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AN INTRODUCTION TO MODEL THEORY

NOVI SAD
1987

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This publication was suported by Self-Menagement Community of Interest for Scientific Research of Vojvodina

PUBLISHED BY: University of Novi Sad
Faculty of Science
Institute of Mathematics
Dr Ilije Djuričića 4
21000 Novi Sad - Yugoslavia

Printed by: "Prepisi-umnožavanja", Vidosav Novaković, Belgrade.

Number of copies: 500

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PREFACE

The present book has grown out of a selection of lectures on model theory I have given in recent years at the Mathematical Institute and the Faculty of Science in Belgrade. These lectures were mainly attended by graduate students and senior undergraduates. The book is therefore intended for those who study model theory and its applications. The book is designed as an excursion through the main topics of classical model theory. The most important constructions and theorems of model theory and their proofs are presented.

Boolean algebras play an important role in this book. The use of Boolean algebras in model theory is prolific. We have applied them in many model-theoretic constructions, but we also have applied model theory in the proofs of certain properties of Boolean algebras.

Basic constructions of models are presented in the book, such as the method of constants, ultraproducts, and elementary chains of models. Notions such as realizing and omitting types, saturated models, as well as their applications are also given. A few words are devoted to abstract model theory. An explanation is given why the first order logic has a distinguished position among all the types of logics (Lindström's theorem). Some extensions of the first order logic are considered in more detail. Special care is given to the first order logic with additional quantifiers. For example, Keisler's completeness theorem for the first order logic with the quantifier "there exist uncountable many" is given, and some applications of this theorem.

The book contains sufficient material to cover a first course in model theory. However, we could not cover all the important topics in model theory, since the selection of material reflects, in a way, the taste of the author. Anyhow, there are books of an encyclopedic nature on this subject, and the reader is directed to consult them whenever he needs more details.

After reading the book, one can proceed to advanced topics, such as nonstandard analysis, models of set theory, models of arithmetic, infinitary logic, model-theoretic algebra, etc.

We suppose that the reader is acquainted with some parts of the naive set theory. This includes the basic properties of ordinal and cardinal numbers, and, partially, their arithmetic. We have adopted Von Neuman's representation of ordinals, so we have taken that every ordinal is the set of all the smaller ordinals, therefore $0=\emptyset$, $1=\{0\}$, $2=\{0,1\}$, ..., $\omega_0=\{0,1,2,\dots\}$, ... Here $\emptyset$ denotes the empty set. The set of all natural numbers is denoted by ω, i.e. $\omega=\omega_0=\{0,1,2,\dots\}$. We do not distinguish ω_α and $\aleph_\alpha$. If $f:A\longrightarrow B$ is a mapping from A into B and $X\subseteq A$, then

$f|X$ denotes the restriction of f to the set X,

$f[X]=\{f(x):x\in X\}$, but sometimes we write $f(X)$ for $f[X]$ as well,

$f\underline{x}$ or $f(\underline{x})$ stands for the sequence $fx_1, fx_2, \dots, fx_n$, where $\underline{x}$ denotes a sequence $x_1, x_2, \dots, x_n$.

The cardinal number of a set X is denoted by $|X|$, and the set of all the subsets of X by P(X). Our metatheory is based on the ZFC set-theory, and we shall not point out explicitly when we use, for example, the Axiom of Choice or its equivalents. However, all exceptions will be indicated, as the use of the Continuum Hypothesis or some weaker variants of the Axiom of Choice.

Finally, I would like to express gratitude to my colleagues and friends who have helped somehow this book to appear; to Professor Slaviša Prešić, who stimulated me to write a Serbo-croatian version of this text, to Milan Grulović for his assistance in obtaining support and his valuable comments on the text, and to Djordje Čubrić, Miodrag Rašković, Milan Božić, Kosta Došen and Željko Sokolović for their help in reading the manuscript and remarks.

Final remarks are on usage and signs. The word "iff" is often used instead of the phrase "if and only if". The end of a proof is indicated by ▌.

Belgrade, November 1986.

1. FIRST NOTIONS OF MODEL THEORY

Some logicians often define model theory as a union of formal logic and universal algebra. By more detailed analysis, one can see that the main subject of model theory is the relationship between syntactical objects on the one hand, and the structures of a set-theoretical nature on the other hand, or more precisely, between formal languages and their interpretations. Therefore, two areas of logic, syntax and semantics, both have a role to play in this subject. While syntax is concerned mainly with the formation rules of formulas, sentences and other syntactical objects, semantics bears on the meaning of these notions. One of the most important concepts is the satisfaction relation, denoted by $\models$, a relation between mathematical structures and sentences. Model theory was recognized as a separate subject during the thirties in the works of Tarski, Gödel, Skolem, Malcev and their followers. Since then, this field, has developed vigorously, and has received many applications in other branches of mathematics: algebra, set-theory, nonstandard analysis, and even computer science. We shall first study model theory of first-order predicate calculus.

1.1. First-order languages

We shall define a first order language as any set L of constant symbols, function symbols and relation symbols. Each of the relation and function symbols has some definite, finite number of argument places. Sometimes it is convenient to consider constant symbols as function symbols with zero argument places. According to our classification, we have

$$L = Fnc_L \cup Rel_L \cup Const_L,$$

where

$Fnc_L = \{s\in L$: s is a function symbol of L$\}$,
$Rel_L = \{s\in L$: s is a relation symbol of L$\}$,
$Const_L = \{s\in L$: s is a constant symbol of L$\}$.

All these three sets are pairwise disjoint, and each of them may be an empty set. Namely, we shall deal only with logic with equality. The function $ar: L \longrightarrow \omega$ assigns to each $s\in L$ its length, i.e. the number of argument places. By the remark above, if $s\in Const_L$, we define $ar(s)=0$, while for $s\in Fnc_L \cup Rel_L$, we have $ar(s)\geq 1$. In most cases it will be clear from the context what the lengths of the symbols of L are, so in such cases the arity function will not be mentioned explicitly.

For example, we may take that $L = \{+,\cdot,-,\leq,0,1\}$ is the language of ordered fields, where
$Fnc_L = \{+,\cdot,-\}$, $ar(+)= 2$, $ar(\cdot)= 2$, $ar(-)= 1$,
$Rel_L = \{\leq\}$, $ar(\leq)= 2$,
$Const_L = \{0,1\}$.

If L and L' are first order languages, and $L\subseteq L'$, then L' is called an *expansion* of the language L, while L is called a *reduct* of L'. If $L'\setminus L$ is a set of constant symbols, then we say that L' is a simple expansion of L.

1.2. Terms and formulas

The terms and formulas of a first-order language L are special finite sequences of the symbols of L, and the logical symbols of the first-order predicate calculus (which shall be abbreviated PR^1). The logical symbols of PR^1 are the so-called connectives: $\wedge$ (and), $\vee$ (or), $\rightarrow$ (implication), $\leftrightarrow$ (equivalence), $\neg$ (negation), then the equality sign $\equiv$, quantifiers $\forall$ (universal quantifier), $\exists$ (existential quantifier), and finally an infinite sequence of variables v_0, v_1, v_2, ... In order to enable a unique readability of terms and formulas, some auxiliary signs are used: the left and right parenthesis, and the comma: ( ) ,. For easier discussion, we shall use metasymbols. Metavariables are x, y, z, x_0, y_0, z_0, x_1, y_1, z_1, ..., and they may denote any variable v_i, $i\in\omega$, i.e.

the domain of metavariables is the set Var= $\{v_0, v_1, v_2, \ldots\}$. Metaequality is another important such sign and it will be denoted by =.

Terms, or algebraic expressions of the language L can be described inductively:

1° Variables and constant symbols are terms.

2° If $F \in Fnc_L$ is of length n, and $t_1, \ldots, t_n$ are terms of L, then $F(t_1, \ldots, t_n)$ is a term of L.

3° Every term of L can be obtained by a finite number of applications of rules 1° and 2°.

A somewhat more formal definition of the terms of the language L is as follows:

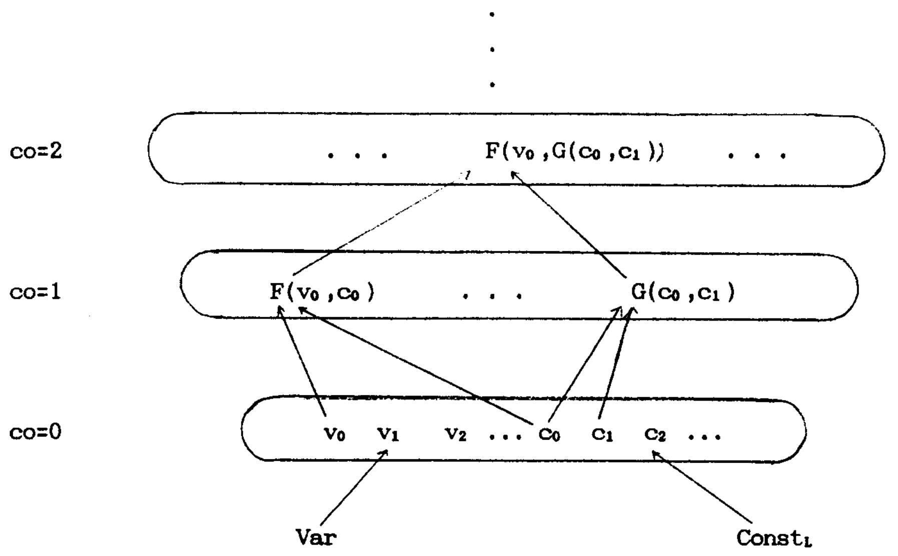
$$Term^0 = Var \cup Const_L,$$
$$Term^{m+1} = \{F(t_1, \ldots, t_n): n \in \omega,\ F \in Fnc_L,\ ar(F)=n,\ t_1, \ldots, t_n \in Term^m\}, \quad m \in \omega,$$
$$Term_L = \bigcup_n Term^n.$$

Then the terms of L are exactly the elements of the set $Term_L$. This definition allows further definitions of related notions, as well as simple (inductive) proofs of the basic properties of the terms. It is not difficult to see that elements of the set $Term_L$ satisfy conditions 1°,2°,3°.

The complexity function $co: Term_L \longrightarrow \omega$ of the terms is a measure of the complexity of the terms, and it is defined in the following way:

If $t \in Term^0$, then $co(t)=0$.
If $t \in Term^n \setminus Term^{n-1}$, then $co(t)= n$, $n \in \omega$

The complexity of terms can be visualized from the following diagram. Letters F and G here are binary function symbols.



We shall suppose that the reader is already familiar with basic properties of the terms and various conventions which have been introduced for the easier use of this notion (rules about deleting parenthesis, special notation for binary function symbols, possible priority of function symbols, etc.).

Formulas of the first-order language L are defined in a similar manner. First, the atomic formulas are defined:

A string φ is an ***atomic formula*** of the language L, if and only if φ has one of the following forms:

$u\equiv v$, u,v are terms of L,

$R(t_1,t_2,\dots,t_n)$, R is an n-placed relation symbol of L, and $t_1, t_2, \dots, t_n$ are terms of L.

Let At_L denote the set of the atomic formulas of L. Then by the previous definition we have

$$At_L = \{u\equiv v:\ u,v\in Term_L\} \cup \{R(t_1,\dots,t_n):\ n\in\omega,\ R\in Rel_L,\ ar(R)=n,\ t_1,\dots,t_n\in Term_L\}.$$

Formulas of the language L are also defined inductively by the use of an auxiliary sequence For^n, $n\in\omega$, of sets of strings of L:

$For^0 = At_L,$

$For^{n+1} = For^n \cup \{(\varphi\wedge\psi):\ \varphi,\psi\in For^n\} \cup$
$\{(\varphi\vee\psi):\ \varphi,\psi\in For^n\} \cup$
$\{\neg\varphi:\ \varphi\in For^n\} \cup$
$\{(\varphi\rightarrow\psi):\ \varphi,\psi\in For^n\} \cup$
$\{(\varphi\leftrightarrow\psi):\ \varphi,\psi\in For^n\} \cup$
$\{\forall x\varphi:\ x\in Var, \varphi\in For^n\} \cup$
$\{\exists x\varphi:\ x\in Var,\ \varphi\in For^n\},$

$For_L = \bigcup_n For^n.$

Then the elements of the set For_L are defined as formulas of the language L. It is not difficult to see that the formulas satisfy the following conditions:

1° Atomic formulas are formulas.

2° If φ,ψ are formulas of L , and x is a variable, then $(\varphi\wedge\psi)$, $(\varphi\vee\psi)$, $\neg\varphi$, $(\varphi\rightarrow\psi)$, $(\varphi\leftrightarrow\psi)$, $\forall x\varphi$, $\exists x\varphi$ are also formulas of L.

3° Every formula of L is obtained by the finite number of use of rules 1° and 2°

In order to measure the complexity of formulas, we shall extend the complexity function co to formulas as well. Therefore, $co: For_L \longrightarrow \omega$ is defined inductively in the following way:

If $\varphi\in At_L$, then $co(\varphi) = 0$,

If $\varphi\in For^n \backslash For^{n-1}$, $n\in\omega\backslash\{0\}$, then $co(\varphi) = n$.

As in the case of terms, we suppose that the reader is familiar with the basic conventions about formulas (rules on deleting parenthesis, priority of logical connectives, etc.). In addition, we shall shrink blocks of quantifiers, for example instead of $\forall x_0\forall x_1\ldots\forall x_n\varphi$ we shall write $\forall x_0x_1\ldots x_n\varphi$, whenever it is appropriate.

The notion of the free occurrence of variables allows us to describe precisely the variables of a formula φ which are not in the scope of the quantifiers.

1.2.1. Definition The set $Fv(\varphi)$ of variables which have free occurrences in a formula φ of L is introduced inductively by the complexity of φ in the following way:

1° If $\varphi \in At_L$, then $Fv(\varphi)$ is the set of variables which occur in φ.

2° $Fv(\neg\varphi) = Fv(\varphi)$.

3° $Fv(\varphi\wedge\psi) = Fv(\varphi\vee\psi) = Fv(\varphi\rightarrow\psi) = Fv(\varphi\leftrightarrow\psi) = Fv(\varphi) \cup Fv(\psi)$.

4° $Fv(\exists x\varphi) = Fv(\forall x\varphi) = Fv(\varphi)\setminus\{x\}$.

The elements of the set $Fv(\varphi)$ are called free variables of the formula φ, while the other variables which occur in φ are called bound. For example, if $\varphi = (\neg x \equiv 0 \rightarrow \exists y(x\cdot y \equiv 1))$, then $Fv(\varphi) = \{x\}$, so x is a free variable of φ, and y is a bound variable of φ.

If $\varphi \in For_L$, then the notation $\varphi(x_0, x_1, \ldots, x_n)$, or $\varphi x_0 x_1 \ldots x_n$ is used to denote that free variables of φ are among the variables $x_0, x_1, \ldots, x_n$.

Formulas φ which do not contain free variables, i.e. $Fv(\varphi) = \emptyset$, are called sentences. The formulas

$$0 \equiv 1, \quad \forall x(\neg x \equiv 0 \rightarrow \exists y(x\cdot y \equiv 1))$$

are examples of sentences of the language $L = \{\cdot, 0, 1\}$, where $\cdot$ is a binary function symbol. The set of all sentences of L is denoted by $Sent_L$.

The cardinal number of For_L is denoted by $\|L\|$, therefore $\|L\| = |For_L|$. It is not difficult to see that for every first-order language L we have

$$\|L\| = \max(|L|, \aleph_0).$$

1.3. Theories

The definition of the notion of a first-order theory is simple:

> ***The theory of a first order language L is any set of sentences of L.***

Therefore, a set T is a theory of L iff $T \subseteq Sent_L$. In this case elements of T are called ***axioms*** of T. The main notion connected with the concept of a theory is the notion of proof in the first-order logic. There are several approaches to formalizing the notion of proof. For example, Gentzen's systems are very useful for the analysis of the proof-theoretical strength of mathematical theories. The emphasis in Gentzen's approach is on deduction rules, as distinct from Hilbert-oriented systems, where the stress is on the axioms. Hilbert style formal systems are more convenient in model theory, so we shall confine our attention to them. Now we shall list the axioms and rules of inference for a first order language L:

1° **Sentential axioms.**

These axioms are derived from propositional tautologies by the simultaneous substitution of propositional letters by formulas of L.

2° **Identity axioms.**

If $\varphi \in For_L$, $t \in Term_L$, $x \in Var$, then $\varphi(t/x)$ denotes the formula obtained from φ by substituting the term t for each free occurrence of x in φ. Sometimes, we shall use the abridged form $\varphi(t)$ or φt, instead of $\varphi(t/x)$. Now we shall list the identity axioms:

$x=x$

$x_1=y_1 \wedge \ldots \wedge x_n=y_n \rightarrow t(x_1,\ldots,x_n)=t(y_1,\ldots,y_n)$, $n \in \omega$, $t \in Term_L$.

$x_1=y_1 \wedge \ldots \wedge x_n=y_n \rightarrow (\varphi x_1 \ldots x_n \leftrightarrow \varphi y_1 \ldots y_n)$, $\varphi \in At_L$.

3° **Quantifier axioms**

$\forall x \varphi x \rightarrow \varphi t$, $\varphi \in For_L$, $t \in Term_L$, $x \in Var$.

$\varphi t \rightarrow \exists x \varphi x$,

where φt is obtained from φx by freely substituting each free occurrence of x in φx by the term t.

Rules of inferences:

Let φ and ψ be formulas of L.

1° **Modus Ponens:** $\dfrac{\varphi, \quad \varphi \rightarrow \psi}{\psi}$

2° **Generalization rules:** $\dfrac{\varphi \rightarrow \psi}{\varphi \rightarrow \forall x \psi}$ provided x does not occur free in φ

$\dfrac{\psi \rightarrow \varphi}{\exists x \psi x \rightarrow \varphi}$ provided x does not occur free in φ

A *proof* of a sentence φ in a theory T of a language L is every sequence $\psi_1, \psi_2, \ldots, \psi_n$ of formulas of the language L such that $\varphi = \psi_n$, and each formula ψ_i, $i\leq n$, is a logical axiom, or an axiom of T, or it is derived by inference rules from preceding members of the sequence. If there exists a proof of φ in T, then φ is called a *theorem* of T, and in this case we use the notation $T\vdash\varphi$. The relation $\vdash$ between theories and formulas of a language L is the *provability relation*. If $T=\emptyset$, then we simply write $\vdash\varphi$ instead of $\emptyset\vdash\varphi$, and φ is called a theorem of the first-order predicate calculus. If φ is not a theorem of T, then we shall write $\neg T\vdash\varphi$ for short.

Formulas of the form $\varphi\wedge\neg\varphi$ are called *contradictions*. A theory T is *consistent* if there is no contradiction ψ such that $T\vdash\psi$. Another important property which theories may have is *completeness*. A theory T of a language L is *complete* if for each sentence φ of L either $T\vdash\varphi$ or $T\vdash\neg\varphi$. Finally, T is *deductively closed* if T contains all its theorems.

There is group of first-order notions which are related to effective computability. We shall suppose that the reader has some basic ideas of effective computability and arithmetical coding. So, if $\varphi\in For_L$, then $\ulcorner\varphi\urcorner$ denotes the code of formula φ. A similar notation is applied to other syntactical objects (terms, elements of L, etc.).

A first order language L is recursive, if the set $\ulcorner L\urcorner=\{\ulcorner s\urcorner : s\in L\}$ is recursive. Similarly, L is recursively enumerable if $\ulcorner L\urcorner$ is a recursively enumerable set. A theory T of the language L is finitely axiomatizable, if T is a finite set of axioms. A generalization of this notion is the concept of an axiomatic theory. A theory T is *axiomatic* or *recursive* if T i.e. $\{\ulcorner\varphi\urcorner : \varphi\in T\}$ is a recursive set of sentences. The definitions of notions introduced in this way can be broadened. Namely, two theories T and S of the same language L are equivalent, if they have the same theorems. Then a theory T is considered to be also finitely axiomatizable (axiomatic), if there is a theory S equivalent to T which has a finite set of axioms. It is interesting that the assumption of recursive enumerability does not bring a generalization, as the following theorem shows.

1.3.1. Theorem (Craig's trick). Suppose T is a theory of a language L with a recursively enumerable set of axioms. Then there is a recursive theory S of the language L equivalent to T.

Proof Since T is recursively enumerable, there exists a map $\tau:\omega\longrightarrow Sent_L$ such that $T=\{\tau_n:\ n\in\omega\}$, and $f:n\longmapsto\ulcorner\tau_n\urcorner$ is a recursive function. Let $\psi:\omega\longrightarrow Sent_L$ be defined by $\psi_n=\tau_0\wedge\tau_1\wedge\ldots\wedge\tau_n$, $n\in\omega$, and $S=\{\psi_n:\ n\in\omega\}$. Then T and S have the same theorems, i.e. T and S are equivalent theories. Furthermore, the mapping $g:n\longmapsto\ulcorner\psi_n\urcorner$ is also a recursive function, because we may take, for example,

$$\ulcorner\psi_n\urcorner=2^{\ulcorner\tau_0\urcorner}3^{\ulcorner\tau_1\urcorner}\ldots p_n^{\ulcorner\tau_n\urcorner}\ ,$$

where p_n is the n-th prime. Also, g is a monotonously increasing function, since $n\leq m$ obviously implies $\ulcorner\psi_n\urcorner\leq\ulcorner\psi_m\urcorner$. Yet, from elementary recursion theory, it is well known that the set of all values of a monotonous increasing recursive function is a recursive set, therefore,

$$\ulcorner S\urcorner=\{g_n:\ n\in\omega\}=\{\ulcorner\psi_n\urcorner:\ n\in\omega\}$$

is a recursive set.

A first-order theory T is ***decidable***, if the set of all the theorems of T is decidable (i.e. recursive) set, otherwise T is ***undecidable***. The most interesting mathematical theories are undecidable. However, the following proposition gives a test of decidability for certain theories.

1.3.2. Theorem Suppose T is an axiomatic and complete theory of a recursive language L. Then T is decidable.

Proof Let T' be the set of all the theorems of theory T. Since T is complete, for each $\varphi\in Sent_L$ we have $\varphi\in T'$ or $\neg\varphi\in T'$. If for some sentence φ it holds that $\varphi,\neg\varphi\in T'$, then $T'=Sent_L$, and since $Sent_L$ is a recursive set, it follows that T' is recursive as well.

Suppose the second, more interesting case holds, i.e. that T is a consistent theory. Since T is recursive, the set (of all the codes) of proofs may be effectively listed. By the completeness of T, for each sentence φ of L either φ or $\neg\varphi$ should occur as the last member of a proof in the list. In the first case, φ is a theorem of T, and in the second case, φ is not a theorem of T by the consistency of T. The property of T just described, defines an algorithm for decidability of $T\vdash\varphi$, where $\varphi\in Sent_L$:

Generate all the proofs of theory T, and look at the end of each proof, until one of the formulas φ, $\neg\varphi$ appears. If φ occurs then $T\vdash \varphi$; otherwise $\sim T\vdash \varphi$.

We know that the search will stop with this algorithm, since either $T\vdash \varphi$ or $T\vdash \neg\varphi$.

Now we shall list several elementary, but important, theorems from logic without proofs.

1.3.3. Deduction Theorem Suppose T is a theory of a language L and $T\vdash \varphi$ where $\varphi \in For_L$. Then, there are sentences $\theta_0, \theta_1, \ldots, \theta_n \in T$ such that

$$\vdash \theta_0 \wedge \theta_1 \wedge \ldots \wedge \theta_n \rightarrow \varphi.$$

As a consequence of this theorem we have that a first order theory T is consistent iff every finite subset of T is consistent.

1.3.4. Lemma on the New Constant Let T be a theory of a language L, and assume c is a constant symbol which does not belong to L. Then for every formula $\varphi(x)$ of L we have: if $T\vdash \varphi(c)$, then $T\vdash \forall x\varphi(x)$.

The proof of this lemma is very easy: if in the proof of $\varphi(c)$ from T, the constant symbol c is replaced by a variable y, which does not occur in that proof, then we shall obtain a proof of $\varphi(y)$ from T, and by the inference rule of generalization, the lemma follows at once.

A formula φ of a first order language L is in a ***prenex normal form***, if φ is of the form $Q_1 y_1 Q_2 y_2 \ldots Q_n y_n \psi$, where ψ is a formula without quantifiers, and $Q_1, Q_2, \ldots, Q_n$ are some of the quantifiers $\forall$, $\exists$. In this case the formula ψ is called a ***matrix***.

1.3.5. Prenex Normal Form Theorem For every formula φ of a first order language L, there exists a formula ψ of L in a prenex normal form, such that $\vdash \varphi \leftrightarrow \psi$.

Another important notion is related to the last theorem. This is the so-called proof-theoretical hierarchy of formulas of a language L.

1.3.6. Definition Let L be a first order language L. Then:

$\Sigma_0{}^0 = \Pi_0{}^0 = \{\varphi \in For_L : \varphi$ does not contain quantifiers$\}$,

$\Sigma_{n+1}{}^0 = \{\exists x_0 x_1 \ldots x_k \varphi : k \in \omega,\ \varphi \in \Pi_n{}^0\}$,

$\Pi_{n+1}{}^0 = \{\forall x_0 x_1 \ldots x_k \varphi : k \in \omega,\ \varphi \in \Sigma_n{}^0\}$.

If $\varphi \in \Sigma_n{}^0$, then φ is a $\Sigma_n{}^0$-formula, and if $\varphi \in \Pi_n{}^0$, then φ is a $\Pi_n{}^0$-formula. If φ is a $\Sigma_1{}^0$-formula, then φ is also called an existentional formula, while if φ is a $\Pi_1{}^0$-formula, then φ is called a universal formula. The sequences $\Sigma_n{}^0$ and $\Pi_n{}^0$ of formulas of L define the ***proof-theore tical hierarchy*** of formulas of L. By Theorem 1.3.5 every formula φ of L is equivalent to a formula ψ, such that either $\psi \in \Sigma_n{}^0$ or $\Pi_n{}^0$. If $\psi \in \Sigma_n{}^0$, then φ is also called a $\Sigma_n{}^0$-formula, and analogously, we shall define the $\Pi_n{}^0$- formulas. If φ is a formula of L and for some $n \in \omega$ there is a $\psi \in \Sigma_n{}^0$ and a $\theta \in \Pi_n{}^0$ both equivalent to φ, then φ is called a $\Delta_n{}^0$-formula. The main properties of the proof-theoretical hierarchy are described in the following proposition.

1.3.7. Theorem

$$\Sigma_0{}^0 = \Pi_0{}^0 \subseteq \Delta_1{}^0 \begin{array}{c} \subsetneq \; \Sigma_1{}^0 \; \subsetneq \\ \\ \subsetneq \; \Pi_1{}^0 \; \subsetneq \end{array} \Delta_2{}^0 \begin{array}{c} \subsetneq \; \Sigma_2{}^0 \; \subsetneq \\ \\ \subsetneq \; \Pi_2{}^0 \; \subsetneq \end{array} \Delta_3{}^0 \; \ldots\ .$$

1.4. Examples of theories

In this section we shall give several examples of first-order theories. Most examples are from working mathematics, and we shall consider some cases in greater detail. For every example, we shall exhibit explicitly the corresponding language L in which the axioms of the theory are written down.

1.4.1. Example Pure predicate calculus with identity, J_0. For this theory we have: $L=\emptyset$, $T=\emptyset$.

Therefore, the theorems of theory T are exactly the theorems of the first-order predicate calculus which contain logical symbols only. Here are several interesting examples of sentences which can be written down in L:

$\sigma_1 = \exists x_1 \forall x (x \equiv x_1)$,

$\sigma_2 = \exists x_1 x_2 (\neg(x_1 \equiv x_2) \wedge \forall x (x \equiv x_1 \vee x \equiv x_2))$

$\vdots$

$$\sigma_n = \exists x_1 \ldots x_n \Big(\big(\bigwedge_{1 \leq i < j \leq n} \neg(x_i \equiv x_j) \big) \wedge \forall x \big(\bigvee_{i \leq n} x = x_i \big) \Big)$$

$\tau_1 = \exists x_1 (x_1 \equiv x_1)$

$\tau_2 = \exists x_1 x_2 \neg(x_1 \equiv x_2)$

$\vdots$

$$\tau_n = \exists x_1 \ldots x_n \big(\bigwedge_{1 \leq i < j \leq n} \neg(x_i \equiv x_j) \big)$$

We see that σ_n says " there are exactly n elements", and τ_n says "there are at least n elements". Observe that

$\vdash \sigma_n \leftrightarrow (\tau_n \wedge \neg\tau_{n+1})$, $\qquad \vdash \tau_n \leftrightarrow (\neg\sigma_1 \wedge \ldots \wedge \neg\sigma_{n-1})$.

In the following examples we shall often write open formulas instead of their universal closures.

1.4.2. Example The theory of linear ordering, LO. In this case we have: $L_{LO} = \{\leq\}$, $\leq$ is a binary relation symbol. Axioms of T are:

LO.1.	$x \leq x$	reflexivity,
LO.2.	$x \leq y \wedge y \leq z \to x \leq z$	transitivity,
LO.3.	$x \leq y \wedge y \leq x \to x \equiv y$	antisymmetricity,
LO.4.	$x \leq y \vee y \leq x$	linearity.

A theory PO whose axioms are LO.1-3. is called a theory of partial ordering. The binary relation symbol $<$ is introduced by the definition axiom: $x < y \leftrightarrow x \leq y \wedge \neg x \equiv y$.

1.4.3. Example The theory of dense linear ordering without endpoints, DLO. The language of this theory is the same as in the case of LO, and the axioms are the axioms of LO plus the following sentences:

$\forall x\exists y(x<y)$, $\forall x\exists y(y<x)$,
$\forall xy\exists z(x<y \rightarrow x<z \wedge z<y)$, $\exists xy\neg(x\equiv y)$.

It is not difficult to see that for each $n\in\omega\backslash\{0\}$, DLO$\vdash \tau_n$, where τ_n is the sentence from Example 1.4.1.

1.4.4. Example The theory of Abelian groups, Ab. In this case we have: $Rel_{AB}=\emptyset$, $Fnc_L=\{+,-\}$, where + is a binary function symbol, and - is a unary function symbol. Further, $Const_L=\{0\}$. The axioms of Ab are the following formulas:

Ab.1.	$(x+y)+z\equiv x+(y+z)$	the associative identity,
Ab.2.	$x+y\equiv y+x$	the commutative identity,
Ab.3.	$x+0\equiv x$	the identity of the neutral element,
Ab.4.	$x+(-x)\equiv 0$	the identity of the inverse element.

It is easy to prove by induction on the complexity of terms: If $t\in Term_L$, then there is a $k\in\omega$ and integers $m_1,\dots,m_k$ such that
Ab$\vdash t\equiv m_1x_1+\dots+m_kx_k$, where $x_1,\dots,x_k$ are variables.

1.4.5. Example Field theory, F. The language of this theory is the language of Abelian groups plus some additional symbols, i.e. $L_F=L_{AB}\cup\{\cdot,1\}$ where $\cdot$ is a binary function symbol, and 1 is a constant symbol. Axioms of F are those of Ab plus the following sentences:

$(x\cdot y)\cdot z\equiv x\cdot(y\cdot z)$, $x\cdot y\equiv y\cdot x$, $x\cdot 1\equiv x$,
$\neg(x\equiv 0)\rightarrow\exists y(x\cdot y\equiv 1)$, $x\cdot(y+z)\equiv(x\cdot y)+(x\cdot z)$.
$\neg(0\equiv 1)$

It is possible to introduce a new function symbol $^{-1}$ in the theory F by the defining axiom: $\forall xy(\neg(x\equiv 0)\rightarrow(x\cdot y\equiv 1\leftrightarrow y\equiv x^{-1}))$. Then F proves:
$\forall x(\neg(x\equiv 0)\rightarrow x\cdot x^{-1}\equiv 1)$.

1.4.6. Example The theory of ordered fields, FO. The language of this theory is $L_{FO}=L_{LO}\cup L_F$, and the axioms are the axioms of theory F plus the following formulas:

$x\leq y\rightarrow(x+z\leq y+z)$, $x\leq y\wedge 0<z\rightarrow x\cdot z\leq y\cdot z$.

We note that the formula

$x_1^2 + \ldots + x_n^2 = 0 \rightarrow x_1 = 0 \wedge \ldots \wedge x_n = 0$

is a theorem of the theory FO.

1.4.7. Example The theory of Boolean algebras, BA. The language of this theory is $L_{BA} = \{+, \cdot, ', \leq, 0, 1\}$, where + and · are binary function symbols, ' is a unary function symbol, ≤ is a binary relation symbol, and 0, 1 are constant symbols. The axioms of BA are:

BA.1,2.	$(x+y)+z \equiv x+(y+z)$,	$(x \cdot y) \cdot z \equiv x \cdot (y \cdot z)$,
BA.3,4.	$x+y \equiv y+x$,	$x \cdot y \equiv y \cdot x$,
BA.5,6.	$x+0 \equiv x$,	$x \cdot 1 \equiv x$,
BA.7,8.	$x+x' \equiv 1$,	$x \cdot x' \equiv 0$,
BA.9.	$\neg(0 \equiv 1)$,	
BA.10.	$x \leq y \leftrightarrow x \equiv x \cdot y$.	

It is easy to show that in BA we have

1° The relation symbol < satisfies the axioms of PO. With respect to this ordering, the following holds:

$\sup\{x_1, \ldots, x_n\} \equiv \Sigma_{i<n} x_i$, $\inf\{x_1, \ldots, x_n\} \equiv \Pi_{i<n} x_i$.

2° For each $t \in Term_L$,

$$BA \vdash t(x_1, \ldots, x_n) \equiv \sum_{\alpha \in 2^n} t(\alpha_1, \ldots, \alpha_n) x_1^{\alpha_1} \ldots x_n^{\alpha_n},$$

where $2^n = \{\alpha : \alpha : n \longrightarrow 2\}$, $\alpha_i = \alpha(i+1)$, $0 \leq i < n$, and $x^0 = x'$, $x^1 = x$. This fact is proved by induction on the complexity of terms. Instead of + and · in the sequel we shall use the signs ∧ and ∨.

1.4.8. Example Peano arithmetic, PA. This theory has the same language as theory BA, i.e. $L_{PA} = L_{BA}$. Axioms of PA are the following formulas:

P.1.	$\neg(x' \equiv 0)$,	P.6.	$x \cdot y' \equiv (x \cdot y)+x$,
P.2.	$x' \equiv y' \rightarrow x \equiv y$,	P.7.	$\neg(x<0)$,
P.3.	$x+0 \equiv x$,	P.8.	$x<y' \rightarrow x<y \vee x \equiv y$,
P.4.	$x+y' \equiv (x+y)'$,	P.9.	$x<y \vee x \equiv y \vee y<x$,
P.5.	$x \cdot 0 \equiv 0$,	P.10.	$1 \equiv 0'$.

(I) *Induction scheme*: Let $\varphi x y_1 \ldots y_n$ be a formula of L. Then the universal closure of

$\varphi 0 y_1 \ldots y_n \wedge \forall x(\varphi x y_1 \ldots y_n \rightarrow \varphi x' y_1 \ldots y_n) \rightarrow \forall x \varphi x y_1 \ldots y_n$

is an axiom of PA.

This theory is also called the formal arithmetic. It contains several interesting subtheories. At this moment we shall mention two of them.

The first theory is P^-. This theory consists of the axioms P1-P10. Therefore, PA= P^- + (I).

Another example is the Presburger arithmetic. It consists of those axioms of PA which are expressed in the language $\{+, ', 0\}$, i.e. in language L_{PA} without the symbols $\cdot$, $\leq$, 1.

All the examples we have listed are axiomatic theories, i.e. with recursive sets of axioms. Also, all except the last example, are finitely axiomatizable theories. Theories J_0, LO, DLO, Ab, BA are decidable theories, while F, FO and PA are not. The Pressburger arithmetic is also decidable and a complete theory.

1.5. Models

We have dealt in the previous sections mainly with syntactical notions. On the other hand, the most important concept in model theory is the idea of an operational-relational structure, or simply a ***model*** of a first-order language L. Customary mathematical structures such as groups, fields, ordered fields, and the structure of natural numbers, are examples of models. When studying the properties of models, a distinctively important role is played by the concept of formal language used to make precise the set of symbols and rules used to build formulas and sentences. The main reason for introducing formulas is to describe properties of models. Therefore, it is not astonishing that some properties of models are often consequences of the structure of sentences or classes of sentences. The proofs of such features of models are often called model-theoretical proofs.

By the methods of model theory many open mathematical problems have been solved. One such famous problem is the consistent foundation of Leibnitz Analysis, a problem which stood open for 300 years. Abraham Robinson gave a simple but ingenious solution, and thanks to him there is now a whole new methodology which is equally well applied to topology, algebra, probability theory, and practically to all mathematical fields where infinite objects appear.

1.5.1. Definition A model is every structure $\mathbf{A}=(A,\mathcal{R},\mathcal{F},\mathcal{C})$ where A is a nonempty set (the domain of **A**), $\mathcal{R}$ is a set of relations over A, $\mathcal{F}$ is a family of operations over A, and $\mathcal{C}$ is a set of constants of A.

By this definition of model we have:

If $R\in\mathcal{R}$, then there is an $n\in\omega$, such that $R\subseteq A^n$, i.e. R is a relation over A of length n. The length of R is denoted by ar(R).

If $F\in\mathcal{F}$, then there is an $n\in\omega$ such that $F:A^n\longrightarrow A$, i.e. F is an n-ary operation over A. To denote the length of F we write ar(F)= n.

Finally, $\mathcal{C}\subseteq A$.

If $\mathcal{R}$, $\mathcal{F}$, $\mathcal{C}$ are finite sets, for example $\mathcal{R}=\{R_0,\ldots,R_n\}$, $\mathcal{F}=\{F_0,\ldots,F_n\}$, $\mathcal{C}=\{a_0,\ldots,a_k\}$, then **A** may be denoted as
$\mathbf{A}=(A,F_0,\ldots,F_n,R_0,\ldots,R_m,a_0,\ldots,a_k)$.

If these sets are indexed, i.e. $\mathcal{R}=\langle R_j: j\in J\rangle$, $\mathcal{F}=\langle F_i: i\in I\rangle$, $\mathcal{C}=\langle a_k: k\in K\rangle$, we can also use the notation:

$\mathbf{A}=(A,F_i,R_j,a_k)_{i\in I,\, j\in J,\, k\in K}$.

1.5.2. Example 1° The ordered field of real numbers:
$\mathbf{R}=(R,+,\cdot,-,\leq,0,1)$.
Here, $\mathcal{F}=\{+,\cdot,-\}$, ar(+)= ar(·) = 2, ar(−)= 1, and
$\mathcal{R}=\{\leq\}$, ar(≤)= 2, and $\mathcal{C}=\{0,1\}$.

2° The structure of natural numbers: $\mathbf{N}=(N,+,\cdot,{}',\leq,0)$.

3° The field of all subsets of a set X: $\mathbf{P}(X)=(P(X),\cup,\cap,{}^c,\emptyset,X)$
where $P(X)=\{Y: Y\subseteq X\}$, and for $Y\in P(X)$, $Y^c=X\setminus Y$.

Models are interpretations of first-order languages. To see that, let L be a first-order language and A a non-empty set. An interpretation of L into the domain A is every mapping I with the domain L, and values determined as follows:

If $R \in Rel_L$, then $I(R)$ is a relation of A of length ar(R).
If $F \in Fcn_L$, then $I(F)$ is an operation of A of length ar(F).
If $c \in Const_L$ then $I(c) \in A$.

Therefore, every interpretation I of a language L into a domain A determines a unique model $\mathbf{A} = (A, I(Rel_L), I(Fcn_L), I(Const_L))$. The model so introduced is written simply as $\mathbf{A} = (A, I)$, or $\mathbf{A} = (A, s^{\mathbf{A}})_{s \in L}$, where for $s \in L$, $s^{\mathbf{A}} = I(s)$.

We see that in Example 1.5.2. **R** is a model of the language of ordered fields, while **N** is a model of the language of Peano arithmetic, and finally **P**(X) is a model of the language of the theory of Boolean algebras.

From now on by the letters **A**, **B**, **C**, ... we shall denote models, and by A, B, C, ... their domains, respectively. If L is a language and **A** a model of L, then $s \in L$ and $s^{\mathbf{A}}$ denote objects of a quite different nature. However, if the context allows, we shall use the same sign to denote a symbol of L and its interpretation in **A**. This means the superscript $^{\mathbf{A}}$ will be often omitted from $s^{\mathbf{A}}$. The circumstance under which s appears will determine if $s \in L$ or if s is in fact an interpretation of a symbol of L. Very often a structure **A** is introduced without explicit mention of the related language. But, from the definition of structure **A** it will be clear what is the corresponding language, and in that case we shall denote the language in question by $L_{\mathbf{A}}$. A similar situation may appear for a theory T; the corresponding language will be denoted by L_T.

Assume $L \subseteq L'$ are first-order languages, and let **A** be a model of L'. Omitting $s^{\mathbf{A}}$ for $s \in L' \setminus L$ from the model **A**, we obtain a new model **B** of L with domain B= A. In this case, **A** is called an ***expansion*** of model **B**, while **B** is called a *reduct* of model **A**. If I and I' are interpretations which determine **B** and **A**, respectively, we see that $I = I' | L$.

1.5.3. Definition Let **A** and **B** be models of a language L. Then **B** is a *submodel* of **A**, if and only if $B \subseteq A$ and
if $R \in Rel_L$ is of length k, then $R^{\mathbf{B}} = R^{\mathbf{A}} \cap B^k$,
if $F \in Fcn_L$ is of length k, then $F^{\mathbf{B}} = F^{\mathbf{A}} | B^k$,
if $c \in Const_L$, then $c^{\mathbf{B}} = c^{\mathbf{A}}$.

The fact that B is a submodel of A, we shall denote by $B \subseteq A$. For example $(N,+,\cdot,\leq,0,1) \leq (R,+,\cdot,\leq,0,1)$, but for $Y \subseteq X$, $Y \neq X$, it is not true that $(P(Y),\cup,\cap,^c,\emptyset,Y) \leq (P(X),\cup,\cap,^c,\emptyset,X)$.

Algebras are also special types of models; they are models of languages L, such that $Rel_L = \emptyset$. As in the case of algebras, it is possible to introduce the notions of homomorphism and isomorphism for arbitrary models.

1.5.4. Definition Let A and B be models of a language L, and $f:A \longrightarrow B$. The map f is a ***homomorphism*** from A into B, which is denoted by $f:A \longrightarrow B$, if and only if:

1° For $R \in Rel_L$, $ar(R) = k$, for all $a_1,\dots,a_k \in A$, $R^A(a_1,\dots,a_k)$ implies $R^B(fa_1,\dots,fa_k)$; in this case we say that f is *concurrent* with relations R^A, R^B.

2° For $F \in Fcn_L$ of length k, for all $a_1,\dots,a_k \in A$, $f(F^A(a_1,\dots,a_k)) = F^B(fa_1,\dots,fa_k)$; in this case we say that f is concurrent with operations F^A, F^B.

3° For $c \in Const_L$, $f(c^A) = c^B$.

Similarly to the case of algebraic structures, we have the following classification of homomorphisms:

f is an ***embedding***, if f is 1-1.

f is an ***onto-homomorphism*** (or epimorphism), if f is onto.

f is a ***strong homomorphism***, if for every k-ary relation symbol R of L, and $a_1,\dots,a_k \in A$, $R^A(a_1,\dots,a_k)$ holds iff $R^B(fa_1,\dots,fa_k)$ holds.

f is an ***isomorphism***, if f is 1-1 and a strong epimorphism.

f is an ***automorphism***, if f is an isomorphism and $A = B$.

Suppose $f:A \longrightarrow B$ is a homomorphism. Then we shall use the following conventions:

If f is an embedding, we shall say that A is embedded into B.

If f is an onto map, we shall say that B is a homomorphic image of A, and we shall occasionally note this fact by $B = f(A)$.

If f is an isomorphism between models A and B, then we shall write $f:A \approx B$. The notation $A \approx B$ is used to indicate that there is an isomorphism $f:A\approx B$, and in this case we shall say that A and B are isomorphic.

The set of all the automorphisms of a model **A** is denoted by Aut **A**. It is not difficult to see that Aut **A** is a group under function multiplication; this group will be denoted by **Aut A**. The set of all automorphisms of a countable model has the following interesting property.

1.5.5. **Theorem** (Kueker) Let **A** be a countable model. Then
$|\text{Aut } A| > \aleph_0$ implies $|\text{Aut } A| = 2^{\aleph_0}$.

Proof First let us introduce some notations. A finite permutation of the set A is every permutation of a finite subset of A. Further, let G be a subgroup of **Aut A**. A finite permutation p of A is extendible (with respect to G), if there exists a $g\in G$ such that $p\subseteq g$. Finally, G is a complete group, if it satisfies the following condition:

> If $p_0 \subseteq p_1 \subseteq p_2 \subseteq \ldots$ is a chain of extendible finite permutations, and $f = \bigcup_n p_n$ is a permutation of A, then $f\in G$.

Now we shall proceed to the proof of the theorem.

Claim 1. **Aut A** is a complete group.

Proof of **Claim 1** Let $f = \bigcup_n p_n$ where $p_0 \subseteq p_1 \subseteq \ldots$ is a chain of extendable finite permutations of A, and suppose f is a permutation of A. We shall show that f is concurrent with operations of model **A**. So, let F be an n-ary operation of model **A** and choose $a_1, \ldots, a_n \in A$. As $A = \text{dom } f = \bigcup_n \text{dom } p_n$, there is an $m\in\omega$ such that $a_1, \ldots, a_n, F(a_1, \ldots, a_n) \in \text{dom } p_m$. Further, p_m is extendible, hence there exists a $g\in \text{Aut } A$ such that $p_m \subseteq g$. Thus

$$f(F(a_1, \ldots, a_n)) = p_m(F(a_1, \ldots, a_n)) = g(F(a_1, \ldots, a_n)) =$$
$$F(ga_1, \ldots, ga_n) = F(p_m a_1, \ldots, p_m a_n) = F(fa_1, \ldots, fa_n).$$

i.e. $f\in \text{Aut } A$. In a similar way one can show that f is concurrent with the relations of model **A**.

Claim 2. If G is an uncountable and complete group of permutations of a countable set A, then $|G| = 2^{\aleph_0}$.

Proof of Claim 2 First we shall show

(1) For every finite sequence $a_1, \ldots, a_n$ of elements of A, there is an element $g \in G \setminus \{i_A\}$ such that $g(a_i) = a_i$, $1 \le i \le n$.

To see this, remark that the set $\{(fa_1, \ldots, fa_n) : f \in G\}$ is countable (since A^n is countable), therefore since G is uncountable, there are $f, h \in G$, so that $f \ne h$, and

$(fa_1, \ldots, fa_n) = (ha_1, \ldots, ha_n)$.

Then, $g = f^{-1}h$ satisfies condition (1).

Further, we shall show

(2) If p is a finite extendible permutation of A, then there are different finite extendible permutations q, r of A such that $p \subseteq q, r$, and $p \ne q, r$.

In order to see (2), let us suppose $p(a_i) = b_i$, $1 \le i \le n$. By (1), there is a $g \in G$, so that $g(a_i) = a_i$, $1 \le i \le n$, and for some $a \in A$, $g(a) \ne a$. Since p is an extendable permutation, there is $h \in G$, such that $p \subseteq h$. Let us define finite permutations q, r as follows

$q(a_i) = b_i$, $1 \le i \le n$, $q(a) = ha$, $r(a_i) = b_i$, $1 \le i \le n$, $r(a) = hga$.

Then q, r are obviously finite permutations of set A, $p \subseteq q, r$, and q, r are extendable, since $q \subseteq h$, $r \subseteq hg$. Therefore (2) holds.

Finally, we shall prove:

(3) If p is a finite extendible permutation of A and $a \in A \setminus \mathrm{dom}(p)$ then there is a finite extendable permutation q such that $p \subseteq q$ and $a \in \mathrm{dom}(q)$ ∪ codom q

Really, if p is extendible and p is given by $p(a_i) = b_i$, $1 \le i \le n$, then there is $g \in G$ such that $p \subseteq g$. Then $q = p \cup \{(a, ga), (g^{-1}a, a)\}$ satisfies the required conditions.

Now, let $A = \{a_1, a_2, \ldots, \}$. Since $\emptyset$ is a permutation of the empty set, and $\emptyset$ is extendible (every $g \in G$ extends $\emptyset$), by (2) and (3) we can build an infinite binary tree T which satisfies the following conditions:

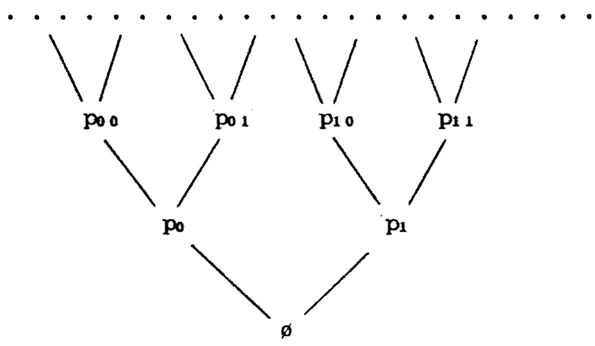
1° Every member of T is a finite extendible permutation of A.

2° The ordering of T is the inclusion relation.

3° If $p_\alpha \in T$, then $p_{\alpha 0} \neq p_{\alpha 1}$, $\alpha \in 2^n$.

4° If $\alpha \in 2^n$, then $a_n \in \mathrm{dom}(p_\alpha) \cup \mathrm{codom}(p_\alpha)$

Hence, tree T looks as follows:



Therefore, if τ is a branch of T and $f_\tau = \bigcup_{p \in \tau} p$, then f_τ is a per- by 4°, $A = \mathrm{dom} f_\tau$, $A = \mathrm{codom} f_\tau$. mutation of set A. Since G is a complete group, it follows that $f_\tau \in G$. On the other hand, tree T has $2^{\aleph_0}$ branches, hence, $\{f_\tau : \tau \text{ is a branch of } T\}$ is of the cardinality $2^{\aleph_0}$, i.e. $|G| = 2^{\aleph_0}$.

Then, by claims 1 and 2, the theorem follows.

We have employed a method in the proof of the theorem which is often used in model theory: First build a binary tree, and then the problem of counting the members of a given set (in this case Aut A) is reduced to counting the branches of that tree. We shall later see other examples of a similar nature.

1.6. Satisfaction relation

When introducing syntactical objects of PR^1, as terms, formulas and sentences are, we had in mind certain meanings related to these notions. Tarski's definition of the satisfaction relation $\models$ determines these ideas precisely. The introduction of this relation also solves the problem of mathematical truth. Namely, a sentence φ will be true in a structure **A**, if $A \models \varphi$. Finally, this formalization of mathematical truth enables a mathematical analysis of metamathematical notions.

We shall first define the values of the terms in models. Let A be a model of a first-order language L. A valuation or an assignment of the domain A is every map $\mu: Var \longrightarrow A$. Therefore, valuations assign values to variables. The value of a term $u(x_0, \ldots, x_n) \in Term_L$ in model A, denoted by $u^A[\mu]$, is defined by induction on the complexity of terms, assuming that $\mu(v_i) = a_i$, $i \in \omega$.

If co(u)= 0, then we can distinguish two cases:

1° If u is a variable v_i, then $u^A[\mu] = a_i$.

2° If u is a constant symbol c, then $u^A[\mu] = c^A$.

Suppose now co(u)= n+1, and assume that the values of the terms of the complexity $\le n$ are determined. Then there is an $F \in Fcn_L$, $ar(F)=k$, such that $u = F(u_1, \ldots, u_k)$ where $u_1, \ldots, u_k$ are terms of complexity $\le n$. Then, by definition,

$$u^A[\mu] = F^A[u_1{}^A[\mu], \ldots, u_k{}^A[\mu]].$$

Instead of $u^A[\mu]$, it is common to write $u^A[a_1, a_2, \ldots, a_r]$ or $u[a_1, a_2, \ldots, a_r]$, or $u(a_1, a_2, \ldots, a_r)$, if it is clear which model is in question. Here, r is the number of variables which occur in term u.

If A is a model of a language L, an operation F of domain A is ***derived*** if there is a $t(x_1, \ldots, x_n) \in Term_L$ such that for all $a_1, \ldots, a_n \in A$, $F(a_1, \ldots, a_n) = t^A[a_1, \ldots, a_n]$. The following proposition says that homomorphisms of a model remain concurrent with respect to derived operations.

1.6.1. Theorem Let A and B be models of a language L, and $h: A \longrightarrow B$ a homomorphism. Then for every term $u(x_1,\dots,x_n)$ of L and all $a_1,\dots,a_n \in A$ the following holds:

$$h(u^A[a_1,\dots,a_n]) = u^B[ha_1,\dots,ha_n].$$

Proof The proof will be performed by induction on the complexity of terms. So, let $u \in Term_L$, and suppose that the variables $v_0, v_1, \dots$ have the values $a_0, a_1, \dots$ under valuation μ. First assume $co(u)=0$. We have two cases:

1° $u \in Const_L$. Then: $h(u^A[\mu]) = h(u^A) = u^B = u^B[ha_1,\dots,ha_n]$.

2° u is a variable x_i. Then: $h(u^A[\mu]) = h(a_i) = u^B[ha_1,\dots,ha_n]$.

Now suppose the statement is true for some fixed $n \in \omega$, and let $co(u) = n+1$. Then there is an $F \in Fcn_L$ of length k and there are some terms $u_1,\dots,u_k$, such that $u = F(u_1,\dots,u_k)$. Then the terms u_i are of complexity $\le n$ and hence, by the inductive hypothesis, we have

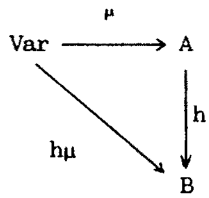
$$\begin{aligned} h(u^A[\mu]) &= h(F^A(u^A[\mu],\dots,u^A[\mu])) \\ &= F^B(hu^A[\mu],\dots,hu^A[\mu]) \\ &= F^B(u^B[h\mu],\dots,u^B[h\mu]) \end{aligned}$$

so the theorem follows by induction.

Note The above theorem can be obviously restated in the following way:

For every valuation $\mu: Var \longrightarrow A$ the displayed diagram commutes, i.e.

$$h(u^A[\mu]) = u^B[h\mu].$$



An algebraic identity of a language L is every formula $u \equiv v$, where $u, v \in Term_L$. We say that an algebra of language L satisfies the identity $u \equiv v$, if and only if for all $a_1,\dots,a_n \in A$, $u^A[a_1,\dots,a_n] = v^A[a_1,\dots,a_n]$.

1.6.2. Corollary Let A and B be algebras of a language L, and assume that B is a homomorphic image of A. Then every identity true in A also holds in B.

Proof Let $h: A \longrightarrow B$ be onto, and suppose identity $u \equiv v$ holds in A. Then, for arbitrary $b_1, \ldots, b_n \in B$, there are $a_1, \ldots, a_n \in A$ such that $ha_1 = b_1$, ..., $ha_1 = b_1$, ..., $ha_n = b_n$, so

$$\begin{aligned} u^B[b_1, \ldots, b_n] &= u^B[ha_1, \ldots, ha_n] \\ &= hu^A[a_1, \ldots, a_n] \\ &= hv^A[a_1, \ldots, a_n] \\ &= v^B[ha_1, \ldots, ha_n] \\ &= v^B[b_1, \ldots, b_n]. \end{aligned}$$

This corollary is an example of a preservation theorem. Namely, it says that some properties are preserved under homomorphisms, and in this case these properties are those which can be described by identities. Some examples are the associativity and the commutativity of algebraic operations. This is probably one of the places where one can see the algebraic nature of model theory.

Now we shall turn to the most important concept of model theory. This is the notion of the ***satisfaction relation***, or the definition of mathematical truth.

1.6.3. Definition Let A be a model of a language L. We define the relation

$$A \models \varphi[\mu]$$

for all formulas φ of L and all valuations $\mu = \langle a_i : i \in \omega \rangle$ of the domain A by induction on the complexity of formulas φ:

If $\varphi = (u \equiv v)$, $u, v \in Term_L$, then $A \models \varphi[\mu]$ iff $u^A[\mu] = v^A[\mu]$.

If $\varphi = R(u_1, \ldots, u_n)$, $R \in Rel_L$, $u_1, \ldots, u_n \in Term_L$, then $A \models \varphi[\mu]$ iff $(u_1^A[\mu], \ldots, u_n^A[\mu]) \in R^A$, i.e. $R^A(u_1^A[\mu], \ldots, u_n^A[\mu])$

If $\varphi = \neg\psi$, then $A \models \varphi[\mu]$ iff not $A \models \psi[\mu]$.

If $\varphi = (\psi \wedge \theta)$, then $A \models \varphi[\mu]$ iff $A \models \psi[\mu]$ and $A \models \theta[\mu]$.

If $\varphi = (\psi \vee \theta)$, then $A \models \varphi[\mu]$ iff $A \models \psi[\mu]$ or $A \models \theta[\mu]$.

If $\varphi = (\psi \rightarrow \theta)$, then $A \models \varphi[\mu]$ iff not $A \models \psi[\mu]$ or $A \models \theta[\mu]$.

If $\varphi = \exists x_i \psi x_0 x_1 \ldots x_n$, $i \le n$, then $A \models \varphi[\mu]$ iff there is an $a \in A$ such that $A \models \psi[a_0, a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_n, \ldots]$.

If $\varphi = \forall x_i \psi x_0 x_1 \ldots x_n$, $i \le n$, then $A \models \varphi[\mu]$ iff for all $a \in A$, $A \models \psi[a_0, a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_n, \ldots]$.

By the definition of the satisfaction relation, we see that the value of $A\models\varphi[\mu]$ depends only on the free variables which occur in φ. A rigorous proof of this fact can be derived by induction on the complexity of formulas. An archetype of this kind of proof is the proof of Theorem 1.6.1. This enables us to introduce the following conventions.

If $\varphi=\varphi(v_0, v_1, \ldots, v_n)$ and $\mu=\langle a_i : i\in\omega\rangle$, then we shall simply write $A\models\varphi(a_0, a_1, \ldots, a_n)$ instead of $A\models\varphi[\mu]$. Sentences do not have free variables, so their values do not depend on the choice of a valuation, i.e. if $\varphi\in Sent_L$ and $A\models\varphi[\mu]$, then for all valuations σ we have $A\models\varphi[\sigma]$. Thus, we shall use the abbreviated form $A\models\varphi$ instead of $A\models\varphi[\mu]$.

The definition of the satisfaction relation permits us to introduce new model-theoretic concepts. One of these is the theory of model A

$ThA=\{\varphi\in Sent_L : A\models\varphi\}$, A is a model of L.

It is easy to see that for each formula φ of L and every valuation either $A\models\varphi[\mu]$ or $A\models\neg\varphi[\mu]$, thus, ThA is a complete theory. For example, the theory of the structure of natural numbers, ThN, is complete, and hence it is called a complete arithmetic. As N is a model of theory PA, it follows that $PA\subseteq ThN$, but by the Gödel's Second Incompleteness Theorem, the set of theorems of PA is a proper subset of ThN. Moreover, ThN is not an axiomatic theory, i.e. it does not have a recursive set of axioms. One of the tasks of model theory is to solve the problem whether a given theory is axiomatic.

Let T be a theory of a language L. A model A of L is a model of theory T, if every axiom of T holds in A, i.e. $T\subseteq ThA$. In such a case, we write $A\models T$. For example, every ordered field, like the ordered fields of rationals and reals, is a model of theory FO. Similarly, every Boolean algebra is a model of theory BA. Every model A of a language L satisfies all the axioms of first-order predicate calculus for L. Rules of inferences (Modus Ponens and Generalization rules) are preserved by the satisfaction relation, i.e. if μ is a valuation of domain A, and $A\models\varphi_1[\mu], \ldots, \varphi_n[\mu]$, where $\varphi_1, \ldots, \varphi_n\in For_L$, and ψ is derived by applications of these rules, then $A\models\psi[\mu]$. Therefore, the following theorem is easily proved by induction on the length of proofs in T.

1.6.4. Soundness Theorem Assume A is a model of a language L and T is a theory of L. If $A \models T$ and $T \vdash \varphi$, where $\varphi \in Sent_L$, then $A \models \varphi$

Two models A and B of a language L are ***elementary equivalent*** if A and B satisfy the same sentences of L, i.e. ThA= ThB. This relation between models is denoted by $A \equiv B$. It is also said that A and B have the same first-order properties. By induction on the complexity of formulas, it is easy to show:

1.6.5. Theorem Let $g:A \approx B$ be an isomorphism of the models A and B of a language L. Then, for every formula $\varphi v_0 \ldots v_n$ of L and every valuation $\mu = \langle a_i : i \in \omega \rangle$ of the domain A, the following holds:

$$A \models \varphi[a_0, \ldots, a_n] \quad \text{if and only if} \quad B \models \varphi[ga_0, \ldots, ga_n].$$

Since the value of a sentence in a model does not depend on the choice of a valuation, we have the following consequence.

1.6.6. Corollary If A and B are isomorphic models of a language L, then $A \equiv B$.

Therefore isomorphisms preserve first-order properties. Embeddings of models which preserve first-order properties are called ***elementary embeddings***. Therefore, an elementary embedding between models A and B of a language L is every map $g:A \longrightarrow B$, such that for all $\varphi \in For_L$, all valuations μ of domain A, it satisfies

$$A \models \varphi[a_0, \ldots, a_n] \quad \text{if and only if} \quad B \models \varphi[ga_0, \ldots, ga_n].$$

In this case we can use the notation $g:A \xrightarrow{\prec} B$. If $A \subseteq B$ and the inclusion map $i_A : A \longrightarrow B$, $i_A : x \longmapsto x$ $(x \in A)$, is elementary, then we can write $A \prec B$. observe that $A \prec B$ implies $A \equiv B$.

A class of $\mathcal{M}$ of models of a language L is ***axiomatic***, if there is a theory T of L such that $\mathcal{M} = \{A : A \models T\}$. For example, the class of all the ordered fields is axiomatic, and so is the class of all Boolean algebras. Later we shall see that, for example, the class of all cyclic groups is not an axiomatic class. The class of all models of a theory T is denoted by $\mathcal{M}(T)$. The crucial theorem of model theory says that for every consistent theory T, $\mathcal{M}(T) \neq \emptyset$.

1.7. Method of new constants

The introduction of new linguistic constants is a dual procedure to the process of interpretations. Namely, to every nonempty set A there corresponds a certain language L_A. If R is a k-ary relation over A, then let **R** be a relation symbol of length k which belongs to L_A. Similarly, if g is an n-ary operation over domain A, we can introduce a function symbol $\mathbf{g} \in L_A$ of arity k. Finally, if $a \in A$ then $\mathbf{a} \in Const_{L(A)}$. The symbols **R**, **g**, **a** are called ***names*** of R, g, a, respectively. We have a natural interpretation of language L_A so defined: If $s \in L_A$, then $s^A = s$. In this way we have built a model $A = (A, R, F, C)$, where R is the set of all relations over A, F is the set of all the operations with domain A, and $C = A$. Of course, it is not always necessary to consider the full expansion of set A. For example, if A is any model of a language L, and $a_1, \ldots, a_n \in A$, then $A' = (A, a_1, \ldots, a_n)$ is a simple expansion of A, and A' is a model of the language $L = L \cup \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$.

The following proposition is interesting for two reasons. The first one relates to the inductive nature of the satisfaction class. Secondly, this proposition shows that the satisfaction relation can be defined only for sentences, if, of course, the starting model is modified.

1.7.1. Theorem Let A be a model of a language L and $\varphi v_0 v_1 \ldots v_n \in For_L$. Then, for all $a_0, a_1, \ldots, a_n \in A$, we have $A \models \varphi[a_0, a_1, \ldots, a_n]$, if and only if $(A, a_0, \ldots, a_n) \models \varphi \mathbf{a}_0 \ldots \mathbf{a}_n$.

Remark that $\varphi \mathbf{a}_0 \ldots \mathbf{a}_n$ is a sentence of $L \cup \{\mathbf{a}_0, \ldots, \mathbf{a}_n\}$.

Proof Let us first prove an auxiliary statement:

(1) If $t v_0 \ldots v_n \in Term_L$ and $A' = (A, a_0, \ldots, a_n)$, then $t^{A'} \mathbf{a}_0 \ldots \mathbf{a}_n = t^A[a_0, \ldots, a_n]$.

This claim is proved by induction on the complexity of term t:

if $co(t) = 0$, then:
if $t = x_i$, then $t^{A'} \mathbf{a}_0 \ldots \mathbf{a}_n = \mathbf{a}_i^{A'} = a_i = t^A[a_0, \ldots, a_n]$,
if $t \in Const_L$, then $t^{A'} \mathbf{a}_0 \ldots \mathbf{a}_n = t^{A'} = t^A = t^A[a_0, \ldots, a_n]$.

If $co(t)= k+1$, then for some $m\in\omega$ and for $f\in Fcn_L$ of length m, there are $t_1,\dots,t_m\in Term_L$ such that $t= f(t_1,\dots,t_m)$, so

$$t^{A'}a_0\dots a_n = f^{A'}(t_1{}^{A'}a_0\dots a_n,\dots,t_m{}^{A'}a_0\dots a_n)$$

(by the inductive hypothesis)

$$= f^A(t_1{}^A[a_0,\dots,a_n],\dots,t_m{}^A[a_0,\dots,a_n])$$

$$= t^A[a_0,\dots,a_n].$$

Therefore, (1) follows by induction.

Now we shall proceed to the proof of the theorem. This proof is also by induction on the complexity of formulas. So let us assume $co(\varphi)=0$, where $\varphi\in For_L$. Then we have two possibilities:

$\varphi= (u\equiv v)$, $u,v\in Term_L$. Then

$$\begin{aligned} A\models \varphi[a_0,\dots,a_n], \quad & \text{iff} \quad u^A[a_0,\dots,a_n]= v^A[a_0,\dots,a_n] \\ & \text{iff} \quad u^{A'}a_0\dots a_n= v^{A'}a_0\dots a_n \\ & \text{iff} \quad A'\models \varphi a_0\dots a_n \ . \end{aligned}$$

$\varphi= R(u_1,\dots,u_m)$, $R\in Rel_L$, $ar(R)= m$, $u_1,\dots,u_m\in Term_L$. Then

$$\begin{aligned} A\models \varphi[a_0,\dots,a_n], \quad & \text{iff} \quad R^A(u_1{}^A[a_0,\dots,a_n],\dots,u_m{}^A[a_0,\dots,a_n]) \\ & \text{iff} \quad R^{A'}(u_1{}^{A'}a_0\dots a_n,\dots,u_m{}^{A'}a_0\dots a_n) \\ & \text{iff} \quad A'\models \varphi a_0\dots a_n \ . \end{aligned}$$

Now, let φ be a formula of complexity $k+1$. Then, we can distinguish the following cases:

$\varphi= (\psi\wedge\theta)$. then the formulas ψ and θ are of complexity $\leq k$, so

$$\begin{aligned} A\models \varphi[a_0,\dots,a_n] \quad & \text{iff} \quad A\models \psi[a_0,\dots,a_n] \text{ and } A\models \theta[a_0,\dots,a_n], \\ & \text{iff} \quad A'\models \psi a_0\dots a_n \text{ and } A'\models \theta a_0\dots a_n \\ & \text{iff} \quad A'\models \varphi a_0\dots a_n \ . \end{aligned}$$

$\varphi= \neg\psi$. Then the formula ψ is of complexity $\leq k$, so

$$\begin{aligned} A\models \varphi[a_0,\dots,a_n] \quad & \text{iff} \quad A\models \psi[a_0,\dots,a_n] \\ & \qquad \text{(using the inductive hypothesis)} \\ & \text{iff} \quad \text{not } A'\models \psi a_0\dots a_n \\ & \text{iff} \quad A'\models \varphi a_0\dots a_n \ . \end{aligned}$$

The proof is similar for other logical connectives.

Let $\varphi = \exists v_i \psi$. Then we may take that $i=0$, $\varphi = \varphi v_1 \ldots v_n$ and $\psi = \psi(v_0, v_1, \ldots, v_n)$. Then

$$\begin{aligned} A \models \varphi[a_1,\ldots,a_n] \quad & \text{iff} \quad \text{for some } b \in A, \ A \models \psi[b,a_1,\ldots,a_n] \\ & \qquad \text{(using the inductive hypothesis)} \\ & \text{iff} \quad \text{for some } b \in A, \ (A',b) \models \psi b a_1 \ldots a_n \\ & \text{iff} \quad \text{for some } b \in A, \ A' \models \theta[b], \\ & \qquad \text{where } \theta x = \psi x a_1 \ldots a_n, \text{ so} \\ & \text{iff} \quad A' \models \exists x \theta x \\ & \text{iff} \quad A' \models \varphi a_0 \ldots a_n. \end{aligned}$$

We shall apply the previous proposition in the following theorem which says that there is no satisfactory model theory for finite structures. The reason is that the relation of elementary equivalence and the isomorphisms of models coincide for finite structures.

1.7.2. Theorem Let A and B be models of a language L. If A is finite and $A \equiv B$, then $A \approx B$.

Proof Let $|A|=n$. By Example 1.4.1, we have $A \models \sigma_n$, therefore $B \models \sigma_n$, i.e. $|B|=n$ as well. Let us now prove the following fact:

(1) If A nd B are finite models and $A \equiv B$, then, for each $a \in A$ there is a $b \in B$ such that $(A,a) \equiv (B,b)$.

Really, let $a \in A$ and suppose $B=\{b_1,\ldots,b_n\}$. Assume there is no $b \in B$ such that $(A,a) \equiv (B,b)$, and choose a constant symbol $c \in L$ (the so-called new constant symbol). Then, for all $i \leq n$ there is a formula $\varphi_i x$ of language L and there is $b_i \in B$ such that $(A,a) \models \varphi_i c$ and $(B,b_i) \models \neg\varphi_i c$, where c is interpreted by a in model (A,a), while in (B,b_i) it is interpreted by b_i. Hence, $(A,a) \models \wedge_{j \leq n} \varphi_j c$, so by Theorem 1.7.1, it follows that $A \models \exists x \wedge_{j \leq n} \varphi_j x$. Since $A \equiv B$, we have $B \models \exists x \wedge_{j \leq n} \varphi_j x$, thus for some $k \leq n$, $B \models (\wedge_{j \leq n} \varphi_j x)[b_k]$. By Theorem 1.7.1, it follows that $(B,b_k) \models \wedge_{j \leq n} \varphi_j b_k$, hence, $(B,b_k) \models \wedge_{j \leq n} \varphi_j c$, if c is interpreted by a_k, and this is a contradiction to the choice of the formula φ_k. This finishes the proof of (1).

By repeated use of (1), we shall find an enumeration $\{a_1,\ldots,a_n\}$ of domain A, so that

(2) $(\mathbf{A}, a_1, \ldots, a_n) \equiv (\mathbf{B}, b_1, \ldots, b_n)$,
where $(\mathbf{A}, a_1, \ldots, a_n)$, $(\mathbf{B}, b_1, \ldots, b_n)$ are models of a language $L \cup \{c_1, \ldots, c_n\}$.

Then, the map $f:A \longrightarrow B$ defined by $f: a_i \longmapsto b_i$, $i \leq n$, is an isomorphism of models **A** and **B**. To see this, suppose that $*$ is a binary operation symbol (we have made this assumption for simplicity) of L. If $a_i, a_j, a_k \in A$ satisfy $a_k = a_i *^{\mathbf{A}} a_j$, then $(\mathbf{A}, a_1, \ldots, a_n) \models a_k = a_i * a_j$, so by (2), $(\mathbf{B}, b_1, \ldots, b_n) \models b_k = b_i * b_j$, i.e. $b_k = b_i *^{\mathbf{B}} b_j$. Therefore, by the definition of function f, we have $f(a_i *^{\mathbf{A}} a_j) = f(a_i) *^{\mathbf{B}} f(a_j)$, i.e. f is concurrent in respect to the operations $*^{\mathbf{A}}$, $*^{\mathbf{B}}$. Obviously, f is onto. This map is also 1-1, since

$$(\mathbf{A}, a_1, \ldots, a_n) \models a_i \equiv a_j \quad \text{iff} \quad (\mathbf{B}, b_1, \ldots, b_n) \models b_i \equiv b_j .$$

In a similar way one can show that f is concurrent with relations of models **A** and **B**. Thus $f: \mathbf{A} \approx \mathbf{B}$.

The idea of constructing an isomorphism as it has been done in the previous theorem is often exploited in model theory. It is summarized in the following theorem.

1.7.3. Theorem Let **A** and **B** be models of a language L, $A = \{a_i : i \in I\}$, $B = \{b_i : i \in I\}$, and $\mathbf{A}' = (\mathbf{A}, a_i)_{i \in I}$, $\mathbf{B}' = (\mathbf{B}, b_i)_{i \in I}$ be models of the language $L \cup \{c_i : i \in I\}$ with c_i interpreted in $\mathbf{A}'$ by a_i, and in $\mathbf{B}'$ by b_i. Then,

$$(\mathbf{A}, a_i)_{i \in I} \equiv (\mathbf{B}, b_i)_{i \in I} \quad \text{implies} \quad \mathbf{A} \approx \mathbf{B}.$$

As can be expected, the map $f: a_i \longmapsto b_i$, $i \in I$, is an isomorphism of models **A** and **B**.

Exercises

1.1. The set of positive propositional formulas is defined as the least set P of propositional formulas such that :

Every propositional letter belongs to P.

If $\varphi, \psi \in P$ then $(\varphi \wedge \psi), (\varphi \vee \psi), (\varphi \rightarrow \psi) \in P$.

Show: (a) A propositional formula $\varphi(p_1, \ldots, p_n)$, $n \in \omega$, is equivalent to a positive propositional formula iff $\varphi(1, \ldots, 1) = 1$.

(b) For every propositional formula φ there exists a positive propositional formula ψ such that $\models \varphi \leftrightarrow \psi$ or $\models \varphi \leftrightarrow \neg\psi$.

1.2. If φ is a propositional formula, let $\Gamma(\varphi)$ denote the set of all propositional letters which occur in φ. If φ and ψ are propositional formulas such that φ is not a contradiction, and ψ is not a tautology, then prove:

(a) If $\models \varphi\to\psi$ then there is a propositional formula θ such that $\models \varphi\to\theta$, $\models \theta\to\psi$, and $\Gamma(\theta) \subseteq \Gamma(\varphi)\cap\Gamma(\psi)$.

(b) If $\models \varphi\to\psi$ then $\Gamma(\varphi)\cap\Gamma(\psi)\neq\emptyset$.

1.3. A sequence of propositional formulas $\psi_1,\ldots,\psi_n$ is increasing iff for all $1\leq i<n$, $\models \psi_i \to\psi_{i+1}$. If $\varphi(p_1,\ldots,p_n)$ is a propositional formula, then for every increasing sequence of propositional formulas $\psi_1,\ldots,\psi_n$:

$\models \varphi(p_1, p_1\vee p_2,\ldots,p_1\vee\ldots\vee p_n)$ implies $\models \varphi(\psi_1,\ldots,\psi_n)$,

where $\varphi(\psi_1,\ldots,\psi_n)$ is obtained from $\varphi(p_1,\ldots,p_n)$ by simultaneous substitutions of propositional letters $p_1,\ldots,p_n$ of φ by $\psi_1,\ldots,\psi_n$.

1.4. Let φ and ψ be propositional formulas such that $\models \varphi\to\psi$ but not $\models \psi\to\varphi$. Show that there is a formula θ such that $\models \varphi\to\theta$ and $\models \theta\to\psi$, but $\tilde{}\models \theta\to\varphi$ and $\tilde{}\models \psi\to\theta$.

1.5. Prove : (a) The Deduction Theorem.
(b) The Disjunctive Normal Form Theorem.
(c) The Prenex Normal Form Theorem.

1.6. Assume T and S are theories of a language L. If $T\cup S$ is an inconsistent theory, then there is a $\varphi\in Sent_L$ such that $T\vdash \varphi$ and $S\vdash \neg\varphi$.

1.7. If L is an at most countable language, show that $\|L\| = \aleph_0$. If L is an infinite language, show that $\|L\| = |L|$. (Hint: if k is na infinite cardinal, then $k^2=k$).

1.8. Compute the number of theories over a language L if $|L|=k$.

1.9. Let $V_0=\emptyset$, $V_{n+1}= P(V_n)$, $n\in\omega$, and $V= \bigcup_n V_n$. Prove that $(V,\in)$ is a model of ZFC set theory without the Axiom of Infinity.

1.10. If $T_0 \subseteq T_1 \subseteq \ldots$ is a sequence of theories of a language L such that
(1) Theories T_n and T_{n+1} are not equivalent for any $n \in \omega$,
then $T = \bigcup_n T_n$ is not finitely axiomatizable.

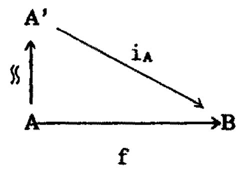
Show that the condition (1) can be replaced by: For each $n \in \omega$ there is a model A of T_n which is not a model of T_{n+1}.

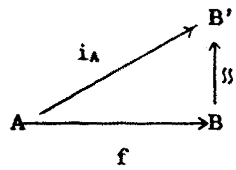
1.11. Show that the following theories are not finitely axiomatizable:
(a) The theory of infinite models of theory J_0.
(b) The theory of fields of the characteristic 0.
(c) The theory of algebraically closed fields.

1.12. Let T be a first-order theory with a recursive set of axioms. If T has only finitely many complete extensions, then T is decidable.

1.13. Construct models A and B such that A is embedded into A and B is embedded into A, but A and B are not isomorphic.

1.14. Let A and B be models of a language L. Show:
(a) If $f: A \longrightarrow B$ is an embedding then there is a model A' such that diagram (1) commutes.
(b) If $f: A \longrightarrow B$ is an embedding then there is a model B' such that diagram (2) commutes.

(1)


(2)


Here $i_A: x \longmapsto x$, $x \in A$.

1.15. Let A be an Abelian group in which every element is of finite order, and assume that the system

$$(S)\quad \begin{matrix} m_{11}x_1 + \ldots + m_{1n}x_n = b_1 \\ \vdots \\ \vdots \\ m_{n1}x_1 + \ldots + m_{nn}x_n = b_n \end{matrix} \qquad \begin{matrix} m_{ij} \text{ are integers, } b \in A^n, \\ 1 \le i,j \le n,\ b = (b_1, \ldots, b_n). \end{matrix}$$

has an unique solution in unknowns $x_1, \ldots, x_n$ for $b = (0, \ldots, 0)$. Show that the system (S) has an unique solution for all $b \in A^n$. (Hint: every finitely generated subgroup of A is finite).

1.16. Show that there are fields $F= (F,+,\cdot,0,1)$ and $H= (H,+,\cdot,0,1)$ such that $(F,+,0) \approx (H,+,0)$ and $(F,\cdot,1) \approx (H,\cdot,1)$ but not $F \approx H$.

1.17. If A and B are countable densely ordered sets without end-points, then $A \approx B$.

1.18. A linearly ordered set $(X,\leq)$ is ***well ordered*** iff every nonempty subset of X has the least element. Now, let $A= (A,\leq)$ be a countable linearly ordered set with the property: Every countable well-ordered set can be embedded into A. Prove that every countable linearly ordered set can be embedded into A. (Hint: Show that the ordering of rationals can be embedded into A).

1.19. Prove the identities $(xy)z= x(yz)$ and $x+(y+z)= (x+y)+z$ in PA.

1.20. Let $N= (N,+,\cdot,\leq,',0)$ denote the standard model of arithmetic.
(a) If M is a model of PA then there is an unique embedding of N into an initial segment of M.
(b) Numerals are defined as follows: $\underline{0}$ is constant symbol 0 of L_{PA}, $\underline{1}=\underline{0}'$, $\underline{2}=\underline{1}'$, $\underline{3}=\underline{2}'$, ... Show that for any term $tx_1\ldots x_k$ of L_{PA}, and $n,n_1,\ldots,n_k\in N$, $PA\vdash \underline{n}= t\underline{n_1}\ldots\underline{n_k}$ iff $n= t^N n_1\ldots n_k$.

1.21. If $M= (M,+,\cdot,',\leq,0)$ is a nonstandard model of PA (i.e. M is not isomorphic to the structure of natural numbers), then the order-type of $(M,\leq)$ is $\omega+(\omega^*+\omega)\theta$, where ω denotes the order-type of natural numbers, $\omega^*+\omega$ is the order type of integers, and θ is the order-type of a dense linear ordering without end-points.

1.22. If A is a model, prove that $(\mathrm{Aut}A,\cdot,i_A)$ is a group, where $\cdot$ is the function multiplication.

1.23. A model A of a language L is ***finitely generated*** if there is a finite set $S\subseteq A$ such that $A= \{t^B[\mu]:\ \mu$ is a valuation of domain A, and t is a term of language $L'\}$, where $L'= L \cup \{\underline{d}:\ d\in S\}$. If A is a finitely generated model and L is countable, show that $|\mathrm{Aut}A| \leq \aleph_0$.

1.24. Construct infinite models A such that:
(a) AutA is finite. (b) $|\mathrm{Aut}A|= \aleph_0$. (c) $|\mathrm{Aut}A|= 2^{\aleph_0}$. (d) $|\mathrm{Aut}A|> 2^{\aleph_0}$.

1.25. Let A and B be models of a language L, and assume $A \subseteq B$. If $\varphi x_1 \ldots x_n$ is a formula of L, and $a_1, \ldots, a_n \in A$, then

(a) If φ is universal then $B \models \varphi[a_1, \ldots, a_n]$ implies $A \models \varphi[a_1, \ldots, a_n]$.

(b) If φ is existential then $A \models \varphi[a_1, \ldots, a_n]$ implies $B \models \varphi[a_1, \ldots, a_n]$.

1.26. Let A and B be models of a language L, and assume $A \subseteq B$. Then $A \prec B$ iff for all $\psi x_1 \ldots x_n \in L$, all $a_1, \ldots, a_n \in A$
$B \models (\exists x \psi x)[a_1, \ldots, a_n]$ implies there is $a \in A$ such that $B \models \psi[a, a_1, \ldots, a_n]$.

1.27. Let A and B be models of a language L, and assume $A \subseteq B$. If for all $a_1, \ldots, a_n \in A$ and $b \in B$ there is an $f \in \text{Aut}B$ such that $fa_1 = a_1, \ldots, fa_n = a_n$, and $fb \in A$, then $A \prec B$.

1.28. If $(Q, \leq)$ is the ordering of rationals, and $(R, \leq)$ the ordering of reals, show that $(Q, \leq) \prec (R, \leq)$.

1.29. Let $(Q, +, \leq, 0)$ be the ordered additive group of rationals and $(R, +, \leq, 0)$ the ordered additive group of reals. Show:

(a) $(Q, +, \leq, 0) \prec (R, +, \leq, 0)$.

(b) Let (S) be a system of linear equations and inequalities over rationals. Show that (S) has a solution in rationals iff (S) has a solution over reals.

2. BOOLEAN ALGEBRAS AND MODELS

In algebraic considerations of metamathematics, Boolean algebras play an important role. Many statements from model theory are nothing but translated facts about Boolean algebras. In addition, Boolean algebras are used for building special models: one of the most important examples are Boolean models of the set theory. In this section we shall consider some basic properties of Boolean algebras and some related constructions from model theory.

2.1. Finite Boolean algebras

The most simple example of a Boolean algebra is the two-element Boolean algebra $2 = (2, \wedge, \vee, ', \leq, 0, 1)$. This algebra is sometimes called a propositional algebra. Powers 2^n of this algebra are also Boolean algebras. These are in fact up to isomorphism, the only finite Boolean algebras. On the other hand, infinite Boolean algebras have much more complex structure; therefore, their theory is more involved and far from trivial. Another important example of Boolean algebras is the field of all subsets of a set X. Namely, $P(X) = (P(X), \cup, \cap, {}^c, \subseteq, \emptyset, X)$ is also a Boolean algebra, where $P(X)$ denotes the power-set of X, and $A^c = X \setminus A$ for $A \in P(X)$.

The relation $\leq$ of a Boolean algebra $B = (B, \vee, \wedge, ', \leq, 0, 1)$ is a partial ordering of domain B, and operations $\vee$ and $\wedge$ are supremum and infimum in respect to this ordering i.e. for all $x, y \in B$ we have:

$x \vee y = \sup(x,y)$, $x \wedge y = \inf(x,y)$.

We have the following classification of Boolean algebras:

B is *complete* iff every $X \subseteq B$ has the supremum.

Minimal elements of $B \setminus \{0\}$ are called ***atoms***. Then

B is ***atomic*** iff for every $b\in B\setminus\{0\}$ there is an atom of B below b.
B is ***atomless*** iff B has no atoms.

When speaking about finite Boolean algebras, the following fact plays an important role:

$$BA\vdash tx_1\ldots x_n = \sum_{\alpha\in 2^n} t(\alpha_1,\ldots,\alpha_n)x_1^{\alpha_1}\ldots x_n^{\alpha_n}, \quad t\in Term_{L(BA)}$$

This theorem is known in propositional calculus as a Theorem on Disjunctive Normal Form. Remark that for $\alpha\in 2^n$ the value of $t(\alpha_1,\ldots,\alpha_n)$ is 0 or 1. A simple consequence of this theorem is that every finitely generated Boolean algebra is finite. Really, if B is a Boolean algebra generated by a set $X=\{a_1,\ldots,a_n\}$, then $B=\{t^B a_1\ldots a_n:\ t\in Term_{L(BA)}\}$. Thus for every $b\in B$ there are elements $b_1,\ldots,b_m\in B$, $m\le 2^n$, such that $b= b_1\vee\ldots\vee b_m$, and every b_i is of the form $a_1^{\alpha_1}\ldots a_n^{\alpha_n}$, $\alpha\in 2^n$. Since there are at most 2^n elements of the form $a_1^{\alpha_1}\ldots a_n^{\alpha_n}$, we can conclude that $|B|\le 2^{2^n}$, i.e. B is finite. In the following, we shall use signs +, · for Boolean operations $\vee$ and $\wedge$.

Let B be a finite Boolean algebra, $a\in B\setminus\{0\}$ and $B_a=\{x\in B:\ x\le a\}$. Then it is easy to see that $\mathbf{B}_a=(B_a,+,\cdot,'^a,\le,0,a)$, where $x'^a=x'\cdot a$, $x\in B_a$, is also a Boolean algebra (but not a subalgebra of B, as long as $0\ne 1$).

2.1.1. Lemma If B is a Boolean algebra and $a\in B$, $a\ne 0,1$, then $\mathbf{B}\approx\mathbf{B}_a\times\mathbf{B}_{a'}$.

Proof Let $f:B_a\times B_{a'}\longrightarrow B$ be defined by $f(x,y)=x+y$. We shall show that f is an isomorphism between algebras B and $B_a\times B_{a'}$.

(1) f is 1-1.

If $f(x_1,y_1)=f(x_2,y_2)$, then $x_1+y_1=x_2+y_2$, thus $x_1a+y_1a=x_2a+y_2a$. Since $x_1,x_2\le a$, $y_1,y_2\le a'$, we have $x_1a=x_1$, $x_2a=x_2$, $y_1a=0$, $y_2a=0$, so $x_1=x_2$. Similarly, from $x_1a'+y_1a'=x_2a'+y_2a'$, it follows that $y_1=y_2$, hence (1) holds.

(2) f is onto.

If $b\in B$, then from $a+a'=1$ we can infer that $b=ab+a'b$. On the other hand $ab\le a$, $a'b\le a'$, so $ab\in B_a$, $a'b\in B_{a'}$, i.e. $f(ab,a'b)=b$, so (2) holds.

(3) f is a homomorphism.

The identities $f(0,0)=0$, $f(1^{B_a},1^{B_{a'}})=f(a,a')=a+a'=1$ are obvious.

Further,

$f((x_1,y_1)+(x_2,y_2))= f(x_1+x_2,y_1+y_2)= (x_1+x_2)+(y_1+y_2)=$
$(x_1+y_1)+(x_2+y_2)= f(x_1,y_1)+f(x_2,y_2)$. Also,

$f((x_1,y_1)(x_2,y_2))= f(x_1x_2,y_1y_2)= x_1x_2+y_1y_2$.

Since $x_i \leq a$, $y_i \leq a'$, it follows $x_i \cdot y_j \leq a \cdot a'$, i.e. $x_i y_j=0$. Hence,

$f(x_1,y_1)f(x_2,y_2)= (x_1+y_1)(x_2+y_2)= x_1x_2+x_1y_2+y_1x_2+y_1y_2= x_1x_2+y_1y_2$

i.e. f is concurrent in respect to the operations of algebras B and $B_a x B_{a'}$. Finally,

$f((x,y)')= f(x'^a,y'^a)= ax'+a'y'= x'y'= (x+y)'= f(x,y)'$.

since $x \leq a$, $y \leq a'$. Therefore (3) holds.

A simple consequence of this lemma is the representation theorem for finite Boolean algebras.

2.1.2. Theorem Every finite Boolean algebra is isomorphic to some algebra 2^n, $n \in \omega$.

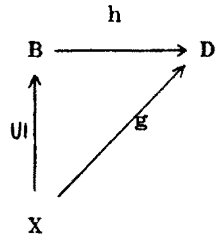
Proof We shall derive the proof by induction on $|B|$. If $|B|=2$, then obviously $B \approx 2$. Assume $|B|>2$, and suppose the statement for all the Boolean algebras of the cardinality $<|B|$. Then, there is $a \in B \setminus \{0,1\}$, so by Lemma 2.1.1, $B \approx B_a x B_{a'}$. But $|B_a|$, $|B_{a'}|<|B|$, therefore by the inductive hypothesis, $B_a \approx 2^m$, $B_{a'} \approx 2^n$ for some $m,n \in \omega$, thus $B \approx 2^{m+n}$.

If X is a finite set, say $|X|=n$, then, by the previous theorem, the field of sets P(X) is isomorphic to 2^n. An isomorphism $g:P(X) \approx 2^n$ is defined by $k:Y \longmapsto k_Y$, $Y \subseteq X$, where k_Y is the characteristic function of set Y. From Theorem 2.1.2, we also have the following consequence.

2.1.3. Corollary If identity $u \equiv v$ holds in a two-element Boolean algebra, then $u \equiv v$ is true in all the Boolean algebras.

In fact, if $u \equiv v$ holds in Boolean algebra 2, then $u \equiv v$ also holds in all the powers 2^n, so does it by Theorem 2.1.1, in all the finite Boolean algebras. If B is an arbitrary Boolean algebra, then $u \equiv v$ holds in all the finitely generated subalgebras of B, since all these subalgebras are finite; hence, $u \equiv v$ holds in B.

The representation theorem for Boolean terms enables us to consider in great detail the structure of free Boolean algebras. We would remind the reader that a Boolean algebra B is free over a set X of (free) generators, if and only if for every Boolean algebra D, and every map $g:X\longrightarrow D$, there is a homomorphism $h:B\longrightarrow D$ such that $g\subseteq h$, i.e. that the displayed diagram commutes. Finally, if one-element Boolean algebras are adopted, i.e. if the axiom $0\neq 1$ is dropped, then all the Boolean algebras make a variety. By the fundamental theorem of the theory of universal algebras on the existence of free algebras in algebraic varieties, there exists a free Boolean algebra over every set. The following theorem gives a condition for a set to be a set of free generators of Boolean algebra.



2.1.4. Theorem A Boolean algebra B is freely generated by a set X, if and only if B is generated by X and for all the different n elements $a_1,\ldots,a_n\in X$, and all $\alpha\in 2^n$, $a_1{}^{\alpha_1}\cdots a_n{}^{\alpha_n}\neq 0$.

Proof ($\rightarrow$) Let $a_1,\ldots,a_n\in X$ be different, and $\alpha\in 2^n$. Further, let $I=[0,1]$ be a real interval and F a field of subsets of I^n generated by sets

$$A_i=\{(x_1,\ldots,x_n):\ 0\leq x_i\leq 1/2\}.$$

Finally, let g be a map defined by $g:a_i\longmapsto A_i$, $i\leq n$. Since B is free over X, there is a homomorphism $h:B\longrightarrow F$, $g\subseteq h$. Let us define for $A\subseteq I^n$, $A^0=I^n\backslash A$, $A^1=A$, and $\beta'=1-\beta$ for $\beta\in\{0,1\}$. Then,

$$(\alpha_1,\ldots,\alpha_n)\in A_1{}^{\alpha_1}\cap\ldots\cap A_n{}^{\alpha_n}$$

i.e. $A_1{}^{\alpha_1}\cap\ldots\cap A_n{}^{\alpha_n}\neq 0$. On the other hand

$$h(a^{\alpha_1}\ldots a^{\alpha_n})=A_1{}^{\alpha_1}\cap\ldots\cap A_n{}^{\alpha_n}$$

so $a_1{}^{\alpha_1}\ldots a_n{}^{\alpha_n}\neq 0$.

($\leftarrow$) Let Ω be a free Boolean algebra generated by set Y of free generators, where $|Y|=|X|$. Suppose g maps set Y 1-1 and onto set X. Since Ω is free, there is a homomorphism $h:\Omega\longrightarrow B$, $g\subseteq h$. Algebra B is generated by set X, and $X=h(Y)$, thus h is onto. Let us see that h is 1-1, so assume $w\in\Omega$, $w\neq 0$. Set Y generates Ω, hence $w=w_1+\ldots+w_k$, where each w_i is of the form $b_1{}^{\alpha_1}\ldots b_n{}^{\alpha_n}$, $b_i\in Y$. Since $w\neq 0$, the sum $w_1+\ldots+w_k$ is nonempty, thus $w\geq b_1{}^{\alpha_1}\ldots b_n{}^{\alpha_n}$ for some choice of different elements $b_1,\ldots,b_n\in Y$ and $\alpha\in 2^n$. Therefore,

$$h(w)\geq h(b_1{}^{\alpha_1}\ldots b_n{}^{\alpha_n})=h(b_1)^{\alpha_1}\ldots h(b_n)^{\alpha_n}=g(b_1)^{\alpha_1}\ldots g(b_n)^{\alpha_n}.$$

Map g is 1-1, so $g(b_i)$ are mutually different for different i's.

Further, $g(b_i) \in X$, hence, by the assumed condition, we have

$$h(b_1)^{\alpha_1} \ldots h(b_n)^{\alpha_n} \neq 0, \text{ i.e. } h(w) \neq 0.$$

In other words,

(1) for all $w \in \Omega \setminus \{0\}$, $h(w) \neq 0$.

Now, let $h(a) = h(b)$, and let $a \Delta b = ab' + a'b$ be the symmetrical difference of elements $a, b \in \Omega$. Then, $h(a) \Delta h(b) = 0$, so $h(a \Delta b) = 0$, thus by (1), $a \Delta b = 0$, i.e. $a = b$.

If $a_1, a_2, \ldots, a_n$ are free generators of a Boolean algebra B, and

$$u^B(a_1, \ldots, a_n) = v^B(a_1, \ldots, a_n),$$

then, for every map $g: X \longrightarrow 2$, there is a homomorphism $h: B \longrightarrow 2$ such that $g \subseteq h$. Therefore, by Theorem 1.6.1, we have

$$u^2(ga_1, \ldots, ga_n) = u^2(ha_1, \ldots, ha_n) = hu^B(a_1, \ldots, a_n) =$$
$$hv^B(a_1, \ldots, a_n) = v^2(ha_1, \ldots, ha_n = v^2(ga_1, \ldots, ga_n).$$

Since g was chosen arbitrarily, identity $u=v$ holds in algebra 2, and, therefore, in all Boolean algebras. For example, if three circular areas K_1, K_2, K3 are so chosen in the plane that $K^{\alpha_1} \cap K^{\alpha_2} \cap K^{\alpha_3} \neq \emptyset$ for all $\alpha \in 2^3$, then, by the last theorem, these circles , as subsets of the plane, generate a free Boolean algebra. Hence, every identity $u(x,y,z) = v(x,y,z)$ in three variables x, y, z, which is satisfied by these circles, holds on all the Boolean algebras. This remark is in fact a proof of the validity of Venn's rules for checking set-theoretical identities in three letters. We also have the following consequence.

2.1.5. Theorem Let B_n be a free Boolean algebra generated by n free generators. Then, $B_n \approx 2^{2^n}$.

Proof The proof we shall present is by induction on the number of free generators. So, let $b_1, \ldots, b_n \in B_n$ be free generators of algebra B_n, and suppose the statement for Boolean algebras generated with fewer number of free generators. The subalgebra $B_{n-1} \subseteq B_n$ generated by elements $b_1, \ldots, b_{n-1}$ is also free, so, by the inductive hypothesis, $B_{n-1} \approx 2^{2^{n-1}}$.
Further, by Theorem 2.1.4, we have for all $\alpha \in 2^n$

$$b_1^{\alpha_1} \ldots b_{n-1}^{\alpha_{n-1}} b_n^{\alpha_n} \neq 0,$$

Using this fact, it is easy to show that the map

$$g: (x,y) \longmapsto xb_n + yb_n', \quad x, y \in B_{n-1}$$

is an isomorphism between algebras B_n and $B_{n-1} \times B_{n-1}$. Thus,

$$B_n \approx 2^{2^{n-1}} \times 2^{2^{n-1}} \approx 2^{2^n}.$$

2.2. Filters

Filters of Boolean algebras do not only have applications in analyzing properties of Boolean algebras, but also in logic, set-theory and topology. Very often, a topological statement has a natural translation into the language of Boolean algebras or model theory. The main reason for this lies in the Stone Representation Theorem for Boolean algebras. Ultrafilters make an important class of filters, whose significance comes from the extreme properties of these objects. In Boolean algebras, ultrafilters define maximal congruences, in logic, they provide logical verification of statements, in set theory they have interesting combinatorial properties, and in topology, ultrafilters give a method for the description of convergence at infinity, as well as the compactification of spaces. In this section, we shall speak about filters of Boolean algebras only as much as we need them for model theory.

2.2.1. Definition Let B be a Boolean algebra. A filter of B is every subset $F \subseteq B$ which satisfies the following conditions:

$1 \in F$.

For all $x \in F$ and all $y \in B$, $x \leq y$ implies $y \in F$.

For all $x, y \in F$, $x \cdot y \in F$.

A filter F is an ***ultrafilter*** of a Boolean algebra B, if F is a maximal proper filter of B ("proper" means $F \neq B$). A simple example of a filter of B is $F_a = \{x \in B:\ a \leq x\}$, where $a \in B$. Filter F_a is called *principal*. Here are some other examples of filters.

2.2.2. Example 1° Filters of the field of sets P(X) are also called filters over set X. For example, the set $F = \{Y \subseteq X:\ Y^c \text{ is finite}\}$ is a filter over X.

2° In finite Boolean algebras every filter is principal; F is generated by $\wedge F$.

3° If $h: B \longrightarrow C$ is a homomorphism of Boolean algebras B and C, then the set $F = \{x \in B:\ hx = 1\}$ is a filter of B. If $C = 2$, then F is an ultrafilter.

The following proposition gives the equivalent conditions for a filter to be an ultrafilter.

2.2.3. Theorem Let F be a proper filter of a Boolean algebra B. Then, the following are equivalent:

1° F is an ultrafilter of B.

2° For all $x \in B$ either $x \in F$ or $x' \in F$.

3° For all $x, y \in B$, $x+y \in B$ implies $x \in F$ or $y \in F$.

Proof (1°→2°) Let F be an ultrafilter, and suppose $x \notin F$, $x \in B$. Then the set $D = F \cup \{z \in B:$ there is $y \in F$ such that $x' y \leq z\}$, is a proper filter of B, and this filter contains $F \cup \{x'\}$ as a subset. But F is an ultrafilter, hence, D=F. Since $F \cup \{x'\} \subseteq D$, we have $x' \in F$.

(2°→3°) Assume 2° and let $x+y \in F$, $x, y \notin F$. Then $x', y' \in F$, hence, $x' \cdot y' \in F$, so $(x+y)x'y' \in F$ i.e. $0 \in F$, which is a contradiction. Therefore, $x \in F$ or $y \in F$.

(3°→1°) Let $F \subseteq D$, $F \neq D$, where D is a filter, and take $x \in D \setminus F$. Since $x+x'=1$ it follows that $x+x' \in F$. So $x \in F$ or $x' \in F$, hence $x' \in F$ i.e. $x' \in D$. Thus $x \cdot x' = 0$, i.e. D= B.

Nonprincipal ultrafilters are of special interest in model theory. If F is an ultrafilter over a set I, then we have the following possibilities:

1° F is principal, i.e. it is generated by a set $A \subseteq I$. If $|A| \geq 2$, then there are nonempty subsets $B, C \subseteq I$ such that $A = B \cup C$, $B \cap C = \emptyset$, so $B \in F$ or $C \in F$. This would mean that F is not generated by A, hence, $A = \{a\}$ for some $a \in I$, and $F = \{x \subseteq I: a \in X\}$.

2° F is a nonprincipal ultrafilter, so, for every $a \in I$, $\{a\} \notin F$, thus $\{a\}^c \in F$.Filter F is closed for finite intersections, thus, F contains the filter from Example 2.2.2.

An easy consequence of the following theorem is that ultrafilters exist. Let us first introduce the so-called finite intersection property (abr. FIP) of subsets X of the domain of a Boolean algebra B:

A subset X of B has FIP iff

for all $n \in \omega$ and all $x_1, x_2, \ldots, x_n \in X$, $x_1 \cdot x_2 \cdot \ldots \cdot x_n \neq 0$.

It is easy to see that every subset X of B which has FIP generates the proper filter $F_X = \{y \in B: x_1 \ldots x_n \leq y,\ n \in \omega,\ x_1, \ldots, x_n \in X\}$.

2.2.4. Theorem If F is a filter over a Boolean algebra and $\mathcal{F}$ is the set of all the ultrafilters of B which contain F, then $F = \cap \mathcal{F}$.

Proof It suffices to prove $\cap \mathcal{F} \subseteq F$. Suppose the opposite. Then there is $a \in \cap \mathcal{F}$, $a \notin F$. Let $\mathcal{S}$ be the set of all the filters of B which contain F but not a. Then, by Zorn's Lemma set $\mathcal{S}$ has a maximal member, say U, and then $F \subseteq U$, $a \notin U$. Now, we shall show that U is an ultrafilter of B. Let us first show that $a' \in U$. To see that, let V be a filter of B generated by $U \cup \{a'\}$. This set has FIP, since if there is an $x \in U$, $a'x=0$, then $a \geq x$, so $a \in U$, and this is a contradiction. Thus $V \in \mathcal{S}$, $U \subseteq V$, and by the choice of filter U we have U=V. Further, let $b \in B$ be any element and suppose $b \notin U$. Then the set $U \cup \{b'\}$ has FIP, so, let W be the filter generated by this set. Then $b' \in W$ and $U \subseteq W$. Also $a \notin W$, since $a' \in W$. Therefore, $W \in \mathcal{S}$, so, by the choice of filter U, we again have W=U, i.e. $b \in U$.

So we have proved that for all $x \in B$, either $x \in U$ or $x' \in U$, i.e. U is an ultrafilter. Since $F \subseteq U$, it follows that $U \in \mathcal{F}$; thus, by the choice of element a, we have $a \in U$, which is a contradiction.

Corollary Every filter of a Boolean algebra B is a subset of an ultrafilter of B.

Now, we shall consider the example of a ***Lindenbaum algebra*** of the propositional calculus. Let $\mathcal{F}$ be the set of all formulas of the propositional calculus, and $\sim$ the equivalence relation of $\mathcal{F}$ defined by

$\varphi \sim \psi$ iff $\varphi \leftrightarrow \psi$ is a tautology, $\varphi, \psi, \in \mathcal{F}$.

Let $B=\mathcal{F}/\sim$, and $+, \cdot, '$ be operations of set B defined as follows: Suppose $x, y \in B$ and $\varphi, \psi \in \mathcal{F}$ be such that $x = \varphi/\sim$, $y = \psi/\sim$. Then,

$x+y = (\varphi \vee \psi)/\sim$, $x \cdot y = (\varphi \wedge \psi)/\sim$, $x' = (\neg\varphi)/\sim$.

Further, let 0 be the equivalence clas of a contradiction, and 1 the equivalence class of a tautology. It can easily be shown that the operations $+, \cdot, '$ and constants 0,1 are well-defined, and $\Omega_{\mathcal{F}} = (\mathcal{F}/\sim, +, \cdot, ', 0, 1)$ is a Boolean algebra. This Boolean algebra is called the Lindenbaum algebra of the propositional calculus. Assume the set of all propositional letters is of the cardinality k, where k is an infinite cardinal. For every infinite set A, the set of all the finite sequences of elements of A is also of cardinality $|A|$, therefore, $\mathcal{F}$ is of cardinality k. From this fact we can easily conclude that $\Omega_{\mathcal{F}}$ is of cardinality k, as well. Namely, the classes of equivalences of propositional letters differ from each other, thus, there are at least as many classes as propositional letters, i.e. k.

Let $p_1, p_2, \dots, p_n$ be distinct propositional letters. Then, for every $\alpha \in 2^n$, $p_1^{\alpha_1} \wedge p_2^{\alpha_2} \wedge \dots \wedge p_n^{\alpha_n}$ is not a contradiction, i.e.

$(p_1/\sim)^{\alpha 1}\cdot(p_2/\sim)^{\alpha 2}\cdot\ldots\cdot(p_n/\sim)^{\alpha n}\neq 0.$

Hence, by Theorem 2.1.4, the set $\{p/\sim: p$ is a propositional letter$\}$ is a set of free generators of Ω_F. Therefore we have the following theorem.

2.2.5. Theorem Ω_F is a free Boolean algebra with $p_\alpha/\sim$, $\alpha<k$, as free generators, where p_α are propositional letters.

Let us describe the ultrafilters of Ω_F. If G is an ultrafilter of Ω_F, then for every $x\in B$ either $x\in G$ or $x'\in G$. Therefore, G determines a function $\tau:k\longrightarrow 2$ such that $\tau_\alpha=1$, if $p_\alpha\in G$ and $\tau_\alpha=0$ if $p_\alpha\notin G$. Then the set $X_\tau=\{p_\alpha{}^{\tau\alpha}/\sim: \alpha<k\}$ determines the filter G, i.e. if D is a proper filter and $X_\tau\subseteq D$, then D=G. On the other hand, for each $\tau:k\longrightarrow 2$ the set X_τ has FIP, so, it is contained in an ultrafilter. Further, if $\tau,\mu:k\longrightarrow 2$ are different functions, then, for some $\alpha<k$, $\tau_\alpha\neq\mu_\alpha$, say $\tau_\alpha=1$ and $\mu_\alpha=0$. Thus, if D_τ and D_μ are ultrafilters which correspond, respectively, to τ and μ then $p_\alpha\in D_\tau\backslash D_\mu$, i.e. $D_\tau\neq D_\mu$. Hence, keeping in mind the well-known fact that any two free Boolean algebras with sets of free generators of the same cardinality are isomorphic, we have:

2.2.6 Theorem A free Boolean algebra with k free generators has 2^k ultrafilters.

Using this theorem, we can compute the number of set-theoretical ultrafilters over any infinite set X.

2.2.7. Theorem (Kantorovic, Pospisl) The number of ultrafilters over an infinite set X of cardinality k is 2^{2^k}.

Proof First, let us prove:

(1) If B is a Boolean algebra, and $P_0,\ldots,P_m$, $Q_0,\ldots,Q_n$, are distinct ultrafilters over B, then, $P_0\cap\ldots\cap P_m\cap Q_0{}^c\cap\ldots\cap Q_n{}^c\neq\emptyset$.

Really, if P is an ultrafilter over B which differs from $Q_0,\ldots$ $\ldots,Q_n$, then, there are elements a_i, $i=0,1,\ldots,n$ such that $a_i\in P$, $a_i'\in Q_i$. If $a=a_0\ldots a_n$ then $a\in P\cap(\cap_{j\leq n}Q_j{}^c)$. So, there are elements b_i, $i=0,\ldots,m$, such that $b_i\in P_i\cap(\cap_{j\leq n}Q_j{}^c)$, thus (1) holds.

A family $\mathcal{X}$ of subsets of X is independent, if for every finite sequence of distinct elements $X_0,\ldots,X_n\in\mathcal{X}$, and every $\tau:\{0,1,\ldots,n\}\longrightarrow 2$, we have $X_0{}^{\tau 0}\cap\ldots\cap X_n{}^{\tau n}\neq\emptyset$. Then,

(2) For every set X of cardinality k, there is an independent family $\mathcal{X}$

By Theorem 2.2.6, the Lindenbaum algebra Ω_F of the propositional calculus with k propositional letters has 2^k ultrafilters. Let J be the set of all these ultrafilters. By (1), J is an independent family. Further, Ω_F is of cardinality k, so, there is a bijective map $f: F/\sim \longrightarrow X$. Then, $X = \{f[P]: P\in J\}$ is an independent family as well, thus, (2) holds. Observe that $|X| = 2^k$.

For every $Z \subseteq X$, the set $Z \cup \{A^c: A\in X \setminus Z\}$ has FIP, so, Z is contained in an ultrafilter D_Z. If $Z \neq Z'$, then for some $A\in X$ we have $A\in Z$ and $A^c \notin Z$, so, $A\in D_Z$, $A^c \in D_{Z'}$, i.e. $D_Z \neq D_{Z'}$. Therefore, there are ultrafilters over X, as many as there are subsets of X, i.e. 2^{2^K}.

2.3. Boolean-valued models

By the definition of the satisfaction relation, the possible logical values of a formula φ in a model belongs to the set $\{0,1\}$, the domain of the propositional algebra. The notion of Boolean structure, or a B-model, where $B = (B,+,',\leq,0,1)$, is a Boolean algebra, is obtained if it is allowed that formulas may have logical values in B. If one wants to compute the Boolean value of a formula, it is necessary to suppose some assumptions. For example, the completeness of a Boolean algebra ensures the correctness of the definition of a B-value of a formula. Therefore, we shall assume in this section that B is a complete Boolean algebra, if not stated otherwise (as in Example 2.3.3).

2.3.1. Definition Let L be a first-order language. A B-model of a language L is every structure $A = (A, J)$, where A is a nonempty set and,

if $c\in Const_L$, then $J(c) \in A$,

if $F\in Fnc_L$, then $J(f)$ is an operation of length $k = ar(F)$ of domain A^k,

if $R\in Rel_L$, then $J(R): A^n \longrightarrow B$, where $n = ar(R)$.

As before, we shall write s^A instead of $J(s)$ for $s\in L$.

We see that the notion of B-models differs from the concept of standard models in the definition of the satisfaction relation. Namely, the logical values can be arbitrary elements of a Boolean algebra B. To make this definition precise, we would remind the reader that the supremum and the infimum of a subset $X \subseteq B$ are denoted by $\sum_{x \in X} x$, $\prod_{x \in X} x$, respectively. Further we shall enlarge language L by the names of elements of domain A, i.e. we shall introduce the language $L_A = L \cup \{\underline{a}: a \in A\}$. The B-value of a sentence φ of language L_A will be denoted by $\|\varphi\|_B$. Sometimes, if the context allows, the subscript will be omitted. The B-value of sentences then is every map $\| \ \|_B: Sent_L \longrightarrow B$ which satisfies the following conditions (in the next, we shall omit the subscript $_B$):

Equality conditions 1° $\|c \equiv c\| = 1$.

2° $\|c_1 \equiv c_2\| = \|c_2 \equiv c_1\|$.

3° $\|c_1 \equiv c_2\| \cdot \|c_2 \equiv c_3\| \leq \|c_1 \equiv c_3\|$.

4° If $R \in Rel_L$ is of length n, then
$\|c_1 \equiv c_1'\| \ldots \|c_n \equiv c_n'\| \cdot \|Rc_1 \ldots c_n\| \leq \|Rc_1' \ldots c_n'\|$.

The definition of $\| \ \|$ goes further, inductively, as follows:

1° If $R \in Rel_L$ is of length n, then for all $\underline{a}_1, \ldots, \underline{a}_n \in Const_L$
$\|R\underline{a}_1 \ldots \underline{a}_n\| = R^A(a_1 \ldots a_n)$.

2° $\|\varphi \wedge \psi\| = \|\varphi\| \cdot \|\psi\|$.

3° $\|\varphi \vee \psi\| = \|\varphi\| + \|\psi\|$.

4° $\|\neg\varphi\| = \|\varphi\|'$.

5° $\|\forall x \varphi x\| = \prod_{a \in A} \|\varphi a\|$.

6° $\|\exists x \varphi x\| = \sum_{a \in A} \|\varphi a\|$.

We can see that by this definition the 2-values of formulas φ coincide with the logical values in the sense of Definition 1.6.3. If $\|\varphi\|_B = 1$, then we say that formula φ is true, or satisfied in structure A (or more exactly: B-satisfied in A). A B-model is nondegenerate, if $0 \neq 1$ in B. In this section, by two models we shall mean the standard models.

The most important applications of B-models can be found in constructing models of formal set-theory, and they are used mainly for proving independence results. At this time we shall consider models of a simpler nature.

2.3.2. Example (Boolean product of models) Let A_i, $i \in I$, be a family of standard models of a language L. The product of models A_i is a structure A of language L, where $A = \prod_i A_i$, and for $f_1, \ldots, f_n \in A$:

if $c \in Const_L$, then $c^A = \langle c_i{}^A : i \in I \rangle$,

if $F\in Fnc_L$ is of length n, then
$F^A(f_1,\ldots,f_n)=\langle F^{A_i}(f_1(i),\ldots,f_n(i)): i\in I\rangle$,
if $R\in Rel_L$ is of length n, then
$R^A(f_1,\ldots,f_n)$, iff for all $i\in I$, $R^{A_i}(f_1(i),\ldots,f_n(i))$.

This is the standard definition of products of models, and, by this construction from 2-models, a B-model is obtained. By a simple modification of the part refering to relations, a B-model is obtained, where $B= 2^I$:

$R^A(f_1,\ldots,f_n)= \langle R^{A_i}(f_1(i),\ldots,f_n(i)): i\in I\rangle$.

Therefore, we have that $R^A:A^n\longrightarrow 2^I$ in this case. Products modified in this way are called Boolean products of models.

2.3.3 Example (Lindenbaum algebras of rich theories). Let T be a theory and $\sim$ a binary relation defined on $Sent_L$ by: $\varphi\sim\psi$ iff $T\vdash \varphi\leftrightarrow\psi$. It is easy to see that $\sim$ is an equivalence relation and:

1° If $\varphi\sim\psi$, then $\neg\varphi\sim\neg\psi$.

2° If $\varphi_1\sim\psi_1$, $\varphi_2\sim\psi_2$ then $(\varphi_1\wedge\varphi_2)\sim(\psi_1\wedge\psi_2)$, $(\varphi_1\vee\varphi_2)\sim(\psi_1\vee\psi_2)$, $(\varphi_1\rightarrow\varphi_2)\sim(\psi_1\rightarrow\psi_2)$.

Properties 1° and 2° enable us to define the following operations on the quotient set $B_T= Sent_L/\sim = \{[\varphi]: \varphi\in Sent_L\}$:

$[\varphi]'=[\neg\varphi]$, $[\varphi]\cdot[\psi]=[\varphi\wedge\psi]$, $[\varphi]+[\psi]=[\varphi\vee\psi]$

If we define $0=[\varphi\wedge\neg\varphi]$, $1=[\varphi\vee\neg\varphi]$, and $[\varphi]\le[\psi]$ iff $[\varphi\rightarrow\psi]=1$, we have that $\mathbf{B}_T=(B_T,+,\cdot,',\le,0,1)$ is a Boolean algebra.

The construction of algebra B_T is similar to that of the Lindenbaum algebra of propositional calculus. This is the reason why B_T is also called the Lindenbaum algebra of theory T. Here are some properties of algebra B_T:

1° T is a consistent theory iff B_T is nondegenerate, i.e. $|B_T|\ge 2$.

2° T is a consistent and complete iff $B_T\approx 2$.

3° A sentence φ of L is a theorem of T iff $[\varphi]=1$. Also, $T\vdash\varphi\rightarrow\psi$ iff $[\varphi]\le[\psi]$.

4° If S is a consistent theory of L such that $T\subseteq S$, then the set $F_S=\{[\varphi]: S\vdash\varphi\}$ is a filter of B_T. S is a maximal consistent theory iff F_S is an ultrafilter of B_T.

5° If $|L|\le k$, then B_T is at most of the cardinality $\aleph_0+k$.

An interesting case arises when starting from a theory T we can construct a B-model of T. Under some assumptions on L(T) and T such a construction is possible. Let $L = L_0 \cup C$ where C is a set of constant symbols. A theory T of L is *rich*, if and only if for every sentence $\exists x\varphi x$ of L, there is a constant symbol $c \in C$ such that $(\exists x\varphi x \rightarrow \varphi c) \in T$. A B-model of a rich theory T, denoted by A is built in the following way:

Let $B = B_T$, i.e. B is a Lindenbaum algebra of theory T. Let us define a binary relation $\approx$ of C by $c_1 \approx c_2$ iff $[c_1 \equiv c_2] = 1$. The domain of A is $A = \{\mathbf{c}: c \in C\}$, where $\mathbf{c} = c/\approx$.

If $c \in Const_L$, then obviously $\exists x(c=x)$ is a theorem of T. Since T is a rich theory, there is $d \in C$ such that $\exists x(c=x) \rightarrow c=d$ is also a theorem of T, so $T \vdash c=d$. Then we define $c^A = \mathbf{d}$.

If $F \in Fnc_L$ of length n and $c_1, \ldots, c_n \in C$, then the sentence

$$\exists x(F(c_1, c_2, \ldots, c_n) = x)$$

is a theorem of T, and, as above we find $c \in C$, such that $T \vdash Fc_1 \ldots c_n = c$. Then, we define $F^A \mathbf{c}_1 \ldots \mathbf{c}_n = \mathbf{c}$.

Finally, let $R \in Rel_L$ be of length n and $c_1, \ldots, c_n \in C$. Then, we define $R^A \mathbf{c}_1 \ldots \mathbf{c}_n = [Rc_1 \ldots c_n]$.

Of course, we should check that objects so introduced are well-defined. For example, let us show that the interpretation of an n-ary function symbol $F \in L$ is well-defined. So let $c_1, \ldots, c_n, d_1, \ldots, d_n, c, d \in C$ be such that $c_1 \approx d_1$, $c_2 \approx d_2, \ldots, c_n \approx d_n$, and $T \vdash Fc_1 \ldots c_n = c$, $Fd_1 \ldots d_n = d$. Then, $[c_i = d_i] = 1$, thus, by property 3' of B_T, we have $T \vdash c_i \equiv d_i$. Using the identity axioms of PR^1, it follows that $T \vdash Fc_1 \ldots c_n = Fd_1 \ldots d_n$, so $T \vdash c \equiv d$, i.e. $c \approx d$. Therefore, the function F^A is well-defined. In a similar way, it is proved that c^A is correctly defined for $c \in Const_L$.

Let us prove the correctness of R^A, where R is an n-ary relation symbol of L. So, assume that $c_1, \ldots, c_n, d_1, \ldots, d_n \in C$ are such that $c_1 \approx d_1$, $c_2 \approx d_2, \ldots, c_n \approx d_n$. Then $T \vdash c_i \equiv d_i$, $1 \le i \le n$, so by the identity axioms of PR^1, we have $T \vdash Rc_1 \ldots c_n \leftrightarrow Rd_1 \ldots d_n$, i.e. $[Rc_1 \ldots c_n] = [Rd_1 \ldots d_n]$.

For each element $\mathbf{c}$ of domain A, we shall suppose that the name of the element $\mathbf{c}$ is the symbol c. Then A becomes a B-model if we define:

for $c_1, c_2 \in C$, $\| c_1 \equiv c_2 \| = [c_1 \equiv c_2]$,

if $R \in Rel_L$ is of length n, and $c_1, \ldots, c_n \in C$, then

$\| Rc_1 c_2 \ldots c_n \| = R^A \mathbf{c}_1 \mathbf{c}_2 \ldots \mathbf{c}_n$.

It is easy to see that for so defined structure the conditions of Definition 2.3.1 are fulfilled. For a B-model constructed in this way, we have some additional properties:

6° For each sentence φ of L, $\|\varphi\| = [\varphi]$.

The proof of this fact can be deduced by induction on the complexity of sentence φ.

7° A sentence φ of L is a theorem of T, iff $\|\varphi\|=1$.

The proof of this fact follows from 3° and 6°.

The B-model of T constructed in the way described above is called a *canonical model*. As a consequence of the preceding, we have

2.3.4. Theorem Let T be a first-order consistent rich theory. Then T has a B-model.

This statement is part of the Completeness Theorem of PR^1. More details about this theorem will be given in following section.

2.4. The Completeness of PR^1

Theorem 2.3.4. says that a first-order theory has Boolean models under certain conditions. We shall prove that every consistent first-order theory is semantically consistent, i.e. it has a model.

2.4.1. Lemma Let T be a consistent rich theory. Then T has a standard model.

Proof By Theorem 2.3.4 theory T has a nondegenerated B-model **A**, where B is a Lindenbaum algebra of T. Let D be an ultrafilter of B, and $k:B\longrightarrow B/D$ the canonical homomorphism. Observe that $B/D\approx 2$. Further, let A' be a standard model, where $A'=A$, and for $s\in Const_L \cup Fnc_L$, $s^{A'}=s^A$, while for $R\in Rel_L$ of length n, $R^{A'}c_1c_2\ldots c_n=k(R^A c_1c_2\ldots c_n)$. Then A' is a 2-model, where $\|\varphi\|_2=k(\|\varphi\|)_B$, $\varphi\in Sent_L$. Since T is a rich theory, for every sentence $\exists x\varphi x$ there is $c\in C$ such that the following holds:

$$\|\exists x\varphi x\|_2 = k(\|\exists x\varphi x\|_B) = k(\|\varphi c\|_B) = \|\varphi c\|_2$$

By this and Theorem 1.7.1, we can prove by induction on the complexity of formulas that the following conditions are equivalent for sentences $\varphi c_1\ldots c_n$ of L:

(1) $A' \models \varphi[c_1,c_2,\ldots,c_n]$.

(2) $A' \models \varphi(c_1, c_2, \dots, c_n)$.

(3) $\|\varphi c_1 c_2 \dots c_n\|_2 = 1$.

Since $T \vdash \varphi$, we have $\|\varphi\|_B = 1$, so $k(\|\varphi\|_B) = 1$, i.e. $\|\varphi\|_2 = 1$. Thus, $A' \models \varphi$, and A' is a standard model of T.

The following lemmas will be used to show that every consistent theory may be expanded to a consistent rich theory. On the ground of the proofs of these statements is the method of constants. This method was introduced by L.Henkin in 1949, when he gave a new proof of the Completeness Theorem for PR^1.

2.4.2. Lemma Let S be a set of formulas of L and $c \in Const_L$ which does not occur in either S or φx. If $S \vdash \varphi c$, then $S \vdash \forall x \varphi x$, and there is a proof of $\forall x \varphi x$ in S in which symbol c does not appear. In other words, $S \vdash \forall x \varphi x$ in $L \setminus \{c\}$.

The proof of this lemma was outlined in the proof of Lemma 1.3.4, so we omit it.

2.4.3. Lemma Let C be a set of new constant symbols for a language L (i.e. $C \cap L = \emptyset$), and let S be a set of formulas of L. If S is a consistent theory in L, then S is consistent in $L \cup C$, also.

Proof Suppose, on the contrary, that S is inconsistent in $L \cup C$, and let $\{c_1, \dots, c_n\} \subseteq C$ be a minimal subset, such that there is a proof $\varphi_0, \dots, \varphi_k$ for $x \neq x$ in S and $L \cup \{c_1, \dots, c_n\}$. Further, $x \neq x$ is $(x \neq x)(y/c_n)$, so by the last lemma, $S \vdash \forall x(x \neq x)$ in $L \cup \{c_1, \dots, c_{n-1}\}$, which is a contradiction by the choice of the set $\{c_1, \dots, c_n\}$.

2.4.4. The Witness Lemma (Henkin) Let T be a consistent theory of a language L. Then, there is a set C of new constant symbols for L, and a theory S of the language $L \cup C$, such that:

(1) $T \subseteq S$.

(2) S is a consistent theory.

(3) For every sentence $\exists x \varphi x$ of $L \cup C$, there is $c \in C$ such that

$$(\exists x \varphi x \rightarrow \varphi c) \in S,$$

i.e. S is a rich theory of $L \cup C$. Set C is called the set of ***witnesses*** for S.

Proof Let C_0 be the set of new constant symbols c_σ, where σ is a sentence of L of the form $\exists x\theta x$. For given C_n, let C_{n+1} be a set of constant symbols c_σ, where σ is a sentence of $L\cup C_n$ of the form $\exists x\theta x$. Thus,

$$C_0 \subseteq C_1 \subseteq C_2 \subseteq \ldots .$$

Let $C=\bigcup_n C_n$, and define

$$S= T \cup \{\exists x\theta x\rightarrow\theta c_{\exists x\theta}: \exists x\theta x \text{ is a sentence of } L\cup C\}.$$

Then, C is the set of witnesses of S, and $T\subseteq S$. By Lemma 2.4.3, T is a consistent theory in $L\cup C$. Now, let us show that S is a consistent theory. Suppose , on the contrary, that S is inconsistent. Then there is a deduction $\varphi_0, \varphi_1, \ldots, \varphi_m$ of $x\neq x$ in S. Let $S_0 = S\cap\{\varphi_0,\ldots,\varphi_m\}$. Then, $S_0 \vdash x\neq x$. Without a loss of generality, we may assume that S_0 is such a minimal set, i.e. every proper subset of S_0 is consistent. Since S_0 is not a subset of T because T is consistent, there is the greatest $n\in\omega$, such that S_0 contains a formula $\exists x\theta x\rightarrow\theta c_{\exists x\theta}$, and $c_{\exists x\theta}\in C_{n+1}\setminus C_n$. Therefore, each sentence in S_0 is a sentence of $L\cup C_{n+1}$, and if it differs from $\exists x\theta x\rightarrow\theta c_{\exists x\theta}$, then it does not contain the symbol $c_{\exists x\theta}$. Further, let $S_1 = S_0\setminus\{\exists x\theta x\rightarrow\theta c_{\exists x\theta}\}$. Then, using the axioms of PR^1, we have

$S_1,\ \exists x\theta x\rightarrow\theta c_{\exists x\theta} \vdash x\neq x,$

$S_1 \vdash \neg(\exists x\theta x\rightarrow\theta c_{\exists x\theta}),$

(by a tautology and The Deduction Theorem)

$S_1 \vdash \exists x\theta x,\ \neg\theta c_{\exists x\theta}$

$S_1 \vdash \neg\forall\neg x\theta x,\ \neg\theta c_{\exists x\theta}$

(by Lemma 2.4.2, since $c_{\exists x\theta}$ does not occur in either $\neg\theta$ or S_1)

$S_1 \vdash \neg\forall\neg x\theta x,\ \forall x\neg\theta x$

Therefore, S_1 is an inconsistent theory, and this is a contradiction to the choice of S_1.

2.4.5. Completeness Theorem for PR^1 Every consistent first-order theory has a (standard) model.

Proof Let T be a consistent theory of a language L. By Lemma 2.4.4, there is a rich consistent extension S of T. By Lemma 2.4.1, theory S has a standard model A. Since $T\subseteq S$, a reduct of A is a model of T.

Here are some consequences of the Completeness Theorem. From now on, when we speak about models, we mean standard models, i.e. 2-models, while, for Boolean models, we shall keep the old name: the B-models. If T is a theory, and φ a sentence of language L, we should remember that

$T \models \varphi$ denotes that φ is true in all the models of T. In this case, we say that φ is a semantical consequence of T.

2.4.6. Completeness Theorem, another form Let T be a theory of a language L , and φ a sentence of L. Then: $T \vdash \varphi$ iff $T \models \varphi$.

Proof $(\rightarrow)$ Suppose $T \vdash \varphi$ and let A be a model of theory T. It is easy to see that A satisfies all the axioms of PR^1, and preserves all the rules of inference. Therefore, one can show that A satisfies all the consequences of T by induction on length of the proof.

$(\leftarrow)$ Suppose not $T \vdash \varphi$. Then, we can conclude that $T \cup \{\neg\varphi\}$ is a consistent theory. Really, if $T, \neg\varphi \vdash x \neq x$, then by the Deduction theorem, $T \vdash \neg\varphi \rightarrow x \neq x$ follows, so $T \vdash \varphi$, a contradiction. Hence, by the Completeness Theorem, there is a model A of the theory $T \cup \{\neg\varphi\}$, so $A \models \varphi$, and $A \models \neg\varphi$, a contradiction.

From Theorem 2.4.6, we can easily deduce Theorem 2.4.5. First observe that Theorem 2.4.6. can be stated as follows:

φ is not a theorem of T iff φ is not a semantical consequence of T. Thus, if T is a consistent theory, and φ is a contradiction, then φ is not a semantical consequence of T. Therefore, there is a model of T.

By Theorem 2.3.4, and Lemma 2.4.4, we have these connections between standard models and B-models:

2.4.7. Theorem The following conditions are equivalent for a first-order theory T:

(1) T is a consistent theory.

(2) T has a B-model.

(3) T has a standard model.

Let B be a Booelan algebra, and T a theory of a language L. If $\varphi \in Sent_L$, and $\|\varphi\|_B = 1$ in all the B-models of T, then, since $2 \subseteq B$, $\|\varphi\|_2 = 1$ in every 2-model of T, i.e. $T \models \varphi$. On the other hand, if $T \models \varphi$, by the Completeness Theorem, $T \vdash \varphi$. Using the inductive definition of $\| \ \|_B$, it follows easily that $\|\varphi\|_B = 1$, i.e. φ holds in all the B-models of T. Thus,

2.4.8. Corollary Let B and B' be Boolean algebras, and T a theory of a language L. If $\varphi \in Sent_L$, then φ holds in all the B-models of T iff φ holds in all the B'-models of T.

A simple, but important consequence of the Completeness Theorem is the Compactness Theorem. This theorem has many applications not only in logic, but in other areas of mathematics, as well: algebra, analysis, etc.

2.4.9. Compactness Theorem Let T be a theory of a language L. If every finite subset of T has a model, then T has a model.

Proof Suppose T has no models. Then, by the Completeness Theorem, T is an inconsistent theory. Let $\varphi_0,\dots,\varphi_n$ be a proof of a contradiction in T, and take that $S= T\cap\{\varphi_0,\dots,\varphi_n\}$. Then, S is a finite subset of T, and it is an inconsistent theory, so S has no models, a contradiction.

Here are some applications of the Compactness Theorem.

2.4.10. Example If T has arbitrary large finite models, then T has an infinite model. To see this, consider theory

$$S= T\cup\{\forall x_0\dots x_n\exists y(y\neq x_0\ \wedge\dots\wedge\ y\neq x_n):\ n\in\omega\}.$$

By assumption, every finite $S_0\subseteq S$ has a model; this is a finite model A of T, such that $|A|$ is greater than any n which occurs in S_0. Thus, S has a model B, and B is infinite.

By the use of the Compactness Theorem, we can show that certain theories are not finitely axiomatizable.

2.4.10. Lemma If $T\models\varphi$, then there is a finite $T_0\subseteq T$ such that $T_0\models\varphi$.

Proof If $T\models\varphi$, then by the Completeness Theorem $T\vdash\varphi$, thus, by the Deduction Theorem there is a finite $T_0\subseteq T$, such that $T_0\vdash\varphi$, so, again by the Completeness Theorem we have $T_0\models\varphi$.

2.4.11. Theorem Let $T_0\subseteq T_1\subseteq T_2\subseteq\dots$ be an increasing chain of theories such that for every $n\in\omega$, there is a model of T_n which is not a model of T_{n+1}. Then, the theory $T=\bigcup_n T_n$ is not finitely axiomatizable.

Proof Suppose T is finitely axiomatizable, and let S be it's finite set of axioms. Since S is a finite set, there is $m\in\omega$ such that for all $\varphi\in S$, $T_m\models\varphi$. On the other hand, for all $\varphi\in T$, we have $S\models\varphi$, so, for all $\varphi\in T_{m+1}$ we have $T_m\models\varphi$, and this means that every model of T_{m+1} is a model of T_m, a contradiction.

By the previous theorem, the following examples are not finitely axiomatizable theories.

2.4.12. Example 1° The theory of fields of characteristic 0. The axioms of this theory are the axioms of the field theory plus sentences

$$1+1\neq 0,\ 1+1+1\neq 0, \dots .$$

2° If T has arbitrary large finite models, then theory T_∞ of all the infinite models of T is not finitely axiomatizable. Observe that

$$T_\infty = T \cup \{\forall x_1 \dots x_n \exists y (y\neq x_1 \wedge \dots \wedge y\neq x_n) : n=1,2,\dots\}.$$

3° The theory of torsion free Abelian groups. The axioms of this theory are axioms of the theory of Abelian groups, plus

$$\forall x(2x=0 \rightarrow x=0),\ \forall x(3x=0 \rightarrow x=0),\ \dots .$$

4° The theory of divisible Abelian groups. The axioms are the axioms for Abelian groups, plus

$$\forall x \exists y(2y=x),\ \forall x \exists y(3y=x),\ \dots .$$

A class M of models of a language L is ***elementary***, if M is the class of all the models of a theory T of a language L. Class M^c is the class of all the models of L which do not belong to M. The following proposition gives the conditions under which an elementary class is finitely axiomatizable, i.e. when it has a theory with a finite set of axioms.

2.4.13. Theorem If both classes M and M^c are elementary, then M and M^c are finitely axiomatizable classes of models.

Proof Let T and S be theories of M and M^c, respectively. Then, $T\cup S$ obviously does not have a model, i.e. $T\cup S$ is an inconsistent theory. So, there are finite subsets $S_0 \subseteq S$, $T_0 \subseteq T$ such that $T_0 \cup S_0$ is also inconsistent. If $A \models T_0$, then A is not a model of S_0, i.e. $A \notin M^c$, thus $A \in M$ and $A \models T$. Therefore, for all $\varphi \in T$ we have $T_0 \models \varphi$. On the other hand , $T_0 \subseteq T$, so T is a finitely axiomatizable theory, since T and T_0 are equivalent. In a similar way, one can prove that S is finitely axiomatizable.

We shall exhibit in the following examples classes of models which are not elementary. In all the cases the Compactness Theorem is used.

2.4.14. Example 1° The class of all the fields of the prime characteristic is not elementary.

Proof Suppose this class is elementary with S as a set of axioms. Let $\mathcal{M}$ be the class of all the fields of the characteristic 0, and let φ be the finite conjunction of all the axioms of the field theory. Further, define $T=\{\varphi \to \theta: \theta\in S\}$. Then, $A\models T$ if and only if $A\models S$, or A is not a field. Thus, $A\models T$ iff $A\in\mathcal{M}^c$. But, by Theorem 2.4.11, $\mathcal{M}^c$ is not elementary since $\mathcal{M}$ is not finitely axiomatizable, a contradiction.

2° If T is a theory which has arbitrary large finite models, then the class of all the finite models of T is not elementary.

Proof Suppose this class is elementary with S as a set of axioms. Then, S has arbitrarily large finite models, so, S has an infinite model (see Example 2.4.10), a contradiction.

3° The class $\mathcal{M}$ of all the well-ordered sets $(A,\le)$ is not elementary.

Proof Let $\mathbf{A}=(A,\le)$ be an infinite well-ordered set, and let us introduce the new constant symbols $c_0, c_1, c_2, \ldots$. If

$$S=\mathrm{Th}\mathbf{A}\cup\{c_0>c_1, c_1>c_2, \ldots\},$$

then every finite $S_0\subseteq S$ has a model; this is an expansion of A, so by the Compactness Theorem S has a model $(B,\le,b_0,b_1,\ldots)$. Then, $(B,\le)$ is not well-ordered, but $(B,\le)\equiv(A,\le)$. If T were a theory of $\mathcal{M}$, we would have $(B,\le)\models T$ i.e. $(B,\le)\in\mathcal{M}$, a contradiction. Thus, $\mathcal{M}$ is not elementary.

Now we shall give an application of the Completeness Theorem to the decidability problem of first-order theories. In fact, this is a generalization of Theorem 1.3.2, see also Problem 1.12. First, we shall introduce some notation.

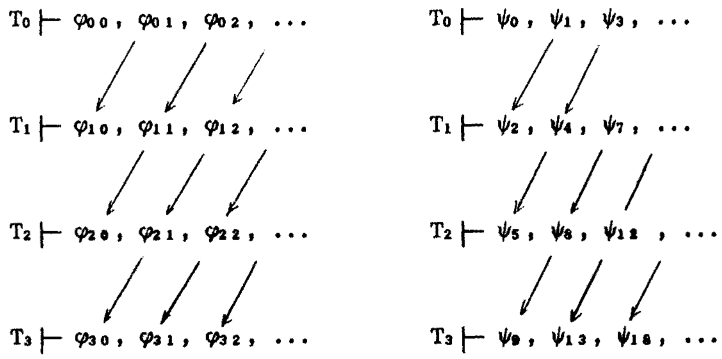
Let T be an axiomatic theory of a language L. We shall say that all complete extensions of T can be effectively and uniformly listed, if there is a sequence of complete and axiomatic theories T_n, $n\in\omega$, such that:

1° For each $n\in\omega$, $T\subseteq T_n$.

2° For every complete extension S of T there is $n\in\omega$ such that $S=T_n$.

3° All theories T_n's, $n\in\omega$, can be listed effectively, uniformly and simultaneously.

This last notion can be made more precise. By assumption every theory T_i is axiomatic, so all theorems of each T_i can be effectively enumerated, say φ_{ij}, $j\in\omega$, is such an enumeration (i.e. the mapping $i:\mapsto\ulcorner\varphi_{ij}\urcorner$, $j\in\omega$, is a recursive function). Then we shall say that 3° holds by definition, if there is an effective enumeration ψ_k, $k\in\omega$, (i.e. $k:\mapsto\ulcorner\psi_k\urcorner$, $k\in\omega$, is a recursive function) of all theorems of all theories T_n, $n\in\omega$, as displayed on diagram (D). That is $\varphi_{ij}=\psi_{c(i,j)}$, where



(D)

$c(i,j)= (i+j)(i+j+1)/2+i$, $i,j,\in\omega$. This function is known as Cantor's enumeration function, and it's main property is that it is an effective (i.e. recursive) pairing of natural numbers; that is, c maps ω^2 1-1 and onto ω.

2.4.15. Theorem Assume T is an axiomatic theory of a language L, and suppose all complete extensions of T can be listed in an effective and uniform way. Then T is decidable.

Proof Suppose T_n, $n\in\omega$, is an effective and uniform listing of all complete extensions of T, and let $\theta\in Sent_L$. By assumption, all theorems of theories T_n can be effectively listed uniformly and simultaneously as displayed on diagram (D). Also, there is a recursive enumeration of all theorems of T: $\varphi_0, \varphi_1, \varphi_2, \ldots$. Therefore,

(1) $\varphi_0, \psi_0, \varphi_1, \psi_1, \varphi_2, \psi_2, \ldots$

is also a recursive enumeration of some sentences of L. Thus, if $T\vdash\theta$, then $\theta=\varphi_n$ for some n. If $\sim T\vdash\theta$, then $T\cup\{\neg\theta\}$ is a consistent theory, so there is a model A of T such that $A\models\neg\theta$. By the choice of theories T_n, there is $m\in\omega$, such that $ThA= T_m$, hence $T_m\models\neg\theta$. Thus, there is j such that $\neg\theta=\varphi_{ij}=\psi_{c(i,j)}$. So, θ must appear at the even stage, or $\neg\theta$ must appear at the odd stage of the sequence (1). Hence an algorithm for the enumeration of this sequence gives a decision procedure for T.

Here are some applications of the last theorem. Therefore, all the examples of theories bellow are decidable theories.

2.4.16. Example 1° All complete extensions of J_0, the pure predicate calculus with equality, see Example 1.4.1, are:

$$T_n = \{\sigma_n\},\ n\in\omega\setminus\{0\},\ \text{and}\ T_\infty = \{\tau_1, \tau_2, \tau_3, \ldots\}.$$

2° The axioms of the theory of algebraically closed fields, which we denote by AF, are the axioms of the field theory plus the axioms which say that every polynomial of a degree ≥1 has a root. All complete extensions of this theory are:

$AF_p = AF \cup \{p\cdot 1=0\}$, p is a prime number,

$AF_0 = AF \cup \{n\cdot 1\neq 0\colon n\in\omega\setminus\{0\}\}$.

Later on, we shall prove this fact. Now, we shall mention two, less trivial examples, but without proofs.

3° All complete extensions of the theory of Boolean algebras were described by Tarski. An exposition on this matter can be found in [Chang, Keisler].

4° A description of all complete extensions of the theory of Abelian groups one can find in [Cherlin].

In both cases, the idea of the proof is to find certain "numerical invariants" (and these are effectively listed), such that two algebras A and B have the same invariant iff $A \equiv B$. Therefore, numerical invariants determine effectively all complete extensions of the corresponding theory.

2.5. Reduced products of models

Reduced product of models is a substantial construction of models owing to its model-theoretic properties. By this construction, new models are obtained starting from some of those already given models. The main theorem related to it is the Łoš Theorem on ultraproducts, a special case of reduced products.

Let $\{A_i : i\in I\}$ be a nonempty family of nonempty sets, and $A = \prod_i A_i$. By the Axiom of Choice, A is also a nonempty set. Further, let D be a filter over I, and $=_D$ a relation over A, defined in the following way:

If $f,g\in A$, then $f =_D g$ iff $\{i\in I\colon f(i)=g(i)\} \in D$.

Instead of $f =_D g$, we shall sometimes write $f = g \bmod D$, or $f(i)= g(i)$ D a.e. This notation is justified, if D is an ultrafilter, since, in this case, D induces a finite additive two-valued measure over I: a set $X \subseteq I$ has a measure 1, if $X \in D$, otherwise, it has a measure 0. The class of equivalence of $f \in A$ is denoted by f_D.

2.5.1. Lemma Let D be a filter over I. Then $=_D$ is an equivalence relation of the domain $A= \prod_I A_i$.

Proof Let $f,g,h \in A$. Then

(1) $f=f_D$, since $I \in D$ and $\{i \in I: f(i)=f(i)\} \in I$.

(2) $f=_D g$ implies $g=_D f$, since $\{i \in I: f(i)=g(i)\}= \{i \in I: g(i)=f(i)\}$.

(3) Suppose $f =_D g$ and $g =_D h$. Then, for $X= \{i \in I: f(i)=g(i)\}$, and $Y=\{i \in I: g(i)=h(i)\}$, we have $X,Y \in D$, so $X \cap Y \in D$. Since $X \cap Y \subseteq \{i \in I: f(i)=h(i)\}$ it follows that $\{i \in I: f(i)=h(i)\} \in D$, i.e. $f=_D h$.

The quotient set $\prod_I A_i /=_D$ is denoted by $\prod_D A_i$, and it is called the ***reduced product*** of sets A_i. An interesting case arises when A_i are domains of some models. So, let $\mathbf{A}_i$, $i \in I$, be models of a language L, and let D be a filter over I. By the last lemma, relation $=_D$ is an equivalence relation, but for models, we have something more.

2.5.2. Lemma Let $\mathbf{A}_i$, $i \in I$, be models of a language L, and $\mathbf{A}'= \prod_I \mathbf{A}_i$. Then,

1° Relation $=_D$ is concurrent with all the operations of $\mathbf{A}'$, i.e. if $F \in Fnc_L$, $ar(F)=n$, then, for all $f_1,\ldots,f_n,g_1,\ldots,g_n \in A'$, $f_1=_D g_1,\ldots,f_n=_D g_n$ implies $F^{A'}(f_1,\ldots,f_n)=_D F^{A'}(g_1,\ldots,g_n)$.

2° If $R \in Rel_L$, $ar(R)=n$, then for all $f_1,\ldots,f_n,g_1,\ldots,g_n \in A'$ $f_1=_D g_1,\ldots,f_n=_D g_n$ implies $\{i \in I: R^{A_i}(f_1(i),\ldots,f_n(i))\} \in D$, iff $\{i \in I: R^{A_i}(g_1(i),\ldots,g_n(i))\} \in D$.

Proof Let $f_1,f_2,\ldots,f_n,g_1,g_2,\ldots,g_n \in A'$ be such that $f_i =_D g_i$, $i \le n$. If $X_j= \{i \in I: f_j(i)=g_j(i)\}$, $j=1,\ldots,n$, then $X_j \in D$, so $\bigcap_j X_j \in D$. Thus,

1° $\bigcap_j X_j \subseteq \{i \in I: F^{A_i}(f_1(i),\ldots,f_n(i))= F^{A_i}(g_1(i),\ldots,g_n(i))\}$, so, $\langle F^{A_i}(f_1(i),\ldots,f_n(i)) \mid i \in I\rangle =_D \langle F^{A_i}(g_1(i),\ldots,g_n(i)) \mid i \in I\rangle$.

2° Let $Y=\{i \in I: R^{A_i}(f_1(i),\ldots,f_n(i))\}$ and $Z=\{i \in I: R^{A_i}(g_1(i),\ldots,g_n(i))\}$. Then, $Y \cap (\bigcap_j X_j) \subseteq Z$ and $Z \cap (\bigcap_j X_j) \subseteq Y$, so, $Y \in D$ iff $Z \in D$.

This lemma facilitates defining a model with domain $A= \prod_D A_i$ in the following way:

If $c \in Const_L$, then $c^A = \langle c^{A_i} : i \in I \rangle / D$.
If $F \in Fnc_L$, $ar(F)=n$, then for all $f_1, \ldots, f_n \in \Pi_i A_i$,
$F^A(f_{1D}, \ldots, f_{nD}) = F^{A'}(f_1, \ldots, f_n)_D$.
If $R \in Rel_L$, $ar(R)=n$, then for all $f_1, \ldots, f_n \in \Pi_i A_i$,
$R^A(f_{1D}, \ldots, f_{nD})$ iff $\{i \in I : R^{A_i}(f_1(i), \ldots, f_n(i))\} \in D$.

Such a constructed model **A** is called the *reduced product* of models A_i, and it is denoted by $\Pi_i A_i / D$ or $\Pi_D A_i$. It is easy to see that the map $k: f \longrightarrow f_D$ is a homomorphism of model $A' = \Pi_i A_i$ onto $\Pi_D A_i$. For example, if R is a relation symbol of length n, and $R^{A'}(f_1, \ldots, f_n)$ is true for $f_1, \ldots, f_n \in A'$, then, for each $i \in I$, $R^{A_i}(f_1(i), \ldots, f_n(i))$, so
$\{i \in I : R^{A_i}(f_1(i), \ldots, f_n(i))\} \in D$, i.e. $R^A(f_{1D}, \ldots, f_{nD})$.

2.5.3. Example 1° If $D=\{I\}$, then $\Pi_D A_i \approx \Pi_i A_i$. Thus, the products of models are special cases of reduced products.
2° Let D be a principal filter over I generated by $i_0 \in I$. Then, the map $\tau: f_D \longrightarrow f(i_0)$ is an isomorphism between $\Pi_D A_i$ and A_{i_0}.
3° Let $X \subseteq I$, and D be a principal filter over I generated by $X \subseteq I$, i.e. $D = \{Y \subseteq I : X \subseteq Y\}$. Then the map $f_D \longmapsto \langle f(i) : i \in X \rangle$ is an isomorphism of models $\Pi_D A_i$ and $\Pi_{i \in X} A_i$.

Suppose that all the models A_i, $i \in I$, are mutually equal, say $A_i = A$. Then the reduced product $\Pi_D A_i$ is called the reduced power of A, and is denoted by A^I/D.

Now, we shall turn to the most interesting case of reduced products, to ultraproducts.

2.5.4. Definition Let D be an ultrafilter over a set I, and suppose that A and A_i, $i \in I$, are models of a language L. Then, the reduced product $\Pi_D A_i$ is called the ultraproduct of models A_i. The reduced power A^I/D is called an ultrapower of A.

Example 2.5.3 shows that ultraproducts of interest are those constructed using nonprincipal ultrafilters. The ultraproduct construction preserves the first-order properties. This is the content of the Łoš theorem and this theorem has many applications in model theory.

2.5.5. Lemma Let $A = \Pi_D A_i$ be a reduced product of models A_i, $i \in I$, of a language L, $t_1, \ldots, t_n \in Term_L$ and R be an m-ary relation symbol of L. Then

1° $t^A[f_{1D},\dots,f_{nD}]= \langle t^{A_i}[f_1(i),\dots,f_n(i)]: i\in I\rangle_D$.

2° $R^A(t_1{}^A[f_{1D},\dots,f_{nD}],\dots,t_m{}^A[f_{1D},\dots,f_{nD}])$ iff
$\{i\in I: R^{A_i}(t_1{}^{A_i}[f_1(i),\dots,f_n(i)],\dots,t_m{}^{A_i}[f_1(i),\dots,f_n(i)])\} \in D$.

Proof First, let us observe that $k:\Pi_I A_i \longrightarrow \Pi_D A_i$, $k(f)= f_D$, $f\in\Pi_I A_i$, is a homomorphism. By induction on the complexity of terms t, taking $B= \Pi_I A_i$, it is easy to prove that

$$kt^B[f_1,\dots,f_n]= t^A[kf_1,\dots,kf_n]= t^A[f_{1D},\dots,f_{nD}].$$

On the other hand,

$$kt^B[f_1,\dots,f_n]= \langle t^{A_i}[f_1(i),\dots,f_n(i)]: i\in I\rangle_D,$$

so 1° holds.

2° Let $g_j\in\Pi_I A_i$ be defined by $g_j(i)= t_j{}^{A_i}[f_1(i),\dots,f_n(i)]$, $i\in I$, $1\le j\le m$. Then, by 1°, $g_{jD}= t_j{}^A[f_{1D},\dots,f_{nD}]$, so

$$R^A[g_{1D},\dots,g_{nD}] \text{ iff } \{i\in I: R^{A_i}[g_1(i),\dots,g_m(i)]\}\in D \text{ iff}$$
$$\{i\in I: R^{A_i}(t_1{}^{A_i}[f_1(i),\dots,f_n(i)],\dots,t_m{}^{A_i}[f_1(i),\dots,f_n(i)])\} \in D. \blacksquare$$

2.5.6. Theorem (J. Łoš) Let A_i, $i\in I$, be models of a language L, and let D be an ultrafilter over I. Then for every formula $\varphi v_0\dots v_n$ of L and $f_0,\dots,f_n\in\Pi_I A_i$, we have

$$\Pi_D A_i \models \varphi[f_{0D},\dots,f_{nD}] \text{ iff } \{i\in I: A_i \models \varphi[f_0(i),\dots,f_n(i)]\} \in D.$$

Proof We shall prove this theorem by induction on the complexity of formula φ. So, let $A= \Pi_D A_i$, and $\varphi v_0\dots v_n$ be a formula of L. Then, we can distinguish the following cases:

1° $cl(\varphi)=0$. Then we have two subcases:

(1) φ is of the form $t_1v_0\dots v_n= t_2v_0\dots v_n$, where $t_1,t_2\in Term_L$. Then,

$$\Pi_D A_i \models \varphi[f_{0D},\dots,f_{nD}] \text{ iff } t_1{}^A[f_{0D},\dots,f_{nD}]= t_2{}^A[f_{0D},\dots,f_{nD}],$$

so by Lemma 2.5.5,

iff $\langle t_1{}^{A_i}[f_0(i),\dots,f_n(i)]: i\in I\rangle =_D \langle t_2{}^{A_i}[f_0(i),\dots,f_n(i)]: i\in I\rangle$,
iff $\{i\in I: t_1{}^{A_i}[f_0(i),\dots,f_n(i)]= t_2{}^{A_i}[f_0(i),\dots,f_n(i)]\}\in D$,
iff $\{i\in I: A_i \models \varphi[f_0(i),\dots,f_n(i)]\} \in D$.

(2) Formula φ is of the form $R(t_1v_0\dots v_n,\dots,t_mv_0\dots v_n)$, where R is a relation symbol of L of length m, and $t_1,\dots,t_m\in Term_L$. So, the statement holds for this case by Lemma 2.5.5.

2° Suppose the statement is true for all the formulas ψ, $cl(\psi)<cl(\varphi)$, and let $cl(\varphi)>1$. Then, we can distinguish the following cases:

(1) $\varphi= (\psi \wedge \theta)$. Then $cl(\psi)$, $cl(\theta) < cl(\varphi)$, so, by the inductive hypo-

thesis,

$A \models \varphi[f_{0D},\dots,f_{nD}]$ iff $A \models \psi[f_{0D},\dots,f_{nD}]$ and $A \models \theta[f_{0D},\dots,f_{nD}]$

iff $\{i\in I\colon A_i \models \psi[f_0(i),\dots,f_n(i)]\}\in D$ and $\{i\in I\colon A_i \models \theta[f_0(i),\dots,f_n(i)]\}\in D$

iff $\{i\in I\colon A_i \models \psi[f_0(i),\dots,f_n(i)]\}\cap\{i\in I\colon A_i \models \theta[f_0(i),\dots,f_n(i)]\} \in D$

iff $\{i\in I\colon A_i \models \varphi[f_0(i),\dots,f_n(i)]\}\in D$.

(2) $\varphi = \neg\psi$. Then $cl(\psi) < cl(\varphi)$, so

$A \models \varphi[f_{0D},\dots,f_{nD}]$ iff not $A \models \varphi[f_{0D},\dots,f_{nD}]$,

so, using the inductive hypothesis,

iff not $\{i\in I\colon A_i \models \psi[f_0(i),\dots,f_n(i)]\}\in D$,

D is an ultrafilter, so, by Theorem 2.2.3,

iff $\{i\in I\colon A_i \models \psi[f_0(i),\dots,f_n(i)]\}^c\in D$

iff $\{i\in I\colon \text{not } A_i \models \psi[f_0(i),\dots,f_n(i)]\}\in D$

iff $\{i\in I\colon A_i \models \varphi[f_0(i),\dots,f_n(i)]\}\in D$.

This is the only place where we have used the assumption that D is an ultrafilter, not just a filter.

(3) $\varphi = \exists y\psi y v_0\dots v_n$. Then $cl(\psi)<cl(\varphi)$.

Assume $A \models \varphi[f_{0D},\dots,f_{nD}]$. Then, for some $g\in\prod_I A_i$

$A \models \psi[g_D,f_{0D},\dots,f_{nD}]$. By the inductive hypothesis,

$\{i\in I\colon A_i \models \psi[g(i),f_0(i),\dots,f_n(i)]\}\in D$. Since

$\{i\in I\colon A_i \models \psi[g(i),f_0(i),\dots,f_n(i)]\}\subseteq\{i\in I\colon A_i \models \varphi[f_0(i),\dots,f_n(i)]\}$

it follows that

$\{i\in I\colon A_i \models \varphi[f_0(i),\dots,f_n(i)]\}\in D$.

On the other hand, suppose $\{i\in I\colon A_i \models \varphi[f_0(i),\dots,f_n(i)]\}\in D$, and take $X=\{i\in I\colon A_i \models \varphi[f_0(i),\dots,f_n(i)]\}$. Then, for $i\in X$ we have $A_i \models \varphi[f_0(i),\dots,f_n(i)]$, i.e. for some $c_i\in A_i$, $A_i \models \psi[c_i,f_0(i),\dots,f_n(i)]$ Let $g\in\prod_I A_i$ be defined so that $g(i)=c_i$, $i\in X$, otherwise $g(i)$ is arbitrary. Then, $X \subseteq \{i\in I\colon A_i \models \psi[g(i),f_0(i),\dots,f_n(i)]\}$, thus, since $X\in D$, we have $\{i\in I\colon A_i \models \psi[g(i),f_0(i),\dots,f_n(i)]\}\in D$. By the inductive hypothesis, then $A \models \psi[g_D,f_{0D},\dots,f_{nD}]$, i.e. $A \models \varphi[f_{0D},\dots,f_{nD}]$.

Other logical operations are reducible to those considered above, so this finishes the proof of the theorem.

If $\varphi\in Sent_L$, then the logical truth of formula φ does not depend on the choice of a valuation. Therefore, by the Łoš Theorem, we have

2.5.7 Corollary Let φ be a sentence of a language L, and let A_i, $i\in I$, be models of L. Then,

$\prod_D A_i \models \varphi$ iff $\{i\in I\colon A_i \models \varphi\}\in D$.

For example, the ultraproduct of ordered fields is a field. Really, the axioms of FO are first-order sentences, so, Corollary 2.5.7 can be applied. On the other hand, the ordinary product of the fields is a ring but not a field, because there are divisors of zero in the product.

An ultrapower A^I/D is an elementary extension of model **A**. This follows from the fact that the map $\hat{}:A \longrightarrow A^I/D$, $\hat{a} = \langle a: i \in I\rangle/D$, $a \in A$, is an elementary embedding of **A** into A^I/D. This is true, because for $\varphi v_0 \ldots v_n \in For_L$ and $a_0, \ldots, a_n \in A$ by the Łoš Theorem, we have

$$A^I/D \models \varphi[a_{0D}, \ldots, a_{nD}] \text{ iff } \{i \in I: A \models \varphi[a_0, \ldots, a_n]\} \in D$$
$$\text{iff } A \models \varphi[a_0, \ldots, a_n].$$

There are some other interesting applications of the Los Theorem. As an example, we shall give an alternative proof of the Compactness Theorem which does not rely on the Completeness Theorem.

2.5.8. Another proof of the Compactness Theorem Let I be a family of the finite subsets of a theory T, and suppose for each $i \in I$, A_i is a model of i. Further, let $S_\theta = \{i \in I: \theta \in i\}$, where $\theta \in T$. Then, the family $X = \{S_\theta: \theta \in T\}$ has FIP, so, there is an ultrafilter D over I which contains X. Let $A = \Pi_D A_i$. Then, for $\theta \in T$, we have $A \models \theta$. Really,

$$S_\theta \subseteq \{i \in I: A_i \models \theta\} \text{ and } S_\theta \in D$$

so $\{i \in I: A_i \models \theta\} \in D$. By Los Theorem it follows that $A \models \theta$.

There are other constructions similar to ultraproducts. We shall consider such a construction which is useful for studying certain properties of models of formal arithmetic, PA. For example, we can prove in this way that PA is not finitely axiomatizable, and even stronger result that PA has no Σ_n^0 axiomatization for any $n \in \omega$ (A.Mostowski). In a similar manner, one can prove that a certain combinatorial principle (a version of the Ramsey Theorem) is not provable in PA, even if it is true in the standard structure of natural numbers. We shall first introduce some notation.

Let $M \models$ PA and $m_1, \ldots, m_k \in M$. A subset $X \subseteq M^n$, $n \in \omega$, is ***definable*** with parameters $m_1, \ldots, m_k$, if there is $\varphi \underline{xy} \in For_{L(PA)}$ such that

$$X = \{(a_1, \ldots, a_n) \in M: M \models \varphi[a_1, \ldots, a_n, m_1, \ldots, m_k]\}.$$

A function $f:M \longrightarrow M$ is definable (with parameters) iff the graph of f is definable in the above sense. The set X is Σ_n^0-definable if φ is a Σ_n^0-formula. Let D(M) denote the set of all subsets of **M**, definable with parameters, and $F(M)=\{f:M\longrightarrow M:$ f is definable with parameters in $M\}$. Some simple properties of these sets are summarized in the following proposition.

2.5.9. Lemma 1° $D(M)$ is closed for the substitution for elements from $F(M)$, i.e. if $f_1,\ldots,f_n\in F(M)$, $\underline{m}\in M$ and $\theta\in For_{L(PA)}$ then

$$\{a\in M:\ M\models\theta[f_1a,\ldots,f_na,\underline{m}]\}\in D(M).$$

2° The identity function i_M of M belongs to $F(M)$, and if $a\in M$ then $\mathbf{a}\in F(M)$, where $\mathbf{a}(x)=a$, $x\in M$.

3° $X\in D(M)$ iff characteristic function k_X of X belongs to $F(M)$.

4° $\mathcal{D}(M)=(D(M),\cup,\cap,\leq,\emptyset,M)$ is a Boolean algebra.

5° If $f,g\in F(M)$ then $f+g, f\cdot g, f'\in F(M)$, hence $(F(M),+,\cdot,',0)$ is a model. We shall also denote this model by $F(M)$.

Now, we shall introduce the notion of ***definable ultrapower***. Let M be a model of PA, and G an ultrafilter of Boolean algebra $\mathcal{D}=\mathcal{D}(M)$. We define a binary relation $\sim$ on set $F=F(M)$ as follows:

$$f\sim g\ \text{ iff }\ \{i\in M:\ f(i)=g(i)\}\in G.$$

It is easy to see that $\sim$ is an equivalence relation of F, moreover, it is a congruence relation of model $F=F(M)$. Thus, the quotient structure $F/\sim$ is well-defined, and this structure we shall denote by F/G. Now, we have a variant of Łoš Theorem.

2.5.10. Theorem Let M be a model of PA, and G an ultrafilter of $\mathcal{D}$.

1° The map $k:a\longrightarrow\mathbf{a}/G$, $a\in M$, is an embedding of M in F/G.

2° If φ is a formula of language $\{+,\cdot,',0\}$, and $f_1,\ldots,f_n\in F$, then

$$F/G\models\varphi[f_{1G},\ldots,f_{nG}]\ \text{ iff }\ \{i\in M:\ M\models\varphi[f_1(i),\ldots,f_n(i)]\}\in G.$$

Proof The proof is inductive, and similar to the proof of Theorem 2.5.6, so we shall consider only the main inductive step, the case of the existential quantifier. Therefore, let $\varphi x_1\ldots x_n=\exists y\psi yx_1\ldots x_n$, $X=\{i\in M:\ M\models\exists y\psi yf_1(i)\ldots f_n(i)\}$, and suppose $X\in G$. Therefore X is a definable subset of M. Now, let us define function $g:M\longrightarrow M$,

$g(i)=$ the least $d\in M$ in respect to $<^M$, such that

$M\models\psi[d,f_1(i),\ldots,f_n(i)]$, if such d exists, and $g(i)=0$, otherwise.

As $M\models$ PA, and scheme (L) is provable in PA, see Problem 2.16, function g is well-defined and $g\in F$. Thus $\{i\in M:\ M\models\psi[g(i),f_1(i),\ldots,f_n(i)]\}\in M$, so by the inductive hypothesis $F/G\models\psi[g_G,f_{1G},\ldots,f_{nG}]$.

Observe that $\neg F\models$ PA, but $F/G\models$ PA.

Using definable ultrapowers, we shall prove that every countable model M of PA has an elementary end extension, i.e there is N such that $M \prec N$, and M is an initial segment of N in respect to $<^N$. First, we shall define a special ultrafilter over D=D(M).

2.5.11. Lemma There is an ultrafilter G of D such that for every bounded function $f \in F$ there is $X \in G$ such that $f|X$ is a constant function.

Proof First we show

(1) Suppose $A \in D$, $f \in F$, the range of f is bounded, i.e. there is $m \in M$ such that $f(M) \subseteq [0,m]_M$, and A is unbounded in M. Then there is an unbounded $B \in D$ such that $B \subseteq A$ and f is a constant function on B.

If for some $b \in M$, $A \cap f^{-1}(\{b\})$ is an unbounded subset of M, the proof of (1) is finished, so assume the other case, i.e. for each $b \in M$ there is $a \in M$ such that $A \cap f^{-1}(\{b\}) \subseteq [0,a]_M$. The set $A \cap f^{-1}(\{b\})$ is definable in M, so we may write informally

$$M \models \forall x \leq m \exists y (A \cap f^{-1}(\{x\}) \subseteq [0,y]).$$

By Problem 2.16, model M satisfies Collection scheme (B), so there is $b \in M$ such that

$$M \models \forall x \leq m \exists y \leq b (A \cap f^{-1}(\{x\}) \subseteq [0,y]),$$

i.e. $A \cap f^{-1}(\{x\}) \subseteq [0,b]_M$ for all $x \leq^M m$. Therefore,

$$\bigcup_{x \leq m} (A \cap f^{-1}(\{x\}) \subseteq [0,b]_M, \text{ hence, } A \cap (\bigcup_{x \leq m} f^{-1}(\{x\})) \subseteq [0,b]_M.$$

But $\bigcup_{x \leq m} f^{-1}(\{x\}) = M$, thus $A \subseteq [0,b]_M$, which is a contradiction to our hypothesis that A is an unbounded subset of M, and this finishes the proof of (1).

M is countable, so $|F| = \aleph_0$. Therefore, there is an enumeration $f_0, f_1, \ldots$ of all elements of F which are bounded. By (1), there is a sequence $\ldots \subseteq C_2 \subseteq C_1 \subseteq C_0$ of definable subsets of M such that $C_0 = M$, C_{n+1} is an unbounded subset of C_n, and f_n is a constant function on C_{n+1}. Let G be an ultrafilter containing $\{X \in D\colon \text{ for some } n \in \omega\ C_n \subseteq X\}$. Then G has the wanted properties.

2.5.13. Theorem Let M be a countable model of PA. Then M has a proper elementary end extension.

Proof Let G be an ultrafilter with the property as in the previous Lemma, and M'= F/G. Further, for each $a \in M$, let $\mathbf{a} \in F$ be defined by $\mathbf{a}(x) = a$

for all $x \in M$, and $N = \{a_G : a \in M\}$. Then there is a submodel N of M' with domain N, and $k: M \approx N$, where $k: a \longrightarrow a_G$. Therefore, we may identify models M and N, and then we have $M \prec F/G$.

Let $f \in F$ and suppose $f_G \leq a_G$. Then for some $Y \in G$, for all $x \in Y$, we have $fx \leq a$. If $gx = fx$ for $x \in Y$, and $gx = 0$ for $x \in M \setminus Y$, then $f_G = g_G$, and for all $x \in M$, $gx \leq a$, i.e. g is a bounded function. By our assumption on G, function g is a constant function on an $X \in G$, i.e. there is $d \in M$ such that for all $x \in X$ $gx = d$. Thus, $g_G = d_G$, i.e. $g_G \in N$, and M' is an elementary end extension of N.

If M and N are models of PA, and N is an elementary end extension of M, then we shall write $M \prec_e N$. Then, if we iterate the previous construction ω_1 times, we can build a chain of models

$$M = M_0 \prec_e M_1 \prec_e M_2 \prec_e \ldots \prec_e M_\delta \prec_e \ldots, \quad \delta < \omega_1,$$

and there is a model K of PA such that $K = \bigcup_\delta M_\delta$ and for all $\delta < \omega_1$, $M_\delta \prec K$. The model K has the interesting property that every proper initial segment of K is countable, even if K itself is uncountable, i.e. the ordering of K resembles to the ordering of ordinal ω_1. So, such models of PA are called *ω_1-like models*.

Now, we shall consider, using the same technique, the problem of finite axiomatizability of PA. First, we shall introduce some refinements. In the following M denotes a model of PA. A subset $A \subseteq M^k$ is $\Sigma_n(M)$ definable, if there is a $\Sigma_n{}^0$-formula $\varphi x_1 \ldots x_k y_1 \ldots y_m$ of L_{PA} such that

$$A = \{(a_1, \ldots, a_k) : M \models \varphi[a_1, \ldots, a_k, b_1, \ldots, b_m]\},$$

where $b_1, \ldots, b_m \in M$. A function $f: M \longrightarrow M$ is $\Sigma_n(M)$ definable, if the graph of f is a $\Sigma_n(M)$ definable subset of M^2. Now let D be the set of all definable subsets of M, and $F_{\Sigma n}(M)$ $\Sigma_n(M)$-definable functions of M. For example, if c is the Cantor enumeration function, then $c \in F_{\Sigma 1}(\omega)$, where ω is the standard structure of natural numbers. In the following, we shall write $\langle x,y \rangle$ instead of $c(x,y)$. Further, if G is an ultrafilter of D, then similarly as in the previous paragraph, we can construct a model $F_{\Sigma n}(M)/G$. We have also a Łoš-type theorem for such a model.

2.5.14. Lemma Let $n \geq 1$, $\theta x_1 \ldots x_k$ be a Σ_n-formula and $f_1, \ldots, f_k \in F_{\Sigma_n}(M)$. Then, $F_{\Sigma n}(M)/G \models \theta[f_{1G}, \ldots, f_{kG}]$ iff $\{m \in M : M \models \theta[f_1 m, \ldots, f_k m]\} \in G$.

Proof As usually, the proof is performed by induction on the complexity of formulas, but in this case up to the complexity of n. We shall consider the only interesting case, the existential quantifier step. So let $\theta x = \exists y_1 \dots y_p \psi x y_1 \dots y_p$, where ψ is a Π_{n-1}-formula, $f \in F_{\Sigma n}(M)$ and assume $A \in G$, where $A = \{m \in M: M \models \theta[fm]\}$. Now, we shall define $h_1, \dots, h_p \in F_{\Sigma n}(M)$ such that

(1) $\{m \in M: M \models \psi[fm, h_1 m, \dots, h_p m]\} \in G$.

Since f is $\Sigma_n(M)$ definable, there is a Σ_n-formula $\varphi(x,y)$ of L_{PA} such that $fm=k$ iff $M \models \varphi mk$. Defining h_i by

$$h_i m = k \text{ iff } M \models \exists t(\varphi mt \wedge \exists z(\exists y_1 \dots y_p \leq z(\langle y_1, \dots, y_p \rangle = z \wedge y_i = k \wedge \psi t y_1 \dots y_p) \wedge \forall s < z \forall y_1 \dots y_p \leq s(\langle y_1, \dots, y_p \rangle = s \rightarrow \neg \psi t y_1 \dots y_p))),$$

for $i=1,\dots,p$, we have $A = \{m \in M: M \models \psi[fm, h_1 m, \dots, h_p m]\}$, and so (1) holds. Using the schemes in Problem 2.16 one can show that $h_1, \dots, h_p$ are $\Sigma_n(M)$ definable, so the inductive hypothesis can be applied, i.e.

$$F_{\Sigma n}(M)/G \models \psi[f_G, h_{1G} \dots, h_{pG}],$$

$$F_{\Sigma n}(M)/G \models \exists y_1 \dots y_p \psi[f_G, y_1 \dots, y_p].$$

As the notions of formulas, sentences and proofs can be arithmetizied, i.e. formalized in PA, the notion of truth of particular formulas of PA can be formalized as well. In the proof that PA has no Σ_n axiomatization for any $n \in \omega$, we are going to use such a result. The proof of this fact can found for example in [Smorinski].

2.5.15. Theorem There exist a Σ_n-formula $Sat_{\Sigma n}(x, y_1, \dots, y_n)$ of language L_{PA} such that for every Σ_n-formula φ of L_{PA}, and the code $e = \ulcorner \varphi \urcorner$ of φ (the Gödel number of φ),

$$PA \vdash \forall x_1 \dots x_n (\varphi x_1 \dots x_n \leftrightarrow Sat_{\Sigma n}(e, x_1, \dots, x_n)).$$

Using that scheme (L) is provable in PA, see Problem 2.16, it is easy to see that in any nonstandard model M of PA holds the following:

2.5.16. Overspill Lemma (A.Robinson) If θ is a formula of L_{PA}, and $M \models \theta n$ for all $n \in \omega$, then there is an infinite $a \in M$ such that $M \models \theta[a]$.

Proof Indeed, if there is not such an a, then the set $X = \{m \in M: M \models \neg\theta m\}$ is nonempty, so by scheme (L) there is the least $b \in X$. Then b is infinite, and $(b-1) \notin X$, hence $M \models \theta[b-1]$, a contradiction.

2.5.17. Theorem The theory PA has no Σ_n axiomatization for any $n \in \omega$.

Proof Suppose there is a Σ_n-set S of sentences of L_{PA} which is equivalent to PA. Let $F_{\Sigma n}(\omega)/G$ be a definable ultraproduct constructed as in

Lemma 2.5.14, where G is a nonprincipal ultrafilter of D. Since $\omega\models S$, by this lemma and our assumption that $S\subseteq\Sigma_n$, we have $F_{\Sigma_n}(\omega)/G\models S$.

Now, we shall prove that $F_{\Sigma_n}(\omega)/G$ is not a model of PA; this will mean that theory S is not equivalent to PA. Suppose, in contrary, that $F_{\Sigma_n}(\omega)/G$ is a model of PA. Since G is nonprincipal, if i is the identity function of ω, then for all $a\in\omega$, $a_G<i_G$. Thus $F_{\Sigma_n}(\omega)/G$ is a nonstandard model of PA, hence, by the Overspill Lemma

(1) If for all $n\in\omega$, $F_{\Sigma_n}(\omega)/G\models\theta n$, then there is an infinite element $a\in F_{\Sigma_n}(\omega)/G$ such that $F_{\Sigma_n}(\omega)/G\models\theta a$.

Further, let $g\in F_{\Sigma_n}(\omega)$ and $\theta\in\Sigma_n$ be such that $gm=k$ iff $\omega\models\theta mk$. If e is the Gödel number of formula θ, then

$gm=k$ iff $\omega\models Sat_{\Sigma_n}(e,m,k)$, so

$\{i\in\omega\colon \omega\models Sat_{\Sigma_n}(e,i,gi)\}\in G$,

$\{i\in\omega\colon \omega\models \forall x(Sat_{\Sigma_n}(e,i,x)\rightarrow x=gi)\}\in G$.

Since Sat_{Σ_n} is a Σ_n-formula, by Lemma 2.5.14,

$F_{\Sigma_n}(\omega)/G\models Sat_{\Sigma_n}(e,i_G,g_G)$, $F_{\Sigma_n}(\omega)/G\models \forall x(Sat_{\Sigma_n}(e,i_G,x)\rightarrow x=g_G)$.

Thus, we see that g_G is the unique element in $F_{\Sigma_n}(\omega)/G$ which satisfies these formulas, so

For every $g\in F_{\Sigma_n}(\omega)$ there is $e\in\omega$ such that

$F_{\Sigma_n}(\omega)/G\models Sat_{\Sigma_n}(e,i_G,g_G)\wedge\forall x(Sat_{\Sigma_n}(e,i_G,x)\rightarrow x=g_G)$,

thus

(2) For every infinite $\alpha\in F_{\Sigma_n}(\omega)/G$,

$F_{\Sigma_n}(\omega)/G\models \forall z\exists y<\alpha(Sat_{\Sigma_n}(y,i_G,z)\wedge\forall x(Sat_{\Sigma_n}(y,i_G,x)\rightarrow x=z)$.

Also, for every $e\in\omega$ there is at most one $g\in F_{\Sigma_n}$ such that

$F_{\Sigma_n}(\omega)/G\models Sat_{\Sigma_n}(e,i_G,g_G)\wedge\forall x(Sat_{\Sigma_n}(e,i_G,x)\rightarrow x=g_G)$.

Therefore, for every $j\in\omega$, set

$\{g_G\colon$ there is $e<j$, $F_{\Sigma_n}(\omega)/G\models Sat_{\Sigma_n}(e,i_G,g_G)\wedge$
$\forall x(Sat_{\Sigma_n}(e,i_G,x)\rightarrow x=g_G)\}$

is finite, hence different from $F_{\Sigma_n}(\omega)/G$, and so

(3) for every $j\in\omega$,

$F_{\Sigma_n}(\omega)/G\models \neg\forall z\exists y<j(Sat_{\Sigma_n}(y,i_G,z)\wedge\forall x(Sat_{\Sigma_n}(y,i_G,x)\rightarrow x=z)$.

But (2) and (3) contradicts Overspill Lemma, i.e. (1).

If S is a finite set, then obviously $S\subseteq\Sigma_n$ for some $n\in\omega$. Hence, we have the following

2.5.18. Corollary (Skolem) The theory PA is not finitely axiomatizable.

Exercises

2.1. Show that every finite Boolean algebra is atomic.

2.2. Let B be a Boolean algebra, and $n\in\omega$, $n\geq 1$. A map $f:B^n\longrightarrow B^n$ is a term-mapping iff there are Boolean terms $t_1,\dots,t_n$ such that for all $b_1,\dots,b_n\in B$, $f(b_1,\dots,b_n)=(t_1^B(b_1,\dots,b_n),\dots,t_n^B(b_1,\dots,b_n))$. If f is a Boolean term-mapping, show that f is 1-1 iff f is onto.

2.3. If B is a finite Boolean algebra of the cardinality 2^n, then B has exactly n ultrafilters.

2.4. Prove that every infinite Boolean algebra has a nonprincipal ultrafilter.

2.5. Show that for every infinite cardinal k, there is a Boolean algebra of the cardinality k with exactly k ultrafilters.

2.6. Let k be an infinite cardinal number. A set $C\subseteq k$ is closed and unbounded, shortly cub, if

1° C is closed, i.e. if $X\subseteq C$ and $\sup X<k$, then $\sup X\in C$.

2° C is unbounded i.e. $\sup C=k$.

Show that $\{X\in P(k)$: X conatains a cub subset of $k\}$ is a proper filter of $P(k)$.

2.7. If B is a complete and atomic Boolean algebra, then $B\approx P(X)$ for some set X.

2.8. Let X be a topological space.

1° If B is the a family of all clopen subsets of X ($A\subseteq X$ is ***clopen*** if A is open and closed in X), then $(B,\cup,\cap,',\subseteq,0,X)$ is a Boolean algebra.

2° Show that the Boolean algebra of clopen subsets of the Cantor space 2^k is isomorphic to the free Boolean algebra with k free generators.

3° A subset $Y\subseteq X$ is ***regular open*** in X, if Y is open and the closure of Y is open in X. If R is the family of all regular open subsets of X, show that the Boolean algebra of all regular open subsets of X is a complete Boolean algebra.

2.9 If B is a Boolean algebra, then $S \subseteq B$ is a ***pairwise-disjoint*** set iff for all distinct $x,y \in S$, $x \cdot y = 0$. Prove that every infinite Boolean algebra has an infinite pairwise-disjoint set.

2.10. We say that a Boolean algebra B satisfies the ***countable chain condition*** (abr. CCC) iff every pairwise-disjoint set of non-zero elements of B is countable.

1° Prove that a Boolean algebra B satisfies CCC iff every subset $Z \subseteq B$ has a countable subset Y such that Y and Z have the same set of upper bounds.

2° A Boolean algebra B is a ***σ-algebra*** if every countable subset of B has the least upper bound and the greatest lower bound. Show that every σ-algebra satisfying CCC is complete.

3° Prove that the regular open algebra of a topological space with a countable base satisfies CCC.

2.11. Let Fin be the filter of all cofinite subsets of ω, and let us define a binary realtion ~ on P(ω) as follows:

$$X \sim Y \text{ iff } (X \cap Y) \cup (X^c \cap Y^c) \in \text{Fin}.$$

1° Show that ~ is an equivalence relation of P(ω).

2° If $\omega^* = P(\omega)$, then $(\omega^*, \cup^*, \cap^*, \subseteq^*, 0/\sim, \omega/\sim)$ is a Boolean algebra, where

$$X/\sim \cup^* Y/\sim = (X \cup Y)/\sim, \quad X/\sim \cap^* Y/\sim = (X \cap Y)/\sim, \quad X \subseteq^* Y \text{ iff } Y \setminus X \in \text{Fin}.$$

3° Show that in ω^* there are uncountable chains.

4° Show that there is an uncountable pairwise-disjoint set $S \subseteq \omega^*$.

2.12. Prove the Completeness Theorem for formulas of language L:

If Γ is formally consistent set of formulas of L, then there is a model A and an assignment μ of domain A such that $A \models \varphi[\mu]$ for all $\varphi \in \Gamma$.

2.13. A partially ordered set $(I, \leq)$ is upward directed iff

$$\forall i,j \in I \, \exists k \in I \;\; i,j \leq k.$$

Suppose, now, that $\langle A_i : i \in I \rangle$ is a family of models such that

$$\forall i,j \in I \; (i \leq j \rightarrow A_i \subseteq B_i).$$

1° Show that there is a unique model A such that

a) $A = \bigcup_i A_i$, b) For every $i \in I$, $A_i \subseteq A$.

A model A is called the ***direct limit*** of models $\langle A_i : i \in I \rangle$.

2° Show that every model is a direct limit of finitely generated models.

3° If φ is Π_2^0, i.e a universal-existential sentence, and if it holds

in all A_i , $i\in I$, then φ holds in the direct limit **A**.
4° If φ is a $\Pi_2{}^0$ sentence of the theory of Boolean algebras which holds on finite Boolean algebras, then φ holds on all Boolean algebras.

2.14. Prove that the following theories are decidable:
1° The theory of pure predicate calculus with equality.
2° The theory of dense linear ordering.
3° The theory of equivalence relation with infinitely many equivalence classes, and in which every class is infinite.
4° If **A** is a finite model of a recursive language, then **ThA** is a decidable theory.

2.15. Let **M** be a model of Peano arithmetic and **N** the standard model of natural numbers. Show that there is unique embedding $f:\mathbf{N}\longrightarrow\mathbf{M}$, and that $f(N)$ is an initial segment of **M** (in respect to $\le^{\mathbf{M}}$).

2.16. Let φ be a formula of L_{PA} of the form $\psi(x,\underline{y})$, where $x,\underline{y}$ are free variables of ψ. Prove that the following schemes (the universal closure of φ) are provable in Peano arithmetic

The scheme of finite induction

FI $\quad \varphi(0) \wedge (\forall x<w)(\varphi(x) \rightarrow \varphi(x')) \rightarrow (\forall x<w')\varphi(x).$

The course-of-value induction

J $\quad \forall x((\forall y<x)\varphi(y) \rightarrow \varphi(x)) \rightarrow \forall x\varphi(x).$

The least element principle

L $\quad \exists x\varphi(x) \rightarrow \exists x(\varphi(x) \wedge (\forall y<x)\neg\varphi(y)).$

The greatest element principle

G $\quad \exists x\varphi(x) \wedge \exists y\forall x(\varphi(x) \rightarrow x\le y) \rightarrow \exists x(\varphi(x) \wedge (\forall y>x)\neg\varphi(y)).$

The collection scheme

B $\quad (\forall x<z)\exists u\varphi(x,u) \rightarrow \exists v(\forall x<z)(\exists u<v)\varphi(x,u).$

The regularity scheme

R $\quad (\forall x<v)\exists z\forall u(\varphi(x,u) \rightarrow u<z) \rightarrow \exists z(\forall x<v)\forall u(\varphi(x,u) \rightarrow u<z).$

3. COMPACTNESS OF PR^1

The Compactness Theorem of the first-order predicate calculus is also considered as a "local" theorem or a theorem of a finitistic nature. However, the term "compactness" is now often used since it's formulation resembles the well-known topological theorem, and, in fact, it has a topological reformulation. As we shall see later, the Compactness Theorem is an essential feature of PR^1 (Lindström Theorem). This theorem is also important because of it's applications outside of logic, for example, in algebra and analysis. It is a significant historical fact that the first application of mathematical logic to other parts of mathematics were in fact applications of the Compactness Theorem (Malcev 1936).

3.1. Statements equivalent to the Compactness theorem

When we were proving the theorems on the existence of ultrafilters in Boolean algebras, we used the Axiom of Choice. On the other hand, it is known that in the Zermelo-Freankel set theory, the existence theorem of ultrafilters does not imply the Axiom of Choice. So, it might be of interest to consider statements equivalent to the existence of ultrafilters. First, we shall introduce some notations.

AC (Axiom of Choice) Every family of nonempty sets has a choice function, i.e. if $\langle X_i : i\in I\rangle$ is a family of nonempty sets, then there is a function f whith domain I, such that for all $i\in I$, $f(i)\in X_i$.

UF (The existence theorem on ultrafilters) Every Boolean algebra has an ultrafilter.

Further, we recall the reader that Zermelo-Freankel set theory is denoted by ZF, and ZF together with the Axiom of Choice by ZFC. Then we have the following connections between statements AC and ZF:

1° ZFC$\vdash$ UF.

2° ZF$\nvdash$ UF.

3° ZF+UF$\nvdash$ AC.

So, if ZF is taken as a basic set theory, then UF is a weaker statement than AC, but nonprovable in ZF. Therefore, ZF+UF is a stronger theory than ZF, but weaker than ZFC. However, many mathematical propositions which are otherwise proved in ZFC are already proved in ZF+UF.

In this section, we shall first consider some statements equivalent under ZF to UF, and then prove some propositions which are simple consequences of the UF hypothesis. In some formulations of these propositions, the notion of a model of the propositional calculus is used.

3.1.1. Definition Let P be a set of propositional letters and S a set of propositional formulas whose propositional variables are from P. A model of S is every map $\mu:P\longrightarrow 2$, such that for each formula $\varphi\in S$, the logical value of φ, denoted by φ^{μ}, is equal to 1.

The logical values of propositional formulas are computed in the following way, taking φ, ψ to be propositional formulas: $(\neg\varphi)^{\mu}=(\varphi^{\mu})'$, $(\varphi\wedge\psi)^{\mu}=\varphi^{\mu}\cdot\psi^{\mu}$, $(\varphi\vee\psi)^{\mu}=\varphi^{\mu}+\psi^{\mu}$, $1^{\mu}=1$, $0^{\mu}=0$, where $+,\cdot,',0,1$ are operations and constants of the two-elements Boolean algebra 2. Other objects such as $\models$, $\vdash$, etc, are defined similarly as in the case of PR^1.

We shall use here the concept of a dual space of a Boolean algebra. To introduce it, assume B is a Boolean algebra, and let

$B^* = \{p: p$ is an ultrafilter of of $B\}$.

Then B^* becomes a topological space, if for the basis is taken the set

$\{a^* : a\in B\}$

where $a^*=\{p\in B^* : a\in p\}$ for $a\in B$. By Theorem 2.2.3, we see that every set a^* is a so called clopen set, i.e. a closed and open set, since $a^{*c}= a'^*$. A space defined in this way is called the ***Stone space***, or the dual space of B. As we shall see, this space is compact, i.e. every open cover of this space is reducible to a finite cover. Finally, let us remember that 2^X is the Cantor space, i.e. $2^X=\prod_{i\in X}Y_i$ is a product space, where for all $i\in X$, Y_i is a discrete two-element space. Now, we are able to assert statements equivalent in ZF to the Compactness Theorem of PR^1.

3.1.2 Theorem The following statements are equivalent in ZF:

1° Compactness Theorem of PR^1.

2° Completeness Theorem of PR^1.

3° Compactness Theorem of Propositional Calculus: If Σ is a set of propositinal formulas, and every finite $\Delta\subseteq\Sigma$ has a model, then Σ has a model.

4° Completeness Theorem of propositional calculus: IF Σ is a consistent set of formulas, then Σ has a model.

5° Every Boolean algebra has an ultrafilter.

6° Every filter of a Boolean algebra is contained in an ultrafilter of B.

7° For every set P, the Cantor space $\mathbf{2}^P$ is compact.

8° For every Boolean algebra **B**, the dual space $\mathbf{B}^*$ is compact.

Proof Some implications have already been proved, for example 2°⟶1°, (Theorem 2.4.9), and 5°⟶2° (Lemma 2.4.2 and Theorem 2.4.5). Further, the implication 4°⟶3° is proved in a similar way as 2°⟶1°. So, up to now, we have the following:

(1) 5°⟶2°⟶1°, 4°⟶3°.

We shall proceed to prove the other implications.

(6°⟶5°) The assertion 5° holds trivially when 6° is applied to filter {1}.

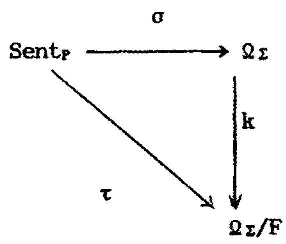
(5°⟶4°) Assume 5° and let Σ be a consistent set of propositional formulas. Further, let Ω_Σ be a Lindenbaum algebra of theory Σ. Hence,

$$\Omega_\Sigma = \{[\varphi]: \varphi \text{ is a propositional formula}\},$$

where $[\varphi]=\{\psi: \Sigma\vdash \varphi\leftrightarrow\psi\}$, while the operations af this algebra are defined as in the case of the Lindenbaum algebra of the predicate calculus. Then, Ω_Σ is a Boolean algebra. Let $\mu:P\longrightarrow\Omega_\Sigma$, $\mu(p)=[p]$, where P is the set of propositional letters. Map μ can be extended in a natural way to a function $\sigma:Sent_P\longrightarrow\Omega_\Sigma$, where $Sent_P$ is the set of all the propositional formulas in P, as follows:

$$\mu\subseteq\sigma,\quad \sigma(\varphi\wedge\psi)=\sigma(\varphi)\cdot\sigma(\psi),\ \sigma(\varphi\vee\psi)=\sigma(\psi)+\sigma(\psi),\ \sigma(\neg\psi)=\sigma(\psi)'.$$

Then, it is easy to see that for every $\varphi\in Sent_L$, $\sigma(\varphi)=[\varphi]$. Let F be an ultrafilter of Ω_F, k the canonical map, $k:\Omega_\Sigma\longrightarrow\Omega_\Sigma/F$, and $\tau=k\cdot\sigma$. Then, for every $\varphi\in\Sigma$, we have $\tau(\varphi)=1$, so τ is a model of Σ. Remark that for $\varphi\in\Sigma$, $[\varphi]=1_{\Omega\Sigma}$, so $\tau([\varphi])=k(1_{\Omega\Sigma})=1$.



($3^\circ\longrightarrow6^\circ$) Let F be a filter of a Boolean algebra B and assume that the Compactness Theorem of propositional calculus holds. Let a propositional letter $p_a\in P$ correspond to each $a\in B$, so that if $a\neq b$, then p_a and p_b differs. For the set

$$\Sigma_F=\{\neg p_0\}\cup\{(p_a\wedge p_b)\rightarrow p_{a\cdot b}: a,b\in B\}\cup$$
$$\{\neg p_a\rightarrow p_{a'}: a\in B\}\cup\{p_a\rightarrow p_b: a\leq b,\ a,b\in B\}\cup\{p_a: a\in F\}.$$

we have

(1) Every finite subset of Σ_F has a model.

Further, if $\Delta\subseteq\Sigma_F$ is finite, then, let

$$A=\{a\in B: a \text{ occur in a formula of } \Delta\}$$

and B' be the Boolean subalgebra generated by set A. Since A is finite, by Theorem 2.1.2, B' is finite too, and $F\cap B'$ has FIP. So, there is an ultrafilter V over B', such that $F\cap B'\subseteq V$. This is provable already in ZF, since B' is finite (this assertion can be proved by induction on the number of elements of B'). Then, the canonical homomorphism $k:B'\longrightarrow B'/V$ determines a model of Δ. By the compactness Theorem and (1), the set Σ_F has a model, say $\mu:\Sigma_F\longrightarrow2$. Then, $G=\{a\in B: \mu(p_a)=1\}$ is an ultrafilter of B and $F\subseteq G$.

($1^\circ\longrightarrow6^\circ$) Suppose 1° and let F be a proper filter of B. Define a first order theory T_F of the language $L=L_{BA}\cup\{P\}\cup\{a: a\in B\}$, where P is a unary predicate symbol, with the following axioms:

$Px\wedge Py\rightarrow P(x\cdot y)$, $Px\wedge x\leq y\rightarrow Py$, $Px\vee Px'$, $\neg P0$,

Pa, $a\in F$,

φ, $\varphi\in\Delta(B)$.

Here, $\Delta(B)$ denotes the ***diagram*** of the model $(B,b)_{b\in B}$, i.e. D(B) is the set of all the atomic and negations of atomic sentences of the language $L_{BA}\cup\{a: a\in B\}$ that hold in $(B,b)_{b\in B}$.

If $S \subseteq T_F$ is a finite subset, then, similarly as in the part ($3^\circ \to 6^\circ$), we can see that S is finitely satisfiable in a finite subalgebra $B' \subseteq B$, where B' is generated by constants a, whose names **a** appear in S, while P is interpreted as an ultrafilter of B'. Therefore, the conditions of the compactness theorem are fulfilled, so the theory T_F has a model, say $C = (C,+,\cdot,',\leq,0,1,G,c_a)_{a\in B}$. Here, G is an interpretation of the relation symbol P, while c_a is an interpretation of symbol **a**. Since C is a model of T_F, we have $C \models \Delta(B)$, so, from this fact we can infer that $h: a \longrightarrow c_a$, $a \in B$, is an embedding of B into $(C,+,\cdot,',\leq,0,1,c_a)_{a\in B}$. Therefore, without loss of generality, we may assume that B is a submodel of $(C,+,\cdot,',\leq,0,1,c_a)_{a\in B}$. Then $B \cap G$ is an ultrafilter of B which contains filter F.

($6^\circ \longrightarrow 8^\circ$) Let B be a Boolean algebra and B^* the dual space of B, i.e, $B^* = \{p: p$ is an ultrafilter of $B\}$. If $a \in B$ and $a \neq 0$, then by hypothesis 6°, the filter $F_a = \{x \in B: a \leq x\}$ is contained in an ultrafilter p, so $p \in a^*$, i.e. $a^* \neq 0$. Remark that by hypothesis 6°, the set

$$F^* = \{p \in B^*: F \subseteq p\}$$

is nonempty for every proper filter F of B. Now, let us prove

(1) $K \subseteq B^*$ is a closed subset iff there is a filter F of B such that $K = F^*$.

First, suppose K is closed. Then, K is an intersection of basic closed sets, i.e. $K = \bigcap_{a\in I} a^*$ for some $I \subseteq B$ since $(x^*)^c = (x')^*$. Let F be the filter generated by the set I. Then, by Theorem 2.2.4, $F^* = \bigcap_{a\in I} a^*$, thus $K = F^*$.

Second, if $K = F^*$, then since $F^* = \bigcap_{a\in I} a^*$, it follows that K is an intersection of closed sets, so K is closed. This finishes the proof of assertion (1).

Now, we shall show that B^* is a compact space. Let Γ be a collection of closed subsets of space B^* which has FIP. By (1), we may assume that $\Gamma = \{F_i^*: i \in I\}$, where F_i are filters of B. Since

$$F_1^*, \ldots, F_n^* \in \Gamma \quad \text{implies} \quad F_1^* \cap \ldots \cap F_n^* \neq \emptyset,$$

it follows that $F_1 \cup \ldots \cup F_n$ has FIP as well, whenever $F_1^*, \ldots, F_n^* \in \Gamma$. Therefore, $\bigcup_i F_i$ has FIP, so by hypothesis 6°, there is an ultrafilter p of B such that $\bigcup_i F_i \subseteq p$, i.e. for all $i \in I$, $F_i \subseteq p$. Hence, $p \in \bigcap_i F_i^*$, so $\bigcap_i F_i^* \neq \emptyset$. Thus, $\bigcap \Gamma \neq \emptyset$, and this proves that B^* is a compact space.

($8^\circ \longrightarrow 7^\circ$) Let Ω_P be the free Boolean algebra with P as a set of free generators. Further, let us introduce the map $\mu:2^P \longrightarrow \Omega_P{}^*$, where for $\alpha\in 2^P$, $\mu(\alpha)$ is a filter of Ω_P generated by the set $\{p^{\alpha(p)}: p\in P\}$. By the proof of Theorem 2.2.5, $\mu(\alpha)$is an ultrafilter of Ω_P, hence, the codomain of μ is $\Omega_P{}^*$ indeed. According to the proof of Theorem 2.2.5, map μ is one to one and onto. Now, we shall prove that μ is continuous. It suffices to show that inverse images, $\mu^{-1}(a^*)$, $a\in\Omega_P$, of basic sets of the space $\Omega_P{}^*$ are open subsets of 2^P. In fact, if $a\in\Omega_P$, then by the theorem on representation of Boolean terms, there are finite $P'\subseteq P$ and $I\subseteq 2^{P'}$, so that

$$a= \Sigma_{\alpha\in I}\, a_\alpha,\quad P'= \{p_1,\dots,p_n\},\quad a_\alpha= p_1{}^{\alpha(p_1)}\dots p_n{}^{\alpha(p_n)}.$$

Hence, we have $a^*= \cup_{\alpha\in I}\, a_\alpha{}^*$, since

$p\in a^*$ iff $a\in p$
iff $\Sigma_\alpha a_\alpha \in p$
iff there is an $\alpha\in I$ such that $a_\alpha\in p$
iff there is an $\alpha\in I$ such that $p\in a_\alpha{}^*$
iff $p \in \cup_\alpha a_\alpha{}^*$.

Thus, $\mu^{-1}(a^*)=\cup_{\alpha\in I}\,\mu^{-1}(a_\alpha{}^*)$. Further, if $\beta\in 2^P$ and $\alpha\in 2^{P'}$, then $\beta\in\mu^{-1}(a_\alpha{}^*)$ iff $\mu(\beta)\in a_\alpha{}^*$ iff $a_\alpha\in\mu(\beta)$. Since $\mu(\beta)$ is an ultrafilter generated by the set $\{p^{\beta(p)}: p\in P\}$, and $a_\alpha= p_1{}^{\alpha(p_1)}\dots p_n{}^{\alpha(p_n)}$, we have $a_\alpha\in\mu(\beta)$ if and only if $p_1{}^{\alpha(p_1)}\dots p_n{}^{\alpha(p_n)}\in\mu(\beta)$ iff $\alpha\subseteq\beta$. Therefore,

$$\mu^{-1}(a_\alpha{}^*)= \{\beta\in 2^P:\ \alpha\subseteq\beta\},$$

i.e. $\mu^{-1}(a_\alpha{}^*)$ is a basic subset of space 2^P, so μ is a continuous map. on the other hand, each basic subset of 2^P is of the form $S_\alpha=\{\beta\in 2^P:\alpha\subseteq\beta\}$ for some finite $P'\subseteq P$ and $\alpha\in 2^{P'}$. Therefore, $\mu(S_\alpha)= \mu\cdot\mu^{-1}(a_\alpha{}^*)= a_\alpha{}^*$, i.e. $\mu(S_\alpha)$ is an open subset of $\Omega_P{}^*$. Hence, μ is a homeomorphism of space 2^P onto $\Omega_P{}^*$, and since $\Omega_P{}^*$ is compact (by assumption 8°), it follows that 2^P is compact.

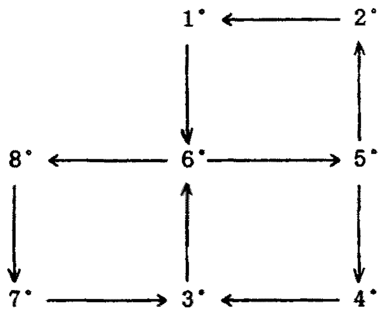
(7°$\longrightarrow$3°) Suppose space 2^P is compact, where P is the set of propositional letters, and let Σ be a set of propositional formulas, such that every finite subset $\Delta\subseteq\Sigma$ has a model. Let us introduce, for every propositional formula τ, the set $A_\tau= \{\mu\in 2^P:\ \tau^\mu=1\}$, i.e. A_τ is the set of all the models of formula τ. It is easy to see that

(1) $A_0=\emptyset$, $A_1= 2^P$, $A_{\tau\wedge\sigma}= A_\tau\cap A_\sigma$, $A_{\neg\tau}= A_\tau{}^c$,

and, also, for every $p\in P$, A_p is a clopen subset of 2^P.

By (1), the set A_τ is obtained from the sets A_p, $p\in P$, by the finite applications of set-theoretical operations of intersection, union, and complementation. Thus, A_τ is clopen, as well. Additionally, let $\Phi=\{A_\tau:\tau\in\Sigma\}$. By the assumption on Σ, the set Φ has FIP, thus, by the hypothesis that 2^P is a compact space, we have $\bigcap\Phi\neq\emptyset$, i.e. there is $\alpha\in\bigcap\Phi$. Then, for all $\varphi\in\Sigma$, $\varphi^\alpha=1$, i.e. α is a model of Σ.

Accordingly to the preceding, we have proved the following implications in ZF:



This proves the theorem.

By the last theorem, the list of statements equivalent to the Compactness theorem under ZF is not exhausted. Let us mention the following proposition without proof.

3.1.3. Theorem The Compactness Theorem of PR^1 implies in ZF that a product of nonempty compact Hausdorff spaces is a nonempty compact Hausdorff space.

The next theorem is called the Reflection theorem, and it relates the Compactness Theorem to the infinitary propositional calculus.

3.1.4. Theorem Let Σ be a set of propositional formulas with the property that every map $\mu\in 2^P$ is a model of a formula $\varphi\in\Sigma$. Then, there is a finite set $\{\varphi_1,\ldots,\varphi_n\}\in\Sigma$ such that $\varphi_1 \vee \ldots \vee \varphi_n$ is a tautology.

If infinite disjunctions of propositional formulas are allowed, this theorem can be restated as follows:

If Σ is an infinite set of propositional formulas, then, $\models \bigvee_{\tau\in\Sigma} \tau$ implies the existence of finite $\Delta\subseteq\Sigma$, such that $\models \bigvee_{\tau\in\Delta} \tau$.

The proof in ZF of the equivalence of the Reflection Theorem and the Compactness Theorem can be derived in a similar manner as the proof of $(7^\circ\longrightarrow 3^\circ)$ in Theorem 3.1.2.

According to the proof of Theorem 3.1.2 and the existence theorem of ultrafilters, some interesting consequences can be obtained. One is the representation theorem on Boolean algebras.

3.1.5. Theorem Every Boolean algebra is isomorphic to a field of sets.

Proof Using the notation introduced in Theorem 3.1.2, one can see that $h:a\longrightarrow a^*$, $a\in B$, is an isomorphism of the algebra B and the field of clopen subsets of the dual space B^*. Indeed,

$$h(a\cdot b)=(a\cdot b)^*=\{p\in B^*: a\cdot b\in p\}=\{p\in B^*: a\in p \text{ and } b\in p\}=$$
$$\{p\in B^*: a\in p\}\cap\{p\in B^*: b\in p\}=a^*\cap b^*=h(a)\cdot h(b).$$

The appropriate identities for other Boolean operations are proved in a similar way.

Using the same technique, we can prove many other interesting properties of Boolean algebras. In the following, we shall consider the so-called splitting property of Boolean algebras. Also, we shall use some ideas which have been already been employed in the proof of Theorem 1.1.5. First, we shall introduce some terminology.

In the following, let $B=(B,+,\cdot,',\leq,o,1)$ be a Boolean algebra.

A subset $T\subseteq B$ is a ***dual normal tree*** in B iff T has the following properties:

1° T is a binary tree in respect to the dual ordering $\leq$ of B i.e.

1°.1. There is the greatest element a in T in respect to $\leq$.

1°.2. For every $t\in T$, the set $[t,a]_T=\{x\in T: t\leq x\}$ is linearly ordered and every nonempty $X\subseteq[t,a]_T$ has the greatest element.

1°.3. Every $t\in T$ has exactly two predecessors $t_1,t_2\in T$ such that $t=t_1+t_2$.

2° If $x,y\in T$ and x,y are incomparable, in respect to $\leq$, i.e. neither $x\leq y$ nor $y\leq x$, then $x\cdot y=0$. Such a tree will be called for short a d.n.b. tree, and this tree is countable if $|T|=\aleph_0$.

Now, a subset $C\subseteq B$ is ***splitting***, iff the following holds:

1° $0\notin C$.

2° If $a\in C$, then there are $a_1,a_2\in C$ such that $a=a_1+a_2$, and $a_1\cdot a_2=0$.

Here are some examples of splitting subsets of B.

3.1.6. Example 1° Every d.n.b. tree of B is a splitting subset.
2° An element $a\in B\setminus\{0\}$ is ***atomless*** if a is not an atom and there are no atoms below a. Then the set $C=\{x\in B: x$ is an atomless element of $B\}$ is

a splitting subset. Really, if $a\in C$, then, there is $b\in B$, such that $0 < b < a$, so $b,ab'\in C$, $b(ab')=0$ and $a=b+ab'$.

3.1.7. Lemma Every nonempty splitting subset C of B contains as a subset a d.n.b. tree of B.

Proof We shall construct, by induction, a sequence of finite dual trees which are subsets of C, as follows. Let $T_0=\{a\}$, where $a\in C$, and suppose T_n has been defined for some $n\in\omega$. Let $a_1,\dots,a_k$ $(k=2^n)$ be minimal elements of T_n. By the inductive hypothesis, we have $a_1,\dots,a_k\in C$. So for every $1\le i\le k$, there are $a_{i0},a_{i1}\in C$ such that $a_i=a_{i0}+a_{i1}$ and $a_{i0}\cdot a_{i1}=0$. Then we take that $T_{n+1}=T_n\cup\{a_{ij}: 1\le i\le k, 0\le j\le 1\}$. Finally, we define $T=\bigcup_n T_n$. It is easy to see that T is a d.n.b. tree of B and $T\subseteq C$.

By this lemma and the last example, we can see that a subset C of B contains a splitting set iff C contains a d.n.b. tree of B. We have also the following simple consequence of the above lemma.

3.1.8. Lemma Assume B contains a nonempty splitting subset C. Then $|B^*|\ge 2^{\aleph_0}$ (we should remember that B^* denotes the set of all the ultrafilters of B).

Proof By the last lemma, there is a d.n.b. tree $T\subseteq C$. Every branch (i.e. a maximal chain) g of T has FIP, so it is contained in an ultrafilter p_g of B. By the normality of T, it follows that for different branches g and g', we have $p_g\ne p_{g'}$. On the other hand, T has $2^{\aleph_0}$ branches, so $|B^*|\ge 2^{\aleph_0}$.

Now, we shall state the main theorem on splitting set. As we shall see, this theorem has interesting applications in model theory, as well.

3.1.9. Theorem Assume $|B| < |B^*|$. Then there is an d.n.b. tree of B such that for all $a\in T$, $|B|<|a^*|$.

We remind the reader that a^* is the set of all the ultrafilters of B which contain a.

Proof In the proof of the theorem, we need the folowing assertion of a purely set-theoretical nature.

Claim 1. If $\langle X_i : i\in I\rangle$ is a family of sets and $X = \bigcup_i X_i$, then,

$|X| \leq |I|\cdot\sup_i |X_i|$.

Proof of Claim 1. If $X = \bigcup_i X_i$, then

$|X| \leq \Sigma_{i\in I} |X_i| \leq \Sigma_{i\in I} \sup_j |X_j| = |I|\cdot\sup_i |X_i|$.

Claim 2. $|B| < |B^*|$ implies there is $a\in B$ such that $|B| < |a^*|$.

Proof of Claim 2. Suppose

(1) $|B| < |B^*|$.

If B were a finite Boolean algebra, then $|B| = 2^n$ for some $n\in\omega$. Thus, by (1), B is infinite. Since $B^* = \bigcup_{a\in B} a^*$, by Claim 1 it follows that $|B^*| \leq |B|\cdot\sup_{a\in B}|a^*|$. If $\sup_{a\in B}|a^*| \leq |B|$, then

$|B^*| \leq |B|\cdot|B| = |B|$, i.e. $|B^*| \leq |B|$.

But this is a contradiction to our assumption (1), therefore, $|B| < \sup_{a\in B}|a^*|$. From this inequality, we can immediately deduce that there is $a\in B$ such that $|B|<|a^*|$, and this finishes the proof of Claim 2.

Claim 3. The set $C = \{a\in B: |B|<|a^*|\}$ is splitting in B.

Proof of Claim 3. Let $a\in C$ and $I = \{b\in B_a : |B|\geq|b^*|\}$, where $B_a = \{x\in B : x\leq a\}$, and take $S = \bigcup_{b\in I} b^*$. Since $b\in I$ implies $|b^*|\leq|B|$, and since $I\subseteq B_a\subseteq B$, it follows that $|S| \leq |I|\cdot\sup_{b\in I}|b^*| \leq |B|\cdot|B|$. Therefore, for $|B|<|a^*|$, we have $|a^*\setminus S| = |a^*|$ i.e. $a^*\setminus S$ is an infinite set. Thus, there are $p_1, p_2 \in a^*\setminus S$, $p_1\neq p_2$. Then there is $c\in p_1$ such that $c\notin p_2$ i.e. $c'\in p_2$. Since $a\in p_1$, $a\in p_2$, for elements $a_1 = ac$ and $a_2 = ac'$, we have $a_1\in p_1$, $a_2\in p_2$, $a = a_1 + a_2$, $a_1\cdot a_2 = 0$ and $a_1, a_2 \neq 0$. Further, if $|a_1{}^*|\leq|B|$ then from $a_1\leq a$ we have $a_1\in I$ and $p_1\in a_1{}^*$, so $p_1\in S$, contradicting $p_1\in a^*\setminus S$. Hence, $|B| < |a_1{}^*|$ i.e. $a_1\in C$. In a similar way, we can prove that $a_2\in C$, and this finishes the proof of Claim 3.

Now, by Lemma 3.1.7 and Claim 3, the statement of the theorem follows.

3.1.10.Corollary 1° If $|B|<|B^*|$ then $|B^*|\geq 2^{\aleph_0}$.

2° If B is a Boolean algebra such that for all countable subalgebras $C\subseteq B$, $|C^*|\leq\aleph_0$, then $|B^*|\leq|B|$.

Later, we shall see some applications of this corollary.

Theorem 3.1.5 was proved by M. Stone. Applying the compactness Theorem, one can prove many other statements of set theory which are usually proved assuming the Axiom of Choice.

usually proved assuming the Axiom of Choice.

3.1.11 Theorem Every set can be linearly ordered.

Proof Let A be a set, $\leq$ a binary relation symbol, and T a theory with axioms:

The axioms of linear ordering for $\leq$, LO.
For all a,b,c$\in$A and their names **a,b,c**,
$\mathbf{a} \leq \mathbf{a}$,
$\mathbf{a} \leq \mathbf{b} \wedge \mathbf{b} \leq \mathbf{a} \rightarrow \mathbf{a}=\mathbf{b}$,
$\mathbf{a} \leq \mathbf{b} \wedge \mathbf{b} \leq \mathbf{c} \rightarrow \mathbf{a} \leq \mathbf{c}$,
$\mathbf{a} \leq \mathbf{b} \vee \mathbf{b} \leq \mathbf{a}$.

Every finite subset Δ of T contains only finitely many constant symbols, say $\mathbf{a}_1,\ldots,\mathbf{a}_n$. We can define an ordering on the set $\{a_1,\ldots,a_n\}$ by $a_i \leq a_j$ iff $i \leq j$. This means that the set Δ has a model. Therefore, by the Compactness Theorem T has a model too, say $\mathbf{B}= (B,\leq,b_a)_{a\in A}$, where $b_a = \mathbf{a}^{\mathbf{B}}$. Then we can define the ordering over A in the following way:

$a_1 \leq^A a_2$ iff $b_{a_1} \leq b_{a_2}$.

Even if the Axiom of Choice cannot be inferred from the compactness Theorem in ZF, some weaker forms of the Axiom of Choice can be obtained.

3.1.12. Theorem (in ZF) The Compactness heorem implies the Axiom of Choice for families of finite sets.

Proof Let $\langle X_i : i\in I\rangle$ be a family of nonempty finite sets, and let τ_i be the discrete topology on X_i. Then, $\mathcal{X}_i = (X_i, \tau_i)$ is a compact space, so, by Theorem 3.1.3.we have $\prod_i X_i \neq \emptyset$. Then every map $f \in \prod_i X_i$ is a choice function for the family $\langle X_i : i\in I\rangle$.

We shall see some more applications of the Compactness Theorem in the next section.

3.2. The combinatorial universe

It is an interesting question whether it is possible to introduce in a natural way, a domain in which we can formulate all the notions of finite combinatorics, and even some of infinite combinatorics. When we think about finite combinatorics, we deal mainly with finite sets. So, in such considerations about a finite set x, we may suppose that all the sets in every chain $x_0 \in x_1 \in \ldots \in x_n \in x$, $n \in \omega$, are finite. Thus, we come, as anticipated, to the following sequence of sets V_n, $n \in \omega$, and the universe $V\omega$.

3.11. Definition The sequence V_n, $n \in \omega$, of sets is defined inductively in the following way:

$V_0 = \emptyset$, $V_{n+1} = V_n \cup P(V_n)$, $n \in \omega$.

Further, we introduce: $V\omega = \bigcup_n V_n$, $\mathbf{V}\omega = (V\omega, \in)$.

The set $V\omega$ will be called the domain of finite combinatorics, while $\mathbf{V}\omega$ will be called the model of finite combinatorics. At the first glance, $V\omega$ might look sparse, however, $V\omega$ is sufficiently rich to admit definitions of virtually all the basic notions of finite combinatorics. Let us denote by ZF^f the Zermelo-Freankel set theory but with the negation of the axiom of infinity, instead of this axiom. It is then easy to see that the following proposition holds:

3.2.2. Theorem $\mathbf{V}\omega$ is a model of ZF^f.

From this theorem, it follows that $V\omega$ is closed under usual set-theoretical operations. Namely, we have:

$x_1, \ldots, x_n \in V\omega \rightarrow \{x_1, \ldots, x_n\} \in V\omega$, $n \in \omega$,

$x, y \in V\omega \rightarrow (x,y) \in V\omega$, where $(x,y) = \{\{x,y\},\{x\}\}$,

$x \in V\omega \rightarrow \bigcup x \in V\omega$,

$x, y \in V\omega \rightarrow x \times y \in V\omega$,

if $x, y \in V\omega$ and $f: x \longrightarrow y$, then $f \in V\omega$,

if $x, y \in V\omega$ then $x^y \in V\omega$, where $x^y = \{f: f: y \longrightarrow x\}$.

Every set in $V\omega$ is strictly finite , i.e. if $y_0 \in y_1 \in \ldots \in y_n \in V\omega$, then all the sets $y_0, \ldots, y_n$ are finite, as well. Remember that $0 = \emptyset$, $1 = \{0\}$, $2 = \{0,1\}, \ldots, n = \{0,1,\ldots,n-1\}, \ldots$, for all $n \in \omega$, hence $\omega \subseteq V\omega$. The notion of natural number can be introduced already in ZF^f, by the definition axiom

$$N(x) \leftrightarrow (\forall y,z\in x)(y\in z \vee z\in y \vee y=z) \wedge (\forall y\in x)(\forall z\in y)(z\in x)$$

where N is a unary predicate symbol. The interpretation of N in $V\omega$ is unique and it is ω, i.e. the set of natural numbers. Further, we see that the usual linear ordering of ω coincides with $\in$.

In a similar manner, one can introduce other elementary set-theoretical notions, for example, the notion of function is introduced by

$$Fn(x) \leftrightarrow \forall y,z,u\ ((u,y)\in x \wedge (u,z)\in x \rightarrow y=z) \wedge \forall y\in x \exists u,v(y=(u,v)).$$

If for some $f,a,b\in V\omega$, $(a,b)\in f$ and $V\omega \models Fn[f]$, the we can write $b=f(a)$ as is customary. In this case, the set $\{x\in V\omega:\ \exists y\in V\omega\ (x,y)\in f\}$ will be called the domain of f, and this set is denoted by dom(f). The codomain of f is the set $codom(f)=\{y\in V\omega:\ \exists x\in dom(f)\ (x,y)\in f\}$.

We see that all these notions are definable in ZF^f, i.e. to each of these concepts a predicate is related, which is definable in ZF^f. For example, we can introduce the predicate $z:x\longrightarrow y$, by the definition axiom

$$(z:x\longrightarrow y) \leftrightarrow (Fn(z) \wedge x=dom(z) \wedge codom(z)\subseteq y).$$

If $a,b\in V\omega$, then it is easy to see that f maps a into b iff $V\omega \models f:a\longrightarrow b$.

Now, we are able to state the theorems of finite combinatorics in ZF^f. As an example, let us present a finite version of the Ramsey theorem.

For a set X and $k\in\omega$, define $[X]^k=\{y\subseteq X:\ |y|=k\}$. Therefore, $[X]^2$ is the set of all the two-element subsets of X. By a partition of X, we consider any onto map $P:X\rightarrow n$ for some $n\in\omega$. Remark that every map of this sort determines a collection $\mathcal{P}$ of disjoint subsets of X, a partition of X: $\mathcal{P}=\{P^{-1}(\{i\}):\ i\in n\}$. For values of P one can imagine colors. Namely, assuming that P maps X onto n, then there are n colors, and if $P(a)=i$, then we say that element a is colored by color i. According to this notation, elements of one class of partition P are colored by the same color. So, there is another name for partitions of the set X - the coloring of X.

By the finite version of the Ramsey Theorem we mean the sentence

$$RT^f \quad \forall k,t,r\in\omega\ \exists m\in\omega\ \forall n>m\ \forall\pi((\pi:[n]^k\longrightarrow r) \rightarrow (\exists e\in[n]^t\ \pi|[e]^k=const)).$$

In other words, this formula says:

For all the natural numbers k,t,r there is a natural number m such that for all $n\in\omega$, $n>m$, all partitions $\pi:[n]^k\longrightarrow r$, there is an $e\subseteq n$ of size t in respect to which π is homogeneous, i.e. π takes only one value on $[e]^k$.

3.2.3. Theorem $V\omega\models RT^f$.

We shall later give a nonconstructive proof of this theorem based on the infinite version of Ramsey theorem.

3.2.4. Theorem (Infinite version of the Ramsey Theorem) Let S be an infinite set and $k,r\in\omega$, $k,r>0$. If $\pi:[S]^k\longrightarrow r$, then there is an infinite $T\subseteq S$ homogeneous for π, i.e. π is constant on $[T]^k$.

Proof We shall give the proof by induction on k. Obviously, we may suppose that $r>1$. Without loss of generality, we may assume $S\subseteq\omega$.

Case 1. For $k=1$ the statement holds trivially since finite unions of finite sets are finite.

Case 2. Let k be fixed, and suppose the statement for $k-1$. Let us introduce a sequence of sets X_i and functions π_i, $i\in\omega$, in the following way:

Let x_0 be the least element in S, and $\pi_0:[S\setminus\{x_0\}]^{k-1}\longrightarrow r$, where $\pi_0(\{y_1,\dots,y_{k-1}\})=\pi(\{x_0,y_1,\dots,y_{k-1}\})$, $y_1<\dots<y_{k-1}$.

By the induction hypothesis, there is an infinite set $X_0\subseteq S\setminus\{x_0\}$ homogeneous for π_0, i.e. for some $r_0<r$, the function $\pi_0\restriction[X_0]^{k-1}$ takes a constant value r_0. We shall briefly write $\pi_0\restriction[X_0]^{k-1}=r_0$.

Assume we have constructed sets $X_0,\dots,X_n$, and let x_{n+1} be the least element in X_n. Now, we shall define the map

$\pi_{n+1}:[X_n\setminus\{x_{n+1}\}]^{k-1}\longrightarrow r$, where

$\pi_{n+1}(\{y_1,\dots,y_{k-1}\})=\pi(\{x_{n+1},y_1,\dots,y_{k-1}\})$, $y_1<\dots<y_{k-1}$,

$y_1,\dots,y_{k-1}\in X_n\setminus\{x_{n+1}\}$.

By the induction hypothesis, there is an infinite $X_{n+1}\subseteq X_n\setminus\{x_{n+1}\}$ homogeneous for π_{n+1}, i.e. for some $r_{n+1}<r$, $\pi_{n+1}\restriction[X_{n+1}]^{k-1}=r_{n+1}$. Therefore, an infinite sequence of sets X_n, maps π_n, and numbers $r_n<r$ are defined in this way. Since there are only finitely many values in the se-

quence r_n, $n\in\omega$, there is a sequence $n_k\in\omega$ and $s<r$, such that for all $k\in\omega$, $r_{n_k}=s$. Then,

If $X=\{x_{n_0},x_{n_1},\ldots\}$, then for all $Y\in[X]^k$ we have $\pi(Y)=s$, i.e. X is an infinite subset of S homogeneous for π.

Let us consider some applications of the Ramsey Theorem.

3.2.5. Example 1° Suppose $(X,\leq)$ is a linearly ordered set, and let $\langle x_n : n\in\omega\rangle$ be a sequence of elements of X. then the sequence x_n contains a monotonous or constant subsequence. Indeed, consider the following partition $\pi:[\omega]^2\longrightarrow 3$

$$\pi(\{m,n\})=\begin{cases} 0 \text{ if } a_m<a_n, \\ 1 \text{ if } a_m>a_n, \\ 2 \text{ if } a_m=a_n. \end{cases} \qquad \{m,n\}\in[\omega]^2,\ m<n,$$

Let $T=\{n_0,n_1,\ldots\}\subseteq\omega$ be an infinite set homogeneous fot π. Then the subsequence $\langle x_{n_i} : i\in\omega\rangle$ is monotonous or constant.

2° Let $(X,\leq)$ be an infinite partially ordered set. Then X contains an infinite chain or an infinite antichain. Really, consider the partition:

$$\pi(\{m,n\})=\begin{cases} 0 \text{ if } m\leq n \text{ or } n\leq m \\ \\ 1 \text{ if } m \text{ and } n \text{ are incomparable} \end{cases} \qquad \{m,n\}\in[X]^2$$

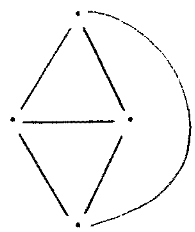
Then any infinite subset $Y\subseteq X$ homogeneous for π satisfies the required conditions.

Using the Ramsey Theorem and König's Lemma, it is not difficult to show that RT^f is true in $V\omega$. However, we shall prove a combinatorial statement on hypergraphs from which RT^f follows directly. The proof of this statement is also an example of an application of the Compactness Theorem.

A *hypergraph* is every pair $G=(G,E)$ where G is a nonempty set, and E is a set of nonempty subsets $e\subseteq G$, such that $|e|\geq 2$. As it was already been explained, a coloring of graph G is any partition $\pi:G\longrightarrow n$, $n\in\omega$. A number $r\in\omega$ is a subchromatic number of G if and only if for every coloring $\pi:G\longrightarrow r$, there is $e\in E$, such that $\pi|e=$ const. A number $R\in\omega$ is a superchromatic number of G if and only if R is not a subchromatic number of G. Finally, the chromatic number h_G of G is the least superchromatic number of G. Therefore,

$h_G > r \leftrightarrow \forall\pi : G \longrightarrow r\ \exists e\in E\ \pi|e=$ const

3.2.6. Example If P is a class of graphs, then the chromatic number for P is $\sup\{h_G : G\in P\}$. The displayed graph shows that for the class of planar graphs, this number is ≥ 4.



Hypergraph $G'=(G',E')$ is a subgraph of a hypergraph G if and only if $G'\subseteq G$ and $E'\subseteq E$. Lastly, G is k-bounded if for all $e\in E$, $|e|<k$, where k is a cardinal number.

3.2.7. The Compactness Theorem for Hypergraphs (N.Bruijin, P.Erdös) Let G be an infinite ω-bounded hypergraph and t it's subchromatic number. Then G has a finite subgraph which has t as a subchromatic number.

Proof Without loss of generality, we may assume that $G\subseteq V\omega$, thus $E\subseteq V\omega$. Since we have taken G to be countable, we may suppose that G is linearly ordered with an order type of ω. So, let G_n be the set of the first n elements of G, and $E_n=\{e\in E: e\subseteq G_n\}$. Hence, G_n and E_n are finite and $G_n, E_n\subseteq V\omega$. Now, suppose, on the contrary, that G has no finite subgraph with a subchromatic number t. Thus, there are mappings $\pi_n : G_n\longrightarrow t$, such that: $\forall e\in E_n\ \pi_n|e\neq$ const.

Consider the structure $V\omega' = (V\omega,\omega,\in,\mu,\tau,\pi)$, where $\mu:\omega\longrightarrow V\omega$, $\mu_n=G_n$, $\tau:\omega\longrightarrow V\omega$, $\tau_n=E_n$, and $\pi:\omega\longrightarrow V\omega$, π_n are the above introduced functions. Then, theory: $\mathrm{Th}V\omega \cup \{c>n : n\in\omega\} \cup \{\omega(c)\}$
is finitely consistent, so, this theory has a model

$^*V\omega=(^*V\omega, {}^*\omega, {}^*\in, {}^*\mu, {}^*\tau, {}^*\pi, H)$, H is an interpretation of c in $^*V\omega$. Since every element of $V\omega$ is definable in $V\omega$, it follows that $V\omega'$ is elementary embeddable in $^*V\omega$, so, we may take $V\omega \prec {}^{\cdot}V\omega$, where $^{\cdot}V\omega$ is the corresponding reduct of $^*V\omega$. Then,

(1) $^*V\omega \models \pi : G_c \longrightarrow t \wedge \forall e(e\in E_c \rightarrow \pi_c|e\neq \mathrm{const})$.

Let us define sets $\underline{E}=\{x\in {}^*V\omega : x^*\in E_H\}$, $\underline{\pi}=\{(x,y): {}^*\pi_H(x)=y\}$, $\underline{G}=\{x: x^*\in G_H\}$. Then, it is easy to see that $\underline{\pi}:\underline{G}\longrightarrow t$ and $E\subseteq\underline{E}$, $G\subseteq\underline{G}$. Let $\sigma=\underline{\pi}|G$. Then, by (1), it follows that

for all $e\in E$ $\sigma|e\neq$ const,
which is a contradiction to the assumption that t is a subchromatic number of G.

This proof is in the spirit of nonstandard analysis, and it could be simplified if we had at our disposal the so called nonstandard universe.

3.2.8. Corollary $V\omega\models RT^f$.

Proof Let us choose the hypergraph $G=(G,E)$, such that
$G=[\omega]^k$, $E=\{[S]^k: S\subseteq\omega, |S|\geq t\}$. By the infinitary version of the Ramsey Theorem, r is a subchromatic number of G, so, by the Compactness theorem for hypergraphs, there is a finite hypergraph $G'\subseteq G$ with r as a subchromatic number. For $G'=(G',E')$ take $n=\max\bigcup G'$. If $e\in E'$, then for some $S\subseteq\omega$ $e=[S]^k$, where $|S|\geq t$. Let $H=([n]^k,E')$. Then, $\pi:[n]^k\longrightarrow r$ implies that $\pi|G'$ is an r-coloring of G', so there is $e\in E'$, such that $\pi|e=$ const, i.e. for some $S\subseteq n$, $|S|\geq t$ there is $s\subseteq S$ such that $s\in[n]^t$ and $\pi|[s]^k=$ const.

Almost without any change in the proof, one can show that a Paris-Harrington version of RT^f holds in $V\omega$:
(PH) $\forall k,t,r\in\omega\exists m\in\omega\forall n>m\forall\pi((\pi:[n]^k\longrightarrow r)\longrightarrow$

$$(\exists e\in[n]^t)(\pi|[e]^k=\text{const}\wedge|e|>\min e).$$

A set $e\in V\omega$ is ***relatively large*** if $|e|\geq\min e$. Therefore, the only difference between (PH) and the statement RT^f is that PH asserts the existence of a relatively large set homogeneous for partition π. The fact that makes (PH) interesting is the nonprovability of (PH) in ZF^f. This is the first example (given by J.Paris and L.Harrington in 1978) of a finitary statement of a "pure mathematical content" with such a property. We shall discuss other statements of this kind in some of the following sections.

Now, we shall consider one more application of the Compactness Theorem in $V\omega$. We would remind the reader that a *tree* is every structure $T=(T,\leq,0)$, where $(T,\leq)$ is a partial ordered set with the least element 0, and with the property that for every $a\in T$ the set $[0,a]_T=\{x\in T: x\leq a\}$ is well-ordered by the relation $\leq$. If T is a tree, then, it is possible to introduce the following map ot with domain T and values in a set of ordinal numbers. Function ot is defined inductively: $ot(x)=\{ot(y):y<x\}$. Then for every ordinal number α, the α-level of T is $T_\alpha=\{x\in T:\ ot(x)=\alpha\}$.

3.2.8. Theorem (König Lemma). Let T be an infinite tree in which every level is finite. Then T has an infinite branch , i.e. there is an infinite maximal chain in T.

Proof Without loss of generality, we may assume that T is countable and $T\subseteq V\omega$. Let $^*V\omega=(^*V\omega,{}^*\omega,{}^*\in,{}^*T,{}^*\le,0)$ be an elementary extension of the model $(V\omega,\omega,\in,T,\le,0)$ constructed similarly as in Theorem 3.2.7. Thus, there is an infinite number $H\in{}^*\omega\backslash\omega$. Let $a\in{}^*T$, $ot(a)=H$. Such an element exists, since $V\omega\models\forall n\in\omega\ T_n\ne 0$, i.e. $^*V\omega\models\forall n\in\omega\ T_n\ne 0$. Therefore, since $H\in{}^*\omega\backslash\omega$, it follows that $T_H\ne 0$. Further, since for all $n\in\omega$, T_n is finite, we have $T_n={}^*T_n$, so, by

$$V\omega\models\forall x\forall n\in\omega(n<ot(x)\rightarrow\exists y\in T_n\ y\le x),$$

it follows that for every $n\in\omega$ there is $b\in T_n$, $b^*\le a$. Thus, $g=\{b\in T:\ b^*\le a\}$ is an infinite chain in T.

In the same way, one can "construct" a non-principal ultrafilter over ω. To see this, let for every $X\subseteq\omega$, the sign $\underline{X}$ denote a unary predicate symbol (the name of X). Now, using the Compactness Theorem, we can construct a proper elementary extension $(^*\omega,{}^*X)_{X\subseteq\omega}$ of the model $(\omega,X)_{X\subseteq\omega}$. Finally, let $H\in{}^*\omega\backslash\omega$ and $F=\{X\subseteq\omega:\ H\in{}^*X\}$. Then, it is easy to see that F is a non-principal ultrafilter over ω.

3.3. Diagrams of models

One of the problems which model theory solves successfully is the problem of the embeddings of structures. In an analysis of this problem, the notion of diagrams of models has an important function. We would remind the reader that this concept has been already used , explicitly or implicitly, in some of the previous sections.

3.3.1. Definition 1° Let A be a model of a language L, and

$$L_A=L\cup\{a:\ a\in A\}.$$

The diagram of model A is the theory Δ_A of the language L_A whose axioms are the atomic and negations of the atomic sentences of the language L_A true in $(A,a)_{a\in A}$.

2° The elementary diagram of model A is the theory $Th(A,a)_{a\in A}$.

As examples of the use of this notion, we have the following propositions.

3.3.2. Theorem Let A and B be models of a language L. then,
1° The model A can be embedded in B iff there exists a simple expansion $(B,b_a)_{a\in A}$ of B which is a model of Δ_A.
2° A is elementary embedded in B iff there is a simple expansion $(B,b_a)_{a\in A}$ which is a model of the elementary diagram $Th(A,a)_{a\in A}$.

Proof 1° If $f:A\longrightarrow B$ is an embedding, then, $(B,f(a))_{a\in A}$ is a model of the theory Δ_A. For example, if $Ra_1\ldots a_n \in \Delta_A$, then $(A,a_1,\ldots,a_n)$ satisfies $Ra_1\ldots a_n$, hence, $(B,fa_1,\ldots,fa_n)\models Ra_1\ldots a_n$, since the map f is an embedding. On the other hand, if $(B,b_a)_{a\in A}\models \Delta_A$, then the map $f:a\longrightarrow b_a$ is a an embedding of A into B. Indeed, if for example $a_1\neq a_2$, $a_1,a_2\in A$, then $(\neg a_1=a_2)\in\Delta_A$, so $(B,b_a)_{a\in A}\models \neg a_1=a_2$, i.e. $b_{a1}\neq b_{a2}$ thus $fa_1\neq fa_2$. In a similar way, one can prove that f has other properties of embeddings.
2° The proof is similar to the proof of 1°.

As the next example, we have the following proposition.

3.3.3. Theorem Let A and B be models of a language L, and suppose $A\equiv B$. Then, there is a model C in which A and B are elementary embedded.

Proof Let $T= Th(A,a)_{a\in A} \cup Th(B,b)_{b\in B}$ and let us prove that T is a consistent theory. Suppose, on the contrary, that there is a finite $\Delta\subseteq T$, such that Δ has no model. Let $a_1,\ldots,a_m,b_1,\ldots,b_n$ be all the names of elements from the set $A\cup B$ which occur in Δ. We shall assume from now on that $\{a: a\in A\}\cap\{b: b\in B\}=\emptyset$. Since Δ is an inconsistent set, we have $\Delta\vdash \forall x(x\neq x)$. Since Δ is a finite set, and by the choice of symbols a,b, there are sentences $\varphi a_1\ldots a_m$ of L_A, $\psi b_1\ldots b_n$ of L_B, and θ of L which are conjunctions of some sentences from Δ and which also satisfy

$$\theta,\ \varphi a_1\ldots a_m,\ \psi b_1\ldots b_n \vdash \forall x(x\neq x).$$

By the Deduction Theorem, it follows that

$$\theta\vdash \neg\varphi a_1\ldots a_m,\ \neg\psi b_1\ldots b_n\ .$$

By the New Constant Lemma and the fact that a_i,b_j do not appear in θ, it follows that

$$\theta\vdash \forall x_1\ldots x_m\forall y_1\ldots y_n(\ \neg\varphi x_1\ldots x_m \vee \neg\psi y_1\ldots y_n).$$

Since $\{x_1,\dots,x_m\}\cap\{y_1,\dots,y_n\}=\emptyset$, we have

$$\theta\vdash \neg\exists x_1\dots x_m\varphi x_1\dots x_m \vee \neg\exists y_1\dots y_n\psi y_1\dots y_n \ .$$

The sentence θ belongs to language L, so $A\models\theta$ and $B\models\theta$, hence

$$A\models \neg\exists x_1\dots x_m\varphi x_1\dots x_m \vee \neg\exists y_1\dots y_n\psi y_1\dots y_n \ .$$

Further, $A\models\varphi[a_1,\dots,a_m]$, hence $A\models\neg\exists y_1\dots y_n\psi y_1\dots y_n$. Therefore, using the assumption $A\equiv B$, we can conclude $B\models\neg\exists y_1\dots y_n\psi y_1\dots y_n$, and this is a contradiction to $B\models\psi[b_1,\dots,b_n]$.

In a similar way, or by the iterate use of the previous theorem, one can prove

3.3.4. Theorem Let $\langle A_i : i\in I\rangle$ be a family of models of a language L which are mutually elementary equivalent. Then there is a model A of L, in which all the models A_i, $i\in I$, are elementary embedded.

The following proposition describes when two models of a first-order theory T are embedded in a model of T.

3.3.5 Theorem Let T be a theory of a first-order language L and let A and B be models of T. Then, A and B are embedded into a model C of T if and only if for all Σ_1° sentences φ,ψ of L such that $A\models\varphi$, $B\models\psi$, there is a model C of T, such that $C\models\varphi\wedge\psi$.

Proof ($\leftarrow$) Suppose the conditions of the theorem, and let $S=T\cup\Delta_A\cup\Delta_B$. Now, we shall prove that S is a consistent theory. Assume the contrary. Then, there is a finite set $\Delta\subseteq S$ such that $\Delta\models\forall x(x\neq x)$. Further, there are sentences $\theta a_1\dots a_m$ of language L_A, $\sigma b_1\dots b_n$ of language L_B, and the sentence τ of L which are conjunctions of some sentences from T, and which also satisfy:

$$\tau,\theta a_1\dots a_m,\sigma b_1\dots b_n\vdash\forall x(x\neq x).$$

From this fact, it follows that $\tau\vdash\neg\theta a_1\dots a_m\vee\neg\sigma b_1\dots b_n$. By the New Constant Lemma, and since $a_1,\dots,a_m,b_1,\dots,b_n$ do not occur in L, we have

$$\tau\vdash\forall x_1\dots x_m\neg\theta x_1\dots x_m\vee\forall y_1\dots y_n\neg\sigma y_1\dots y_n.$$

By the choice of formulas $\theta a_1\dots a_m$, $\sigma b_1\dots b_n$, we have

$$A\models\exists x_1\dots x_m\theta x_1\dots x_m,\ B\models\exists y_1\dots y_n\sigma y_1\dots y_n.$$

By the assumption that there is a model C of T such that

(1) $C\models\exists x_1\dots x_m\theta x_1\dots x_m\wedge\exists y_1\dots y_n\sigma y_1\dots y_n.$

On the other hand, since C is a model of T, we have

$$C \models \forall x_1 \ldots x_m \neg\theta x_1 \ldots x_m \vee \forall y_1 \ldots y_n \neg\sigma y_1 \ldots y_n,$$

and this is a contradiction to (1).

($\rightarrow$) If φ, ψ are Σ_1^0 sentences and $A \models \varphi$, $B \models \psi$, then by the hypothesis, there is a model C of T, such that A and B are embedded into C. But Σ_1^0 sentences are preserved under embeddings, so, $C \models \varphi \wedge \psi$.

If φ is a Σ_1^0 senetence then there is a formula ψ without quantifiers, such that $\vdash \varphi \leftrightarrow \exists x_1 \ldots x_m \psi$. Further, let

$$\vdash \psi \leftrightarrow (\psi_1 \vee \ldots \vee \psi_n),$$

where ψ_i are conjunctions of atomic formulas and negations of atomic ones. Then $\vdash \varphi \leftrightarrow (\varphi_1 \vee \ldots \vee \varphi_n)$, where $\varphi_i = \exists x_1 \ldots x_m \psi_i$. Thus, we have the following variant of Theorem 3.3.5.

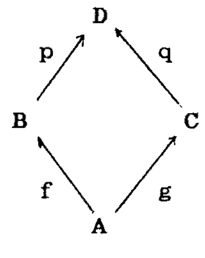
3.3.6. Theorem Let L be an algebraic language L. Further, let T be a theory of L, and let A,B be models of T. Then, A and B are embedded into a model C of T if and only if for every two finite systems of equations and negations of equations S_1, S_2 which are consistent respectively in A and B, there is a model C of T in which the system $S_1 \cup S_2$ is consistent.

Theories of linearly ordered structures give an interesting possibility. Namely, we can replace, in the previous theorem, the term "negations of equations" by "inequalities".

3.3.7. Example The amalgamation property of Boolean algebras.

We shall prove that the class of all Boolean algebras has the amalgamation property, namely, the following holds:

(1) If A,B,C are Boolean algebras and $f:A \rightarrow B$, $g:A \rightarrow C$, where f and g are embeddings, there is an algebra D and embeddings $p:B \rightarrow D$ and $q:C \rightarrow D$, such that the displayed diagram commutes.



First, let us prove the following proposition.

Claim Let A be a finite Boolean algebra, B an atomless Boolean algebra, and $h:A \rightarrow B$ an embedding. Further, suppose $A \subseteq A'$ and $a \in A'$. If A(a) denotes Boolean algebra generated by the subalgebra A and the element a, then h can be extended to an embedding $f:A(a)\longrightarrow B$.

Proof of Claim Without loss of generality, we may suppose that h is the inclusion map, i.e. $h:x\mapsto x$, $x\in A$, and so $A\subseteq B$. Let $\{a_1,\dots,a_n\}$ be the set of all the atoms of the algebra A. Since B is atomless, for every a_i there is $c_i\in B$ such that $a_i>c_i>0$. Let

$$I=\{i\leq n:\ a\cdot a_i\neq 0,\ a'\cdot a_i\neq 0\},\quad J=\{i\leq n:\ a\cdot a_i\neq 0,\ a'\cdot a_i=0\},$$

and $b=\Sigma_{i\in I}c_i+\Sigma_{j\in J}a_j$. then the map $f:A(a)\longrightarrow B$ defined by

$$f(a\cdot x+a'\cdot y)=b\cdot x+b'\cdot y\ ,\quad x,y\in A,$$

is an embedding which extends h, and this finishes the proof of Claim.

Now, we shall turn to the proof of statement (1). Let T be the theory of Boolean algebras, and let us show that the theory $T+\Delta_A+\varphi+\psi$ is consistent, where A is a Boolean algebra, and φ, ψ are $\Sigma_1{}^0$ sentences of the language of T expanded with the names of elements from A such that the theories $T+\Delta_A+\varphi$, $T+\Delta_A+\psi$ are consistent. Let $\Sigma\subseteq\Delta_A$ be finite, and $E\subseteq A$ be a finite Boolean subalgebra generated by names in $\Sigma\cup\{\varphi,\psi\}$. Further, let B_1, C_1 be finite Boolean algebras such that $E\subseteq B_1$, $E\subseteq C_1$, and

$$B_1\models T+\Sigma+\varphi,\quad C_1\models T+\Sigma+\psi.$$

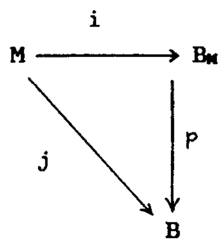
Since B_1 and C_1 are finite Boolean algebras, there are embeddings $\alpha:B_1\longrightarrow\Omega$, $\beta:C_1\longrightarrow\Omega$ where Ω is a countable atomless Boolean algebra. If α' and β' are restrictions of α and β to E, then, since Ω is homogeneous (see Claim), there is $\mu\in\mathrm{Aut}\Omega$ such that $\mu\cdot\alpha'=\beta'$. Further, there is a simple expansion Ω' of Ω, such that $\Omega'\models T+\Delta_A+\varphi+\psi$.

Thus the theory $T+\Delta_A+\varphi+\psi$ is consistent, so, by Theorem 3.3.5, every two models of $T+\Delta_A$ are embedded into a model of $T+\Delta_A$. This proves that the theory of Boolean algebras has the amalgamation property.

Using the amalgamation property of Boolean algebras, it is possible to show that the same property has the class of distributive lattices with the greatest and the least element. In addition, we shall use the following theorem which says that every distributive lattice with the greatest and the least element is embedded into the least Boolean algebra.

3.3.8. Theorem Let M be a distributive lattice with the greatest and the least element. Then, there is a Boolean algebra B_M which contains M, and has the property:

If B is any Boolean algebra, such that there is an embedding $j: M \longrightarrow B$ (which preserves 0 and 1), then, there is an embedding $p: B_M \longrightarrow B$ such that the displaed diagram commutes.



i is the inclusion map.

Proof We shall give only an outline of the proof of this theorem. By the representation theorem for distributive lattices, there is a ring of sets S isomorphic to M. Then, let B_M be the least field of sets containing S, i.e. B_M is generated by S and Boolean operations. It is easy to see that B_M has the desired properties.

Now, Theorem 3.3.8. is used to transfer the basis of the lattice (Fig.1) amalgam into the class of the Boolean algebra (Fig.2), to amalgamate there, and then, to transfer into the class of distributive lattices (Fig.3). This process is shown by the following diagrams.

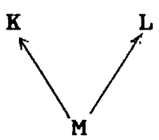


Fig.1

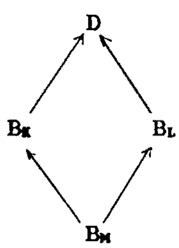


Fig.2

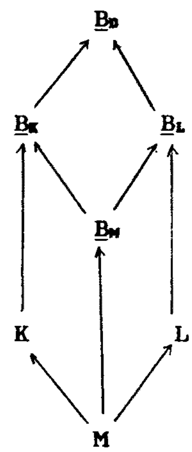


Fig.3

Here M, K, and L are distributive lattices with the greatest element and the least element, while B_M, B_K, and B_L are the least Boolean algebras containing the lattices M, K, and L, respectively. The first diagram is the amalgam basis of lattices, and the second is the amalgam of Boolean algebras. Finally, $\underline{B}_M$, $\underline{B}_K$, $\underline{B}_L$, $\underline{B}_D$ are reducts of Boolean algebras B_M, B_K, B_L, B_D to the language of the lattices with 0 and 1.

Corollary The class of all the distributive lattices with the greatest element andthe least element has the amalgamation property.

As one more application of the compactness argument, we shall prove the Löwenheim-Skolem theorems. These theorems are specific features of the first-order predicate calculus.

3.3.9 Theorem (Downward Löwenheim-Skolem Theorem). Let T be a theory of a first order language L and A an infinite model of this theory such that $|A| \geq \|L\|$, and let $X \subseteq A$. Then, there is an elementary submodel $B \prec A$, such that $X \subseteq B$ and $|B| \leq |X| + \|L\|$.

Proof We going to construct a sequence of sets $X \subseteq S_0 \subseteq S_1 \subseteq \ldots \subseteq A$ in the following way:

S_0 is a set with property: if φx is a formula of the language L_X and $A \models \exists x \varphi x$, then there is $a \in S_0$, such that $A_0 \models \varphi a$, where $A_0 = (A,a)_{a \in X}$. By the Axiom of Choice, such a set exists, and we may take $|S_0| \leq \|L\| + |X|$.

Further, the construction is repeated, i.e. S_{i+1} is constructed from S_i as it was S_0 from X. If $B = \bigcup_i S_i$, then it is not difficult to see that B satisfies the conditions of the theorem.

3.3.10. Theorem (Upward Löwenheim-Skolem Theorem) Let A be an infinite model of a language L. Then, for every infinite cardinal $k \geq |A| + \|L\|$, there is a model B of cardinality k such that $A \prec B$.

Proof Let $C = \{c_\alpha : \alpha < k\}$ be the set of new constant symbols. By the compactness Theorem, it is easy to see that

$T = Th(A,a)_{a \in A} \cup \{c_\alpha \neq c_\beta : \alpha < \beta < k\}$

is a consistent theory. Namely, every finite subset Σ of this theory is satisfied in a simple expansion of model A. Thus, T is a consistent theory, so there is a model $B' = (B, b_a, d_\alpha)_{a \in A,\ \alpha < k}$, where B is the reduct of B' to the language L. since there k different constants in B, we can infer that $|B| \geq k$. Further, the map $\tau : a \longrightarrow b_a$ is elementary, so we may take $A \prec B$, identifying a and b_a. Note that $A' = \{b_a : a \in A\}$ is an elementary submodel of B isomorphic to A. By Theorem 3.3.9, there is a model C of the language L, such that $A \subseteq C$, $C \prec B$, $|C| = k$. Then, it follows that $A \prec C$.

The following examples illustrate applications of the Löwenheim-Skolem theorems.

3.3.11. Theorem 1° Let **R** be the ordered field of real numbers. By the Löwenheim-Skolem Theorems, there are ordered fields ***R** and **F**, such that

$$F \prec R \prec {}^*R \quad \text{and} \quad |F| = \aleph_0, \ |{}^*R| = 2^c.$$

2° For every infinite cardinal k, there is a model of formal arithmetic, which is an elementary equivalent to the standard structure of natural numbers.

3° We shall exhibit a nonstandard model of real numbers in which there are nonstandard elements of **N** as many as there are real numbers. To see this, let $R = (R, N, +, \cdot, \leq, 0, 1)$ be the field of real numbers expanded by unary relation N, the set of natural numbers. By the Downward Löwenheim-Skolem Theorem, there is a countable model $S \prec R$, where $S = (S, N, +, \cdot, \leq, 0, 1)$. So, there is a map $f : S \longrightarrow N$ which is 1-1 and onto. Using the Upward Löwenheim-Skolem Theorem, it can be easily seen that there is a model with the desired properties for every infinite cardinal.

Exercises

3.1. Let A and B be Boolean algebras and $f : A \longrightarrow B$. Let $f^* : B^* \longrightarrow A^*$ be defined by $f^*(p) = \{f(b) : b \in p\}$, $p \in B^*$. Show:

1° f is a homomorphism from A into B iff f^* is continuous map from B^* into A^*.

2° f is an embedding of A into B iff f^* is continuous and maps B^* onto A^*.

3° f is an epimorphism iff f^* is an embedding of space B^* into A^*.

3.2. Let A and B be Boolean algebras. Show:

1° If $A \subseteq B$, then A^* is homomorphic to a quotient space of B^*.

2° If $A = B/J$, where J is an ideal of B, then $A^* = B^* \setminus J^*$, where $J^* = \{p \in B^* : p \cap B^* \neq \emptyset\}$.

3.3. If B is a Boolean algebra, then B^* is a metric space iff $|B| \leq \aleph_0$.

3.4. If B is a Boolean algebra and $T \subseteq B$ an infinite binary tree, then subalgebra C of B generated by T is an atomless Boolean algebra.

3.5. If B is a Boolean algebra and $|B| < |B^*|$ then B contains as an subalgebra an atomless Boolean algebra.

3.6. If T is a first order theory of a countable a language, and C is the set of all complete extensions of T, then $|C| \in \omega \cup \{\aleph_0, 2^{\aleph_0}\}$.

3.7. If B is a countable Boolean algebra then $|B^*| \in \omega \cup \{\aleph_0, 2^{\aleph_0}\}$.

3.9. If X is an uncountable metric space, then the Cantor space is a quotient space of X.

3.10. Show that for every Boolean algebra B, $|B| \leq |B^*|$.

3.11. Show that there are $2^{\aleph_0}$ nonisomorphic countable Boolean algebras.

3.12. Assuming that every finite planar graph has coloring in four colors, show that every infinite planar graph has coloring in four colors, as well.

3.13. Show that for every positive integer n there is a positive integer m such that in every subset S of plain with m points, there are n points which are vertices of a convex polygon.

3.14. Let $P_i : [\omega]^{e_i} \longrightarrow r_i$, $1 \leq i \leq n$, be n partitions. show that there is a partition $P : [\omega]^e \longrightarrow r$ such that for all $X \subseteq \omega$, $|X| \geq e$, X is homogeneous for P iff X is homogeneous for all the partitions P_i.

3.15. Show that the following theories are not finitely axiomatizable:
1° The theory of infinite models of pure predicate calculus with equality.
2° The theory of fields of characteristic 0.
3° The theory of algebraically closed fields.

3.16. Let T be a theory of a language l and φ a sentence of L. If φ

3.16. Let T be a theory of a language l and φ a sentence of L. If φ holds in all infinite models of T, show that there is $n\in\omega$ such that φ holds in all models $A\models T$ such that $|A|\geq n$.

3.17. Let φ be a sentence in the language of fields. Suppose φ is true in every field of characteristic 0. Show that there exists an integer n such that φ is true in every field of characteristic greater than n.

3.18. Let F be an algebraically closed field. If $f:F^n \longrightarrow F^n$ is polynomial, i.e. $f(\underline{x})=(f_1(\underline{x}),\dots,f_n(\underline{x}))$ where $f_1,\dots,f_n$ are polynomials over F, then: if f is 1-1 then f is onto.

3.19. Suppose that every existential sentence φ of a language L which holds in a model A of L, also holds in a model of T. Show that A is a submodel of a model of T.

3.20. A linearly ordered set $A=(A,\leq)$ is homogeneous if for all finite sequences $a_1<a_2<\dots<a_n$, $b_1<b_2<\dots<b_n$ there is $f\in \mathrm{Aut}A$ such that $fa_1=b_1,\ \dots,\ fa_n=b_n$. Show:

1° The ordering of rational numbers is homogeneous.

2° For every infinite cardinal k there is a homogeneous linearly ordered dense set of the cardinality k.

4. REALIZING AND OMITTING TYPES

Saturated structures may be useful in an analysis of the model-theoretic versions of syntactical notions, as they are, for example, the elimination of quantifiers. On the other hand, saturated models have many properties of universal objects in categories, and this enables us to characterize some model-theoretic properties by arrow diagrams. The best known example of this kind is the model-completion of theories. There is one more aspect of saturated models. Namely, the applications of these structures enables one to avoid, in many cases, the call of transfinite induction, because it is absorbed in the construction of saturated models. In other words, when a model is built, say in α steps (α is an infinite cardinal), we can use the existence theorem of saturated models in some cases.

On the other hand, there are few saturated structures; for a given infinite cardinal α up to the elementary equivalence. The existence of these models is certainly provided only under some set-theoretical assumptions, for example under GCH, or the existence of inaccessible cardinals. Therefore, there are several generalizations of this notion, and usually it is not necessary to assume an additional set-theoretical hypothesis for their existence. These generalized concepts are mainly reduced to the partial saturativness of models, and the most important are k-saturated models, special models and recursively saturated models.

There is another class of results in model theory which employs constructions in which structures are extended in such a way that selected sets of formulas are omitted, i.e. not realized. Usually, it is more difficult to omit a certain set of formulas then to realize it, since the process of omitting requires that every element of the extension be "worried over". There are many applications of omitting types, not only in the first order logic, but beyond it, as well. We shall study later such an application in logic with additional quantifiers.

4.1. Partially saturated models

Intuitively speaking, saturated models realize all the consistent types. This notion was introduced by B. Jonsson, M. Morley, and R. Vaught, and it made it possible to unify and simplify a large part of model theory. Some ramifications of this concept have also been considered. J. Keisler introduced the notion of a k-saturated model, while recursively saturated models were defined by J. Barwise and J. Schlipf in 1975.

The notion of type plays a main role in the definition of saturated models. In the following, by $\Sigma(x)$ we shall denote a set of formulas with x as the only free variable. The set $\Sigma(x)$ is ***satisfiable*** in a model **A**, if there is an element $a\in A$, such that $A\models \varphi[a]$ for all $\varphi x\in\Sigma(x)$.

4.1.1. Example 1° Let $\Sigma(x)=\{1<x,\ 1+1<x,\ 1+1+1<x,\dots\}$. Then an ordered field **F** realizes $\Sigma(x)$, iff **F** is a nonarchimedian field. A model **M** of Peano arithmetic realizes $\Sigma(x)$ iff **M** is a nonstandard model, i.e. **M** is not isomorphic to the structure of natural numbers.
2° If $\Sigma(x)=\{p(x)\neq 0:\ p(x)\in \mathbf{Q}[x]\}$, where $\mathbf{Q}[x]$ is the set of all rational polynomials, then a field **F** of characteristic 0 realizes $\Sigma(x)$ iff **F** has a transcendental element over **Q**.

Now, let **A** be a model of a language L, and $A_X=(A,a)_{a\in X}$, where $X\subseteq A$. A ***type*** over A_X is every set of formulas p(x) of the language L_X, which is consistent with the theory ThA_X. We shall keep the notion introduced in this way in the following definition.

4.1.2. Definition 1° A model A is saturated over $X\subseteq A$ iff every type p(x) over A_X is realized in the model A_X.
2° A model A is saturated iff A is saturated over every $X\subseteq A$, $|X|<|A|$.
3° A model A is k-saturated, where k is a cardinal number, iff A is saturated over every $X\subseteq A$, $|X|<k$.

The assumption $|X|<|A|$ in the previous definition (part two) is necessary since the set of formulas $\Gamma=\{x\neq a: a\in A\}$ obviously cannot be realized in A, even if Γ is a type for infinite A. Therefore, A is saturated iff A is an $|A|$-saturated model. By countably saturated models, we mean countable ω-saturated models. Saturated models are usually obtained by iterating the compactness argument. Another way for obtaining saturated models is the ultraproduct construction. Here is an example of this second kind.

4.1.3. Theorem Let A_i, $i\in\omega$, be models of a language L, and let D be a nonpricipal ultrafilter over ω. Then $A=\Pi_D A_i$ is an $\aleph_1$-saturated model.

Proof First, let us note that D contains a decreasing chain of sets $\ldots\subseteq J_2\subseteq J_1\subseteq J_0=\omega$, such that $\cap_n J_n=\emptyset$. Further, for any simple expansion $B=(A,f_{1D},f_{2D},\ldots)$, there are simple expansions $B_i=(A_i,a_1,a_2,\ldots)$, such that $B=\Pi_D B_i$. Therefore, it suffices to check that A realizes types over A, i.e. we may assume $X=\emptyset$.

So, let $\Sigma(x)=\{\varphi_1(x),\varphi_2(x),\ldots\}$ be a set of formulas over L such that every finite subset of $\Sigma(x)$ is realized in A. Further, let us introduce the sequence of sets

$X_n=\{i\in J_n: A_i\models\exists x(\varphi_1 x\wedge\ldots\wedge\varphi_n x)\}$, $n>0$, $n\in\omega$.

Then, $\cap_n X_n=\emptyset$, and X_n is a decreasing sequence of sets in D, so, for every $i\in\omega$, there is the greatest $n_i\in\omega$, such that $i\in X_{n_i}$. Let $f\in\Pi_i A_i$ be a function such that

if $n_i>0$, then $A_i\models(\varphi_1\wedge\ldots\wedge\varphi_{n_i})[f(i)]$.

Thus, if $i\in X_n$, then $A_i\models\varphi_n[f(i)]$, therefore $A\models\varphi_n[f_D]$ by the Łoš Theorem. Hence, f_D realizes the type $\Sigma(x)$ in A.

Assume $p(x)$ is a type over A_X, $X\subseteq A$. Since $p(x)$ is finitely satisfiable, by the Compactness Theorem it follows that $p(x)$ is satisfied in a model B_X, where $A_X\prec B_X$.

4.1.4. Example Let A be an algebraically closed field. Then, A is a saturated model iff A is of an infinite transcendence degree over its prime field.

We would remind the reader that a field is prime, if it is isomorphic to a finite field Z_p, or to the field of rational numbers. A field A is of the finite transcendence degree over F, if there is $n\in\omega$ and $a_1,\ldots,a_n\in A$, such that every element $a\in A$ is algebraic over the field of rational expressions $F(a_1,\dots,a_n)$, otherwise, it is of the infinite transcendence degree.

Now, we shall prove the statement itself. We shall use the fact:

(1) The theory of algebraically closed fields T admits the elimination of quantifiers, i.e. for every formula $\varphi x_1\dots x_n$ of L_T there is a formula $\psi x_1\dots x_n$ of L_T without quantifiers, such that

$$T\vdash \forall x_1\dots x_n(\varphi\leftrightarrow\psi)$$

Let A be an algebraically closed field of the infinite transcendence degree, and let $p(x)$ be a type over A_X, where $|X|<|A|$. Then $p(x)$ is realized by an element b in a model B, where $A\prec B$. If b is algebraic over X, then $b\in A$. Suppose b is transcendental over X, and let C be the least subfield of the field A which contains X. Choose an element $a\in A$ which is transcendental over X. Such an element exists, since $|X|<|A|$ and $|A|$ is equal to the transcendence degree of field A. Let G be the algebraic closure of the field C(a) (C(a) is the field of rational expressions over $C\cup\{a\}$) in the field A. Let $h:G\longrightarrow B$ be the embedding, such that $h|C$ is the identity function, and $ha=b$. By (1), the embedding h is elementary, so, the element a realizes $p(x)$ in G, and this means that a realizes $p(x)$ in A, as well, since $G\prec A$.

By this example, it follows that every uncountable algebraically closed field is saturated.

The theory of dense linearly ordered sets without end-points admits the elimination of quantifiers, and, using this fact, one can show that saturated dense linearly ordered sets are exactly η_α sets.

4.1.5. Example For a dense linearly ordered set $A=(A,\le)$ is said that it is an η_α set iff for all $X,Y\subseteq A$, such that $|X\cup Y|<\aleph_\alpha$ and $X<Y$ (i.e. for all $x\in X$, all $y\in Y$, $x<y$), there is an $a\in A$, such that $X<a<Y$. Then, a dense linearly ordered set A is k-saturated iff A is an η_k-set.

The following theorems are related to the existence and uniqueness of saturated structure.

4.1.6. Theorem If A and B are saturated models of the same cardinality, then $A\equiv B$ implies $A\approx B$.

Proof Let $|A|=k$, $A=\{a_\alpha: \alpha<k\}$ and $B=\{b_\alpha: \alpha<k\}$. We shall define sequences $\langle c_\alpha: \alpha<k\rangle$ and $\langle d_\alpha: \alpha<k\rangle$ by the so-called back-and-forth argument, so that

(1) $(A,c_\alpha)_{\alpha<\beta} \equiv (B,d_\alpha)_{\alpha<\beta}$

holds for all $\beta<k$. Let us note that (1) is reduced to the already given condition $A\equiv B$ for $\beta=0$. Further, we shall distinguish two cases, when β is an even and when β is an odd ordinal.

β is an even ordinal: Let c_β be the first element in the sequence a_α which differs from all the c_α, $\alpha<\beta$, which were constructed in the first β steps, and define

$$p(x)=\{\varphi x: (A,c_\alpha)_{\alpha<\beta} \models \varphi[c_\beta]\}.$$

Then $p(x)$ is a type over A_X, $X=\{c_\alpha: \alpha<\beta\}$. Let $q(x)$ be a type obtained from $p(x)$, so that each c_α is substituted by a symbol d_α for $\alpha<\beta$. Then $q(x)$ is a type over B_Y, where $Y=\{d_\alpha: \alpha<\beta\}$. As B is a saturated model, $q(x)$ is realized in B by an element b, so, let $d_\beta=b$. Then,

$$(A,c_\alpha)_{\alpha<\beta} \equiv (B,d_\alpha)_{\alpha<\beta}$$

β is an odd ordinal: The construction is similar to the case when β is even, but the roles of models **A** and **B** are reversed.

Now, we can define the map $h:A\longrightarrow B$, taking $h:c_\alpha\longmapsto d_\alpha$, $\alpha<k$. It is not difficult to see that h is an isomorphism of models **A** and **B**.

If A_i, $i\in\omega$, are infinite models of the cardinality less than or equal to the continuum, then for every nonprincipal ultrafilter D over ω, the ultraproduct $\Pi_D A_i$ is of the cardinality continuum. Then, by Theorem 4.1.3, and Łoš Theorem, we have

4.1.7. Corollary (under CH) 1° Let A_i and B_i, $i\in\omega$, be infinite models of the same language of the cardinality of at most $c=2^{\aleph_0}$, and let D be a nonprincipal ultrafilter over ω. Then

$$\Pi_D A_i \equiv \Pi_D B_i \quad \text{implies} \quad \Pi_D A_i \approx \Pi_D B_i .$$

2° If A and B are models of the same language of the cardinality at the most continuum, then, for every nonprincipal ultrafilter D over ω, $A \equiv B$ implies $\Pi_D A_i \approx \Pi_D B_i$.

By Example 4.1.4, we also have

4.1.8. Corollary Let A and B be algebraically closed fields of the same characteristic, the same cardinality and the infinite transcendence degree. Then $A \approx B$.

A similar statement is true under GCH for η_k-sets, namely, every two η_k-sets of the same cardinality are isomorphic (Hausdorff).

The following theorem asserts the existence of saturated models.

4.1.9. Theorem Let A be an infinite model of a language L, $\|L\| \le k$. Then for every infinite cardinal k there is a k^+-saturated model B such that $A \prec B$ and $|B| \le |A|^k$.

Here k^+ denotes the least cardinal greater than k.

Proof The construction of model B goes as follows. First, model A is extended to a model A_1 which realizes every type over A_X for every $X \subseteq A$, $|X| < k$. Such a model A_1 exists by the Compactness Theorem since the theory

$Th(A,a)_{a \in A} \cup \bigcup\{p(c_p): p(x)$ is a type over A_X, for some $X \subseteq A$, $|X| \le k\}$
is finitely consistent. By the downward Löwenheim-Skolem Theorem, we may take that A_1 has the cardinality at most $|A| + \|L_1\|$, where

$L_1 = L \cup \{c_p: p(x)$ is a type over A_X for some $X \subseteq A$, $|X| \le k\}$.
Since $|\{X: X \subseteq A, |X| \le k\}| \le |A|^k$, we may assume that $|A_1| \le |A|^k$.

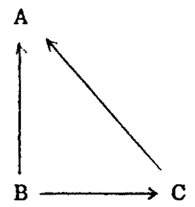
We can now construct an elementary chain of models A_α, $\alpha < k^+$, taking $A_0 = A$, and $A_{\alpha+1}$ is a model of the cardinality $\le |A|^k$ constructed from A_α in the same way as A_1 was constructed from A. If β is a limit ordinal, then $A_\beta = \bigcup_{\alpha<\beta} A_\alpha$. Now, defining $B = \bigcup_{\alpha<k^+} A_\alpha$ and using the regularity of the cardinal k^+, it can be easily seen that model B satisfies the conditions of the theorem.

4.1.10. Corollary Let $\|L\| \le k$, and assume GCH. Then every theory of L having infinite models has a saturated model in each regular power $\tau \ge k$.

4.2. Property of universality of saturated models

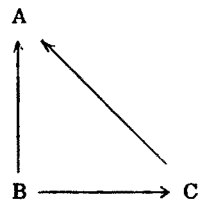
In this section, we shall discuss some universality features of saturated models. The first characteristic concerns a diagram property of saturated models.

4.2.1 Theorem Let $k > \aleph_0$ and suppose $\|L\| < k$. A model A of a language L is k-saturated iff the displayed diagram has the indicated completion i.e. for all elementary embeddings $\mu : B \longrightarrow A$, $\tau : B \longrightarrow C$ there is an elementary embedding $\alpha : C \longrightarrow A$ such that $\mu = \alpha\tau$.



B and C are models of language L, $|B| < k$, $|C| \leq k$.

Proof Suppose A has the diagram property, and let $Y \subseteq A$, $|Y| < k$. Let p be a type over B_Y, where $B \prec A$, $Y \subseteq B$, $|B| < k$. The existence of model B_Y is provided by the downward Löwenheim-Skolem Theorem. Type p is realized in a model C, where $B \prec C$, by an element $c \in C$, and $|C| \leq k$. Then, type p is realized in A by αc.



B and C are models of language L, $|B| < k$, $|C| \leq k$, $B \prec C$, $B \prec A$

Suppose, now, that model A is k-saturated. Without loss of generality, we may assume the situation as indicated in diagram. Let $C \setminus B = \{c_\delta : \delta < k\}$. A map α is defined as follows by using the back and forth argument.

$\alpha | B = i_B$.

Values αc_δ are defined inductively. Suppose

$(C, b, c_\sigma)_{b \in B,\, \sigma < \delta} \equiv (A, b, \alpha c_\sigma)_{b \in B,\, \sigma < \delta}$.

Let p be a type of the element c_δ in model $(B, b, c_\sigma)_{b \in B,\, \sigma < \delta}$. Then p is a type over $(A, b, \alpha c_\sigma)_{b \in B,\, \sigma < \delta}$, so, since A is k-saturated, it follows that A realizes p by an element a. Then we define $\alpha c_\delta = a$.

A model A is *homogeneous* iff every partial automorphism of A is extendable to an automorphism of A. More details on this notion will be given in Chapter 5.2. By the last theorem, we have the following result.

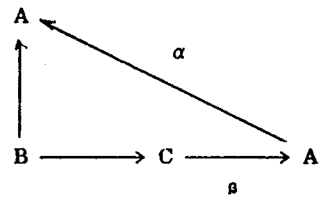
4.2.2. Corollary Every saturated model is homogeneous.

A model A of a language L is ***universal*** for a class M of models of language L if and only if for every model $B \in M$ of the cardinality $\leq |A|$ there is an embedding $\alpha: B \longrightarrow A$. If the embedding α is elementary, then we say that A is an ***elementary universal*** model.

4.2.3. Theorem (J.Keisler) A model A is saturated if A is homogeneous and elementary universal.

Proof ($\rightarrow$) Suppose A is saturated. By Corollary 4.2.2, A is homogeneous. We shall now prove that A is an elementary universal model for the class of models elementary equivalent to model A. The proof is similar to the one in Theorem 4.2.1, namely by "half" of the back and forth argument a sequence of partial isomorphisms from B into A is built, and the union of these maps gives an isomorphism from B into A.

($\leftarrow$) If A is a homogeneous and elementary universal model, then, on the displayed diagram, we have

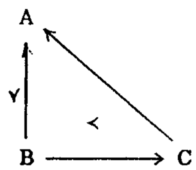


B is an arbitrary model of L, $|B| < |A|$, $|C| \leq |A|$.

Map β exists since A is universal, and map α exists since A is homogeneous. Therefore, model A has the diagram property on which Theorem 4.2.1 reflects, and, therefore, A is a saturated model.

A model B is ***finitely generated***, if there is $Y \subseteq B$, $|Y| < \aleph_0$ such that B is the least substructure of B which contains Y. Then, we have for ω-saturated models the following variant of theorem 4.2.1.

4.2.4. Theorem A model A is ω-saturated iff the displayed diagram has a completion as indicated.



B is finitely generated, $|C| \leq \aleph_0$.

4.2.5. Theorem Let C be a saturated model of a regular cardinality k, $|L_C| \leq k$. Then for any consistent theory T, such that $Th(C) \subseteq T$, $\|L_T\| \leq k$, there is an expansion C^* of C to the language L_T which is a model of T.

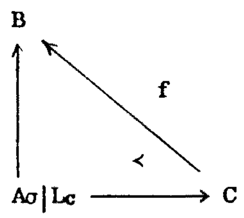
Proof Assume k is a regular cardinal, A is a k^+-saturated model of T, and B is a reduct of A to L_C. Thus, B is a k^+-saturated model as a reduct of a k^+-saturated model. We shall now construct sequences of models

$$A_0 \prec A_1 \prec \ldots \prec A_\sigma \prec \ldots \prec A,$$
$$B_0 \prec B_1 \prec \ldots \prec B_\sigma \prec \ldots \prec B, \qquad \sigma < k,$$

such that B_σ is a reduct of A_σ to L_C, $|A_\sigma| = |B_\sigma| = k$, $B_\sigma \subseteq A_{\sigma+1}$, and for all $\sigma < k$, B_σ is a saturated model.

For limit ordinals α, we shall take $A_\alpha = \bigcup_{\sigma<\alpha} A_\sigma$, $B_\alpha = \bigcup_{\sigma<\alpha} B_\sigma$. A model $A_0 \prec A$ is chosen arbitrarily so that $|A_0| = k$.

Now, suppose the model A_σ has been constructed. Then the reduct $A_\sigma|L_C$ is a model of the theory ThC, therefore, by the universality of model C, model $A_\sigma|L_C$ is elementary embedded into C



$|A_\sigma| = k$, $|C| = k$,
B is k^+-saturated.

By diagram property of saturated models, this diagram is completed as indicated. Let $B_\sigma = fC$. Thus $B_\sigma \approx C$, therefore, B_σ is a saturated model as well. Suppose, now, that B_σ has been constructed. Let $A_{\sigma+1}$ be such that $A_\sigma \prec A_{\sigma+1}$, $B_\sigma \subseteq A_{\sigma+1}$, and $|A_{\sigma+1}| = k$. Observe that such a model $A_{\sigma+1}$ exists by the Downward Löwenheim-Skolem Theorem. Then $A_{\sigma+1}|L_C \prec B$, so $B_\sigma \prec A_{\sigma+1}|L_C$, i.e. $B_\sigma \prec A_{\sigma+1}$. Let $D = \bigcup_{\sigma<k} A_\sigma$. Then,

1° $D|L_C$ is a saturated model of the theory ThC, so $D|L_C \approx C$.
2° D is a model of T.

Here we have used the regularity condition on cardinal k:
If $X \subseteq D$, $|X| < k$, then for some $\sigma < k$ it follows that $X \subseteq B_\sigma$.

4.3. Applications of saturated models

As the first example of an application of saturated structures, we shall consider the interpolation theorem and propositions to it related. These theorems are of arithmetical character, i.e. one can consider them statements about the combinatorial universe $V\omega$, but in the proofs of these assertions, we shall use very nonconstructive arguments, such as GCH for example. However, these proofs can be transformed into proofs of an arithmetical character.

4.3.1. Robinson Consistency Theorem Let T_1 and T_2 be theories of countable languages L_1 and L_2, respectively. Further, suppose these theories are deductively closed, and let $T= T_1 \cap T_2$ (thus, $L_T \subseteq L_1 \cap L_2$). If T is a complete theory and if T_1 and T_2 are consistent, then $T_1 \cup T_2$ is consistent, too.

Proof Let A be a saturated model of T. Then, $ThA \subseteq T_1, T_2$, so by Theorem 4.2.5, model A has expansions $A^{\#_1}$, $A^{\#_2}$ to models of theories T_1 and T_2, respectively. Thus, $A^{\#}$ is a model of $T_1 \cup T_2$, where $\# = \#_1 \cup \#_2$.

Using the previous theorem, it is easy to deduce the interpolation theorem of PR^1.

4.3.2. Craig Interpolation Theorem Let φ and ψ be sentences of PR^1. If $\models \varphi \rightarrow \psi$, then there exists a sentence θ, such that for the language $L(\theta)$ of θ, we have $L(\theta) \subseteq L(\varphi) \cap L(\psi)$, $\models \varphi \rightarrow \theta$, and $\models \theta \rightarrow \psi$.

Proof Suppose $\models \varphi \rightarrow \psi$ and let $L_1 = L(\varphi)$, $L_2 = L(\psi)$. Further, define $\Sigma = \{\theta \in Sent(L_1 \cap L_2) : \models \varphi \rightarrow \theta\}$. Then,

(1) There is $\theta \in \Sigma$ such that $\models \theta \rightarrow \psi$.

Suppose the contrary. Then $\Sigma \cup \{\neg\psi\}$ is a consistent theory in L_2, so, this theory has a model A. Let $T_2 = ThA$. Further, let

$T = T_2 \cap Sent(L_1 \cap L_2)$, i.e. $T = Th(A|(L_1 \cap L_2))$.

Thus, T is complete. Now, define $T_1 = T \cup \{\varphi\}$. Then T_1 is a consistent theory. If not, then there are senetences $\sigma_1, \ldots, \sigma_n \in T$ such that $\varphi \models \neg(\sigma_1 \wedge \ldots \wedge \sigma_n)$. But $\neg(\sigma_1 \wedge \ldots \wedge \sigma_n)$ is a sentence in $L_1 \cap L_2$ and belongs to Σ, hence it holds in A. However, this is a contradiction, since each σ_i is true in A.

Further, we see that $T = T_1 \cap T_2$. By the Robinson Consistency Theorem, there is a model of $T_1 \cup T_2$ which is a model of φ and $\neg\psi$ too, but this is a contradiction, since $\models \varphi \rightarrow \psi$.

Let P be a new n-ary relation symbol which does not belong to a language L, and let T be a theory of the language $L\cup\{P\}$. In this case, we shall write T(P) instead of T. Further, we shall say that P is ***explicitly definable*** in L by T(P) iff there is a formula $\varphi x_1 x_2 \ldots x_n$ of L, such that

$$T(P) \models \forall x_1 \ldots x_n (Px_1 \ldots x_n \leftrightarrow \varphi x_1 \ldots x_n).$$

A necessary condition for T(P) to define P explicitly is:

(1) For any model **A** of language L there is at most one relation R of length n, and with domain A such that the expansion (**A**,R) is a model of T(P).

Really, if (**A**,R) and (**A**,S) are models of T(P), then for all $a_1, \ldots, a_n \in A$ we have

$Ra_1 \ldots a_n$ iff $\mathbf{A} \models \varphi[a_1, \ldots, a_n]$,

$Sa_1 \ldots a_n$ iff $\mathbf{A} \models \varphi[a_1, \ldots, a_n]$,

and from these follows that R=S. Padoa was the first to observe this fact, and it led him to the conclusion

(2) If there is a model **A** with two different interpretations R and S of a relation symbol P, such that (**A**,R) and (**A**,S) are models of T(P), then T(P) does not define P explicitly.

Fact (2) is known as the Padoa method, and it is often used in order to show that a certain set of axioms T(P) is not sufficient to define a predicate P explicitly. It is interesting that the necessary condition (1) for T(P) to define P explicitly is also sufficient, as the following theorem says.

4.3.3. Theorem (E.W.Beth) A theory T(P) defines P explicitly if and only if

(1) For every model **A** of language L, there exists at most one relation R over A, such that $(\mathbf{A},R) \models T(P)$.

Proof Obviously, we need to show only the sufficiency of condition (1). Thus, let Q be a relation symbol of length n which is not in L∪{P}, and let T(Q) be the appropriate theory of the language L∪{Q}, i.e. every sentence $\varphi(Q)$ of T(Q) is obtained from $\varphi(P)$ of T(P) by replacing P by Q. Then, condition (1) implies

$$T(P), T(Q) \models \forall x_1 \ldots x_n (Px_1 \ldots x_n \leftrightarrow Qx_1 \ldots x_n).$$

By the Compactness Theorem there are a finite conjunction $\psi(P)$ of sentences of T(P) and a finite conjunction $\theta(Q)$ of sentences of T(Q), such that

$$\psi(P), \theta(Q) \models \forall x_1 \ldots x_n (Px_1 \ldots x_n \leftrightarrow Qx_1 \ldots x_n).$$

Replacing the variables by new constant symbols $c_1, \ldots, c_n$, we have

$$\psi(P), \theta(Q) \models Pc_1 \ldots c_n \leftrightarrow Qc_1 \ldots c_n .$$

Hence, we obtain

$$\psi(P) \wedge Pc_1 \ldots c_n \models \theta(Q) \rightarrow Qc_1 \ldots c_n .$$

Applying the Craig Interpolation Theorem, we can find an interpolant $\alpha c_1 \ldots c_n$, where $\alpha x_1 \ldots x_n$ is a formula of L so that

$$\psi(P) \wedge Pc_1 \ldots c_n \models \alpha c_1 \ldots c_n ,$$

$$\alpha c_1 \ldots c_n \models \theta(Q) \rightarrow Qc_1 \ldots c_n .$$

Replacing Q by P in the last line, we have

$$\psi(P) \models Pc_1 \ldots c_n \rightarrow \alpha c_1 \ldots c_n ,$$

$$\theta(P) \models \alpha c_1 \ldots c_n \rightarrow Pc_1 \ldots c_n ,$$

$$\psi(P),\ \theta(P) \models Pc_1 \ldots c_n \longleftrightarrow \alpha c_1 \ldots c_n ,$$

$$\psi(P),\ \theta(P) \models \forall x_1 \ldots x_n (Px_1 \ldots x_n \leftrightarrow \alpha x_1 \ldots x_n),$$

$$T(P) \models \forall x_1 \ldots x_n (Px_1 \ldots x_n \leftrightarrow \alpha x_1 \ldots x_n).$$

We have supposed in the proof of the Robinson Consistency Theorem that for sufficiently large cardinals there are saturated models. However, one cannot assert the existence of these objects without additional set-theoretical assumptions, such as GCH. On the other hand, when one discusses the syntactical properties of theories, that is, those which can be stated in the combinatorial universe $V\omega$, then the question naturally arises whether the proofs are correct, i.e. is it necessary to assume such a strong set-theoretical hypothesis. The scope of model theory would be more narrow if it were not be possible to eliminate the strong set-theoretical hypothesis from the proofs of syntactical properties such as the completeness, the decidability, and the elimination of quantifiers of theories are. But, fortunately, these eliminations are possible.

Let φ be a formula of the language of the set theory, i.e. of $\{\in\}$, and let φ^* be a formula obtained from φ by relativising all the quantifiers in φ to ω. This means that all occurrences of quantifiers $\forall x$, $\exists x$ in φ are replaced by $\forall x\in\omega$, $\exists x\in\omega$. A set-theoretical statement which can be expressed in the form φ^* is called an arithmetical statement. For example, if a theory T has a recursive set of axioms, then the statement ψ= "the theory T has a model" is not arithmetical, but by the completeness theorem we have $ZFC\vdash \psi \leftrightarrow Con_T$. That a first-order theory T is formally consistent is often proved by exhibiting a model of T, therefore, in such a case we have $ZFC\vdash Con_T$. But Con_T is an arithmetical statement, so the question is whether AC is a surplus, that is whether we may assert $ZF\vdash Con_T$. This and similar questions are fully resolved by the Levy-Schoenfield theorem. Here we shall give only a fragmnet of this theorem.

4.3.4. Absolutness Theorem (Levy-Schoenfield) Let φ be an arithmetical statement such that $ZF+\theta\vdash \varphi$, where θ is one of the hypothesis AC, BPI (the theorem on the existence of ultrafilters), GCH, V=L (the constructability axiom). Then, $ZF\vdash \varphi$.

As a consequence of this theorem, we have that all three previous theorems (Robinson's, Craig's and Beth's) are valid for arithmetical theories already in ZF. However, another approach is possible. Namely, instead of using arbitrary saturated structures, we may employ their refinements-recursively saturated models.

4.3.5. Definition A model **A** is recursively saturated iff for every finite subset $X\subseteq A$, every recursive set of formulas $\Sigma(x)$ of L_X which is finitely satisfiable in $\mathbf{A}_X$, is satisfiable in $\mathbf{A}_X$.

Since we have bounded ourselves on recursive sets of formulas, we are now able to prove the next theorem on the existence of recursively saturated models.

4.3.6. Theorem Every consistent theory T of a countable language L has a finite or countable recursively saturated model.

Proof Let $\mathbf{A}_0$ be a finite or countable model of T. then, there are at most countably many finite subsets $X\subseteq A_0$, and for every X there are only countably many recursive sets $\Sigma(x)$ of formulas of language L_X. Therefore, by the Compactness Theorem, there is a countable elementary extension A_1 of A_0, such that every recursive $\Sigma(x)$ which is finitely satisfiable in A_0x is satisfiable in A_1x. Repeating this construction countably many times, we obtain an elementary chain $A_0 \prec A_1 \prec A_2 \prec \dots$ in which for every n, A_{n+1} is related to A_n as A_1 is to A_0. Then, $A = \bigcup_n A_n$ is a recursively saturated model of T.

Many properties of saturated models are also true for recursively saturated models. For example, it is easy to see that every recursively saturated model A is also ω-homogeneous, i.e. every finite partial isomorphism can be extended to an automorphism of A. Theorem 4.1.6. does not hold for recursively saturated models, but there is a modification of it. By a model-theoretic pair of models A and B, we mean a model C of the language $L_1 \cup L_2$, such that A=B, $C|L_1 = A$ and $C|L_2 = B$. We then use the notation C= (A,B).

4.3.7. Theorem Let (A,B) be a countable recursively saturated model-theoretic pair, where A and B are models of a language L, such that $A \equiv B$. Then $A \approx B$.

Proof The construction of an isomorphism between models A and B is also by the back and forth argument. We can build enumarations

$A = \{a_0, a_1, \dots\}$, $B = \{b_0, b_1, \dots\}$

realizing recursive types of the forms

$\Gamma = \{\varphi a_0 \dots a_k \longleftrightarrow \varphi b_0 \dots b_{k-1} x : \varphi \in For_L\}$,

$\Sigma = \{\varphi a_0 \dots a_{k-1} x \longleftrightarrow \varphi b_0 \dots b_k : \varphi \in For_L\}$.

If k is even we choose a_k, and for k odd we take b_k. Then we shall have

$(A, a_0, a_1, \dots) \equiv (B, b_0, b_1, \dots)$,

and therefore $A \approx B$.

Here is a new proof of the Robinson Consistency Theorem, but, now, using recursively saturated models. We shall suppose that theories T_1 and T_2 are theories of countable languages L_1 and L_2. By the Löwenheim-Skolem theorem and Theorem 4.3.6, we can find the finite or countable recursively saturated pair (A,B), where $A \models T_1$ and $B \models T_2$. Reducts A_0 and B_0 to the language $L_0 = L_1 \cap L_2$ of respectively models A and B , are models of the theory $T_0 = T_1 \cap T_2$, so $A_0 \equiv B_0$ since T_0 is complete.Further,

(A_0, B_0) is also recursively saturated , so by Theorem 4.3.7, it follows that $A_0 \approx B_0$. This isomorphism gives an expansion of A_0 to a model A_1 of language L_1 which is a model of theory T_1, as well. Changing the interpretation of symbols from $L_1 \setminus L_2$ in model A, we obtain a model of $T_1 \cup T_2$.

As the second example of the use of recursively saturated models, we shall prove a preservation theorem for homomorphisms. We shall say that a theory T is preserved under homomorphic images, if for every model A of T and every homomorphism $f: A \longrightarrow B$, f(A) is a model of T. In the following we need the notion of ***positive*** sentence. A sentence φ is called positive iff it is constructed from atomic formulas using only logical symbols $\forall$, $\exists$, $\wedge$, $\vee$.

4.3.8. Theorem (Lyndon) A theory T is preserved under homomorphic images iff T is equivalent to a set of positive sentences.

Proof That the positive sentences are preserved under homomorphic images is proved by induction on the complexity of formulas. Suppose, now, that T is preserved under homomorphic images, and let T_P be the set of all the positive consequences of theory T. Let A be a model of T_P. Then, there is a model B of T such that every positive sentence which is true in B, also holds in A. Choose a recursively saturated pair (A', B') such that $A \equiv A'$ and $B \equiv B'$. By the back and forth argument, we find that A' is a homomorphic image of B', so $A' \models T$. Therefore, theory T_P is equivalent to T.

4.4. Omitting a type

This construction may be considered as a refinement of the method employed in the proof of the completeness theorem. The notion of type also plays a main role in this discussion. In this section, we shall consider countable languages only. So let L be a countable language and A a model of L. The model A omits a set of formulas $\Sigma x_1 \ldots x_n$ of L iff there is no n-tuple $(a_1, \ldots, a_n) \in A^n$ such that for all $\varphi \in \Sigma$, $A \models \varphi a_1 \ldots a_n$. In other words, the (possible infinite) conjunction $\wedge_{\theta \in \Sigma} \theta x_1 \ldots x_n$ does not hold in A for any values of variables $x_1, \ldots, x_n$ in A.

Now, let T be a consistent theory T of language L. According to the terminology introduced in section 4.1, a type of T is every set of formulas $\Sigma x_1 \ldots x_n$ consistent with T. We say that the theory T ***realizes*** $\Sigma x_1 \ldots x_n$ ***locally***, if there is a formula $\varphi x_1 \ldots x_n$ of L such that for all $\sigma \in T$, $T \vdash \forall x_1 \ldots x_n(\varphi \rightarrow \sigma)$, and φ is consistent with T (i.e. there is a model A of T and $a_1, \ldots, a_n \in A^n$ such that $A \models \varphi[a_1, \ldots, a_n]$). Therefore, T realizes $\Sigma x_1 \ldots x_n$ by the formula $\varphi x_1 \ldots x_n$, if every model of T, every n-tuple $(a_1, \ldots, a_n)$, such that $A \models \varphi[a_1, \ldots, a_n]$, this n-tuple realizes $\Sigma x_1 \ldots x_n$.

A type $\Sigma x_1 \ldots x_n$ of T is ***locally omitted*** by T iff T does not locally realize $\Sigma x_1 \ldots x_n$. Therefore, T locally omits $\Sigma x_1 \ldots x_n$ iff for every formula φ of L consistent with T, there is $\sigma \in \Sigma$ such that $\varphi \wedge \neg\sigma$ is consistent with T.

It is easy to find some algebraic and topological equivalents to the notions "locally realized" and "locally omitted" in the Lindenbaum algebra and the Stone space of theory T. First, let us introduce a refinement of a Lindenbaum algebra of a theory. Let Φ_n be the set of all the formulas of L with only free variables $x_1, \ldots, x_n$ and consider the relation of equivalence $\sim$ over Φ_n defined by

$$\varphi \sim \psi \text{ iff } T \vdash \varphi \leftrightarrow \psi, \quad \varphi, \psi \in \Phi_n.$$

Let $B_n(T)$ be the set of all the equivalence classes of this equivalence relation, i.e. $B_n(T) = \{[\varphi]: \varphi \in \Phi_n\}$, where $[\varphi]$ denotes the class of equivalence of a formula $\varphi x_1 \ldots x_n$ of L. Then, as in the case of Lindenbaum algebra of theories, see Example 2.3.3, in natural way we can supply this set with Boolean operations $\cdot, +, ', 0, 1$ so that

$$\mathbf{B}_n = (B_n(T), +, \cdot, ', 0, 1)$$

becomes a Boolean algebra. Let us denote by $S_n(T)$ the Stone space of this Boolean algebra. Then the points of this space correspond to the maximal sets of formulas in variables $x_1, \ldots x_n$ consistent with T, that is, the so-called maximal n-types of T.

If $\Sigma \subseteq \Phi_n x_1 \ldots x_n$, let $\underline{\Sigma} = \{[\varphi]: \varphi \in \Sigma\}$, and $\Sigma^* = \bigcap_{\sigma \in \Sigma}[\sigma]^*$, i.e.

$\Sigma^* = \{p \in S_n(T): \underline{\Sigma} \subseteq p\}$.

Now, we can state the mention algebraic and topological equivalents. First observe that every maximal type p in variables $x_1, \ldots, x_n$ determines a unique ultrafilter $\underline{p} = \{[\varphi]: \varphi \in p\}$ in $B_n(T)$ i.e. a point in $S_n(T)$. Therefore, we can speak of principal or nonprincipal types, depending on whether p is a principal or nonprincipal ultrafilter of $B_n(T)$. These correspond in $S_n(T)$ to the isolated and nonisolated points, respectively.

4.4.1. Proposition Let L be a first-order language and let $\Sigma \subseteq \Phi_n$ be a type of a theory of L. then,
1° Σ is locally realized in T iff $\underline{\Sigma}$ is contained in a principal filter of $B_n(T)$.
2° Σ is locally omitted by T iff Σ^* is a nowhere dense set in $B_n{}^*(T)$.
3° If $\Sigma_{n1}, ,\Sigma_{n2}, \ldots$ are sets of formulas of T, in variables $x_1,\ldots,x_{ni}$ respectively, which are locally omitted by T, then $\bigcup_i \Sigma^*{}_{ni}$ is a set of the first category in $B_n{}^*(T)$.

Proof 1° Σ is locally realized by T iff there is a formula $\varphi x_1\ldots x_n$ such that $[\varphi]>0$ and for all $\sigma\in\Sigma$ $[\sigma]\geq[\varphi]$, i.e. if and only if Σ is contained in filter F of $B_n(T)$ generated by $[\varphi]$.
2° By 1°, Σ is locally realized iff there is $[\varphi]\neq\emptyset$ such that $[\varphi]^*\subseteq\Sigma^*$, i.e. there is a basic open set contained in Σ^*. Thus, Σ is locally realized iff $\mathrm{Int}\Sigma^*\neq\emptyset$. As $\Sigma^*=\bigcap_{\sigma\in\Sigma}[\sigma]^*$, Σ^* is closed, so the statement follows.
3° This is an immediate consequence of 1°.

Now, we shall discuss the connection between locally omitted and omitted. First, we shall see that we may concentrate on rich theories. Namely, by Lemma 2.4.4 (the Witness Lemma), every theory T has a conservative rich extension S. In this case, we have:

4.4.2. Lemma Let T be a theory of L and S it's conservative rich extension to $L\cup C$, where C is the set of witnesses for S. If T locally omits $\Sigma x_1\ldots x_n$, then S locally omits $\Sigma x_1\ldots x_n$, as well.

Proof Assume, on the contrary, that locally S realizes $\Sigma x_1\ldots x_n$, i.e. that there is a formula $\varphi x_1\ldots x_n c_1\ldots c_m$, where $c_1,\ldots,c_m\in C$, consistent with S such that

$$S\vdash \forall\underline{x}(\varphi\underline{x}c_1\ldots c_m \rightarrow \sigma\underline{x}) \quad \text{for all } \sigma\in\Sigma x_1\ldots x_n.$$

So,

$$T\vdash \exists y_1\ldots y_m\forall x_1\ldots x_n(\varphi x_1\ldots x_n y_1\ldots y_m \rightarrow \sigma x_1\ldots x_n),$$
$$T\vdash \forall x_1\ldots x_n\exists y_1\ldots y_m(\varphi x_1\ldots x_n y_1\ldots y_m \rightarrow \sigma x_1\ldots x_n),$$

$$T \vdash \forall x_1 \ldots x_n (\forall y_1 \ldots y_m \varphi x_1 \ldots x_n y_1 \ldots y_m \to \sigma x_1 \ldots x_n),$$

thus, locally T realizes $\Sigma x_1 \ldots x_n$ by $\forall y_1 \ldots y_m \varphi x_1 \ldots x_n y_1 \ldots y_m$.

If T is a complete theory and A is a model of T which omits a type $\Sigma x_1 \ldots x_n$, then there is no formula φ such that T realizes Σ locally by φ, since otherwise we would have $T \vdash \exists x \varphi$, and so A would realize Σ. Thus, T omits Σ locally in this case. The omitting types theorem is a converse of this fact.

4.4.3 Omitting Types Theorem (Ehrenfeucht, Ryll-Nardzewski). Let T be a consistent theory in a countable language L, and let $\Sigma x_1 \ldots x_n \subseteq \Phi_n$ be a set of formulas of L. If T omits Σ locally, then T has a countable model which omits Σ.

Proof By Lemma 4.4.2, we can suppose that T is a rich theory with a countable set of witnesses $C = \{c_0, c_1, \ldots\}$. Also, we shall simplify the notation by assuming $\Sigma \subseteq \Phi_1$, i.e. that $\Sigma = \Sigma(x)$. Suppose T omits $\Sigma(x)$ locally. The set of formulas of LUC is countable, so, let $\varphi_0, \varphi_1, \ldots$ be the sequence of all the sentences of LUC. We shall construct a sequence

$$T = T_0 \subseteq T_1 \subseteq T_2 \subseteq \ldots$$

of theories such that

1° Each T_m is a consistent, finite extension of T in LUC.

2° Either $\varphi_m \in T_{m+1}$ or $\neg\varphi_m \in T_{m+1}$.

3° There is a formula $\sigma(x) \in \Sigma(x)$, such that $\neg\sigma(c_m) \in T_{m+1}$.

We can construct T_{m+1}, assuming T_m is given. Suppose

$$T_m = T \cup \{\psi_1, \ldots, \psi_n\},$$

and let $\psi = \psi_1 \wedge \psi_2 \wedge \ldots \wedge \psi_n$. Further, suppose $c_0, \ldots, c_k$ are all the constants from C occuring in ψ. Let $\psi x_0 \ldots x_k$ be a formula of L, obtained from $\psi c_0 \ldots c_k$ by replacing c_i by x_i, $0 \le i \le k$, where x_i are variables which do not occur in $\psi c_0 \ldots c_k$. Then $\theta = \psi x_0 \ldots x_k$ is consistent with T, so, by assumption, there is $\sigma \in \Sigma$ such that $\theta \wedge \neg\sigma$ is consistent with T. Then, we can put $\neg\sigma c_m$ in T_{m+1}, and this assures 3°. To provide 2°, we can put φ_m in T_{m+1}, if φ_m is consistent with $T_m \cup \{\neg\sigma c_m\}$, otherwise, we can put $\neg\varphi_m$ in T_{m+1}. Hence, T_{m+1} is a consistent finite extension of T_m, and so also of T in LUC.

Now, define $T' = \cup_n T_n$. By 1° T' is a consistent theory, and, by 2°

complete as well. Let A be a canonical model of T. Then, the universe of A consists of interpretations of constant symbols from C, and by 3°, A omits $\Sigma(x)$. Since A is a model of T, the theorem follows.

For complete types, we have the following consequence.

4.4.4 Corollary The complete n-type p is omitted by T iff p is a non-principal ultrafilter of $B_n(T)$.

There is a simple generalization of the Omitting types theorem which says when a model omits countably many types. We shall refer to this theorem as to the Omitting types theorem as well.

Omitting types theorem for countably many types Let T be a consistent theory of a countable language L, and assume that $\Sigma_n(x_1, \ldots x_{k_n})$ is a set of formulas in k_n variables. If T omits locally each Σ_n, then T has a countable model which omits each Σ_n.

Now, we shall turn to some applications of the Omitting type theorem. The first application will be in the study of the so-called elementary end-extensions of models. Here we shall present a rather general case which covers not only specific instances of the first-order logic, but will go beyond it, as we shall see in logics with additional quantifiers. So, let us introduce first some notation.

Let R be a binary relation symbol of a countable language L and A a countable model for L. Let us define $R_a{}^A = \{x \in A: A \models Rxa\}$ for every element $a \in A$. We shall write R_a instead of $R_a{}^A$, if there is no ambiguity. A model B is an elementary R-end-extension of A if and only if the following hold

1° B is a proper elementary extension of A i.e. $A \prec B$ and $A \neq B$.

2° For every $a \in A$, $R_a{}^A = R_a{}^B$.

We write $A \prec_{eR} B$ iff B is an elementary R-end-extension of A. Our aim is to give sufficient conditions for R under which A has an elementary R-end-extension.

4.4.5. Definition A binary relation R is regular in a model A of a language L iff the following hold in A:

1° For every $a \in A$, $R_a \neq A$.

2° For all $a,b\in A$, there is $c\in A$ such that $a,b\in R_c$.

In the formulation of the third condition, we shall use the following notation: For a formula φxy of L with two free variables x,y and $a\in A$, let $\varphi_a{}^A = \{b\in A: A\models \varphi[a,b]\}$.

3° Let $a\in A$. If for every $c\in R_a$ there is $b\in A$ such that $\varphi_c{}^A \subseteq R_b$, then there is $d\in A$ such that $\bigcup_{c\in R_a}\varphi_c{}^A \subseteq R_d$.

The above conditions are the first order properties of R, since R is regular in **A** iff the following sentences hold in **A**:

S1. $\forall x\exists y\neg Ryx$.

S2. $\forall x\forall y\exists z(Rxz \wedge Ryz)$.

In the formulation of the third condition, we shall use the following abbreviations: $(\forall x\in_R y)\varphi$ and $(\exists x\in_R y)\varphi$ will stand for $\forall x(Rxy \rightarrow \varphi)$ and $\exists x(Rxy \wedge \varphi)$, respectively, where φ is a formula of L.

S3. $\forall v[(\forall x\in_R v)\exists y\forall u(\varphi xu \rightarrow Ruy)) \rightarrow \exists y(\forall x\in_R v)\forall u(\varphi xu \rightarrow Ruy)]$

So, sentence S3 states that the bounded quantifier $(\forall x\in_R v)$ and the existential quantifier $\exists y$ commute in an appropriate way. Here are some examples of regular relations:

4.4.6. Example If k is a regular cardinal and $<$ is the natural ordering on k, then $<$ is regular in $A=(k,<,\dots)$. Observe that if $B=(B,<^B,\dots)$ is an elementary submodel of **A**, then $<^B$ is regular in **B**.

4.4.7. Example If $M=(M,<,\dots)$ is a model of Peano arithmetic, then $<$ is a regular relation in **M**. This is true since sentence S3 can be restated as $\forall v\psi v$, where ψv is

$$(\forall x<v)\exists y\forall u(\varphi xu \rightarrow u<y) \rightarrow \exists y(\forall x<v)\forall u(\varphi xu \rightarrow u<y).$$

It is easy to prove by induction that $\forall v\psi v$ is a theorem of Peano arithmetic.

There are other regular relations in models of Peano arithmetic:

1° The divisibility relation: $x|y$ iff $\exists z\ y=zx$.

2° The relation θ defined by: $x\theta y$ iff "x occurs in the binary expansion of y", i.e. if $y=\Sigma_{i\le n}2^{x_i}$, $x_1<\dots<x_n$, then for some $i\le n$, $x=x_i$.

4.4.8. Example In any model $M=(M,R)$ of the ZF-set theory, where R is an interpretation of the membership relation $\in$, R is a regular relation. In fact, in any model **M** of the Axiom of pairs, Axiom of regularity (or

$\forall x\ x \notin x$), Axiom of union and Collection scheme, R is a regular relation. for example, if **M** is a model of Peano arithmetic and θ is the relation defined in the previous example, then (M,θ) is a model of ZF-∞+$\neg\infty$, where ∞ denotes the axiom of infinity; and, thus, θ is regular in $(\mathbf{M},\theta)$.

4.4.9. Example If r is the rank-function in a model **M** of ZF, then $R^M xy$, defined by $rx < ry$, is a regular relation.

4.4.10. Example If k is a strongly inaccessible cardinal, then, the relation of inclusion $\subseteq$ is regular in $(S_k(k),\subseteq,\ldots)$, where

$S_k(X) = \{Y \subseteq X: |Y| < k\}$.

Since the regularity of a binary relation is a first order property, we have: If **A** $\equiv$ **B** and R is regular in **A**, then R is regular in **B**, too. Now we are going to describe the main property of regular relations in respect to end-extensions. In the following, we shall assume that R is regular in a model **A**.

4.4.11. Lemma 1° $\bigcup_{a \in A} R_a = A$.

2° For every $a_1,\ldots,a_n \in A$, $n \in \omega$, there is $b \in B$, such that

$R_{a_1} \cup \ldots \cup R_{a_n} \subseteq R_b$

and, hence, $R_{a_1} \cup \ldots \cup R_{a_n} \neq A$.

Proof 1° This follows since R satisfies S2.

2° It is sufficient to prove that for any $a,b \in A$ there is $c \in A$, such that $R_a \cup R_b \subseteq R_c$, since we can obtain the general case easily by induction. Let $a,b \in A$. By S2, there is $d \in A$ such that $a,b \in R_d$. Let φxy be Ryx. As for every $x \in R_c$, for $\varphi_x{}^A = R_x$, by S3 there is $c \in A$, such that $\bigcup_{x \in R_d} \subseteq R_c$. Further, $a,b \in R_d$, so $R_a \cup R_b \subseteq R_c$.

Assume A has an elementary R-end extension. Then, by Lemma 4.4.11 and the compactness theorem the following theory is consistent:

$T = Th(A,a)_{a \in A} \cup \{\neg Rca: a \in A\}$

where c is a new constant symbol.

4.4.12. Lemma Let ψx be a formula with free variable x of the language $L_A = L \cup \{a: a \in A\}$, and c is a new constant symbol. Then ψc is inconsistent with T iff for some $a \in A$, $\{d \in A: A \models \psi[d]\} \subseteq R_a$, i.e. $A \models \forall x(\psi x \rightarrow Rxa)$.

Proof (→) Assume ψc is inconsistent with T. Then $T \models \neg\psi c$, hence, for some $a_1, \ldots a_n \in A$, $Th(A,a)_{a \in A} \models \neg Rca_1 \wedge \ldots \wedge \neg Rca_n \rightarrow \neg\psi c$. Therefore, $A \models \forall y(\psi y \rightarrow Rya_1 \vee \ldots \vee Rya_n)$. By Lemma 4.4.11, there is $b \in A$, such that $R_{a_1} \cup \ldots \cup R_{a_n} \subseteq R_b$. Then $\{d \in A: A \models \psi[d]\} \subseteq R_b$.
(←) Assume $\psi(c)$ is consistent with T, and for some $a \in A$,

$$A \models \forall x(\psi x \rightarrow Rxa).$$

Let B be a model of $T \cup \{\psi c\}$, and $c^B = c_0$. Then, $B \models \psi[c_0]$. Since $A \prec B$ (i.e. A is an elementary submodel of B; we can identify a^B with a for every $a \in A$), it follows that $B \models \forall y(\psi y \rightarrow Rya)$. Hence, $B \models Rca$, and this is a contradiction to $B \models \neg Rca$ (we recall that $\neg Rca \in T$).

Remark that $A \prec_{eR} B$ if and only if $A \prec B$, and there are no $a \in A$ and $b \in B$, such that $b \in R_a{}^B$, and $b \neq a'$ for all $a' \in R_a{}^A$. Hence $A \prec_{eR} B$ iff $A \prec B$ and B omits $\Sigma_a = \{Rxa\} \cup \{x \neq a' : a' \in R_a\}$ for every $a \in A$.

4.4.13. Lemma If $T = Th(A,a)_{a \in A} \cup \{\neg Rca: a \in A\}$, then T omits Σ_a locally for every $a \in A$.

Proof Let φxc be consistent with T. Assume there is no $\sigma \in \Sigma_a$ such that $\exists x(\varphi xc \wedge \neg\sigma)$ is consistent with T. Hence,
(1) $\exists x(\varphi xc \wedge \neg Rxa)$ is inconsistent with T, so by Lemma 4.4.13, for an element b_1,

$$\{d \in A: A \models \exists x(\varphi xd \wedge \neg Rxa)\} \subseteq R_{b_1}.$$

(2) Let $e \in R_a$. then, $\exists x(\varphi xc \wedge x=e)$ is inconsistent with T, i.e. φec is inconsistent with T, so by Lemma 4.4.12, there is $b \in A$, such that $\{d \in A: A \models \varphi ed\} \subseteq R_b$. Therefore, we have proved that for every $e \in R_a$, there is $b \in A$, such that $\varphi_e{}^A \subseteq R_b$, hence, by the regularity of R, there is $b_2 \in A$, such that $\bigcup_{e \in R_a} \{d \in A: A \models \varphi ed\} \subseteq R_{b_2}$.

Let $b \in A$ be such that $R_{b_1} \cup R_{b_2} \subseteq R_b$. Then:

$$\{d \in A: A \models \exists x(\varphi xd \wedge \neg Rxa)\} \cup \{d \in A: A \models \exists x(\varphi xd \wedge Rxa)\} \subseteq R_b,$$

i.e. $\{d \in A: A \models \exists x\varphi xd\} \subseteq R_b$. Thus, by Lemma 4.4.12, the sentence $\exists x\varphi xc$ is inconsistent with T, and this is a contradiction.

Now, we shall state the main theorem of this section.

4.4.14. Theorem Let A be a countable model of a countable language L and R a regular relation in A. Then A has an elementary countable R-end extension.

Proof By Lemma 4.4.13, the theory $T=\mathrm{Th}(A,a)_{a\in A} \cup \{\neg Rca: a\in A\}$ omits $\Sigma_a x = \{Rxa\} \cup \{x\neq b: b\in R_a\}$ locally for every $a\in A$. Thus by the Omitting types theorem, there is a model B of T which omits each Σ_a. Then, model B is an elementary R-end extension, if we identify a^B with a; in fact, the mapping $a \longrightarrow a^B$ is an elementary embedding of A into B.

We infer the following corollary as an immediate consequence of the above theorem. It is obtained by the repeated use of the theorem.

4.4.15. Corollary If R is a regular relation in a countable model A, then there is a model B such that $A \prec_{eR} B$ and $|B| = \omega_1$.

Proof In view of Theorem 4.4.14, we can construct an elementary chain of countable models

$$A = A_0 \prec A_1 \prec \dots \prec A_\alpha \prec \dots,\ \alpha<\omega_1,$$

so that $A_\alpha \prec_{eR} A_{\alpha+1}$, $|A_\alpha| = \omega$. If μ is a limit ordinal, then we take $A_\mu = \bigcup_{\alpha<\mu} A_\alpha$, and it is easy to check that for $\alpha<\mu$, $A_\alpha \prec_{eR} A_\mu$. Then, the required model B is $\bigcup_{\alpha<\omega_1} A_\alpha$.

Now, we are going to consider some examples which illustrate Theorem 4.4.14.

4.4.16. Example 1° Every countable model N of Peano arithmetic has an elementary end extension M in respect to standard ordering. By Corollary 4.4.15, it follows that M may be chosen to be uncountable. If M is constructed as in this corollary, then M has an additional property:

Every initial segment is countable.

Every linear ordering with this property is called an ω_1-like ordering; an archetype of such an ordering is ω_1 itself. In view of the above discussion, every countable model has an uncountable ω_1-elementary end-extension. We shall see later that there are $2^{\aleph_1}$ such extensions for every countable model of PA.

The statement that every countable model of PA has an elementary end-extension is part of MacDowel-Specker theorem, which states that *every* model of Peano arithmetic has an elementary <-end extension of the same cardinality.

2° (Keisler, Morley) Every countable model of ZF set theory, and it's variants as well, has an elementary $\in$-end extension.

Now, we shall see that the assumption of property S3 (the regularity property) is crucial for a linear ordered model to have an elementary end-extension. Namely, we have the following assertion.

4.4.17. Proposition The countable linearly ordered model $\mathbf{A}= (A,\leq^A,\ldots)$ has an elementary $\leq$-end extension if and only if A satisfies conditions S1 and S3.

Proof The if-part follows from Theorem 4.4.14. So let $\mathbf{B}= (B,\leq^B,\ldots)$ be an elementary $\leq$-end extension of A. Then, for every $a\in A$, $a<^B b$ for $b\in B\setminus A$. Therefore, $\mathbf{B}\models \exists y(x<y)[a]$, hence $\mathbf{A}\models \forall x\exists y(x<y)$, i.e. model A satisfies S1. Property S3 is checked similarly.

We are going to give an additional application of Theorem 4.4.14. We shall prove a theorem from series of the so-called two-cardinal theorems. These theorems are related to theories T of language L which contain a unary predicate symbol P. Let us denote a model A of such a theory by $\mathbf{A}= (A,V,\ldots)$, where $V= P^A$. We see that V may be considered a "second" domain or subdomain of A. The cardinal type of A is the pair $(|A|,|V|)$. For example, if the set of natural numbers N is distinguished in the field of real numbers, say $\mathbf{R}= (R,N,+,\cdot,0,1)$, then the cardinal type of R is $(2^{\aleph_0},\aleph_0)$. The main question which is associated with such theories is what are the cardinal types of models of theory T? For example, it is easy to see (by the Löwenheim-Skolem theorem) that if a theory has a model of the cardinal type (k,θ), $\theta<k$, then T admits a model of the cardinal type (k',θ) for any $\theta\leq k'\leq k$. The following theorem is called Keisler's two cardinal theorem.

4.4.18. Theorem (Keisler) Let $\mathbf{A}= (A,V,\ldots)$ be a model of a countable language L such that $\omega\leq|V|<|A|$. Then, there are models $\mathbf{B}= (B,W,\ldots)$ and $\mathbf{C}=(C,W,\ldots)$, such that $\mathbf{B} \prec \mathbf{A}$, $|B|= \aleph_0$, $\mathbf{B} \prec \mathbf{C}$ and $|C|= \aleph_1$.

Proof By the downward Löwenhim-Skolem theorem, we can assume that $|A|= |V|^+$. Let us consider the expansion $(\mathbf{A},<)$, where $<$ is the ordering of the cardinal $|A|$ and let $(\mathbf{B},<^B) \prec (\mathbf{A},<)$, where $|B|= \aleph_0$. We have remarked already that $<^B$ is still regular. Since $V \subseteq A$, $|V|<|A|$ and $|A|$ is a regular cardinal, there is $a\in A$, such that $V \subseteq \{x\in A: x<a\}$. Hence, there is $b\in B$, such that for $W= V^B$, $W \subseteq \{x\in B: x<^B b\}$. Since $<^B$ is a regular relation, by Corollary 4.4.15, it follows that there is a model $(\mathbf{C},<^C)$

which is an elementary $<$-end extension of $(B,<^B)$, such that $|C|=\aleph_1$. Since $W \subseteq \{x\in B: x<^B b\}$, it follows that $V^C \subseteq \{x\in B\ x<^B b\}$ and, hence, $V^C=W$.

4.4.19. Corollary Assume a theory T has a model of the type (k,θ), $\theta<k$. Then, T has a model of the type $(\aleph_1,\aleph_0)$.

Exercises

4.1. Let $F=(F,+,\cdot,\leq,0,1)$ be an ordered field. Show that F is a saturated model iff $(F,\leq)$ is an η_α-set.

4.2. Let $M=(M,+,\cdot,',\leq,0)$ be a nonstandard model of Peano arithmetic. Prove that $(M,+,',\leq,0)$ is recursively saturated.

4.3. Assume $M=(M,+,',\leq,0)$ is a nonstandard, recursively saturated model of $Th(\omega,+,',\leq,0)$. Show that M has an expansion to a model of Peano arithmetic.

4.4. Show that every countable atomless Boolean algebra is a saturated model.

4.5. Suppose T is a complete theory of a countable language L. Prove:
1° If T has no countable saturated model, then T has $2^{\aleph_0}$ nonisomorphic countable models.
2° If T has a countable elementary universal model, then T has a countable saturated model.

4.6. Let B be an atomless Boolean algebra and k an infinite cardinal number. Show that the following are equivalent:
1° B is a k-saturated model.
2° B satisfies the following property:
H_k For all nonempty subsets $X,Y\subseteq B$ such that
X is directed upwards, i.e. $x,y\in X$ implies $x+y\in X$,
Y is directed downwards, i.e. $x,y\in Y$ implies $x\cdot y\in Y$,
$X<Y$ i.e. for all $x\in X$, $y\in Y$, we have $x<y$,
there is an element $a\in B$ such that for all $x\in X$, $y\in Y$, $x<a<y$.
3° B is k-injective in the class of all Boolean algebras i.e. for any Boolean algebras B and C such that $|B|<k$, $|C|\leq k$, and any embeddings

$f:B \longrightarrow A$, $g:B \longrightarrow C$ there is an embedding $h:C \longrightarrow A$ such that $f=hg$.

4.7. Assume a filter F over ω has the following, so called Cantor separation property: Foe every decreasing chain $\ldots <a_2<a_1<a_0$ in Boolean algebra $P(\omega)/F$ there is $b\in P(\omega)/F$ such that $0<b<\ldots<a_2<a_1<a_0$. If $P(\omega)/F$ is an atomless Boolean algebra, show that $P(\omega)/F$ is a saturated model that $P(\omega)$ is a saturated

4.8. If Fin is a filter of cofinite subsets of ω then the reduced product $\Pi_i A_i/F$ is ω_1-saturated.

4.9. If $F_0=\{a\subseteq\omega: \Sigma_{n\in a}1/(n+1)<\infty\}$, show that F_0 is a filter over ω, and $P(\omega)/F$ is a saturated model.

4.10. If F is one of the filters from the previous two problems, show, assuming CH, that for any family of Boolean algebras $\langle B_i : i\in\omega\rangle$ such that for all $i\in\omega$ $|B_i|\le\aleph_1$, $\Pi B_i/F \approx P(\omega)/F$.

4.11. Let I be an ideal of a Boolean algebra $P(\omega)$. We say that I has the property (M) if for any sequence $\langle a_i : i\in\omega\rangle$ such that $a_i\in P(\omega)\setminus I$ for all i there is there is a sequence $\langle b_i : i\in\omega\rangle$ such that $b_i\le a_i$ for all i and $\bigcup\{b_i : i\in\omega\}\in P(\omega)\setminus I$. Further, assume $P(\omega)/I$ is an atomless Boolean algebra. Show that $P(\omega)/I$ is an ω_1-saturated model iff I has property (M).

4.12. A topological space X is called a Parovicenko space iff it satisfies:

1° X is a Stone space of weight $2^{\aleph_0}$ without isolated points.

2° Every two disjoint open F_σ sets have disjoint closures.

3° Every nonempty $G\delta$ set in X has a nonempty interior.

Show:

(1) A topological space X is a Parovicenko space iff it is a Stone space of an $\aleph_1$-saturated atomless Boolean algebra of cardinality $2^{\aleph_0}$.

(2) If CH is assumed, then every two Parovicenko spaces are homeomorphic. Particularly, every Parovicenko space is homeomorphic to the space $P(\omega)/\mathrm{Fin}$.

4.13. A P-point of a topological space X is an element $a\in X$ with the property: If S is any countable family of neighborhoods of a then $\bigcap S$ is a neighborhood of a, too. Let ω^* be the Stone space of $P(\omega)/\mathrm{Fin}$. Show

1° If p is an ultrafilter over ω, then p is a P-point of ω^* iff $(P(\omega),p)/Fin$ is an $\aleph_1$-saturated model (p is considered as an unary relation in $(P(\omega),p)$).
2° Let p and q be P-points of ω^*. Assuming CH, show that there is an autohomeomorphism α of ω^* such that $\alpha(p)=q$.

4.14. If $\mathbf{M}=(M,\in^M)$ is a countable model of ZF set theory, then $\mathbf{M}$ has a proper elementary end extension, i.e. there is a model $\mathbf{K}=(K,\in^K)$ such that $\mathbf{M} \prec \mathbf{K}$, $\mathbf{M}\neq\mathbf{K}$, and for all $a\in M$, $b\in K$m $b\in^M a$ implies $b\in M$.

5. ABSTRACT MODEL THEORY

The Compactness theorem and the Löwenheim-Skolem theorem are fundamental properties of the first-order predicate calculus. In fact, these two properties characterize PR^1 completely, as the Lindström theorem shows. To state the theorem itself, we need some general notions of a logic.

The Lindström theorem is part of the so-called soft or abstract model theory. The most important task of the abstract model theory is, perhaps the discovering and investigation of useful extensions of the first-order predicate calculus. A large amount of effort has been spent on the study of logics which enhance PR^1 in some way. It appears that some of the logics have proved profitable, while others have been abandoned for obscurity. We shall see that a logic can be understood as an operation which assigns to each set L of symbols a syntax and semantics, so that elementary syntactical operations are performable, on one side, and isomorphic structures satisfy the same senetences, on the other side.

5.1. Abstract logics

By an abstract logic, we shall consider a class pair $(L^*, \models^*)$ where L^* is called a class of sentences, and $\models^*$ a satisfaction relation. We shall adopt the following axioms for abstract logics.

Occurrence axiom For each formula φ of L^*, there is a finite language $L(\varphi)$, which is called the set of symbols that occur in φ. Further, $A \models^* \varphi$ is a relation between sentences φ and models A of a language L which contains $L(\varphi)$.

Expansion axiom If $A \models^* \varphi$ and B is an expansion of model A to a larger language, then $B \models^* \varphi$.

Isomorphism axiom If $A \approx B$ and $A \models^* \varphi$, then $B \models^* \varphi$.

Closure axiom Class L^* contains all the atomic sentences, and it is closed under logical connectives $\wedge$, $\vee$, $\neg$. Also, the usual rules for the satisfaction class hold in the case of atomic formulas and logical connectives.

Quantifier axiom For every constant symbol $c \in L(\varphi)$, there are sentences $\forall v_c \varphi$ and $\exists v_c \varphi$ with the set of symbols $L(\varphi) \setminus \{c\}$, such that

$A \models^* \forall v_c \varphi$ iff for all $a \in A$, $(A,a) \models^* \varphi$,

$A \models^* \exists v_c \varphi$ iff there is $a \in A$, $(A,a) \models^* \varphi$.

Relativisation axiom For every sentence φ of language L^* and a relation $Rvb_1 \ldots b_k$, such that $R,b_1,\ldots,b_k \notin L(\varphi)$, there is a new sentence φ^R, read φ relativised $Rvb_1 \ldots b_k$, such that whenever there is a submodel B of A with the domain $B = \{a: A \models Rab_1 \ldots b_k\}$, then

$(A,R,b_1,\ldots,b_k) \models^* \varphi^R$ iff $B \models^* \varphi$.

We shall list several important examples of abstract logics. Each example is an extension of the first order predicate calculus, and it is usually obtained by adding one or more formation rules for syntactically well-defined formulas of the logic. Then, L^* will be the class of sentences formed under these rules, but having only finitely many relation symbols. All these logics can be divided into two large groups. In the first, the ranges of quantifiers are changed, or new quantifiers are added. In the second group, the formation rules allow the construction of formulas of the infinite size. The most general cases arises when both kinds of formation rules are allowed, i.e. adding new quantifiers and building formulas of infinite size.

5.1.1. Example The logic PR^1 or $L_{\omega\omega}$, the usual first-order predicate calculus. The syntax and semantics have been explained in detail in the previous sections.

5.1.2. Example The logic $L\omega_1\omega$ - logic with countable conjunctions and disjunctions.

We shall add a new formation rule to PR^1: If S is a countable set of formulas, then $\wedge S$ and $\vee S$ are formulas. Logical sings of PR^1 are interpreted in usual way, with additions:

If A is a structure of a language L, and S a countable set of formulas of L, then

$A \models^* \wedge S$ means that $A \models^* \varphi$, for all $\varphi \in S$,

$A \models^* \vee S$ means that $A \models^* \varphi$, for some $\varphi \in S$.

This logic has a greater power than $L\omega\omega$. For example, the class of all cyclic groups is axiomatizable in this logic; we shall see that a group G is cyclic if and only if G satisfies φ, where

$\varphi = \exists x \forall y (y=1 \vee y=x^2 \vee y=x^{-2} \vee y=x^3 \vee y=x^{-3} \vee \ldots)$, i.e.

$\varphi = \exists x \forall y \vee \{y=x^n : n \in Z\}$.

It is easy to see that the class of finite models of a given first-order theory is also axiomatizable in $L\omega_1\omega$. This example show that the logic $L\omega_1\omega$, as well as its extensions, does not satisfy the Compactness theorem.

5.1.3. Example The logic $L_{\infty}\omega$ - logic with infinite conjunctions and disjunctions.

Its syntax is similar to the syntax of $L\omega_1\omega$, except that it does not require S to be countable, while the semantics is defined exactly like that of $L\omega_1\omega$.

There is an obvious refinement of the previous two examples, i.e. it is possible to define logic $L_k\omega$, where k is an infinite cardinal number. The new formation rule is then: if $\mu < k$ is a cardinal number, and S a set of formulas of the cardinality μ, then $\wedge S$ and $\vee S$ are formulas of $L_k\omega$. It can be shown that the power of logic $L_k\omega$ increases as k grows.

5.1.4. Example The logic $L\omega_1\omega_1$ - logic with countable conjunctions, disjunctions and countable blocks of quantifiers.

The syntax rules of this logic are those of $L_{\omega_1\omega}$ with the addition of the following formation rule: If φ is a formula and X a countable set of variables, then $\forall X\varphi$ and $\exists X\varphi$ are formulas of $L_{\omega_1\omega_1}$ as well. Therefore, this logic admits countable conjunctions and disjunctions and the "gluing" of countable sequences of quantified variables to formulas. For example, in the ordered field of reals, we have

$$R \models^* \forall xy(x<y \rightarrow \exists x_1 x_2 x_3 \cdots (x<x_1 \wedge x_1<y \wedge x_1<x_2 \wedge x_2<y \wedge \cdots)).$$

The next sentence of $L_{\omega_1\omega_1}$ expresses "there are at most denumerably many elements":

$$\sigma = \exists v_0 v_1 v_2 \cdots \forall x \bigvee_{n\in\omega} x=v_n .$$

The possible refinement of previous examples are $L_{k\mu}$ logics, where k and μ are cardinal numbers. In addition to the formation rules of $L_{\infty k}$, there is the syntax rule according to which, as in the case of $L_{\omega_1\omega_1}$, we have the formation of formulas with sequences of quantified variables of the length $\tau<\mu$. Thus, if φ is a formula of $L_{k\mu}$ and X a set of variables such that $|X|<\mu$, then $\forall X\varphi$ and $\exists X\varphi$ are formulas of $L_{k\mu}$, too.

5.1.5. Example $L(Q_0)$ - logic with an additional quantifier "there exist infinitely many".

To the syntax of $L_{\omega\omega}$ a new symbol Q_0 is added, by means of the formation rule: If φ is a formula, then $Q_0 x\varphi$ is a formula too. The interpretation of the new quantifier is as follows:

$A \models^* Q_0 x\varphi x$ iff the set $\{a\in A: A \models^* \varphi[a]\}$ is infinite.

This logic is also more expressive than $L_{\omega\omega}$. For example, a simple sentence of $L(Q_0)$ axiomatizes the class of all the finite models of language L: $\neg Q_0 x(x=x)$.

5.1.6. Example $L(Q_k)$ - logic with an additional quantifier "there exists $\aleph_k$ many elements".

The syntax of this logic is similar to that of $L(Q_0)$, and the interpretation of Q_k is:

$A \models^* Q_k x\varphi x$ iff $|\{a\in A: A \models^* \varphi[a]\}| \geq \aleph_k$.

The most interesting case is the logic $L(Q_1)$, i.e. PR^1 with additional quantifier $Q_1 x$= "there exist uncountably many x". Later on we shall discuss this logic in greater detail.

5.1.7. **Example** L^{II} - the second order logic.

Besides individual variables $v_0, v_1, \ldots$ of $L\omega\omega$, variables $X_0, X_1, \ldots$ are added and a new logical symbol $\in$. The intended range of new variables consists of sets. If t is a term of $L\omega\omega$, new atomic formulas $t \in X_i$ are allowed. The following formation rule is added to those of $L\omega\omega$:

If φ is a formula, so are $\forall X_i\, \varphi$ and $\exists X_i\, \varphi$.

The semantics of this logic is determined in the following way: The sign $\in$ is interpreted as a set-theoretical membership relation. Also, if A is a model of a language L and φ is a formula of L, then

$A \models^* \forall X\varphi(X)$ iff for all $S \subseteq A$, $(A,S) \models^* \varphi(S)$, $A \models^* \exists X\varphi(X)$ iff there is $S \subseteq A$, $(A,S) \models^* \varphi(S)$.

5.1.8. **Example** L^W - weak second order logic.

The syntax of this logic is exactly like that of L^{II}. The symbol $\in$ is also interpreted by the set membership relation, but the range of quantifiers $\forall X$, $\exists X$ is changed:

$A \models^* \forall X\varphi(X)$ means that for all finite $S \subseteq A$, $(A,S) \models^* \varphi(S)$,
$A \models^* \exists X\varphi(X)$ means that for some finite $S \subseteq A$, $(A,S) \models^* \varphi(S)$.

The list of examples is potentially infinite, but these examples suffices for isolating some of the crucial features of the notion of a logic. We see in all the examples that a logic is a certain mapping, which in a uniform way assigns to any set L of symbols a set of formulas. This mapping may be considered part of syntax of this logic. Further more, a logic correlates a semantics with the syntax, whether or not φ is true in a given structure. This second feature is the heart of the notion of a logic: the examples above show that radically different logics may often have exactly the same syntax. In conclusion, we may consider a logic an operation which assigns to each set L of symbols a syntax and semantics, such that elementary syntactical operations are performable, and isomorphic structures satisfy the same sentences.

We shall consider several properties of abstract logics. The introduction of these notions is mainly motivated by the fundamental theorems of the first order predicate calculus. These results may be regarded as entirely "soft", in that they use only very general properties of logic, properties that are valid for a large number of extensions of $L_{\omega\omega}$.

5.1.10. Definition An abstract logic $(L^*, \models^*)$ is countably compact iff for every countable $T \subseteq L^*$ holds:

If every finite subset of T has a model, then T has a model, too.

The following two notions are connected with the validity of the Löwenheim-Skolem theorem. The ***Löwenheim number*** of a logic $(L^*, \models^*)$ is the least cardinal number k, such that every sentence of L^* which has a model, also has a model of cardinality at most k. The ***Hanf number*** of a logic $(L^*, \models^*)$ is the least cardinal number μ such that every sentence of L which has a model of the cardinality at least μ, has models of arbitrary large power. These cardinal numbers we shall denote respectively by $Lö(L^*)$ and $Ha(L^*)$.

Since the Compactness theorem holds for $L_{\omega\omega}$, we see that $L_{\omega\omega}$ is countably compact. Further, by the Löwenheim-Skolem theorems for $L_{\omega\omega}$, it follows that the Löwenheim and Hanf numbers coincide, and they are equal to $\aleph_0$. In general case, these numbers do not exist for a logic $(L^*, \models^*)$, for example, if L^* is a proper class; an illustration is $L_{\infty\omega}$.

The following table shows the status of some logics in respect to the introduced notions:

Logic	countably compact	$Lö(L^*)$	$Ha(L^*)$
$L_{\omega_1\omega}$	no	$\aleph_0$	exists
$L_{\infty\omega}$	no	none	none
$L_{\omega_1\omega_1}$	no	$2^{\aleph_0}$	exists
$L(Q_0)$	no	$\aleph_1$	exists
$L(Q_1)$	yes	$\aleph_1$	exists

The following theorems tell us when the Hanf number and Löwenheim number exist.

5.1.11. Theorem (Hanf) Assume $(L^*, \models^*)$ is an abstract logic. If L^* is a set, then the Löwenheim number and the Hanf number of this logic exist.

Proof Let us introduce a mapping k, which assigns to every consistent sentence φ of L^* the least cardinal number $k(\varphi)$, such that there is a models of φ of the cardinality $k(\varphi)$. Further, we shall define a mapping θ on sentences of L^* in the following way: if φ has arbitrary large models, then $\theta(\varphi)=\aleph_0$, otherwise $\theta(\varphi)$ is the supremum of the cardinal numbers of models of φ. Then the Löwenheim number and Hanf one of logic $(L^*, \models^*)$ are, respectively,

$Lö(L^*) = \sup\{k(\varphi): \varphi \text{ is a formula of } L^*\}$,

$Ha(L^*) = \sup\{\theta(\varphi): \varphi \text{ is a formula of } L^*\}$.

5.2. A characterization of PR^1

This section is devoted to the Lindström theorem and related subjects. We shall start with definition of the notion of a partial isomorphism which is due to Carol Karp, and plays an important role in infinitary logics. We shall see that this notion is very closely connected to the so-called back-and-forth construction.

Before proceeding further, we shall introduce some notions to be used throughout. Let A and B be models of a language L. A map f is called a ***partial isomorphism*** from A into B, denoted by $f:A \approx_p B$, if f is an isomorphism from a submodel of A onto a submodel of B. If Dom f is generated by less than μ elements, where μ is an infinite cardinal, then f will be called a μ-partial isomorphism. The empty map will also be considered a partial isomorphism.

A nonempty set I of partial isomorphisms from A into B is said to have the back-and-forth property, if:

For every $f \in I$ and $x \in A$ (respectively, $y \in B$), there is a $g \in I$ with $f \leq g$ and $x \in \text{Dom } g$ (respectively, $y \in \text{Im } g$). We write

$$I:A \approx_p B,$$

if I is a nonempty set of partial isomorphisms from A into B having back-and-forth-property. We say that A and B are partially isomorphic, in notation $A \approx_p B$, if there is an $I:A \approx_p B$.

The following theorem of C. Karp enables one to show that certain countable structures are isomorphic.

5.2.1. Theorem Assume A and B are countably generated models of a first-order language L. Then $A \approx B$ iff $A \approx_p B$. In fact, if $I:A \approx_p B$ and $f_0 \in I$, then f_0 can be extended to an isomorphism of A onto B

Proof If $f:A \approx B$, then $\{f\}:A \approx_p B$. To see the converse, let A be generated by $\{a_0, a_1, \ldots\}$, and B by $\{b_0, b_1, \ldots\}$, and assume $I:A \approx_p B$. Thus, $I \neq \emptyset$ so let $f_0 \in I$. Now, we shall use the back-and-forth argument to extend f_0 to an isomorphism from A onto B. We define a sequence

$$f_0 \subseteq f_1 \subseteq f_2 \subseteq \ldots$$

of functions as follows, $n \in \omega$:

f_{2n+1} is some function $g \in I$ such that $f_{2n} \subseteq g$ and $a_n \in \mathrm{Dom}\ g$,

f_{2n+2} is some function $g \in I$ such that $f_{2n+1} \subseteq g$ and $b_n \in \mathrm{Dom}\ g$.

This sequence of functions exists by the back-and-forth property of the family I, so the limit $f = \bigcup_n f_n$ of these functions clearly has for its domain the whole of A, and for its range the whole of B. It also preserves all the operations and relations from A into B, that is, f is an isomorphism of A onto B.

5.2.2. Example (Cantor) We shall show that any countable, dense linearly ordered set without end points is isomorphic to the set of rational numbers with a natural ordering.

The proof of this statement is based on Theorem 5.2.1, so let $A = (A, \leq_A)$ and $B = (B, \leq_B)$ be dense linearly ordered sets without end points, not necessarily countable, and define the following set of partial isomorphisms

$I = \{f$: f is a isomorphism of a finite subordering of A onto a finite subordering of B$\}$.

To see that I has the back-and-forth property, let $f \in I$ and $a \in A$. Assume $\mathrm{Dom}\ f = \{a_0 <_A a_1 <_A a_2 <_A \ldots <_A a_n\}$, and $a_{i-1} < a < a_i$, $i \leq n$. Then, by the density of B, there is $b \in B$, such that $f(a_{i-1}) < b < f(a_i)$. If we define $g = f \cup \{(a,b)\}$, then $g \in I$. The proof is similar, if $a < a_0$ or $a_n < a$, so, for any $f \in I$ and $a \in A$ there is $g \in I$, such that $f \subseteq g$ and $a \in \mathrm{Dom}\ g$. In a similar manner, one can show that for every $f \in I$ and $b \in B$, there is $g \in I$ such that $f \subseteq g$ and $b \in \mathrm{Im}\ g$.

It shold be noted that the example above shows that for any two dense linearly ordered sets A and B, regardless of their cardinality, we have $A \approx_p B$.

A further refinement of the notion of partial isomorphism is possible. We shall write $I{:}A \approx_p{}^\theta B$ whenever I is a non-empty family of partial isomorphisms from A into B with the following two properties:
1° *The θ-extension property* Given $F \subseteq I$ totally ordered by the extension order $\subseteq$, of a power less than θ, and $|\mathrm{Dom} \bigcup_{f \in F} f| < \theta$, there is $g \in I$ which extends all the $f \in F$.
2° *The back-and-forth property.*

A similar theorem to Theorem 5.2.1 holds:

5.2.3. Theorem Assume A and B are models of the same language, and suppose they are generated by sets of the power $\leq \theta$. Then,
1° $A \approx B$ iff $A \approx_p{}^\theta B$.
2° If $I{:}A \approx_p{}^\theta B$, then every $f_0 \in I$ can be extended to an isomorphism between A and B.

The proof of this theorem is just "a θ-extension" of the proof of Theorem 5.2.1, using now induction on even and odd ordinals, so we omit it. As an example of the application of the last theorem, we can consider η_k-sets. We have the following Hausdorff extension of Cantor's theorem.

5.2.4. Example If $\aleph_k$ is a regular cardinal number, then any two η_k-sets of power k are isomorphic.

The proof of this statement is based on Theorem 5.2.3. Namely, we consider

$I = \{f\colon f$ is an order preserving map defined on a subset of A of a power $< k$, with values in B$\}$.

Now, we shall turn to one more application of Theorem 5.2.1.

5.2.5. Example Let A and B be atomless Boolean algebras. Then $A \approx_p B$, and so if A and B are countable, then $A \approx B$.

To see the validity of this statement, consider the family

I= {f: f ia an isomorphism from a finite subalgebra of A onto a subalgebra of B}.

According to the proof of Claim in Example 3.3.7, it follows that the family I has the back-and-forth property.

The content of this example can be restated as follows, see also Example 3.3.7.

5.2.6. Lemma Let A be a finite Boolean algebra, B an atomless Boolean algebra, and $h: A \longrightarrow B$ an embedding. Then, h can be extended to an embedding $f: A(a) \longrightarrow B$.

We have also the following conclusion of Example 5.2.5.

5.2.7. Theorem The theory of atomless Boolean algebras is ω-categorical i.e. every two countable atomless Boolean algebras are isomorphic.

The proof of Theorem 4.1.6 shows the validity of the following statement.

5.2.8. Theorem Let A and B be elementary equivalent, μ-saturated, infinite models of the same language. Then, $A \approx_p^{\mu} B$.

There is a surprising link between logic $L_{\infty\omega}$ and the relation $\approx_p$. We shall first introduce some additional notations. Given the structures A and B for language L, we write $A \equiv_\infty B$, if A and B are models of the same sentences of $L_{\infty\omega}$. The following basic result belongs to C. Karp.

5.2.9. Theorem Given the models A and B of language L, the following are equivalent.

1° $A \equiv_\infty B$.

2° $A \approx_p B$.

3° There is a set $I: A \approx_p B$ such that every $f \in I$ has finitely generated domain and range.

Proof In the following we shall assume that there are no function symbols in L. Obviously 3° implies 2°. So, we shall show that 2° implies 1°. To prove this fact, let $I: A \approx_p B$. We can show by induction on the complexity of formulas $\varphi x_1 \ldots x_n$ of $L_{\infty\omega}$, that if $f \in I$ and $a_1, \ldots, a_n \in \mathrm{Dom}\, f$, then

$$A \models \varphi[a_1, \ldots, a_n] \quad \text{iff} \quad B \models \varphi[f(a_1), \ldots, f(a_n)].$$

If φ is atomic, this holds because f is an partial isomorphism. The case for propositional connectives is easy. So, suppose by the induction hypothesis, that the equivalence for $\varphi x_1 \ldots x_n$ holds, and let $a_1, \ldots, a_{n-1}$ belong to Dom f, f∈I. Then,

$A \models \exists x \varphi a_1 \ldots a_{n-1}$ implies there is an a∈A, such that

$$A \models \varphi[a, a_1, \ldots, a_{n-1}]$$

By the back-and-forth property of family I, there is g∈I such that $f \subseteq g$ and a∈Dom g. Then, by the induction hypothesis we have

$B \models \varphi[ga, ga_1, \ldots, ga_{n-1}]$

i.e. $B \models \varphi[ga, fa_1, \ldots, fa_{n-1}]$, thus $B \models \exists x_n \varphi[fa_1 \ldots fa_{n-1}]$.

The proof of the converse, i.e. that $B \models \exists x \varphi[fa_1, \ldots, fa_{n-1}]$ implies $A \models \exists x \varphi[a_1, \ldots, a_{n-1}]$ is similar to the proof above, again by using the back-and-forth property of I.

There is no need to check the case $\forall x \varphi$, because this is equivalent to $\neg \exists x \neg \varphi$.

Now, we shall prove that 1° implies 3°. We shall suppose a simplification, namely, that A and B are models of a language L which contains only relation symbols and at most finitely many constant symbols. So, let us assume $A \equiv_\infty B$, and construct a set I satisfying 3°. We define that

I= {f: f is an isomorphism of a finitely generated submodel A_0 of A onto a finitely generated submodel B_0 of B, such that for all $a_1, \ldots, a_n \in A_0$ and all $\varphi x_1 \ldots x_n$ of $L_{\infty\omega}$, $A \models \varphi[a_1, \ldots, a_n]$ iff $B \models \varphi[fa_1, \ldots, fa_n]$}.

The hypothesis that $A \equiv_\infty B$ guarantees that the submodels of A and B generated by the empty set are isomorphic, and that the isomorphism belongs to I. Further, we note that finitely generated means finite. Let us show that I has the back-and-forth property. So, let f∈I and a∈A. The case where b∈B is similarly done, so we can omit it. Let

$D = \text{Dom } f = \{a_1, a_2, \ldots, a_{n-1}\}, \quad n \geq 1.$

If a∈D, then we can take g=f, so assume a∉D. We have to find an element b∈B, such that for all the formulas $\varphi x_0 \ldots x_n$,

if $A \models \varphi[a, a_1, a_2, \ldots, a_{n-1}]$, then $B \models \varphi[b, fa_1, fa_2, \ldots, fa_{n-1}]$.

Now, suppose there is no such b. Then, for every b∈B there is a formula $\varphi_b x_0 \ldots x_n$ such that

$A \models \varphi_b[a, a_1, \ldots, a_{n-1}]$ and $B \models \neg\varphi_b[b, fa_1, \ldots, fa_{n-1}]$.

If $\psi x_0 x_1 \ldots x_n = \bigwedge_{b \in B} \varphi_b x_0 \ldots x_n$, then $A \models \psi[a, a_1, \ldots, a_{n-1}]$, hence

$A \models \exists x_0 \psi(x_0)[a_1, \ldots, a_{n-1}]$.

But $B \models \neg\exists x_0 \psi(x_0)[fa_1, \ldots, fa_{n-1}]$, and this is a contradiction to the definition of I, since $f \in I$. Hence, there is an element b with wanted property, so, we can define an extension of f by $g = f \cup \{(a,b)\}$.

The previous theorem is the first theorem which illustrates the connection between infinitary languages and the back-and-forth argument. Karp's theorem has a natural generalization to the $L_{\infty k}$ logic, but in order to state it, we need some more notations. However, the proof of the last theorem gives the core of the back and forth method, so, we shall not enter into generalizations.

Before we state and prove the characterization theorem for PR^1, we shall introduce one more notation of abstract logics. An abstract logic $(L^*, \models^*)$ has the *Karp property,* if any two models which are partially isomorphic are elementary equivalent, with respect to $(L^*, \models^*)$; in other words, $A \approx_p B$ implies $A \equiv_{L^*} B$. For example, Karp's theorem shows that $L_{\infty\omega}$ has Karp's property. Since the sublogic inherits Karp's property, we see that $L\omega\omega$ and $L_{\omega_1\omega}$ have this property, as well. Examples of logics which do not have this property are $L_{\omega_1\omega_1}$ and $L(Q_{\omega_1})$. The following Barwise theorem gives a sufficient condition for a logic to have Karp's property.

5.2.10. Theorem If an abstract logic $(L^*, \models^*)$ has the Löwenheim number $\aleph_0$, then it has Karp's property.

Proof Suppose $I: A \approx_p B$ but not $A \equiv_{L^*} B$. Thus, there is a sentence φ, such that $A \models^* \varphi$ and $B \models^* \neg\varphi$. Let A_∞ be the set of finite sequences of elements of domain A, and let $F: A_\infty \times A \longrightarrow A_\infty$ be the map

$$F: ((x_1, \ldots, x_m), y) \longrightarrow (x_1, \ldots x_m, y).$$

Further, define set B_∞ and map G analogously. Then using coding function, we obtain an expanded model $(A, B, A_\infty, B_\infty, F, G, I)$. Observe that we can make $A \cap B$ be the set of constants, since $A \approx_p B$. By the closure, quantifier, and relativisation properties, there is a sentence θ of L^*, which states that

$$A \models^* \varphi, \quad B \models^* \neg\varphi \quad \text{and} \quad I: A \approx_p B.$$

By assumption, the Löwenheim number of the given logic is $\aleph_0$, so, θ has a countable model $D' = (A', B', A_\infty, B_\infty, F', G', I')$. But, then $I': A' \approx_p B'$ and A' and B' are countable, so they are isomorphic. Since θ holds in D', it follows that $A' \models^* \varphi$, $B' \models^* \neg\varphi$, contradicting the isomorphism property. Hence, the logic has Karp's property.

The converse of this theorem does not hold, for instance logic $L_{\infty\omega}$ has Karp's property, but not the Löwenheim-Skolem number $\aleph_0$. Now, we shall state the main theorem of this section.

5.2.11. Theorem (Lindström) The logic $L_{\omega\omega}$ is the only abstract logic which has the Löwenheim number $\aleph_0$, and for which one of the following holds

1° Countable compactness,

2° The Hanf number is $\aleph_0$.

Proof We shall prove case 1°. So, let $(L^*, \models^*)$ be an abstract logic with $Lö(L^*) = \aleph_0$, and for which countable compactness holds. We can show that every sentence φ of L^* is equivalent to a sentence ψ of $L_{\omega\omega}$, i.e. that for every model A, we have

$A \models^* \varphi$ iff $A \models^* \psi$.

It suffices to prove this fact for a finite language L which has only relation symbols. The given models A and B of a language L and sequences $\underline{a} = a_1, \ldots, a_m$ and $\underline{b} = b_1, \ldots, b_m$, we can define a relation $\underline{a} I_n \underline{b}$ inductively, in the following way:

$\underline{a} I_0 \underline{b}$ means that $\underline{a}$ and $\underline{b}$ satisfy the same atomic formulas,

$\underline{a} I_{n+1} \underline{b}$ holds iff

1° For each $c \in A$, there is $d \in B$, such that $\underline{a}, c I_n \underline{b}, d$.

2° As in 1°, but changing the roles of A and B.

We write $A \equiv_n B$ to denote $\emptyset I_n \emptyset$ ($\emptyset$ denotes the empty sequence). Then, there is a finite set Φ_n of sentences of $L_{\omega\omega}$, such that $A \equiv_n B$ iff A and B satisfy the same sentences of Φ_n.

Let φ be a sentence of L^*, where the language of φ is a subset of L. It suffices to show:

3° For some n, $A \equiv_n B$ and $A \models^* \varphi$ implies $B \models^* \varphi$

since then φ is equivalent to a Boolean combination of sentences of Φ_n. Suppose, on the contrary, that 3° does not hold for any $n \in \omega$. Thus, there are models A_n and B_n, such that

4° $A_n \equiv_n B_n$, $A_n \models^* \varphi$ and $B_n \models^* \neg\varphi$.

Consider the expanded model $(C, D, R, S, \omega, \leq, \ldots)$, such that for each $n \in \omega$

A_n is a submodel of C with the domain $\{a\colon Ran\}$,

B_n is a submodel of D with the domain $\{b\colon Sbn\}$.

Then, there is a sentence ψ of L^* which indicates that 4° holds for every $n\in\omega$. By countable compactness, ψ has a model

$(C', D', R', S', \omega', \leq', \ldots)$

so that $(\omega', \leq')$ has a nonstandard element H, i.e. $H\in\omega'\setminus\omega$. Then,

5° $A_H \models^* \varphi$, $B_H \models^* \neg\varphi$ and $A_H \equiv_H B_H$.

From this, it follows that the relation between m-tuples, given by

$\underline{a}J\underline{b}$ iff $\underline{a}I_{H-m}\underline{b}$,

is an isomorphism between models A_H and B_H. But the logic $(L^*, \models^*)$ has Karp's property by Theorem 5.2.10. Hence, we have a contradiction. Therefore, φ is equivalent to a sentence of $L\omega\omega$.

5.3. Model theory for L(Q)

In this section we shall go beyond the classical first-order logic. Namely, we shall illustrate some methods and problems of abstract logics by an example: the first order logic with additional quantifier "there exist uncountably many". We shall see that some model-theoretic constructions which are important in classical model theory are available in this logic, as well. From now on, this logic will be denoted by L(Q). The first reason why we have decided to consider this logic in more detail is that it has a well behaved model theory. The other reason is that L(Q), after PR^1 has probably the most applications.

We have already described the syntax of L(Q) in Example 5.1.6. Now, we shall present a proof of the completeness theorem for L(Q). Keisler proved that this theorem holds for very simple set of axiom schemes:

All axioms of PR^1.

K1. $\neg Qx(x\equiv y \vee x\equiv z)$.

K2. $\forall x(\varphi \rightarrow \psi) \rightarrow (Qx\varphi \rightarrow Qx\psi)$.

K3. $Qx\varphi(x) \leftrightarrow Qy\varphi(y)$, y does not occur in $\varphi(x)$, and $\varphi(y)$ is obtained by replacing each free occurrence of x by y.

K4. $Qy\exists x\varphi \rightarrow \exists xQy\varphi \vee Qx\exists y\varphi$.

Here, φ and ψ are arbitrary formulas of L(Q). Occasionally, we shall consider the following axiom, too:

K5. $Qx(x\equiv x)$.

As has been indicated, the intended meaning of $Qx\varphi x$ in a model A is "for uncountably many $a \in A$, φa is true in A". The negation of the quantifier Qx is denoted by $\rotatebox{180}{Q}x$, so, the meaning of $\rotatebox{180}{Q}x\varphi x$, which is read "for few x, φx" is given by $\rotatebox{180}{Q}x\varphi \leftrightarrow \neg Qx\varphi$. Then, the axioms K1-K5 can be restated as follows:

K1'. $\rotatebox{180}{Q}x(x=y \vee x=z)$.
K2'. $\forall x(\varphi \rightarrow \psi) \rightarrow (\rotatebox{180}{Q}x\psi \rightarrow \rotatebox{180}{Q}x\varphi)$.
K3'. $\rotatebox{180}{Q}x\varphi x \leftrightarrow \rotatebox{180}{Q}y\varphi y$.
K4'. $\forall x\rotatebox{180}{Q}y\varphi \wedge \rotatebox{180}{Q}x\exists y\varphi \rightarrow \rotatebox{180}{Q}y\exists x\varphi$.
K5'. $\neg\rotatebox{180}{Q}x(x=x)$.

The intuitive contents of these axioms are:

K1. Every set of power ≤ 2 is countable.
K2. Every set which has an uncountable subset is uncountable.
K4. If $\bigcup_{x \in X} A_x$ is an uncountable family, then either some A_x is uncountable or X is uncountable. This is equivalent to: The union of countably many of countable sets is countable.
K5. The universe itself is uncountable.

So, for the axioms of L(Q) we take all the first-order axiom scheme for L plus the axioms K1-K4 (the axiom K5 is optional). The rules of inference of L(Q) are the same as for L in PR^1. Now, we shall consider models of L(Q) in more detail.

There are, in fact, two interpretations of L(Q): ***standard*** models and ***weak*** models. The standard models are the intended interpretations, where Qx means "there are uncountably many x", while the weak models are used as a main tool for constructing standard models.

By a weak model for L(Q), we mean a pair (A,q), such that A is a model of the first-order language L, and q is a set of subsets of the universe A of A. The definition of the satisfaction relation

$$(A,q) \models \varphi[a_1, \ldots, a_n]$$

for a formula $\varphi x_1 \ldots x_n$ of L(Q) and $a_1, \ldots, a_n \in A$ is defined in the usual way, by induction on the complexity of φ. The Qx case in the definition for $Qx\varphi x x_1 \ldots x_n$ is:

$$(A,q) \models Qx\varphi[a_1, \ldots, a_n] \quad \text{iff} \quad \{b \in A : (A,q) \models \varphi[b, a_1, \ldots, a_n]\} \in q.$$

Many simple propositions and definitions of the first-order predicate calculus can be transferred directly to L(Q). For example, the truth of a formula $\varphi x_1 \ldots x_n$ whose free variables are $x_1, \ldots, x_n$, depends only on $x_1, \ldots, x_n$ i.e. if μ and μ' are valuations of the domain A, which coincide on $x_1, \ldots, x_n$, then $\varphi[\mu]$ holds in A iff $\varphi[\mu']$ holds in A. Further, we can define, similarly as in PR^1, the notions as: "realizing a type", and "omitting a type".

Let A be a model for L. We shall write $A \models \varphi[a_1, \ldots, a_n]$ iff $(A,q) \models \varphi[a_1, \ldots, a_n]$, where q is the set of all the uncountable subsets of A. We say that A is a standard model of a sentence φ iff $A \models \varphi$, in the above sense. Therefore, A is a standard model of φ just in case φ holds in A with Qx interpreted by "there exist uncountably many x".

We shall particularly use following formulas which are provable in L(Q).

5.3.1. **Lemma** (Keisler) Let φ and ψ be formulas of L(Q) and x,y distinct variables. Then, the following formulas are provable in L(Q):

1° $Qx\varphi \to \exists x\varphi$.

2° $\exists xQy\varphi \to Qy\exists x\varphi$.

3° $Qx(\varphi \wedge \psi) \to Qx\varphi \wedge Qx\psi$.

4° $Qx(\varphi \vee \psi) \leftrightarrow Qx\varphi \vee Qx\psi$.

5° $Qx\varphi \wedge \neg Qx\psi \to Qx(\varphi \wedge \neg\psi)$.

Proof 1° Assuming that y,z do not occur in φ, we have

$$\vdash \neg\exists x\varphi \to \forall x(\varphi \to (x\equiv y \vee x\equiv z)).$$

Further, by Axiom 2,

$$\vdash \neg\exists x\varphi \to (Qx\varphi \to Qx(x\equiv y \vee x\equiv z)).$$

By Axiom 1, $\vdash \neg\exists x\varphi \to \neg Qx\varphi)$, and the statement 1° follows by propositional logic.

The proofs of 2° and 3° are not difficult, so we omit them.

4° Obviously $\vdash \forall x(\varphi \to \varphi \vee \psi)$, so by Axiom 2, $\vdash Qx\varphi \to Qx(\varphi \vee \psi)$. In a similar way we obtain $\vdash Qx\psi \to Qx(\varphi \vee \psi)$, thus

(1) $\vdash Qx\varphi \vee Qx\psi \to Qx(\varphi \vee \psi)$.

Now, we shall show that the converse of (1) also holds. So let u,v be distinct variables which do not occur in $\varphi \vee \psi$, and take

$$\sigma = ((y\equiv u \wedge \varphi) \vee (y\equiv v \wedge \psi)).$$

Then by predicate logic $\vdash \forall x((\varphi \vee \psi) \rightarrow \exists y\sigma)$, so by Axiom 2°

(2) $\vdash Qx(\varphi \vee \psi) \rightarrow Qx\exists y\sigma$.

Further, using Axiom 2,

(3) $\vdash Qx\exists y\sigma \rightarrow \exists yQx\sigma \vee Qy\exists x\sigma$.

We have $\vdash \sigma \rightarrow y{\equiv}u \vee y{\equiv}v$, thus $\vdash \exists x\sigma \rightarrow y{\equiv}u \vee y{\equiv}v$, and by Axiom 2,

$\vdash Qy\exists x\sigma \rightarrow Qy(y{\equiv}u \vee y{\equiv}v)$.

Then by Axiom 1, we have

(4) $\vdash \neg Qy\exists x\sigma$.

From (3) and (4) it follows that

(5) $\vdash Qx\exists y\sigma \rightarrow \exists yQx\sigma$.

Now, we have to consider two cases: $\forall uv(u{\equiv}v)$ and $\exists uv(u{\not\equiv}v)$.

Case 1 By predicate logic,

$\forall uv(u{\equiv}v) \vdash \forall x((\varphi \vee \psi) \rightarrow x{\equiv}u \vee x{\equiv}v)$.

Further, by Axiom 2,

$\forall uv(u{\equiv}v) \vdash Qx(\varphi \vee \psi) \rightarrow Qx(x{\equiv}u \vee x{\equiv}v)$,

thus, by Axiom 1,

(6) $\forall uv(u{\equiv}v) \vdash \neg Qx(\varphi \vee \psi)$.

It follows from (1) and (6) that the formula 4° is deducible from $\forall uv(u{\equiv}v)$.

Case 2 We have $\vdash \neg y{\equiv}u \rightarrow \forall x(\sigma \rightarrow \psi)$, thus by Axiom 2,

$\vdash \neg y{\equiv}u \rightarrow (Qx\sigma \rightarrow Qx\psi)$.

In a similar way, one can obtain $\vdash \neg y{\equiv}v \rightarrow (Qx\sigma \rightarrow Qx\varphi)$. Therefore,

$\vdash \neg y{\equiv}u \vee \neg y{\equiv}v \rightarrow (Qx\sigma \rightarrow Qx\varphi \vee Qx\psi)$.

Hence by predicate logic,

$\vdash u{\not\equiv}v \rightarrow (Qx(\varphi \vee \psi) \leftrightarrow Qx\varphi \vee Qx\psi)$,

therefore 4° is deducible from $\exists uv(u{\not\equiv}v)$.

5° Since $\vdash \forall x(\varphi \rightarrow \psi \vee (\varphi \wedge \neg\psi))$, by Axiom 2 it follows that

$\vdash Qx\varphi \rightarrow Qx(\psi \vee (\varphi \wedge \neg\psi))$.

By 4°,

$\vdash Qx(\psi \vee (\varphi \wedge \neg\psi)) \rightarrow Qx\psi \vee Qx(\varphi \wedge \neg\psi)$.

Then formula 5° follows from the above two formulas by propositional logic.

Now, we shall show that there is a close connection between regular relations and quantifiers Q, ∃. In most applications of Logic L(Q), the quantifier Q is eliminated by the use of some binary (or ternary) relation. This approach was first used by Fuhrken, and it is known as

Fuhrken's reduction technique. In that way Vaught proved the first form of the completeness theorem for the logic L(Q). Keisler also used this approach in several papers, either implicitly or explicitly. Jervell gave a new proof of Keisler's completeness theorem, taking as a binary relation a special kind of ordering. The method, we shall use here is a utilization of all the mentioned methods.

As we shall see, the notion of regular relation discussed in Section 4.4, i.e. properties S1, S2, and S3 of a binary relation plays an important role in this approach. So, we shall recall that L(R) denotes the language $L\cup\{R\}$, where R is a binary relation. Now, we can define a map $*$ from $L(\text{Ɉ})$ into L(R). This map preserves the classical logical conjunctivae and eliminates quantifier Ɉ in terms of R.

5.3.2. Definition Let φ and ψ be formulas of $L(\text{Ɉ})$. Then,

$\varphi^* = \varphi$, for φ atomic.

$(\neg\varphi)^* = \neg\varphi^*$, $(\varphi \wedge \psi)^* = \varphi^* \wedge \psi^*$, $(\exists x\varphi)^* = \exists x\varphi^*$,

$(\text{Ɉ}x\varphi)^* = \exists y\forall x(\varphi^* x \rightarrow Rxy)$, y is not free in φ.

Of considerable importance is language L^*, where

$L^* = \{\varphi^* : \varphi \text{ is a formula of } L(\text{Ɉ})\}$.

Thus, L^* is the set of all the $*$-transforms of the formulas of $L(\text{Ɉ})$, and, so $For_L \subseteq L^* \subseteq For_{L(R)}$. Observe that if $R\in L$, then $L^* = For_{L(R)}$.

5.3.3. Lemma Let $*$ be from the previous definition. Then,

1° $(Qx(x\equiv x))^*$ is $\forall y\exists x\neg Rxy$.

2° $(\text{Ɉ}w(w\equiv x \vee w\equiv y))^*$ is $\exists z(Rxz \wedge Ryz)$.

3° If $R\in L$ and φ is a formula of L, then, the condition of regularity of R (i.e. the property S3 in section 4.4.) is the $*$-transform of the formula

$Qy(\exists x\in_R u)\varphi xy \rightarrow (\exists x\in_R u)Qy\varphi xy$,

where $(\exists x\in_R u)\varphi$ stands for $\exists x(Rxy \wedge \varphi)$ (se Section 4.4).

Proof The proofs of 1° and 2° are easy, so we shall prove only 3°. By the first order predicate calculus and the definition of $*$, we have

$$\begin{aligned}\exists z(\forall x\in_R u)\forall y(\varphi xy \rightarrow Ryz) &\leftrightarrow \exists z\forall y\forall x(Rxu \rightarrow (\varphi xy \rightarrow Ryz))\\ &\leftrightarrow \exists z\forall y(\exists x(Rxu \wedge \varphi xy) \rightarrow Ryz)\\ &= (\text{Ɉ}y(\exists x\in_R u)\varphi xy)^* .\end{aligned}$$

Thus, S3 can be restated as the $*$-transform of

$$(\forall x \in_R u) Ⅎ y \varphi xy \rightarrow Ⅎ y(\exists x \in_R u)\varphi xy.$$

Taking a contraposition of this formula, we have that S3 is a *-transform of

$$Qy(\exists x \in_R u)\varphi xy \rightarrow (\exists x \in_R u)Qy\varphi xy.$$

By (A,q) we shall denote a weak model of $L(Q)$, and by (A,q^c) a weak model of $L(Ⅎ)$. If (A,q) is a weak model of $L(Q)$, then it is easy to see that the set q^c of all the subsets of A definable in (A,q), which do not belong to q, ma y be taken for the weak model (A,q^c) of $L(Ⅎ)$. Now, let (A,R^A) be a model of $L(R)$. A model (A,q^c), where

$$q^c = \{X: \text{ for some } a \in A,\ X \subseteq R_a{}^A\},$$

is said to be *induced* by relation R^A. We shall recall that

$$R_a{}^A = \{b \in A: R^A ba\}.$$

5.3.4 Lemma Assume that (A,q^c) is a model induced by a binary relation R^A on A. Then, for every formula $\varphi x_1 \ldots x_n$ of $L(Ⅎ)$ and every valuation $a_1, \ldots, a_n \in A$:

$$(A,q^c) \models \varphi[a_1,\ldots,a_n] \quad \text{iff} \quad (A,R^A) \models \varphi^*[a_1,\ldots,a_n].$$

Proof The proof is by induction on the complexity of φ. We shall give the proof only for the main step, when φ is of the form $Ⅎ x\psi x x_1 \ldots x_n$. We have:

$(A,q^c) \models \varphi$ iff $(A,q^c) \models Ⅎ x\psi[a_1,\ldots,a_n]$

iff $\{d \in A: (A,q^c) \models \psi[d,a_1,\ldots,a_n]\} \in q^c$

iff (by definition of q^c) there is $a \in A$, such that $\{d \in A: (A,q^c) \models \psi[d,a_1,\ldots,a_n]\} \subseteq R_a{}^A$

iff (by induction hypothesis), there is $a \in A$ such that $\{d \in A: (A,R^A) \models \psi^*[d,a_1,\ldots,a_n]\} \subseteq R_a{}^A$

iff there is $a \in A$ such that for all $d \in A$, $(A,R^A) \models (\psi^*[d,a_1,\ldots,a_n] \rightarrow R[a,d])$

iff $(A,R^A) \models \exists y \forall x(\psi^* x \rightarrow Rxy)[a_1,\ldots,a_n]$

iff $(A,R^A) \models \varphi^*[a_1,\ldots,a_n]$.

Now we shall introduce a refinement of the notion of a regular relation. So, let (A,R^A) be a model of $L(R)$. Relation R^A is ***-regular*** in model $A' = (A,R^A)$ iff A satisfies the conditions S1, S2, and S3 restricted to the formulas φ which are *-transforms of formulas of $L(Ⅎ)$. We can see that every regular relation is also *-regular.

5.3.5 Lemma Let $\underline{A}$ be a model of a language L, and assume that R^A is an $*$-regular relation in $(\underline{A}, R^A)$. If $(\underline{A}, q^c)$ is induced by R^A, then $(\underline{A}, q^c)$ verifies all the axioms K1'-K5' of logic $L(\Finv)$.

Proof Since R^A satisfies S1 and S2, $(\underline{A}, q^c)$ satisfies axioms K1' and K5' by Lemma 5.3.3 and Lemma 5.3.4. Further, it is easy to see that K2' and K3' are verified, as well. We shall show that K4' holds, too. In order to avoid working with assignments, we shall assume that formulas φ, $\psi, \ldots$ are sentences of the language $L(\Finv)$ expanded with names of the elements of the domain A, and that $\underline{A}$ is the simple expansion of A. Suppose

(1) $\quad (\underline{A}, q^c) \models \forall x \Finv y \varphi xy \wedge \Finv x \exists y \varphi xy$.

From $(\underline{A}, q^c) \models \Finv x \exists y \varphi xy$, by Lemma 5.3.4,

$$(\underline{A}, R^A) \models \exists w \forall x (\exists y \varphi^* xy \rightarrow Rxw).$$

Let $w \in A$ be such that $(\underline{A}, R^A) \models \forall xy(\varphi^* xy \rightarrow Rxw)$. Hence,

(2) $\quad$ for all $a, b \in A$, if $(A, R^A) \models \varphi^* ab$ then $a \in R_w^A$.

By (1), we also have $(\underline{A}, q^c) \models \forall x \Finv y \varphi xy$. Thus, by Lemma 5.3.4,

$$(\underline{A}, R^A) \models \forall x \exists w \forall y (\varphi^* xy \rightarrow Ryw),$$

and, so, for each $a \in R_w^A$, there is $d \in A$, such that

$$(\underline{A}, R^A) \models \forall y (\varphi^* ay \rightarrow Ryd).$$

By assumption , R^A is $*$-regular, therefore

(3) $\quad$ for some $d \in A$ and for all $a \in R_w^A$, $(\underline{A}, R^A) \models \forall y (\varphi^* ay \rightarrow Ryd)$.

Let $a, b \in A$ and assume that $(\underline{A}, R^A) \models \varphi^* ab$. Then, by (2), $a \in R_w^A$. By (3) it follows $(\underline{A}, R^A) \models Rbd$, i.e. $b \in R_d^A$. Thus we have proved

$$(\underline{A}, R^A) \models \exists z \forall xy (\varphi^* xy \rightarrow Ryz).$$

Hence, by Lemma 5.4.4, $(\underline{A}, q^c) \models \Finv y \exists x \varphi xy$.

5.3.6. Corollary Let A be a model of a language L, $R \in L$ and R^A a regular relation in A. If $(\underline{A}, q^c)$ is induced by R^A, then $(\underline{A}, q^c)$ verifies all the axioms K1'-K5' of logic $L(\Finv)$.

Now, we shall show that the converse of Theorem 5.3.5 also holds. Let $(\underline{A}, q^c)$ be a weak model of $L(\Finv)$. By axioms K1, K5 and Lemma 5.3.1.4° it follows that A is infinite, so let τ be a map from A onto the set of all the formulas of $L_A(\Finv)$ with one free variable. Therefore, $\{\tau(a): a \in A\}$ is the set of all the formulas of $L_A(\Finv)$ with exactly one free variable. In the following, we shall often write τ_a instead of $\tau(a)$.

A binary relation on the domain A is *induced* by (A, q^c) iff for all $a, b \in A$:

$$Rab \quad \text{iff} \quad (\underline{A}, q^c) \models \tau_b(a) \wedge \Finv x \tau_b(x)$$

Therefore, for every $a \in A$, we have

$$(\underline{A}, R^A, q^c) \models \forall x(Rxa \leftrightarrow \tau_a(x) \wedge \exists x \tau_a(x)).$$

5.3.7. Lemma Let a binary relation R^A on A be induced by a weak model $(\underline{A}, q^c)$ of K1-K5. Then,

1° The sentence $(\exists \varphi x \leftrightarrow \exists y \forall x(\varphi x \rightarrow Rxy)$ holds in $(\underline{A}, R^A, q^c)$ for all the formulas φx of $L_A(\exists)$ with one free variable x.

2° For any sentence φ in $L_A(\exists)$, $(\underline{A}, q^c) \models \varphi$ iff $(\underline{A}, R^A) \models \varphi^*$.

Proof 1° Let φx be a formula of $L_A(\exists)$ with one free variable x. Then, there is $d \in A$ such that φ is τ_d. Assume $(\underline{A}, q^c) \models \exists x \varphi x$. Since

$$(\underline{A}, R^A, q^c) \models (\exists x \tau_d(x) \wedge \tau_d(\mathbf{a}) \rightarrow \mathbf{Rad}),$$

it follows that $(\underline{A}, R^A, q^c) \models \tau_d(\mathbf{a}) \rightarrow \mathbf{Rad}$, hence

$$(\underline{A}, R^A, q^c) \models (\exists x \varphi x \rightarrow \exists y \forall x(\varphi x \rightarrow Rxy)).$$

Now, assume

$$(\underline{A}, R^A, q^c) \models \exists y \forall x(\varphi x \rightarrow Rxy).$$

So there is $e \in A$, such that

(1) $(\underline{A}, R^A, q^c) \models \forall x(\varphi x \rightarrow Rxe)$.

Since Rxe may be considered as the formula $(\tau_e(x) \wedge \exists x \tau_e(x))$ of $L_A(\exists)$, we may apply K2' to (1). then, we get

(2) $(\underline{A}, q^c) \models (\exists x(\tau_e(x) \wedge \exists x \tau_e(x)) \rightarrow \exists x \varphi x)$.

By Lemma 5.3.1, we have the following theorem of L(Q) for any formula ψ

$$Qx(\psi x \wedge \neg Qx\psi x) \rightarrow Qx\psi x \wedge \neg Qx\psi x,$$

hence $\neg Qx(\psi x \wedge \neg Qx\psi x)$, i.e.

(3) $\exists x(\psi x \wedge \exists x \psi x)$.

Taking for ψx the formula $\tau_e(x)$, by (2) and (3), it follows that

$$(\underline{A}, q^c) \models \exists x \varphi x$$

so we have proved

$$(\underline{A}, R^A, q^c) \models \exists y \forall x(\varphi x \rightarrow Rxy) \rightarrow \exists x \varphi x.$$

2° This assertion may be proved by induction on the complexity of formula φ, using 1°.

5.3.8. Lemma Let a binary relation R^A on A be induced by a weak model (A, q^c) of K1-K5. Then R^A is a *-regular relation in (A, R^A).

Proof the model (A, R^A) satisfies conditions S1 and S2 by Lemma 5.3.3. Therefore, we have to check only if condition S3, restricted to *-transforms of formulas of $L(\exists)$, holds in (A, R^A). So let φxy be a formula of $L(\exists)$ with free variables x,y. Then for $a \in A$:

$$(\underline{A},R^A) \models \forall x \in_A a \exists z \forall y(\varphi^* xy \rightarrow Ryz) \rightarrow \exists z(\forall x \in_R a)\forall y(\varphi^* xy \rightarrow Ryz)$$

iff

$$(\underline{A},R^A,q^c) \models (\forall x \in_R a)\text{⅃}y\varphi xy \rightarrow \text{⅃}y(\exists x \in_R a)\varphi xy$$

iff

(1) $(\underline{A},R^A,q^c) \models Qy(\exists x \in_R a)\varphi xy \rightarrow (\exists x \in_R a)Qy\varphi xy.$

Therefore, it suffices to prove (1).

The following holds in $(\underline{A},R^A,q^c)$

(2) $Qy(\exists x \in_R a)\varphi xy \rightarrow Qy\exists x(\tau_a(x) \wedge \text{⅃}x\tau_a(x) \wedge \varphi xy)$ (by definition of R^A)

$\rightarrow \exists xQy(\tau_a(x) \wedge \text{⅃}x\tau_a(x) \wedge \varphi xy) \vee$

$Qx\exists y(\tau_a(x) \wedge \text{⅃}x\tau_a(x) \wedge \varphi xy)$ (by Axiom K4).

Further, by Lemma 5.3.1,

$$Qx\exists y(\tau_a(x) \wedge \text{⅃}x\tau_a(x) \wedge \varphi xy) \rightarrow Qx\tau_a(x) \wedge Qx\exists y\varphi xy \wedge \neg Qx\tau_a(x).$$

So,

(3) $\vdash \neg Qx\exists y(\tau_a(x) \wedge \text{⅃}x\tau_a(x) \wedge \varphi xy).$

From (2),(3) and a propositional tautology, it follows that

$$(\underline{A},R^A,q^c) \models Qy(\exists x \in_R a)\varphi xy \rightarrow \exists xQy(\tau_a(x) \wedge \text{⅃}x\tau_a(x) \wedge \varphi xy),$$

thus (1) holds.

The following technical lemma will be used later.

5.3.9 Lemma Let R be a binary relation on A induced by a weak model (A,q^c) of K1-K5. Then,

1° $\bigcup_{a \in A} R_a = A.$

2° For all $a \in A$, $R_a \neq A$.

3° For all $a_1,\ldots,a_n \in A$, there is $b \in A$ such that $R_{a_1} \cup \ldots \cup R_{a_n} \subseteq R_b$.

Proof Relation R satisfies conditions S1 and S2 by the previous lemma, so 1° and 2° follow immediately. We shall prove clause 3° for n=2, since the general case follows easily by induction. Let φ and ψ be the formulas of L(⅃). By (3), from the proof of Lemma 5.3.3, we have

$$\vdash \text{⅃}x(\varphi x \wedge \text{⅃}x\varphi x) \wedge \text{⅃}x(\psi x \wedge \text{⅃}x\psi x)$$

and by Lemma 5.3.1,

$$\vdash \text{⅃}x((\varphi x \wedge \text{⅃}x\varphi x) \vee (\psi x \wedge \text{⅃}\psi x)).$$

Hence, for $a,b \in A$

$$(\underline{A},q^c) \models \forall x((\tau_a(x) \wedge \text{⅃}x\tau_a(x)) \vee (\tau_b(x) \wedge \text{⅃}x\tau_b(x)) \rightarrow \tau_c(x) \wedge \text{⅃}x\tau_c(x)),$$

where $c \in A$ is such that τ_c is the formula

$$(\tau_a(x) \wedge \text{⅃}x\tau_a(x) \vee (\tau_b(x) \wedge \text{⅃}x\tau_b(x)).$$

therefore, for $a,b \in A$, there is $c \in A$ such that

$(\underline{A},R^A) \models \forall x(Rxa \wedge Rxb \rightarrow Rxc)$,

i.e. $R_a \cup R_b \subseteq R_c$.

5.3.10. Definition Let (A,R^A) and (B,R^B) be models of $L(R)$. Then

1° A is a *-elementary submodel of B iff $A \subseteq B$ and for all formulas φ of L^* and all $a_1,\ldots a_n \in A$, $(A,R^A) \models \varphi[a_1,\ldots,a_n]$ iff $(B,R^B) \models \varphi[a_1,\ldots,a_n]$.

For this notion we shall use the notation $A \prec_R^* B$ (sometimes the subscript R is omitted).

2° B is an *-elementary R-end extension of A iff $A \prec_R^* B$, $A \neq B$ and for all $a \in A$, $R_a^B \subseteq A$. We shall use the notation $A \prec_{eR}^* B$. (If $(A,R^A) \subseteq (B,R^B)$, then the condition $R_a^B \subseteq A$ may be replaced by $R_a^B = R_a^A$.)

Before we state and prove the main lemma used in the proof of the completeness theorem of $L(Q)$, we shall present an *-version of the omitting types theorem.

5.3.11. Lemma (*-version of the omitting types theorem). Let T be a consistent set of sentences of L^*, and for each $n \in \omega$, let $\Sigma_n(x_n)$ be a set of formulas of L^* (with only x_n free). Assume that for every $n \in \omega$ and every formula φx_n of L^*, if $\exists x_n \varphi$ is consistent with T, then there exists $\sigma \in \Sigma_n$, such that $\exists x_n(\varphi \wedge \neg\sigma)$ is consistent with T (i.e. T locally *-omits Σ_n). Then, T has a countable model (A,R^A), which omits each Σ_n and R^A is *-regular in (A,R^A).

This lemma may be proved in a similar way as the omitting types theorem, if we observe that L^* is closed under logical conjunctivas $\neg$, $\wedge$ and quantifier $\exists$. Thus, we can omit the proof of this lemma.

The following lemma is the main step in Keisler's proof of the completeness theorem of $L(Q)$. A close look at the proof will show many steps similar to those in the proof of Theorem 4.4.3.

5.3.12. Lemma Let (A,q) be a countable weak model of $L(Q)$ in which all the axioms of $L(Q)$ hold, and let θx be a formula of $L_A(Q)$, such that $(\underline{A},q) \models Qx\theta x$. Then, there is a countable elementary extension (B,r) of (A,q), such that:

1° For some $b \in B \backslash A$, $(B,r) \models \theta[b]$.

2° For every formula ψy of $L_A(Q)$, such that $(\underline{A},q) \models \neg Qy\psi y$, we have $\{a \in B: (\underline{B},r) \models \psi[a]\} \subseteq A$.

Proof Let R^A be a binary relation induced by (A,q^c), and

$$T = Th^*(\underline{A}) \cup \{\neg R^* ca\colon a\in A\} \cup \{\theta^* c\},$$

where c is a new constant symbol and R^*ca stands for an $*$-transform of $\tau_a(c) \wedge \exists x \tau_a(x)$. In the following, we shall use the fact

(1) $(\underline{A},R^A) \models Rab \leftrightarrow R^*ab$ for all $a,b\in A$.

Claim 1 T is a consistent theory.

Proof of Claim 1 We shall show that every finite subset Σ of T has a model. Let $a_1,\ldots,a_n$ be all the elements of A, such that $\neg R^*ca_i$, $1\le i\le n$, belong to Σ. By Lemma 5.3.9, here is $a\in A$, such that $R^A{}_{a_1}\cup\ldots\cup R^A{}_{a_n}\subseteq R^A a\neq A$. Let $d\in A\setminus R_a$. Since $(\underline{A},q)\models Qx\theta x$, we have $(\underline{A},R^A)\models \forall y\exists x(\theta^* x \wedge \neg Rxy)$. So, there is $e\in A$, such that $(\underline{A},R^A)\models (\theta^* e \wedge \neg Red)$. Hence, $(\underline{A},R^A,e)$ is a model of Σ, where c is interpreted by e.

Claim 2 Let ψx be a formula with one free variable in the language $L_A{}^*$. Then, ψc is inconsistent with T iff there is $a\in A$, such that

$$(\underline{A},R^A)\models \forall x(\theta^* x \wedge \psi x \to Rxa).$$

Proof of Claim 2 Assume that ψc is inconsistent with T. Then, $T\vdash \neg\psi c$, hence for some $a_1,\ldots,a_n\in A$,

$$Th\underline{A}^* \models \neg R^* ca_1 \wedge \ldots \wedge \neg R^* ca_n \wedge \theta^* c \to \neg\psi c.$$

Therefore,

$$Th\underline{A}^* \models \forall y(\psi y \to \neg\theta^* y \vee R^* ya_1 \vee \ldots \vee R^* ya_n).$$

Using (1) and Lemma 5.3.9, we obtain for some $b\in A$,

$$Th\underline{A}^* \models \forall y(\psi y \to \neg\theta^* y \vee R^* yb), \quad \text{i.e.}$$

$$Th\underline{A}^* \models \forall y(\psi y \wedge \theta^* y \to Ryb).$$

Now, assume that ψc is consistent with T and that for some $a\in A$,

$$(\underline{A},R^A)\models \forall x(\psi x \wedge \theta^* x \to R^* xa).$$

Let (B,R^B,b) be a model of $T\cup\{\psi c\}$. Then, $(B,R^B,b)\models (\psi c \wedge \theta^* c)$. Since, $A\prec_{R^*}B$ (we identify a^B with a), it follows that

$$(B,R^B,b)\models \forall x(\psi x \wedge \theta^* x \to R^* xa).$$

Hence,

$$(B,R^B,b)\models R^* ca, \quad \text{contradicting} \quad (B,R^B,b)\models \neg R^* ca.$$

Claim 3 For every $a\in A$, T locally $*$-omits $\Sigma_a(x) = \{R^* xa\} \cup \{x\neq d\colon d\in R_a{}^A\}$.

Proof of Claim 3 Let $\exists x\varphi xc$ be a formula of $(L_A \cup \{c\})^*$ which is consistent with T. Assume that there is no $\sigma\in\Sigma_a$, such that $\exists x(\varphi xc \wedge \neg\sigma)$ is consistent with T. Then, $\exists x(\varphi xc \wedge \neg R^* xa)$ is inconsistent with T, so, by Claim 2 and (1), for some $b_1\in A$:

(2) $(\underline{A},R^A)\models \forall y(\exists x(\varphi xy \wedge \neg R^* xa) \wedge \theta^* y \to Ryb_1)$.

Let $d\in R_a{}^A$. Then, $\exists x(\varphi xc \wedge x=d)$ is inconsistent with T, i.e. φdc is inconsistent with T, so, by Claim 2, there is $b\in A$ such that

$$(\underline{A},R^A)\models \forall y(\theta^* y \wedge \varphi dy \rightarrow Ryb).$$

Therefore, we have proved

$$(\underline{A},R^A)\models (\forall x\in_R a)\exists z\forall y(\theta^* y \wedge \varphi xy \rightarrow Ryz).$$

Since the relation R^A is $*$-regular in $(\underline{A},R^A)$, it follows that

$$(\underline{A},R^A)\models \exists z(\forall x\in_R a)\forall y(\theta^* y \wedge \varphi xy \rightarrow Ryz).$$

Thus, for some $b_2\in A$

(3) $(\underline{A},R^A)\models (\forall x\in_R a)\forall y(\varphi xy \wedge \theta^* y \rightarrow Ryb_2)$.

By Lemma 5.3.9, there is $b_3\in A$ such that $R_{b_1}{}^A \cup R_{b_2}{}^A \subseteq R_{b_3}{}^A$. Then, using (2) and (3), we have

$$(\underline{A},R^A)\models \forall y((\theta^* y \wedge \exists x(\varphi xy \wedge \neg Rxa)) \vee (\theta^* y \wedge \exists x(\varphi xy \wedge Rxa)) \rightarrow Ryb_3).$$

Thus,

$$(\underline{A},R^A)\models \forall y(\exists x\varphi xy \wedge \theta^* y \rightarrow Ryb_3).$$

So, by Claim 2, $\exists x\varphi xc$ is inconsistent with T, which is a contradiction. This finishes the proof of Claim 3.

Since T locally $*$-omits each Σ_a, by the $*$-omitting theorem, it follows that there is a countable model $(B,b_a,R^B,d)_{a\in A}$ of T omitting each Σ_a $(a\in A)$, in which R^B is $*$-regular. Identifying b_a with a, we have:

1° $A<_R{}^* B$.

2° R^B is $*$-regular in (B,R^B).

3° $(\underline{B},R^B)\models \theta^* d$ and $d\in B\backslash A$.

Really, if $d\in A$, then for all $a\in A$, $(A,R^A)\models \neg R^* da$, contradicting to $\bigcup_{a\in A} R_a{}^A = A$ (see Lemma 5.3.9). Let (B,r^c) be induced by (B,R^B). Since R^B is $*$-regular in (B,R^B), by Theorem 3.6. (B,r^c) is a weak model of axioms K1 - K5 for L(Q). By 1°, 3° and Lemma 5.3.4, we have

4° $(B,q) < (B,r)$ and $(\underline{B},r)\models \theta[d]$.

Now, assume that formula ψx of $L_A(Q)$ is such that $(\underline{A},q)\models \neg Qx\psi x$. Let $a\in A$ be such that τ_a is ψ (where τ is the map from the definition of relation R^A induced by (A,q^c)). By Lemma 5.3.7, $(\underline{A},R^A)\models (\neg Qx\tau_a(x))^*$. Since $A <_R{}^* B$, it follows that $(\underline{B},R^B)\models (\neg Qx\tau_a(x))^*$, hence

$$(\underline{B},R^B)\models R^* ba, \quad \text{i.e.} \quad (\underline{B},R^B)\models Rba.$$

Since $(B,b_a,R^B,d)_{a\in A}$ omits Σ_a, we have $b\in R_a{}^A$, hence $b\in A$. Thus,

$$\{b\in B\colon (\underline{B},R^B)\models (\psi b)^*\} \subseteq A,$$

therefore, by Lemma 5.3.11, $\{b\in B\colon (\underline{B},r)\models \psi[b]\} \subseteq A$.

In the following, we shall need the notion of elementary chains of models of $L(Q)$, and their unions. A sequence (A_α,q_α), $\alpha<\delta$, of weak models is said to be elementary iff we have $(A_\alpha,q_\alpha) \prec (A_\beta,q_\beta)$ for all $\alpha<\beta<\delta$. The union of an elementary chain (A_α,q_α), $\alpha<\delta$, is the weak model

$$(A,q)= \cup_{\alpha<\delta}(A_\alpha,q_\alpha),$$

such that $A= \cup_{\alpha<\delta}A_\alpha$, and $q= \{S\subseteq A\text{: For some } \beta<\delta,\ \beta\leq\alpha<\delta \text{ implies } S\cap A_\alpha\in q_\alpha\}$. That is, q is the set of all $S\subseteq A$, such that $S\cap A_\alpha$ is eventually in q_α.

5.3.13. Let (A_α,q_α), $\alpha<\delta$, be an elementary chain, and let (A,q) be the union of this models. Then, for all $\alpha<\delta$, $(A_\alpha,q_\alpha) \prec (A,q)$.

Proof We shall show by induction on the complexity of $\varphi x_1\ldots x_n$ that
(1) For all $\alpha<\delta$ and all $a_1,\ldots,a_n\in A$,

$$(A_\alpha,q_\alpha)\models \varphi[a_1,\ldots,a_n] \text{ iff } (A,q)\models \varphi[a_1,\ldots,a_n].$$

We shall consider the main step, i.e. when φ is of the form $Qx\psi x$, assuming that (1) holds for ψ.

So assume $\alpha<\delta$, and $a_1,\ldots,a_n\in A$ and let

$$S= \{a\in A\colon (A,q)\models \psi[a,a_1,\ldots,a_n]\}.$$

Let $(A,q)\models Qx\psi[a_1,\ldots,a_n]$. Then $S\in q$, hence, for some β, $\alpha\leq\beta<\delta$ and $S\cap A_\beta\in q_\beta$. Since (1) holds for ψ, we have

$$S\cap A_\beta= \{a\in A_\beta\colon (A_\beta,q_\beta)\models \psi[a,a_1,\ldots,a_n]\}.$$

Thus $(A_\beta,q_\beta)\models Qx\psi[a_1,\ldots,a_n]$, and since, $(A_\alpha,q_\alpha) \prec (A_\beta,q_\beta)$, it follows that $(A_\alpha,q_\alpha)\models Qx\psi[a_1,\ldots,a_n]$.

Now, assume $(A,q)\models \neg Qx\psi[a_1,\ldots,a_n]$. Then, $S\notin q$, so, there exists β such that $\alpha\leq\beta<\delta$ and $S\cap A_\beta\notin q_\beta$. Reasoning as above, we obtain

$$(A_\alpha,q_\alpha)\models \neg Qx\psi[a_1,\ldots,a_n].$$

Therefore, $\varphi= Qx\psi x$ satisfies (1).

By iterating the main lemma (Lemma 5.3.12), we have the following stronger result

5.3.14. **Lemma** Let (A,q) be a countable weak model of K1 - K5 let L_A be the language of $\underline{A}= (A,a)_{a\in A}$. Then, (A,q) has a countable elementary extension (B,r), such that for all the formulas $\varphi(x)$ of $L_A(Q)$, $(A,q)\models Qx\varphi x$ iff there exists $b\in B\setminus A$, such that $(B,r)\models \varphi[b]$.

Proof Since $(A,q)\models Qx(x=x)$, and set A is countable, we may list all the formulas φx of $L_A(Q)$, such that $Qx\varphi x$ holds in $(\underline{A},q)$, in a sequence:

$$\varphi_0(x_0),\ \varphi_1(x_1),\ \ldots\ .$$

Then, using Lemma 5.3.12 countably many times, we get a countable elementary chain $(A_0, q_0) \prec (A_1, q_1) \prec \dots$, such that

(1) $(A_0, q_0) = (A, q)$.

(2) There exists $a_n \in A_{n+1} \setminus A_n$, such that $(\underline{A}_{n+1}, q_{n+1}) \models \varphi_n[a_n]$, $n \in \omega$.

(3) For all the formulas ψ of $L_A(Q)$, such that $(\underline{A}_n, q) \models \neg Qy\psi y$, and all $n \in \omega$, $\{a \in A_{n+1} : (\underline{A}_{n+1}, q_{n+1}) \models \psi[a]\} \subseteq A_n$.

Then, model $(B, r) = \bigcup_n (A_n, q_n)$ is an elementary extension of (A, q), and by (1) - (3), (B, r) satisfies the conclusion of the lemma.

Now, we are going to prove the Completeness theorem for L(Q). This form of Completeness theorem for L(Q) was proved by J.Keisler.

5.3.15. Completeness theorem for L(Q) Let T be a set of sentences of L(Q). Then T has a standard model iff T is consistent in L(Q).

Proof If T has a standard model, then, it is easy to see that T is consistent; this follows simply from the fact that standard models satisfy axioms K1 - K5. So, we shall consider only the hard direction.

The proof is divided into two parts. The first part in most details is similar to the proof of the completeness theorem of PR^1, i.e. a weak model of L(Q) is constructed. Thus, we shall only outline this part of the proof. Then, by appropriate application of Lemma 5.3.14, we obtain a standard model of L(Q).

Claim 1 (Weak completeness theorem) Let T be a consistent set of sentences of L(Q). Then, T has a countable weak model.

Proof of Claim 1 We can enlarge language L to $L_C = L \cup C$, where C is a set of new constant symbols. Then, T is still consistent in $L_C(Q)$. By the method of Henkin, T can be extended to a maximal set S of sentences of $L_C(Q)$, such that C is a set of witnesses for S. Now, we shall show that S has a weak model.

Let $S_0 = \{\varphi \in S : \varphi \in Sent_L\}$. Then, S_0 is a maximal and consistent in the sense of L. By the Henkin construction, it follows that S_0 has a model A, such that $A = \{c^A : c \in C\}$. Let us define for each formula φx of L(Q) the set

$$Y_\theta = \{c^A : c \in C \text{ and } S \vdash \varphi c\}.$$

Further, we shall introduce the family

$$q = \{Y_\theta : \varphi \text{ has only one free variable, say } x, \text{ and } S \vdash Qx\varphi x\}.$$

Then, obviously, q is a set of subsets of A. Now, we shall show by induction on the complexity of φ, that for all sentences of L(Q)

(1) $(A,q)\models\varphi$ iff $S\vdash\varphi$.

We shall consider only the main step, when φ is $Qx\theta x$. So, assume (1) holds for all sentences θc, $c\in C$. Then

(2) $Y_\theta=\{c^A: S\vdash\theta c\}=\{c^A: (A,q)\models\theta c\}=\{c^A: (A,q)\models\theta[c^A]\}$.

If $S\vdash Qx\theta x$, then $Y_\theta\in q$, by definition of q, so, $(A,q)\models Qx\theta x$ by (2).

Now assume $(A,q)\models Qx\psi x$. Then, $Y_\theta\in q$ by (2). from the definition of q, we can see that there is a formula σy such that $Y_\theta=Y_\sigma$ and $S\vdash Qy\sigma y$. Then by (2), $c^A\in Y_\theta$ implies $S\vdash\theta c$, for all $c\in C$, and similarly for σ. Therefore, for all $c\in C$, $S\vdash\theta c$ iff $S\vdash\sigma c$, thus for all $c\in C$, $S\vdash\sigma\leftrightarrow\theta$. Let v be a variable occurring in neither θx nor σx. Then since C is a set of witnesses for S, we have $S\vdash\forall v(\theta v\leftrightarrow\sigma v)$. Then, by K2,

$$S\vdash Qv\theta v\leftrightarrow Qv\sigma v,$$

and so, by K3,

$$S\vdash Qx\theta x\leftrightarrow Qv\theta v,\quad S\vdash Qy\theta y\leftrightarrow Qv\theta v,$$

and thus

$$S\vdash Qx\theta x\leftrightarrow Qy\sigma y.$$

Since $S\vdash Qy\sigma y$, we have $S\vdash Qx\theta x$. So, φ satisfies (1), i.e. (A,q) is a model of L(Q).

Claim 2 If T is a consistent set of sentences, then T has a a standard model.

Proof of Claim 2 Suppose T is a consistent set of sentences of L(Q). By Claim 1, there is a countable weak model (A_0,q_0) of T. By repeated applications of Lemma 5.3.14 ω_1 times, we obtain an elementary chain of countable weak models: (A_α,q_α), $\alpha<\omega_1$, such that:

(1) If α is a limit ordinal then $(A_\alpha,q_\alpha)=\bigcup_{\beta<\alpha}(A_\beta,q_\beta)$.

(2) For any $\alpha<\omega_1$ and each formula φx of $L_A(Q)$

$(\underline{A}_\alpha,q_\alpha)\models Qx\varphi x$ iff there is $a\in A_{\alpha+1}\setminus A_\alpha$ such that $(\underline{A}_{\alpha+1},q_{\alpha+1})\models\varphi[a]$.

Now, take $B=\bigcup_{\alpha<\omega_1}A_\alpha$. We shall show that B is a standard model of T.

First, consider the weak model $(B,r)=\bigcup_{\alpha<\omega_1}(A_\alpha,q_\alpha)$. Then, by Lemma 5.3.13, $(A_0,q_0)\prec(B,r)$, i.e. (B,r) is a weak model of T. Further, observe that B is an uncountable model. So, it suffices to prove that

(3) For all the formulas $\varphi x_1\ldots x_n$ of L(Q) and all $b_1,\ldots,b_n\in B$,
$(B,r)\models\varphi[b_1,\ldots,b_n]$ iff $B\models\varphi[b_1,\ldots,b_n]$.

This statement will be proved by induction on the complexity of formulas φ. We shall consider only the main step, when $\varphi = Qx\psi xy_1\ldots y_n$. So, let $b_1,\ldots,b_n \in B$. Then for some $\alpha<\omega_1$, $b_1,\ldots,b_n \in A_\alpha$.

First, assume $(B,r) \models Qx\psi[b_1,\ldots,b_n]$. Since $(A_\beta,q_\beta) \prec (B,r)$ for all $\beta<\omega_1$, we have $(A_\beta,q_\beta) \models Qx\psi[b_1,\ldots,b_n]$ for all $\beta<\omega_1$. By (2), if $\alpha<\beta<\omega_1$, there is $a_\beta \in A_{\beta+1} \setminus A_\beta$ such that $(A_{\beta+1},q_{\beta+1}) \models \psi[a_\beta,b_1,\ldots,b_n]$, so $(B,r) \models \psi[a_\beta,b_1,\ldots,b_n]$. Since all the elements from the sequence $\langle a_\beta : \beta<\omega_1 \rangle$ are distinct, the set $S = \{b\in B: (B,r) \models \psi[b,b_1,\ldots,b_n]\}$ is of the cardinality $\aleph_1$. By the induction hypothesis, ψ satisfies (3), so,

$$S = \{b\in B: B \models \psi[b,b_1,\ldots,b_n]\}.$$

Therefore, $B \models Qx\psi[b_1,\ldots,b_n]$.

Now, suppose $B \models \neg Qx\psi[b_1,\ldots,b_n]$. Then, whenever $\alpha\leq\beta<\omega_1$, we have $(A_\beta,q_\beta) \models \neg Qx\psi[b_1,\ldots,b_n]$. Thus, from (1) and (2), we can infer that

$$S = \{b\in B: (B,r) \models \psi[b,b_1,\ldots,b_n]\} \subseteq A_\alpha.$$

The set A_α is countable, so, S is countable, as well. Again, using the induction hypothesis for ψ, we have $B \models Qx\psi[b_1,\ldots,b_n]$, so B satisfies the conclusion of Claim 2.

We have the following consequence of the completeness theorem for L(Q).

5.3.16. Corollary A sentence φ of L(Q) is provable iff it is valid.

5.3.17. Corrolary (Compactness theorem for L(Q)) Assume T is a set of sentences of L(Q). If every finite subset of T has a standard model, then T has a standard model.

We shall note that neither the completeness theorem nor the compactness theorem are valid for uncountable L. For example, the theory

$$T = \{\exists xPx\} \cup \{Pc_\alpha \ \alpha<\omega_1\} \cup \{c_\alpha \neq c_\beta : \alpha<\beta<\omega_1\}$$

is finitely consistent (i.e. every finite subset of T has a standard model), but T itself obviously does not have a standard model.

A form of the completeness theorem was proved by R. Vaught. Namely, he proved the following assertion.

5.3.18. Proposition The set of all the valid sentences of L(Q) is recursively enumerable in the set of symbols of L.

We can see that this statement is an immediate consequence of Completeness theorem for L(Q). We would note that proposition 5.3.18 can be proved using Proposition 4.4.17 without appealing to the Completeness theorem for L(Q). In fact, if M is the class of all the countable linearly ordered models which satisfy S1 and S3, then for any formula φ of L(Q) we have: $M \models \varphi$ iff $M \models \varphi^*$. Here φ^* denotes the $*$-transform of φ, if we take for R the symbol <. Then, the statement follows from the Löwenheim-Skolem theorem, i.e. that LO, S1, S3 $\vdash \varphi^*$ iff $M \models \varphi^*$.

As an example of a theory in L(Q) consider Peano arithmetic, but with the induction scheme applied to all the formulas of $L_{PA}(Q)$. Let us denote this theory by PA(Q). By induction, it is easy to show that

(1) $PA(Q) \vdash \forall y \lnot\exists x(x<y)$.

Now, let us describe the standard models , in the sense of L(Q), of PA(Q). So assume A is a standard model of PA(Q). Then, since

$PA(Q) \vdash Qx(x=x)$,

it follows that A is uncountable. By (1), $A \models \forall y \lnot\exists x(x<y)$, so every initial segment in A is countable.

Thus, if A is a standard model of PA(Q), then A is an ω_1-like model of PA of cardinality ω_1. Further, it is possible to show that theory PA(Q) admits the elimination of the quantifier Q, i.e.

(2) $PA(Q) \vdash Qx\varphi \leftrightarrow \forall y \exists x{>}y\varphi$, φ is a formula of $L_{PA}(Q)$.

So, if A is an ω_1-like uncountable model of PA, then A obviously satisfies the $*$-transform of the induction scheme of PA(Q), and by virtue of (2), A also satisfies the induction scheme of PA(Q) in the sense of logic L(Q). Thus, A is a standard model of PA(Q). Hence, we can prove

5.3.19 Proposition Let A be a model of $L_{PA}(Q)$. Then A is a standard model of PA(Q) iff A is an uncountable ω_1-like model of PA.

Therefore, Example 4.4.16.1° may be considered a particular case of Keisler's Completeness theorem. It is possible to show (for example, by using so-called end-extension types) that there are $2^{\aleph_1}$ ω_1-like, uncountable models of PA, i.e. $2^{\aleph_1}$ standard models of PA(Q). Proposition 5.3.19 also shows that Löwenheim-Skolem theorems do not hold for L(Q).

By analyzing the proof of the completeness theorem of L(Q), it is possible to obtain improvements which leads to the application of various kinds. Such an improvement is the omitting types theorem for L(Q).

5.3.20. Theorem Let T be a consistent set of sentences of $L(Q)$, and for each $n\in\omega$ assume that $\Sigma_n(x_n)$ is a set of formulas, such that

1° For some $\sigma_n\in\Sigma_n$, $T\vdash \text{⅃}x_n\sigma_n$.

2° If $\exists x_n\varphi$ is consistent with T, then there exists $\sigma\in\Sigma_n$, such that $\exists x_n(\varphi \wedge \neg\sigma)$ is consistent with T.

Then T has a standard model which omits each Σ_n, $n\in\omega$.

Proof By the Completeness theorem for $L(Q)$, the theory T has a standard model, say **A**. By Lemma 5.3.7, and Lemma 5.3.8, we can expand A to a model (A,R^A) such that:

(1) R^A is *-regular in (A,R^A).

(2) $(A,R^A)\models \text{⅃}x\varphi x \leftrightarrow \exists y\forall x(\varphi x \rightarrow Rxy)$, φx is a formula of $L_A(\text{⅃})$.

(3) For all the sentences φ of $L_A(\text{⅃})$, $A\models\varphi$ iff $(A,R^A)\models\varphi^*$.

Thus, theory T' of the language $L\cup\{R\}$, R is a binary relation symbol, which consists of:

the axioms of theory T,

$S1^*$, $S2^*$ and scheme $S3^*$,

$\text{⅃}x\varphi x \leftrightarrow \exists y\forall x(\varphi x \rightarrow Rxy)$, where f is a formula of $L(Q)$,

is a consistent theory, and obviously T' satisfies the conditions of the *-version of the Omitting types theorem (Lemma 5.3.11). By this lemma, there is a countable model $(B,R_1{}^B)$ of T' which omits each $\Sigma_n{}^*$, and $R_1{}^B$ is *-regular in $(B,R_1{}^B)$. Then the induced model (B,q) is a weak model of $L(Q)$ which omits each Σ_n. From the proof of the Completeness theorem, it follows that there is an elementary extension (C,r) of (B,q), such that

$(C,r)\models\varphi[b_1,\dots,b_n]$ iff $(B,q)\models\varphi[b_1,\dots,b_n]$, $b_1,\dots,b_n\in B$,

and for each formula ψx of $L_B(Q)$,

$(B,q)\models\text{⅃}x\psi$ iff $\{b\in C\colon (C,r)\models\psi[b]\}\subseteq A$.

Therefore, C is a standard model of T. Also, for all $n\in\omega$, $(B,q)\models\text{⅃}x_n\sigma_n$, so $\{b\in C\colon (C,r)\models\sigma_n[b]\}\subseteq B$.

Since $\sigma_n(x_n)\in\Sigma_n$, and no element of B satisfies Σ_n in (B,q), it follows that (C,r) omits Σ_n. Therefore, C omits Σ_n.

As an application of the Omitting types theorem for $L(Q)$, we shall give a new proof of Keisler's Two-cardinal theorem, Theorem 4.4.18

5.3.21. Example (Proof of Theorem 4.4.18) So, let T be a theory in a countable language $L=\{P,\dots\}$ with a model $A=(A,U,\dots)$ of type (k^+,k). Let $<$ be an ordering of the domain A of the ordering type of ordinal k^+. Then, relation $<$ is regular, and so the quantifier $\text{⅃}x$, introduced by

$$\exists x\varphi x \leftrightarrow \exists y\forall x(\varphi x \rightarrow x<y),$$

satisfies the axioms K1 - K5. Let $B<A$ be countable, $B=(B,W,\dots)$, and define $\Sigma(x)=\{P(x)\}\cup\{x\neq b:b\in B\}$. Then, for $\Gamma=\mathrm{Th}B\cup\{\exists xPx\}$ we have:

1° $\Gamma\vdash \exists xPx$.

2° If $\exists x\varphi x$ is consistent with Γ, by the Completeness of Γ follows $\Gamma\vdash \exists x\varphi x$, so, for some $b\in B$, we have $\Gamma\vdash \varphi b$, i.e. $\exists x(\varphi x \wedge \neg x\neq b)$ is consistent with Γ.

Therefore, by the Omitting types theorem for $L(Q)$, it follows that there is a standard model $C=(C,V',\dots)$ of Γ which omits $\Sigma(x)$. Since C omits $\Sigma(x)$, it follows that $V'=W$. Also, $B<C$, and C has the type (ω_1,ω_0).

Exercises

1. Show that the class of Archimedian fields has a characterization in $L\omega_1\omega$, i.e. there is a sentence φ in $L\omega_1\omega$ where L is the language of ordered fields, such that for a model A of L,

$$A\models\varphi \text{ iff } A \text{ is an Archimedian field.}$$

2. Let A be a countable model of L. Show that there is a sentence φ of $L\omega_1\omega$ such that for all countable models B of L, $B\models\varphi$ iff $A\approx B$.

3. Assume A and B are countable models of a countable L, and let $A\equiv B$ in $L\omega_1\omega$. Show that $A\approx B$.

4. Let A and B be elementary equivalent, k-saturated, infinite models. Show that A and B are elementary equivalent in $L\infty k$.

5. Prove

1° If (A,q^c) is a weak model of axioms for $L(\exists)$, R^A is induced by (A,q^c) and (A,r^c) is induced by (A,R^A), then $(\underline{A},q^c)\equiv(\underline{A},r^c)$.

2° If (A,q^c) is induced by (A,R^A), where R^A is *-regular in (A,R^A), and S^A is induced by (A,q^c), then for all sentences φ of $L_A{}^*$:

$$(\underline{A},R^A)\models\varphi \quad\text{iff}\quad (\underline{A},S^A)\models\varphi.$$

6. Show that Tarski-Vaught lemma holds for weak models of the language $L(Q)$ (which need not satisfy axioms K1, K2, K4, K5): Let (A,q) and (B,r)

be weak models of the language $L(Q)$. Then $(A,q)\prec(B,r)$ iff

1° For any formula $\varphi(x,x_1,\dots,x_n)$ of $L(Q)$ and $a_1,\dots,a_n\in A$, if $(B,r)\models\exists x\varphi[a_1,\dots,a_n]$ then there is $a\in A$ such that $(B,r)\models\varphi[a,a_1,\dots,a_n]$.

2° A set $X\in q$ is definable in (A,q) iff $X=Y\cap A$ for some $Y\in r$, which is definable in (B,r) with "parameters" from A.

If Qx is interpreted as "there exist uncountably many x", instead of clause 2° we have

3° If $X\subseteq B$ is uncountable and definable in (B,r) with "parameters" from A, then $X\cap A$ is uncountable.

7. Let L be a first order language, R a binary relation symbol, and $L(R)=L\cup\{R\}$. Further, let * be the map introduced be Definition 5.3.2. We shall say that a first order theory T of $L(R)$ is regular (*-regular) if S1, S2 and S3 (S1, S2, and S3 restricted to *-transforms of formulas of $L(Q)$) are provable in T. $T(Q)$ denotes a theory in $L(Q)$ obtained from T by expanding all schemes, if there are any, of T to all formulas of $L(Q)$. Finally, let ($\daleth$) denote the following scheme:

($\daleth$) $\quad \daleth\varphi x \leftrightarrow \exists y\forall x(\varphi x\to Rxy)$.

Prove

1° Let $R\in L$, T be a theory of L and assume that the axioms K1-K5 for $L(Q)$ are provable in T by use of axioms of PR^1 and the scheme ($\daleth$). Then T is a regular theory.

2° Let T be a *-regular theory in $L(R)$. Then for each formula φ of $L(Q)$ the following holds:

(1) $T(Q)\vdash\varphi$ implies $T\vdash\varphi^*$.

(2) $T+(\daleth)\vdash$ K1-K5.

3° Let T be a *-regular theory in $L(R)$. If

$T(Q)\vdash_K \forall y\daleth xRxy, \qquad T(Q)\vdash_K \forall x\daleth y\neg Rxy,$

then

(1) $T(Q)\vdash_K (\varphi\leftrightarrow\varphi^*)$, φ is a formula of $L(Q)$.

(2) $T(Q)\vdash_K\varphi$ iff $T\vdash\varphi^*$.

(3) $T(Q)\vdash_K \daleth x\varphi x\leftrightarrow\exists y\forall x(\varphi x\to Rxy)$, y does not occur in φ.

Here $\vdash_K$ denotes the provability relation with addition of Keisler axioms K1-K5.

8. Show

1° There are $2^{\aleph_1}$ ω_1-like nonisomorphic models of PA.

2° There are $2^{\aleph_1}$ ω_1-like nonisomorphic recursively saturated models of PA.

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