\\ &v = 0 && \text{on } \omega \times \{0\} ,\\ &v = P u_0 && \text{on } \gamma \times \overline{\theta}_\tau ; \end{aligned}$$

where,

$$\mathcal{L}_h v = -0.5 \sum_{i,j=1}^{2} \left[(a_{ij}\, v_{\bar{x}_j})_{x_i} + (a_{ij}\, v_{x_j})_{\bar{x}_i} \right] + (T_1^2 T_2^2\, a)\, v ,$$

and

$$P u = \begin{cases} u , & 2 < s \le 3 \\ T_1^2 T_2^2\, u , & 1 < s \le 2 \end{cases} .$$

The scheme (2) is a standard symmetric implicit difference scheme (see Samarskiĭ [84]) with the averaged right–hand side and the lowest coefficient. The scheme without averaging cannot be used for $s \le 4$, since, in that case, $f(x, t)$ is not a continuous function. (The coefficient $a(x)$ becomes discontinuous for $s \le 3$).

The convergence of the finite difference scheme. Let u be the solution of the initial–boundary–value problem (1) and v — the solution of the difference scheme (2). For $1 < s \le 2$, $u(x, t)$ needs not be a continuous function, but it has the integrable traces for $t = const$. In the following assume that for $1 < s \le 2$, the solution $u(x, t)$ is oddly extended in x_1 and x_2 outside Q. (For the above indicated values of s such an extension preserves the class $W_2^{s,s/2}$).

Define the error in the following manner

$$z = P u - v .$$

The error thus defined satisfies the following relations

$$
(3)\qquad
\begin{aligned}
&z_{\bar t} + \mathcal{L}_h z = \sum_{i,j=1}^{2} \eta_{ij,\bar x_i} + \eta + \psi_{\bar t} \quad \text{in} \quad Q_{h\tau}^{+}, \\
&z = 0 \quad \text{on} \quad \omega \times \{0\}, \\
&z = 0 \quad \text{on} \quad \gamma \times \bar\theta_\tau ,
\end{aligned}
$$

where,

$$
\begin{aligned}
\eta_{ij} &= T_i^{+} T_{3-i}^{2} T_t^{-} \left(a_{ij}\, D_j u\right) - 0.5 \left[a_{ij}\,(P\,u)_{x_j} + a_{ij}^{+i}\,(P\,u)_{\bar x_j}^{+i}\right], \\
\eta &= (T_1^2 T_2^2\, a)\,(P\,u) - T_1^2 T_2^2 T_t^{-}\,(a\,u), \quad \text{and} \\
\psi &= P\,u - T_1^2 T_2^2\, u .
\end{aligned}
$$

Introduce the discreete inner product

$$
(v,\, w)_{Q_{h\tau}} = (v,\, w)_{L_2(Q_{h\tau})} = h^2\,\tau \sum_{x\in\omega} \sum_{t\in\theta_\tau^+} v(x,\,t)\, w(x,\,t),
$$

and norms and seminorms such as

$$
\begin{aligned}
\|v\|_{Q_{h\tau}}^2 &= (v,\, v)_{Q_{h\tau}}, \\
\|v\|_i^2 &= h^2\,\tau \sum_{x\in\omega_i} \sum_{t\in\theta_\tau^+} v^2(x,\,t), \\
|v|_{1/2}^2 &= h^2\,\tau^2 \sum_{x\in\omega} \sum_{\substack{t,\,t'\in\bar\theta_\tau \\ t\neq t'}} \left[\frac{v(x,\,t) - v(x,\,t')}{t - t'}\right]^2, \\
\|v\|_{W_2^{1,\,1/2}(Q_{h\tau})}^2 &= \|v\|_{1,\,1/2}^2 = \sum_{i=1}^{2} \|v_{x_i}\|_i^2 + |v|_{1/2}^2 + \|v\|_{Q_{h\tau}}^2 .
\end{aligned}
$$

LEMMA 1. *The finite difference scheme*

$$
(4)\qquad z_{\bar t} + \mathcal{L}_h z = \sum_{i,j=1}^{2} \eta_{ij,\bar x_i} \quad \text{in} \quad Q_{h\tau}^{+}, \qquad z = 0 \quad \text{on} \quad \gamma \times \bar\theta_\tau ,
$$

satisfies the a priori estimate

$$
(5)\qquad \|z\|_{W_2^{1,\,1/2}(Q_{h\tau})}^2 \le C\left(\|z(\,.\,,0)\|_\omega^2 + \tau \sum_{i=1}^{2} \|z_{x_i}(\,.\,,0)\|_{\omega_i}^2 + \sum_{i,j=1}^{2} \|\eta_{ij}\|_i^2\right).
$$

Proof. Multiplying (4) by τz and summing over the mesh ω one obtains

$$
\frac{1}{2}\left(\|z\|_\omega^2 - \|z^-\|_\omega^2\right) + \frac{1}{2}\|z - z^-\|_\omega^2 + \tau\,(\mathcal{L}_h z,\, z)_\omega = \sum_{i,j=1}^{2} \tau\,(\eta_{ij,\bar x_i},\, z)_\omega .
$$

Hence, using the relations

$$(\mathcal{L}_h z,\, z)_\omega \geq c_0 \sum_{i=1}^{2} \|z_{x_i}\|^2_{\omega_i} \quad \text{and}$$

$$(\eta_{ij,\bar{x}_i},\, z)_\omega = -(\eta_{ij},\, z_{x_i})_{\omega_i} \leq \frac{1}{c_0}\|\eta_{ij}\|^2_{\omega_i} + \frac{c_0}{4}\|z_{x_i}\|^2_{\omega_i}$$

one obtains

$$\|z\|^2_\omega - \|z^-\|^2_\omega + c_0 \sum_{i=1}^{2} \tau \|z_{x_i}\|^2_{\omega_i} \leq \frac{2}{c_0} \sum_{i,j=1}^{2} \tau \|\eta_{ij}\|^2_{\omega_i}\,.$$

Finally, performing the summation over the mesh θ_τ^+, and using the discrete Friedrichs's inequality,

$$\|z\|^2_{Q_{h\tau}} + \sum_{i=1}^{2} \|z_{x_i}\|^2_i \leq C\Big(\|z(\,.\,,0)\|^2_\omega + \sum_{i,j=1}^{2} \|\eta_{ij}\|^2_i\Big)\,. \tag{6}$$

To estimate $|z|_{1/2}$ expand the function z in sine and cosine series of t, similarly as in Paragraph 2.1

$$z(x,\,t) = \frac{a_0(x)}{2} + \sum_{k=1}^{m-1} a_k(x)\cos\frac{k\pi t}{T} + \frac{a_m(x)}{2}\cos\frac{m\pi t}{T}\,, \qquad t \in \bar{\theta}_\tau\,,$$

$$z(x,\,t) = \sum_{k=1}^{m-1} b_k(x)\sin\frac{k\pi t}{T}\,, \qquad t \in \theta_\tau\,,$$

where,

$$a_k = a_k[z] = \frac{2}{T}\tau\left[\frac{z(x,0)}{2} + \sum_{t\in\theta_\tau} z(x,\,t)\cos\frac{k\pi t}{2} + \frac{z(x,\,T)}{2}(-1)^k\right],$$

$$b_k = b_k[z] = \frac{2}{T}\tau \sum_{t\in\theta_\tau} z(x,\,t)\sin\frac{k\pi t}{2}\,.$$

Define the norms

$$A(z) = \Big(\sum_{k=1}^{m-1} k\,\|a_k[z]\|^2_\omega + \frac{1}{2}m\,\|a_m[z]\|^2_\omega\Big)^{1/2}, \quad \text{and}$$

$$B(z) = \Big(\sum_{k=1}^{m-1} k\,\|b_k[z]\|^2_\omega\Big)^{1/2}.$$

Using the results of Lemma 2.1.1 it is easy to show that

$$c_3\,|z|_{1/2} \leq A(z) \leq c_4\,|z|_{1/2}\,, \quad \text{and}$$

$$c_3\,|z|_{1/2} \leq B(z) \leq c_4\left[|z|^2_{1/2} + \tau\sum_{t\in\theta_\tau}\Big(\frac{1}{t} + \frac{1}{T-t}\Big)\|z(\,.\,,t)\|^2_\omega\right]^{1/2}. \tag{7}$$

One also readily verify that

$$\sum_{k=1}^{m-1} \|b_k[z]\|_\omega^2 = \frac{2}{T}\tau \sum_{t\in\theta_\tau} \|z(\,.\,,t)\|_\omega^2 ,$$

(8)
$$\frac{1}{2}\|a_0[z]\|_\omega^2 + \sum_{k=1}^{m-1} \|a_k[z]\|_\omega^2 + \frac{1}{2}\|a_m[z]\|_\omega^2$$
$$= \frac{2}{T}\tau \left[\frac{1}{2}\|z(\,.\,,0)\|_\omega^2 + \sum_{t\in\theta_\tau} \|z(\,.\,,t)\|_\omega^2 + \frac{1}{2}\|z(\,.\,,T)\|_\omega^2\right].$$

Multiply equation (4) by $\frac{2}{T}\tau \sin\frac{k\pi(t-\tau/2)}{T}$, and perform the summation over the mesh θ_τ^+. Using the partial summation, aditive trigonometric formulae and the above expansions,

$$-\frac{\sin\frac{k\pi\tau}{2T}}{\frac{k\pi\tau}{2T}}\,\frac{\pi}{T}k\,a_k[z] = \cos\frac{k\pi\tau}{2T}\Big\{-b_k[\mathcal{L}_h z] + \sum_{i,j=1}^{2} b_k[\eta_{ij,x_i}]\Big\}$$
$$-\sin\frac{k\pi\tau}{2T}\Big\{-a_k[\mathcal{L}_h z] + \sum_{i,j=1}^{2} a_k[\eta_{ij,x_i}]\Big\} + \frac{\tau}{T}\sin\frac{k\pi\tau}{2T}\Big\{-\mathcal{L}_h z(x,0)$$
$$+\sum_{i,j=1}^{2} \eta_{ij,x_i}(x,0) + (-1)^k\mathcal{L}_h z(x,T) - (-1)^k \sum_{i,j=1}^{2}\eta_{ij,x_i}(x,T)\Big\}.$$

Multiplying this relation by $a_k[z]$, summing over the mesh ω and k, using the fact that $\frac{\sin t}{t}$ is bounded for $0\le t\le \pi/2$, the relations (7) and (8), realations

$$\big(a_k[\varphi_{x_i}],\,a_k[z]\big)_\omega = -\big(a_k[\varphi],\,a_k[z_{x_i}]\big)_{\omega_i}, \quad \text{and}$$
$$\big(b_k[\varphi_{x_i}],\,a_k[z]\big)_\omega = -\big(b_k[\varphi],\,a_k[z_{x_i}]\big)_{\omega_i}$$

and the Cauchy–Schwartz inequality, one obtains

(9)
$$|z|_{1/2}^2 \le C\tau \sum_{t\in\bar\theta_\tau}\Big(\sum_{i,j=1}^{2}\|\eta_{ij}\|_{\omega_i}^2 + \|z\|_\omega^2 + \sum_{i=1}^{2}\|z_{x_i}\|_{\omega_i}^2\Big).$$

Since the values of $\eta_{ij}(x,0)$ do not appear in (4), without loss of generality one may set them to zero. Thus, from (6) and (9) one derives inequality (5). ∎

Similarly, one can prove

LEMMA 2. *The finite difference scheme*

(10)
$$z_{\bar t} + \mathcal{L}_h z = \psi_{\bar t} \quad \text{in } Q_{h\tau}^+, \qquad z = 0 \quad \text{on } \gamma\times\bar\theta_\tau,$$

satisfies the a priori estimate

$$\|z\|^2_{W_2^{1,1/2}(Q_{h\tau})} \le C\left[\|z(\,.\,,0)\|^2_\omega + \tau\sum_{i=1}^{2}\|z_{x_i}(\,.\,,0)\|^2_{\omega_i} + A^2(\psi) + B^2(\psi)\right]. \tag{11}$$

From (5) and (11), using relations (7), one can conclude that the difference scheme (3) satisfies the following a priori estimate

$$\begin{aligned}\|z\|^2_{W_2^{1,1/2}(Q_{h\tau})} &\le C\Big[\sum_{i=1}^{2}\|\eta_{ij}\|^2_i + \|\eta\|^2_{Q_{h\tau}}\\ &+ |\psi|^2_{1/2} + \tau\sum_{t\in\theta_\tau}\Big(\frac{1}{t}+\frac{1}{T-t}\Big)\|\psi(\,.\,,t)\|^2_\omega\Big].\end{aligned} \tag{12}$$

The problem of deriving the convergence rate estimate for the difference scheme (2) is now reduced to the estimation of the right–hand–side terms of the inequality (12).

First, we represent η_{ij} in the following manner

$$\begin{aligned}
\eta_{ij} &= \eta_{ij1} + \eta_{ij2} + \eta_{ij3} + \eta_{ij4} + \eta_{ij5}\,, \quad \text{where,}\\
\eta_{ij1} &= T_i^+T_{3-i}^2\,(a_{ij}\,T_t^-\,D_j u) - (T_i^+T_{3-i}^2\,a_{ij})\,(T_i^+T_{3-i}^2T_t^-\,D_j u)\,,\\
\eta_{ij2} &= \left[T_i^+T_{3-i}^2\,a_{ij} - 0.5\,(a_{ij}+a_{ij}^{+i})\right](T_i^+T_{3-i}^2T_t^-\,D_j u)\,,\\
\eta_{ij3} &= 0.5\,(a_{ij}+a_{ij}^{+i})\left\{T_i^+T_{3-i}^2T_t^-\,D_j u - 0.5\left[(P\,u)_{x_j} + (P\,u)_{\bar{x}_j}^{+i}\right]\right\},\\
\eta_{ij4} &= -\,0.25\,(a_{ij}-a_{ij}^{+i})\left[(T_t^-u)_{x_j} - (T_t^-u)_{\bar{x}_j}^{+i}\right], \quad \text{and}\\
\eta_{ij5} &= -\,0.25\,(a_{ij}-a_{ij}^{+i})\left[(P\,u - T_t^-u)_{x_j} - (P\,u - T_t^-u)_{\bar{x}_j}^{+i}\right].
\end{aligned}$$

For $1 < s \le 2$ set $\eta = \eta_0 + \eta_1 + \eta_2 + \eta_3 + \eta_4 + \eta_5$, where,

$$\begin{aligned}
\eta_0 &= (T_1^2T_2^2\,a_0)\,(T_1^2T_2^2T_t^-u) - T_1^2T_2^2\,(a_0\,T_t^-u)\,,\\
\eta_1 &= (T_1^2T_2^2\,a_0)\,T_1^2T_2^2\,(u - T_t^-u)\,,\\
\eta_{2i} &= (T_1^2T_2^2\,D_i\,a_i)\,(T_1^2T_2^2T_t^-u) - T_1^2T_2^2\left[(T_t^-u)\,D_i a_i\right],\\
\eta_{2i+1} &= (T_1^2T_2^2\,D_i\,a_i)\,T_1^2T_2^2\,(u - T_t^-u)\,, \qquad i = 1,\,2\,.
\end{aligned}$$

For $2 < s \le 3$ set $\eta = \eta_6 + \eta_7 + \eta_8 + \eta_9$, where,

$$\begin{aligned}
\eta_6 &= (T_1^2T_2^2\,a)\,(T_t^-u - T_1^2T_2^2T_t^-u)\,,\\
\eta_7 &= (T_1^2T_2^2\,a)\,(T_1^2T_2^2\,u - T_1^2T_2^2T_t^-u)\,,\\
\eta_8 &= (T_1^2T_2^2\,a)\,(u - T_1^2T_2^2\,u - T_t^-u + T_1^2T_2^2T_t^-u)\,, \quad \text{and}\\
\eta_9 &= (T_1^2T_2^2\,a)\,(T_1^2T_2^2T_t^-u) - T_1^2T_2^2\,(a\,T_t^-u)\,.
\end{aligned}$$

Introduce now the elementary rectangles (see Paragraph 2.2) $e_0 = e_0(x) = \{y : |y_j - x_j| < h,\ j = 1, 2\}$, $e_i = e_i(x) = \{y : x_i < y_i < x_i + h,\ |y_{3-i} - x_{3-i}| < h\}$, $i = 1, 2$, and parallelepipeds $g_0 = g_0(x, t) = e_0 \times (t - \tau, t)$, $g_i = g_i(x, t) = e_i \times (t - \tau, t)$.

For $2 < s \leq 3$, η_{ij1} satisfies the conditions for which the estimate of the form (2.2.7) is valid

$$\|\eta_{ij1}(\cdot, t)\|_{\omega_i} \leq C h^{s-1} \|a_{ij}\|_{W_2^{s-1}(\Omega)} \|T_t^- u(\cdot, t)\|_{W_2^s(\Omega)}, \qquad 2 < s \leq 3.$$

Then, performing the summation over the mesh θ_τ^+, and using the obvious majorisation,

$$(13) \qquad \|\eta_{ij1}\|_i \leq C h^{s-1} \|a_{ij}\|_{W_2^{s-1}(\Omega)} \|u\|_{W_2^{s, s/2}(Q)}, \qquad 2 < s \leq 3.$$

Analogously, for $1 < s \leq 2$, from (2.2.6) if follows that

$$(14) \qquad \|\eta_{ij1}\|_i \leq C h^{s-1} \|a_{ij}\|_{W_p^{s-1}(\Omega)} \|u\|_{W_2^{s, s/2}(Q)}, \qquad 1 < s \leq 2.$$

Similarly, using estimates (2.2.7) and (2.2.6) one obtains for η_{ij2} and η_{ij4} the estimates of the form (13) and (14).

For $s > 1$, $\eta_{ij3}(x, t)$ is a bounded bilinear functional of $(a_{ij}, u) \in C(\overline{e}_i) \times W_2^{s, s/2}(g_i)$ which vanishes if u is a polynomial of the second degree in x_1 and x_2 and of an arbitrary degree in t (with constant coefficients). Applying Lemma 1.6.3 one obtains the estimate

$$|\eta_{ij3}(x, t)| \leq C h^{s-3} \|a_{ij}\|_{C(\overline{e}_i)} |u|_{\widehat{W}_2^{s, s/2}(g_i)}, \qquad 1 < s \leq 3.$$

After a summation over the mesh $Q_{h\tau}^+$ one obtains

$$\|\eta_{ij3}\|_i \leq C h^{s-1} \|a_{ij}\|_{C(\overline{\Omega})} \|u\|_{W_2^{s, s/2}(Q)}, \qquad 1 < s \leq 3.$$

Using imbeddings

$$W_2^{s-1}(\Omega) \subseteq C(\overline{\Omega}) \quad \text{for} \quad 2 < s \leq 3 \qquad \text{and} \qquad W_p^{s-1}(\Omega) \subseteq C(\overline{\Omega}) \quad \text{for} \quad 1 < s \leq 2,$$

wherefrom one obtains estimates of the form (13) and (14). The same holds for η_{ij5}.

η_0 satisfies the conditions which allow an estimate of the form (2.2.9)

$$\|\eta_0(\cdot, t)\|_\omega \leq C h^{s-1} \|a_0\|_{L_{2+\epsilon}(\Omega)} \|T_t^- u(\cdot, t)\|_{W_2^s(\Omega)}, \qquad 1 < s \leq 2.$$

By a summation over the mesh θ_τ^+, after an evident majorization, one obtains

$$(15) \qquad \|\eta_0\|_{Q_{h\tau}} \leq C h^{s-1} \|a_0\|_{L_{2+\epsilon}(\Omega)} \|u\|_{W_2^{s, s/2}(Q)}, \qquad 1 < s \leq 2.$$

Analogously, from (2.2.6), one obtains an estimate of the form (14) for η_2 and η_4, while from (2.2.8) for η_6 and η_9 one obtains

$$(16) \qquad \|\eta_6\|_{Q_{h\tau}}, \ \|\eta_9\|_{Q_{h\tau}} \le C\,h^{s-1}\,\|a\|_{W_2^{s-2}(\Omega)}\,\|u\|_{W_2^{s,s/2}(Q)}, \qquad 2 < s \le 3.$$

For $s > 1$ and $q \ge 1$, $\eta_1(x, t)$ is a bounded bilinear functional of $(a, u) \in L_q(e_0) \times W_2^{s,s/2}(g_0)$ which vanishes if u is a polynomial of the first degree in x_1 and x_2 (with constant coefficients). Applying Lemma 1.6.3 one arrives to

$$|\eta_1(x, t)| \le C(h)\,\|a_0\|_{L_q(e_0)}\,|u|_{W_2^{s,s/2}(g_0)}, \qquad 1 < s \le 2,$$

where $C(h) = C\,h^{s-2-2/q}$. By a further majorisation one has

$$|\eta_1| \le C\,h^{s-2-2/q}\,\|a_0\|_{L_q(\Omega)}\,|u|_{W_2^{s,s/2}(g_0)}.$$

The summation over the mesh yields

$$\|\eta_1\|_{Q_{h\tau}} \le C\,h^{s-2/q}\,\|a_0\|_{L_q(\Omega)}\,|u|_{W_2^{s,s/2}(Q)}.$$

Setting $q = 2 + \varepsilon$, after an evident majorisation, one obtains the desired estimate

$$(17) \qquad \|\eta_1\|_{Q_{h\tau}} \le C\,h^{s-1}\,\|a_0\|_{L_{2+\varepsilon}(\Omega)}\,\|u\|_{W_2^{s,s/2}(Q)}, \qquad 1 < s \le 2.$$

For $s > 1$, $\eta_{2i+1}(x, t)$ $(i = 1, 2)$ is a bounded bilinear functional of $(a_i, u) \in L_\infty(e_0) \times W_2^{s,s/2}(g_0)$, which vanishes if u is a polynomial of the first degree in x_1 and x_2 (with constant coeficients). As in the previous case,

$$|\eta_{2i+1}| \le C\,h^{s-3}\,\|a_i\|_{L_\infty(\Omega)}\,|u|_{W_2^{s,s/2}(g_0)}, \qquad 1 < s \le 2,$$

and

$$\|\eta_{2i+1}\|_{Q_{h\tau}} \le C\,h^{s-1}\,\|a_i\|_{L_\infty(\Omega)}\,\|u\|_{W_2^{s,s/2}(Q)}, \qquad 1 < s \le 2.$$

Using the imbedding $W_p^{s-1}(\Omega) \subseteq L_\infty(\Omega)$, one obtains the estimate

$$(18) \qquad \|\eta_{2i+1}\|_{Q_{h\tau}} \le C\,h^{s-1}\,\|a_i\|_{W_p^{s-1}(\Omega)}\,\|u\|_{W_2^{s,s/2}(Q)}, \qquad 1 < s \le 2.$$

For $\lambda > 1/2$, $\eta_7(x, t)$ is a bounded bilinear functional of $(a, T_1^2T_2^2 u) \in L_q(e_0) \times W_2^\lambda(t - \tau, t)$, which vanishes if $T_1^2T_2^2 u$ is a constant. Applying Lemma 1.6.3,

$$|\eta_7(x, t)| \le C\,h^{2\lambda-1-2/q}\,\|a\|_{L_q(e_0)}\,|T_1^2T_2^2 u|_{W_2^\lambda(t-\tau,t)}, \qquad 1/2 < \lambda \le 1.$$

For $1/2 < \lambda < 1$,

$$|T_1^2T_2^2 u|_{W_2^\lambda(t-\tau,t)} = \left\{ \int_{t-\tau}^t \int_{t-\tau}^t \frac{|T_1^2T_2^2 u(\,.\,, t') - T_1^2T_2^2 u(\,.\,, t'')|^2}{|t' - t''|^{1+2\lambda}}\, dt'\, dt'' \right\}^{1/2}$$
$$\le C\,h^{-2/r} \left\{ \int_{t-\tau}^t \int_{t-\tau}^t \frac{\|u(\,.\,, t') - u(\,.\,, t'')\|_{L_r(e_0)}^2}{|t' - t''|^{1+2\lambda}}\, dt'\, dt'' \right\}^{1/2}.$$

Setting $r = 2q/(q-2)$,

$$|\eta_7(x, t)| \le C\, h^{2\lambda-2}\, \|a\|_{L_q(e_0)} \left\{ \int_{t-T}^{t} \int_{t-T}^{t} \frac{\|u(\,.\,, t') - u(\,.\,, t'')\|^2_{L_{2q/(q-2)}(e_0)}}{|t'-t''|^{1+2\lambda}}\, dt'\, dt'' \right\}^{1/2}$$

The summation over the mesh, using Hölder's inequality, yields

$$\|\eta_7\|_{Q_{h\tau}} \le C\, h^{2\lambda}\, \|a\|_{L_q(\Omega)} \left\{ \int_0^T \int_0^T \frac{\|u(\,.\,, t') - u(\,.\,, t'')\|^2_{L_{2q/(q-2)}(\Omega)}}{|t'-t''|^{1+2\lambda}}\, dt'\, dt'' \right\}^{1/2}.$$

Adopt the value of q such that the following imbeddings hold

$$W_2^{s-2}(\Omega) \subseteq L_q(\Omega) \qquad \text{and} \qquad W_2^1(\Omega) \subseteq L_{2q/(q-2)}(\Omega)\,.$$

For $2 < s \le 3$ it can be done by using $2 < q < 2/(3-s)$. Then,

$$\begin{aligned} \|\eta_7\|_{Q_{h\tau}} &\le C\, h^{2\lambda}\, \|a\|_{W_2^{s-2}(\Omega)} \left\{ \int_0^T \int_0^T \frac{\|u(\,.\,, t') - u(\,.\,, t'')\|^2_{W_2^1(\Omega)}}{|t'-t''|^{1+2\lambda}}\, dt'\, dt'' \right\}^{1/2} \\ &\le C\, h^{2\lambda}\, \|a\|_{W_2^{s-2}(\Omega)}\, \|u\|_{\widehat{W}_2^{2\lambda+1,\,\lambda+1/2}(Q)}\,, \qquad 2 < s \le 3, \qquad 1/2 < \lambda < 1\,. \end{aligned}$$

The same result holds for $\lambda = 1$ (then, the term $\displaystyle\int_{t-T}^{t} \int_{t-T}^{t} \frac{dt'\, dt''}{|t'-t''|^{1+2\lambda}}$ is substitued by $\displaystyle\int_{t-T}^{t} dt'$, and $u(\,.\,, t') - u(\,.\,, t'')$ by $\dfrac{\partial u(\,.\,, t')}{\partial t'}$). Setting $s = 2\lambda + 1$ one finally obtains the desired estimate for η_7

$$\|\eta_7\|_{Q_{h\tau}} \le C\, h^{s-1}\, \|a\|_{W_2^{s-2}(\Omega)}\, \|u\|_{W_2^{s,\,s/2}(Q)}\,, \qquad 2 < s \le 3\,. \tag{19}$$

For $a \in L_2(\Omega)$ and $s > 2$, $\eta_8(x, t)$ is a bounded linear functional of $u \in W_2^{s,\,s/2}(g_0)$ which vanishes on the polynomials of the second degree in x_1 and x_2 and the first degree in t (with constant coefficients). Applying Lemma 1.6.3, one obtains

$$\begin{aligned} |\eta_8| &\le C\, h^{s-3}\, \|a_i\|_{L_2(e_0)}\, |u|_{\widehat{W}_2^{s,\,s/2}(g_0)} \\ &\le C\, h^{s-3}\, \|a_i\|_{L_2(\Omega)}\, |u|_{\widehat{W}_2^{s,\,s/2}(g_0)}\,, \qquad 2 < s \le 3\,. \end{aligned}$$

Summating over the mesh $Q_{h\tau}^{+}$, and using the imbedding $W_2^{s-2}(\Omega) \subseteq L_2(\Omega)$, one obtains the estimate

$$\|\eta_8\|_{Q_{h\tau}} \le C\, h^{s-1}\, \|a\|_{W_2^{s-2}(\Omega)}\, \|u\|_{W_2^{s,\,s/2}(Q)}\,, \qquad 2 < s \le 3\,. \tag{20}$$

Next, estimate the value of ψ. Obviously,

$$\psi = 0 \qquad \text{za} \qquad 1 < s \le 2\,. \tag{21}$$

For $2 < s \le 3$,

$$|\psi|_{1/2} \le |T_t^- \psi|_{1/2} + |\psi - T_t^- \psi|_{1/2}. \tag{22}$$

Furthermore, for $0 < \lambda \le 1/2$

$$|T_t^- \psi|_{1/2}^2 \le \tau^{2\lambda-1} \tau^2 h^2 \sum_{x\in\omega} \sum_{\substack{t,\, t' \in \overline{\theta}_\tau \\ t \ne t'}} \frac{\left[T_t^- \psi(x,\, t) - T_t^- \psi(x,\, t')\right]^2}{|t-t'|^{1+2\lambda}}.$$

For $t,\, t' \in \overline{\theta}_\tau$ and $t \ne t'$,

$$|T_t^- \psi(x,\, t) - T_t^- \psi(x,\, t')| = \left|\tau^{-2} \int_{t-\tau}^{t} \int_{t'-\tau}^{t'} \left[\psi(x,\, \sigma) - \psi(x,\, \sigma')\right] d\sigma\, d\sigma'\right|$$

$$\le \left\{\tau^{-2}\, 2^{1+2\lambda} |t-t'|^{1+2\lambda} \int_{t-\tau}^{t} \int_{t'-\tau}^{t'} \frac{\left[\psi(x,\, \sigma) - \psi(x,\, \sigma')\right]^2}{|\sigma - \sigma'|^{1+2\lambda}} d\sigma\, d\sigma'\right\}^{1/2}.$$

The last two inequalities yield

$$|T_t^- \psi|_{1/2}^2 \le 2^{1+2\lambda} \tau^{2\lambda-1} h^2 \sum_{x\in\omega} \int_0^T \int_{-\tau}^{T-\tau} \frac{\left[\psi(x,\, \sigma) - \psi(x,\, \sigma')\right]^2}{|\sigma - \sigma'|^{1+2\lambda}} d\sigma\, d\sigma'.$$

Using the relation

$$\psi(x,\, t) = u(x,\, t) - T_1^2 T_2^2 u(x,\, t)$$

$$= h^{-2} \int_{x_1-h}^{x_1+h} \int_{x_2-h}^{x_2+h} \left(1 - \frac{|y_1 - x_1|}{h}\right) \left(1 - \frac{|y_2 - x_2|}{h}\right) \left[u(x,\, t) - u(y,\, t)\right] dy_2\, dy_1$$

$$= h^{-2} \int_{x_1-h}^{x_1+h} \int_{x_2-h}^{x_2+h} \int_{s_1}^{x_1} \int_{s_2}^{x_2} \left(1 - \frac{|s_1 - x_1|}{h}\right) \left(1 - \frac{|s_2 - x_2|}{h}\right) \frac{\partial^2 u(y,\, t)}{\partial y_1\, \partial y_2} dy_2\, dy_1\, ds_2\, ds_1$$

$$- h^{-2} \int_0^h \int_0^s \int_{x_1-r}^{x_1+r} \int_{x_2-h}^{x_2+h} \left(1 - \frac{s}{h}\right) \left(1 - \frac{|y_2 - x_2|}{h}\right) \frac{\partial^2 u(y,\, t)}{\partial y_1^2} dy_2\, dy_1\, dr\, ds$$

$$- h^{-2} \int_{x_1-h}^{x_1+h} \int_0^h \int_0^s \int_{x_2-r}^{x_2+r} \left(1 - \frac{|y_1 - x_1|}{h}\right) \left(1 - \frac{s}{h}\right) \frac{\partial^2 u(y,\, t)}{\partial y_2^2} dy_2\, dr\, ds\, dy_1$$

and the Cauchy–Schwartz inequality, one obtains

$$|T_t^- \psi|_{1/2} \le C h^{1+2\lambda} |u|_{\widehat{W}_2^{2+2\lambda,\, 1+\lambda}(Q)}, \qquad 0 < \lambda \le 1/2.$$

Finally, setting $s = 2 + 2\lambda$,

$$|T_t^- \psi|_{1/2} \le C h^{s-1} \|u\|_{W_2^{s,\, s/2}(Q)}, \qquad 2 < s \le 3. \tag{23}$$

The second term in (22) can be estimated by

$$|\psi - T_t^- \psi|_{1/2}^2 \le \frac{4}{3}\pi^2 h^2 \sum_{x\in\omega}\sum_{t\in\bar{\theta}_\tau} (\psi - T_t^- \psi)^2 .$$

Applying Lemma 1.6.3,

$$|\psi - T_t^- \psi|_{1/2} \le C h^{s-1} |u|_{\widehat{W}_2^{s,s/2}(Q)}, \qquad 2 < s \le 3. \tag{24}$$

Using Lemma 1.6.3 and the trace Theorem 1.4.1,

$$\begin{aligned} & h^2 \tau \sum_{x\in\omega}\sum_{t\in\theta_\tau} \Big(\frac{1}{t} + \frac{1}{T-t}\Big) \psi^2(x,t) \\ & \le C h^{2s-2} \tau \sum_{t\in\theta_\tau} \Big(\frac{1}{t} + \frac{1}{T-t}\Big) |u(\,.\,,t)|_{W_2^{s-1}(\Omega)}^2 \\ & \le C h^{2s-2} \ln\frac{1}{h} \|u\|_{W_2^{s,s/2}(Q)}^2, \quad \text{for } 2 < s \le 3 . \end{aligned} \tag{25}$$

Combining (12) with (13)–(25) the following result is obtained.

THEOREM 1. *The difference scheme* (2) *converges in the norm* $W_2^{1,1/2}(Q_{h\tau})$, *if* $c_1 h^2 \le \tau \le c_2 h^2$, *and the following estimates hold*

$$\begin{aligned} \|u - v\|_{W_2^{1,1/2}(Q_{h\tau})} \le C h^{s-1} \Big(& \max_{i,j} \|a_{ij}\|_{W_2^{s-1}(\Omega)} \\ & + \|a\|_{W_2^{s-2}(\Omega)} + \sqrt{\ln \tfrac{1}{h}}\Big) \|u\|_{W_2^{s,s/2}(Q)}, \quad \textit{for } 2 < s \le 3, \end{aligned} \tag{26}$$

and

$$\begin{aligned} \|T_1^2 T_2^2 u - v\|_{W_2^{1,1/2}(Q_{h\tau})} \le C h^{s-1} \big(& \max_{i,j} \|a_{ij}\|_{W_p^{s-1}(\Omega)} \\ & + \max_i \|a_i\|_{W_p^{s-1}(\Omega)} + \|a_0\|_{L_{2+\epsilon}(\Omega)}\big) \|u\|_{W_2^{s,s/2}(Q)}, \quad \textit{for } 1 < s \le 2. \end{aligned} \tag{27}$$

REMARK 1. *The estimate* (27) *is consistent with the smoothness of data, while the estimate* (26) *is "almost consistent" — the consistency is disturbed by the term* $\sqrt{\ln \frac{1}{h}}$, *which is slowly increasing for* $h \to 0$.

REMARK 2. *If the coefficients* a_{ij} *and* a *depend of* t, *the similar result holds. The derivation of the a priori estimate is more difficult (see Samarskiĭ* [84]*).*

REMARK 3. *It is possible to derive similar estimates in other discrete norms (e.g.* L_2 *and* $W_2^{2,1}$*). For equations with constant coefficients such estimates were obtained by Jovanović, Ivanović & Süli* [29], [50].

The factorised scheme. The finite difference scheme (2) is not efficient, because a system of linear equation with a block–threediagonal matrix has to be solved at every time-level. Instead, consider the factorised scheme

$$(28) \qquad (I+\sigma\tau\Lambda_1)(I+\sigma\tau\Lambda_2)\,v_{\bar t} + \mathcal{L}_h v^- = T_1^2T_2^2T_t^- f$$

with the same initial- and boundary-data as in (2). Here, σ is a positive real parameter, $\Lambda_i v = -v_{x_i\bar x_i}$ $(i=1,2)$ and I is the unit operator. The difference scheme (28) is stable, if the operator

$$(I+\sigma\tau\Lambda_1)(I+\sigma\tau\Lambda_2) - \frac{\tau}{2}\mathcal{L}_h$$

is positive definite (see Samarskiĭ [84], Jovanović [41]). This condition holds, for example, if

$$\sigma \ge \max_{i,j} \|a_{ij}\|_{C(\bar\Omega)},$$

and if the step h is sufficiently small

$$h \le 3\left(c_2\,\|a\|_{L_2(\Omega)}\right)^{-1}, \qquad \text{for} \quad 2<s\le 3,$$

i.e.

$$h \le \left[c_2\left(\|a_0\|_{L_{2+\epsilon}(\Omega)} + \|a_1\|_{L_p(\Omega)} + \|a_2\|_{L_p(\Omega)}\right)\right]^{-p/(p-2)}, \qquad \text{for} \quad 1<s\le 2.$$

The error $z = Pu - v$ satisfies the conditions

$$(I+\sigma\tau\Lambda_1)(I+\sigma\tau\Lambda_2)\,z_{\bar t} + \mathcal{L}_h z^- = \sum_{i,j=1}^{2} \eta'_{ij,\bar x_i} + \eta' + \psi_{\bar t} \qquad \text{in} \qquad Q^+_{h\tau},$$

$$z = 0 \qquad \text{on} \qquad \omega\times\{0\} \quad \text{and} \quad \gamma\times\bar\theta_\tau,$$

where,

$$\eta'_{ij} = \eta_{ij} + \frac{\tau}{2}\left[a_{ij}\,(Pu)_{x_j} + a_{ij}^{+i}\,(Pu)_{\bar x_j}^{+i}\right]_{\bar t}$$
$$-\frac{\sigma\tau}{2}\delta_{ij}\left[(Pu)_{x_j} + (Pu)_{\bar x_j}^{+i}\right]_{\bar t} + \frac{\sigma^2\tau^2}{2}(1-\delta_{ij})(Pu)_{x_j\bar x_j x_i\bar t},$$
$$\eta' = \eta - \tau(T_1^2T_2^2\,a)(Pu)_{\bar t}, \quad \text{and}$$
$$\delta_{ij} \text{ — the Kronecker's symbol}.$$

The a priori estimate (12) still holds with η'_{ij} and η' instead of η_{ij} and η, respectively. Applying the same routine as above, it is easy to prove that the factorised scheme (28) satisfies the error etimates (26) and (27).

2. The Hyperbolic Problem

The formulation of the problem. Let the domain Q be defined in the same manner as in the previous paragraph. Consider the following initial-boundary-value problem

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} + \mathcal{L}u = f, && (x,t) \in Q, \\ (1) \qquad &u = 0, && (x,t) \in \Gamma \times [0,T], \\ &u(x,0) = u_0(x), && \frac{\partial u(x,0)}{\partial t} = u_1(x), \qquad x \in \Omega, \end{aligned}$$

where,

$$\mathcal{L}u = -\sum_{i,j=1}^{2} D_i(a_{ij} D_j u) + a\,u.$$

Suppose that the generalised solution to the problem (1) belongs to the Sobolev space $W_2^s(Q)$, $2 < s \le 4$, and that the coefficients $a_{ij} = a_{ij}(x)$ and $a = a(x)$ satisfy the conditions

$$a_{ij} \in W_2^{s-1}(\Omega), \quad a_{ij} = a_{ji},$$

$$\sum_{i,j=1}^{2} a_{ij}\, y_i\, y_j \ge c_0 \sum_{i=1}^{2} y_i^2, \qquad c_0 > 0, \qquad \forall x \in \Omega, \qquad \forall y \in \mathbf{R}^n,$$

$$a \in W_2^{s-2}(\Omega), \qquad a(x) \ge 0 \quad \text{in the sense of distributions in } \Omega.$$

These conditions show that the coefficients belong to the corresponding multiplier spaces

$$a_{ij} \in M(W_2^{s-1}(Q)), \qquad a \in M(W_2^s(Q) \to W_2^{s-2}(Q)).$$

Also assume that the solution $u(x,t)$ of (1) is extended on $(-d, 1+d)^2 \times (-d, T]$, where $d > 0$, preserving the class.

The finite difference scheme. Define the mesh $Q_{h\tau}$ in Q in the same manner as in the previous paragraph. However, assume that the following condition holds

$$c_1 h \le \tau \le c_2 h, \qquad c_1, c_2 = const > 0.$$

Approximate the problem (1) on the mesh $\overline{Q}_{h\tau}$ by the following finite difference scheme

$$\begin{aligned} &v_{t\bar{t}} + \frac{1}{4}\mathcal{L}_h(v^+ + 2v + v^-) = T_1T_2T_t f \quad \text{in} \quad Q_{h\tau}^+, \\ (2) \qquad &v = 0 \quad \text{on} \quad \gamma \times \overline{\theta}_\tau, \\ &v = u_0 \quad \text{on} \quad \omega \times \{0\}, \\ &v^+ = u_0 + \tau T_1T_2 u_1 + \frac{\tau^2}{2}(-\mathcal{L}_h u_0 + T_1T_2T_t f) \quad \text{on} \quad \omega \times \{0\}, \end{aligned}$$

where,

$$\mathcal{L}_h v = -0.5 \sum_{i,j=1}^{2} \left[(a_{ij}\, v_{\bar{x}_j})_{x_i} + (a_{ij}\, v_{x_j})_{\bar{x}_i} \right] + (T_1 T_2 a)\, v\,.$$

The scheme (2) is a standard symmetric implicit difference scheme (see Samarskiĭ [84]) with the averaged right–hand–side and lowest coefficient.

The convergence of the finite difference scheme. Let u be the solution to the problem (1) and v — the solution of the difference scheme (2). The error $z = u - v$ satisfies the conditions (see Jovanović, Ivanović & Süli [53])

$$(3)\qquad \begin{aligned} & z_{t\bar{t}} + \frac{1}{4}\mathcal{L}_h(z^+ + 2z + z^-) = \varphi \quad \text{in} \quad Q_{h\tau}^+\,, \\ & z = 0 \quad \text{on} \quad \gamma \times \bar{\theta}_\tau\,, \\ & z = 0\,, \quad z^+ = \tau\, \upsilon + 0.5\,\tau^2\,(\varphi - \chi) \quad \text{on} \quad \omega \times \{0\}\,, \end{aligned}$$

where,

$$\begin{aligned} \varphi &= \sum_{i,j=1}^{2} \xi_{ij} + \xi + \chi + \zeta\,, \\ \xi_{ij} &= T_1 T_2 T_t\, D_i(a_{ij}\, D_j u) - 0.5\left[(a_{ij}\, u_{x_j})_{\bar{x}_i} + (a_{ij}^{+i}\, u_{\bar{x}_j})_{x_i} \right], \\ \xi &= u_{t\bar{t}} - T_1 T_2 T_t\, \frac{\partial^2 u}{\partial t^2}\,, \\ \chi &= \frac{\tau^2}{2}\,\mathcal{L}_h u_{t\bar{t}}\,, \\ \zeta &= a\,u - T_1 T_2 T_t\,(a\,u)\,, \quad \text{and} \\ \upsilon &= 0.5\,(u_t + u_{\bar{t}}) - T_1 T_2\, \frac{\partial u}{\partial t}\,. \end{aligned}$$

The energy method leads to the a priori estimate

$$(4)\qquad \|z\|_{2,\infty}^{(1)} \le C\left(\|\upsilon(\,.\,,0)\|_\omega + \tau \sum_{t\in\theta_\tau^-} \|\varphi(\,.\,,t)\|_\omega \right),$$

where,

$$\|z\|_{2,\infty}^{(1)} = \max_{t\in\theta_\tau^-} \left(\|z_t\|_\omega^2 + \sum_{i=1}^{2} \left\| 0.5\,(z^+ + z)_{x_i} \right\|_{\omega_i}^2 \right)^{1/2}.$$

The problem of deriving the convergence rate estimate for the difference scheme (2) is reduced to the estimation of the right–hand–side of inequality (4).

First, we represent ξ_{ij} in the following manner

$$\begin{aligned}
\xi_{ij} &= \xi_{ij1}+\xi_{ij2}+\xi_{ij3}+\xi_{ij4}+\xi_{ij5}+\xi_{ij6}+\xi_{ij7}, \quad \text{where,}\\
\xi_{ij1} &= T_1T_2T_t\,(a_{ij}\,D_iD_ju)-(T_1T_2\,a_{ij})\,(T_1T_2T_t\,D_iD_ju),\\
\xi_{ij2} &= (T_1T_2\,a_{ij})\left[T_1T_2T_t\,D_iD_ju-0.5\,(u_{\bar{x}_ix_j}+u_{x_i\bar{x}_j})\right],\\
\xi_{ij3} &= 0.5\,(T_1T_2\,a_{ij}-a_{ij})\,(u_{\bar{x}_ix_j}+u_{x_i\bar{x}_j}),\\
\xi_{ij4} &= T_1T_2T_t\,(D_ia_{ij}\,D_ju)-(T_1T_2\,D_ia_{ij})\,(T_1T_2T_t\,D_ju),\\
\xi_{ij5} &= \left[T_1T_2\,D_ia_{ij}-0.5\,(a_{ij,x_i}+a_{ij,\bar{x}_i})\right]\,(T_1T_2T_t\,D_ju),\\
\xi_{ij6} &= 0.5\,(a_{ij,x_i}+a_{ij,\bar{x}_i})\left[T_1T_2T_t\,D_ju-0.5\,(u_{x_j}^{-i}+u_{\bar{x}_j}^{+i})\right],\\
\xi_{ij7} &= 0.25\,(a_{ij,x_i}-a_{ij,\bar{x}_i})\,(u_{x_j}^{-i}-u_{\bar{x}_j}^{+i}).
\end{aligned}$$

Also set

$$\chi=\sum_{i,j=1}^{2}(\chi_{ij1}+\chi_{ij2})+\chi_0 \qquad \text{and} \qquad \zeta=\zeta_i+\zeta_2,$$

where,

$$\begin{aligned}
\chi_{ij1} &= -\frac{\tau^2}{8}\left(a_{ij}^{-i}\,u_{\bar{x}_ix_jt\bar{t}}+a_{ij}^{+i}\,u_{x_i\bar{x}_jt\bar{t}}\right),\\
\chi_{ij2} &= -\frac{\tau^2}{8}\left(a_{ij,\bar{x}_i}\,u_{x_jt\bar{t}}+a_{ij,x_i}\,u_{\bar{x}_jt\bar{t}}\right),\\
\chi_0 &= \frac{\tau^2}{4}\,(T_1T_2\,a)\,u_{t\bar{t}},\\
\zeta_1 &= (T_1T_2\,a)\,(u-T_1T_2T_t\,u), \quad \text{and}\\
\zeta_2 &= (T_1T_2\,a)\,(T_1T_2T_t\,u)-T_1T_2T_t\,(a\,u).
\end{aligned}$$

The values ξ_{ij1}, ξ_{ij6}, ξ_{ij3} and ξ_{ij7} at the node $(x,t)\in Q_{h\tau}^-$ are bounded bilinear functionals of $(a_{ij},u)\in W_q^\lambda(e_0)\times W_{2q/(q-2)}^\mu(g)$, where e_0 is the elementary square introduced in the previous paragraph and $g=e_0\times(t-\tau,t+\tau)$. Here, for ξ_{ij1} — $\lambda\geq 0$, $\mu\geq 2$ and $q\geq 2$, while for ξ_{ij6}, ξ_{ij3} and ξ_{ij7} — $\lambda>2/q$, $\mu>3/2-3/q$ and $q\geq 2$. Moreover, ξ_{ij1} and ξ_{ij6} vanish if either a_{ij} is a constant or u is a second-degree polynomial; ξ_{ij3} and ξ_{ij7} vanish if either a_{ij} or u is a first-degree polynomial. By Lemma 1.6.4 one obtains the estimate

$$|\xi_{ij1}(x,t)|\leq C(h)\,|a_{ij}|_{W_q^\lambda(e_0)}\,|u|_{W_{2q/(q-2)}^\mu(g)},$$

where, $C(h)=C\,h^{\lambda+\mu+1/q-7/2}$, $0\leq\lambda\leq 1$, $2\leq\mu\leq 3$.

A summation over the mesh $Q_{h\tau}^-$ yields

$$\tau\sum_{t\in\theta_\tau^-}\|\xi_{ij1}\|_\omega\leq C\,h^{\lambda+\mu-2}\,\|a_{ij}\|_{W_q^\lambda(\Omega)}\,\|u\|_{W_{2q/(q-2)}^\mu(Q)}, \tag{5}$$

$$0\leq\lambda\leq 1, \qquad 2\leq\mu\leq 3.$$

The following imbeddings are satisfied

$$W_2^{\lambda+\mu-1}(\Omega) \subseteq W_q^{\lambda}(\Omega) \quad \text{for} \quad \mu \geq 2 - 2/q, \tag{6}$$

and

$$W_2^{\lambda+\mu}(Q) \subseteq W_{2q/(q-2)}^{\mu}(Q) \quad \text{for} \quad \lambda \geq 3/q. \tag{7}$$

Setting $\lambda + \mu = s$, $q > 3$, from (5)–(7), one obtains,

$$\tau \sum_{t\in\theta_\tau^-} \|\xi_{ij1}\|_\omega \leq C h^{s-2} \|a_{ij}\|_{W_2^{s-1}(\Omega)} \|u\|_{W_2^s(Q)}, \quad \text{for} \quad 2 + 3/q \leq s \leq 4. \tag{8}$$

The estimate (8) is valid for arbitrary $q > 3$; setting $q \to \infty$ one can readily conclude that it holds for $2 < s \leq 4$.

In the same manner, ξ_{ij6} satisfies an estimate of the form (5) for $2/q < \lambda \leq 1$ and $3/2 - 3/q < \mu \leq 3$. Setting $\lambda + \mu = s$ and taking into account the imbeddings (6) and (7),

$$\tau \sum_{t\in\theta_\tau^-} \|\xi_{ij6}\|_\omega \leq C h^{s-2} \|a_{ij}\|_{W_2^{s-1}(\Omega)} \|u\|_{W_2^s(Q)}, \quad \text{for} \quad 2 + 1/q \leq s \leq 4, \tag{9}$$

or, due to the arbitrariness of q, for $2 < s \leq 4$. In the same manner we can estimate χ_{ij2}.

ξ_{ij3} and ξ_{ij7} satisfy an estimate of the form (5) for $2/q < \lambda \leq 2$ and $3/2 - 3/q < \mu \leq 2$. Hence, as in the previous cases, one obtains

$$\tau \sum_{t\in\theta_\tau^-} (\|\xi_{ij3}\|_\omega + \|\xi_{ij7}\|_\omega) \leq C h^{s-2} \|a_{ij}\|_{W_2^{s-1}(\Omega)} \|u\|_{W_2^s(Q)}, \tag{10}$$

for $2 + 1/q \leq s \leq 4$, or $2 < s \leq 4$.

For $s \geq 2$, $\xi_{ij2}(x, t)$ is a bounded bilinear functional of $(a_{ij}, u) \in L_\infty(e_0) \times W_2^s(g)$, which vanishes if u is a third-degree polynomial. Applying Lemma 1.6.3,

$$|\xi_{ij2}(x, t)| \leq C h^{s-7/2} \|a_{ij}\|_{L_\infty(e_0)} |u|_{W_2^s(g)}, \qquad 2 \leq s \leq 4.$$

A summation over the mesh $Q_{h\tau}^-$ yields

$$\tau \sum_{t\in\theta_\tau^-} \|\xi_{ij2}\|_\omega \leq C h^{s-2} \|a_{ij}\|_{L_\infty(\Omega)} \|u\|_{W_2^s(Q)}, \qquad 2 \leq s \leq 4.$$

Finally, using the imbedding

$$W_2^{s-1}(\Omega) \subseteq L_\infty(\Omega) \quad \text{for} \quad s > 2,$$

one obtains the desired estimate

$$(11)\qquad \tau \sum_{t\in\theta_\tau^-} \|\xi_{ij2}\|_\omega \le C\, h^{s-2} \|a_{ij}\|_{W_2^{s-1}(\Omega)} \|u\|_{W_2^s(Q)}, \qquad 2 < s \le 4.$$

In the same manner one can estimate χ_{ij1}.

$\xi_{ij4}(x,t)$ and $\xi_{ij5}(x,t)$ are bounded bilinear functionals of $(a_{ij}, T_t u) \in W_q^\lambda(e_0) \times W_{2q/(q-2)}^\mu(e_0)$, for $\lambda \ge 1$, $\mu \ge 1$, $q \ge 2$. Moreover, ξ_{ij4} vanishes if either a_{ij} or $T_t u$ is a first-degree polynomial; ξ_{ij5} vanishes if either a_{ij} is a second-degree polynomial or $T_t u$ is a constant. Using Lemma 1.6.4,

$$|\xi_{ij4}(x,t)| \le C\, h^{\lambda+\mu-3} |a_{ij}|_{W_q^\lambda(e_0)} |T_t u|_{W_{2q/(q-2)}^\mu(e_0)}, \qquad 1 \le \lambda, \mu \le 2,$$

and, by a summation over the mesh

$$(12)\qquad \|\xi_{ij4}\|_\omega \le C\, h^{\lambda+\mu-2} \|a_{ij}\|_{W_q^\lambda(\Omega)} \|T_t u\|_{W_{2q/(q-2)}^\mu(\Omega)}, \qquad 1 \le \lambda, \mu \le 2.$$

The following imbedding holds

$$(13)\qquad W_2^{\lambda+\mu}(\Omega) \subseteq W_{2q/(q-2)}^\mu(\Omega) \qquad \text{for} \qquad \lambda \ge 2/q.$$

Setting $\lambda + \mu = s$, one obtains from (12), (6) and (13)

$$(14)\qquad \|\xi_{ij4}\|_\omega \le C\, h^{s-2} \|a_{ij}\|_{W_2^{s-1}(\Omega)} \|T_t u\|_{W_2^s(\Omega)}, \qquad 3 - 2/q \le s \le 4.$$

Switching to the limit $q \to 2+0$ one can conclude that the estimate (14) holds for $2 < s \le 4$. Since

$$\tau \sum_{t\in\theta_\tau^-} \|T_t u\|_{W_2^s(\Omega)} \le T^{1/2} \Big(\tau \sum_{t\in\theta_\tau^-} \|T_t u\|^2_{W_2^s(\Omega)}\Big)^{1/2} \le C \|u\|_{W_2^s(Q)},$$

from (14), summing over $t \in \theta_\tau^-$, one obtains

$$(15)\qquad \tau \sum_{t\in\theta_\tau^-} \|\xi_{ij4}\|_\omega \le C\, h^{s-2} \|a_{ij}\|_{W_2^{s-1}(\Omega)} \|u\|_{W_2^s(Q)}, \qquad 2 < s \le 4.$$

Similarly, applying Lemma 1.6.4, one obtains, for ξ_{ij5}, an estimate of the form (12), where $\mu = 1$ and $1 \le \lambda \le 3$. Setting $q = 2$, $\lambda = s - 1$ and using the imbedding

$$W_2^s(\Omega) \subseteq W_\infty^1(\Omega) \qquad \text{for} \qquad s > 2,$$

follows

$$\|\xi_{ij5}\|_\omega \le C\, h^{s-2} \|a_{ij}\|_{W_2^{s-1}(\Omega)} \|T_t u\|_{W_2^s(\Omega)}, \qquad 2 < s \le 4,$$

and

$$(16)\qquad \tau \sum_{t\in\theta_\tau^-} \|\xi_{ij5}\|_\omega \le C\, h^{s-2} \|a_{ij}\|_{W_2^{s-1}(\Omega)} \|u\|_{W_2^s(Q)}, \qquad 2 < s \le 4.$$

$\chi_0(x,t)$ and $\zeta_1(x,t)$ are bounded bilinear functionals of $(a,u) \in L_q(e_0) \times W^{\mu}_{2q/(q-2)}(g)$, for $\mu > 3/2 - 3/q$, $q \geq 2$, which vanish if u is a first-degree polynomial. Applying Lemma 1.6.3,

$$|\chi_0|, |\zeta_1| \leq C h^{\mu+1/q-3/2} \|a\|_{L_q(e_0)} |u|_{W^{\mu}_{2q/(q-2)}(g)}, \qquad 3/2 - 3/q < \mu \leq 2.$$

Hence, by a summation over the mesh $Q^-_{h\tau}$, one obtains

$$\tau \sum_{t \in \theta^-_\tau} \left(\|\chi_0\|_\omega + \|\zeta_1\|_\omega\right) \leq C h^{\mu} \|a\|_{L_q(\Omega)} \|u\|_{W^{\mu}_{2q/(q-2)}(Q)}, \quad 3/2 - 3/q < \mu \leq 2.$$

Setting $\mu = s - 2$ and applying the imbeddings

$$W^s_2(Q) \subseteq W^{s-2}_{2q/(q-2)}(Q) \quad \text{for} \quad q > 2 \quad \text{and} \quad W^{s-2}_2(\Omega) \subseteq L_q(\Omega) \quad \text{where}$$

$2 \leq q < 2/(3-s)$ for $2 < s \leq 3$, $2 \leq q < \infty$ for $s = 3$ and q — arbitrary for $s > 3$, one obtains the estimate

$$\tau \sum_{t \in \theta^-_\tau} \left(\|\chi_0\|_\omega + \|\zeta_1\|_\omega\right) \leq C h^{s-2} \|a\|_{W^{s-2}_2(\Omega)} \|u\|_{W^s_2(Q)}, \quad 7/2 - 3/q < s \leq 4. \tag{17}$$

Since $q > 2$ is arbitrary, switching to the limit $q \to 2+0$, one concludes that the estimate (17) holds for $2 < s \leq 4$.

$\zeta_2(x,t)$ is a bounded bilinear functional of $(a,u) \in W^{\lambda}_q(e_0) \times W^{\mu}_{2q/(q-2)}(g)$, for $\lambda \geq 0$, $\mu \geq 0$, $q \geq 2$, which vanishes if either a or u are constant. As in the previous case, one obtains

$$\tau \sum_{t \in \theta^-_\tau} \|\zeta_2\|_\omega \leq C h^{\lambda+\mu} \|a\|_{W^{\lambda}_q(\Omega)} \|u\|_{W^{\mu}_{2q/(q-2)}(Q)}, \qquad 0 \leq \lambda, \mu \leq 1.$$

Setting $\lambda + \mu = s - 2$ and using the imbeddings

$$W^{\lambda+\mu}_2(\Omega) \subseteq W^{\lambda}_q(\Omega) \quad \text{for} \quad \mu \geq 1 - 2/q, \quad \text{and} \quad W^{\lambda+\mu+2}_2(Q) \subseteq W^{\mu}_{2q/(q-2)}(Q),$$

one obtains

$$\tau \sum_{t \in \theta^-_\tau} \|\zeta_2\|_\omega \leq C h^{s-2} \|a\|_{W^{s-2}_2(\Omega)} \|u\|_{W^s_2(Q)}, \quad 3 - 2/q \leq s \leq 4. \tag{18}$$

Switching to the limit $q \to 2+0$, one concludes that the estimate (18) holds for $2 < s \leq 4$.

$\xi(x, t)$ and $v(x, t)$ are bounded linear functionals of $u \in W_2^s(g)$, for $s > 2$ (or rather, for $s > 3/2$). Moreover, ξ vanishes on the third-degree polynomials, and v — on the second-degree polynomials. Using Lemma 1.6.3, after a summation over the mesh, one obtains

$$\tau \sum_{t \in \theta_\tau^-} \|\xi\|_\omega \le C\, h^{s-2} \|u\|_{W_2^s(Q)}, \qquad 2 < s \le 4, \tag{19}$$

and

$$\|v(\cdot, 0)\|_\omega \le C\, h^{s-3/2} |u|_{W_2^s(Q_\tau)}, \qquad 2 < s \le 3, \tag{20}$$

where $Q_\tau = \Omega \times (-\tau, \tau)$. Applying Theorem 1.3.5, from (20) it follows that

$$\|v(\cdot, 0)\|_\omega \le C\, h^{s-2} \|u\|_{W_2^s(Q)}, \qquad 2 < s \le 4. \tag{21}$$

Combining (4), (8)–(11), (15), (16)–(19) and (21) one obtains the following result.

THEOREM 1. *The finite difference scheme* (2) *converges in the norm* $\|\cdot\|_{2,\infty}^{(1)}$, *if* $c_1 h \le \tau \le c_2 h$, *and the error estimate*

$$\|u - v\|_{2,\infty}^{(1)} \le C\, h^{s-2} \big(\max_{i,j} \|a_{ij}\|_{W_2^{s-1}(\Omega)} + \|a\|_{W_2^{s-2}(\Omega)} \big) \|u\|_{W_2^s(Q)}, \qquad 2 < s \le 4 \tag{22}$$

holds.

REMARK 1. *An analogous estimate can be obtained if the coefficients a_{ij} and a depend on t. The derivation of the a priori estimate of the type* (4) *is more difficult (see Samarskiĭ* [84]*).*

REMARK 2. *Using the factors* $1/12$, $10/12$, $1/12$ *instead of* $1/4$, $1/2$, $1/4$, *and* $T_1T_2T_t\, f + \frac{\tau^2}{12} T_1T_2T_t \frac{\partial^2 f}{\partial t^2}$ *instead of* $T_1T_2T_t\, f$ *in* (2), *the order of convergence can be increased (for smoother solutions).*

The factorised scheme. Consider the efficient factorised finite difference scheme

$$(23) \qquad (I+\sigma\tau^2\Lambda_1)(I+\sigma\tau^2\Lambda_2)\,v_{t\bar{t}}+\mathcal{L}_h v = T_1T_2T_t f \quad \text{in} \quad Q_{h\tau}\,,$$

with the same initial- and boundary-data as in (2). The scheme (23) is stable if the operator

$$(I+\sigma\tau^2\Lambda_1)(I+\sigma\tau^2\Lambda_2)-\frac{\tau^2}{4}\mathcal{L}_h$$

is positive definite (see Samarskiĭ [84], Jovanović [41]). This condition is satisfied, for example, if

$$\sigma \geq \frac{1}{2}\max_{i,j}\|a_{ij}\|_{C(\bar{\Omega})}$$

and

$$h \leq 4\,c_2^{-2}\,\|a\|^{-1}_{L_2(\Omega)}\,.$$

The error $z=u-v$ satisfies the conditions

$$\begin{aligned}
&(I+\sigma\tau^2\Lambda_1)(I+\sigma\tau^2\Lambda_2)\,z_{t\bar{t}}+\mathcal{L}_h z=\varphi' && \text{in} \quad Q_{h\tau}\,,\\
&z=0 \qquad \text{on} \quad \gamma\times\bar{\theta}_\tau\,,\\
&z=0\,, \qquad z^+=\tau v+0.5\,\tau^2(\varphi'-\chi') && \text{on} \quad \omega\times\{0\}\,,
\end{aligned}$$

where,

$$\varphi'=\sum_{i,j=1}^{2}\xi_{ij}+\xi+\chi'+\zeta\,,$$

and

$$\chi'=-\sigma\tau^2\left(u_{x_1\bar{x}_1t\bar{t}}+u_{x_2\bar{x}_2t\bar{t}}\right)+\sigma^2\tau^4\,u_{x_1\bar{x}_1x_2\bar{x}_2t\bar{t}}\,.$$

The a priori estimate (4) still holds with φ' instead of φ. One can easily prove that

$$\tau\sum_{t\in\theta_\tau^-}\|\chi'\|_\omega \leq C\,h^{s-2}\|u\|_{W_2^s(Q)}\,, \qquad 2<s\leq 4\,,$$

which implies that the convergence rate estimate (22) also holds for the factorised difference scheme (23).

3. The Problem History and Comments

In this chapter we have derived the convergence rate estimates for the finite difference method for some basic initial-boundary-value problems for parabolic and hyperbolic linear partial differential equations. The procedure was based on the Bramble-Hilbert lemma and its generalisations (see Paragraph 1.6), and represents further developement of the methodology presented in Chapter II.

As it was already mentioned in Paragraph 1.8, in the case of parabolic linear partial differential equations of the second order, a complete theory of existence and uniqueness of the solution of basic initial-boundary-value problems is built in the anisotropic Sobolev spaces $W_2^{s,s/2}(Q)$. Thus, analogous difference norms were used for the convergence rate estimates.

Analogously to the elliptic case, for a convergence rate estimate of a parabolic difference scheme of the form

$$\|u-v\|_{W_2^{r,r/2}(Q_{h\tau})} \le C\left(h+\sqrt{\tau}\right)^{s-r} \|u\|_{W_2^{s,s/2}(Q)}, \qquad s>r \tag{1}$$

is said to be consistent with the smoothness of the solution of the initial-boundary-value problem. If steps h and τ satisfy the obvious relation

$$c_1 h^2 \le \tau \le c_2 h^2 ,$$

the estimate (1) reduces to

$$\|u-v\|_{W_2^{r,r/2}(Q_{h\tau})} \le C h^{s-r} \|u\|_{W_2^{s,s/2}(Q)}, \qquad s>r. \tag{2}$$

In case of equations with variable coefficients, the constant C depends on the norms of the coefficients. For example, if the coefficients are not functions of t, one obtains estimates of the form

$$\begin{aligned} \|u-v\|_{W_2^{r,r/2}(Q_{h\tau})} &\le C h^{s-r} \big(\max_{i,j} \|a_{ij}\|_{W_p^{s-1}(\Omega)} \\ &+ \|a\|_{W_p^{s-2}(\Omega)}\big) \|u\|_{W_2^{s,s/2}(Q)}, \qquad s>r. \end{aligned} \tag{3}$$

(compare (1.26) and (1.27)).

For equations with constant coefficients, estimates of the form (1) were obtained by Lazarov [65] for $r=0$, $s=2$. A similar estimate in the discrete L_p-norm (for $s=2$) was derived by Godev & Lazarov [25].

The case of fractional values of s was studied by Ivanović, Jovanović & Süli [29], [50]. Estimates of the form (2) were obtained for $2\le s\le 4$, $r=0,2$. For $r=1$ the estimate was derived in the $W_2^{1,0}$-norm.

In the paper [13] by Dražić, estimates of the form (1) and (2) were obtained; also the conditions under which steps h and τ may be independent of each other.

In the papers by Scott & Seward [87] and Seward, Kasibhatla & Fairweather [88] the role of the averaging of the initial data on the convergence rate of the difference scheme was examinated.

The problem of the optimal control with the parabolic equations was studied by Ivanović & Jovanović [28].

In all these publications the Bramble–Hilbert lemma was used in the derivation of the convergence rate estimates. Note that some convergence rate estimates of the difference schemes for problems with weak solutions were obtained earlier, using different techniques (see e.g. Juncosa & Young [56]).

Equations with variable coefficients were studied by Weinelt, Lazarov & Streit [98], and Kuzik & Makarov [58] — for integer values of s, and by Jovanović [39], [40], [42], [47] — for fractional values of s.

Paragraph 1 is mainly following the ref. [47] by Jovanović. A simpler problem was considered in refs. [39], [40] and [42], with the coefficients $a_{ij} \in W_\infty^{s-1}(\Omega)$ and $a \in W_\infty^{s-2}(\Omega) \cap L_\infty(\Omega)$.

The variational–difference schemes also satisfy estimates of the form (1–3) (see Jovanović [33]). However, more common are the estimates with a continuous, rather than discrete, $W_2^{r,r/2}$–norm on the right–hand–side (see Zlotnik [114], [115], Hackbusch [27], Amosov & Zlotnik [3]).

Besides the above described estimates, parabolic problems are also characterised by the estimates in the norms $L_\infty((0,T);\, L_2(\Omega))$ and $L_\infty((0,T);\, W_2^1(\Omega))$ (see Douglas & Dupont [11], Douglas, Dupont & Wheeler [12], Ranacher [82], Thomée & Wahlbin [95], Wheeler [110], Zlamal [113]), and in the "negative" norms (see Thomée [93]).

A review of the more recent results related to the variational–difference methods of solving parabolic partial differential equations was given by Thomée in ref. [94].

In the hyperbolic case, the theory of existence and uniqueness of the solutions of the contour problems was not developed as much as for the elliptic and parabolic problems. As an example, consider the first initial–boundary–value problem for the second–order linear hyperbolic partial differential equation. The results given in Paragraph 1.8 imply that, if the right–hand–side of the equation belongs to the W_2^{s-2} space, with the corresponding smoothness of the initial and boundary data, then the solution belongs to the W_2^{s-1} space, but need not belong to the W_2^s. Consequently, in order to obtain the convergence rate estimates, one must assume a smoother solution than in the previous cases. That is why the hyperbolic problems are characterised by convergence rate estimates which are inconsistent with the smoothnes of the solution.

For the estimation of the convergence rates, usualy complex norms of the form

$$(4) \qquad \left\| \left(\|u(\,.\,,t)\|^2_{W_2^r(\Omega)} + \left\| \frac{\partial u(\,.\,,t)}{\partial t} \right\|^2_{W_2^{r-1}(\Omega)} \right)^{1/2} \right\|_{L_\infty(0,T)}$$

or

$$\left\| \left(\|u(\,.\,,t)\|^2_{W_2^s(\Omega)} + \left\| \frac{\partial^r u(\,.\,,t)}{\partial t^r} \right\|^2_{L_2(\Omega)} \right)^{1/2} \right\|_{L_\infty(0,T)} \tag{5}$$

are used.

For the difference schemes, approximating the first initial–boundary–value problem for second–order linear hyperbolic partial differential equations with constant coefficients, with

$$c_1 h \le \tau \le c_2 h,$$

the convergence rate estimates of the form

$$\|u - v\|^{(r)}_{2,\infty} \le C\, h^{s-r-1}\, \|u\|_{W_2^s(Q)}, \tag{6}$$

were obtained by Jovanović, Ivanović & Süli [49], [52] for $r = 0, 1, 2$ and $r+1 < s \le r+3$. Here $\|\cdot\|^{(r)}_{2,\infty}$ is the discrete analogue of the norm (4). Since by transition from the function $u(x, t)$ to its $t = const$ trace in Sobolev spaces W_2^s one loses $1/2$ order of smoothness, so we may conclude that in the estimates of the form (6) an additional $1/2$ order of smoothness is lost.

Estimates in discrete norms of the form (5) for $r = 1$ and $r = -1$, with the interpolation for $-1 < r < 1$, were derived by Dzhuraev & Moskal'kov [19].

Equations with variable coefficients $a_{ij} \in W_\infty^{s-1}(\Omega)$, $a \in W_\infty^{s-2}(\Omega)$ were examined by Jovanović, Ivanović & Süli [53], and an estimate of the form (6) was derived for $r = 1$ and $2 < s \le 4$. Here the constant C depends on the norms of the coefficients.

The paper by Dzhuraev, Kolesnik & Makarov [18] also considers an equation with variable coefficients, using the method of straight lines for its solution. An estimate of the form (6) was obtained for $r = 0$ and a fixed, integer value $s = 2$.

The Paragraph 2 was written following refs. [53] by Jovanović, Ivanović & Süli, and [48] by Jovanović.

List of Symbols

Numbers and Operations

$\mathbb{N}$ — set of natural numbers	7	$[\alpha] = \big([\alpha_1], [\alpha_2], \ldots, [\alpha_n]\big)^T$	15
$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$	7	$[\alpha]^- = \big([\alpha_1]^-, [\alpha_2]^-, \ldots, [\alpha_n]^-\big)^T$	15
$\mathbf{R}$ — set of real numbers	7	$\lvert\alpha\rvert = \alpha_1 + \alpha_2 + \ldots + \alpha_n$	7
C, C_i — constants	9	$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$	7
$[s], [s]^-$ — integer part s	7	$\lvert x\rvert = \big(x_1^2 + x_2^2 + \ldots + x_n^2\big)^{1/2}$	7
$x = (x_1, x_2, \ldots, x_n)^T \in \mathbf{R}^n$	7	δ_{kl} — Kronecker's delta	28
$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T \in \mathbb{N}_0^n$	7		

Domains

Ω — domain in $\mathbf{R}^n$	8	Ω_h	14
also: $\Omega = (0, 1)^2$	32	Ω^h	14
$\overline{\Omega}$ — closure of the domain Ω	8	$\Omega_{hi} = \Omega_{hi}(x)$	46
$\Gamma = \partial\Omega$ — boundary of Ω	8, 32	$\Omega_{i0} = \Omega_{hi}(0)$	51
S — hypersurface in $\mathbf{R}^n$	9	Γ_{ik}	32
$\mathcal{B}_1$ — unit ball in $\mathbf{R}^n$	21	Q	16, 55
$\Omega' \Subset \Omega$ — subdomain	8		

Functions, Distributions and Operations

$f(x) = f(x_1, x_2, \ldots, x_n)$	8	$\mathcal{L}u$	22, 47, 55, 68

Function and Distribution Spaces

Inner Products and Norms

Meshes and Mesh Domains

Mesh Functions and Difference Operators

Discrete Inner Products and Norms

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