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MIROSLAV PAVLOVIĆ

INTRODUCTION TO FUNCTION SPACES ON THE DISK

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BEOGRAD

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To Mirjana and Pavle

Contents

Preface vi

1 Quasi-Banach spaces **1**
- 1.1 Quasinorm and p-norm . . . 1
- 1.2 Linear operators . . . 4
- 1.3 Open mapping, closed graph . . . 6
- 1.4 F-spaces . . . 8
- 1.5 The spaces ℓ^p . . . 10

2 Interpolation and maximal functions **12**
- 2.1 The Riesz/Thorin theorem . . . 12
- 2.2 Weak L^p-spaces and Marcinkiewicz's theorem . . . 16
- 2.3 Maximal function and Lebesgue points . . . 19
- 2.4 The Rademacher functions . . . 22
- 2.5 Nikishin's theorem . . . 24
- 2.6 Nikishin and Stein's theorem . . . 28
- 2.7 Banach's principle . . . 30

3 Poisson integral **32**
- 3.1 Harmonic functions . . . 32
- 3.2 Borel measures and the space h^1 . . . 36
- 3.3 Radial limits of the Poisson integral . . . 39
- 3.4 The spaces h^p and $L^p(\mathbb{T})$ . . . 43
- 3.5 The Littlewood/Paley theorem . . . 45
- 3.6 Harmonic Schwarz lemma . . . 47

4 Subharmonic functions **49**
- 4.1 Basic properties . . . 49
- 4.2 Properties of the mean values . . . 54
- 4.3 Integral means of univalent functions . . . 56
- 4.4 The subordination principle . . . 58
- 4.5 The Riesz measure . . . 61
- 4.6 A Littlewood/Paley theorem . . . 63

5 Classical Hardy spaces **67**
- 5.1 Basic properties . . . 67
- 5.2 The space H^1 . . . 72
- 5.3 Blaschke product . . . 74
- 5.4 Inner and outer functions . . . 79
- 5.5 Composition with inner functions . . . 82

6 Conjugate functions **86**
6.1 Harmonic conjugates 86
6.2 Riesz projection theorem 90
6.3 Applications of the projection theorem 93
6.4 Aleksandrov's theorem 94
6.5 Strong convergence in H^1 95
6.6 Quasiconformal harmonic homeomorphisms 98

7 Maximal functions, interpolation, coefficients **104**
7.1 Maximal theorems 104
7.2 Maximal characterization of H^p 108
7.3 "Smooth" Cesàro means 110
7.4 Interpolation of operators on Hardy spaces 114
7.5 On the Hardy/Littlewood inequality 118
7.6 On the dual of H^1 121

8 Bergman spaces: Atomic decomposition **123**
8.1 Bergman spaces 123
8.2 Reproductive kernels 124
8.3 The Coifman/Rochberg theorem 127
8.4 Coefficients of vector-valued functions 131

9 Subharmonic behavior **137**
9.1 Subharmonic behavior and Bergman spaces 137
9.2 The space h^p, $p < 1$ 141
9.3 Subharmonic behavior of smooth functions 142

10 Lipschitz spaces **148**
10.1 Lipschitz spaces of first order 148
10.2 Lipschitz condition for the modulus 152
10.3 Lipschitz spaces of higher order 154
10.4 Growth of derivatives 156

11 Lacunary series **164**
11.1 Lacunary series in H^p 164
11.2 Karamata's theorem and Littlewood's theorem 166
11.3 Lacunary series in $C[0,1]$ 170
11.4 L^p-integrability of lacunary series on $(0,1)$ 172

BIBLIOGRAPHY **176**

INDEX **181**

Preface

This text contains some facts, ideas, and techniques that can help or motivate the reader to read books and papers on various classes of functions on the disk and the circle. The reader will find several well known, fundamental theorems as well as a number of the author's results, and new proofs or extensions of known results. Most of assertions are proved, although sometimes in a rather concise way. A number of assertions are named by *Exercise*, while certain assertions are collected in *Miscellaneous* or *Remarks*; most of them can be treated by the reader as exercises.

The reader is assumed to have good foundation in Lebesgue integration, complex analysis, functional analysis, and Fourier series, which means in particular that he/she had a good training through these areas. It is of some importance that the reader can accept the following:

Throughout this text, constants are often given without computing their exact values. In the course of a proof, the value of a constant C may change from one occurrence to the next. Thus, the inequality $2C \leqslant C$ is true even if $C > 0$.

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1 Quasi-Banach spaces

In this text we mention only two examples of locally convex spaces: $h(\Omega)$, the space of all complex-valued functions harmonic in $\Omega \subset \mathbb{C}$, and its subspace $H(\Omega)$ consisting of analytic functions. For our purposes, the class of locally bounded spaces is more important. By Kolmogorov's theorem, the intersection of this class with the class of locally convex spaces consists precisely of normable spaces. The topology of a locally bounded space can be described by a "quasinorm"; conversely, a "quasinormable" space is locally bounded.

In the class of quasi-Banach spaces, there hold the "basic principles of functional analysis." A concise discussion of these principles is contained in Section 1.3 and, in the context of F-spaces, in 1.4. Some properties of ℓ^p are stated, without proofs, in Section 1.5.

1.1 Quasinorm and p-norm

Let X be a (complex) vector space. A functional $\|\cdot\|\colon X \mapsto [0,\infty)$ is called a **quasinorm** if the following conditions hold:

$$\|f+g\| \leqslant K(\|f\|+\|g\|), \tag{1.1}$$

where K ($\geqslant 1$) is a constant independent of $f, g \in X$; and

$$\|f\| > 0 \quad (f \neq 0), \qquad \|\lambda f\| = |\lambda|\,\|f\| \quad (\lambda \in \mathbb{C}). \tag{1.2}$$

The couple $(X, \|\cdot\|)$ is then called a **quasinormed space.** The standard example are Lebesgue spaces: if μ is a positive measure defined on a sigma-algebra of subsets of a set S, then the space $L^p(\mu) = L^p(S,\mu) = L^p(S)$ $(0 < p \leqslant \infty)$ consists of all measurable complex-valued functions f on S for which

$$\|f\| = \|f\|_p = \left(\int_S |f|^p\, d\mu\right)^{1/p} < \infty,$$

with the usual interpretation in the case $p = \infty$. When $p < 1$, this functional is not a norm but satisfies (1.1) with $K = 2^{1/p-1}$ and, moreover,

$$\|f+g\|^p \leqslant \|f\|^p + \|g\|^p. \tag{1.3}$$

A functional satisfying (1.3) and (1.2) is called a **p-norm.**

From (1.3) it follows that

$$\|f_1 + f_2 + \cdots + f_n\|^p \leqslant \|f_1\|^p + \|f_2\|^p + \cdots + \|f_n\|^p.$$

A similar inequality holds in the general case although a quasinorm need not be a p-norm for any p.

1.1.1 Lemma *If $\|\cdot\|$ is a quasinorm on X, then there exist constants $p \in (0,1)$ and $C \leqslant 4$ such that*

$$\|f_1 + f_2 + \cdots + f_n\|^p \leqslant C \left(\|f_1\|^p + \|f_2\|^p + \cdots + \|f_n\|^p \right) \tag{1.4}$$

for every finite sequence $f_1, \ldots, f_n \in X$.

From this one can deduce that X is "p-normable" for some $p > 0$.

1.1.2 Theorem (Aoki/Rolewicz) *If $\|\cdot\|$ is a quasi-norm on X, then there is $p > 0$ and a p-norm $|\!|\!|\cdot|\!|\!|$ on X such that $\|f\|/C \leqslant |\!|\!|f|\!|\!| \leqslant \|f\|$, $f \in X$, where C is independent of f.*

The p-norm is defined by

$$|\!|\!|f|\!|\!| = \inf\Big\{\Big(\sum_{j=1}^{n}\|f_j\|^p\Big)^{1/p} : f = \sum_{j=1}^{n} f_j\Big\},$$

where the infimum is taken over all finite sequences $\{f_j\} \subset X$.

Proof of Lemma. Take p so that $(2K)^p = 2$, where K is the constant from (1.1), and define the functional H on X in the following way: $H(0) = 0$ and

$$H(f)^p = 2^k \quad \text{if} \quad 2^{k-1} \leqslant \|f\|^p < 2^k \quad \text{for some integer } k.$$

Since

$$\|f\|^p \leqslant H(f)^p \leqslant 2\|f\|^p, \tag{1.5}$$

inequality (1.4) is a consequence of the inequality

$$\|f_1 + \cdots + f_n\|^p \leqslant 2 \left(H(f_1)^p + \cdots + H(f_n)^p \right).$$

The latter holds for $n = 1$. If $n \geqslant 2$, we consider two cases.

(i) Let the summands $H(f_j)$ be mutually distinct and arranged in decreasing order. Then we have

$$H(f_j)^p \leqslant 2^{1-j} H(f_1)^p \qquad (1 \leqslant j \leqslant n). \tag{1.6}$$

From (1.1) it follows that $\|f + g\| \leqslant 2K \max\{\|f\|, \|g\|\}$, whence, by (1.5),

$$\|f_1 + \cdots + f_n\| \leqslant \max\{(2K)^j H(f_j) \,:\, 1 \leqslant j \leqslant n\}.$$

Because of (1.6) and the choice of p, it turns out that $\|f_1 + \cdots + f_n\|^p \leqslant 2H(f_1)^p$, which implies the required inequality.

(ii) Assume that the sequence $H(f_j)$ contains at least two equal elements; for example, let $H(f_1) = H(f_2) = 2^m$. Then $2^{m-1} \leqslant \|f\|_1^p, \|f\|_2^p < 2^m$. Since

$$\|f_1 + f_2\|^p \leqslant (2K)^p \max\{\|f_1\|^p, \|f_2\|^p\} = 2\max\{\|f_1\|^p, \|f_2\|^p\} \leqslant 2^{m+1},$$

we have $H(f_1+f_2)^p \leqslant 2^{m+1} = H(f_1)^p+H(f_2)^p$. This and the induction hypothesis imply

$$\begin{aligned}\|(f_1 + f_2) + \cdots + f_n\|^p &\leqslant 2\ (\ H(f_1 + f_2)^p + \cdots + H(f_n)^p\ ) \\ &\leqslant 2\ (\ H(f_1)^p + H(f_2)^p + \cdots + H(f_n)^p\ ) \\ &\leqslant 2\ (\ \|f_1\|^p + \|f_2\|^p + \cdots + \|f_n\|^p\ )\ . \qquad \square\end{aligned}$$

The space X is endowed with the structure of a topological vector space by declaring "a neighborhood of zero" to mean "a set containing $\{f : \|f\| < 1/n\}$ for some $n = 1, 2, \ldots$."(*) This topology is metrizable, according to the Aoki/Rolewicz theorem; namely, if a p-norm $⦀\cdot⦀$ is equivalent to the original quasinorm, then the formula $d(f,g) = ⦀f - g⦀^p$ defines a metric that induces the same topology.

The space X need not be locally convex(†); it is locally bounded because the neighborhoods $\{f : \|f\| < 1/n\}$ are bounded in the sense of theory of topological vector spaces. On the other hand, it is known that a locally bounded vector topology can be described by a quasinorm (cf. [87]).

1.1.3 Exercise On the space $L^p(0,1)$ $(0 < p < 1)$, there is not an equivalent q-norm for $1 \geqslant q > p$. The same holds for the sequence space ℓ^p.

Quasi-Banach and p-Banach spaces A quasinormed space X is called a quasi-Banach space if it is complete, which means that a sequence $\{f_n\} \subset X$ is convergent if (and only if) $\|f_m - f_n\| \to 0$ as $m, n \to \infty$. If X is p-normed and complete, then X is said to be p-Banach.

1.1.4 Proposition *Let X be p-normed. Then X is complete iff convergence of the series $\sum \|f_n\|^p$ implies convergence of $\sum f_n$. If X is complete and $\sum f_n$ converges, then there holds the inequality $\left\| \sum_{n=1}^{\infty} f_n \right\|^p \leqslant \sum_{n=1}^{\infty} \|f_n\|^p$.*

1.1.5 Proposition *Let $\{f_{jk}\}$ $(j, k \geqslant 1)$ be a double sequence in a p-Banach space X. If $\sum_{j,k} \|f_{jk}\|^p < \infty$, then the iterated series $\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty} f_{jk}\right)$ and $\sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty} f_{jk}\right)$ converge and have the same sum.*

1.1.6 Exercise (Peck [81]) Let $(X, \|\cdot\|)$ be a complex p-normed space of dimension $n < \infty$. By a theorem of Carathéodory, every point from the convex hull of the unit ball can be represented as a convex combination of $2n$ points from the ball (because the real dimension is $2n$). This can be used to show that there exists a *norm* $\|\cdot\|_n$ on X such that $\|f\|_n \leqslant \|f\| \leqslant (2n)^{1/p-1}\|f\|_n$. Note that $(2n)^{1/p-1}$ is not the best constant, at least for $n = 1$.

(*)The "ball" $\{f : \|f\| < 1\}$ need not be an open set. Therefore a quasinorm, in contrast to a p-norm, need not be continuous.

(†)For example, the space $L^p(0,1)$, $0 < p < 1$, is not locally convex.

1.2 Linear operators

In the class of quasinormed spaces, continuity and boundedness of linear operators are equivalent. In the space $L(X,Y)$, of continuous linear operators from X to Y, the quasinorm is defined by $\|T\| := \sup_{\|f\|\leqslant 1} \|Tf\|$.

The space $L(X,Y)$ is complete iff so is Y.

An operator $T \in L(X,Y)$ is said to be **invertible** if it is bijective and its inverse is *continuous*.

1.2.1 Proposition *Let X be a quasi-Banach space and $T \in L(X,X)$ such an operator that $\|I-T\| < 1$, where I is the identity operator. Then T is invertible and there holds the inequality $\|T^{-1}\|^p \leqslant C\big(1-\|I-T\|^p\big)^{-1}$, where C and p are the constants from Lemma* 1.1.1.

Proof. Consider the series $\sum_{k=0}^{\infty}(I-T)^k$. From inequality (1.4), applied to the space $L(X,X)$, we get

$$\left\|\sum_{k=m}^{n}(I-T)^k\right\|^p \leqslant C\sum_{k=m}^{n}\|I-T\|^{pk}.$$

Therefore the series converges; denote its sum by S. Then we have $ST = TS = I$ and $\|S\|^p \leqslant C\sum_{k=0}^{\infty}\|I-T\|^{pk}$, which was to be proved. □

The following statement is important although its proof is very simple.

1.2.2 Theorem *Let X and Y be quasi-Banach spaces and E a dense subset of X. Let $T_n \in L(X,Y)$ be a sequence such that $\sup_n \|T_n\| < \infty$. If the limit $\lim_{n\to\infty} T_n f$ exists for all $f \in E$, then it exists for all $f \in X$ and the operator $Tf := \lim_{n\to\infty} T_n f$ is linear and continuous.*

1.2.3 Exercise Let T be a continuous linear operator from a quasi-normed space X to quasi-normed space Y, and let E be a subset of X such that the linear hull of E is dense in X. If Y_0 is a closed subspace of Y such that $T(E) \subset Y_0$, then $T(X) \subset Y_0$.

q-Banach envelope

In the general case, a quasi-Banach space is embedded into many q-Banach spaces; the "smallest" of them is called the q-Banach envelope of X. To be more precise, define the functional N_q $(0 < q \leqslant 1)$ on X in the following way:

$$N_q(f) = \inf\left\{\left(\sum_j \|f_j\|^q\right)^{1/q} : \sum_j f_j = f\right\}, \tag{1.7}$$

where the infimum is taken over the set of finite sequences $\{f_j\} \subset X$. This functional is a " q-seminorm", i.e., satisfies the conditions

$$\{N_q(f+g)\}^q \leqslant \{N_q(f)\}^q + \{N_q(g)\}^q, \qquad N_q(\lambda f) = |\lambda|\, N_q(f).$$

The set $\{f \in X : N_q(f) = 0\} =: \operatorname{Ker} N_q$ is a closed subspace of X. If $\operatorname{Ker} N_q = \{0\}$, i.e., if N_q is a q-norm, then the "completion" of the space (X, N_q) is a q-Banach space and is called the **q-Banach envelope** of X; denote it by $[X]_q$. According to the Aoki/Rolewicz theorem, always there exists a q such that $X = [X]_q$, with equivalent quasinorms. A simple but illustrative example is $X = \ell^p$; then $[X]_q = \ell^q$ $(p < q \leqslant 1)$ and the corresponding quasinorms are equal (see 1.2.8 and 1.2.6). It is much more difficult to identify the envelops of the Hardy space H^p (see Theorem 8.3.5).

The importance of the space $[X]_q$ lies in the fact that every operator from X to an arbitrary q-Banach space extends to an operator on $[X]_q$; more precisely:

1.2.4 Proposition *Let X possess the q-Banach envelope (i.e., let N_q be a q-norm) and let Y be an arbitrary q-Banach space. If $T \in L(X,Y)$, then there exists a unique operator $S \in L([X]_q, Y)$ such that $Sf = Tf$ for all $f \in X$.*

The following fact is useful in identifying the envelope:

1.2.5 Proposition *Let X be continuously embedded into a q-Banach space Y in such a way that every $f \in Y$ can be represented as $f = \sum_{n=1}^{\infty} f_n$, $f_n \in X$, with $\sum_{n=1}^{\infty} \|f_n\|_X^q \leqslant C\|f\|_Y^q$, where C does not depend of f. Then $Y = [X]_q$ (with equivalent quasinorms).*

Proof. The space X is a dense subset of Y. Since X is dense in $[X]_q$, we see that it suffices to prove that the q-norms $\|\cdot\|_Y$ and N_q are equivalent on X.

Let $f = \sum f_j$, where $\{f_j\}$ is a finite sequence in X. Then

$$\|f\|_Y^q \leqslant \sum \|f_j\|_Y^q \leqslant C^q \sum \|f_j\|_X^q.$$

Taking the infimum over $\{f_j\} \subset X$, we get $\|f\|_Y \leqslant CN_q(f)$. (Incidentally this shows that N_q is a q-norm.)

To prove the reverse inequality, let $f \in X$. Then $f = \sum_{n=1}^{\infty} f_n$, where $\sum_{n=1}^{\infty} \|f_n\|_X^q \leqslant C\|f\|_Y^q$. Since $N_q(f_n) \leqslant \|f_n\|_X$, we get $\sum_{n=1}^{\infty} N_q(f_n)^q \leqslant C\|f\|_Y^q$. Hence $\sum f_n$ converges to f in $[X]_q$ to f, and $N_q(f)^q \leqslant \sum N_q(f_n)^q \leqslant C\|f\|_Y^q$, which completes the proof. □

Miscellaneous

1.2.6 The functional N_q is a q-norm on X iff there is a q-Banach space Y such that $L(X,Y)$ *separates points* in X. The latter means that for every $f \neq 0$ there is $T \in L(X,Y)$ with $Tf \neq 0$.

1.2.7 The dual of a quasi-Banach space X is $X^* = L(X, \mathbb{C})$. If X^* separates points in X, then the Banach envelope of X is equal to the completion of the normed space (X, N), where $N(f) = \sup\{|\Lambda f| : \Lambda \in X^*,\ \|\Lambda\| \leqslant 1\}$.

1.2.8 If $X = L^p(0,1)$, $0 < p < 1$ and $1 \geqslant q > p$, then $N_q(f) = 0$ for all $f \in X$. This is connected with the relation $L(X,Y) = \{0\}$, where Y is an arbitrary q-Banach space.

1.3 Open mapping, closed graph

Let X, Y be a pair of complete spaces such that X is a dense subset of Y, which means that each member of Y can be approximated by members of X. This does not imply that members of a ball $K_1 \subset Y$ can be approximated by members of any fixed ball $K_2 \subset X$, i.e., that $\overline{K}_2 \supset K_1$. ($\overline{K}_2 =$ the closure of K_2 in the topology of Y.) Namely, as the following theorem states, if $\overline{K}_2 \supset K_1$, then $X = Y$.

1.3.1 Theorem *Let X and Y be quasi-Banach spaces. Let $T \in L(X,Y)$ be such that the closure of $T(B)$, where $B = \{f \in X : \|f\| < 1\}$, contains a neighborhood of zero in Y. Then the mapping $T\colon X \mapsto Y$ is open and the operator $\widehat{T} : X/\operatorname{Ker} T \mapsto Y$ is invertible.*

A mapping is open if it maps open sets onto open sets. If $T \in L(X,Y)$, then the operator $\widehat{T} \in L(X/\operatorname{Ker} T, Y)$ is defined by $\widehat{T}(f + \operatorname{Ker} T) = Tf$. The quasinorm in X/Z is defined by $\|f + Z\| = \inf\{\|f - g\| : g \in Z\}$.

Proof. Because of the Aoki/Rolewicz theorem, we can suppose that X and Y are p-normed for some $p < 1$. Let $\delta > 0$ and $U = \{f \in X : \|f\|^p < \delta\}$. From the hypotheses of the theorem it follows that there are balls $U_n = \{f \in X : \|f\|^p < \delta_n\}$ and $V_n = \{g \in Y : \|g\|^p < \varepsilon_n\}$, $n \geqslant 1$, $\lim_n \varepsilon_n = 0$, such that

$$V_n \subset \overline{T(U_n)}, \tag{1.8}$$

$$\sum_{n=1}^{\infty} \delta_n < \delta. \tag{1.9}$$

We will prove that $V_1 \subset T(U)$; then it will be easy to complete the proof.

Let $g \in V_1$. It follows from (1.8) that there exists $f_1 \in U_1$ such that $g - Tf_1 \in V_2$. Similarly, there is $f_2 \in U_2$ such that $(g - Tf_1) - Tf_2 \in V_3$. Continuing in this way, we get the sequence of relations $g - \sum_{k=1}^{n} Tf_k \in V_{n+1}$, $f_k \in U_k$. It follows that $g = \sum_{n=1}^{\infty} Tf_n$. And since $\|f_k\|^p < \delta_k$, inequality (1.9) implies that the series $\sum_k f_k$ converges; denote its sum by f. Thus we have $g = Tf$ and $\|f\|^p \leqslant \sum_{n=1}^{\infty} \|f_n\|^p < \delta$, which was to be proved. □

The open mapping theorem

1.3.2 Theorem *Let X and Y be complete spaces and $T \in L(X,Y)$. If T is onto, then T is open. In particular, T is invertible if it is onto and one-to-one.*

Proof. Let U be the unit ball in X. Choose a zero-neighborhood W so that $W - W \subset U$. By the hypothesis, the space Y is the union of the sets $T(nW)$

$(n \geqslant 1)$. By Baire's category theorem, the closure at least of one of them has nonempty interior, which implies that $\overline{T(W)}$ contains an open set $V \neq \emptyset$. Then $V - V$ is a neighborhood of zero and there hold the inclusions

$$V - V \subset \overline{T(W)} - \overline{T(W)} \subset \overline{T(W) - T(W)} \subset \overline{T(U)}.$$

Now the desired result follows from Theorem 1.3.1. □

1.3.3 Exercise A subspace E of a quasi-Banach space X is said to have the Hahn/Banach extension property (HBEP) if each $\lambda \in E^*$ has an extension $\Lambda \in X^*$. If E has HBEP, then Λ can be chosen so that $\|\Lambda\|_{X^*} \leqslant C\|\lambda\|_{E^*}$, where C is independent of λ.

As a special case of the open mapping theorem we have:

1.3.4 Theorem (on equivalent norms) *Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be quasinorms on a a vector space X, and let $\|f\|_1 \leqslant \|f\|_2$ for every $f \in X$. If X is complete with respect to both quasinorms, then there exists a constant $C < \infty$ such that $\|f\|_2 \leqslant C\|f\|_1$ for all $f \in X$.*

The uniform boundedness principle

1.3.5 Theorem (Banach/Steinhauss) *Let X and Y be quasi-Banach spaces, and let $\{A_s\} \subset L(X, Y)$ be a family of operators. If $\sup_s \|A_s f\| < \infty$, for all $f \in X$, then $\sup_s \|A_s\| < \infty$. In particular the limit of an everywhere convergent sequence of bounded operators is a bounded operator.*

Proof. Let $\|f\|_2 = \|f\|_X + \sup_s \|A_s f\|_Y$ $(f \in X)$. From the hypotheses it follows that the functional $\|\cdot\|_2$ is a quasinorm on X. It is not hard to prove that the space $(X, \|\cdot\|_2)$ is complete and therefore the conclusion follows from Theorem 1.3.4. □

1.3.6 Corollary *Let $B : X \times Y \mapsto Z$ be a separately continuous bilinear operator, where X, Y, Z are quasi-Banach spaces. Then there is a constant $C < \infty$ such that $\|B(f,g)\|_Z \leqslant C\|f\|_X\|g\|_Y$ for all $f \in X$, $g \in Y$.*

"Separately continuous" means that every operator of the form $f \mapsto B(f,g)$ $(f \in X)$ or $g \mapsto B(f,g)$ $(g \in Y)$ is continuous.

Shauder basis

A sequence $\{e_n : n \geqslant 1\}$ in a quasi-Banach space X is called a Shauder basis of X if to each $f \in X$ there corresponds a unique scalar sequence $\{\lambda_n(f)\}$ such that $f = \sum_{n=1}^{\infty} \lambda_n(f) e_n$, the series converging in the topology of X.

1.3.7 Proposition *If $\{e_n : n \geqslant 1\}$ is a Shauder basis of X, then the functionals λ_n are continuous and the linear operators $S_n : X \mapsto X$ defined by $S_n f = \sum_{k=1}^{n} \lambda_k(f) e_k$ are uniformly bounded.*

Proof. Let $|||f||| = \sup_n \|S_n f\|$. Since $\|f - S_n f\| \to 0$, we have $\|f\| \leqslant K|||f|||$, where K is the constant from (1.1), and therefore, by Theorem 1.3.4, it is enough to prove that X is complete with respect to the quasinorm $|||\cdot|||$. Let $\{f_j\}_{j=1}^{\infty}$ be a Cauchy sequence in $|||\cdot|||$. This implies, because of the completeness of $\|\cdot\|$, that there is a sequence g_n such that

$$\sup_{n\geqslant 1} \|S_n f_j - g_n\| \to 0 \quad \text{as } j \to \infty, \tag{1.10}$$

and that for every k the sequence $\{\lambda_k(f_j)\}_{j=1}^{\infty}$ converges; let $\gamma_k = \lim_j \lambda_k(f_j)$. Since the functional λ_k is linear and the space $S_n(X)$ is finite-dimensional, it follows that $\lambda_k(g_n) = \lim_j \lambda_k(S_n f_j) = \lim_j \lambda_k(f_j) = \gamma_k$ for $k \leqslant n$, and $\lambda_k(g_n) = 0$ for $k > n$. Hence $g_n = \sum_{k=1}^{n} \gamma_k e_k$. On the other hand, (1.10) implies that $\{g_n\}$ converges in $\|\cdot\|$ to some g. Thus $g = \sum_{n=1}^{\infty} \gamma_n e_n$, whence $g_n = S_n g$. Returning to (1.10) we see that $|||f_j - g||| \to 0$ as $j \to \infty$, which was to be proved. □

1.3.8 Exercise A sequence $\{e_n : n \geqslant 1\}$ of nonzero vectors in a quasi-Banach space X is a Shauder basis of X if and only if the following conditions are satisfied: (a) The closed linear span of $\{e_n\}$ is X; (b) There is a constant K such that $\|\sum_{j=0}^{m} a_j e_j\| \leqslant K \|\sum_{j=0}^{n} a_j e_j\|$ for all scalar sequences $\{a_j\}$ and $m < n$.

The closed graph theorem

1.3.9 Theorem *Let $T : X \mapsto Y$ be a linear operator, where X and Y are complete spaces. Then T is continuous if the following condition is satisfied: For every sequence $\{f_n\} \subset X$ such that f_n tends to $0 \in X$ and Tf_n tends to some $g \in Y$ we have $g = 0$.*

Proof. It follows from the hypotheses that X is complete with respect to the quasinorm $\|f\|_2 = \|f\|_X + \|Tf\|_Y$ so we can apply Theorem 1.3.4. □

1.4 F-spaces

The closed graph theorem remains valid in a wider class of spaces, the so called F-spaces. By the term "F-norm" on a vector space X we mean a functional $N : X \mapsto [0, \infty)$ satisfying: (a) $N(f) = 0 \Longrightarrow f = 0$; (b) $N(f+g) \leqslant N(f) + N(g)$; (c) $N(\lambda f) \leqslant N(f)$ for $|\lambda| \leqslant 1$, and

$$\lim_{\lambda \to 0} N(\lambda f) = 0. \tag{1.11}$$

The formula $d(f,g) = N(f - g)$ defines an invariant metric on X and the topology induced by this metric is vectorial, which means in particular that multiplication by scalars is continuous on $\mathbb{C} \times X$. In the case where the metric d is complete, the space X is called an **F-space.**

A p-Banach space can be treated as an F-space by introducing the F-norm $N(f) = \|f\|_X^p$.

Besides, if X is a locally convex space whose topology is given by a sequence of seminorms p_n $(n = 1, 2, \dots)$, then the formula

$$N(f) = \sum_{n=1}^{\infty} \frac{2^{-n} p_n(f)}{1 + p_n(f)}$$

defines an F-norm on X that induces the same topology. As an example one can take the space $h(\mathbb{D})$ consisting of all harmonic functions on the unit disk $\mathbb{D} \subset \mathbb{C}$ as well as its analytic analogue $H(\mathbb{D})$. These spaces are endowed with the topology of uniform convergence on compact subsets of $\mathbb{D}$. This topology can be given by the sequence of norms $p_n(f) = \max_{|z| \leqslant r_n} |f(z)|$, where $r_n \in (0, 1)$ is an arbitrary sequence tending to 1.

Concerning the requirement (1.11), which guarantees the continuity of scalar multiplication, it is useful to consider the case of the Nevanlinna class $\mathcal{N}(\mathbb{D})$. This class consists of the functions $f \in H(\mathbb{D})$ for which

$$N(f) := \sup_{0<r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \log\left(1 + |f(re^{i\theta})|\right) d\theta < \infty. \tag{1.12}$$

The functional N induces a complete invariant metric on the vector space $\mathcal{N}(\mathbb{D})$ but (1.11) is not satisfied. For example,

$$\text{if} \quad f(z) = \exp \frac{1+z}{1-z}, \quad \text{then} \quad \lim_{\varepsilon \to 0} N(\varepsilon f) = 1.$$

The set on which (1.11) holds coincides with *the Smirnov class* $\mathcal{N}^+(\mathbb{D})$ consisting of those $f \in \mathcal{N}(\mathbb{D})$ for which the family $\theta \mapsto \log\left(1 + |f(re^{i\theta})|\right)$, $0 < r < 1$, is uniformly integrable. This and other topological properties of the Nevanlinna class are discussed in [90]. The Nevanlinna theory is exposed in, e.g., [100, 18, 22].

In the class of F-spaces, there holds the open mapping theorem 1.3.2 as well (the formulation is the same). Theorem 1.3.1 is now stated as follows:

Let X and Y be F-spaces and $T \in L(X, Y)$. If for each neighborhood U of $0 \in X$ the set $\overline{T(U)}$ contains a neighborhood of $0 \in Y$, then $T(X) = Y$ and T is open.

The proof is identical to the proof of Theorem 1.3.1 up to the obvious changes of notation. The property (1.11) is used only in proving that $T(X) = Y$.

As a special case of the open mapping theorem, we have the following generalization of Theorem 1.3.4, from which the closed graph theorem is obtained immediately.

1.4.1 Theorem *Let N_1 and N_2 be F-norms on a vector space X, and let $N_1(f) \leqslant N_2(f)$ for all $f \in X$. If X is complete with respect to both of them, then every sequence $\{f_n\} \subset X$ satisfies the condition: $N_1(f_n) \to 0 \Longrightarrow N_2(f_n) \to 0$.*

1.4.2 Exercise Let X and Y be quasi-Banach spaces "continuously" contained in $H(\mathbb{D})$. A scalar sequence $\mu = \{\mu_n\}_0^\infty$ is called a **multiplier** from X to Y if for every $f \in X$ the series $(\mu * f)(z) = \sum_{n=0}^{\infty} \mu_n \widehat{f}(n) z^n$ converges in $\mathbb{D}$ and $\mu * f$ belongs to Y. If μ is a multiplier, then there is a constant $C < \infty$ such that $\|\mu * f\|_Y \leqslant C\|f\|_X$ for all $f \in X$. In particular if $Y = X$ and X contains all the polynomials, then the sequence μ is bounded.

1.4.3 Exercise Let $T_s \colon X \mapsto Y$ be a family of operators between F-spaces X and Y. If $\sup_s N_Y(T_s f/n) \to 0$, as $n \to \infty$, for every $f \in X$, then the following holds: If $N_X(f_n) \to 0$, where $f_n \in X$, then $\sup_s N_Y(T_s f_n) \to 0$.

1.5 The spaces ℓ^p

The simplest examples of infinite-dimensional (quasi-)Banach spaces are ℓ^p $(0 < p \leqslant \infty)$ and $\mathfrak{c}_0 = \{a \in \ell^\infty \colon \lim a_n = 0\}$. They play an exceptional role in many areas of analysis and especially in geometry of Banach spaces; we refer the reader to [53]. Their importance for the theory of spaces of analytic functions lies in the following fact:

1.5.1 Theorem *For every* $p \in (0, \infty)$ *the Bergman space* A^p,

$$A^p = \{f \in L^p(\mathbb{D}) \colon f \text{ analytic in } \mathbb{D}\}, \qquad \mathbb{D} = \{z \colon |z| < 1\},$$

is isomorphic with ℓ^p.

In the case $1 < p < \infty$, this was proved by Lindenstrauss and Pełczyński [50] by using Theorem 1.5.5(b) below; an explicit isomorphism was constructed in [59] $(p > 1)$ and [97] $(p \leqslant 1)$. The case of mixed norm spaces was considered in [97, 59].

Here we list a few properties of ℓ^p and $\mathfrak{c}_0$. A slight modification of the proof of Theorem 1.3.1 yields the following:

1.5.2 Theorem *If* X *and* Y *are* p*-Banach spaces and* $T \in L(X, Y)$ *is such that* $\|T\| \leqslant 1$ *and* $\overline{T(B_X)} \supset B_Y$*, where* B *indicates the open unit ball, then the spaces* Y *and* $X/\operatorname{Ker} T$ *are isometrically isomorphic.*

In particular:

1.5.3 Theorem *Every separable* p*-Banach space* $(0 < p \leqslant 1)$ *is isometrically isomorphic to some quotient space of* ℓ^p.

Proof. Define the operator $T \colon \ell^p \mapsto X$, by $T(\{a_n\}_1^\infty) = \sum_{n=1}^{\infty} a_n f_n$, where $\{f_n\}$ is a dense subset of the unit ball of X. □

1.5.4 Exercise Let Y be a subspace of ℓ^p such that $L^p(0,1)$ is isometric to ℓ^p/Y. If a functional $\Lambda \in (\ell^p)^* = \ell^\infty$ vanishes on Y, then it vanishes everywhere on ℓ^p.

The proof of the following is much more delicate (cf. [35, 52]).

1.5.5 Theorem *Let X be either c_0 or ℓ^p, $0 < p \leqslant \infty$. Then:* (a) *Every closed subspace (of infinite dimension) of X contains an isomorphic copy of X. And:*

(b) *Every complemented subspace of X is isomorphic to X.*

In the case $1 \leqslant p < \infty$ both assertions were proved Pełczyński [82], while the case $p < 1$ was discussed by Stiles [96, 95]. Assertion (b) for $p = \infty$ was proved by Lindenstrauss [49].

The fact that the spaces ℓ^p and ℓ^q $(p \neq q)$ are not isomorphic is contained in the following assertion of Pitt ($p \geqslant 1$, cf. [52, Theorem I.2.7]) and Stiles [96] ($p < 1$, cf. [35, Proposition 2.9]):

1.5.6 Theorem *Every bounded linear operator from ℓ^q to ℓ^p $(0 < p < q < \infty)$ is compact; the same is true for linear operators from c_0 to ℓ^p. (Moreover, if $p < q \leqslant 1$, then every operator from a q-Banach space to ℓ^p $(p < q)$ is compact.) Consequently, no space of the class ℓ^p, $0 < p < \infty$, and c_0 is isomorphic to a subspace of another member of this class.*

On the other hand, if $0 < q < p \leqslant 2$, then $L^p(0,1)$ is isometrically isomorphic to a subspace of $L^q(0,1)$ (see [52, Theorem II.3.4]).

2 Lebesgue spaces: Interpolation and maximal functions

Thorin's proof of two variants of the Riesz/Thorin theorem is in Section 2.1; as an example, we prove the Hausdorff/Young theorem. The simplest version of Marcinkiewicz's interpolation theorem is proved in Section 2.2; as an example, we prove Paley's theorem on Fourier coefficients (Theorem 2.2.5). Section 2.3 contains the Hardy/Littlewood maximal theorem with application to Lebesgue points. In Section 2.4 we prove Khintchine's inequality, which says that the subspace of $L^p(0,1)$, $0 < p < \infty$, spanned by the Rademacher functions is isomorphic with ℓ^2. The rest of this chapter is devoted to the proof of Nikishin's theorem. This theorem states, in particular, that if a bounded linear operator T maps $L^p(\mathbb{T})$ into $L^q(\mathbb{T})$, where $0 < q < p \leqslant 2$, then actually T maps $L^p(\mathbb{T})$ into the weak Lebesgue space $L^{p,\infty}(\mathbb{T} \setminus A)$, where $\mathbb{T} \setminus A$ is of arbitrarily small measure (see Theorem 2.5.2). If in addition T "commutes with rotations", then we can take $A = \emptyset$. Also, we prove the so called Banach's principle and the theorem on a.e. convergence (Theorems 2.7.1 and 2.7.2).

2.1 The Riesz/Thorin theorem

The proof of various variants of the Riesz/Thorin (convexity) theorem can be found, e.g., in the books [7, Ch. IV §2] and [8].

The case of bilinear forms

Let $\gamma = (\gamma_1, \ldots, \gamma_m)$ and $\delta = (\delta_1, \ldots, \delta_n)$ be sequences of positive real numbers. For $a \in \mathbb{C}^m$, $b \in \mathbb{C}^n$ and $p, q > 0$ let

$$\|a\|_{\gamma,p} = \left(\sum_{j=1}^{m} |a_j|^p \gamma_j\right)^{1/p}, \quad \|b\|_{\delta,q} = \left(\sum_{k=1}^{n} |b_k|^p \delta_k\right)^{1/q}.$$

2.1.1 Theorem *Let $0 < p_0, q_0, p_1, q_1 \leqslant \infty$. Let*

$$B(a,b) = \sum_{j,k=1}^{m,n} B_{jk} a_j b_k, \qquad a \in \mathbb{C}^m,\, b \in \mathbb{C}^n,$$

where $B_{jk} \in \mathbb{C}$, and suppose that

$$|B(a,b)| \leqslant M_0 \|a\|_{\gamma,p_0} \|b\|_{\delta,q_0}, \quad |B(a,b)| \leqslant M_1 \|a\|_{\gamma,p_1} \|b\|_{\delta,q_1}$$

for all $a \in \mathbb{C}^m$, $b \in \mathbb{C}^n$. If

$$\frac{1}{p} = \frac{1-\eta}{p_0} + \frac{\eta}{p_1}, \quad \frac{1}{q} = \frac{1-\eta}{q_0} + \frac{\eta}{q_1} \quad \text{and} \quad 0 < \eta < 1,$$

then

$$|B(a,b)| \leqslant M_0^{1-\eta} M_1^{\eta} \|a\|_{\gamma,p} \|b\|_{\delta,q}.$$

In other words, if

$$M(\alpha,\beta) = \sup\{\, |B(\alpha,\beta)| : \|a\|_{\gamma,1/\alpha} \leqslant 1, \|b\|_{\delta,1/\beta} \leqslant 1\},$$

then the function $\log M(\alpha,\beta)$ is convex in the quadrant $\alpha \geqslant 0, \beta \geqslant 0$.

Proof. (Throughout the proof we omit the indices γ, δ.) Let $p, q < \infty$, and

$$\|a\|_p = 1 \quad \text{and} \quad \|b\|_q = 1. \tag{2.1}$$

Define $a(z) \in \mathbb{C}^m$ and $b(z) \in \mathbb{C}^n$ by

$$a(z)_j = |a_j|^{p/p(z)} e^{i \arg a_j} \quad \text{and} \quad b(z)_k = |b_k|^{q/q(z)} e^{i \arg b_k},$$

where

$$\frac{1}{p(z)} = \frac{1-z}{p_0} + \frac{z}{p_1} \quad \text{and} \quad \frac{1}{q(z)} = \frac{1-z}{q_0} + \frac{z}{q_1}.$$

The function $F(z) := M_0^{z-1} M_1^{-z} B(a(z), b(z))$ is entire, as a sum of exponential functions. We consider the restriction of F to the strip $\Pi = \{z : 0 \leqslant \operatorname{Re} z \leqslant 1\}$. We have, for $t \in \mathbb{R}$,

$$|a(it)_j| = |a_j|^{p/p_0}, \qquad |b(it)_k| = |b_k|^{q/q_0},$$
$$|a(1+it)_j| = |a_j|^{p/p_1}, \qquad |b(1+it)_k| = |b_k|^{q/q_1},$$

whence, in view of (2.1),

$$\|a(it)\|_{p_0} = 1, \qquad \|b(it)\|_{q_0} = 1,$$
$$\|a(1+it)\|_{p_1} = 1, \qquad \|b(1+it)\|_{q_1} = 1.$$

It follows that $|F(it)| = M_0^{-1} |B(a(it), b(it))| \leqslant 1$ and similarly $|F(1+it)| \leqslant 1$ for every $t \in \mathbb{R}$. Thus the function F is analytic and *bounded* on Π and $|F| \leqslant 1$ on the boundary of Π. It follows that $|F| \leqslant 1$ on Π and in particular $|B(a,b)| \leqslant M_0^{1-\eta} M_1^{\eta}$, which completes the proof in the case $p, q < \infty$. The remaining case is similar. □

The case of linear operators

2.1.2 Theorem *Let (R,μ) and (S,ν) be sigma-finite measure spaces and let $1 \leqslant p_0, q_0, p_1, q_1 \leqslant \infty$. Let T be a (complex-)linear operator defined on μ-simple functions on R and taking values in the set of all complex ν-measurable functions, and let $\|Tf\|_{q_0} \leqslant M_0\|f\|_{p_0}$ and $\|Tf\|_{q_1} \leqslant M_1\|f\|_{p_1}$ for all μ-simple functions f on R. If*

$$\frac{1}{p} = \frac{1-\eta}{p_0} + \frac{\eta}{p_1}, \quad \frac{1}{q} = \frac{1-\eta}{q_0} + \frac{\eta}{q_1} \quad \text{and} \quad 0 < \eta < 1,$$

then $\|Tf\|_q \leqslant M_0^{1-\eta} M_1^{\eta}\|f\|_p$ for all μ-simple functions f.

Remark. Concerning the validity of this theorem in the entire first quadrant, that is, for $0 < p_k, q_k \leqslant \infty$, see [7, page 281].

Proof of Theorem. We consider the bilinear form $B(f,g) = \int_S (Tf)g\,d\nu$, where f, g are simple functions on R, S, respectively, and use the formula

$$\|Tf\|_q = \sup\{\,|B(f,g)| : \|g\|_{q'} = 1,\ g \text{ simple}\},$$

where $1/q' + 1/q = 1$. Let $f = \sum_{j=1}^m a_j K_j$, $g = \sum_{k=1}^n b_k H_k$, where K_j and H_k are sequences of "pairwise disjoint" characteristic functions. Then straightforward calculation shows that we can apply Theorem 2.1.1 with the indices p_0, q_0', p_1, q_1',

$$B_{j,k} = \int_S (TK_j)H_k\,d\nu, \quad \gamma_j = \int_R K_j\,d\mu, \quad \delta_k = \int_S H_k\,d\nu.$$

The details are left to the reader. □

The following form of the preceding theorem is perhaps more convenient in application.

2.1.3 Theorem *Let (R,μ) and (S,ν) be sigma-finite measure spaces and let $1 \leqslant p_0, q_0, p_1, q_1 \leqslant \infty$. Let T be a linear operator defined on the complex space $L^{p_0}(R,\mu) + L^{p_1}(R,\mu)$ and taking values in the set of all complex ν-measurable functions, and let $\|Tf\|_{q_0} \leqslant M_0\|f\|_{p_0}$, $\|Tf\|_{q_1} \leqslant M_1\|f\|_{p_1}$ for all $f \in L^{p_0}$, $f \in L^{p_1}$, respectively. If*

$$\frac{1}{p} = \frac{1-\eta}{p_0} + \frac{\eta}{p_1}, \quad \frac{1}{q} = \frac{1-\eta}{q_0} + \frac{\eta}{q_1} \quad \text{and} \quad 0 < \eta < 1,$$

then T is a bounded operator from $L^p(R,\mu)$ into $L^q(S,\nu)$ and

$$\|Tf\|_q \leqslant M_0^{1-\eta} M_1^{\eta}\|f\|_p.$$

Proof. We shall consider the case where the measure μ is finite and $p_0 < p_1 \leqslant \infty$. Then $L^p(\mu) \subset L^{p_0}(\mu)$. With the above notation, let $g \in L^{q'}(\nu)$ be a simple function and let $f \in L^p(\mu)$ be arbitrary. Choose a sequence f_n of simple functions on R such that $\|f_n - f\|_p \to 0$. Then $\|f_n - f\|_{p_0} \to 0$ and therefore

$$\int_S T(f_n - f)g\,d\nu \to 0,$$

because $T\colon L^{p_0} \mapsto L^{q_0}$ is continuous on $L^{p_0}(\mu)$ and $g \in L^{q_0'}$. Hence

$$\left|\int_S T(f) g \, d\nu\right| \leqslant M_0^{1-\eta} M_1^{\eta} \|f\|_p \|g\|_{q'}.$$

The result follows. □

The case of real-linear operators

For the validity of the conclusion $\|Tf\|_q \leqslant M_0^{1-\eta} M_1^{\eta} \|f\|_p$ in the Riesz/Thorin theorem, it is essential that the operator T be complex-linear, i.e., that $T(\lambda f) = \lambda T f$ for all complex scalars λ (see [7, Ch. 4, Example 1.3]). For real spaces, the conclusion of the theorem remains valid if $p_k \leqslant q_k$ $(k = 0, 1)$. Otherwise, we have $\|Tf\|_q \leqslant 2M_0^{1-\eta} M_1^{\eta} \|f\|_p$. However, if T is a *positive* linear operator, then Theorem 2.1.3 remains valid for any p_k, $q_k \geqslant 1$.

The Hausdorff/Young theorem

Let T denote the unit circle of the complex plane. For a function $f \in L^1(\mathbb{T})$, let $\widehat{f}(n)$ be the Fourier coefficients of f,

$$\widehat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} \, d\theta.$$

2.1.4 Theorem *If $f \in L^p(\mathbb{T})$, $1 \leqslant p \leqslant 2$,*

$$\left(\sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^{p'} \right)^{1/p'} \leqslant \|f\|_p, \tag{2.2}$$

where $1/p + 1/p' = 1$. In other words: If $f \in L^p(\mathbb{T})$ and $\{b_n\}$ is a two-sided ℓ^p-sequence, $1 \leqslant p \leqslant 2$, then the series $\sum b_n \widehat{f}(n)$ is absolutely convergent and there holds the inequality

$$\left| \sum_{n=-\infty}^{\infty} b_n \widehat{f}(n) \right| \leqslant \|f\|_p \, \|\{b_n\}\|_p. \tag{2.3}$$

Proof. The theorem is true for $p = 1$ because $|\widehat{f}(n)| \leqslant \|f\|_1$, and is true for $p = 2$ because of Parseval's formula. Then the result is obtained by the Riesz/Thorin theorem. □

2.1.5 Exercise If $p > 2$ and $\{b_n\}_{-\infty}^{\infty} \in \ell^{p'}$, then there exists a unique function $g \in L^p(\mathbb{T})$ such that $\widehat{g}(n) = b_n$ for all n and $\|g\|_p \leqslant \left(\sum_{n=-\infty}^{\infty} |b_n|^{p'}\right)^{1/p'}$.

2.2 Weak L^p-spaces and Marcinkiewicz's theorem

The space $L^{p,\infty}$

Let Ω be a measure space with a (positive) sigma-finite measure μ. **The weak L^p space $L^{p,\infty}(\mu)$**, $0 < p < \infty$, consists of those measurable functions f on Ω for which

$$\|f\|_{p,\infty} := \sup_{0<\lambda<\infty} \lambda \cdot \big(\mu(f,\lambda)\big)^{1/p} < \infty,$$

where $\mu(f,\lambda) = \mu\{\omega\colon |f(\omega)| > \lambda\} = \mu\big(\{\omega \in \Omega\colon |f(\omega)| > \lambda\}\big)$.

Chebyshev's inequality,

$$\mu(g,\lambda) \leqslant \frac{1}{\lambda}\int_\Omega |g|\, d\mu,$$

shows that $L^p \subset L^{p,\infty}$, while the formula

$$\int_\Omega |g|^q\, d\mu = \int_0^\infty \mu(g,\lambda)\, d(\lambda^q) \tag{2.4}$$

(proved by means of Fubini's theorem) implies $L^{p,\infty} \subset L^q$ for $q < p$, if μ is finite. The quantity $\|\cdot\|_{p,\infty}$ is a norm for no p, but we have

$$\|f+g\|_{p,\infty} \leqslant C_p\,(\|f\|_{p,\infty} + \|g\|_{p,\infty}) \qquad (C_p = 2^{\max(1/p,1)}),$$

and hence $\|\cdot\|_{p,\infty}$ is a (complete) quasinorm. It is interesting, however, that if $p = 1$, then the space need not be locally convex (if, for example, $\Omega = [0,1]$ with Lebesgue measure), although it can be q-renormed for every $q < 1$. For $p > 1$ the space is locally convex, and for $p < 1$ it is p-convex, i.e., there is an equivalent p-norm on it.(*)

2.2.1 Exercise There hold the inequalities

$$\mu(f_1+f_2, \lambda_1+\lambda_2) \leqslant \mu(f_1,\lambda_1) + \mu(f_2,\lambda_2),$$
$$\mu(f_1 f_2, \lambda_1\lambda_2) \leqslant \mu(f_1,\lambda_1) + \mu(f_2,\lambda_2).$$

Marcinkiewicz's theorem

Quasilinear operators Let T be an operator acting from a vector space X to the set of all nonnegative measurable functions defined on a measure space (Ω,μ). Then T is called a quasilinear operator if there exists a constant K such that

$$T(f+g) \leqslant K(Tf + Tg) \qquad (f, g \in X).$$

If $K = 1$, then T is said to be **subadditive.** If an operator S with values in the set of finite measurable functions on Ω is linear, then the operator $Tf = |Sf|$ is subadditive.

(*)For further information see Kalton [36]. See also [7] for the general theory of weak L^p spaces.

2.2.2 Theorem *Let μ and σ be sigma-finite measures on Ω and S, respectively, let $0 < p < q \leqslant \infty$ and let T be a quasilinear operator from $L^p(\sigma) + L^q(\sigma)$ to the set of all nonnegative μ-measurable functions. Assume there exist constants C_1 and C_2, independent of f, such that*

$$\|Tf\|_{p,\infty} \leqslant C_1\|f\|_p\,, \tag{2.5}$$

$$\|Tf\|_{q,\infty} \leqslant C_2\|f\|_q\,. \tag{2.6}$$

Then for every $s \in (p,q)$ there exists a constant C independent of f such that

$$\|Tf\|_s \leqslant C\|f\|_s. \tag{2.7}$$

In the case $q = \infty$ inequality (2.6) should be interpreted as $\|Tf\|_\infty \leqslant C_2\|f\|_\infty$.

Weak type and strong type If T satisfies (2.5), i.e., if T maps L^p into $L^{p,\infty}$ and is continuous at zero, then we say that T is of weak type (p,p); if (2.7) holds, then T is of strong type (s,s).

Proof of Theorem 2.2.2

We consider the case where $K = C_1 = C_2 = 1$ and $q < \infty$, leaving the remaining cases to the reader. We have to deduce the inequality

$$\int_\Omega |Tf|^s\,d\mu \leqslant C\int_S |f|^s\,d\sigma$$

from two "weak" inequalities:

$$\mu(Tf,\lambda) \leqslant \frac{1}{\lambda^p}\int_S |f|^p\,d\sigma, \tag{2.8}$$

$$\mu(Tf,\lambda) \leqslant \frac{1}{\lambda^q}\int_S |f|^q\,d\sigma. \tag{2.9}$$

To show this we represent the function f in the form $f = g_\lambda + h_\lambda$, where

$$g_\lambda(\zeta) = \begin{cases} f(\zeta), & \text{if } |f(\zeta)| \geqslant \lambda, \\ 0, & \text{if } |f(\zeta)| < \lambda. \end{cases}$$

Since $Tf \leqslant T(g_\lambda) + T(h_\lambda)$, we have $\mu(Tf,\lambda) \leqslant G(\lambda) + H(\lambda)$, where

$$G(\lambda) = \mu(Tg_\lambda, \lambda/2) \quad \text{and} \quad H(\lambda) = \mu(Th_\lambda, \lambda/2).$$

It follows from (2.8) and (2.9) that

$$G(\lambda) \leqslant (2/\lambda)^p \int_S |g_\lambda|^p\,d\sigma = (2/\lambda)^p \int_{|f|>\lambda} |f|^p\,d\sigma \tag{2.10}$$

and

$$H(\lambda) \leqslant (2/\lambda)^q \int_S |h_\lambda|^q \, d\sigma = (2/\lambda)^q \int_{|f| \leqslant \lambda} |f|^q \, d\sigma.$$

Now we use the formula

$$\int_\Omega |Tf|^s d\mu = s \int_0^\infty \mu(Tf, \lambda)\lambda^{s-1} d\lambda \leqslant s \int_0^\infty (G(\lambda) + H(\lambda))\lambda^{s-1} d\lambda.$$

Multiplying inequality (2.10) by $s\lambda^{s-1}$ and then integrating over $\lambda \in (0, \infty)$ we get

$$\begin{aligned} s\int_0^\infty G(\lambda)\lambda^{s-1}\, d\lambda &\leqslant s2^p \int_0^\infty \Big(\lambda^{s-p-1} \int_{|f|>\lambda} |f|^p \, d\sigma\Big)\, d\lambda \\ &= s2^p \int_S \Big(\int_0^{|f|} \lambda^{s-p-1}\, d\lambda\Big)|f|^p \, d\sigma \\ &= \frac{s2^p}{s-p}\int_S |f|^s \, d\sigma. \end{aligned}$$

The analogous inequality for $H(\lambda)$ is proved in a similar way. □

Marcinkiewicz's theorem for $L\log^+ L$

2.2.3 Theorem *Let μ and σ be finite measures on Ω and S, respectively, let $1 < q \leqslant \infty$ and let T be a quasilinear operator from $L^1(\sigma)$ to the set of all nonnegative μ-measurable functions. If T satisfies* (2.5)($p = 1$) *and* (2.6), *then*

$$\int_\Omega Tf \, d\mu \leqslant K_1 + K_2 \int_S |f| \log^+ |f| \, d\sigma, \tag{2.11}$$

where K_1 and K_2 are independent of f.

The class of those σ-measurable functions f on S for which the integral on the right hand side of (2.11) is finite is denoted by $L\log^+ L(S)$.

The proof is similar to that of Theorem 2.2.2. Other variants of Marcinkiewicz's theorem can be found in Zygmund [100, Ch. XII§4]; the proof of some of them is much more difficult.

Paley's theorem

The implication

$$f \in L^p(\mathbb{T}) \Longrightarrow \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^{p'} < \infty \qquad (1 < p < 2),$$

which is a weak form of the Hausdorff/Young theorem, was improved by Hardy and Littlewood; namely:

2.2.4 Theorem *If $f \in L^p(\mathbb{T})$, $1 < p < 2$, then $\sum_{n=0}^{\infty}(n+1)^{p-2}(c_n^*)^p < \infty$, where $\{c_n^*\}$ is the decreasing rearrangement of the sequence $\{\widehat{f}(n)\}$.*

An application of Marcinkiewicz's theorem yields a more general result, due to Paley:

2.2.5 Theorem *Let (Ω, μ) be a finite measure space and let $\{\varphi_n\}_1^\infty$ be an orthonormal sequence in $L^2(\Omega, \mu)$ such that $\sup_n \|\varphi_n\|_\infty < \infty$. Then*

$$\sum_{n=1}^{\infty} n^{p-2}|a_n|^p \leqslant C\|f\|_p^p, \qquad f \in L^p(\Omega, \mu),$$

where $1 < p < 2$ and $a_n = \int_\Omega f\overline{\varphi_n}\, d\mu$.

Proof. Let $\mu(\Omega) = 1$ and $\sup_n \|\varphi_n\|_\infty = K$. Define the measure σ on $\mathbb{N}$, the set of positive integers, by $\sigma(\{n\}) = n^{-2}$. Define the operator $T\colon L^1(\Omega, \mu) \mapsto \mathcal{L}_0(\mathbb{N}, \sigma)$ by $(Tf)(n) = na_n$. Bessel's inequality implies that T is of strong type $(2, 2)$. To prove that T is of weak type $(1, 1)$ and therefore to conclude the proof (by Marcinkiewicz's theorem), observe that $|a_n| \leqslant K\|f\|_1$. Hence, if $\|f\|_1 = 1$, we have

$$\sigma\{n\colon |Tf(n)| > \lambda\} \leqslant \sigma\{n\colon Kn > \lambda\} \leqslant \sum_{n>\lambda/K} n^{-2} \leqslant CK\min(1, 1/\lambda),$$

which concludes the proof. □

2.2.6 Exercise Let (Ω, μ) be a sigma-finite measure space and let $\{\varphi_n\}_1^\infty$ be an orthonormal sequence in $L^2(\Omega, \mu)$ such that $\|\varphi_n\| \leqslant Mn^\gamma$, where M and γ are positive constants. Then for $1 < p < 2$ there holds the inequality

$$\sum_{n=1}^{\infty} n^{(\gamma+1)(p-2)}|a_n|^p \leqslant C\|f\|_p^p.$$

2.3 Maximal function and Lebesgue points

The maximal function of a 2π-periodic function $\phi \in L^1(-\pi, \pi)$ is the (2π-periodic) function $\mathcal{M}\phi$ defined as

$$(\mathcal{M}\phi)(\theta) = \sup_{0<h<\pi} \frac{1}{2h}\int_{\theta-h}^{\theta+h} |\phi(t)|\, dt. \tag{2.12}$$

The function $\mathcal{M}\phi$ is above semicontinuous (and, consequently, measurable) as the supremum of a family of continuous functions. The (sublinear) operator $\mathcal{M}$ taking ϕ to $\mathcal{M}\phi$ is called the **maximal operator** of Hardy and Littlewood. If $g \in L^1(\mathbb{T})$, then we define

$$(\mathcal{M}g)(e^{i\theta}) = (\mathcal{M}\phi)(\theta), \quad \text{where } \phi(\theta) = g(e^{i\theta}). \tag{2.13}$$

The main maximal theorem

2.3.1 Theorem (a) *If ϕ is in $L^1(-\pi,\pi)$, then there exists an absolute constant C such that $\left|\{\theta \in (-\pi,\pi) : \mathcal{M}\phi(\theta) > \lambda\}\right| \leqslant \frac{C}{\lambda}\|\phi\|_1$.*

(b) *If $\phi \in L^p(-\pi,\pi)$, $p > 1$, then $\mathcal{M}\phi \in L^p(-\pi,\pi)$ and $\|\mathcal{M}\phi\|_p \leqslant C_p\|\phi\|_p$, where C_p depends only of p.*

By $|\cdots|$ we denote the Lebesgue measure on the line.

Proof. Assertion (b) is obtained from (a) by Marcinkiewicz's theorem. To prove (a), let $\phi \in L^1(-\pi,\pi)$, let $E = \{\theta \in (-\pi,\pi) : \mathcal{M}\phi(\theta) > 1\}$ and let K be a compact subset of E. It suffices to find an absolute constant C such that $|K| \leqslant C\|\phi\|_1$. By the definition of $\mathcal{M}\phi$ and the compactness of E, there are intervals I_i $(i = 1,\dots,n)$ such that $I_i \subset (-2\pi, 2\pi)$, $K \subset \bigcup I_i$ and $|I_i| \leqslant \int_{I_i} |\phi(t)|\,dt$. Assume that the sequence $|I_i|$ is decreasing. Let $J_1 = I_1$. Let $J_2 = I_k$, where k is the smallest i for which $I_i \cap J_1 = \emptyset$. Then let $J_3 = I_m$, where m is the smallest $i > k$ such that $I_i \cap (J_1 \cup J_2) = \emptyset$. Continuing in this way we find a sequence $J_j \subset (-2\pi, 2\pi)$ of pairwise disjoint intervals such that $\bigcup I_i \subset \bigcup J_j^*$, where, for each j, J_j^* is the interval "concentric" with J_j and $|J_j^*| = 3|J_j|$. It follows that

$$(1/3)|K| \leqslant \sum_j |J_j| \leqslant \sum_j \int_{J_j} |\phi(t)|\,dt,$$

which gives the desired inequality with $C = 6$. □

Lebesgue points

The maximal theorem has many important applications. It is useful, for example, in proving almost everywhere convergence. Usually, we can easily prove a.e. convergence for a dense set of functions, and then use the maximal theorem to interchange the limits. Here we consider the existence of Lebesgue points.

The Lebesgue point of a function ϕ is a point $x \in \mathbb{R}$ such that

$$\lim_{h\to 0} \frac{1}{2h} \int_{-h}^{h} \left|\phi(t+x) - \phi(x)\right| dt = 0.$$

The set of all Lebesgue points of f is called the **Lebesgue set** of f.

2.3.2 Theorem *If a 2π-periodic function ϕ is integrable on $(-\pi,\pi)$, then almost every point in $\mathbb{R}$ is a Lebesgue point for ϕ.*

2.3.3 Corollary *The inequality $|\phi(\theta)| \leqslant (\mathcal{M}\phi)(\theta)$ holds almost everywhere.*

Proof of Theorem. The operator

$$T\phi(x) = \limsup_{h\to 0} \frac{1}{2h} \int_{-h}^{h} \left|\phi(t+x) - \phi(x)\right| dt$$

satisfies: (a) $T(\phi_1 + \phi_2) \leqslant T\phi_1 + T\phi_2$; (b) $T\phi \leqslant |\phi| + M\phi$; (c) $Tg = 0$ if g is continuous.

Let $\phi \in L^1(-\pi, \pi)$, $\lambda > 0$ and $\varepsilon > 0$. Choose a continuous function g so that $\|\phi - g\|_1 < \varepsilon$. From (a) we get $T\phi \leqslant Tg + T(\phi - g) = T(\phi - g)$ and then, from (b), by Theorem 2.3.1 and Chebyshev's inequality, we get

$$\begin{aligned}\left|\{\theta : T(\phi - g)(\theta) > \lambda\}\right| &\leqslant \left|\{\theta : |\phi - g|(\theta) > \lambda/2\}\right| + \left|\{\theta : \mathcal{M}(\phi - g)(\theta) > \lambda/2\}\right| \\ &\leqslant \frac{4\pi}{\lambda}\|\phi - g\|_1 + \frac{2C}{\lambda}\|\phi - g\|_1 \leqslant \frac{2(2\pi + C)\varepsilon}{\lambda}.\end{aligned}$$

Thus $\left|\{\theta : T(\phi - g)(\theta) > \lambda\}\right| = 0$, for every $\lambda > 0$, because ε is arbitrary. □

Non-periodic case

Let ϕ be a locally integrable function defined on $\mathbb{R}^n$, $n \geqslant 1$. The maximal function $M\phi$ is defined on $\mathbb{R}^n$ by

$$(M\phi)(z) = \sup_{r>0} \frac{1}{r^n} \int_{|\xi - z| < r} |\phi(\xi)|\, dV_n(\xi), \tag{2.14}$$

where dV_n is the Lebesgue measure on $\mathbb{R}^n$ normalized so that the measure of the unit ball is equal to one. The Lebesgue point of a function $\phi \in L^1(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is open, is a point $x \in \Omega$ such that

$$\lim_{h \to 0} \frac{1}{h^n} \int_{|t| < h} |\phi(t + x) - \phi(x)|\, dV_n(t) = 0.$$

Theorem 2.3.1 extends in the obvious way. As a consequence, we have that *if $\phi \in L^1(\Omega)$, then the Lebesgue set of ϕ is "almost equal" to Ω.*

Density of rational functions in L^p

2.3.4 Theorem *If D is a bounded subdomain of $\mathbb{C}$, then the set of rational functions with simple poles is dense in $L^p(D)$, for $0 < p < 2$.*

Observe that the function $1/(z - a)$, where $a \in \mathbb{D}$, belongs to $L^p(D)$ if and only if $p < 2$.

Proof. It suffices to consider the case $1 \leqslant p < 2$. Let $\varphi \in L^q(D)$, $1/p + 1/q = 1$, and let

$$\int_D \varphi(w) R(w)\, dA(w) = 0, \quad \text{where } dA = dV_2,$$

for every rational function R. Then

$$\int_D \frac{\varphi(w)}{z - w}\, dA(w) = 0$$

for every $z \in \mathbb{C}$, whence

$$\int_{|z-z_0|=r} \left(\int_D \frac{\varphi(w)}{z-w} \, dA(w) \right) dz = 0,$$

where $z_0 \in D$ and $r < \operatorname{dist}(z_0, \partial D)$. Here we can apply Fubini's theorem because

$$\int_D \left| \frac{\varphi(w)}{z-w} \right| dA(w) \leqslant \|\varphi\|_q \left(\int_D |z-w|^{-p} dA(w) \right)^{1/p} \leqslant C,$$

for $|z - z_0| = r$, where C is independent of z. Hence, by Cauchy's integral formula,

$$\int_{|w-z_0|<r} \varphi(w) \, dA(w) = 0.$$

If z_0 is a Lebesgue point of φ, then we obtain

$$0 = \lim_{r \to 0} \frac{1}{r^2} \int_{|w-z_0|<r} \varphi(w) \, dA(w) = \varphi(z_0),$$

and this concludes the proof. □

2.3.5 Exercise An interesting fact, observed in [10], can be deduced from Theorem 2.3.4 and Runge's theorem.

If f is a Lebesgue measurable function defined on $\mathbb{C}$, then there is a sequence P_n of (holomorphic) polynomials such that $P_n \to f$ a.e.

2.3.6 Exercise Let $\mathcal{R}$ denote the set of all rational functions. If $p < 2$ then the set $\mathcal{R} \cap L^p(\mathbb{C})$ is dense in $L^p(\mathbb{C})$.

2.4 The Rademacher functions

The Rademacher functions $r_j(t)$ are defined by $r_j(t) = \operatorname{sign}\sin(2^j t\pi)$ ($j \geqslant 0, t \in \mathbb{R}$). For example, $r_0(t) = 1$ for $0 < t < 1$,

$$r_1(t) = \begin{cases} 1, & 0 < t < 1/2, \\ -1, & 1/2 < t < 1, \\ 0, & t = 0, 1/2, 1, \end{cases}$$

and $r_n(t) \equiv r_{n-1}(2t)$. These functions form an orthonormal sequence in $L^2(0, 1)$ and therefore

$$\int_0^1 \left| \sum_{j=0}^n a_j r_j(t) \right|^2 dt = \sum_{j=0}^n |a_j|^2, \qquad \{a_k\}_0^n \subset \mathbb{C}. \tag{2.15}$$

This generalization of the parallelogram law can also be written as

$$\frac{1}{2^n}\sum_{k=0}^{n}\left|a_0+\sum_{(\varepsilon_j)\in\{-1,1\}^n}\varepsilon_j a_k\right|^2=\sum_{j=0}^{n}|a_j|^2,$$

or in a more symmetric form

$$\frac{1}{2^{n+1}}\sum_{k=0}^{n}\left|\sum_{(\varepsilon_j)\in\{-1,1\}^{n+1}}\varepsilon_j a_k\right|^2=\sum_{j=0}^{n}|a_j|^2.$$

It was proved by Rademacher that $\sum_{k=0}^{\infty}|a_k|^2<\infty$, then $\sum_{k=0}^{\infty}a_k r_k(t)$ converges a.e. On the other hand, if $\sum_{k=0}^{\infty}|a_k|^2=\infty$, then $\sum_{k=0}^{\infty}a_k r_k(t)$ diverges a.e. (Khintchine and Kolmogorov). The proof of these facts can be found in Duren [18] and, in a stronger form, in Zygmund [100].

Khintchine's inequality

2.4.1 Theorem *For every $p\in(0,\infty)$ there are positive constants c_p and C_p such that*

$$c_p\Big(\sum_{j=0}^{n}|a_j|^2\Big)^{1/2}\leqslant\Big(\int_0^1\Big|\sum_{j=0}^{n}a_j r_j(t)\Big|^p dt\Big)^{1/p}\leqslant C_p\Big(\sum_{j=0}^{n}|a_j|^2\Big)^{1/2}\tag{2.16}$$

for every finite sequence $\{a_j\}_1^n$ of complex scalars.

Note that we can take $c_p=1$ for $p\geqslant 2$, and $C_p=1$ for $p\leqslant 2$.

Proof. Suppose we have proved the theorem for $p\geqslant 2$. Let $\phi_n(t)=\sum_{j=0}^{n}a_j r_j(t)$. Then

$$\begin{aligned}\|\phi_n\|_2^2&=\int_0^1|\phi_n(t)|^2dt=\int_0^1|\phi_n(t)|^{1/2}|\phi_n(t)|^{3/2}dt\\&\leqslant\|\phi_n\|_1^{1/2}\,\|\phi_n\|_3^{3/2}\leqslant C_3^{3/2}\,\|\phi_n\|_1^{1/2}\,\|\phi_n\|_2^{3/2},\end{aligned}$$

and hence $\|\phi_n\|_2\leqslant C_3^3\,\|\phi_n\|_1$, which proves the left-hand side inequality in (2.16) for $p=1$. Using this we can prove that $\|\phi_n\|_1\leqslant\text{const}\,\|\phi_n\|_{1/2}$, and so on.

To discuss the case $p>2$, we can suppose that p is an even integer. Using the binomial formula we find that there holds the inequality

$$\frac{|x+y|^p+|x-y|^p}{2}\leqslant\left(|x|^2+\kappa_p\,|y|^2\right)^{p/2},\qquad x,y\in\mathbb{C},\tag{2.17}$$

where $\kappa_p\geqslant 1$ is a constant. We shall prove that $\|\phi_n\|_p\leqslant\kappa_p^{1/2}\,\|\phi_n\|_2$ by induction on the length of $\{a_k\}_0^n$. In the case $n=1$ we have

$$\|\phi_n\|_p=\Big(\frac{|a_0+a_1|^p+|a_0-a_1|^p}{2}\Big)^{1/p}\leqslant\kappa_p^{1/2}\,\|\phi_n\|_2,$$

because of (2.17). Let $n \geqslant 2$. Then, as is easily verified,

$$\int_0^1 \left| \sum_{j=0}^{n} a_j r_j(t) \right|^p dt = \frac{1}{2} \int_0^1 \left(|a_0 + \psi(t)|^p + |a_0 - \psi(t)|^p \right) dt, \tag{2.18}$$

where $\psi(t) = \sum_{k=1}^{n} a_k r_{k-1}(t) = \sum_{k=0}^{n-1} a_{k+1} r_k(t)$. From this and (2.17) it follows that

$$\int_0^1 \left| \sum_{j=0}^{n} a_j r_j(t) \right|^p dt \leqslant \int_0^1 \left(|\psi(t)|^2 + \kappa_p |a_0|^2 \right)^{p/2} dt.$$

The last integral is $\leqslant \left(\|\psi\|_p^2 + \kappa_p |a_0|^2 \right)^{p/2}$, which can be seen by using the binomial formula, or by using Jensen's inequality for the concave function $x \mapsto (x^{2/p} + 1)^{p/2}$, $x \geqslant 0$. On the other hand, by induction hypothesis we have $\|\psi\|_p \leqslant \kappa_p^{1/2} \|\psi\|_2$, which implies

$$\int_0^1 \left| \sum_{j=0}^{n} a_j r_j(t) \right|^p dt \leqslant \left(\kappa_p \|\psi\|_2^2 + \kappa_p |a_0|^2 \right)^{p/2}.$$

This completes the proof because

$$\left(\kappa_p \|\psi\|_2^2 + \kappa_p |a_0|^2 \right)^{1/2} = \kappa_p^{1/2} \left(\sum_{j=0}^{n} |a_j|^2 \right)^{1/2}. \qquad \square$$

Miscellaneous

2.4.2 What is essential in the above proof is that r_n is defined by $r_n(t) = \mathfrak{r}(2^{n-1}t)$, $n \geqslant 1$, where $\mathfrak{r} \in L^\infty(\mathbb{R})$ is a 1-periodic function such that $\mathfrak{r}(t + 1/2) \equiv -\mathfrak{r}(t)$. As an example we can take $\mathfrak{r}(t) = e^{i2\pi t}$ to get Paley's inequality; this inequality will be discussed in a different context (see Theorems 11.1.1 and 11.1.3, page 164).

2.4.3 The L^p-closure of the linear span of the Rademacher functions is complemented in L^p, for $1 < p < \infty$.

2.4.4 Let $\{f_j\}_{j \geqslant 1}$ be a finite sequence in a vector space X, let $F_t = \sum_{j \geqslant 1} r_j(t) f_j$, $F_t = \sum_{j \geqslant 1} \varepsilon_j r_j(t) f_j$, where $\varepsilon_j = \pm 1$. If X_0 is a subset of X, then

$$\left| \{ t \in [0,1] \colon G_t \in X_0 \} \right| = \left| \{ t \in [0,1] \colon F_t \in X_0 \} \right|.$$

2.5 Nikishin's theorem

While Marcinkiewicz's theorem enables us to deduce a strong inequality from two weak inequalities, by using Nikishin's theorem we can obtain a weak inequality from a "very weak" one. Before stating this theorem we introduce some terminology.

The space $\mathcal{L}_0$ and sublinear operators

Let (Ω, μ) be a *finite* measure space. The vector space $\mathcal{L}_0(\mu) = \mathcal{L}_0(\Omega, \mu)$ of all finite measurable functions on Ω becomes an F-space when endowed with the F-norm

$$\mathfrak{N}_0(f) = \int_\Omega \frac{|f|}{1 + |f|} \, d\mu.$$

Convergence in $\mathcal{L}_0(\mu)$ is equivalent to convergence in measure, i.e., $f_n \to 0$ iff $\lim_n \mu(f_n, \lambda) = 0$ for every $\lambda > 0$. This can be deduced from the formula

$$\mathfrak{N}_0(g) = \int_0^\infty \frac{\mu(g, \lambda)}{(1 + \lambda)^2} \, d\lambda, \tag{2.19}$$

which can easily be deduced from (2.4)($q = 1$) by taking $g = |f|/(1 + |f|)$.

An operator T that maps a quasi-normed space X to the set of nonnegative measurable functions is said to be **sublinear** if (almost everywhere):
(a) $T(f + g) \leqslant Tf + Tg$ for $f, g \in X$; (b) $T(\lambda f) = |\lambda| Tf$ for $f \in X$ and $\lambda \in \mathbb{C}$.

If Tf is finite a.e. for all f, then we can treat it as an operator from X to $\mathcal{L}_0$. Since (a) and (b) imply $|Tf - Tg| \leqslant T(f - g)$, we see that continuity of T at the origin implies continuity of T on all of X.

Spaces of type p

A quasi-Banach space X is of type p ($0 < p \leqslant 2$) if there exists a constant K such that

$$\int_0^1 \Big\| \sum_{j \geqslant 0} r_j(t) f_j \Big\|^p dt \leqslant K \sum_j \|f_j\|^p \tag{2.20}$$

for every finite sequence $\{f_j\} \subset X$, where r_j are the Rademacher functions; recall that $r_j(t) = \operatorname{sign} \sin(2^j t\pi)$. Every p-Banach space ($0 < p \leqslant 1$) satisfies (2.20) with $K = 1$. And by application of Khintchine's inequality (2.16), one proves the following:

2.5.1 Theorem *L^p has type* $\min(p, 2)$ ($p > 0$).

From Khintchine's inequality also follows that the notion of type p has no sense for $p > 2$ because if $X = \mathbb{C}$, then the integral in (2.20) is "proportional" to the ℓ^2-norm of the sequence $\{f_j\}$.

Nikishin's theorem

In order to state Nikishin's theorem let $T_B f = \chi_B T f$, where χ_B is the characteristic function of $B \subset \Omega$.

2.5.2 Theorem (Nikishin [66]) *Let X be a quasi-Banach space of type p, $0 < p \leqslant 2$, and let $T : X \mapsto \mathcal{L}_0(\Omega, \mu)$ be sublinear and continuous. Then for every $\varepsilon > 0$ there exists a measurable set $B \subset \Omega$ such that $\mu(\Omega \setminus B) < \varepsilon$ and T_B maps X to $L^{p,\infty}$ and is continuous.*

Lemmas

For the proof we need two lemmas. The first lemma provides a characterization of continuity of T by means of "weak" inequalities.

2.5.3 Lemma *Let $T : (X, \|\cdot\|) \mapsto \mathcal{L}_0(\mu)$ be a sublinear operator on a quasi-Banach space X. Then T is continuous iff there exists a decreasing function $c(\lambda)$, $\lambda \geqslant 0$, such that* $\lim_{\lambda\to\infty} c(\lambda) = 0$ *and*

$$\mu(Tf, \lambda\|f\|) \leqslant c(\lambda) \qquad (\lambda > 0,\, f \in X). \tag{2.21}$$

Therefore, Nikishin's theorem says that under some conditions inequality (2.21) can be improved so that $c(\lambda) = C/\lambda^p$ outside a set of arbitrarily small measure.

Proof. Let T be continuous. Then there exists a decreasing function $\varepsilon(\lambda)$ defined for $\lambda > 0$ such that $\varepsilon(\lambda) \to 0$ $(\lambda \to \infty)$, and that there holds the implication

$$\|f\| = 1/\lambda \Longrightarrow \mathfrak{N}_0(Tf) < \varepsilon(\lambda).$$

On the other hand, (2.19) implies $\mu(Tf, 1) \leqslant 2\mathfrak{N}_0(Tf)$ because the function $\mu(Tf, \lambda)$ is decreasing. Thus if $\|f\| = 1/\lambda$, then $\mu(Tf, \lambda\|f\|) \leqslant 2\varepsilon(\lambda) =: c(\lambda)$, and hence

$$\mu(Tf, \lambda\|f\|) = \mu(Tf/\lambda\|f\|, 1) \leqslant c(\lambda)$$

for every $\lambda > 0$.

Conversely, if (2.21) is satisfied, let

$$\epsilon(\delta) = \int_0^\infty \frac{c(\lambda/\delta)}{(1+\lambda)^2}\, d\lambda.$$

From (2.19) it follows that there holds the implication

$$\|f\| < \delta \Longrightarrow \mathfrak{N}_0(Tf) < \epsilon(\delta),$$

which implies that T is continuous, because $\epsilon(\delta) \to 0$ $(\delta \to 0)$. □

2.5.4 Lemma *Let the hypotheses of Theorem 2.5.2 be satisfied. Then its conclusions hold if (and only if) there exists a decreasing function $c(\lambda)$, $\lambda > 0$, such that* $\lim_{\lambda\to\infty} c(\lambda) = 0$ *and*

$$\mu(\sup_j Tf_j(\omega), \lambda) \leqslant c(\lambda) \tag{2.22}$$

for all sequences $\{f_j\} \subset X$ such that $\sum_j \|f_j\|^p \leqslant 1$.

This lemma can also be interpreted in the following way: The conclusions of Theorem 2.5.2 hold iff the operator $Sf = \sup_j Tf_j$ maps the space $\ell^p(X)$ into $\mathcal{L}_0(\mu)$ and is continuous. The space $\ell^p(X)$ consists of those sequences $\{f_n\}_1^\infty \subset X$ for which

$$\|\{f_n\}\| = \Big(\sum_{n=1}^\infty \|f_n\|^p\Big)^{1/p} < \infty.$$

Proof. We will consider "if" part.[(†)] Let $\varepsilon > 0$ and choose $R > 0$ so that $c(R) < \varepsilon$, where $c(\lambda)$ is the function from (2.22). We say that a measurable set F satisfies condition (W) if there exists $f \in X$, $\|f\| \leqslant 1$, such that

$$\mu(F)\,Tf(\omega)^p > R^p \qquad (\omega \in F). \tag{W}$$

If the collection, say $\mathcal{F}$, of such sets is empty, then for fixed $\lambda > 0$ and $f \in X$, $\|f\| \leqslant 1$, we take $F = \{\omega \in \Omega \colon Tf(\omega) > \lambda\}$ to get $\mu(F)Tf(\omega)^p \leqslant R^p$ $(\omega \in F)$, whence $\mu(F)\lambda^p \leqslant R^p$. i.e., $\mu\{\omega \in \Omega\colon Tf(\omega) > \lambda\} \leqslant (R/\lambda)^p$, and this means that T maps X into $L^{p,\infty}$ and is continuous. Therefore we can suppose that $\mathcal{F}$ is nonempty. Let (F_j) be a maximal collection of pairwise disjoint measurable sets satisfying (W) and let $f_j \in X$ be the corresponding vectors. Putting $c_j = \mu(F_j)^{1/p}$ we have $\sum_j \|c_j f_j\|^p \leqslant 1$ and $\sup_j T(c_j f_j)(\omega) > R$ $(\omega \in \bigcup F_j)$. From (2.22) it follows that $\mu(\cup F_j) \leqslant c(R) < \varepsilon$. Since the collection is maximal, there holds the inequality $\mu\{\omega \in \Omega \setminus \bigcup F_j : Tf(\omega) > \lambda\} \leqslant (R/\lambda)^p$, for every $\lambda > 0$. This means that the operator T_B, where $B = \Omega \setminus \cup F_j$, maps X into $L^{p,\infty}$, which concludes the proof. □

Proof of Theorem 2.5.2.

First observe that Lemma 2.5.4 remains true if we assume that the sequence f_j is finite. Let $f_j \in X$ be such a sequence, and let

$$\sum_{j\geqslant 1} \|f_j\|^p \leqslant 1. \tag{2.23}$$

According to Lemma 2.5.4, it is enough to find a decreasing function $c_1(\lambda)$ defined for $\lambda > 0$ such that $\lim_{\lambda\to\infty} c_1(\lambda) = 0$ and

$$\mu(E_\lambda) \leqslant c_1(\lambda) \quad (\lambda > 0), \tag{2.24}$$

where $E_\lambda = \{\omega \in \Omega : \max_j Tf_j \geqslant \lambda\}$. Let $g_t = \sum_{j\geqslant 1} r_j(t) f_j$ $(0 \leqslant t \leqslant 1)$, where r_j are the Rademacher functions. Then $2r_k(t)f_k = g_t + g_t^k$, where

$$g_t^k = -r_1(t)f_1 - \cdots - r_{k-1}(t)f_{k-1} + r_k(t)f_k - \ldots.$$

Therefore there holds the inequality $2T(f_k) \leqslant T(g_t) + T(g_t^k)$ almost everywhere on $[0,1]$. Since

$$\left|\{\, t \in [0,1] : Tg_t(\omega) \geqslant \eta\}\right| = \left|\{\, t \in [0,1] : Tg_t^k(\omega) \geqslant \eta\}\right|$$

for every $\eta \geqslant 0$ (see 2.4.4), we see that $\left|\{\, t \in [0,1] : Tg_t(\omega) \geqslant Tf_k(\omega)\,\}\right| \geqslant 1/2$ for all $\omega \in \Omega$ and $k \geqslant 1$. Thus we have $\left|\{t \in [0,1] : Tg_t(\omega) \geqslant \lambda\}\right| \geqslant 1/2$, $\omega \in E_\lambda$, so it follows that

$$\mu(E_\lambda)/2 \leqslant \int_{E_\lambda} \left|\{t \in [0,1] : Tg_t(\omega) \geqslant \lambda\}\right| d\mu(\omega).$$

(†) Concerning the opposite direction, which is not needed in our text, one could see Wojtaszczyk [98, Ch. III.H]; the proof of "if" part is essentially the same as in that book.

This inequality is sufficient to conclude the proof in the case when X is p-Banach, $p < 1$. Namely, replacing $\int_{E_\lambda}$ by $\int_\Omega$ and applying the formula

$$\int_\Omega \left|\{t \in [0,1] : Tg_t(\omega) \geqslant \lambda\}\right| d\mu(\omega) = \int_\Omega d\mu(\omega) \int_{Tg_t(\omega)\geqslant\lambda} dt$$
$$= \int_0^1 dt \int_{Tg_t(\omega)\geqslant\lambda} d\mu(\omega) = \int_0^1 \mu\{\omega \in \Omega : Tg_t(\omega) \geqslant \lambda\}\, dt,$$

we get

$$\mu(E_\lambda)/2 \leqslant \int_0^1 \mu\{\omega \in \Omega : Tg_t(\omega) \geqslant \lambda\}\, dt.$$

Now Lemma 2.5.3 and the inequality $\|g_t\| \leqslant 1$, which holds for every t, give $\mu(E_\lambda) \leqslant 2c(\lambda/2)$, where $c(\lambda)$ is the function from Lemma 2.5.3. So we have proved (2.24) in this special case.

Let X be a space of type p. Then the inequality $\|g_t\| \leqslant 1$ can be false for some t, but from (2.23) and (2.20) it follows that

$$\left|\{t : \|g_t\| \geqslant \lambda\}\right| \leqslant K/\lambda^p. \tag{2.25}$$

In order to exploit this fact, we start from the inequality

$$\left|\{t : Tg_t(\omega) \geqslant \lambda\}\right| \leqslant \left|\{t : \|g_t\| \geqslant \sqrt{\lambda}\}\right| + \left|\{t : Tg_t \geqslant \|g_t\|\sqrt{\lambda}\}\right|.$$

Hence, by integration over $\omega \in \Omega$ and using (2.25) and Fubini's theorem as above, we get

$$\mu(E_\lambda)/2 \leqslant \int_\Omega \left|\{t : Tg_t(\omega) \geqslant \lambda\}\right| d\mu(\omega)$$
$$\leqslant K\lambda^{-p/2}\mu(\Omega) + \int_0^1 \mu\{\omega : Tg_t(\omega) \geqslant \|g_t\|\sqrt{\lambda}\}\, dt.$$

The last integral is $\leqslant c(\sqrt{\lambda}/2)$, because of Lemma 2.5.3, so we have (2.24) again. □

2.6 Nikishin and Stein's theorem

This theorem tells us that under additional, algebraic, conditions we can take $B = \Omega$ in Theorem 2.5.2. We restrict ourselves to the case of the multiplicative group $\mathbb{T}$.

2.6.1 Theorem (Nikishin/Stein) *Let X be a space of type $p \in (0,2]$, and let $\zeta \mapsto f_\zeta$ be a mapping of the unit circle $\mathbb{T}$ to the set of all isometric endomorphisms of X. If $T : X \mapsto \mathcal{L}_0(\mathbb{T})$ is sublinear, continuous and "commutes with rotations", i.e., for every $\zeta \in \mathbb{T}$ satisfies the condition $(Tf_\zeta)(\omega) = (Tf)(\omega\zeta)$, $\omega \in \mathbb{T}$ a.e., then $T(X) \subset L^{p,\infty}(\mathbb{T})$ and T is continuous as an operator from X to $L^{p,\infty}(\mathbb{T})$.*

In the case where X is $L^p(\mathbb{T})$ or $L^p(\mathbb{D})$, we put $f_\zeta(z) = f(\zeta z)$.

Proof. Let $d\mu(\zeta) = |d\zeta|/2\pi$. Theorem 2.5.2 guarantees that there is a set B so that $\mu(B) > 0$ and $\mu(A_\lambda \cap B) \leqslant C\,(\|f\|/\lambda)^p$, where $A_\lambda = \{\omega \in \mathbb{T} : |(Tf)(\omega)| > \lambda\}$, and C is independent of f and λ. If we put f_ζ $(\zeta \in \mathbb{T})$ instead of f and apply the hypotheses of the theorem, we get $\mu((\zeta^{-1}A_\lambda) \cap B) \leqslant C\,(\|f\|/\lambda)^p$. Integrating this with respect to $d\mu(\zeta)$ and using the formula

$$\int_{\mathbf{T}} \mu((\zeta^{-1}A_\lambda) \cap B)\, d\mu(\zeta) = \mu(A_\lambda)\mu(B), \tag{2.26}$$

we get $\mu(A_\lambda) \leqslant \dfrac{C}{\mu(B)}\left(\dfrac{\|f\|}{\lambda}\right)^p$, which was to be proved.

To verify formula (2.26), we write the left-hand side as

$$\int_{\mathbf{T}} \mu(\zeta^{-1}A \cap B)\, d\mu(\zeta) = \int_{\mathbf{T}} d\mu(\zeta) \int_B \chi_{\zeta^{-1}A}(\omega)\, d\mu(\omega)$$

(χ is the characteristic function), and then apply Fubini's theorem together with the relation $\chi_{\zeta^{-1}A}(\omega) = \chi_{\omega^{-1}A}(\zeta)$. □

A theorem on multipliers

As a nice application of the Nikishin/Stein theorem and the Marcinkiewicz theorem we have the following.

2.6.2 Theorem *Let $T : L^1(\mathbb{T}) \mapsto \mathcal{L}_0(\mathbb{T})$ be a continuous linear operator that commutes with rotations. Then T is of weak type $(1,1)$ and of strong type (p,p) for every $p \in (1,\infty)$, and there exists a bounded sequence m_n $(-\infty < n < \infty)$ such that*

$$Tf(e^{i\theta}) = \sum_{n=-\infty}^{\infty} m_n \widehat{f}(n) e^{in\theta} \tag{2.27}$$

for every trigonometric polynomial f. Further, if $f \in L^p$, $p > 1$, then

$$\widehat{Tf}(n) = m_n \widehat{f}(n) \tag{2.28}$$

for every integer n.

Proof. Since L^p is of type p for $0 < p \leqslant 2$, Nikishin/Stein theorem tells us that the operator T with the above properties is of weak type (p,p) for $p = 1$ and $p = 2$. Hence, by Marcinkiewicz's theorem, T is of strong type (p,p) for $1 < p < 2$. To prove the rest, let $g(w) = w^n$, $w \in \mathbb{T}$, for a fixed integer n. By the hypothesis, for every $\zeta \in \mathbb{T}$ we have $\zeta^n (Tg)(w) = (Tg)(\zeta w)$ for a.e., $w \in \mathbb{T}$. The function Tg belongs to L^1 because $g \in L^2$ and T is of strong type (p,p) for $p \in (1,2)$. It follows that if $\phi \in L^\infty$, then

$$\int_{\mathbf{T}} (Tg)(w)\phi(w)\, |dw| = \int_{\mathbf{T}} \zeta^{-n} (Tg)(\zeta w)\phi(w)\, |dw|$$

for every $\zeta \in \mathbb{T}$. Integrating this with respect to ζ and using Fubini's theorem, we get

$$\begin{aligned}\int_{\mathbf{T}}(Tg)(w)\phi(w)\,|dw| &= \frac{1}{2\pi}\int_{\mathbf{T}}\phi(w)\,|dw|\int_{\mathbf{T}}\zeta^{-n}(Tg)(\zeta w)\,|d\zeta| \\ &= \frac{1}{2\pi}\int_{\mathbf{T}}\phi(w)w^n\,|dw|\int_{\mathbf{T}}\zeta^{-n}(Tg)(\zeta)\,|d\zeta|.\end{aligned}$$

Hence

$$(Tg)(w) = w^n\int_{\mathbf{T}}\zeta^{-n}(Tg)(\zeta)\,|d\zeta| =: m_n w^n \quad \text{for a.e. } w \in \mathbb{T};$$

this proves formula (2.27). The validity of (2.28) can then be deduced from the Weierstrass theorem that the trigonometric polynomials are dense in L^p.

It remains to prove that T is of strong type (q,q) for $q \geqslant 2$. By Marcinkiewicz's theorem (or by the Riesz/Thorin theorem), we can assume that $q > 2$. Let f, g be trigonometric polynomials. Then, in view of (2.27), we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(Tf)(e^{i\theta})g(e^{-i\theta})\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}f(e^{i\theta})(Tg)(e^{-i\theta})\,d\theta.$$

Using this and the fact that T is of strong type (p,p) for $p = q/(q-1)$, we conclude that

$$\|Tf\|_q \leqslant C\|f\|_q, \tag{2.29}$$

where C is independent of f. Now let $f \in L^q$ be arbitrary, and let f_n be a sequence of trigonometric polynomials such that $\|f_n - f\|_q \to 0$. The validity of (2.29) for trigonometric polynomials implies

$$\|Tf_n\|_q \leqslant C_q\|f\|_q. \tag{2.30}$$

Since $\|f_n - f\|_1 \leqslant \|f_n - f\|_q$ and T is continuous from L^1 to $\mathcal{L}_0$, we see that $Tf_n \to Tf$ in measure; after extracting a subsequence we can assume that $Tf_n \to Tf$ almost everywhere. Now Fatou's lemma and (2.30) give $\|Tf\|_q \leqslant C_q\|f\|_q$, which was to be proved. □

2.7 Banach's principle

The following fact, known as Banach's principle, plays an important role in applications of Theorem 2.6.1 to maximal operators.

2.7.1 Theorem *Let X be a quasi-Banach space, let T_n $(n \geqslant 1)$ be a sequence of continuous linear operators from X to $\mathcal{L}_0(\Omega,\mu)$, and let*

$$T_{\max}f(\omega) := \sup_{n\geqslant 1}|T_n f(\omega)| < \infty$$

for almost all $\omega \in \Omega$. Then the operator $T_{\max} : X \mapsto \mathcal{L}_0(\Omega,\mu)$ is continuous.

Proof.[‡] Let $\mathcal{L}_0(\ell^\infty)$ denote the set of all functions $F = (f_1, f_2, \dots) : \Omega \mapsto \ell^\infty$ with measurable coordinates. The following two facts, the proof of which is left to the reader, imply the validity of the theorem.

(a) With the F-norm

$$\int_\Omega \frac{\|F(\omega)\|_\infty}{1 + \|F(\omega)\|_\infty}\, d\mu(\omega),$$

the set $\mathcal{L}_0(\ell^\infty)$ is an F-space.

(b) The operator $Tg = (T_1 g, T_2 g, \dots)$ maps X into $\mathcal{L}_0(\ell^\infty)$ and has closed graph. □

Theorem on a.e. convergence

2.7.2 Theorem *Suppose that the conditions of Theorem 2.7.1 are satisfied. If the limit*

$$\lim_{n\to\infty} T_n f(\omega) := Tf(\omega) \qquad \text{a.e.}$$

exists and is finite for every f from a dense subset of X, then it exists for every $f \in X$, and T is continuous as an operator from X to $\mathcal{L}_0$.

Proof. Let X_0 denote the dense subset. Consider the following sublinear operator on X :

$$Sf(\omega) = \limsup_{m,n\to\infty} |T_m f(\omega) - T_n f(\omega)| \qquad (\omega \in \Omega).$$

By Banach's principle, this operator is continuous because $Sf \leqslant 2T_{\max} f$. By Lemma 2.5.3, we have

$$\mu(Sf, \varepsilon) \leqslant c(\varepsilon/\|f\|) \qquad (\varepsilon > 0,\ f \in X), \tag{2.31}$$

where $c(\lambda) \to 0$ as $\lambda \to \infty$. On the other hand, since $Sg = 0$ for $g \in X_0$, we have $S(f) = S(f - g)$ for all $f \in X$, $g \in X_0$. From this and (2.31) it follows that

$$\mu(Sf, \varepsilon) \leqslant c(\varepsilon/\|f - g_k\|) \quad (\varepsilon > 0),$$

where, for a fixed $f \in X$, we have chosen a sequence $g_k \in X_0$ so that $\|f - g_k\| \to 0$ $(k \to \infty)$. Thus $\mu(Sf, \varepsilon) = 0$ for every $\varepsilon > 0$. The result follows. □

[‡] See [7, theorem IV.5.7], where a proof is given based on Baire's theorem.

3 Poisson integral

Harmonic functions occur in Fourier analysis naturally. If $\sum c_n e^{in\theta}$ is the Fourier series of a function $\phi \in L^1(\mathbb{T})$, then the function $f(re^{i\theta}) = \sum_{-\infty}^{\infty} c_n r^{|n|} e^{in\theta}$ is harmonic in $\mathbb{D}$. The function $f = P[\phi]$ is called the Poisson integral of ϕ.

In this chapter we are mainly concerned with some classical results on radial limits of $P[\phi]$, or, what is the same, on the Abel summability of the Fourier series of ϕ, where ϕ belongs to $L^p(\mathbb{T})$, $1 \leqslant p \leqslant \infty$, or $C(\mathbb{T})$. However, there are much stronger results in this direction, and the reader should consult Zygmund's book. For example, in many cases we can replace Abel's method of summability by the method (C, α), $\alpha > 0$. But, in contrast to the case of (C, α), Abel's method leads us to considering not only the radial but also the non-tangential limits of $f(z)$. At the end we prove the Littlewood/Paley inequality and a variant of Schwarz lemma for harmonic functions (Theorems 3.5.1 and 3.6.1)

3.1 Harmonic functions

A complex-valued function f, defined on a domain $\Omega \subset \mathbb{C}$, is said to be harmonic if it is of class C^2 and $\Delta f \equiv 0$ in Ω, where Δ denotes the Laplacian,

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4\frac{\partial^2 f}{\partial z \partial \bar{z}} \qquad (z = x + iy);$$

here

$$\frac{\partial f}{\partial z} = \partial f(z) = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \qquad \frac{\partial f}{\partial \bar{z}} = \bar{\partial} f(z) = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

Thus f is harmonic iff the function $\partial f / \partial z$ is analytic. This implies the following.

3.1.1 Theorem *A harmonic function f defined on a simply connected domain Ω can be represented in the form $f(z) = h(z) + \overline{g(z)}$, $z \in \Omega$, where h and g are analytic and uniquely determined up to an additive constant; conversely, if $f = h + \bar{g}$, where h and g are analytic, then f is harmonic.*

Using this theorem one can deduce various properties of harmonic functions from tbe corresponding properties of analytic functions. For instance,

the composition of a harmonic function with an analytic function is harmonic.

Uniqueness theorem As a further consequence of Theorem 3.1.1 we have:

If f is harmonic in a simply connected domain Ω and $f = 0$ in an open subset of Ω, then $f = 0$ in Ω.

Series expansion *If f is harmonic in $D = \{z : |z| < R\}$, then there are unique complex numbers $\widehat{f}(k)$, $-\infty < k < \infty$, such that $f(re^{i\theta}) = \sum_{k=-\infty}^{\infty} \widehat{f}(k) r^{|k|} e^{ik\theta}$, with the series converging uniformly and absolutely on every compact subset of D. Conversely, if such a series converges in D, then it converges uniformly on compact subsets of D and its sum is harmonic in D.*

Heine/Borel property The set of all functions harmonic in a domain Ω and endowed with the topology of uniform convergence on compact subsets of Ω is denoted by $h(\Omega)$. By $H(\Omega)$ we denote the subspace of $h(\Omega)$ consisting of analytic functions. Both $h(\Omega)$ and $H(\Omega)$ are complete and have the Heine/Borel property. The latter means:

If a sequence $f_n \in h(\Omega)$, resp. $f_n \in H(\Omega)$, is uniformly bounded on compact subsets, then there is a subsequence tending uniformly on compact subsets to a harmonic, resp. analytic, function.

Mean value property A characteristic property of harmonic functions is the mean value property on circles:

$$f(a) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(a + Re^{i\theta})\, d\theta, \qquad \{z : |z-a| \leqslant R\} \subset \Omega. \tag{3.1}$$

Green's formula If f is harmonic, then (3.1) can be deduced from, say, Green's formula, a special case of which can be stated as follows:

If F is a C^2-function in $D = \{z : |z| < R\}$, then

$$\frac{d}{dr} \frac{1}{2\pi} \int_{-\pi}^{\pi} F(re^{i\theta})\, d\theta = \frac{1}{2\pi r} \int_{|z|<r} (\Delta F)(z)\, dm(z) \tag{3.2}$$

or, what is the same,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} F(\rho e^{i\theta})\, d\theta - F(0) = \frac{1}{2\pi} \int_{|z|<\rho} (\Delta F)(z) \log \frac{\rho}{|z|}\, dm(z), \tag{3.3}$$

where $0 < r$, $\rho < R$.

Here dm denotes the Lebesgue measure in $\mathbb{C}$.

That a continuous function having the mean value property must be harmonic can be deduced from the maximum principle for subharmonic functions and the existence of solution of the Dirichlet problem for the disk (see 4.1.10).

Exercises

3.1.2 If f is harmonic in $\mathbb{D}$ and $f(z) = g(|z|)$ in a "rectangle" $r_1 < |z| < r_2$, $\theta_1 < \arg z < \theta_2$, then $f = \text{const}$ in $\mathbb{D}$. This can be shown by using the formula

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{\partial f}{r\, \partial r} + \frac{\partial^2 f}{r^2\, \partial \theta^2}.$$

3.1.3 The mean value of f over a rectifiable Jordan curve Γ is defined as

$$I(f,\Gamma) = \frac{1}{|\Gamma|}\int_\Gamma f(z)\,|dz|, \quad \text{where } |\Gamma| = \text{length of } \Gamma,$$

provided the integral is somehow defined. If $[a,b]$ is the straight line joining a and b and f is analytic in a domain containing Γ, then

$$I(f,[a,b]) = \frac{F(b)-F(a)}{b-a},$$

where F is a primitive function of f. Using this one can prove the following:

Let Π_n be a regular polygon centered at c with vertices $a_1, a_2, \ldots, a_n$, $n \geqslant 4$. Let f be a function harmonic in a domain containing the curve Π_n and its interior. Then

$$I(f,\Pi_n) = \sum_{k=1}^{n} I(f,[c,a_k]).$$

In particular, $I(f,\Pi_4) = I(f,d_1) + I(f,d_2)$, where d_1, d_2 are the diagonals of the square. For the case $n=4$ see [7, Ch. 5, Lemma 6.12].

3.1.4 (maximum modulus principle) If $f \in h(\Omega)$, where Ω is simply connected, and $|f| = \text{const}$ in Ω, then $f = \text{const}$.

The Poisson integral of a continuous function

Poisson kernel One of everywhere occurring harmonic functions is the Poisson kernel:

$$P(z) = \frac{1-|z|^2}{|1-z|^2} = \operatorname{Re}\frac{1+z}{1-z}.$$

There hold the formulas

$$\begin{aligned} P(r,\theta) := P(re^{i\theta}) &= \frac{1-r^2}{1+r^2-2r\cos\theta} \\ &= \sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta} = 1 + 2\sum_{n=1}^{\infty} r^n \cos n\theta, \end{aligned} \tag{3.4}$$

$$P(r,t) \geqslant 0 \quad \text{and} \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,t)\,dt = 1. \tag{3.5}$$

The Poisson kernel has the *reproducing property*:

3.1.5 Theorem *If a function f continuous on $\overline{\mathbb{D}}$ is harmonic in $\mathbb{D}$, then*

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta-t)f(e^{it})\,dt \qquad (re^{i\theta} \in \mathbb{D}). \tag{3.6}$$

Proof. The equality

$$f(0) = \frac{1}{2\pi}\int_{\mathbb{T}} f(\zeta)\,|d\zeta| \qquad (\mathbb{T} = \partial\mathbb{D})$$

follows from the mean value property on the circles $|z| = \rho < 1$ and the continuity of f. Applying this equality to the *harmonic* function $f \circ \varphi_a$, where

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}, \qquad |a| < 1,\ |z| \leqslant 1,$$

we get

$$f(a) = \frac{1}{2\pi}\int_{\mathbb{T}} f(\varphi_a(\zeta))\,|d\zeta|.$$

Now introduce the substitution $\zeta = \varphi_a(w)$, i.e., $w = \varphi_a(\zeta)$. Since

$$|d\zeta| = \frac{1 - |a|^2}{|1 - \bar{a}w|^2}\,|dw|,$$

we get

$$f(a) = \frac{1}{2\pi}\int_{\mathbb{T}} P(\bar{a}w) f(w)\,|dw|,$$

which is another form of (3.6). □

The Poisson integral of a function The Poisson integral of a function $\phi \in L^1(\mathbb{T})$ is the harmonic function $P[\phi]$ defined by

$$P[\phi](re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r, \theta - t)\phi(e^{it})\,dt \qquad (re^{i\theta} \in \mathbb{D}).$$

The proof of Theorem 3.1.5 shows that

$$P[\phi](z) = \frac{1}{2\pi}\int_{\mathbb{T}} \phi(\varphi_z(w))\,|dw| \qquad (z \in \mathbb{D}). \tag{3.7}$$

The Poisson integral can be used to solve the **Dirichlet problem** for the disk:

3.1.6 Theorem *If ϕ is a continuous function defined on $\mathbb{T}$, then ϕ has a unique continuous extension to $\overline{\mathbb{D}}$ that is harmonic in $\mathbb{D}$; this extension equals $P[\phi]$.*

An immediate consequence is the well known **Weierstrass theorem** on approximation by trigonometric polynomials:

The set of all trigonometric polynomials is dense in each of the spaces $C(\mathbb{T})$, $L^p(\mathbb{T})$ $(0 < p < \infty)$.

Proof of Theorem 3.1.6. The uniqueness follows from Theorem 3.1.5. Let $f = P[\phi]$. From (3.7) and the continuity of the function ϕ, by using, for instance, the dominated convergence theorem, we get $\lim_{\mathbb{D}\ni z\to 1} f(z) = \phi(1)$, which proves that the extension is continuous at the point $z = 1$; and so on. □

The above proof, although very short, has a disadvantage in that it is based on very special properties of the Poisson kernel. The standard proof depends on (3.5) and the following: $\lim_{r\to 1} \sup_{\delta<|t|<\pi} P(r,t) = 0$ for all $\delta \in (0,\pi)$ (see [86, 100, 22, 46]).

Theorems 3.1.5 and 3.1.6 yield the following.

3.1.7 Theorem *The Poisson integral acts as an isometric isomorphism from $C(\mathbb{T})$ onto $hC(\overline{\mathbb{D}}) \subset L^\infty(\mathbb{D})$, the space of functions that are defined and continuous on $\overline{\mathbb{D}}$ and harmonic in $\mathbb{D}$.*

3.2 Borel measures and the space h^1

The space $M(\mathbb{T})$

Let $M(\mathbb{T})$ denote the space of all complex Borel measures on the circle $\mathbb{T}$; the norm is given by $\|\mu\| = (1/2\pi)|\mu|(\mathbb{T})$, where $|\mu|$ is the variation of μ. If $d\mu(e^{i\theta}) = \phi(e^{i\theta})\,d\theta$ with $\phi \in L^1(\mathbb{T})$, then $\|\mu\| = \|\phi\|_1$, which means in particular that $M(\mathbb{T})$ contains an isometric copy of $L^1(\mathbb{T})$. By one of Riesz' representation theorems,

$M(\mathbb{T})$ is isometrically isomorphic to the dual of $C(\mathbb{T})$ with respect to the bilinear form

$$(\varphi,\mu) = \frac{1}{2\pi}\int_{\mathbb{T}} \varphi(\bar\zeta)\,d\mu(\zeta), \qquad \varphi \in C(\mathbb{T}),\ \mu \in M(\mathbb{T}).$$

(See [86, Theorem 6.19] for a general representation theorem.)

The Poisson integral of a measure The Poisson integral of $\mu \in M(\mathbb{T})$ is defined by

$$P[\mu](z) = \frac{1}{2\pi}\int_{\mathbb{T}} P(z\bar\zeta)\,d\mu(\zeta). \tag{3.8}$$

Again, if $d\mu(e^{i\theta}) = \phi(e^{i\theta})\,d\theta$ with $\phi \in L^1(\mathbb{T})$, then $P[\mu] = P[\phi]$.

3.2.1 Proposition *The coefficients of the function $P[\mu]$ are equal to the corresponding coefficients of the measure μ, i.e., there holds*

$$\widehat{P[\mu]}(n) = \widehat{\mu}(n) := \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int}\,d\mu(e^{it}).$$

Proof. The formula can be deduced from (3.8) by term-by-term integration of the series (3.4) with respect to μ; this is possible because the series (3.4) converges uniformly on the circles $|z| = r < 1$. □

3.2.2 Theorem *The Poisson integral acts as a continuous and injective linear operator from $M(\mathbb{T})$ into $h(\mathbb{D})$.*

Proof. To prove the injectivity assume $P[\mu] = 0$. From Proposition 3.2.1 we get $\widehat{\mu}(n) = 0$ for every n, and hence $(\phi, \mu) = 0$ for every trigonometric polynomial ϕ. And since trigonometric polynomials are dense in $C(\mathbb{T})$ we have $(\phi, \mu) = 0$ for every $\phi \in C(\mathbb{T})$. Now the Riesz representation theorem gives $\mu = 0$. □

The Riesz/Herglotz theorem

The space h^1 consists of the functions $f \in h(\mathbb{D})$ for which

$$\|f\|_1 := \sup_{r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|\, d\theta < \infty.$$

The Poisson integral of a measure $\mu \in M(\mathbb{T})$ belongs to h^1 because

$$\| P[\mu] \|_1 \leqslant \|\mu\|, \tag{3.9}$$

which is deduced from (3.5). The converse is true as well: every function belonging to h^1 is equal to the Poisson integral of some (unique) measure. More precisely:

3.2.3 Theorem (Riesz/Herglotz) *The Poisson integral acts as an isometric isomorphism from $M(\mathbb{T})$ onto h^1.*

Proof. Let $f \in h^1$. Then the "sequence" f_r $(r \to 1)$, where $f_r(\zeta) = f(r\zeta)$ for $\zeta \in \mathbb{T}$, is bounded in $L^1(\mathbb{T})$. This means that the "sequence" of measures $d\mu_r(e^{i\theta}) = f(re^{i\theta})\, d\theta$ is bounded in $M(\mathbb{T})$. Since $M(\mathbb{T})$ is the dual of $C(\mathbb{T})$, we can apply the Banach/Alaoglu theorem; there exists a sequence $r_n \to 1$ such that μ_{r_n} converges weakly (star) to a measure $\mu \in M(\mathbb{T})$, which means that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(r_n e^{it}) g(e^{-it})\, dt \to \frac{1}{2\pi} \int_{-\pi}^{\pi} g(e^{-it})\, d\mu(e^{it})$$

for every $g \in C(\mathbb{T})$. Take $g(e^{-it}) = P(ze^{-it})$, with fixed $z \in \mathbb{D}$. Then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(r_n e^{it}) g(e^{-it})\, dt = f(r_n z)$$

because the function $f(r_n z)$ is harmonic in a neighborhood of the closed disk. From the last two relations we get $f(z) = P[\mu](z)$. And because of the weak convergence we have $\|\mu\| \leqslant \liminf_n \|\mu_{r_n}\| = \liminf_n \|f_{r_n}\|_1$. Hence $\|\mu\| \leqslant \|f\|_1$, which along with (3.9) implies the desired result. □

3.2.4 Exercise The closure in h^1 of the set of all harmonic polynomials is equal to $\{P[\phi] \colon \phi \in L^1\}$.

Positive harmonic functions

A positive real function $f \in h(\mathbb{D})$ belongs to h^1 because

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{it})|\, dt = f(0).$$

From the proof of Theorem 3.2.3 we obtain the following variant of the Riesz/Herglotz theorem.

3.2.5 Theorem (a) *If a function $f \in h(\mathbb{D})$ is real-valued and positive, then f is equal to the Poisson integral of a finite positive measure.*

(b) *A real harmonic function belongs to h^1 iff it is equal to the difference of two positive harmonic functions.*

The following fact is a consequence of Theorem 3.2.5(a) and the inequality

$$\frac{1-r}{1+r} \leqslant P(r,\theta) \leqslant \frac{1+r}{1-r}.$$

3.2.6 Theorem (Harnack's inequality) *If u is a function harmonic and positive in the disk $|z| < R$, then*

$$\frac{R-r}{R+r}u(0) \leqslant u(re^{i\theta}) \leqslant \frac{R+r}{R-r}u(0) \qquad (0 < r < R)$$
$$\text{and} \qquad |\nabla u(0)| \leqslant 2u(0)/R.$$

3.2.7 Corollary *If u is a positive harmonic function in a disk D, and K a compact subset of D, then $u(z)/u(w) \leqslant C$ for z, $w \in K$, where C is a constant depending only on K.*

3.2.8 Corollary (Hurwitz) *If $\{f_n\}$ is a sequence of analytic functions with no zeros in $\mathbb{D}$ such that $f_n \to f$ uniformly on compact subsets, then f has no zeros in $\mathbb{D}$ unless f vanishes identically.*

Proof. Let D be a closed disk contained in $\mathbb{D}$. Then there is a constant M such that $\sup_D |f_n| < M$ for all n. Now we put $u_n = \log(M/|f_n|)$ and apply Corollary 3.2.7. □

3.2.9 Theorem (Harnack) *Let $\{u_n\}$ be an increasing sequence of real-valued harmonic functions defined in a disk $D \subset \mathbb{C}$. If there exists a point $a \in D$ such that $u_n(a)$ converges, then $\{u_n\}$ converges uniformly on compact subsets of D.*

Proof. Let $K \subset D$ be a compact set containing a. By Corollary 3.2.7 we have $|u_m(z) - u_n(z)| \leqslant C|u_m(a) - u_n(a)|$, where C is independent of m, n and z. □

Poisson/Stieltjes integral

Instead with measures, it is sometimes more convenient to work with functions of bounded variations. Let $BV[a,b]$ denote the set of functions of bounded variation on $[a,b]$. The Poisson/Stieltjes integral of a function $\gamma \in BV = BV[-\pi,\pi]$ is defined as to be the harmonic function

$$PS[\gamma](re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta - t)\,d\gamma(t). \tag{3.10}$$

In view to the well known connection between Borel measures and functions of bounded variation, the above assertions can be stated as follows.

3.2.10 Theorem *A function $f \in h(\mathbb{D})$ belongs to h^1 iff f is equal to the Poisson/Stieltjes integral of a function $\gamma \in BV$. A positive harmonic function is equal to the Poisson/Stieltjes integral of an increasing function.*

It is worthwhile to note that (3.10) can be rewritten in the form

$$PS[\gamma](re^{i\theta}) = k \cdot P(r, \theta + \pi) + \frac{1}{2\pi}\int_{-\pi}^{\pi} P'(r, \theta - t)\gamma(t)\, dt, \tag{3.11}$$

where $k = \dfrac{\gamma(\pi) - \gamma(-\pi)}{2\pi}$ and

$$P'(r,t) = \frac{\partial P}{\partial t} = \frac{-2r \sin t}{1 + r^2 - 2r\cos t} P(r,t), \tag{3.12}$$

or in the form

$$PS[\gamma](re^{i\theta}) = k \cdot P(r, \theta + \pi) + \frac{\partial}{\partial \theta} P[\gamma](re^{i\theta}). \tag{3.13}$$

Exercises

3.2.11 Let f be analytic in $\mathbb{D}$ and $\operatorname{Re} f \geqslant 0$. Then there exists a positive measure $\mu \in M(\mathbb{T})$ and a real constant c such that

$$f(z) = i \operatorname{Im} f(0) + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\zeta + z}{\zeta - z}\, d\mu(\zeta) \qquad (z \in \mathbb{D}).$$

3.2.12 Let $u \in h(\mathbb{D})$ be a *real*-valued function such that $|u(z)| \leqslant 1$ u $\mathbb{D}$. Then

$$|\nabla u(z)| \leqslant 2\frac{1 - |u(z)|}{1 - |z|} \qquad (z \in \mathbb{D}).$$

The assertion does not hold for complex-valued functions even if we replace "2" by an arbitrary constant.

3.2.13 Let $\{u_n\}$ be a sequence of positive harmonic functions in a domain $D \subset \mathbb{C}$. If it is not true that $u_n \to +\infty$ uniformly on compact subsets of D, then there exists a subsequence of $\{u_n\}$ converging uniformly on compact subsets of D.

3.3 Radial limits of the Poisson integral

By the Jordan/Dirichlet test from Fourier analysis, the Fourier series of a 2π-periodic function γ of bounded variation on $[-\pi, \pi]$ converges everywhere and its sum is equal to $\gamma(\theta + 0) + \gamma(\theta - 0)/2$. This implies, via Abel's theorem, that the limit $\lim_{r\to 1} P[\gamma](re^{i\theta})$ exists everywhere and has the same value, so one can expect that

$$\lim_{r\to 1^-} \frac{\partial}{\partial\theta} P[\gamma](re^{i\theta}) = \gamma'(\theta),$$

provided the derivative γ' exists. It follows from (3.13) and the following assertion that this is true.

3.3.1 Proposition *If a function $\gamma \in BV[-\pi,\pi]$ has the finite derivative at $\theta \in (-\pi,\pi)$, then*

$$\lim_{r\to 1^-} PS[\gamma](re^{i\theta}) = \gamma'(\theta). \tag{3.14}$$

Proof. Let $\theta = 0$ and $\gamma'(\theta) = 0$. From (3.11) it follows that

$$L := \limsup_{r\to 1^-} |PS[\gamma](r)| \leqslant \limsup_{r\to 1^-} \frac{1}{2\pi}\int_{-\delta}^{\delta} |P'(r,t)|\cdot|\gamma(t)|\,dt$$

for every (small) $\delta > 0$; this is so because, according to (3.12),

$$|P'(r,t)| \leqslant C(\delta)(1-r) \qquad (|t| > \delta),$$

whence

$$\limsup_{r\to 1^-} \int_{|t|>\delta} |P'(r,t)|\cdot|\gamma(t)|\,dt = 0.$$

Now we apply the inequality $|t\,P'(r,t)| \leqslant 2P(r,t)$, $|t| \leqslant \pi$, to obtain

$$L \leqslant \limsup_{r\to 1^-} \frac{1}{\pi}\int_{-\delta}^{\delta} P(r,t)\left|\frac{\gamma(t)}{t}\right| dt.$$

Finally, (3.14) follows from this and the hypothesis $\gamma(t)/t \to 0$, $t \to 0$. □

3.3.2 (symmetric derivative) In Proposition 3.3.1, the ordinary derivative $\gamma'(\theta)$ can be replaced by the symmetric derivative $\lim_{t\to 0} \dfrac{\gamma(\theta+t)-\gamma(\theta-t)}{2t}$.

Fatou's theorem

Proposition 3.3.1 is of elementary character and only deeper results on differentiability of functions give deeper results on the existence of radial limits. Namely, by the Lebesgue theorem, the derivative $\gamma'(\theta)$ exists for almost all θ (see [86]), which along with Proposition 3.3.1 shows that the Poisson/Stieltjes integral has finite radial limits almost everywhere. Returning to the Poisson integral of a measure, we can state a more precise form of this result as follows.(*)

(*)Rudin considers the case of "M-harmonic" functions on the complex ball, [84, Ch. V]. Much more information on the classical case can be found in Garnett [22, Ch. I] and, of course, Zygmund [100, Ch. III].

3.3.3 Theorem (Fatou) *If $\mu \in M(\mathbb{T})$, then $P[\mu]$ has radial limits almost everywhere. Besides*

$$\lim_{r\to 1^-} P[\mu](re^{i\theta}) = \phi(e^{i\theta})$$

for almost all θ, where $\phi(e^{i\theta})\,d\theta$ is the absolutely continuous part of μ.

Recall that the measure μ is uniquely represented as $d\mu(e^{i\theta}) = \phi(e^{i\theta})\,d\theta + d\mu_s(e^{i\theta})$, where ϕ is an integrable function and μ_s is a singular measure (the theorem of Lebesgue and Radon/Nikodym).

As a special case of Theorem 3.3.3 we have:

3.3.4 Corollary *Every bounded harmonic function on $\mathbb{D}$ has radial limits almost everywhere.*

On the other hand, this special case is sufficient to prove the qualitative part of Fatou's theorem, i.e., to prove the existence of the limits for h^1-functions (see the proof of Corollary 3.3.9).

Another special case:

3.3.5 Corollary *A measure $\mu \in M(\mathbb{T})$ is singular iff $\lim_{r\to 1^-} P[\mu](re^{i\theta}) = 0$ almost everywhere.*

3.3.6 Exercise If u is the Poisson integral of a singular measure, then

$$\lim_{r\to 1} \int_0^{2\pi} |u(re^{i\theta})|^p\,d\theta = 0 \quad \text{for} \quad 0 < p < 1.$$

Theorems of Kolmogorov and Carleson

Since $P[\phi](re^{i\theta}) = \sum_{n=-\infty}^{\infty} \widehat{\phi}(n) r^{|n|} e^{in\theta}$, Theorem 3.3.3 shows that the Fourier series of an integrable function ϕ is almost everywhere summable by the Abel/Poisson method with the sum equal to ϕ. In connection with this, let us mention that Kolmogorov proved the existence of an *everywhere divergent* Fourier series (cf. [44, 100]). Carleson [12] proved that the Fourier series of an L^2-function converges almost everywhere, which is generalized to the case $p > 1$ by Hunt [31].

Nontangential limits

Slightly modifying the proof of Proposition 3.3.1 one can prove the "nontangential" variant of Fatou's theorem:

3.3.7 Theorem *If $f \in h^1$, then for almost all $\zeta \in \mathbb{T}$ there exists the limit*

$$\sphericalangle \lim_{z\to\zeta} f(z) := \lim_{U_\zeta \ni z \to \zeta} f(z),$$

where U_ζ is the convex envelope of the union $\{z : |z| < \rho\} \cup \{\zeta\}$, and $\rho < 1$ is fixed.

Proof. The key property of the set U_ζ is:

$$re^{i\theta} \in U_\zeta \Longrightarrow |\theta - \arg\zeta| \leqslant \mathrm{const}(1-r). \tag{3.15}$$

In order to prove that $\sphericalangle \lim_{z\to 1} PS[\gamma](z) = 0.$ under the hypothesis $\gamma'(0) = 0$, we start from (3.11); we get

$$L := \sphericalangle \limsup_{z\to 1} |PS[\gamma](z)| \leqslant \sphericalangle \limsup_{re^{i\theta}\to 1} \frac{1}{2\pi}\int_{-\delta}^{\delta} |P'(r,\theta-t)| \cdot |\gamma(t)|\, dt$$

$(0 < \delta < 1)$. For a given $\varepsilon > 0$, take δ so that $|\gamma(t)/t| < \varepsilon$ for $|t| < \delta$, whence

$$\int_{-\delta}^{\delta} |P'(r,\theta-t)| \cdot |\gamma(t)|\, dt \leqslant \varepsilon \int_{-\delta}^{\delta} |t\, P'(r,\theta-t)|\, dt.$$

Finally, by using the property (3.15) we find $|t\, P'(r,\theta-t)| \leqslant \mathrm{const}\, P(r,\theta-t)$ $(re^{i\theta} \in U_1)$; thus

$$L \leqslant C\varepsilon \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta-t)\, dt = C\varepsilon, \quad \text{etc.} \qquad \square$$

Radial $\Longrightarrow$ nontangential

The above theorem can also be deduced from the following.

3.3.8 Theorem *Suppose f is bounded and analytic in $\mathbb{D}$, and $\zeta \in \mathbb{T}$. Then the existence of the radial limit of f at ζ implies the existence of the nontangential limit at ζ.*

3.3.9 Corollary *If $f \in H(\mathbb{D})$ and $u = \mathrm{Re}\, f \in h^1$, then f has nontangential limits at almost every point $\zeta \in \mathbb{T}$.*

Proof of Corollary. We can assume that u is positive. Then the function $1/(1+f)$ is analytic and bounded in $\mathbb{D}$. The result follows. $\square$

Proof of Theorem. Let $\zeta = 1$ and $\lim_{r\to 1} f(r) = 0$. If f fails to have the limit 0 within U_ζ, then there exist $\varepsilon > 0$ and sequences $t_n \to 0$, $0 < t_n < 1$, and w_n, $|w_n| < \rho_0 < 1$, such that

$$|f(t_n w_n + 1 - t_n)| > \varepsilon \quad \text{for all} \quad n. \tag{3.16}$$

Consider the functions $f_n(w) = f(t_n w + 1 - t_n)$, $w \in \mathbb{D}$. The sequence f_n is uniformly bounded in $\mathbb{D}$ and therefore there exists a subsequence, denote it by f_n, that converges to a function $g \in H(\mathbb{D})$ uniformly on $|w| < \rho_0$; this means that $f_n(w_n) - g(w_n) \to 0$. By the hypothesis, we have $f(t_n r + 1 - t_n) \to 0$ for $0 < r < 1$, which implies that $g(r) = 0$ for $0 < r < 1$, whence $g(w) = 0$ for all $w \in \mathbb{D}$. Thus $f_n(w_n) \to 0$, which contradicts (3.16). $\square$

Lindelöf's theorem Theorem 3.3.8 is a special case of Lindelöf's theorem:

Let $f \in H^\infty(\mathbb{D})$ and let $\gamma\colon [0,1) \mapsto \mathbb{D}$ be a continuous curve such that $\gamma(t) \to 1$ as $t \to 1$. If there exists $\lim_{t\to 1} f(\gamma(t)) = L$, then f has nontangential limit L at the point 1.

For the proof see [84, Theorem 8.4.1].

3.4 The spaces h^p and $L^p(\mathbb{T})$

The *harmonic* Hardy space h^p $(0 < p \leqslant \infty)$ is defined as

$$h^p = \{\, f \in h(\mathbb{D})\colon \|f\|_p = \sup_{r<1} M_p(r,f) < \infty \,\},$$

where $M_p(r,f)$ is the *integral mean* of order p,

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^p \, d\theta\right)^{1/p}.$$

In the case $p = \infty$ the integral is to be interpreted as a supremum:

$$M_\infty(r,f) = \sup_{\theta\in[-\pi,\pi]} |f(re^{i\theta})|.$$

Therefore $h^\infty = h^\infty(\mathbb{D})$ is the subspace of $L^\infty(\mathbb{D})$ spanned by harmonic functions.

By Parseval's formula we have $M_2^2(r,f) = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2 r^{2|n|}$, and consequently

$$\|f\|_2 = \left(\sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2\right)^{1/2}.$$

3.4.1 Proposition *If $p \geqslant 1$, and $f \in h(\mathbb{D})$, then $M_p(r,f)$ increases with r. Moreover, if $p > 1$, then $M_p(r,f)$ is strictly increasing unless $f =$ const.*

It should be noted, however, that if $p < 1$ and f is positive, then $M_p(r,f)$ is decreasing.

Proof. The "increasing" property can be deduced from the subharmonicity of the function $|f|^p$ (see Theorem 4.2.1), or by application of Minkowski's inequality in continuous form ("norm of integral $\leqslant$ integral of norm") to the formula[(†)]

$$f(\lambda re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(\lambda,t) f_r(\theta+t)\, dt \qquad (0<\lambda<1),$$

where $f_r(t) = f(re^{it})$. To prove the second assertion, let $p > 1$ and suppose that $M_p(r_1,f) = M_p(r_2,f)$ for some $0 \leqslant r_1 < r_2 < 1$. Let

$$u(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(ze^{it})|^p \, dt = M_p^p(|z|,f), \qquad |z| < r_2.$$

[(†)]The case $p = \infty$ is trivial, while in the case $1 \leqslant p < \infty$ we can apply Jensen's inequality as well.

The function u is subharmonic in the disk $|z| < r_2$ and attains its maximum at $z = r_1$. It follows that u=const, by the maximum principle (see Theorem 4.1.7), whence

$$|f(0)|^p = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{it})|^p\,dt, \qquad 0 < r < \rho.$$

Since

$$|f(0)|^p = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(re^{it})\,dt\right|^p \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{it})|^p\,dt$$

(by Jensen's inequality or by Hölder's inequality), we see that

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(re^{it})\,dt\right|^p = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{it})|^p\,dt, \qquad r < r_2.$$

This implies that $f(re^{it})$ depends only on r, when $r < r_2$, and therefore $f(z) =$ const for $|z| < 1$; see 3.1.2. □

Radial limits

Since

$$\|\phi\|_2 = \Big(\sum_{n=-\infty}^{\infty} |\widehat{\phi}(n)|^2\Big)^{1/2},$$

where $\phi \in L^2$, as well as $\widehat{f}(n) = \widehat{\phi}(n)$ provided $f = P[\phi]$, the Poisson integral acts as an isometric isomorphism from L^2 onto h^2. It is very important that this fact extends to the case $1 < p \leqslant \infty$. On the other hand, as we have seen, the operator $P\colon L^1 \mapsto h^1$ is not *onto*.

3.4.2 Theorem *The function f belongs to h^p $(1 < p \leqslant \infty)$ iff it is equal to the Poisson integral of some function $\phi \in L^p(\mathbb{T})$. And if $f = P[\phi]$, $\phi \in L^p(\mathbb{T})$, then*

$$\|f\|_p = \|\phi\|_p \qquad (1 \leqslant p \leqslant \infty),$$

$$\lim_{r\to 1^-} \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta}) - \phi(e^{i\theta})|^p\,d\theta = 0 \qquad (1 \leqslant p < \infty) \tag{3.17}$$

and

$$\lim_{r\to 1^-} f(re^{i\theta}) = \phi(e^{i\theta}) \qquad \textit{almost everywhere.}$$

The proof is very similar to the proof of Theorem3.2.3 and will be omitted here.

The Poisson kernel shows that an h^1-function need not be equal to the Poisson integral of the boundary function. However, (3.17) implies the following.

3.4.3 Corollary *If $f \in h^p$, $p > 1$, then $f = P[f_*]$, where $f_*(e^{i\theta}) = \lim_{r\to 1^-} f(re^{i\theta})$.*

The Poisson kernel also shows that boundedness of f_* does not imply boundedness of f. However, if $p > 1$, we have the following.

3.4.4 Corollary *Let a function $f \in h(\mathbb{D})$ have radial limits $f_*(e^{i\theta})$ almost everywhere and $f_* \in L^\infty(\mathbb{T})$. If $f \in h^p$ for some $p > 1$, then $f \in h^\infty$ and $\|f\|_\infty = \|f_*\|_\infty$.*

Exercises

3.4.5 [72] The Poisson kernel satisfies: $M_p^p(r,P) = M_q^q(r,P)$ $(q = 1-p)$ and

$$\begin{aligned} M_p(r,P) &\asymp (1-r), && \text{for } 0<p<1/2;\\ &\asymp (1-r)\Big(\log\frac{e}{1-r}\Big)^2, && \text{for } p=1/2;\\ &\asymp (1-r)^{1/p-1}, && \text{for } p>1/2. \end{aligned}$$

3.4.6 The inclusion $h^p \subset h(\mathbb{D})$ $(1 \leqslant p \leqslant \infty)$ is compact, i.e., every closed ball of the space h^p are compact in the topology of uniform convergence on compact subsets of $\mathbb{D}$.

3.4.7 Let $z \in \mathbb{D}$. The norm of the linear functional $z \mapsto f(z)$ on the space h^p $(1 \leqslant p < \infty)$ is equal to $K_p(|z|)(1-|z|^2)^{-1/p}$, where

$$K_p(r) = \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |1-re^{it}|^{2q-2}dt\right\}^{1/q} \qquad (1/p+1/q=1).$$

Observe that $K_2(r) = (1+r^2)^{1/2}$.

3.5 The Littlewood/Paley theorem

Here we consider the simplest variant of the Littlewood/Paley theorem.

3.5.1 Theorem (a) *If $u \in h^p$, $2 \leqslant p < \infty$, then*

$$K := \int_{\mathbb{D}} |\nabla u(z)|^p (1-|z|)^{p-1}\, dA(z) < \infty \tag{3.18}$$

and there holds the inequality

$$\int_{\mathbb{D}} |\nabla u(z)|^p (1-|z|)^{p-1}\, dA(z) \leqslant C_p \|u\|_p^p.$$

(b) *If u is harmonic in $\mathbb{D}$ and satisfies condition* (3.18) *for some $1<p<2$, then $u \in h^p$ and we have $\|u\|_p^p \leqslant C_p\,(K + |u(0)|^p)$.*

The case $p > 2$

There are many proofs of (a). For example, it is possible to apply the Riesz/Thorin interpolation theorem; namely, the operators

$$(S_1 u)(z) = (1-|z|)\frac{\partial u}{\partial z} \quad \text{and} \quad (S_2 u)(z) = (1-|z|)\frac{\partial u}{\partial \bar z}$$

map h^p into $L^p(\mathbb{D}, dA/(1-|z|))$ for $p=\infty$ and $p=2$. An elementary but rather long proof, based on local estimates deduced from the Hardy/Stein identity, was found by Luecking [55]. We shall present a proof that is both elementary and short. We only need three simple lemmas on positive harmonic functions.

3.5.2 Lemma *If u is a positive harmonic function in $\mathbb{D}$, then* $|\nabla u(z)| \leqslant \dfrac{2u(z)}{1-|z|}$.

Proof. Applying the inequality $|\nabla u(0)| \leqslant 2u(0)$ (Theorem 3.2.6) to the function $w \mapsto u(z+(1-|z|)w)$, we get the result. □

3.5.3 Lemma *If $u \in h^p$ is a real-valued function and $1<p<\infty$, then there are nonnegative functions h_1 and h_2 from h^p such that*

$$u = h_1 - h_2, \qquad \|u\|_p^p = \|h_1\|_p^p + \|h_2\|_p^p.$$

Proof. Let $g_1(\zeta) = \max\{u(\zeta), 0\}$ and $g_2(\zeta) = \max\{-u(\zeta), 0\}$ for $\zeta \in \mathbb{T}$. Then, because of Theorem 3.4.2, the required conditions are satisfied by the functions $h_1 = P[g_1]$ and $h_2 = P[g_2]$. □

3.5.4 Lemma *Let $u > 0$ belong to h^p, $1<p<\infty$. Then*

$$\|u\|_p^p = |u(0)|^p + \frac{p(p-1)}{2}\int_{\mathbf{D}} u^{p-2}|\nabla u(z)|^2 \log\frac{1}{|z|}\, dA.$$

Proof. This is easily deduced from Green's formula and the formula

$$\Delta(u^p) = p(p-1)u^{p-2}|\nabla u|^2. \qquad \square$$

Proof of Theorem 3.5.1(a). It is enough to consider real-valued functions. Because of Lemma 3.5.3, we can suppose that u is positive. Then Lemmas 3.5.4 and 3.5.2 give

$$\|u\|_p^p \geqslant \frac{p^2-p}{2}\int_{\mathbf{D}} |\nabla u|^2 2^{2-p}|\nabla u|^{p-2}(1-|z|)^{p-1}\, dA,$$

which implies the desired conclusion.

The case $1<p<2$

We shall apply the method of "dualization." Let X_p be the (real) subspace of h^p consisting of real-valued functions. Let Y_p be the space of real-valued harmonic functions that satisfy (3.18) or, equivalently,

$$\int_{\mathbf{D}} |\nabla u(z)|^p\Big(\log\frac{1}{|z|}\Big)^{p-1} dA(z) < \infty;$$

the norm is introduced in the obvious way. From the proof of (a) it follows that $X_q \subset Y_q$ $(1/q+1/p=1)$, the inclusion being continuous. From this we can conclude that $Y_p \subset X_p$ provided that we have the inclusions $Y_p \subset (Y_q)^*$ and $(X_q)^* \subset X_p$ with respect to the same bilinear form. The results of Section 3.4 can be formulated in terms of real L^p and h^p spaces. As a consequence we we get $(X_q)^* = X_p$ with respect to the bilinear form

$$(u,v)_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{i\theta})\, v(e^{i\theta})\, d\theta.$$

On the other hand, by Hölder's inequality, we have $Y_p \subset (Y_q)^*$ with respect to a different form, namely

$$(u,v)_1 = u(0)v(0) + \int_{\mathbb{D}} \nabla u(z)\cdot\nabla v(z)\,\log\frac{1}{|z|}\,dA(z).$$

Fortunately, from Green's formula and the equality $\Delta(uv) = 2\nabla u\cdot\nabla v$ it follows that $(u,v)_0 = (u,v)_1$ provided that u and v are harmonic in a neighborhood of the closed disk. Then, by Hölder's inequality, we get

$$|(u,v)_0| = |(u,v)_1| \leqslant \|u\|_{Y_p}\|v\|_{Y_q}.$$

Now we use assertion (a) to get $|(u,v)_0| \leqslant C\|u\|_{Y_p}\|v\|_{X_q}$, where C depends only on q. And since $\sup\{|(u,v)_0| : \|v\|_{X_q} \leqslant 1\} = \|u\|_{X_p}$, we see that $\|u\|_p \leqslant C\|u\|_{Y_p}$, provided the function u is harmonic in a neighborhood of the closed disk. In the general case, we apply this inequality to the functions $u(\rho z)$, $\rho \to 1$, and this completes the proof for $1 < p < 2$.

3.6 Harmonic Schwarz lemma

If $f \in h(\mathbb{D})$, $|f| \leqslant 1$, and $f(0) = 0$, then there holds the sharp inequality

$$|f(z)| \leqslant \frac{4}{\pi}\arctan|z|\,.$$

See [6]; see also 4.4.6.

3.6.1 Theorem *Let f be a complex-valued function harmonic in $\mathbb{D}$ such that $|f(z)| \leqslant 1$ for $z \in \mathbb{D}$. Then there holds the inequality*

$$\left|f(z) - \frac{1-|z|^2}{1+|z|^2}f(0)\right| \leqslant \frac{4}{\pi}\arctan|z|. \tag{3.19}$$

If equality occurs for some $z \in \mathbb{D}\setminus\{0\}$, then there are real constants α and β such that $f(z) = e^{i\alpha}\kappa(e^{i\beta}z)$ for all $z \in \mathbb{D}$, where

$$\kappa(z) = \frac{2}{\pi}\arg\frac{1-z}{1+z} = \frac{2}{\pi}\arctan\frac{2r\sin\theta}{1-r^2} \qquad (z = re^{i\theta}).$$

Proof. We start from the formula

$$\begin{aligned} f(r) - \frac{1-r^2}{1+r^2}f(0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1-r^2}{1+r^2-2r\cos\theta} - \frac{1-r^2}{1+r^2}\right)f_*(e^{i\theta})\,d\theta \\ &= \frac{(1-r^2)2r}{1+r^2}\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\cos\theta}{1+r^2-2r\cos\theta}f_*(e^{i\theta})\,d\theta. \end{aligned}$$

From this and the hypothesis $|f| \leqslant 1$ we get

$$\left|f(r) - \frac{1-r^2}{1+r^2}f(0)\right| \leqslant \frac{(1-r^2)2r}{1+r^2}\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|\cos\theta|}{1+r^2-2r\cos\theta}\,d\theta.$$

So we have to compute the integral

$$J = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|\cos\theta|}{1+r^2-2r\cos\theta}\,d\theta.$$

We have

$$\begin{aligned} J &= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\Big(\frac{\cos\theta}{1+r^2-2r\cos\theta}+\frac{\cos\theta}{1+r^2+2r\cos\theta}\Big)\,d\theta \\ &= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\frac{2(1+r^2)\cos\theta}{(1+r^2)^2-4r^2\cos^2\theta}\,d\theta \\ &= \frac{1}{\pi}\int_{0}^{\pi/2}\frac{2(1+r^2)\cos\theta}{(1-r^2)^2+4r^2\sin^2\theta}\,d\theta \end{aligned}$$

Since

$$\int \frac{2(1+r^2)\cos\theta}{(1-r^2)^2+4r^2\sin^2\theta}\,d\theta = \frac{1+r^2}{r(1-r^2)}\arctan\frac{2r\sin\theta}{1-r^2},$$

we see that

$$J = \frac{1+r^2}{r(1-r^2)}\arctan\frac{2r}{1-r^2} = \frac{1+r^2}{r(1-r^2)}2\arctan r.$$

Combining all these results we get (3.19) for $z=r$.

If equality holds for a fixed $z=r>0$, then, as the above inequalities show, we have $|f_*|=1$ a.e., and $f_*(e^{i\theta})\cos\theta = e^{i\alpha}g(\theta)$, where α is a constant and $g \geqslant 0$. It follows that

$$e^{-i\alpha}f_*(e^{i\theta}) = \begin{cases} 1, & \theta\in(-\pi/2,\pi/2), \\ -1, & \theta\in(\pi/2,3\pi/2). \end{cases}$$

Hence

$$\begin{aligned} f(re^{i\theta}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta-t)f_*(e^{i\theta})\,dt \\ &= e^{i\alpha}\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\frac{(1-r^2)4r\cos(t-\theta)}{(1-r^2)^2+4r^2\sin^2\theta}\,d\theta \\ &= e^{i\alpha}\frac{2}{\pi}\arctan\frac{2r\cos\theta}{1-r^2} = e^{i\alpha}\kappa(ire^{i\theta}). \qquad \square \end{aligned}$$

3.6.2 With the hypotheses of Theorem 3.6.1 we have $|\partial f(0)|+|\bar\partial f(0)| \leqslant 4/\pi$ with equality iff $f(z)\equiv e^{i\alpha}\kappa(e^{i\beta}z)$ for some $\alpha,\beta\in\mathbb{R}$. Therefore, if f is in addition real-valued, then $|\nabla f(0)|\leqslant 4/\pi$.

4 Subharmonic functions

A real-valued C^2-function u is subharmonic iff $\Delta u \geqslant 0$. If u is not of class C^2, then u is subharmonic iff it is the limit of a decreasing sequence of subharmonic functions of class C^2. The importance of subharmonic functions for spaces of analytic and harmonic functions lies in the fact that if f is analytic (resp. harmonic), then $|f|^p$ is subharmonic for every $p > 0$ (resp. $p \geqslant 1$). This chapter contains concise proofs of the basic properties such as the maximum principle, local integrability, approximation by smooth functions, the subordination principle. The discussion of the integral means (Sections 4.2, 4.3) follows Hörmander's book [30][(*)] and includes Prawitz' theorem 4.3.1 and, as consequences, Koebe's one-quarter theorem, and Bieberbach's theorem. In Section 4.5 we prove a weak version of Riesz' representation theorem. Section 4.6 is devoted to the proof of a Littlewood/Paley theorem for subharmonic functions.

4.1 Basic properties

A function $u : D \mapsto [-\infty, \infty)$, where D is a subdomain of the complex plane, is said to be *subharmonic* if it is upper semicontinuous, i.e.,

$$u(a) \geqslant \limsup_{z \to a} u(z) \quad \text{for all } a \in D, \tag{4.1}$$

and for every $a \in D$ there exists $R > 0$ such that $\{z : |z - a| < R\} \subset D$ and

$$u(a) \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} u(a + \rho e^{i\theta})\, d\theta \quad \text{for every} \quad 0 < \rho < R. \tag{4.2}$$

Upper semicontinuity implies boundedness from above, which guarantees the existence of the integral in (4.2). From (4.2) it follows that for every $a \in D$ there exists a sequence $z_n \to a$ such that $u(a) \leqslant u(z_n)$, which implies $\limsup_{z \to a} u(z) = u(a)$. In particular, u is continuous at a if $u(a) = -\infty$. There are discontinuous subharmonic functions; e.g., the function

$$u(z) = \sum_{n=1}^{\infty} \frac{\log |z - 2^{-n}|}{2^n}$$

is subharmonic in the entire plane and is discontinuous at zero. Because of the mean value property,

every real-valued harmonic function is both subharmonic and superharmonic.

[(*)] The only difference is in the proof of Prawitz' theorem.

A function u is said to be superharmonic if $-u$ is subharmonic.

In the case of C^2-functions there is a simple criterion of subharmonicity deduced from Green's formula:

A function $u \in C^2(D)$ is subharmonic iff $\Delta u \geqslant 0$ in D.

From this and the formula $\Delta(u \circ \varphi)(z) = (\Delta u)(\varphi(z))\,|\varphi'(z)|^2$, where φ is an analytic function, we get:

The composition $u \circ \varphi$ is subharmonic if u is subharmonic and φ is analytic.

In the general case this assertion can be reduced to the "smooth" case by approximating an arbitrary subharmonic function by smooth ones (Theorem 4.1.15, later on).

An important example of a subharmonic function taking the value $-\infty$ is the function $\log|z-a|$. More generally:

4.1.1 Theorem *If f is analytic in D, then the function $\log|f|$ is subharmonic in D, and $|f|^p$ is subharmonic for every $p > 0$.*

New examples can be produced by using the following assertions:

4.1.2 Theorem *The sum and the maximum of a finite sequence of subharmonic functions are subharmonic functions. The same holds for the limit of a decreasing sequence of subharmonic functions.*

4.1.3 Theorem *Let ϕ be an increasing convex function that is defined and continuous on an interval $I \subset [-\infty, +\infty)$. If v is subharmonic and takes its values in I, then the function $u = \phi(v)$ is subharmonic. In particular, u is subharmonic in the following cases:*

(a) *$u = |h|^p$, where $p \geqslant 1$ and h is harmonic;*

(b) *$u = v^p$, $p \geqslant 1$, where v is subharmonic and nonnegative.*

Exercises

4.1.4 [59] If $p > 0$ and $f(z) = \sum_{k=m}^{n} a_k z^k$, then for $0 < p < \infty$ we have

$$r^n M_p(1,f) \leqslant M_p(r,f) \leqslant r^m M_p(1,f) \qquad 0 < r < 1.$$

4.1.5 Let $h \not\equiv 0$ be a real-valued harmonic function and $0 < p < 1$. The function $|h|^p$ is subharmonic iff h is constant. The function $|h|^p$ is superharmonic iff h has no zeros.

4.1.6 Let $u_1, \dots, u_n$ be a sequence of nonnegative functions subharmonic in D. If $p \geqslant 1$, then the function $u := (u_1^p + \cdots + u_n^p)^{1/p}$ is subharmonic in D. If $u_k = |f_k|$, where f_k are analytic, then u is subharmonic for every $p > 0$.

The maximum principle

The simplest variant of the maximum principle says:

4.1.7 Theorem *A nonconstant subharmonic function cannot attain its maximum inside the domain. In particular, a nonconstant harmonic function attains neither the maximum nor the minimum inside the domain.*

Proof. Let u be subharmonic in a domain D and let M denote the set of points in D where u attains its maximum. Because of the semicontinuity, M is closed. Let us prove that M is open as well, which will imply $M = D$ or $M = \emptyset$, so the proof will be finished. Let $a \in M$. Then (4.2) implies that, for R small enough and for all $\rho < R$, we have $u(a) = u(a + \rho\zeta)$ almost everywhere on the circle $|\zeta| = 1$. From this and (4.1) it follows that $u(a) \leqslant u(a + \rho\zeta)$ everywhere; thus, $u(a) = u(a + \rho\zeta)$ everywhere, i.e., $\{z : |z - a| < R\} \subset M$. □

4.1.8 Corollary *If u is upper semicontinuous on $\overline{D}$ and subharmonic in D, then*

$$\max\{u(z) : z \in \overline{D}\} = \max\{u(\zeta) : \zeta \in \partial D\}.$$

4.1.9 Corollary *Let D be a bounded domain, let u be a function subharmonic in D and upper semicontinuous on $\overline{D}$, and let h be a real-valued function harmonic in D and continuous on $\overline{D}$. If $u \leqslant h$ on ∂D, then $u \leqslant h$ in D; besides, $u < h$ if u is not harmonic.*

4.1.10 Corollary *If a function is both subharmonic and superharmonic, then it is harmonic.*

Proof. If u and $-u$ are subharmonic in D, then u is continuous. Let K be a disk, relatively compact in D, and let h be harmonic and $h = u$ on ∂K. Then $u \leqslant h$ and $-u \leqslant -h$ and K. □

Corollary 4.1.9 gives a partial explanation of the term "subharmonic." Moreover, we have:

4.1.11 Theorem *Let u be an upper semicontinuous function in a domain D, with values in $[-\infty, \infty)$. Then u is subharmonic iff for each disk K with $\overline{K} \subset D$, and each function h continuous on $\overline{K}$ and harmonic in K, the condition $u \leqslant h$ on ∂K implies $u \leqslant h$ in K.*

Proof. Necessity follows from Corollary 4.1.8 applied to the function $u - h$. Let $\overline{\mathbb{D}} \subset D$ and let u be an upper semicontinuous function satisfying the above condition. Let ϕ be an arbitrary continuous function on $\partial\mathbb{D}$ such that $\phi \geqslant u$ on $\partial\mathbb{D}$. Then the function

$$h(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(r, \theta - t)\phi(e^{it})\, dt$$

equals the continuous extension of ϕ to $\overline{\mathbb{D}}$ (Theorem 3.1.6). Hence $u \leqslant h$ in $\mathbb{D}$ and in particular $u(0) \leqslant h(0)$, i.e.,

$$u(0) \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(e^{it})\, dt.$$

Being upper semicontinuous, the function $u(e^{it})$ is the infimum of a family of continuous functions; therefore,

$$u(0) \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{it})\, dt.$$

Applying this inequality to the functions $z \mapsto u(a + \rho z)$, we find that there holds (4.2), and this was to be proved. □

Local integrability

If u is subharmonic in D, then

$$u(a) \leqslant \frac{1}{\pi\rho^2} \int_{|z-a|<\rho} u(z)\, dm(z) < \infty, \tag{4.3}$$

whenever $\{z : |z - a| \leqslant r\} \subset D$, and therefore u is integrable in a neighborhood of any point at which u has a finite value. Moreover, u is integrable near an arbitrary point:

4.1.12 Theorem *Every subharmonic function $u \not\equiv -\infty$ is locally integrable.*

4.1.13 Remark It follows from Theorem 4.5.2 that $|u|^p$ is locally integrable for every $p < \infty$.

4.1.14 Corollary *If $u \not\equiv -\infty$ is subharmonic, then the integral in (4.2) is finite.*

Proof of Theorem 4.1.12. Let E be the set of those points in D in a neighborhood of which u is integrable. This set is nonempty because of (4.3). The definition implies that E is open. Let b belong to ∂E and let $G \subset D$ be the disk of radius ε centered at b. Then G contains at least one point a such that $|a - b| < \varepsilon/4$ and $u(a) > -\infty$. (Otherwise, b is in the interior of the complement of E.) The disk $G_0 = \{|z - a| < \varepsilon/2\}$ contains b and u is integrable on G_0 because of (4.3). Hence, $b \in E$, which means that E is closed as well. □

Approximation by smooth functions

4.1.15 Theorem *Let $u \not\equiv -\infty$ be subharmonic in a domain D. Then there exists an increasing sequence of open sets D_n, whose union is D, and a decreasing sequence of subharmonic functions $u_n \in C^\infty(D_n)$ tending to u.*

Proof. Let $\omega(z) = \omega_0(|z|)$ be a nonnegative "radial" function of class $C^\infty(\mathbb{C})$ with compact support in $\mathbb{D}$ such that

$$\int_{\mathbf{D}} \omega(w)\, dm(w) = 1.$$

For $\varepsilon > 0$ small enough consider the sets $D_\varepsilon = \{z : \operatorname{dist}(z, \mathbb{C} \setminus D) > \varepsilon\}$ and the functions

$$u_\varepsilon(z) = \int_{\mathbf{C}} \omega(w) u(z + \varepsilon w)\, dm(w) = \int_{\mathbf{C}} \varepsilon^{-2} \omega((w - z)/\varepsilon)\, u(w)\, dm(w),$$

where $u \equiv 0$ outside of D. Then u_ε is finite (because of local integrability of u), subharmonic and of class C^∞ in D_ε. From the formula

$$u_\varepsilon(z) = 2 \int_0^1 r\omega_0(r)\, dr \int_{-\pi}^{\pi} u(z + r\varepsilon e^{it})\, dt$$

and the inequality

$$\int_{-\pi}^{\pi} u(z + r\varepsilon e^{it})\, dt \leqslant \int_{-\pi}^{\pi} u(z + r\delta e^{it})\, dt, \qquad \delta < \varepsilon,$$

(see Theorem 4.2.1 and its proof), it follows that $u(z) \leqslant u_\delta(z) \leqslant u_\varepsilon(z)$, $\delta < \varepsilon$, $z \in D_\varepsilon$. And since u is bounded from above on compact subsets, we can apply the "limsup" variant of Fatou's lemma; we get

$$\limsup_{\varepsilon \to 0} u_\varepsilon(z) \leqslant 2 \int_0^1 r\omega_0(r)\, dr \int_{-\pi}^{\pi} \limsup_{\varepsilon \to 0} u(z + r\varepsilon e^{it})\, dt.$$

Hence $\limsup_{\varepsilon \to 0} u_\varepsilon(z) \leqslant u(z)$, which completes the proof. □

Miscellaneous

4.1.16 Let $u \not\equiv -\infty$ be upper semicontinuous on $\overline{\mathbb{D}}$ and subharmonic in $\mathbb{D}$. Then

$$u(re^{i\theta}) \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} P(r, \theta - t) u(e^{it})\, dt \qquad (0 < r < 1).$$

4.1.17 If a real-valued function h is semicontinuous and satisfies the condition

$$h(a) = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(a + re^{i\theta})\, d\theta,$$

locally, then h is harmonic.

4.1.18 If an upper semicontinuous function u satisfies the condition

$$u(a) \leqslant \frac{1}{\pi r^2} \int_{|z-a|<r} u(z)\, dm(z),$$

locally, then u is subharmonic.

4.2 Properties of the mean values

Convexity and monotonicity

By definition, a real function $\varphi(r)$, $r > 0$, is convex of $\log r$ if the function $x \mapsto \varphi(e^x)$ is convex. In other words, $\varphi(r)$ is convex of $\log r$ if there holds the inequality

$$\varphi(r_1^{1-\lambda} r_2^{\lambda}) \leqslant (1-\lambda)\varphi(r_1) + \lambda\varphi(r_2), \qquad 0 < \lambda < 1.$$

If φ is of class C^2, then it is convex of $\log r$ iff $\varphi''(r) + \varphi'(r)/r \geqslant 0$.

4.2.1 Theorem *Let u be subharmonic in the disk $|z| < R$. Then the function*

$$I(r,u) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(re^{i\theta})\,d\theta \qquad (0 < r < R)$$

is finite, increasing and convex of $\log r$. *The same holds for the function*

$$I_\infty(r,u) = \max_{0 \leqslant t \leqslant 2\pi} u(re^{it}).$$

Proof. If u is subharmonic, then so is the function

$$I(z) := \frac{1}{2\pi}\int_{-\pi}^{\pi} u(ze^{i\theta})\,d\theta = I(|z|, u) \qquad (|z| < R).$$

For fixed $0 < r_1 < r_2$ define the harmonic function $h(z) = a\log|z| + b$ by $h(r_j) = I(r_j)$. Since $I(z) = h(z)$ on the boundary of the annulus $r_1 \leqslant |z| \leqslant r_2$, we have $I(z) \leqslant h(z)$ for $r_1 < |z| < r_2$. From this it follows that $I(r,u)$ is convex of $\log r$. That $I(r,u)$ increases with r follows from the fact that the function $\varphi(x) = I(r, e^x)$ is convex and bounded for $-\infty < x < \log R$, and this completes the proof in case of $I(r,u)$.

In case of I_∞ the proof is similar; we define h by $h(z) = I_\infty(r_j)$ for $|z| = r_j$. □

The function $I(r,u)$ need not be increasing if u is defined and subharmonic in an annulus; a simple example is the function $u(z) = -\log|z|$ which is subharmonic (and harmonic) in the annulus $\mathbb{C} \setminus \{0\}$.

4.2.2 Theorem *Let u be subharmonic in the annulus $\rho < |z| < R$. Then the function*

$$I(r,u) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(re^{i\theta})\,d\theta \qquad (\rho < r < R)$$

is finite and convex of $\log r$.

Remark. The same holds for the function $I_\infty(r,u)$.

Proof. Since u is locally integrable and

$$\int_{r_0<|z|<r_1} u(z)\,dm(z) = \int_{r_0}^{r_1} r\,dr \int_{-\pi}^{\pi} u(re^{i\theta})\,d\theta,$$

we see that $I(r,u)$ is finite for almost every r; that it is finite for every r will follow from the convexity. In proving that $I(r,u)$ is convex of $\log r$ we can proceed as in the case of Theorem 4.2.1, or use Theorem 4.1.15 to reduce the proof to the case where u is smooth. If u is C^2, then the function $v(z) = \varphi(|z|)$, where $\varphi(r) = I(r,u)$ is subharmonic and C^2. It follows that $0 \leqslant \Delta v(z) = \varphi''(r) + \varphi'(r)/r$, $r = |z|$, and this completes the proof. □

Logarithmic convexity

A positive real function φ is said to be logarithmically convex if $\log\varphi$ is convex. A necessary and sufficient condition for φ to be logarithmically convex in (a,b) is that for every $c \in \mathbb{R}$ the function $e^{cx}\phi(x)$ be convex in (a,b). We say that $\varphi(r)$ is logarithmically convex of $\log r$ if $\varphi(r) > 0$ and $\log\varphi$ is convex of $\log r$, which can be expressed as $\varphi(r_1^{1-\lambda}r_2^{\lambda}) \leqslant \varphi(r_1)^{1-\lambda}\varphi(r_2)^{\lambda}$, $0 < \lambda < 1$. If φ is continuous, then the validity of this for $\lambda = 1/2$ is sufficient for φ to be log-convex.

4.2.3 Theorem *Let u be subharmonic in the annulus $\rho < |z| < R$, $\rho \geqslant 0$. Then the function $I(r,e^u)$ is logarithmically convex of* $\log r$ *in the interval $\rho < r < R$.*

Proof. The function $v(z) = e^{c\log|z|+u(z)}$ is subharmonic in the annulus $\rho < |z| < R$ for every $c > 0$. Hence, the function $I(e^x,v) = e^{cx}I(e^x,e^u)$ is convex for every $c > 0$. The result follows. □

4.2.4 Corollary *If f is a function analytic in the annulus $\rho < |z| < R$, then the function*

$$M_p(r,f) = \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^p\,d\theta\right\}^{1/p}$$

is logarithmically convex of $\log r$ *in the interval $\rho < r < R$, for every $0 < p \leqslant \infty$.*

In the case $p = \infty$ this is Hadamard's three circles theorem. The case $p < \infty$ was discussed by Hardy [25] in the first paper from "theory of Hardy spaces."

Miscellaneous

4.2.5 [30, Theorem 3.2.17] If u is subharmonic in the plane and satisfies the condition

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u(re^{i\theta})\,d\theta = o(\log r) \qquad (r \to \infty),$$

then u harmonic.

4.2.6 (Liouville's theorem [30, Theorem 3.2.24]) If u is subharmonic in all of $\mathbb{C}$ and $u(z) \leqslant o(\log|z|)$ $(z \to \infty)$, then u is a constant.

4.2.7 If $u \geqslant 0$ is subharmonic in the annulus $\rho < |z| < R$ and $p > 1$, then the function $M_p(r,u) = \{I(r,u^p)\}^{1/p}$ is convex of $\log r$ for $\rho < r < R$.

4.3 Integral means of univalent functions

A function f defined on $\mathbb{D}$ is said to be univalent if it is analytic and one-to-one. The leading example is the Koebe function $f(z) = z/1 - z)^2$ mapping $\mathbb{D}$ to $\mathbb{C}$ slit from $-1/4$ to $-\infty$ along the real axis.

Prawitz' theorem

4.3.1 Theorem *Let $f\colon \mathbb{D} \mapsto \mathbb{C}$ be a univalent function and $f(0) = 0$. Then for every $p > 0$ the function*

$$J_p(r) = J_p(r, f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^{-p}\, d\theta, \quad 0 < r < 1,$$

is decreasing.

What is interesting here is that the function $u = |f|^{-p}$ is subharmonic in the annulus $\mathbb{D} \setminus \{0\}$ but not in $\mathbb{D}$, because $u(0) = +\infty$. Also, the function $-u$ is not subharmonic in $\mathbb{D}$ and therefore we cannot apply Theorem 4.2.1.

Proof. We have

$$\begin{aligned}
2\pi J'_p(r) &= -p\int_0^{2\pi} |f(re^{i\theta})|^{-p-2}\operatorname{Re}\{\overline{f(re^{i\theta})} f'(re^{i\theta})e^{i\theta}\}\, d\theta \\
&= -(p/r)\operatorname{Im}\int_{|\zeta|=r} |f(\zeta)|^{-p-2}\overline{f(\zeta)} f'(\zeta)\, d\zeta \\
&= -(p/r)\operatorname{Im}\int_{\Gamma_r} |w|^{-p-2}\bar{w}\, dw \qquad\qquad (w = u + iv) \\
&= -(p/r)\int_{\Gamma_r} |w|^{-p-2}(u\, dv - v\, du),
\end{aligned}$$

where Γ_r is the image under f of the circle $|\zeta| = r$; the curve Γ_r is oriented positively. Now we apply Green's formula to the domain $\Omega_{r,R}$ bounded by Γ_r and the circle $|w| = R$, where $R > \max_{|z|=r} |f(z)|$. Since

$$\frac{\partial(|w|^{-p-2}u)}{\partial u} - \frac{\partial(-|w|^{-p-2}v)}{\partial v} = -p|w|^{-p-2},$$

we have

$$\int_{|w|=R} |w|^{-p-2}(u\,dv - v\,du) - \int_{\Gamma_r} |w|^{-p-2}(u\,dv - v\,du) = -p\iint_{\Omega_{r,R}} |w|^{-p-2} du\, dv$$

The first integral is equal to $2\pi R^{-p}$, and therefore

$$J'_p(r) = -(p/r)R^{-p} - (p^2/2\pi r)\iint_{\Omega_{r,R}} |w|^{-p-2} du\, dv.$$

This concludes the proof. □

Distortion theorems

4.3.2 Theorem (Bieberbach) *If f is a univalent function in $\mathbb{D}$, then $|f''(0)| \leqslant 4|f'(0)|$.*

Proof. [30] We can assume that $f(0) = 0$ and $f'(0) = 1$. Then

$$f(z)^{-1/2} = z^{-1/2}(1 - f''(0)z/4 + z^2h(z)),$$

where h is analytic in $\mathbb{D}$. By Theorem 4.3.1, case $p = 1$, the function

$$J_1(r) = r^{-1}(1 + |f''(0)/4|^2r^2 + r^4M_2^2(r, h))$$

is decreasing. Hence the function $r^{-1} + |f''(0)/4|^2r$ is decreasing, and hence $|f''(0)/4| \leqslant 1$. □

Remark. The famous theorem of de Branges [17] states that if f is univalent in $\mathbb{D}$, then

$$\left|\frac{f^{(n)}(0)}{n!}\right| \leqslant n|f'(0)|, \qquad n \geqslant 1.$$

This was conjectured by Bieberbach.

4.3.3 Theorem (Koebe) *If f is univalent in $\mathbb{D}$, then $f(\mathbb{D})$ contains the disk of radius $|f'(0)|/4$ centered at $f(0)$.*

Proof. [30] Let $f(0) = 0$. If w is not in the range of f, then the function $g(z) = 1/(f(z) - w)$ is univalent in $\mathbb{D}$ and hence $|g''(0)| \leqslant 4|g'(0)|$. It follows that $|f''(0) + 2f'(0)^2/w| \leqslant 4|f'(0)|$, whence $2|f'(0)^2/w| \leqslant 4|f'(0)| + |f''(0)| \leqslant 8|f'(0)|$. Thus $|w| \geqslant |f'(0)/4|$, and this concludes the proof. □

4.3.4 Corollary *If f is a conformal mapping of $D \subset \mathbb{C}$ onto G, then*

$$\frac{|f'(z)|}{4} \leqslant \frac{\operatorname{dist}(f(z), \partial G)}{\operatorname{dist}(z, \partial D)} \leqslant 4|f'(z)|, \qquad z \in \mathbb{D}.$$

4.3.5 Theorem *If f is univalent in $\mathbb{D}$, then*

$$\frac{1-r}{(1+r)^3} \leqslant \frac{|f'(z)|}{|f'(0)|} \leqslant \frac{1+r}{(1-r)^3}, \qquad |z| = r.$$

Proof. If we apply Bieberbach's theorem to the function $g(w) = f\left(\frac{z-w}{1-\bar{z}w}\right)$, we get

$$\left|z\frac{f''(z)}{f'(z)} - \frac{2r^2}{1-r^2}\right| \leqslant \frac{r}{1-r^2}.$$

Since $r\frac{\partial}{\partial r}\log|f'| = \operatorname{Re}\left(z\frac{f''}{f'}\right)$ we see that

$$\frac{2r-4}{1-r^2} \leqslant \frac{\partial}{\partial r}\log|f'| \leqslant \frac{2r+4}{1-r^2}.$$

The desired result is obtained by integration. □

4.3.6 Corollary *If f is univalent in $\mathbb{D}$, then* $\left|\dfrac{f(z)-f(0)}{z}\right| \leqslant \dfrac{|f'(0)|}{(1-|z|)^2}$.

4.3.7 Exercise If f is univalent in $\mathbb{D}$, then $\left|\dfrac{f(z)-f(0)}{z}\right| \geqslant \dfrac{|f'(0)|}{(1+|z|)^2}$.

4.3.8 Exercise If f is univalent in $\mathbb{D}$, then $\dfrac{p}{r}M(r)^{-p} \leqslant -J_p'(r) \leqslant \dfrac{p}{r}m(r)^{-p}$, where $M(r) = \max\limits_{|z|=r}|f(z)|$, $m(r) = \min\limits_{|z|=r}|f(z)|$.

Mean growth

If $p > 0$, then the function $I_p(r) = I_p(r, f) = J_{-p}(r, f)$ is increasing, and this fact does not depends on the hypothesis that f is univalent (Theorem 4.2.1). However, the proof of Theorem 4.3.1 gives additional information on $I_p(r)$, namely: If f is univalent in $\mathbb{D}$, $f(0) = 0$ and $f'(0) = 1$, then

$$I_p'(r) = \frac{p}{r}R^p - \frac{p^2}{2\pi r}\iint_{\Omega_{r,R}} |w|^{p-2}du\,dv.$$

This implies that $I_p'(r) \leqslant (p/r)M(r)^p$. Combining this with Corollary 4.3.6 we get:

4.3.9 Theorem *If f is univalent in $\mathbb{D}$ and $0 < p < 1/2$, then $I_p(r, f)$ is bounded, and*

$$I_{1/2}(r, f) = O\left(\log\frac{1}{1-r}\right), \quad r \to 1.$$

4.4 The subordination principle

Let F be a univalent function defined in $\mathbb{D}$. A function f analytic in $\mathbb{D}$ is said to be **subordinate** to F if $f(\mathbb{D}) \subset F(\mathbb{D})$ and $f(0) = F(0)$. In other words, f is subordinate to F if $f(z) = F(\omega(z))$, where $|\omega(z)| \leqslant |z|$, $z \in \mathbb{D}$, and ω is analytic. In this form the notion of subordination is defined for arbitrary functions. This notion is important because of the following theorem, known as **Littlewood's subordination principle.**

4.4.1 Theorem *If a function u is subordinate to a subharmonic function U, then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u(re^{i\theta})\,d\theta \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} U(re^{i\theta})\,d\theta \qquad (0 < r < 1). \tag{4.4}$$

In the simplest case $\omega(z) = \rho z$, $0 < \rho < 1$, this theorem reduces to Theorem 4.2.1.

Proof. We can assume that U is continuous. Let h be a function harmonic in $D_r = \{|z| \leq r\}$, continuous on the closure and equal to U on the boundary. Then $U \leqslant h$ on $\overline{D_r}$ and, hence, $u(z) = U(\omega(z)) \leqslant h(\omega(z))$ for $|z| = r$. It follows that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u(re^{i\theta})\,d\theta \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} h\big(\omega(re^{i\theta})\big)\,d\theta = h\big(\omega(0)\big) = h(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(re^{i\theta})\,d\theta.$$

This concludes the proof because $h(re^{i\theta}) = U(re^{i\theta})$. □

For various applications of the subordination principle in the theory of univalent functions we refer the reader to Duren [19, Ch. 6]. We will consider two examples which cannot be found in [19].

4.4.2 Theorem (Kolmogorov/Smirnov) *If $f \in H(\mathbb{D})$ and $\operatorname{Re} f \in h^1$, then $f \in h^p$, i.e.,*

$$\sup_{r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p \, d\theta < \infty \quad \textit{for every } p \in (0,1).$$

Proof. We can assume that $f(0) \in \mathbb{R}$. Assume first that $\operatorname{Re} f > 0$. Then f is subordinate to the univalent function $F(z) = c(1+z)/(1-z)$, $c = \operatorname{Re} f(0)$. Applying the subordination principle we get the following:

$$\begin{aligned}\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p \, d\theta &\leqslant c^p \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{1+re^{i\theta}}{1-re^{i\theta}} \right|^p d\theta \leqslant c^p \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{1+e^{i\theta}}{1-e^{i\theta}} \right|^p d\theta \\ &= c^p \frac{2}{\pi} \int_0^{\pi/2} (\cot\theta)^p \, d\theta = \frac{u(0)^p}{\cos(p\pi/2)} \qquad (0 < p < 1).\end{aligned}$$

If f is arbitrary, then we can use Theorem 3.2.5(b) to reduce the proof to the preceding case. □

4.4.3 Remark The Kolmogorov/Smirnov theorem can be proved in the following way. Let $\operatorname{Re} f > 0$. Then $\operatorname{Re}(f^p) = |f|^p \cos(\arg f) \geqslant |f|^p \cos(\pi p/2)$, and hence

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left|f(re^{i\theta})\right|^p d\theta \leqslant \frac{1}{\cos(\pi p/2)} \frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{Re}\left\{f(re^{i\theta})^p\right\} d\theta = \frac{1}{\cos(\pi p/2)} \operatorname{Re}\{f(0)^p\}.$$

Our next example is the case $p \leqslant 1$ of the following theorem of Ahern [1].

4.4.4 Theorem *If $f \in H(\mathbb{D})$ and $0 < |f(z)| < 1$ for all $z \in \mathbb{D}$, then for every $p > 0$ we have*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} (1 - |f(re^{i\theta})|)^p \, d\theta \geqslant c_p (1-r)^{1/2},$$

where c_p is a positive constant.

Ahern's proof is based on a highly nontrivial analysis of singular measures, which enabled him to treat the case $p > 1$. Here we use the subordination principle to discuss the case $p = 1/2$. It turns out that then Ahern's theorem can be improved. On the other hand, it seems that application of the subordination principle is limited to the case $p \leqslant 1$.

4.4.5 Theorem *With the hypotheses of the previous theorem, we have*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left(1 - |f(re^{i\theta})|\right)^{1/2} d\theta \geqslant c(1-r)^{1/2} \log \frac{2}{1-r},$$

where c is a positive constant.

Proof. The analytic function $a(z) = -\log f(z)$ maps $\mathbb{D}$ into the right half-plane and therefore is subordinate to the function $\lambda(1+z)/(1-z)$, where $\lambda = a(0) > 0$. It follows that $f(z)$ is subordinate to

$$S_\lambda(z) = \exp\Big(-\lambda\frac{1+z}{1-z}\Big).$$

The function $-(1-|z|)^{1/2}$ is subharmonic for $|z| < 1$, and therefore, by the subordination principle,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\big(1-|f(re^{i\theta})|\big)^{1/2}\,d\theta \geqslant \frac{1}{2\pi}\int_{-\pi}^{\pi}\big(1-|S_\lambda(re^{i\theta})|\big)^{1/2}\,d\theta.$$

In order to estimate this integral we use the inequality

$$\frac{x}{1+x} \leqslant 1-e^{-x} \leqslant \frac{2x}{1+x}, \qquad x > 0.$$

It follows that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(1-|S_\lambda(re^{i\theta})|)^{1/2}\,d\theta \asymp \int_0^{\pi}\Big(\frac{\lambda P(r,\theta)}{1+\lambda P(r,\theta)}\Big)^{1/2}d\theta \asymp \int_0^{\pi}\frac{(1-r)^{1/2}}{(1-r+\theta^2)^{1/2}}\,d\theta$$

Introducing the change $\theta = t\sqrt{1-r}$ we conclude the proof. □

Miscellaneous

4.4.6 (Harmonic Schwarz lemma) If $u \in h(D)$ is real valued, $|u| \leqslant 1$, and $u(0) = 0$, then u is subordinate to the function

$$U(z) = \frac{2}{\pi}\arg\frac{1+z}{1-z}.$$

Hence $|u(z)| \leqslant \frac{4}{\pi}\arctan|z|$, and, by the subordination principle, $M_2(r,u) \leqslant r$, $0 < r < 1$.

4.4.7 (Rogosinski's theorem [19, §6.2]) Let $f(z) = F(\omega(z))$, where F is analytic in $\mathbb{D}$, and ω is as above. Let $F_n(z) = \sum_{k=0}^n \widehat{F}(k)z^k$ and $f_n(z) = \sum_{k=0}^n \widehat{f}(k)z^k$. Then $F_n(\omega(z)) = f_n(z) + O(z^{n+1})$. Therefore, by the subordination principle and Parseval's formula,

If f is subordinate to $F \in H(\mathbb{D})$, then

$$\sum_{k=0}^{n}|\widehat{f}(k)|^2r^{2k} \leqslant \sum_{k=0}^{n}|\widehat{F}(k)|^2r^{2k}, \quad 0 < r < 1.$$

4.4.8 With the hypotheses of Theorem 4.4.4, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(1-|f(re^{i\theta})|)^p\,d\theta \geqslant c_p(1-r)^p \quad \text{for } 0 < p < 1/2.$$

4.4.9 [19] If $U = |f|^p$, $0 < p < \infty$, then strict equality holds for $0 < r < 1$ in (4.4) unless f is constant or $\omega(z) = \alpha z$, $|\alpha| = 1$.

4.5 The Riesz measure

Let $C_0^2(D)$ be the set of all C^2-functions with compact support in D. If u is subharmonic in D, then the functional

$$\varphi \mapsto \int_D u\Delta\varphi\, dm, \qquad \varphi \in C_0^2(D)$$

is positive and therefore there exists a unique positive Borel measure μ on D such that

$$\int_D \varphi\, d\mu = \int_D u\Delta\varphi\, dm, \qquad \varphi \in C_0^2(D).$$

This measure is called the **Riesz measure** of u. If u is of class C^2, then

$$d\mu = (\Delta u)\, dm.$$

The Riesz measure of the function $\log|z-a|$ is equal to $2\pi\delta_a$, where δ_a denotes the Dirac measure at the point a. The existence and uniqueness of the Riesz measure can be proved by using Riesz' theorem on representation of positive linear functionals on $C_0(D)$.(†)

Green's formula

Here we will consider in some detail a generalization of Green's formula.

4.5.1 Theorem *Let u be subharmonic in $D_R = \{z : |z| < R\}$ and let $u(0) > -\infty$. Then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u(re^{i\theta})\, d\theta - u(0) = \frac{1}{2\pi}\int_{|z|<r} \log\frac{r}{|z|}\, d\mu(z), \tag{4.5}$$

$(0 < r < R)$, where μ is the Riesz measure of u.

Observe that $\log(r/|z|) = 0$ for $|z| = r$, so it does not matter what stands in (4.5) : $|z| < r$ or $|z| \leqslant r$. Besides, the formula holds in the case $u(0) = -\infty$ as well, which shows that the integral on the right-hand side is finite iff $u(0) > -\infty$.

Jensen's formula The well known Jensen's formula is one of special cases of (4.5). Namely, if a function f is analytic in D_R and $f(0) \neq 0$, then the Riesz measure of the function $\log|f(z)|$ is equal to $2\pi\sum_k \delta_{a_k}$, where a_k are the zeros of f. This and (4.5) give Jensen's formula,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{i\theta})|\, d\theta = \log|f(0)| + \sum_{|a_k|<r} \log\frac{r}{|a_k|},$$

which, of course, can easily be proved without appealing to Theorem 4.5.1.

(†)For more information on the Riesz measure we refer to Hörmander [30] and Hayman/Kennedy [27]. For a discussion on Riesz' representation of positive linear functionals on $C_0(D)$, see Rudin [86, Ch. 2].

Proof of Theorem 4.5.1

Fix $R_0 < R$ and let $0 < \rho < r < R_0$. Choose a decreasing sequence $\{u_n\}$ of subharmonic functions of class C^2 in the disk $|z| < R_0$ such that u_n tends to u (Theorem 4.1.15). Then Green's formula for smooth functions gives

$$I(r, u_n) - I(\rho, u_n) = \frac{1}{2\pi} \int_{D_R} G_\rho(z) \Delta u_n(z)\, dm(z),$$

where

$$G_\rho(z) = \min\left\{\log \frac{r}{\rho}, \log \frac{r}{|z|}\right\}, \qquad |z| < r,$$

$G_\rho(z) = 0$ for $|z| > r$. Let g be a C^2-function in D_R such that $g = 0$ in a neighborhood of the circle $|z| = r$ and $g \leqslant G_\rho$ in D_R. Then

$$I(r, u_n) - I(\rho, u_n) \geqslant \frac{1}{2\pi} \int_{D_R} g(z) \Delta u_n(z)\, dm(z) = \frac{1}{2\pi} \int_{D_R} u_n(z) \Delta g(z)\, dm(z).$$

Here we apply the dominated convergence theorem, which is possible because $|u_n| \leqslant |u| + |u_1|$ and the functions u and u_1 are locally integrable (Theorem 4.1.12), to get

$$I(r, u) - I(\rho, u) \geqslant \frac{1}{2\pi} \int_{D_R} u(z) \Delta g(z)\, dm(z) = \frac{1}{2\pi} \int_{D_R} g(z)\, d\mu(z).$$

Now we take an increasing sequence of the functions g which tends to G_ρ and get

$$I(r, u) - I(\rho, u) \geqslant \frac{1}{2\pi} \int_{D_R} G_\rho(z)\, d\mu(z).$$

The reverse inequality is proved in a similar way. Finally, let ρ tend to 0. □

Riesz' representation formula

A slight modification of the above proof yields the following variant of Theorem 4.5.1.

4.5.2 Theorem *If u is subharmonic in a neighborhood of the closed unit disk, then*

$$u(z) = h(z) + \frac{1}{2\pi} \int_{\mathbb{D}} G(z, w)\, d\mu(w), \tag{4.6}$$

for every $z \in \mathbb{D}$, where

$$h(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(r, \theta - t) u(e^{it})\, dt,$$

μ is the Riesz measure of u, and $G(z, w) = \log \left|\dfrac{w - z}{1 - \bar{w} z}\right|$.

Miscellaneous

4.5.3 Let u be subharmonic in $\mathbb{D}$. Then for every $\varepsilon \in (0, 1/4)$ there exists a, $|a| < 1/4$, such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u(a+\varepsilon e^{i\theta})\,d\theta - u(a) \leqslant C\varepsilon^2\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{i\theta})\,d\theta - u(0)\right),$$

where C is an absolute constant.

4.5.4 Let $\varphi : D \mapsto \mathbb{C}$ be a conformal mapping and let u be subharmonic in D. Then

$$\int_D f(z)\,d\mu_{u\circ\varphi}(z) = \int_{\varphi(D)} f(\varphi^{-1}(w))\,d\mu_u(w),$$

where f is an arbitrary positive (Borel) function on D, and $\mu_{\_}$ is the Riesz measure of the corresponding function. For instance, if $v(z) = u(a+rz)$ $(a \in \mathbb{C},\ r > 0)$, then

$$\int_D f(z)\,d\mu_v(z) = \int_{a+rD} f\big((w-a)/r\big)\,d\mu_u(w).$$

4.5.5 If u subharmonic in the unit disk, then there holds the formula

$$\int_{\mathbb{D}} f(z)\,d\mu_{u\circ\varphi_a}(z) = \int_{\mathbb{D}} f\big(\varphi_a(w)\big)\,d\mu_u(w) \qquad (|a| < 1),$$

where $\varphi_a(z) = \dfrac{a-z}{1-\bar{a}z}$ and f is positive and Borel.

4.5.6 If f is analytic and $p > 0$, then the Riesz measure of $|f|^p$ is absolutely continuous and equal to

$$\frac{p^2}{2}|f|^{p-2}|f'|^2\,dm.$$

If $p \geqslant 2$, then the function $|f|^{p-2}|f'|^2$ is subharmonic; from this and Green's formula we can deduce that the function

$$r \mapsto \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\sqrt{r}e^{i\theta})|^p\,d\theta$$

is convex.

4.6 A Littlewood/Paley theorem

For a function u defined in $\mathbb{D}$ we write $I(u) = \sup\limits_{0<r<1} I(r,u)$, where, as above,

$$I(r,u) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{it})\,dt.$$

The following version of the Littlewood/Paley inequality contains an estimate of the subharmonicity of u^q $(q > 1)$ via the Riesz measure of u (not of u^q as in Green's formula).

4.6.1 Theorem [79] *Let $u \geqslant 0$ be subharmonic in $\mathbb{D}$ and let μ be the Riesz measure of u. If $q \geqslant 1$ and $I(u^q) < \infty$, then there holds the inequality*

$$\int_{\mathbf{D}} (1-|z|)^{-1}\{\mu(E_\varepsilon(z))\}^q dm(z) \leqslant C_q\big(I(u^q) - u(0)^q\big), \tag{4.7}$$

where $\varepsilon = 1/6$ and $E_\varepsilon(z) = \{w : |w - z| < \varepsilon(1 - |z|)\}$.

If in addition Δu $(u \in C^2)$ is a subharmonic function, then

$$\mu(E_\varepsilon(z)) = \int_{E_\varepsilon(z)} \Delta u\, dm \geqslant \pi\varepsilon^2(1-|z|)^2 \Delta u(z),$$

which leads to the following:

4.6.2 Theorem *Let $u \geqslant 0$ be a subharmonic function of class $C^2(\mathbb{D})$ such that its Laplacian is a subharmonic function. If $q \geqslant 1$ and $I(u^q) < \infty$, then*

$$\int_{\mathbf{D}} (1-|z|)^{2q-1}(\Delta u(z))^q\, dm(z) \leqslant C_q\big(I(u^q) - u(0)^q\big). \tag{4.8}$$

The classical inequality of Littlewood and Paley (Theorem 3.5.1) is a special case of (4.8). Namely, if $p \geqslant 2$ and $I(|h|^p) < \infty$, where f is a real-valued harmonic function in $\mathbb{D}$, then we take $u = h^2$ and $q = p/2$, and get

$$\int_{\mathbf{D}} (1-|z|)^{p-1}\, |\nabla h|^p\, dm \leqslant C_p\big(I(|h|^p - |h(0)|^p)\big).$$

In the case $q < 1$ we have the following theorem, the proof of which is omitted here (cf. [79]).

4.6.3 Theorem *Let $0 < q < 1$ and let $u \geqslant 0$ be a C^2-function such that both u^q and Δu are subharmonic. If*

$$\int_{\mathbf{D}} (1-|z|)^{2q-1}(\Delta u)^q\, dm < \infty,$$

then $I(u^q) < \infty$ and there holds the inequality

$$I(u^q) - u(0)^q \leqslant C_q \int_{\mathbf{D}} (1-|z|)^{2q-1}(\Delta u)^q\, dm(z).$$

Jevtić [32] extended the above theorems to the case of $\mathcal{M}$-harmonic functions on the unit ball in $\mathbb{C}^n$.

Local estimates for the Riesz measure

In what follows we suppose that u is a nonnegative subharmonic function defined in $\mathbb{D}$ and denote by μ the Riesz measure of u. As we have seen, there holds the formula

$$I(r,u) - u(0) = \frac{1}{2\pi}\int_{r\mathbf{D}} \log\frac{r}{|z|}\,d\mu(z) \qquad (0 < r < 1) \tag{4.9}$$

(see Theorem 4.5.1).

4.6.4 Lemma *There holds the equality*

$$I(u) - u(0) = \frac{1}{2\pi}\int_{\mathbf{D}} \log\frac{1}{|z|}\,d\mu(z).$$

Proof. Write (4.9) as

$$I(r,u) - u(0) = \frac{1}{2\pi}\int_{\mathbf{D}} K_r(z)\log\frac{r}{|z|}\,d\mu(z),$$

where $K_r(z)$ is the characteristic function of the disk $r\mathbb{D}$. Since $K_r(z)\log(r/|z|)$ increases with r, we have

$$\lim_{r\to 1^-} I(r,u) - u(0) = \frac{1}{2\pi}\int_{\mathbf{D}} \lim_{r\to 1^-} K_r(z)\log\frac{r}{|z|}\,d\mu(z).$$

Since $I(r,u)$ increases, we see that $I(u) = \lim_{r\to 1^-} I(r,u)$. □

4.6.5 Lemma *Let $q \geqslant 1$ and let μ_q be the Riesz measure of u^q. Then*

$$\{\mu(E)\}^q \leqslant C_q\,\mu_q(5E) \tag{4.10}$$

for every disk E such that $6E \subset \mathbb{D}$. The constant C_q depends only of q.

If E is a disk of radius R, then rE denotes the concentric disk of radius Rr.

Proof. By translation, the proof reduces to the case where E is centered at zero. Then, since $\mu(E) = \nu((1/r)E)$, where ν is the Riesz measure of the function $u(rz)$, we can assume that the radius of E is fixed, e.g., $E = \varepsilon\mathbb{D}$, $\varepsilon = 1/6$. Using the simple inequalities

$$\left(I(r,u) - u(0)\right)^q \leqslant \left(I(r,u)\right)^q - u(0)^q$$

and $\left(I(r,u)\right)^q \leqslant I(r,u^q)$, which hold because $q > 1$, we see from (4.9) (applied to u and u^q) that

$$\left(\frac{1}{2\pi}\int_{r\mathbf{D}} \log\frac{r}{|z|}\,d\mu(z)\right)^q \leqslant \frac{1}{2\pi}\int_{r\mathbf{D}} \log\frac{r}{|z|}\,d\mu_q(z).$$

Letting $r = 4\varepsilon$, we get

$$\{\mu(2\varepsilon\mathbb{D})\}^q \leqslant C\int_{4\varepsilon\mathbf{D}} |z|^{-1}d\mu_q(z), \tag{4.11}$$

where we have applied the estimate $\log(4\varepsilon/|z|) \geqslant \log 2$ for $|z| < 2\varepsilon$ and $\log(4\varepsilon/|z|) \leqslant 1/|z|$. Therefore, in order to prove (4.10) we have to remove $|z|^{-1}$. To do this, we translate the "center" of (4.11) to get

$$\{\mu(2\varepsilon D_a)\}^q \leqslant C \int_{4\varepsilon D_a} |z-a|^{-1} d\mu_q(z)$$

for $a \in \varepsilon\mathbb{D}$, where $D_a = \{z : |z-a| < 1\}$. Since $\varepsilon\mathbb{D} \subset 2\varepsilon D_a$ and $4\varepsilon D_a \subset 5\varepsilon\mathbb{D}$, we see that

$$\{\mu(\varepsilon\mathbb{D})\}^q \leqslant C \int_{4\varepsilon D_a} |z-a|^{-1} d\mu_q(z).$$

Now we integrate this inequality over the disk $\varepsilon\mathbb{D}$, with respect to $dm(a)$, and apply Fubini's theorem, which finishes the proof because

$$\sup_{z\in\mathbf{D}} \int_{\varepsilon\mathbf{D}} |z-a|^{-1} dm(a) < \infty. \qquad \square$$

Proof of Theorem 4.6.1

From (4.10) it follows that

$$\int_{\mathbf{D}} (1-|z|)^{-1} \{\mu(E_\varepsilon(z))\}^q \, dm(z) \leqslant C \int_{\mathbf{D}} (1-|z|)^{-1} \mu_q(E_{5\varepsilon}(z)) \, dm(z). \tag{4.12}$$

Further, from

$$\mu_q(E_{5\varepsilon}(z)) = \int_{E_{5\varepsilon}(z)} d\mu_q(w)$$

and Fubini's theorem it follows that the right-hand side of (4.12) equals

$$\int_{\mathbf{D}} d\mu_q(w) \int_{G(w)} (1-|z|)^{-1} \, dm(z),$$

where $G(w) = \{z : |z-w| < 5\varepsilon(1-|z|)\}$. Since $z \in G(w)$ implies $|z| - |w| < 5\varepsilon(1-|z|)$, whence $1-|w| < (1+5\varepsilon)(1-|z|)$, we see that

$$\int_{G(w)} (1-|z|)^{-1} dm(z) \leqslant (1+5\varepsilon)\, m(G(w))\, (1-|w|)^{-1}.$$

And since $(1-5\varepsilon)(1-|z|) < 1-|w|$ for $z \in G(w)$, we have $m(G(w)) \leqslant C'(1-|w|)^2$, where $C' = \pi(5\varepsilon/(1-5\varepsilon))^2$. Combining all these results we see that

$$\int_{\mathbf{D}} (1-|z|)^{-1} \{\mu(E_\varepsilon(z))\}^q \, dm \leqslant C_q \int_{\mathbf{D}} (1-|w|) \, d\mu_q(w).$$

This completes the proof of (4.7) because of Lemma 4.6.4 and the inequality $1-|w| \leqslant \log(1/|w|)$.

5 Classical Hardy spaces

There are various equivalent definitions of H^p-spaces. If $p \geqslant 1$, the shortest way to introduce H^p is by identifying it with a subspace of $L^p(\mathbb{T})$,

$$H^p(\mathbb{T}) = \{\phi \in L^p(\mathbb{T}) : \widehat{\phi}(-n) = 0 \text{ for } n \geqslant 1\}.$$

Thus $H^p(\mathbb{T})$ coincides with the closure in $L^p(\mathbb{T})$ of the set of all analytic polynomials; this can be used to define $H^p(\mathbb{T})$ for $0 < p < 1$.

In this text, we define H^p as a subclass of $H(\mathbb{D})$, see (5.1). In view of Riesz' projection theorem 6.2.1, H^p is isomorphic with h^p for $1 < p < \infty$. Because of the theorem of Burkholder, Gundy and Silverstein, see Theorem 7.2.1, one can define H^p as a space of harmonic functions ($\neq h^p$ for $p \leqslant 1$) for every $p > 0$, which is used to extend H^p-theory to several real variables (cf. [93]).

This chapter contains the standard facts on radial limits and factorization; an exception is Section 5.5, where we consider the composition of an H^p-function with an inner function. Our approach slightly differs from that in other texts [18, 22, 46, 83, 86, 99] in that we first prove the Hardy/Littlewood decomposition lemma (Lemma 5.1.7), and then deduce the radial limits theorem and F. and M. Riesz' theorems, without appealing to the Blaschke products.

5.1 Basic properties

The Hardy space H^p $(0 < p \leqslant \infty)$ is defined as the subspace of h^p consisting of analytic functions,

$$H^p = \{f \in H(\mathbb{D}) \colon \|f\|_p = \sup_{r<1} M_p(r,f) < \infty\}. \tag{5.1}$$

Here we can replace "sup" by "lim" because $M_p(r,f)$ increases with r for every $p > 0$. If $p \geqslant 1$, then H^p is a Banach space, and if $0 < p < 1$, it is a p-Banach space. The completeness is proved in the standard way. The first step is the continuity of the inclusion $H^p \subset H(\mathbb{D})$, which follows from the following lemma.

5.1.1 Lemma *If $f \in H^p$, $0 < p \leqslant \infty$, then*

$$|f(z)| \leqslant (1-|z|^2)^{-1/p}\|f\|_p. \tag{5.2}$$

The space H^p is not normable for $p < 1$. On the other hand, it follows from the lemma that the dual of H^p separates points in H^p.

5.1.2 Corollary *If $f \in H^p$, then*

$$M_q(r,f) \leqslant (1-r^2)^{1/q-1/p}\|f\|_p \qquad (q > p). \tag{5.3}$$

Proof. This follows from (5.2) and the following:

$$M_q^q(r,f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^{q-p}\,|f(re^{i\theta})|^p\,d\theta \leqslant \left(\sup_\theta |f(re^{i\theta})|\right)^{q-p} M_p^p(r,f). \qquad \square$$

Taking $q = 1 > p$ and $r = 1-(1/n)$ in (5.3), we get, via the inequality $M_1(r,f) \geqslant |\widehat{f}(n)|r^n$, one of many results of Hardy and Littlewood.

5.1.3 Corollary *If $f \in H^p$ $(0 < p < 1)$, then $|\widehat{f}(n)| \leqslant C_p\,\|f\|_p\,(n+1)^{1/p-1}$, where C_p depends only on p.*

Proof of Lemma 5.1.1. Assume that f is analytic in a neighborhood of the closed disk. Then, for $p < \infty$,

$$\|f\|_p^p = \frac{1}{2\pi}\int_{\mathbf{T}} |f(\zeta)|^p\,|d\zeta|.$$

For fixed $z \in \mathbb{D}$ let $\varphi(w) = \dfrac{z-w}{1-\bar{z}w}$. By the substitution $\zeta = \varphi(\xi)$ we get

$$\|f\|_p^p = \frac{1}{2\pi}\int_{\mathbf{T}} |f(\varphi(\xi))|^p\,|\varphi'(\xi)|\,|d\xi| = \frac{1}{2\pi}\int_{\mathbf{T}} |f(\varphi(\xi))\varphi'(\xi)^{1/p}|^p\,|d\xi|.$$

The function inside the last integral is subharmonic and therefore

$$\|f\|_p^p \geqslant |f(\varphi(0))\varphi'(0)^{1/p}|^p = |f(z)|^p(1-|z|^2),$$

which was to be proved. $\square$

5.1.4 Theorem *For every $p > 0$ the space H^p is complete.*

Proof. Let $\{f_n\}$ be a Cauchy sequence in H^p. This means that for every $\varepsilon > 0$ there exists N such that $M_p(r, f_n - f_m) < \varepsilon$ for every r and $m, n > N$. From (5.1.1) it follows $\{f_n\}$ is a Cauchy sequence in $H(\mathbb{D})$ and hence $\{f_n\}$ converges uniformly on compact subsets to some function $f \in H(\mathbb{D})$. Letting m tend to ∞, we get $M_p(r, f_n - f) \leqslant \varepsilon$ for $n > N$ and all r, which implies $\|f_n - f\|_p \leqslant \varepsilon$ for $n > N$. $\square$

5.1.5 Exercise If $0 < p < \infty$ and f is analytic in $\mathbb{D}$, then $M_p(r,f)$ is strictly increasing unless f =const; see Proposition 3.4.1. In particular, if $M_p(r,f) = |f(0)|$ for some $r > 0$, then f =const.

5.1.6 Exercise For a fixed $z \in \mathbb{D}$, equality occurs in (5.2) iff

$$f(w) = c\left(\frac{1-|z|^2}{(1-\bar{z}w)^2}\right)^{1/p},$$

where c is a constant.

A decomposition lemma

Various properties of a zero-free H^p-function can be deduced from the corresponding properties of the function $f^{2/p} \in H^2$. The following lemma of Hardy and Littlewood is often used to reduce the general case to the zero-free case.

5.1.7 Lemma *If $f \in H^p$, $p > 0$, then there exist functions g and h without zeros in $\mathbb{D}$ such that $f = g + h$, $\|g\|_p \leqslant \|f\|_p$ and $\|h\|_p \leqslant \|f\|_p$.*

For example, in proving an inequality of the form $\|Tf\|_X \leqslant C\|f\|_p$, $f \in H^p$, where X is a quasinormed space and $T : H^p \mapsto X$ a linear operator, we can suppose that f has no zeros in $\mathbb{D}$.

Proof. Assume, at first, that $f \not\equiv 0$ is analytic in a neighborhood of the closed disk and that f has at least one zero in $\mathbb{D}$. Then the number of zeros of f is finite; denote the zeros by $a_1, \dots, a_m$ (counting multiplicity). Let

$$A(z) = \prod_{k=0}^{m} \frac{z - a_k}{1 - \bar{a}_k z}$$

and define g and h as follows: $g = (A-1)f/2A$, $h = (A+1)f/2A$; we have $f = g + h$. Neither g nor h have zeros in $\mathbb{D}$ because $|A| < 1$ in $\mathbb{D}$ and the function f/A has no zeros in $\mathbb{D}$. Both h and g are analytic in a neighborhood of the closed disk because so are $A-1$, $A+1$ and f/A. And since $|A| = 1$ on $\mathbb{T}$, we have $|g| \leqslant |f|$ and $|h| \leqslant |f|$ on $\mathbb{T}$, which proves the lemma in that special case.

If f is arbitrary, let $f_n(z) = f(r_n z)$, where $r_n = 1 - 1/n$ (or any sequence tending to 1). By the preceding, we have a decomposition $f_n = g_n + h_n$ with the the desired properties. Since

$$\|g_n\|_p \leqslant \|f_n\|_p \leqslant \|f\|_p \tag{5.4}$$

(and similarly for h), then, according to Lemma 5.1.1, the sequences $\{g_n\}$ and $\{h_n\}$ are bounded on compact subsets of $\mathbb{D}$. Therefore, passing to subsequences, we can assume that g_n and h_n tend uniformly on compact subsets to analytic functions g and h, respectively. By Hurwitz' theorem, the function g is either without zeros or $g \equiv 0$ because g_n have no zeros. The same holds for h. But it is not true that $g \equiv 0$ because this and $f = g + h$ imply $f \equiv h$, which is impossible because $f \not\equiv 0$ and f has zeros.

Finally, from (5.4) it follows that $M_p(r, g_n) \leqslant \|f\|_p$ for every $r < 1$, and hence $M_p(r, g) \leqslant \|f\|_p$. □

Radial limits

Since $H^1 \subset h^1$, we see from the Riesz/Herglotz theorem and Fatou's theorem that every function $f \in H^1$ has radial limits almost everywhere. However, the hypothesis that f is analytic improves the properties of the boundary function substantially (see, e.g., Theorems 5.1.8 and 5.2.1).

5.1.8 Theorem *Let $f \in H^p$, $0 < p \leqslant \infty$. Then almost everywhere on $\mathbb{T}$ there exist radial limits $f_*(e^{i\theta}) = \lim_{r\to 1^-} f(re^{i\theta})$, and there hold the relations*

$$\|f\|_p = \|f_*\|_p \qquad (p \leqslant \infty) \tag{5.5}$$

$$\lim_{r\to 1^-} \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta}) - f_*(e^{i\theta})|^p\, d\theta = 0 \qquad (p < \infty). \tag{5.6}$$

Proof. When $p > 1$, we may appeal to Theorem 3.4.2. Let $f \in H^p$ and $1/2 < p \leqslant 1$. Observe that (5.5) is implied by (5.6). Next, because on the lemma on decomposition, we may assume that f has no zeros. Then the function $g = f^{1/2} \in H^{2p}$ is well defined. Since $2p > 1$, the function g, so the function $f = g^2$, has radial limits. Let f_* denote the boundary function and $f_r(e^{i\theta}) = f(re^{i\theta})$. Then

$$\|f_r - f_*\|_p = \|(g_r - g_*)(g_r + g_*)\|_p \leqslant \|g_r - g_*\|_{2p}\|g_r + g_*\|_{2p},$$

where we have applied the Cauchy/Schwarz inequality. Since $\|g_r - g_*\|_{2p}$ tends to 0 and $\|g_r + g_*\|_{2p}$ is bounded, we can conclude that there holds (5.6) for $p > 1/2$. In the same way we reduce the case $p > 1/4$ to the case $p > 1/2$, etc. □

The set of all harmonic polynomials is not dense in h^p for $p \leqslant 1$. However, we have:

5.1.9 Theorem *If $0 < p < \infty$, then the set of all (analytic) polynomials is dense in H^p.*

Proof. This follows from (5.6) and the fact that $\|f_r - s_n f_r\|_p \to 0$ $(n \to \infty)$, for every fixed $r \in (0,1)$, where $s_n g$ denotes the partial sum of the Taylor series of g. □

The Poisson integral of $\log|f_*|$

If f is analytic in a neighborhood of the closed disk, then we have $\log|f| \leqslant P[\log|f_*|]$ because of the subharmonicity of $\log|f|$.

5.1.10 Theorem *Let $f \not\equiv 0$ belong to H^p $(p > 0)$. Then $\log|f_*| \in L^1(\mathbb{T})$ and there holds*

$$\log|f(re^{i\theta})| \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r, \theta - t)\log|f_*(e^{it})|\, dt, \quad 0 \leqslant r < 1. \tag{5.7}$$

Before proving this theorem we note two consequences.

Smirnov's maximum principle There are *unbounded* functions $f \in H(\mathbb{D})$ for which the boundary function belongs to $L^\infty(\mathbb{T})$; one of them is $f(z) = \exp\left(\frac{1+z}{1-z}\right)$. However:

5.1.11 Theorem (Smirnov) *If $f \in H^p$, $p > 0$, and if $f_* \in L^\infty$, then $f \in H^\infty$ and $\|f\|_\infty = \|f_*\|_\infty$.*

In the case $p > 1$ this theorem is contained in Corollary 3.4.4, while in the general case it is a consequence of (5.7), or of the weaker inequality

$$|f(re^{i\theta})|^p \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta-t)|f_*(e^{it})|^p\,dt \qquad (0 \leqslant r < 1) \tag{5.8}$$

and (5.5). It is worthwhile to note that (5.8) can be deduced immediately from (5.6) and the inequality $|f(\rho z)|^p \leqslant P[\,|f_\rho|^p\,](z)$.

In fact, inequality (5.7), together with Jensen's inequality for the function $x \mapsto e^x$, implies a more general fact, namely:

5.1.12 Theorem (Smirnov) *If $f \in H^p$ and $f_* \in L^q$ for some $q > p$, then $f \in H^q$.*

Uniqueness theorem An immediate consequence of Theorem 5.1.10:

5.1.13 Theorem *If $f \in H^p$ and $f_*(e^{i\theta}) = 0$ on a set of positive measure, then $f(z) = 0$ for every $z \in \mathbb{D}$.*

Proof of Theorem 5.1.10

Let $f \in H^p$ and, say, $f(0) \neq 0$. Since $\log^+ x \leqslant x^p/p$, $x > 0$, it follows that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+|f(re^{i\theta})|\,d\theta \leqslant C \qquad (0 < r < 1),$$

where C is a constant. We also have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \big|\log|f(re^{i\theta})|\big|\,d\theta = 2\frac{1}{2\pi}\int_{-\pi}^{\pi}\log^+|f(re^{i\theta})|\,d\theta - \frac{1}{2\pi}\int_{-\pi}^{\pi}\log|f(re^{i\theta})|\,d\theta.$$

The last summand is $\leqslant -\log|f(0)|$ because of the subharmonicity of $\log|f|$. Hence,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \big|\log|f(re^{i\theta})|\big|\,d\theta \leqslant 2C - \log|f(0)|, \tag{5.9}$$

and Fatou's lemma concludes the proof that $\log|f_*| \in L^1(\mathbb{T})$.

To prove (5.7) we start from the inequality

$$\log|f(\rho z)| \leqslant P[\log|f_\rho|](z), \quad \text{where } f_\rho(e^{i\theta}) = f(\rho e^{i\theta}), \tag{5.10}$$

$\rho < 1$, $z \in \mathbb{D}$, which holds for an arbitrary $f \in H(\mathbb{D})$ because $\log|f|$ is subharmonic. Since $\log x = \log^+ x - \log^- x$, we have, from (5.10),

$$\log|f(z)| \leqslant \limsup_{\rho\to 1} P[\log^+|f_\rho|](z) - \liminf_{\rho\to 1} P[\log^-|f_\rho|](z). \tag{5.11}$$

And since $|\log^+ x - \log^+ y| \leqslant |x-y|^p/p$, $x, y > 0$, it follows from (5.6) that

$$\lim_{\rho\to 1^-}\frac{1}{2\pi}\int_{-\pi}^{\pi} \big|\log^+|f(\rho e^{i\theta})| - \log^+|f_*(e^{i\theta})|\big|\,d\theta = 0;$$

therefore $\limsup_{\rho\to 1} P[\log^+ |f_\rho|](z) = P[\log^+ |f_*|](z)$. Now we deduce (5.7) from (5.11) by means of Fatou's lemma. □

Remark. For another proof, see 7.1.5.

5.2 The space H^1

The results of this section, due to F. and M. Riesz, Privalov, and Smirnov(*), show how much H^1 differs from h^1.

The Poisson integral of the boundary function

A function belonging to h^1 need not be equal to the Poisson integral of the boundary function. However:

5.2.1 Theorem *If $f \in H^1$, then $f_* \in L^1$ and $f = P[f_*]$.*

This is easily deduced from the relation $f(rz) = P[f_r](z)$ $(r < 1)$, by means of (5.6).

5.2.2 Exercise (Cauchy's integral formula) If $f \in H^1$, then

$$f(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f_*(\zeta)}{\zeta - z} d\zeta \qquad (z \in \mathbb{D}).$$

Now we are in position to prove the famous theorem of F. and M. Riesz:

5.2.3 Theorem *Let μ be a complex Borel measure on $\mathbb{T}$ such that $\int_{\mathbb{T}} \zeta^n \, d\mu(\zeta) = 0$ for every $n = 1, 2, \ldots$. Then μ is absolutely continuous.*

Proof. Let $f = P[\mu]$. Then $f \in h^1$ and $\widehat{f}(k) = \widehat{\mu}(k)$ for every $k \in \mathbb{Z}$ (Proposition 3.2.1). Hence, the condition of the theorem implies that f is analytic. Hence $f \in H^1$ so, according to Theorem 5.2.1, we have $f = P[f_*]$. In view of the injectivity of the Poisson integral (Theorem 3.2.2), it follows that $d\mu(e^{i\theta}) = f_*(e^{i\theta})\, dt$. □

Bounded variation $\Longrightarrow$ absolute continuity

5.2.4 Theorem *If $f \in H^1$ and if the boundary function is almost everywhere equal to a function of bounded variation, then f has absolutely continuous extension to $\overline{\mathbb{D}}$.* (†)

(*)Further information, as well as references and historical comments, can be found in Zygmund [100, Ch. VII§§8–10] and Duren [18, Ch. III]

(†)i.e., a continuous extension that is absolutely continuous on $\mathbb{T}$.

Proof. Let $f_* = \gamma$ a.e., $\gamma \in BV[-\pi,\pi]$. Then $f = P[\gamma]$, by Theorem 5.2.1. We have

$$g(re^{i\theta}) := \frac{\partial f}{\partial \theta} = PS[\gamma](re^{i\theta}) - k \cdot P(r, \theta + \pi),$$

where $PS[\gamma]$ is the Poisson/Stiltjes integral of γ (see (3.10) and (3.13)). Then, by the Riesz/Herglotz theorem, $g \in H^1$ and, by Theorem 5.2.1, $g = P[g_*]$; thus $g = PS[G]$, where

$$G(\theta) = \int_0^\theta g_*(e^{it}) \qquad (\theta \in \mathbb{R}).$$

Applying (3.13) again, with the obvious change of notation, and taking into account that the function G is 2π-periodic because $g(0) = 0$, we get

$$\frac{\partial f}{\partial \theta} = PS[G](re^{i\theta}) = \frac{\partial}{\partial \theta} \frac{1}{2\pi} \int_{-\pi}^{\pi} P(r, \theta - t) G(t)\, dt.$$

It follows that

$$f(re^{i\theta}) = \text{const} + \frac{1}{2\pi} \int_{-\pi}^{\pi} P(r, \theta - t) G(t)\, dt,$$

which concludes the proof. □

In a similar way one proves the following:

5.2.5 Theorem *The derivative of a function $f \in H(\mathbb{D})$ belongs to H^1 iff f has absolutely continuous extension to $\overline{\mathbb{D}}$. The boundary function of the function $(\partial/\partial\theta) f(re^{i\theta}) = ire^{i\theta} f'(re^{i\theta})$, if $f \in H^1$, is equal to $(d/d\theta) f_*(e^{i\theta})$.*

Conformal mappings

5.2.6 Theorem *Let f be a conformal mapping of $\mathbb{D}$ onto a domain G whose boundary, ∂G, is a Jordan curve. Then $f' \in H^1$ iff ∂G is rectifiable. If ∂G is rectifiable, then*

$$|\partial G| = \int_{-\pi}^{\pi} |f'(e^{i\theta})|\, d\theta \tag{5.12}$$

and $|\partial G| \geqslant 2 \sum_{n=0}^{\infty} |\widehat{f}(n)|$, where $|\partial G|$ is the length of ∂G.

The last inequality follows from (5.12) and the inequality

$$\sum_{n=0}^{\infty} \frac{|\widehat{f}(n)|}{n+1} \leqslant \pi \|f\|_1, \tag{5.13}$$

due to Hardy (see 5.3.7).

Proof. Let $f' \in H^1$. The function f can be extended as a continuous function to $\overline{\mathbb{D}}$, and the extended function is a homeomorphism between $\overline{\mathbb{D}}$ and $\overline{G}$ (theorem of Carathéodory, [100, Ch. VII, §10]). By Theorem 5.2.5, this extension is absolutely continuous on $\mathbb{T}$ and therefore ∂G is parameterized by the absolutely continuous function $\Gamma(\theta) = f_*(e^{i\theta})$ ($|\theta| \leqslant \pi$). Hence the length of ∂G is equal to

$$\int_{-\pi}^{\pi} |\Gamma'(\theta)|\, d\theta.$$

Now formula (5.12) follows from theorem 5.2.5.

Conversely, let ∂G be rectifiable, $|\partial G| = 2\pi$, and let $\gamma(\theta)$ ($0 \leqslant \theta \leqslant 2\pi$) be the arclength parameterization of ∂G. The function f has the radial limits $\varphi(\theta) = \lim_{r\to 1-} f(re^{i\theta})$ on a set $S \subset [0, 2\pi]$, $|S| = 2\pi$. Since the extended mapping is homeomorphic, the function φ "increases", i.e., there exists an increasing function $t : S \mapsto [0, 2\pi]$ such that $\varphi(\theta) = \gamma(t(\theta))$, $\theta \in S$. We extend the function $t(\theta)$ as an increasing function on $[0, 2\pi]$, and the corresponding extension of φ is of bounded variation on $[0, 2\pi]$. Now Theorem 5.2.4 shows that f has absolutely continuous extension to $\bar{\mathbb{D}}$. Finally, $f' \in H^1$, by Theorem 5.2.5.(‡) □

5.3 Blaschke product

If a function $f \in H(\mathbb{D})$ has an infinite number of zeros, $a_1, a_2, \ldots$, then $1 - |a_n| \to 0$. If $f \in H^p$, there holds more:

5.3.1 Theorem *If $\{a_n\}$ $(n \geqslant 1)$ is the sequence of zeros of a function $f \in H^p$ $(p > 0)$, then the* ***Blaschke condition*** *is satisfied: $\sum_{n=1}^{\infty}(1 - |a_n|) < \infty$.*

Conversely, if a sequence $\{a_n\} \subset \mathbb{D}$ satisfies the Blaschke condition, then the product

$$B(z) = \prod_{n=1}^{\infty} \frac{a_n - z}{1 - \bar{a}_n z} \frac{|a_n|}{a_n}$$

converges in $\mathbb{D}$ and the function B is analytic and has the properties:

(a) *The sequence of zeros of B, including repetitions for multiplicities, coincides with $\{a_n\}$;* (b) *$|B(z)| \leqslant 1$ for $|z| < 1$;* (c) *$|B(e^{i\theta})| = 1$ almost everywhere.*

The function $B(z)$ is called a **Blaschke product**; in the case $a_n = 0$ the ratio $|a_n|/a_n$ is interpreted as -1. Note that $B(0) = \prod_{n=1}^{\infty} |a_n|$, and this product converges iff the Blaschke condition is satisfied. By the term a Blaschke product we also mean a function of the form

$$B(z) = \prod_{n=1}^{k} \frac{a_n - z}{1 - \bar{a}_n z} \frac{|a_n|}{a_n}$$

with $k \geqslant 1$ as well as the function $B(z) \equiv 1$.

(‡)so we have proved the implication r"... $\Longrightarrow f' \in H^1$" without appealing to the theorem of Carathéodory.

Proof. Let $f \in H^p$ $(p > 0)$ and, say, $f(0) \neq 0$. From the inequality

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{i\theta})|^p \, d\theta \leqslant \log\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^p \, d\theta\right\},$$

which is obtained by an application of Jensen's inequality for the concave function $\log x$, it follows that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{i\theta})| \, d\theta \leqslant \log\|f\|_p.$$

On the other hand, there holds (Jensen's) formula

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{i\theta})| \, d\theta = \log|f(0)| + \sum_{|a_k|<r} \log\frac{r}{|a_k|}.$$

Hence $\sum_{k=1}^{\infty} \log(1/|a_k|) < \infty$, which is equivalent to the Blaschke condition.

If the Blaschke condition holds, then the product $B(z)$ converges uniformly on compact subsets to a function that vanishes exactly at the points a_n because

$$\left|1 - \frac{a_n - z}{1 - \bar{a}_n z}\right| \leqslant (1 - |a_n|)\frac{1 + |z|}{1 - |z|}.$$

(Details are omitted.) It is clear that $B(z)$ has property (b). In order to prove (c), observe that $B(z)/B_k(z)$, where

$$B_k(z) = \prod_{n=1}^{k} \frac{a_n - z}{1 - \bar{a}_n z}\frac{|a_n|}{a_n},$$

is a Blaschke product as well. It follows that

$$\frac{|B(0)|}{|B_k(0)|} \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|B(e^{i\theta})|}{|B_k(e^{i\theta})|} \, d\theta.$$

And since $|B_k(e^{i\theta})| = 1$ and $B(0)/B_k(0) \to 1$ $(k \to \infty)$, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |B(e^{i\theta})| \, d\theta \geqslant 1.$$

Finally, since $|B(e^{i\theta})| \leqslant 1$ almost everywhere, we have (c). □

Riesz' factorization theorem

5.3.2 Theorem *Let $\{a_n\}$ be the sequence of zeros of $f \in H^p$ $(p > 0)$ and $B(z)$ the corresponding Blaschke product. Then f/B belongs to H^p and $\|f/B\|_p = \|f\|_p$. Consequently, every H^p-function can be represented as $f = Bg$, where B is a Blaschke product (finite or infinite), the function g has no zeros in $\mathbb{D}$ and $\|f\|_p = \|g\|_p$.*

Proof. With the above hypotheses, let

$$B_k(z) = \prod_{n=1}^{k} \frac{a_n - z}{1 - \bar{a}_n z} \frac{|a_n|}{a_n}.$$

Then the function f/B_k belongs to H^p because it is analytic in $\mathbb{D}$ and $|B_k(z)| > 1/2$ near $\mathbb{T}$. Since $|f/B_k| = |f|$ on $\mathbb{T}$, we have $\|f/B_k\|_p = \|f\|_p$. Hence

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|f(re^{i\theta})|}{|B_k(re^{i\theta})|} \, d\theta \leqslant \|f\|_p$$

for every $r < 1$. By Fatou's lemma we get

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|f(re^{i\theta})|}{|B(re^{i\theta})|} \, d\theta \leqslant \|f\|_p,$$

which implies $\|f/B\|_p \leqslant \|f\|_p$. The reverse holds because $|f/B| \geqslant |f|$. □

5.3.3 Exercise If $f \in H^p$, then there exist functions g, $h \in H^{2p}$ such that $f = gh$.

Some inequalities

Isoperimetric inequality

Let G be a domain with rectifiable boundary. Then there holds the (isoperimetric) inequality $|G| \leqslant |\partial G|^2/4\pi$, and equality occurs iff G is a disk. This inequality can be rewritten as

$$\int_{\mathbf{D}} |f'|^2 \, dA \leqslant \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |f'(e^{i\theta})| \, d\theta \right\}^2,$$

where f is a conformal mapping of $\mathbb{D}$ onto G. Here dA denotes the Lebesgue measure on $\mathbb{D}$ normalized so that the measure of $\mathbb{D}$ is 1.

Thus the isoperimetric inequality is a consequence of the following theorem of Carleman.

5.3.4 Theorem *If $f \in H^p$, $p > 0$, then*

$$\int_{\mathbf{D}} |f|^{2p} \, dA \leqslant \|f\|_p^{2p}. \tag{5.14}$$

Equality occurs iff $f(z) = c(1 - az)^{-2/p}$, where c and a are complex constants, $|a| < 1$.

Proof. Let $f \in H^p$. Then we can write $f = Bg^{2/p}$, where B is a Blaschke product and g is in H^2 and has no zeros in $\mathbb{D}$. Then

$$\int_{\mathbf{D}} |f|^{2p} \, dA \leqslant \int_{\mathbf{D}} |g^2|^2 \, dA \quad \text{and} \quad \|f\|_p^{2p} = \|g\|_2^2.$$

We have

$$\int_{\mathbb{D}} |g^2|^2 \, dA = \sum_{n=0}^{\infty} \frac{A_n}{n+1}, \text{ where } A_n = \Big|\sum_{k=0}^{n} \widehat{g}(k)\widehat{g}(n-k)\Big|^2.$$

Since

$$A_n \leqslant (n+1) \sum_{k=0}^{n} |\widehat{g}(k)|^2 |\widehat{g}(n-k)|^2,$$

we have

$$\int_{\mathbb{D}} |g^2|^2 \, dA \leqslant \sum_{n=0}^{\infty} \sum_{k=0}^{n} |\widehat{g}(k)|^2 |\widehat{g}(n-k)|^2 = \|g\|_2^2.$$

This proves the inequality. If in (5.14) equality holds, then we see from the preceding that $B = 1$ and that there exists a sequence λ_n such that $\widehat{g}(k)\widehat{g}(n-k) = \lambda_n$ for $0 \leqslant k \leqslant n$. This implies that $g(z) = c(1-az)^{-1}$; etc. □

5.3.5 Exercise [62] If $f \in H^p$, $p > 0$, and $n = 2, 3, \ldots$, then

$$\left\{(n-1) \int_{\mathbb{D}} |f(z)|^{np} (1-|z|^2)^{n-2} \, dA(z)\right\}^{1/np} \leqslant \|f\|_p.$$

It is not known whether this holds for other values of n.

5.3.6 Exercise Let $q > p > 0$ and $r = \sqrt{p/q}$. If q/p is an integer and $f \in H^p$, then

$$\left\{\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^q \, d\theta\right\}^{1/q} \leqslant \|f\|_p.$$

Inequalities of Riesz/Fejér and Hilbert

If g is a function analytic in $\mathbb{D}$, then a special case of the Riesz/Zygmund theorem (see (6.5)) states that

$$\int_{-1}^{1} |g'(r)| \, dr \leqslant \pi \|g'\|_1.$$

Replacing here g' by f and using Riesz' factorization we get the Riesz/Fejer inequality:

$$\int_{-1}^{1} |f(r)|^p \, dr \leqslant \pi \|f\|_p^p \qquad (f \in H^p, \; p > 0). \tag{5.15}$$

In particular, if $p = 2$ and $f(z) = \sum_{n=0}^{\infty} a_n z^{2n}$ ($a_n \geqslant 0$), then (5.15) yields

$$\sum_{m,n \geqslant 0} \frac{a_m a_n}{m + n + (1/2)} \leqslant \pi \sum_{n=0}^{\infty} |a_n|^2. \tag{5.16}$$

This inequality, known as Hilbert's inequality, can be deduced immediately from the equality

$$\int_{-1}^{1} f(r)^2\,dr = i\int_0^{\pi} f(e^{i\theta})^2 e^{i\theta}\,d\theta,$$

a consequence of Cauchy's integral theorem. From (5.16) it follows that

$$\sum_{m,n\geqslant 0} \frac{a_m b_n}{m+n+(1/2)} \leqslant \pi\Big(\sum_{n=0}^{\infty}|a_n|^2\Big)^{1/2}\Big(\sum_{n=0}^{\infty}|b_n|^2\Big)^{1/2}. \tag{5.17}$$

Hardy's inequality

From Hilbert's inequality we can obtain a slightly improved version of Hardy's inequality (5.13). Namely:

5.3.7 Theorem *If $f \in H^1$, then*

$$\sum_{n=0}^{\infty} \frac{|\widehat{f}(n)|}{n+(1/2)} \leqslant \pi\|f\|_1. \tag{5.18}$$

Proof. Let $f \in H^1$ and let $f = Bg$ be the Riesz' factorization of f. Then the functions $F = Bg^{1/2}$ and $G = g^{1/2}$ belong to H^2 and $\|f\|_1 = \|F\|_2^2 = \|G\|_2^2$. Let $a_k = |\widehat{F}(k)|$ and $b_k = |\widehat{G}(k)|$. Then we have

$$\sum_{n=0}^{\infty} \frac{|\widehat{f}(n)|}{n+(1/2)} \leqslant \sum_{n=0}^{\infty} \frac{1}{n+(1/2)}\sum_{k=0}^{n} a_k b_{n-k} = \sum_{m,n\geqslant 0} \frac{a_m b_n}{m+n+(1/2)}.$$

Now we use (5.17) to get

$$\sum_{n=0}^{\infty} \frac{|\widehat{f}(n)|}{n+(1/2)} \leqslant \pi\|F\|_2\|G\|_2 = \pi\|f\|_1. \qquad \square$$

5.3.8 Remark In the case $p = 2$ the isoperimetric inequality (5.3.4) can be written as

$$\sum_{n=0}^{\infty} \frac{|\widehat{f}(n)|^2}{n+1} \leqslant \|f\|_1^2. \tag{5.19}$$

It is interesting to compare this inequality with (5.18). In general, convergence of the series $\sum|\widehat{f}(n)|/(n+1)$, with $f \in H(\mathbb{D})$, does not imply convergence of $\sum|\widehat{f}(n)|^2/(n+1)$. However, if $f \in H^1$, then $|\widehat{f}(n)| \leqslant \|f\|_1$, and therefore (5.18) implies a weak form of (5.19), namely

$$\sum_{n=0}^{\infty} \frac{|\widehat{f}(n)|^2}{n+1} \leqslant \pi\|f\|_1^2.$$

On the other hand, (5.19) implies $\|f\|_1^2 - |f(0)|^2 \geqslant (1/2)|f'(0)|^2$, which cannot be deduced from (5.18).

5.4 Inner and outer functions

Inner-outer factorization

Riesz' factorization theorem can be refined by introducing inner and outer functions. Suppose that a function $f \in H^p$ has no zeros in $\mathbb{D}$. Then $\log|f|$ is harmonic and belongs to h^1, which follows from (5.9). Nevertheless inequality (5.7) may be strict; this is the case if, e.g., f is the so called *atomic function* $\exp\left(-\frac{1+z}{1-z}\right)$. The atomic function satisfies the following:

(a) $|S(z)| \leqslant 1$ for $z \in \mathbb{D}$; (b) $|S_*(e^{i\theta})| = 1$ almost everywhere.

An analytic function satisfying (a) and (b) is called an **inner function.**

Singular inner functions If an inner function has no zeros, then it is called a singular inner function. The following theorem describes a connection between singular inner functions and singular measures.

5.4.1 Theorem *A function $S \in H(\mathbb{D})$ is a singular inner function iff there exists a nonnegative singular measure σ on $\mathbb{T}$ such that*

$$S(z) = e^{ic} \exp\left(-\frac{1}{2\pi}\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\sigma(\zeta)\right), \tag{5.20}$$

where c is a real constant.

Proof. If S is inner, then the function $u = \log|S|$ is negative and, by the Riesz/Herglotz theorem, there exists a nontrivial positive measure $\sigma \in M(\mathbb{T})$ such that $u = -P[\sigma]$. The measure is singular because $u(re^{i\theta}) \to 0$, $r \to 1$ (Corollary 3.3.5). Then, by passing to "analytic completion", we get

$$\log S(z) = -\frac{1}{2\pi}\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\sigma(\zeta) + ic,$$

where c is a real constant, and this implies (5.20). The rest of the proof is simpler and we omit it. □

Outer functions A function F is called an outer function if

$$F(z) = e^{ic} \exp\left(\frac{1}{2\pi}\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log\psi(\zeta)\, |d\zeta|\right), \tag{5.21}$$

where $\psi \geqslant 0$ is a measurable function such that $\log\psi \in L^1(\mathbb{T})$. By the theorems of Fatou and Privalov/Plessner, F has radial limits and we have $|F_*(\zeta)| = \psi(\zeta)$, $\zeta \in \mathbb{T}$ a.e.

If $f \in H^p$, then, in view of (5.7), we can define F by (5.21) with $\psi = |f_*|$. The function $\omega = f/F$ is then inner and we have the factorization $f = \omega F$. This leads to Smirnov's factorization theorem.

5.4.2 Theorem *Every function $f \in H^p$, $p > 0$, admits a representation $f = BSF$, where B is a Blaschke product, S is a singular inner function and F is an outer function. We have $\|f\|_p = \|F\|_p$, $|f(z)| \leqslant |F(z)|$, and $|f_*| = |F_*|$.*

We call F and $I = BS$ the **outer factor** and the **inner factor** of f, respectively. The factorization $f = IF$ is called the inner-outer factorization of f. This factorization is unique if we require, for instance, that $I(0)$ is a positive real number.

Exercises

5.4.3 A positive, nonconstant harmonic function u is equal to the Poisson integral of a singular measure iff there exists an inner function ω such that

$$u(z) = \operatorname{Re} \frac{1 + \omega(z)}{1 - \omega(z)}.$$

5.4.4 If f is a nonconstant inner function, then there holds strict inequality in (5.7). On the other hand, if a function $f \in H^p$ is outer, then

$$\log |f(re^{i\theta})| = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(r, \theta - t) \log |f_*(e^{it})| \, dt \qquad (0 \leqslant r < 1).$$

Conversely, if this equality holds for a fixed $re^{i\theta} \in \mathbb{D}$, then f is outer. In particular, a function $f \in H^p$ is outer iff

$$\log |f(0)| = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f_*(e^{it})| \, dt.$$

5.4.5 Every outer function can be represented as the ratio of two *bounded* outer functions. Consequently, every H^p-function is the ratio of two bounded analytic functions.

5.4.6 If $f \in H^p$ and $1/f \in H^p$ for some $p > 0$, then f is outer. By Theorem 4.4.2, f is in H^p if $\operatorname{Re} f > 0$. Since $\operatorname{Re}(1/f) = \bar{f}/|f|^2$, we see that f is outer if $\operatorname{Re} f > 0$.

Addendum: Riesz' representation theorem

The factorization theorems of Riesz and Smirnov are closely related to the Riesz' representation theorem for subharmonic functions. We formulate this theorem following Hörmander [30, §3.3].

5.4.7 Theorem (a) *A function $u \not\equiv -\infty$, subharmonic in $\mathbb{D}$, can be represented in the form*

$$u(z) = h(z) + \frac{1}{2\pi} \int_{\mathbb{D}} \log \left| \frac{w - z}{1 - \bar{w}z} \right| d\mu(w), \tag{5.22}$$

where h is a function harmonic in $\mathbb{D}$ and μ is a positive measure in $\mathbb{D}$, iff the function $I(r, u)$ $(0 < r < 1)$ is bounded from above, which is equivalent to the requirement *that u possesses a harmonic majorant in $\mathbb{D}$. The function h and the measure μ are uniquely determined by u; h is the smallest harmonic majorant of u and μ is equal to the Riesz measure of u.*

(b) *Let μ be a positive measure on $\mathbb{D}$. If the integral in (5.22) converges for some $z \in \mathbb{D}$, then*

$$\int_{\mathbb{D}} (1 - |z|)\, d\mu(z) < \infty. \tag{5.23}$$

Conversely, this condition implies that the integral in (5.22) defines a subharmonic function $\not\equiv -\infty$, the smallest harmonic majorant of which is $\equiv 0$.

When specialized to the case $u = \log|f|$, $f \in H(\mathbb{D})$, this theorem yields the following extension of Theorem 5.3.2.

5.4.8 Theorem *Let $f \in H(\mathbb{D})$, $f \not\equiv 0$, and*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log|f(re^{i\theta})|\, dt \leqslant C \qquad (0 < r < 1). \tag{5.24}$$

If $\{a_n\}$ is the sequence of zeros of f, then the Blaschke condition is satisfied and $f = Bg$, where B is the corresponding Blaschke product and g has no zeros in $\mathbb{D}$. The smallest harmonic majorant of $\log|B|$ is $\equiv 0$. Further, if f admits a factorization $f = f_1 f_2$ with $|f_1(z)| \leqslant 1$ and $f_2(z) \neq 0$ for $z \in \mathbb{D}$, then (5.24) holds and $|f_1(z)| \leqslant |B(z)|$, $|f_2(z)| \geqslant |g(z)|$.

Here we only note that if $u = \log|f|$, $f \in H(\mathbb{D})$, then condition (5.23) reduces to the Blaschke condition because the Riesz measure of $\log|f|$ is equal to $2\pi \sum \delta_{a_n}$.

Beurling's approximation theorem

This theorem can be viewed as a generalization of the fact that the set, $\mathcal{Q}$, of all polynomials is dense in H^p for $0 < p < \infty$.

5.4.9 Theorem *Let $f \in H^p$, $0 < p < \infty$. Then the closed linear span in H^p of the set $\mathcal{Q}f = \{qf : q \in \mathcal{Q}\}$ is equal to BSH^p, where BS is the inner factor of f. In particular, if f is outer, then the set $\mathcal{Q}f$ is dense in H^p.*

Proof. [30] Since $|BS| \leqslant 1$, we see that it suffices to prove that $\mathcal{Q}F$ is dense in H^p. Proving this reduces to proving that $\mathcal{Q} \subset \overline{\mathcal{Q}F}$ because $\mathcal{Q}$ is dense in H^p. Now we see that the proof of the theorem reduces to the proof that $1 \in \overline{\mathcal{Q}F}$. Let

$$F(z) = e^{ic} \exp\left(\frac{1}{2\pi} \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \lambda(\zeta)\, |d\zeta|\right),$$
$$F_s(z) = e^{ic} \exp\left(\frac{1}{2\pi} \int_{\lambda(\zeta) < s} \frac{\zeta + z}{\zeta - z} \lambda(\zeta)\, |d\zeta|\right).$$

The functions F_s have the properties: (a) $|F_s(\zeta)| \leqslant 1$ for $s < 0$ and $F_s(\zeta) \to 1$ as $s \to -\infty$; (b) $F_s = FG_s$, where $G_s \in H^\infty$.

It follows from (a) that $\|F_s - 1\|_p \to 0$ as $s \to -\infty$ so, by (b), it remains to prove that $F_s \in \overline{QF}$. To prove this let $H_\rho(z) = G_s(\rho z)$, $\rho < 1$. Then

$$\|FH_\rho - F_s\|_p^p = \frac{1}{2\pi}\int_{-\pi}^{\pi} |F(e^{i\theta})|^p\,|G_s(\rho e^{i\theta}) - G_s(e^{i\theta})|^p\,d\theta \to 0, \qquad \rho \to 1,$$

by the dominated convergence theorem. Finally, if $\rho < 1$, then the Taylor series of H_ρ converges uniformly on the unit circle, which implies that $FH_\rho \in \overline{QF}$. The proof is complete. □

5.5 Composition with inner functions

Throughout this section we consider nonconstant inner functions. If ω is such a function, then we put

$$\omega_*(\zeta) = \sphericalangle \lim_{z\to\zeta} \omega(z),$$

for those $\zeta \in \mathbb{T}$ for which this limit exists and belongs to $\mathbb{T}$; then we extend ω_* to a function from $\mathbb{T}$ to $\mathbb{T}$ in an arbitrary way.

Our main purpose is to prove the validity of the relations

$$f \in H^p \iff f\circ\omega \in H^p, \quad \text{and}$$
$$\|f\circ\omega\|_p = \|f\|_p \quad \text{if} \quad \omega(0) = 0,$$

due to Stephenson [94] (see Theorems 5.5.5 and 5.5.6). These relations as well as all other assertions in this section become obvious when specialized to the case $\omega(z) = z^n$, $n \geqslant 1$.

5.5.1 Proposition *Let ω be an inner function. If ϕ and g are Borel measurable functions on $\mathbb{T}$ such that $\phi = g$ a.e., then $\phi\circ\omega_* = g\circ\omega_*$ a.e. Consequently, if ϕ_n are Borel functions on $\mathbb{T}$ such that $\phi_n \to f$ a.e., then $\phi_n\circ\omega_* \to f\circ\omega_*$ a.e.*

Proof. If $\omega(0) = a$, then $\omega = \varphi\circ(\varphi\circ\omega)$, where $\varphi(z) = (a-z)/(1-\bar{a}z)$ and $\varphi\circ\omega(0) = 0$. Therefore we can assume that $\omega(0) = 0$. Let

$$E = \{e^{i\theta} \in \mathbb{T} \colon \phi(e^{i\theta}) \neq g(e^{i\theta})\}.$$

This is a set of measure zero, and we have to prove that the set

$$F = \{e^{i\theta} \in \mathbb{T} \colon \omega_*(e^{i\theta}) \in E\}$$

is of measure zero. To prove this let $\varepsilon > 0$, let $E_\varepsilon = \bigcup_n I_n$, where $I_n \subset \mathbb{T}$ are closed arcs such that $E \subset E_\varepsilon$, $\sum_n |I_n| < \varepsilon$, and let $F_\varepsilon = \{e^{i\theta} \in \mathbb{T} \colon \omega_*(e^{i\theta}) \in E_\varepsilon\}$ We shall prove that

$$\int_0^{2\pi} K_n(\omega_*(e^{i\theta}))\,d\theta = \int_0^{2\pi} K_n(e^{i\theta})\,d\theta = |I_n|, \tag{5.25}$$

where K_n is the characteristic function of I_n. This implies that

$$|F_\varepsilon| \leqslant \sum_n \int_0^{2\pi} K_n(\omega_*(e^{i\theta}))\, d\theta < \varepsilon,$$

and this implies that F is of measure zero, because $F \subset F_\varepsilon$ for all $\varepsilon > 0$.

To prove (5.25), when n is fixed, we choose a sequence $\phi_j \in C(\mathbb{T})$ such that $\phi_j(e^{it})$ tends to $K_n(e^{it})$ for every t and $|\phi_j(e^{it})| \leqslant 1$ for every t. Then $\phi_j(\omega_*(e^{it})) \to K_n(\omega_*(e^{it}))$, as $j \to \infty$, so we can apply the dominated convergence theorem to reduce the proof to the formula

$$\int_0^{2\pi} \phi(\omega_*(e^{i\theta}))\, d\theta = \int_0^{2\pi} \phi(e^{i\theta})\, d\theta, \qquad \phi \in C(\mathbb{T}). \tag{5.26}$$

Finally, this is reduced to the case where ϕ is a trigonometric polynomial. The details are left to the reader. □

The formula (5.26) extends to arbitrary $\phi \in L^1(\mathbb{T})$.

5.5.2 Theorem [85, 94] *If ϕ is a Borel function of class $L^1(\mathbb{T})$ and ω is an inner function with $\omega(0) = 0$, then $\phi \circ \omega_* \in L^1(\mathbb{T})$ and*

$$\int_0^{2\pi} \phi(\omega_*(e^{i\theta}))\, d\theta = \int_0^{2\pi} \phi(e^{i\theta})\, d\theta.$$

Proof. In order to reduce the proof to the case $\phi \in C(\mathbb{T})$, we can suppose that ϕ is a positive real function. The sequence $\min\{\phi(\zeta), n\}$ increases to $\phi(\zeta)$ everywhere, so the proof reduces to the case where ϕ is bounded. If ϕ is bounded, then we choose a bounded sequence $\phi_n \in C(\mathbb{T})$ such that $\phi_n \to \phi$ a.e.; by Proposition 5.5.1, we have $\phi_n \circ \omega_* \to \phi \circ \omega_*$ a.e. The result follows. □

5.5.3 Theorem *If ϕ is a Borel function of class $L^1(\mathbb{T})$ and ω is an inner function, then $P[\phi \circ \omega_*] = P[\phi] \circ \omega$.*

Proof. It suffices to consider the case where $\phi \in C(\mathbb{T})$. Then the functions $P[\phi \circ \omega_*]$ and $P[\phi] \circ \omega$ are harmonic and bounded so it suffices to prove that their boundary functions coincide almost everywhere. Since $P[\phi]$ is continuous on the closed disk, we have $\lim_{r\to 1} P[\phi](\omega(re^{i\theta})) = \phi(\omega_*(e^{i\theta}))$ a.e. On the other hand, $\lim_{r\to 1} P[\phi \circ \omega](re^{i\theta}) = (\phi \circ \omega_*)(e^{i\theta})$ a.e., and this completes the proof. □

5.5.4 Corollary *If ω and I are inner functions, then so is the composition $I \circ \omega$, and $(I \circ \omega)_* = I_* \circ \omega_*$ a.e. If in addition I is singular, then so is $I \circ \omega$.*

Proof. This follows from the relations: $P[I_* \circ \omega_*] = P[I_*] \circ \omega = I \circ \omega$ and $P[(I \circ \omega)_*] = I \circ \omega$. □

Stephenson's theorems

Combining the above results one easily proves the following.

5.5.5 Theorem *If $f \in H^p$, $p > 0$, and ω is inner with $\omega(0) = 0$, then $f \circ \omega \in H^p$, $(f \circ \omega)_* = f_* \circ \omega_*$, and $\|f \circ \omega\|_p = \|f\|_p$. If $f = IF$ is the inner-outer factorization of f, then $f \circ \omega = (I \circ \omega)(F \circ \omega)$ is the inner-outer factorization of $f \circ \omega$.*

We conclude this section by proving the implication $f \circ \omega \in H^p \Longrightarrow f \in H^p$.

5.5.6 Theorem *If $f \in H(\mathbb{D})$ and ω is an inner function, then $f \circ \omega \in H^p$ implies $f \in H^p$.*

Proof. Assume that $\omega(0) = 0$. Let $f \circ \omega \in H^p$, $p > 0$, let $u = |f|^p$ and $v = |f \circ \omega|^p$, and let $h = P[v_*]$. We know that $v \leqslant h$; see (5.8). Let $D = r\mathbb{D}$, where r, $0 < r < 1$, is fixed. Then $u \leqslant M$ on $\overline{D}$ for some constant $M < \infty$. Put $\Omega = \omega^{-1}(D)$. For $0 < \rho < 1$, let $E_\rho = \{\zeta \in \mathbb{T}\colon \omega(\rho\zeta) \in \overline{D}\}$. Since $1 - |\omega(\rho\zeta)| \to 0$ a.e. as $\rho \to 1$, we see, using Egorov's theorem, that $\lim_{\rho\to 1} |E_\rho| = 0$. Hence we can choose ρ so that the following is true:

If φ is the bounded harmonic function in the disk $\rho\mathbb{D}$ whose values are M on ρE_ρ and 0 on the rest of ρE_ρ, then

$$\varphi(0) < 1. \tag{5.27}$$

Since u is subharmonic and continuous in $\mathbb{D}$, there exists a function $u_1 \in C(\overline{D})$ harmonic in D such that $u \leqslant u_1 \leqslant M$ and $u_1 = u$ at every point of ∂D. By the mean value property,

$$u_1(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(re^{i\theta})\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^p\,d\theta. \tag{5.28}$$

Since $u \circ \omega \leqslant h$ in $\mathbb{D}$, we have that $u_1 \circ \omega \leqslant h$ on $\partial\Omega \cap \rho\overline{\mathbb{D}}$. Consider the function $u_1 \circ \omega - h$ on the closure of the set $\Omega_\rho = \Omega \cap \rho\mathbb{D}$. At boundary points of Ω_ρ that lie in $\rho\mathbb{D}$ we have $u_1 \circ \omega - h \leqslant 0$. The other boundary points of Ω_ρ lie in ρE_ρ, and there $u_1 \circ \omega - h \leqslant u_1 \circ \omega \leqslant M$. Thus $u_1 \circ \omega - h \leqslant \varphi$ on $\partial\Omega_\rho$. Since these functions are harmonic in Ω_ρ and $\Omega_\rho \ni 0$, it follows from (5.27) that $u_1(0) = u_1(\omega(0)) \leqslant h(0) + 1$. Now the desired result follows from (5.28). □

Approximation by inner functions

Let ω be an inner function with $\omega(0) = 0$. If $f \in H^2$, then, by Rogosinski's theorem (see 4.4.7) and Stephenson's theorem, we have

$$\sum_{k=n}^{\infty} |\widehat{f}(k)|^2 \geqslant \sum_{k=n}^{\infty} |\widehat{F}(k)|^2,$$

where $f = F \circ \omega$. This fact can be expressed in terms of best approximation.

Let $\mathcal{Q}_n$, $n \geqslant 0$, denote the set of all holomorphic polynomials of degree at most n. For a function $f \in H^p$, let $E_n(f)_p = \inf_{g \in \mathcal{Q}_n} \|f - g\|_p$. Then the above inequality can be stated as $E_n(f \circ \omega)_2 \geqslant E_n(f)_2$, $f \in H^2$. It is interesting that this extends to the case $1 \leqslant p \leqslant \infty$.

5.5.7 Theorem [64] *Let $1 \leqslant p \leqslant \infty$, $f \in H^p$, and let ω be an inner function with $\omega(0) = 0$. Then $E_n(f \circ \omega)_p \geqslant E_n(f)_p$.*

Proof. Let

$$(f, h) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) \overline{h(e^{i\theta})}\, d\theta.$$

We identify H^p with $H^p(\mathbb{T})$. From the general theory of best approximation in Banach spaces we know that

$$E_n(f)_p = \sup\{\,|(f, h)| \colon h \in L^q(\mathbb{T}),\ (\mathcal{Q}_n, h) = 0,\ \|h\|_q \leqslant 1\} \quad (1/p + 1/q = 1),$$

where $(\mathcal{Q}_n, h) = 0$ means that $(g, h) = 0$ for every $g \in \mathcal{Q}_n$. Now we apply this formula to $f \circ \omega$ and use the following facts: (a) If $h \in L^q(\mathbb{T})$, then $h \circ \omega \in L^q(\mathbb{T})$ and $\|h \circ \omega\|_q = 1$, and (b) if $(h, \mathcal{Q}_n) = 0$, then $h \circ \omega = 0$. Fact (a) follows from Theorem 5.5.2, while the proof of (b) is straightforward—it is enough to observe that $(h, \mathcal{Q}_n) = 0$ iff $\widehat{h}(j) = 0$ for $0 \leqslant j \leqslant n$. We get

$$E_n(f \circ \omega)_p \geqslant \sup\{\,|(f \circ \omega, h \circ \omega)| \colon h \in L^q(\mathbb{T}),\ (\mathcal{Q}_n, h) = 0,\ \|h\|_q \leqslant 1\}.$$

This concludes the proof because $|(f \circ \omega, h \circ \omega)| = |(f, h)|$, by Theorem 5.5.2. □

5.5.8 Remark If we change the notation and denote by $E_n(f)_p$ the best L^p approximation of $f \in L^p(\mathbb{T})$ by trigonometric polynomials of degree $\leqslant n$, then Theorem 5.5.7 remains valid. However, the above method heavily depends on the Hahn/Banach theorem and cannot be applied to the case $p < 1$. It would be interesting to study this case.

6 Conjugate functions

The main theorems of this chapter are the Privalov/Plessner theorem on the existence of radial limits of conjugate functions and the existence of the Hilbert operator (Theorem 6.1.1), and the Riesz theorem that if u is in h^p, $1 < p < \infty$, then so is its harmonic conjugate (Section 6.2). The rest of the chapter is devoted to some related results. The equality $L^p(\mathbb{T}) = H^p(\mathbb{T}) + \overline{H^p}(\mathbb{T})$, $0 < p < 1$, due to Aleksandrov, is in Section 6.4. Section 6.5 contains a theorem on strong convergence in H^1. In Section 6.6 we characterize harmonic quasiconformal homeomorphisms of the unit disk via the Hilbert transformation of the derivative of the boundary function.

6.1 Harmonic conjugates

To each $f \in h(\mathbb{D})$ there corresponds the harmonic conjugate $\widetilde{f} \in h(\mathbb{D})$,

$$\widetilde{f}(re^{i\theta}) = -i \sum_{n=-\infty}^{\infty} (\operatorname{sign} n)\, \widehat{f}(n) r^{|n|} e^{in\theta}.$$

If f is real-valued, then $\widetilde{f}$ is uniquely determined by the conditions: (a) $\widetilde{f}$ is real-valued, (b) $f + i\widetilde{f}$ is analytic, and (c) $\widetilde{f}(0) = 0$.

For an arbitrary $f \in h(\mathbb{D})$ there holds

$$\widetilde{\widetilde{f}} = -f + f(0). \tag{6.1}$$

And if f is analytic, then $\widetilde{f} = -i(f - f(0))$, and $\widetilde{\operatorname{Re} f} = \operatorname{Im}(f - f(0))$.

The function conjugate to $P[\phi]$, $\phi \in L^1(\mathbb{T})$, equals

$$\widetilde{P}[\phi](re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \widetilde{P}(r, \theta - t)\phi(e^{it})\,dt = \frac{1}{2\pi}\int_0^{\pi} \widetilde{P}(r,t)\left[\phi(\theta - t) - \phi(\theta + t)\right] dt,$$

where we write $\phi(x)$ instead of $\phi(e^{ix})$. Here $\widetilde{P}$ denotes the **conjugate Poisson kernel,**

$$\widetilde{P}(z) = \operatorname{Im}\frac{2z}{1-z} = \operatorname{Im}\frac{1+z}{1-z}, \quad \text{i.e.,} \quad \widetilde{P}(r,\theta) = \widetilde{P}(re^{i\theta}) = \frac{2r\sin\theta}{1 + r^2 - 2r\cos\theta}.$$

This kernel does not belong to h^1.

The kernels P and $\widetilde{P}$ are connected by the formula

$$\widetilde{P}(r,\theta) - \frac{1}{\tan(\theta/2)} = -\frac{1-r}{\tan(\theta/2)}\frac{P(r,\theta)}{1+r}. \tag{6.2}$$

Note that $1/\tan(\theta/2) = \widetilde{P}(1,\theta)$.

It is known that $\widetilde{P}[\phi]$ need not be in h^1 (see 6.1.4) but the Kolmogorov/Smirnov theorem 4.4.2 says that there holds the implication $f \in h^1 \Longrightarrow \widetilde{f} \in h^p$, $0 < p < 1$.(*)

The Privalov/Plessner theorem

6.1.1 Theorem *If $\phi \in L^1(\mathbb{T})$, then there exist and are equal the following limits (a.e.):*

$$\lim_{r\to 1^-} \widetilde{P}[\phi](re^{i\theta}) \qquad \textit{and} \qquad \lim_{\varepsilon\to 0+} \frac{1}{\pi}\int_\varepsilon^\pi \frac{\phi(\theta-t)-\phi(\theta+t)}{2\tan(t/2)}\,dt.$$

The existence of $\lim_{r\to 1^-} \widetilde{P}[\phi](re^{i\theta})$ is contained in Corollary 3.3.9. This theorem guarantees the existence of the improper integral

$$\widetilde{\phi}(e^{i\theta}) = \frac{1}{\pi}\int_{0+}^\pi \frac{\phi(\theta-t)-\phi(\theta+t)}{2\tan(t/2)}\,dt.$$

There exists such a function $\phi \in C(\mathbb{T})$ that this integral converges absolutely for no θ. It is even more interesting that there exists a function $\phi \in C(\mathbb{T})$ such that the improper integral

$$\int_{0+}^\pi \frac{\phi(\theta+t)-\phi(\theta)}{2\tan(t/2)}\,dt$$

diverges for every θ (see [100, p. 133–4]).

The Hilbert operator The function $\widetilde{\phi}$ is said to be **conjugate** with ϕ and the operator H taking ϕ to $\widetilde{\phi}$ is called the Hilbert operator.(†) The Hilbert operator maps L^1 into L^p, for every $p < 1$, but not into L^1, so in the general case the Poisson integral of $\widetilde{\phi}$ has no sense. However, as we will prove later on (see Theorem 6.1.3), if $\widetilde{\phi} \in L^1$, then $P[\widetilde{\phi}] = \widetilde{P}[\phi]$.

Proof of Theorem 6.1.1. It suffices to prove the relation

$$\lim_{r\to 1^-} \left\{ \widetilde{P}[\phi](re^{i\theta}) - \frac{1}{2\pi}\int_{1-r}^\pi \frac{\phi(\theta-t)-\phi(\theta+t)}{\tan(t/2)}\,dt \right\} = 0, \tag{6.3}$$

under the hypothesis that θ is a Lebesgue point of ϕ. We write the difference under $\lim_{r\to 1^-}$ in (6.3) as $I_1(r) + I_2(r)$, where

$$I_1(r) = \frac{1}{2\pi}\int_0^{1-r} \widetilde{P}(r,t)\big[\phi(\theta-t)-\phi(\theta+t)\big]\,dt.$$

Since $|\widetilde{P}(r,t)| \leqslant 2/(1-r)$, $|t| \leqslant 1-r$, we have

$$|I_1(r)| \leqslant \frac{1}{\pi(1-r)}\int_0^{1-r} \big|\phi(\theta-t)-\phi(\theta+t)\big|\,dt \to 0 \qquad (r\to 1),$$

(*)See also Kolmogorov's theorem 7.1.10.

(†)Usually $2\tan(t/2)$ is replaced by t.

because θ is a Lebesgue point. In the case of the integral

$$I_2(r) = \frac{1}{2\pi}\int_{1-r}^{\pi}\left[\widetilde{P}(r,t) - \frac{1}{\tan(t/2)}\right][\phi(\theta - t) - \phi(\theta + t)]\,dt$$

we use the formula (6.2); it follows that

$$\left|\widetilde{P}(r,t) - \frac{1}{\tan(t/2)}\right| \leqslant \text{const.}\,P(r,t) \qquad (1 - r < |t| < \pi).$$

Thus

$$|I_2(r)| \leqslant C\frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,t)\big|\phi(\theta - t) - \phi(\theta + t)\big|\,dt.$$

Now the hypothesis that θ is a Lebesgue point and the following lemma imply that $I_2(r) \to 0$ $(r \to 1)$, and this completes the proof. □

6.1.2 Lemma *If θ is a Lebesgue point of a function $\psi \in L^1$, then*

$$\limsup_{r\to 1^-} \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,t)|\psi(\theta + t) - \psi(\theta)|\,dt = 0.$$

Proof. This can be deduced from Proposition 3.3.1 by taking

$$\gamma(t) = \int_0^t |\psi(\theta + x) - \psi(\theta)|\,dx. \qquad \square$$

The Poisson integral of the conjugate function

6.1.3 Theorem *If $\phi \in L^1$ and $\widetilde{\phi} \in L^1$, then $P[\widetilde{\phi}] = \widetilde{P}[\phi]$.*

Proof. Assume that ϕ is real-valued. By the Kolmogorov/Smirnov theorem, the function $f = P[\phi] + i\widetilde{P}[\phi]$ belongs to H^p for $p < 1$. Now Smirnov's theorem 5.1.12 tells us that $f \in H^1$, and hence $f = P[f_*]$, by Theorem 5.2.1. Finally, since $f_* = \phi + i\widetilde{\phi}$, by the theorems of Fatou and Privalov/Plessner, we see that

$$P[\phi] + i\widetilde{P}[\phi] = P[f_*] = P[\phi] + iP[\widetilde{\phi}],$$

and the result follows. □

Miscellaneous

6.1.4 If $\{a_k\}$ is a convex sequence tending to 0, then the sum of the series $\sum_{n=1}^{\infty} a_n \cos n\theta$ is positive for every $\theta \in (-\pi, \pi)$, and its sum belongs to $L^1(\mathbb{T})$ (see [42, Theorem 4.1]). In particular, the function

$$\phi(e^{i\theta}) = \sum_{n=2}^{\infty}(\log n)^{-1}\cos n\theta$$

is in L^1, while the function conjugate to ϕ is equal to $\sum_{n=2}^{\infty}(\log n)^{-1}\sin n\theta$ and is not in L^1.

6.1.5 Using (6.1) one can prove the following:

If $\widetilde{\phi} \in L^1$, then

$$\widetilde{\widetilde{\phi}} = -\phi + \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(e^{i\theta})\,d\theta.$$

The Privalov/Plessner theorem can be stated in the following form:

6.1.6 Theorem *Let $\phi \in L^1(\mathbb{T})$ and let $\Phi(\theta)$ be the indefinite integral of the function $\theta \mapsto \phi(e^{i\theta})$. Then the improper integral*

$$-\frac{1}{\pi}\int_{0+}^{\pi} \frac{\Phi(\theta+t)+\Phi(\theta-t)-2\Phi(\theta)}{4\sin^2(t/2)}\,dt$$

exists for all θ and is equal to $\widetilde{\phi}(\theta)$ almost everywhere.

A more general variant states:

Let $\gamma \in BV[-\pi,\pi]$ and let $\gamma(t+2\pi)-\gamma(t) = \mathrm{const}$. Then the integral

$$-\frac{1}{\pi}\int_{0+}^{\pi} \frac{\gamma(\theta+t)+\gamma(\theta-t)-2\gamma(\theta)}{4\sin^2(t/2)}\,dt$$

and the limit

$$\lim_{r\to 1^-} \widetilde{PS}[\gamma](re^{i\theta})$$

exist and are equal almost everywhere.

See Zygmund [100, Ch. III §§7-8, IV §3 and VII §1].

The Riesz/Zygmund inequality

As we have seen, if $g \in h^1$, then the conjugate function $\widetilde{g}$ need not belong to h^1. A result of Riesz and Zygmund [100, Ch. IV, (6.28)] states that if $g \in h^1$, then

$$\int_{-1}^{1} \left|\frac{\widetilde{g}(re^{it})}{r}\right| dr \leqslant \pi \|g\|_1. \tag{6.4}$$

In other words:

6.1.7 Theorem *If $g \in h(\mathbb{D})$ and $(\partial g/\partial\theta) \in h^1$, then*

$$\int_{-1}^{1} \left|\frac{\partial g}{\partial r}(re^{it})\right| dr \leqslant \pi \left\|\frac{\partial g}{\partial \theta}\right\|_1. \tag{6.5}$$

Proof. We can assume that g is harmonic in a neighborhood of $\overline{\mathbb{D}}$. The function $v = r\partial g/\partial r$ is conjugate to the function $u = \partial g/\partial\theta$ and therefore

$$v(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left[\operatorname{Im} F(re^{it})\right] u(e^{i(\theta-t)})\,dt, \tag{6.6}$$

where $F(z) = 2z/(1-z)$. Now (6.5) can be deduced from (6.6) and the equation

$$\int_{-1}^{1} |r^{-1} \operatorname{Im} F(re^{it})| \, dr = \pi \quad (0 < |t| < \pi), \tag{6.7}$$

by using Fubini's theorem.

In order to prove (6.7) let $0 < t < \pi$, and apply Cauchy's integral theorem to the function $F(z)/z$ on the semidisk $\{re^{i\theta} : 0 \leqslant r \leqslant 1, \, t \leqslant \theta \leqslant t + \pi\}$ (which does not contain the point $z = 1$). We get

$$\int_{-1}^{1} \frac{F(re^{it})}{r} \, dr = -i \int_{t}^{t+\pi} F(e^{i\theta}) \, d\theta.$$

Hence

$$\int_{-1}^{1} \frac{\operatorname{Im} F(re^{it})}{r} \, dr = - \int_{t}^{t+\pi} \operatorname{Re} F(e^{i\theta}) \, d\theta.$$

Now (6.7) follows from $\operatorname{Re} F(e^{i\theta}) = -1$ and $\operatorname{Im} F(re^{it})/r > 0$. □

Geometric interpretation Let γ be a function continuous and of bounded variation on $[-\pi, \pi]$, let $\gamma(\pi) = \gamma(-\pi)$, and $f(re^{i\theta}) = P[\gamma]$. The function f can be treated as a continuous mapping of the closed disk to the complex plane.

6.1.8 Corollary *If f is a homeomorphism, then*

$$\textit{length of } f([-1,1]) \leqslant (1/2) \times \textit{length of } \partial f(\mathbb{D}).$$

The hypothesis that the curve $z = \gamma(t), \pi \leqslant t \leqslant \pi$, is a Jordan curve is not sufficient for f to be a homeomorphism. A sufficient condition is described by Choquet's theorem [13]:

6.1.9 Theorem *A sufficient condition for f to be a homeomorphism is that the curve $z = \gamma(t)$ ($|t| \leqslant \pi$) is a* convex *Jordan curve.*

6.2 Riesz projection theorem

The operator R_+ acting from $h(\mathbb{D})$ to $H(\mathbb{D})$ according to the rule $(R_+ u)(z) = \sum_{n=0}^{\infty} \widehat{u}(n) z^n$ is called the **Riesz projector**.

The projection theorem

It is a direct consequence of Parseval's theorem that R_+ acts as an orthogonal projection from h^2 onto H^2. This fact was generalized by M. Riesz in the following way.

6.2.1 Theorem *If $1 < p < \infty$, then R_+ acts as a bounded projection from h^p onto H^p.*

By the Kolmogorov/Smirnov theorem, we can treat R_+ as an operator from $L^1(\mathbb{T})$ to $H^p(\mathbb{T}) = \{f_* : f \in H^p\} \subset \mathcal{L}_0(\mathbb{T})$, for $p < 1$. From the projection theorem and Theorem 3.4.2 it follows that for every $\phi \in L^p(\mathbb{T})$ $(1 < p < \infty)$ there exists a unique function $\psi \in L^p(\mathbb{T})$ such that $\widehat{\psi}(n) = \widehat{\phi}(n)$ for $n \geqslant 0$ and $\widehat{\psi}(n) = 0$ for $n < 0$. This enables us to treat R_+ as an operator from $L^p(\mathbb{T})$ to $H^p(\mathbb{T})$, $1 < p < \infty$. However, the Riesz projection does not map L^1 into $H^1(\mathbb{T})$ (see 6.1.4); furthermore, H^1 is not complemented in L^1, i.e., there is no bounded projection from L^1 to H^1 (see [87]).[‡]

The conjugate functions theorem

Since the Riesz projector is connected with conjugate function in a simple way, namely $R_+u = u(0) + (u + i\widetilde{u})/2$, the Riesz theorem can be stated as follows:

6.2.2 Theorem *If $u \in h^p$, $p > 1$, then $\widetilde{u} \in h^p$ and there exists a constant C_p such that $\|\widetilde{u}\|_p \leqslant C_p\|u\|_p$.*

In view of the connection between conjugate functions and the Hilbert operator (Privalov/Plessner theorem, 6.1.1), we have:

6.2.3 Theorem *The Hilbert operator maps $L^p(\mathbb{T})$ to $L^p(\mathbb{T})$ for $1 < p < \infty$.*

In the case $p = 2$, Theorems 6.2.2 and 6.2.1 follow from Parseval's formula; we have $\|\widetilde{u}\|_2^2 = \|u\|_2^2 - |u(0)|^2$. If a proof is known either for $1 < p < 2$ or for $p > 2$, then the general case can be treated by duality. Let us mention four "short proofs."

(i) The operator $R_+ \colon L^1(\mathbb{T}) \mapsto \mathcal{L}_0(\mathbb{T})$ is of strong type $(2,2)$ and, by Kolmogorov's theorem (Theorem 7.1.10), of weak type $(1,1)$. Therefore we can apply Marcinkiewicz's theorem.

(ii) If $1 < p < 2$, we can use Theorem 2.6.2 and the existence of the radial limits of $\widetilde{u}$.

(iii) If $1 < p < 2$, then, as noted after Theorem 7.2.1, we can use Theorems 7.2.1 and 7.1.2.

(iv) If $p = 2^n$, for some positive integer n, then we can easily deduce the validity of Theorem 6.2.2 from the case $p = 2$; see the proof of Theorem 6.2.6. Then, for arbitrary $p > 2$, we can apply the Riesz/Thorin theorem.

Here we present an elementary proof, due to P. Stein, based on the Hardy/Stein identities.

[‡]If every subspace of a Banach space X is complemented in X, then X is isomorphic to a Hilbert space [51].

Hardy/Stein identities

There is a way to express the H^p-norm as a double integral. For example, from Green's formula (3.3) and the formula $\Delta(|f|^2) = 4|f'|^2$ it follows that

$$\|f\|_2^2 = |f(0)|^2 + 2\int_{\mathbb{D}} |f'(z)|^2 \log\frac{1}{|z|}\, dA(z). \tag{6.8}$$

Concerning other values of p, we consider only the cases that are sufficient to prove Riesz' projection theorem.

6.2.4 Lemma *Let $0 < p < \infty$. A function $f \in H(\mathbb{D})$ belongs to H^p if and only if*

$$HS_p(f) := |f(0)|^p + \frac{p^2}{2}\int_{\mathbb{D}} |f|^{p-2}|f'|^2 \log\frac{1}{|z|}\, dA < \infty.$$

Moreover we have $\|f\|_p^p = HS_p(f)$.

Proof. If f has no zeroes in $\mathbb{D}$, then it suffices to apply (6.8) to the function $f^{p/2}$. If $p \geqslant 2$, then the function $|f|^p$ is of class C^2 so we can apply Green's formula. In the general case one applies Green's formula to the functions $(|f|^2 + \varepsilon)^{p/2}$, $\varepsilon > 0$, and then let ε tend to 0 (see also 4.5.6). □

The formula $\|f\|_p^p = HS_p(f)$ is known as the **Hardy/Stein identity**. There holds an analogous formula for *real-valued* harmonic functions. We only need the case of positive functions.

6.2.5 Lemma *Let $f \in H(\mathbb{D})$ and let $u = \operatorname{Re} f$ belong to h^p, $1 < p < \infty$. Then*

$$\|u\|_p^p = |u(0)|^p + \frac{p(p-1)}{2}\int_{\mathbb{D}} u^{p-2}|f'|^2 \log\frac{1}{|z|}\, dA. \tag{6.9}$$

Proof. In the case where $u > 0$ this reduces to Lemma 3.5.4. In the general case one considers the functions $(u^2 + \varepsilon)^{p/2}$. □

Proof of Riesz' theorems

We shall prove Theorem 6.2.2. We may suppose that u is real-valued, and then the theorem can be stated as follows.

6.2.6 Theorem *Let f be analytic in $\mathbb{D}$, and let $1 < p < \infty$. If $\operatorname{Re} f \in h^p$, then $f \in H^p$ and there holds the inequality*

$$\|f\|_p^p \leqslant C_p(\|\operatorname{Re} f\|_p^p + |f(0)|^p).$$

If f is a conformal mapping of the disk onto the domain $G = \{z : 0 < \operatorname{Re} z < 1\}$, then $\operatorname{Re} f \in h^\infty$ but f is not in H^∞; therefore, the theorem does not hold for $p = \infty$. For the case $p = 1$ see 6.1.4.

Proof of Theorem 6.2.6. Consider first the case $1 < p \leqslant 2$. Let $u = \operatorname{Re} f \in h^p$. In view of Lemma 3.5.3, we may suppose that $u > 0$ and, as in the proof of Lemma 5.1.7, that f is analytic in a neighborhood of $\overline{\mathbb{D}}$. Then from Lemmas 6.2.4 and 6.2.5, together with the inequality $u^{p-2} \geqslant |f|^{p-2}$, it follows that

$$\|f\|_p^p - |f(0)|^p \leqslant \frac{p}{p-1}(\|u\|_p^p - |u(0)|^p),$$

which gives the desired result for $1 < p \leqslant 2$.

Let $2 < p \leqslant 4$ and let f be analytic in a neighborhood $\overline{\mathbb{D}}$. Then the function $g = -if^2$ belongs to H^q, $q = p/2$, and we have $\operatorname{Re} g = 2uv$, where $v = \operatorname{Im} f$, and therefore, by the preceding case, $\|f\|_p^p = \|g\|_q^q \leqslant C_p\|2uv\|_q^q$. On the other hand, Cauchy/Schwarz inequality gives $\|uv\|_q^q \leqslant (\|u\|_p\|v\|_p)^{p/2}$. Hence $\|f\|_p^p \leqslant C_p 2^q \|u\|_p^{p/2}\|f\|_p^{p/2}$. Since $\|f\|_p$ is finite, we can divide both sides by $\|f\|_p^{p/2}$, and this yields the result for $2 \leqslant p \leqslant 4$; etc. □

6.2.7 Remark If $0 < p < 1$, then (6.9) does not hold, but we still have

$$\frac{p(1-p)}{2}\int_{\mathbb{D}} u^{p-2}|f'|^2 \log\frac{1}{|z|}\,dA = u(0)^p - M_p^p(1,u) \leqslant u(0)^p,$$

provided u is positive and continuous on the closed disk. When combined with Lemma 6.2.4, this yields another proof of Theorem 4.4.2.

6.3 Applications of the projection theorem

The projection theorem has many important applications. For example, the trigonometric system is a Shauder basis in $L^p(\mathbb{T})$ for $1 < p < \infty$; in other words, the system of the functions $r^{|n|}e^{in\theta}$ is a Shauder basis in h^p.[§]

6.3.1 Theorem *Let $\phi \in L^p(\mathbb{T})$, $1 < p < \infty$, and $\phi_{m,n}(e^{i\theta}) = \sum_{k=m}^{n} \widehat{\phi}(k)e^{ik\theta}$, where m and n are integers, $m < n$. Then*

$$\|\phi_{m,n}\|_p \leqslant C_p\|\phi\|_p \tag{6.10}$$

$$\|\phi - \phi_{m,n}\|_p \to 0 \quad \text{as} \quad n \to \infty, m \to -\infty. \tag{6.11}$$

Proof. Let $e_k(e^{i\theta}) = e^{ik\theta}$. Then $\phi_{m,n} = e_m R_+(e_{-m}\phi) - e_n R_+(e_{-n}\phi)$. From this and Theorem 6.2.1 we obtain (6.10), and from (6.10) and the Weierstrass approximation theorem we obtain (6.11). □

Now we can determine the dual of H^p for $1 < p < \infty$.

[§]However, there are spaces, e.g., Bergman, in which this system is not a basis although there holds the analogue of the projection theorem.

6.3.2 Theorem *If $1 < p < \infty$, then the dual of H^p is isomorphic to H^q $(1/p + 1/q = 1)$ with respect to the bilinear form*

$$(f,g) = \lim_{r\to 1^-} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(re^{-i\theta})g(re^{i\theta})\,d\theta = \lim_{r\to 1^-}\sum_{n=0}^{\infty}\widehat{f}(n)\widehat{g}(n)r^{2n}.$$

Proof. Let Λ be a bounded linear functional on H^p and Λ_1 be the Hahn/Banach extension to h^p. By the projection theorem and Theorem 3.4.2, there exists a function $g \in h^q$ such that $\|\Lambda_1\| = \|g_1\|_q$ and $\Lambda_1 f = (f, g_1)$ for $f \in h^p$. Hence, $\Lambda f = (f, g_1)$ for $f \in H^p$. The function $g = R_+ g_1$ belongs to h^q and we have $(f,g) = (f,g_1)$ for $f \in H^p$ (because f is analytic), and this proves the inclusion $(H^p)^* \subset H^q$. The reverse inclusion is proved by using Hölder's inequality. □

6.3.3 Exercise (isomorphism L^p to H^p) If $1 < p < \infty$, then the formula

$$(Tu)(z) = \sum_{n=0}^{\infty}\widehat{u}(n)z^{2n} + \sum_{n=1}^{\infty}\widehat{u}(-n)z^{2n-1}$$

defines an isomorphism of h^p onto H^p.

6.3.4 Exercise (Parseval's formula) If $f \in L^p(\mathbb{T})$ and $g \in L^q(\mathbb{T})$, where $1/p + 1/q = 1$ and $1 < p < \infty$, then the series

$$\sum_{n=-\infty}^{\infty} \widehat{f}(n)\overline{\widehat{g}(n)}\,r^{|n|}e^{in\theta}$$

converges uniformly in $\overline{\mathbb{D}}$, and there holds Parseval's formula:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\,d\theta = \sum_{n=-\infty}^{\infty}\widehat{f}(n)\overline{\widehat{g}(n)}.$$

6.4 Aleksandrov's theorem

Relation (5.5) shows that $L^p(\mathbb{T})$ contains an isometric copy of H^p $(p > 0)$; denote this subspace by $H^p(\mathbb{T})$. Thus $H^p(\mathbb{T}) = \{f_* : f \in H^p\}$. If $p \geqslant 1$, then $H^p(\mathbb{T})$ can be described in the following way:

$$H^p(\mathbb{T}) = \{\phi \in L^p(\mathbb{T}) : \widehat{\phi}(-n) = 0 \text{ for } n \geqslant 1\}.$$

In the case $p < 1$, this fact does not hold, simply because the Fourier coefficients are not defined; then $H^p(\mathbb{T})$ is equal to the L^p-closure of

$$\mathcal{T}_+ = \{\phi \in \mathcal{T} : \widehat{\phi}(-n) = 0 \text{ for } n \geqslant 1\},$$

where $\mathcal{T}$ is the set of all trigonometric polynomials.

Let $\overline{H^p}(\mathbb{T}) = \{\overline{\phi} : \phi \in H^p(\mathbb{T})\}$. One of consequences of the projection theorem is that $L^p(\mathbb{T}) = H^p(\mathbb{T}) + \overline{H^p}(\mathbb{T})$, $1 < p < \infty$. This fact was extended by Aleksandrov [4, 3] to the case $p < 1$. However, in that case, the decomposition is not unique (up to an additive constant) because the intersection $H^p \cap \overline{H^p}$ is equal to the linear span of the set of the functions $g_a(\zeta) = 1/(1 - a\zeta)$ ($a \in \mathbb{T}$, $\zeta \in \mathbb{T}$) (see [4, 3]).

6.4.1 Theorem (Aleksandrov) *If $f \in L^p(\mathbb{T})$, $p < 1$, then there are functions $f_1 \in H^p(\mathbb{T})$, $f_2 \in \overline{H^p}(\mathbb{T})$, such that $f = f_1 + f_2$ and $\|f_1\|_p + \|f_2\|_p \leqslant C_p\|f\|_p$.*

Proof. Let X denote the direct sum of the spaces H^p and $\overline{H^p}$. Consider the operator $T : X \mapsto L^p$, $T(f_1, f_2) = f_1 + f_2$. For every trigonometric polynomial f, $\|f\|_p \leqslant 1$, we will find (f_1, f_2) so that $f = T(f_1, f_2)$, where $\|(f_1, f_2)\| \leqslant C_p$ (and C_p depends only of p) and then the result will follow from Theorem 1.3.1.

Let $f = \sum_{|k|\leqslant n} a_k e^{ik\theta}$ and $\gamma(\zeta) = \zeta^n$. Then γf and $\gamma \bar{f}$ belong to H^p. Put $\varphi = \gamma/(\gamma - 1)$. Then $\varphi \in H^p \cap \overline{H^p}$ and $\varphi + \bar{\varphi} = 1$. Now let $\varphi_t(\zeta) = \varphi(\zeta e^{it})$, $t \in \mathbb{R}$, and $g_t = f\varphi_t$, $h_t = \bar{f}\varphi_t$. Then g_t and h_t are H^p and $f = g_t + \bar{h}_t$. Routine calculation shows that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \|g_t\|_p^p \, dt = \|\varphi\|_p^p \, \|f\|_p^p,$$

which means that there exists t such that $\|g_t\|_p \leqslant \|\varphi\|_p$. For this value of t we have $\|h_t\|_p^p \leqslant \|f\|_p^p + \|g_t\|_p^p \leqslant 1 + \|\varphi\|_p^p$. Finally, we take $f_1 = g_t$, $f_2 = \bar{h}_t$, and this completes the proof. □

Kalton's theorem

Let B^p denote the space of functions $f \in H(D)$ such that

$$\|f\|_{B^p} := \left(|f(0)| + \int_D |f'(z)|^p (1 - |z|)^{p-1} \, dA(z)\right)^{1/p} < \infty.$$

By the Littlewood/Paley theorem, we have $B^p \subset H^p$ for $1 < p < 2$. This inclusion remains valid for $p \leqslant 1$, which follows from the inequality

$$M_p^p(r_{n+1}, f) - M_p^p(r_n, f) \leqslant C_p 2^{-np} M_p^p(r_{n+1}, f'), \qquad r_n = 1 - 2^{-n},$$

see Proposition 7.1.6. Therefore the following result of Kalton [37] improves Theorem 6.4.1.

6.4.2 Theorem *If $f \in L^p(\mathbb{T})$, $p < 1$, then there are functions $g \in B^p$, $h \in B^p$, such that $f = g_* + \overline{h_*}$ and $\|g\|_{B^p} + \|h\|_{B^p} \leqslant C_p\|f\|_p$.*

6.5 Strong convergence in H^1

For a function f analytic in $\mathbb{D}$ let

$$P_n f = \frac{1}{A_n}\sum_{j=0}^{n} \frac{1}{j+1} s_j f, \quad \text{where} \quad A_n = \sum_{j=0}^{n} \frac{1}{j+1} \qquad (n = 0, 1, 2, \ldots)$$

and $s_j f$ are the partial sums of the Taylor series of f. It is well known that $\|s_n f\| \leqslant C A_n \|f\|$ and that A_n is "best possible". A direct consequence is that

$$\frac{1}{\log n} \sum_{j=0}^{n} \frac{1}{j+1} \|s_j f\| \leqslant C \|f\| \log n \qquad (n \geqslant 2). \tag{6.12}$$

where C is an absolute constant. It turns out, however, that there holds the stronger inequality

$$\frac{1}{\log n} \sum_{j=0}^{n} \frac{1}{j+1} \|s_j f\| \leqslant C \|f\| \qquad (f \in H^1, \ n \geqslant 2). \tag{6.13}$$

Moreover, we have the following characterization of the space H^1.

6.5.1 Theorem [92, 76] *For a function f analytic in D the following assertions are equivalent:*

$$f \in H^1;$$

$$\sup_n \frac{1}{A_n} \sum_{j=0}^{n} \frac{1}{j+1} \|s_j f\| < \infty; \tag{6.14}$$

$$\sup_n \|P_n f\| < \infty. \tag{6.15}$$

Remark. It follows from the proof that the quantities occurring in (6.14) and (6.15) are "proportional" to the original norm in H^1; in particular there holds (6.13).

Since the polynomials are dense in H^1, we have the following consequence:

6.5.2 Theorem *If $f \in H^1$, then*

$$\lim_n \frac{1}{A_n} \sum_{j=0}^{n} \frac{1}{j+1} \|f - s_j f\| = 0$$

and, consequently,

$$\lim_n \frac{1}{A_n} \sum_{j=0}^{n} \frac{1}{j+1} \|s_j f\| = \|f\|.$$

6.5.3 Corollary [76] *If $f \in H^1$, then $\liminf_{n\to\infty} \|f - s_n f\| = 0$.*

There are functions $\phi \in L^1$ such that $\lim_n \|\phi - s_n \phi\| = \infty$; such an example is given by $\phi(e^{i\theta}) = \sum_{j=2}^{\infty} (\log j)^{-1/2} \cos j\theta$. Since the sequence $(\log j)^{-1/2}$ is convex, the function belongs to L^1 (see 6.1.4). Furthermore, one can show that $\|f - s_n f\| \geqslant c(\log n)^{1/2}$, $c = \text{const.} > 0$. We omit the details.

By means of Fatou's lemma, from Corollary 6.5.3 we obtain:

6.5.4 Corollary [76] *If $\phi \in H^1(\mathbb{T})$, then $\liminf_{n\to\infty} |\phi(e^{i\theta}) - s_n \phi(e^{i\theta})| = 0$ a.e.*

On the other hand, there exists a function $\phi \in H^1(\mathbb{T})$ whose Fourier series diverges almost everywhere (see [100, Ch. VIII, theorem (3.5)]).

Konyagin's theorem

The above corollary does not extend to L^1; Konyagin [45] proved the following improvement of Kolmogorov's theorem:

If $\{\psi(m)\}$ is a sequence of positive numbers such that

$$\psi(m) = o(\sqrt{\ln m}/\sqrt{\ln\ln m}) \quad \text{as} \quad m \to \infty,$$

then there exists a function $\phi \in L^1(\mathbb{T})$ such that

$$\limsup_{m\to\infty} s_m\phi(e^{i\theta})/\psi(m) = \infty \quad \textit{for all} \quad \theta \in \mathbb{T}.$$

Proof of Theorem 6.5.1

It is obvious that (6.14) implies (6.15). To prove that the condition $f \in H^1$ implies (6.14) let $f \in H^1$ and for fixed $n \geqslant 2$ and $w \in \mathbb{D}$ define the function $g \in H^1$ by

$$g(z) = (1 - rz)^{-1} f(rwz) \qquad (|z| \leqslant 1),$$

where $r = 1 - 1/n$. We have $g(z) = \sum_{j=0}^{\infty} s_j f(w) r^j z^j$. Applying Hardy's inequality (Theorem 5.3.7) we get

$$\sum_{j=0}^{\infty} \frac{1}{j+1} |s_j f(w)| r^j = \sum_{j=0}^{\infty} \frac{1}{j+1} |\widehat{g}(j)| \leqslant \pi \|g\|.$$

Since $r^j = (1 - 1/n)^j \geqslant c$ for $0 \leqslant j \leqslant n$, where $c > 0$ is an absolute constant, we have

$$\sum_{j=0}^{n} \frac{1}{j+1} |s_j f(w)| \leqslant (\pi/c)\|g\| = (1/2c) \int_0^{2\pi} |1 - re^{it}|^{-1} |f(rwe^{it})| dt.$$

Integrating this inequality over the circle $|w| = 1$ we find

$$\sum_{j=0}^{n} \frac{1}{j+1} \|s_j f\| \leqslant (1/2c)\|f\| \int_0^{2\pi} |1 - re^{it}|^{-1} dt,$$

where we have used Fubini's theorem. Finally, using the estimate

$$\int_0^{2\pi} |1 - re^{it}|^{-1} dt \leqslant C \log \frac{1}{1-r} = C \log n,$$

we see that (6.13) holds and therefore that (6.14) is implied by $f \in H^1$.

Let f be analytic in $\mathbb{D}$. From the uniform convergence of $s_n f$ on compact sets it follows that $P_n f \mapsto f$ uniformly on compact subsets of $\mathbb{D}$. Assuming that

$\|P_n f\| \leqslant 1$ for each n, we have $M_1(r, P_n f) \leqslant 1$ for all n and $r < 1$. This implies, via the uniform convergence of $P_n f$ on the circles $|z| = r$, that $M_1(r, f) \leqslant 1$ for every $r < 1$, which means that $\|f\| \leqslant 1$. Thus we have proved that (6.15) implies $f \in H^1$, and this completes the proof. □

Remarks

6.5.5 Inequality (6.12) is optimal in L^1 in the sense that $\log n$ cannot be replaced by any $\psi(n)$ (independent of f) such that $\psi(n) = o(\log n)$. To see this one takes f to be the Poisson kernel, then let r tend to 1 and use the norm estimate for the Dirichlet kernel.

6.5.6 Using Fejer's theorem one shows, by summation by parts, that if $f \in h^1$, then $\sup_n \|P_n f\| < \infty$, where P_n is extended to harmonic functions in the obvious way. Conversely, if f is harmonic in D and $\sup_n \|P_n f\| < \infty$, then $f \in h^1$.

6.6 Quasiconformal harmonic homeomorphisms(§)

Throughout this section we denote by φ a continuous increasing function on $\mathbb{R}$ such that $\varphi(t + 2\pi) - \varphi(t) \equiv 2\pi$, so that the function $\gamma(t) = e^{i\varphi(t)}$ is 2π-periodic and continuous, and of bounded variation on $[0, 2\pi]$. We consider the harmonic mapping f defined on $\mathbb{D} = \{z : |z| < 1\}$ by

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(r, \theta - t)\gamma(t)\, dt \qquad (z = re^{i\theta}). \tag{6.16}$$

By Choquet's theorem (Theorem 6.1.9), f is a homeomorphism of $\overline{\mathbb{D}}$ onto $\overline{\mathbb{D}}$. Conversely, every orientation-preserving homeomorhism $f : \overline{\mathbb{D}} \mapsto \overline{\mathbb{D}}$, harmonic in $\mathbb{D}$, can be represented in the form (6.16).(¶) A consequence of Choquet's theorem and a result of Lewy [48] is that the Jacobian of f is *strictly* positive in $\mathbb{D}$, i.e.,

$$J_f(z) = |\partial f(z)|^2 - |\bar{\partial} f(z)|^2 > 0 \quad (z \in \mathbb{D}). \tag{6.17}$$

Being harmonic, the mapping f can be represented as $f(z) = h(z) + \overline{g(z)}$, $g(0) = 0$, where h and g are analytic in $\mathbb{D}$ and uniquely determined by f. We can rewrite (6.17) as

$$\left|\frac{g'(z)}{h'(z)}\right| < 1 \quad (z \in \mathbb{D}). \tag{6.18}$$

Characterization theorem

We characterize those φ for which f is quasiconformal, i.e., for which (6.18) can be improved to

$$k = \sup_{z \in \mathbb{D}} \left|\frac{g'(z)}{h'(z)}\right| < 1. \tag{6.19}$$

(§)This section is almost identical to the paper [80].

(¶)Various properties of harmonic homeomorphisms are described in [14].

6.6.1 Theorem *The mapping f is quasiconformal iff the function φ is bi-Lipschitz and the Hilbert transformation of φ' is essentially bounded on $\mathbb{R}$. In other words, f is quasiconformal iff φ is absolutely continuous and satisfies the conditions:*

$$\operatorname{ess\,inf} \varphi' > 0, \tag{6.20}$$

$$\operatorname{ess\,sup} \varphi' < \infty, \tag{6.21}$$

$$\operatorname*{ess\,sup}_{\theta\in\mathbb{R}} \left| \int_{+0}^{\pi} \frac{\varphi'(\theta+t) - \varphi'(\theta-t)}{t}\, dt \right| < \infty. \tag{6.22}$$

The proof that the three conditions are sufficient is short; we simply compute the radial limits of the modulus of the *bounded* analytic function g'/h' and apply the maximum modulus principle.

The necessity proof is more complicated and depends on Mori's theorem in theory of quasiconformal mappings (cf. Ahlfors [2]), which states that if Φ is a quasiconformal homeomorphism of $\mathbb{D}$, then

$$|\Phi(z_1) - \Phi(z_2)| \leqslant C|z_1 - z_2|^{\alpha} \qquad (z_1,\, z_2 \in \mathbb{D}), \tag{6.23}$$

where

$$\alpha = \frac{1-k}{1+k}, \quad k = \sup_{z\in\mathbb{D}} \left| \frac{\bar\partial\Phi(z)}{\partial\Phi(z)} \right|$$

and C depends only on $f(0)$[(II)]. The mapping $|z|^{\alpha}(z/|z|)$ shows that the exponent α cannot be improved in the class of arbitrary k-quasiconformal homeomorphisms. However, it follows from our proof (see (6.32)) that if Φ is harmonic, then it satisfies the ordinary Lipschitz condition (with Lipschitz constant depending on k). Combining this with Heinz' inequality [29] $|h'(z)|^2 + |g'(z)|^2 \geqslant 1/\pi^2$, $z \in \mathbb{D}$, which holds if $f(0) = 0$, we get the following.

6.6.2 Theorem *If the mapping f is quasiconformal, then it is bi-Lipschitz, i.e., there is a constant $L < \infty$ such that*

$$\frac{1}{L} \leqslant \left| \frac{f(z_1) - f(z_2)}{z_1 - z_2} \right| \leqslant L \quad (z_1,\, z_2 \in \mathbb{D}).$$

Note that an arbitrary bi-Lipschitz homeomorphism is quasiconformal.

Boundary values of the derivatives

In calculating the boundary values of the analytic functions h' and g' it is useful to use the formulas

$$h'(z) = \partial f(z) = \frac{1}{2} e^{-i\theta} \Big(f_r(z) - i \frac{f_\theta(z)}{r} \Big) \tag{6.24}$$

$$\overline{g'(z)} = \bar\partial f(z) = \frac{1}{2} e^{i\theta} \Big(f_r(z) + i \frac{f_\theta(z)}{r} \Big), \tag{6.25}$$

(II) $C = 16$ if $\Phi(0) = 0$.

where $f_\theta = \partial f/\partial\theta$, $f_r = \partial f/\partial r$. The derivatives f_r and f_θ are connected by the simple but fundamental fact that *the function rf_r is equal to the harmonic conjugate of f_θ*. It follows from (6.16) that f_θ equals the Poisson–Stieltjes integral of $\gamma = e^{i\varphi}$:

$$f_\theta(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta-t)\,d\gamma(t).$$

Hence, by Fatou's theorem, the radial limits of f_θ exist almost everywhere and $\lim_{r\to 1^-} f_\theta(re^{i\theta}) = \gamma_0'(\theta)$ a.e., where γ_0 is the absolutely continuous part of γ. It turns out that if γ is absolutely continuous, then $\lim_{r\to 1^-} f_r(re^{i\theta}) = H(\gamma')(\theta)$, a.e.

Absolute continuity The function γ, of course, need not be absolutely continuous. However: *If*

$$\sup_{\rho<1}\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f_r(\rho e^{i\theta})\right|d\theta < \infty,$$

then γ is absolutely continuous and, moreover, the functions $h(e^{i\theta})$ and $g(e^{i\theta})$ are absolutely continuous.

This is one of possible formulations of Theorem 5.2.5.

Using (6.24) and (6.25) we can easily show that (6.19) implies

$$\frac{1-k}{1+k} \leqslant \left|\frac{rf_r(z)}{f_\theta(z)}\right| \leqslant \frac{1+k}{1-k} \quad (z\in\mathbb{D}).$$

Thus: *If f is quasiconformal, then φ is absolutely continuous.*

From now on we suppose that φ is absolutely continuous. Then there hold the formulas $f_\theta(e^{i\theta}) = \gamma'(\theta) = i\varphi'(\theta)e^{i\varphi(\theta)}$ and, by Theorem 6.1.6,

$$f_r(e^{i\theta}) = H(\gamma')(\theta) = -\frac{1}{\pi}\int_{+0}^{\pi}\frac{\gamma(\theta+t)+\gamma(\theta-t)-2\gamma(\theta)}{4\sin^2(t/2)}\,dt.$$

By straightforward computation we find that $e^{-i\varphi(\theta)}f_r(e^{i\theta}) = A(\theta)+iB(\theta)$, where

$$\begin{aligned}
A(\theta) &= \frac{1}{\pi}\int_{+0}^{\pi}\frac{2-\cos\left(\varphi(\theta+t)-\varphi(\theta)\right)-\cos\left(\varphi(\theta-t)-\varphi(\theta)\right)}{4\sin^2(t/2)}\,dt\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{\sin\left(\varphi(\theta+t)/2-\varphi(\theta)/2\right)}{\sin(t/2)}\right)^2 dt,\\
B(\theta) &= -\frac{1}{\pi}\int_{+0}^{\pi}\frac{\sin\left(\varphi(\theta+t)-\varphi(\theta)\right)+\sin\left(\varphi(\theta-t)-\varphi(\theta)\right)}{4\sin^2(t/2)}\,dt.
\end{aligned}$$

Then using (6.24) and (6.25) we get

$$\begin{aligned}
|h'(e^{i\theta})|^2 &= \frac{1}{4}\left((A(\theta)+\varphi'(\theta))^2+B(\theta)^2\right), \qquad (6.26)\\
|g'(e^{i\theta})|^2 &= \frac{1}{4}\left((A(\theta)-\varphi'(\theta))^2+B(\theta)^2\right).
\end{aligned}$$

Since the function g'/h' is analytic and bounded, by (6.18), we see that

$$k^2 = \sup_{z\in\mathbf{D}} \left|\frac{g'(z)}{h'(z)}\right|^2 = \operatorname{ess\,sup}_{\theta} \frac{\varphi'(\theta)^2 + A(\theta)^2 + B(\theta)^2 - 2\varphi'(\theta)A(\theta)}{\varphi'(\theta)^2 + A(\theta)^2 + B(\theta)^2 + 2\varphi'(\theta)A(\theta)}.$$

Hence:

The mapping f is quasiconformal iff

$$K := \operatorname{ess\,sup}_{\theta\in\mathbf{R}} \frac{\varphi'(\theta)^2 + A(\theta)^2 + B(\theta)^2}{2\varphi'(\theta)A(\theta)} < \infty. \tag{6.27}$$

There holds the formula $k = \left(\frac{K-1}{K+1}\right)^{1/2}$.

Proof of the characterization theorem

Now it is easy to show that conditions (6.20), (6.21) and (6.22) imply that f is quasiconformal. We have only to note that condition (6.21) implies

$$\|B - H\varphi'\|_\infty \leqslant C\|\varphi'\|_\infty^2, \tag{6.28}$$

where C is an absolute constant; this inequality is deduced from Theorem 6.1.6 by using the relation $x - \sin x = O(x^3)$.

The necessity proof. Let f be quasiconformal. Then $K < \infty$ (see (6.27)), i.e.,

$$\varphi'(\theta)^2 + A(\theta)^2 + B(\theta)^2 \leqslant 2K\varphi'(\theta)A(\theta). \tag{6.29}$$

It follows that $A(\theta)^2 \leqslant 2K\varphi'(\theta)A(\theta)$ and therefore

$$\varphi'(\theta) \geqslant \frac{1}{2K}A(\theta). \tag{6.30}$$

Since

$$\begin{aligned} A(\theta) &\geqslant \frac{1}{4\pi}\int_{-\pi}^{\pi} (1 - \cos(\varphi(\theta+t) - \varphi(\theta)))\,dt \\ &= \frac{1}{2}\left(1 - \operatorname{Re}\left(e^{-i\varphi(\theta)}f(0)\right)\right) \geqslant \frac{1}{2}(1 - |f(0)|), \end{aligned}$$

we get $\operatorname{ess\,inf}\varphi'(\theta) > 0$. Thus condition (6.20) is satisfied.

In order to verify (6.21) we use the inequality

$$\varphi'(\theta) \leqslant C\int_{-\pi}^{\pi} \left(\frac{\varphi(\theta+t) - \varphi(\theta)}{t}\right)^2 dt \tag{6.31}$$

(C is an absolute constant) which is obtained from (6.29). Assume first that φ is of class C^2 and choose θ so that $\varphi'(\theta) = \max\varphi' =: M$. Let $0 < \beta < 1$. It follows from (6.31) that

$$M \leqslant C\int_{-\pi}^{\pi} \left(\frac{\varphi(\theta+t) - \varphi(\theta)}{t}\right)^{2-\beta} M^\beta\,dt,$$

whence

$$M^{1-\beta} \leqslant C \int_{-\pi}^{\pi} \left(\frac{\varphi(\theta+t)-\varphi(\theta)}{t} \right)^{2-\beta} dt.$$

Now we apply Mori's inequality (6.23) to deduce that

$$M^{1-\beta} \leqslant C_1 \int_0^{\pi} \left(t^{\alpha-1}\right)^{2-\beta} dt, \quad \alpha = \frac{1-k}{1+k}.$$

Choose β so that $(\alpha-1)(2-\beta) > -1$, which is possible because $(\alpha-1)(2-\beta) \to \alpha - 1 > -1$ as $\beta \to 1^-$, to get $\max \varphi' \leqslant C_2$, where C_2 depends only on K. From this and (6.30) we get $A(\theta) \leqslant 2KC_2$ and hence, by (6.29) and (6.26), $|h'(e^{i\theta})| \leqslant C_3$. The function $h'(z)$ is continuous on the closed disk because the function $\gamma = e^{i\varphi}$ is C^2, so we have

$$|h'(z)| \leqslant C_3 \quad (z \in \mathbb{D}), \tag{6.32}$$

and the constant C_3 depends only on K.

In the general case consider the mappings f_n, of $\mathbb{D}$ onto $\mathbb{D}$, defined by

$$f_n(z) = f(w_n(z))/r_n = h_n(z) + \overline{g_n(z)} \quad (r_n = 1 - 1/n,\ n \geqslant 2),$$

where w_n is the conformal mapping of $\mathbb{D}$ onto $G_n = f^{-1}(r_n\mathbb{D})$, $w_n(0) = 0$, $w_n'(0) > 0$. Since the boundary of G_n is an analytic Jordan curve, the mapping w_n can be continued analytically across $\partial\mathbb{D}$, which implies that f_n has a harmonic extension across $\partial\mathbb{D}$. Since also

$$\left|\frac{g_n'}{h_n'}\right| = \left|\frac{(g' \circ w_n)w_n'}{(h' \circ w_n)w_n'}\right| \leqslant k,$$

we can appeal to the preceding special case to conclude that $|h'(w_n(z))|\,|w_n'(z)|/r_n \leqslant C_3$, where C_3 is independent of n and z. And since $G_n \subset G_{n+1}$ and $\cup G_n = \mathbb{D}$, we can apply Carathéodory's convergence theorem (Theorem 6.6.3 below): $w_n(z)$ tends to z, uniformly on compacts, whence $w_n'(z) \to 1$ $(n \to \infty)$. Thus inequality (6.32) holds in the general case. Using this and (6.26) we get $\varphi'(\theta)+|B(\theta)| \leqslant C_4$. Finally, it remains to apply (6.28).

The Carathéodory convergence theorem

Here we prove the simplest variant of Carathéodory's theorem; for the general form see [23, Ch. II§5] as well as [19].

6.6.3 Theorem *Let $f_n \colon \mathbb{D} \mapsto \mathbb{D}$ be a sequence of univalent functions such that $f_n(0) = 0$, $f_n'(0) > 0$, $f_n(\mathbb{D}) \subset f_{n+1}(\mathbb{D})$ for every n, and $\bigcup f_n(\mathbb{D}) = \mathbb{D}$. Then $f_n(z) \to z$ uniformly on compact subsets.*

Proof. The set $\{f_n\}$ is relatively compact in $H(\mathbb{D})$ and hence it is enough to prove that every $H(\mathbb{D})$-convergent subsequence of $\{f_n\}$ converges to the function $\varphi(z) = z$. Therefore we can assume that f_n tends, uniformly on compact subsets, to some function $f \in H(\mathbb{D})$. Clearly $f(\mathbb{D}) \subset \mathbb{D}$ and $f_n'(0) \to f'(0)$. Let $D_\rho =$

$\{z\colon |z| \leqslant \rho\}$, $\rho < 1$. Since $\cup f_n(\mathbb{D}) = \mathbb{D}$ and D_ρ is compact, we see that $f_n(\mathbb{D})$ contains D_ρ for $n > n_0$, where n_0 is large enough. The function $g(w) = f_n^{-1}(\rho w)$ maps $\mathbb{D}$ into $\mathbb{D}$ and $g(0) = 0$ and hence $g'(0) = \rho/f_n'(0) \leqslant 1$, for $n > n_0$. It follows that $f_n'(0) \to 1$ and hence $f'(0) = 1$. Hence $f(z) = z$, by Schwarz' lemma, and the proof is complete. □

A problem

Let $QCH = \{f_*\colon f$ is a q.c. harmonic homeomorhism of $\mathbb{D}\}$. Is QCH a group with respect to composition? The set of all quasiconformal harmonic homeomorphisms of $\mathbb{D}$ is not a group because the composition of two harmonic functions need not be harmonic. On the other hand, the set of all quasiconformal transformations of $\mathbb{D}$ is a group (cf. [2]).

Miscellaneous

6.6.4 [56] The mapping $f = P[e^{i\varphi}]$ is quasiconformal if $\varphi \in C^1(\mathbb{R})$, $\min \varphi' > 0$ and

$$\int_0^\pi \frac{\omega(t)}{t}\, dt < \infty, \tag{6.33}$$

where $\omega(t) = \sup\{\, |\varphi'(x) - \varphi'(y)| \,:\, |x-y| < t\}$ is the modulus of continuity of φ'. Condition (6.33), known as Dini's condition (applied to φ'), is sufficient but not necessary for the Hilbert transformation of φ' to belong to L^∞.

6.6.5 Let $G = \varphi(\mathbb{D})$, where $\varphi\colon \mathbb{D} \mapsto \mathbb{C}$ is a univalent function such that $\varphi'(0) > 0$. Let $f_n\colon \mathbb{D} \mapsto G$ be a sequence of univalent functions such that $f_n(0) = \varphi(0)$, $f_n'(0) > 0$, $f_n(\mathbb{D}) \subset f_{n+1}(\mathbb{D})$ for every n, and $\cup f_n(\mathbb{D}) = G$. Then $f_n(z) \to \varphi(z)$ uniformly on compact subsets.

6.6.6 Martio [56] proved that condition (6.21) implies that $J_f(z)$ is bounded above on $\mathbb{D}$. The converse is true because $J_f(z) = \operatorname{Im}(\,\overline{f_r(z)} f_\theta(z)\,)$, whence $J_f(e^{i\theta}) = \varphi'(\theta)A(\theta)$. Therefore f satisfies a Lipschitz condition on the boundary iff $J_f(z) \leqslant C$ $(z \in \mathbb{D})$, where C is a constant, or, equivalently, $|f(E)| \leqslant C|E|$ for any measurable set $E \subset \mathbb{D}$. Clearly, this does not imply a Lipschitz condition on $\mathbb{D}$.

The mapping f satisfies a Lipschitz condition on $\mathbb{D}$ iff both φ' and $H\varphi'$ are bounded.

7 Maximal functions, interpolation, coefficients

This chapter concerns two fundamental facts:

(1) (*maximal theorem*) The operator M_{rad} defined on $h(\mathbb{D})$ by

$$(M_{\mathrm{rad}}u)(e^{i\theta}) = \sup_{0<r<1} |u(re^{i\theta})|$$

maps h^p to $L^p(\mathbb{T})$ for $p > 1$, and H^p to $L^p(\mathbb{T})$ for $p > 0$.

(2) (*maximal characterization*) Let $p \in (0, \infty)$. An analytic function f belongs to H^p iff $M_{\mathrm{rad}}(\mathrm{Re}\, f)$ belongs to $L^p(\mathbb{T})$.

In Section 7.3 we state a theorem of Hardy and Littlewood on (C, α)-means in H^p and prove an analogous result concerning "smooth" Cesáro means. A Marcinkiewicz theorem for H^p, $0 < p < \infty$, with applications to Taylor coefficients, is in Section 7.4. In Section 7.5 we consider some extensions of Hardy and Littlewood's theorem on Taylor coefficients. Section 7.6 contains some remarks on the dual of H^1.

7.1 Maximal theorems

Radial maximal function

Let u be a complex-valued function defined on $\mathbb{D}$. The radial maximal function of u is the function $M_{\mathrm{rad}}u$ defined on $\mathbb{T}$ by

$$(M_{\mathrm{rad}}u)(\zeta) = \sup_{0<r<1} |u(r\zeta)| \qquad (\zeta \in \mathbb{T}).$$

In other words, $M_{\mathrm{rad}}u$ is the smallest dominant of the family $u_r(\zeta) = u(r\zeta)$.

7.1.1 Proposition *If $\phi \in L^1(\mathbb{T})$ and $f = P[\phi]$, then $M_{\mathrm{rad}}f(\zeta) \leqslant \mathcal{M}\phi(\zeta)$, $\zeta \in \mathbb{T}$, where $\mathcal{M}$ is the maximal operator of Hardy and Littlewood defined by* (2.12) *and* (2.13).

Proof. Let $\zeta = 1$, $\gamma(t) = \int_0^t \phi(e^{ix})\, dx$ and

$$F(t) = \int_0^t \{\phi(e^{ix}) + \phi(e^{-ix})\}\, dx = \int_{-t}^t \phi(e^{ix})\, dx = \gamma(t) - \gamma(-t).$$

From (3.11) ($\theta = 0$), substituting $-t$ for t, we get

$$f(r) = \frac{1}{2\pi} P(r,\pi)F(\pi) - \frac{1}{2\pi}\int_0^\pi P'(r,t)F(t)\,dt,$$

where $P'(r,t) = \partial P/\partial t$. It follows that

$$|f(r)| \leqslant \frac{1-r}{1+r} M\phi(0) + \frac{1}{\pi} M\phi(0)\int_0^\pi |t\,P'(t)|\,dt.$$

Now the result follows from $\int_0^\pi |t\,P'(t)|\,dt = 2r/(1+r)$. □

Radial maximal theorems

The following theorem, due to Hardy and Littlewood, is an immediate consequence of Theorem 2.3.1 and Proposition 7.1.1. This theorem provides additional information on the convergence in (3.17): if $p > 1$, the convergence is dominated.

7.1.2 Theorem (radial maximal) *The operator M_{rad} maps h^1 (resp. h^p, $p > 1$) into $L^{1,\infty}(\mathbb{T})$ (resp. $L^p(\mathbb{T})$), and is continuous.*

It is worthwhile to note that this theorem can be proved without appealing to the main maximal theorem. Namely, we can use Nikishin's theorem, Banach's principle, and Fatou's theorem, but Proposition 7.1.1 is of independent interest.

Using the properties of *sub*harmonic functions, we can easily deduce the "subharmonic" version of Theorem 7.1.2.

7.1.3 Theorem (subharmonic maximal) *Let u ba a nonnegative, continuous, subharmonic function on $\mathbb{D}$, and let*

$$\sup_{0<r<1}\int_{-\pi}^{\pi} u(re^{i\theta})^p\,d\theta =: A_p < \infty, \quad \text{where } p > 1.$$

Then $M_{\mathrm{rad}}u \in L^p(\mathbb{T})$ and we have $\|M_{\mathrm{rad}}u\|_p \leqslant C_p(A_p)^{1/p}$, where C_p is a constant depending only on p.

In the case of analytic functions, Theorem 7.1.2 extends to all $p > 0$; it is enough to take $u = |f|^{p/2}$ and apply Theorem 7.1.3.

7.1.4 Theorem (complex maximal) *For $f \in H^p$, $p > 0$, we have*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left(M_{\mathrm{rad}} f(e^{i\theta})\right)^p d\theta \leqslant C_p\|f\|_p^p.$$

7.1.5 Remark We can use the complex maximal theorem to deduce (5.7) from (5.10). Namely, from the inequality $\log|f(\rho e^{i\theta})| \leqslant \{M_{\mathrm{rad}} f(e^{i\theta})\}^p/p$, and the complex maximal theorem it follows that $\{M_{\mathrm{rad}} f\}^p$ is an integrable majorant of the family $\log|f_\rho|$ and therefore we may apply Fatou's lemma to inequality (5.10).

A further example:

7.1.6 Proposition *If $f \in H(\mathbb{D})$ and $0 < p < 1$, then*

$$M_p^p(\rho, f) - M_p^p(r, f) \leqslant C_p(\rho - r)^p M_p^p(\rho, f') \qquad (0 < r < \rho < 1).$$

Proof. From the inequality

$$|f(\rho e^{i\theta})| - |f(re^{i\theta})| \leqslant (\rho - r)\sup\{|f'(se^{i\theta})| \,:\, r < s < \rho\},$$

it follows that

$$|f(\rho e^{i\theta})|^p - |f(re^{i\theta})|^p \leqslant (\rho - r)^p \sup\{|f'(se^{i\theta})|^p \,:\, r < s < \rho\}.$$

Then the desired inequality is obtained by integration in θ and using the maximal theorem. □

Nontangential maximal function

Theorems 7.1.2, 7.1.3, and 7.1.4 remain true if we replace the radial maximal function by the nontangential maximal function $M_* u$,

$$M_* u(\zeta) = \sup_{z \in U_\zeta} |u(z)| \qquad (\zeta \in \mathbb{T}),$$

where $r_0 < 1$ is fixed and $U_\zeta =$ the convex hull of $\{|z| \leqslant r_0\} \cup \{\zeta\}$. These variants are proved by means of the following.

7.1.7 Proposition *If $\phi \in L^1(\mathbb{T})$ and $f = P[\phi]$, then $M_* f(\zeta) \leqslant C\,\mathcal{M}\phi(\zeta)$, $\zeta \in \mathbb{T}$, where C depends only on r_0.*

The proof is very similar to that of Proposition 7.1.1 and we omit it; see the proof of Theorem 3.3.7.

Although the pointwise estimate $M_* u(e^{i\theta}) \leqslant \text{const}\, M_{\text{rad}} u(e^{i\theta})$ is not valid, we still have:

7.1.8 Theorem (Fefferman/Stein [21]) *If $0 < p < \infty$ and $u \geqslant 0$ is subharmonic in $\mathbb{D}$, then*

$$\int_0^{2\pi} \{M_* u(e^{i\theta})\}^p \, d\theta \leqslant C_p \int_0^{2\pi} \{M_{\text{rad}} u(e^{i\theta})\}^p \, d\theta.$$

Proof. The proof is based on the inequality

$$u(z)^q \leqslant \frac{C_q}{\delta^2} \int_{|w-z|<\delta} u(w)^q \, dm(w),$$

due to Hardy and Littlewood; see Theorem 9.1.1. For $z = re^{i\theta} \in \mathbb{D}$ let

$$\Delta_z = \{\rho e^{it} \colon |\rho - r| < \delta,\, |t - \theta| < \delta\}, \quad \delta = 1 - r.$$

Let $z \in U_\zeta$, $\zeta = e^{i\eta} \in \mathbb{T}$, and $0 < q < p$. It is easy to deduce from the preceding inequality that

$$u(z)^q \leqslant \frac{C_q}{\delta^2} \int_{\Delta_z} u(\rho e^{it})^q \, dm(\rho e^{it}) = \frac{C_q}{\delta^2} \int_{\theta-\delta}^{\theta+\delta} \int_{r-\delta}^{r+\delta} u(\rho e^{it})^q \, \rho d\rho \, dt.$$

Hence

$$u(z)^q \leqslant \frac{2C_q}{\delta} \int_{\theta-\delta}^{\theta+\delta} g(t)\, dt, \quad \text{where} \quad g(t) = \{M_{\text{rad}} u(e^{it})\}^q.$$

Since $z = re^{i\theta} \in U_\zeta$, we have $|\theta - \eta| \leqslant \kappa(1-r)$, $\kappa = \text{const.}$, which implies that

$$(\theta - \delta, \theta + \delta) \subset (\eta - \varepsilon, \eta + \varepsilon), \quad \varepsilon = (1+\kappa)\delta.$$

It turns out that

$$\{M_* u(e^{i\eta})\}^q = \sup_{z \in U_\zeta} u(z)^q \leqslant C_q' \mathcal{M} g(\eta),$$

whence

$$\{M_* u(e^{i\eta})\}^p \leqslant C_q'' \{\mathcal{M} g(\eta)\}^{p/q}.$$

Integrating this inequality from $\eta = 0$ to 2π, and using Theorem 2.3.1 we get the desired result. □

Kolmogorov's theorem

It follows from the Theorem 4.4.2 and the complex maximal theorem that if $f \in H(\mathbb{D})$ and $\operatorname{Re} f \in h^1$, then

$$\|M_{\text{rad}} f\|_p \leqslant C_p \|\operatorname{Re} f\|_1, \quad \text{for every } p \in (0,1).$$

This is improved by the following result of Kolmogorov.

7.1.9 Theorem *If $f \in H(\mathbb{D})$ and $\operatorname{Re} f \in h^1$, then $M_{\text{rad}} f \in L^{1,\infty}$, and there is a constant C such that*

$$\|M_{\text{rad}} f\|_{1,\infty} \leqslant C \|\operatorname{Re} f\|_1.$$

This can also be stated as follows.

7.1.10 Theorem *The operator $u \mapsto M_{\text{rad}} \widetilde{u}$ acts as a continuous operator from h^1 into $L^{1,\infty}(\mathbb{T})$.*

This theorem is easily deduced from Corollary 3.3.9, Banach's principle 2.7.1, and the Nikishin/Stein theorem; details are left to the reader.

Another form of Kolmogorov's theorem concerns the **maximal Hilbert operator.**

7.1.11 Theorem *Let*

$$H_{\max}\phi(e^{i\theta}) = \sup_{0<\varepsilon<\pi} |H_\varepsilon\phi(e^{i\theta})|,$$

where

$$H_\varepsilon\phi(e^{i\theta}) = \frac{1}{\pi}\int_\varepsilon^\pi \frac{\phi(\theta-t)-\phi(\theta+t)}{2\tan(t/2)}\,dt.$$

Then $H_{\max}$ is of weak type $(1,1)$.

7.1.12 Remark The maximal Hilbert operator maps L^p to L^p, $1<p<\infty$. See Garnett [22, Ch. 3, Ex. 11].

7.2 Maximal characterization of H^p

The following theorem, due to Burkholder, Gundy and Silverstein [11] enables us to treat H^p as a space of harmonic functions and can be used to extend H^p-theory to several real variables (cf. [21]).

7.2.1 Theorem *Let $0<p<\infty$. A function $f=u+iv\in H(\mathbb{D})$ belongs to H^p if and only if the function $M_{\mathrm{rad}}u$ belongs to $L^p(\mathbb{T})$; there holds the inequality*

$$(1/C_p)\|f\|_{H^p} \leqslant \|M_{\mathrm{rad}}u\|_{L^p} + |v(0)|^p \leqslant C_p\|f\|_{H^p}. \tag{7.1}$$

Before proving the theorem some remarks are in order. The right-hand side inequality is a consequence of the complex maximal theorem. In the case $p>1$, the left-hand side inequality is a formal consequence of the radial maximal theorem and Riesz' inequality $(1/C_p)\|f\|_{H^p}\leqslant\|u\|_{L^p}+|v(0)|^p$ (Theorem 6.2.6). On the other hand, if $1<p<2$, then Riesz' inequality follows from (7.1) and the radial maximal theorem; the case $p\geqslant 2$ can then be discussed in various ways (for example, as in the proof of Theorem 6.2.6).

Proof. By the preceding remarks, it suffices to prove the inequality

$$\int_0^{2\pi} |v(e^{i\theta})|^p\,d\theta \leqslant C_p\int_0^{2\pi}(u^+(\theta))^p\,d\theta, \quad 0<p<2,$$

where $u^+(\theta)=M_{\mathrm{rad}}u(e^{i\theta})$, supposing that f is a polynomial and $f(0)=0$. By Theorem 7.1.8, we can replace u^+ by u^*, $u^*(\theta)=\sup_{\zeta\in A}|u(\zeta e^{i\theta})|$, where

$$A = \text{convex hull of } \{1\}\cup\{\,\zeta\colon |\zeta|\leqslant 1/\sqrt{2}\,\}.$$

So it suffices to prove

$$\int_0^{2\pi} |v(e^{i\theta})|^p\,d\theta \leqslant C_p\int_0^{2\pi}(u^*(\theta))^p\,d\theta, \quad 0<p<2, \tag{7.2}$$

where $u+iv$ is a polynomial.

Let $E_\lambda = |\{\theta \in [0, 2\pi]: u^*(\theta) \leqslant \lambda\}|$, $G_\lambda = \{\theta \in [0, 2\pi]: u^*(\theta) > \lambda\}$, and $m(\lambda) = |G_\lambda|$. Assume we have proved that

$$\int_{E_\lambda} v^2\, d\theta \leqslant \int_{E_\lambda} u^2\, d\theta + 2\lambda^2 m(\lambda), \qquad \lambda > 0. \tag{7.3}$$

Multiplying this by $q\lambda^{-q-1}$, $q = 2 - p > 0$, and then integrating from $\lambda = 0$ to ∞, and using Fubini's theorem, we get

$$\int_0^{2\pi} v^2 (u^*)^{-q}\, d\theta \leqslant \int_0^{2\pi} u^2 (u^*)^{-q}\, d\theta + \frac{2q}{2-q} \int_0^{2\pi} (u^*)^{2-q}\, d\theta$$

Hence

$$\int_0^{2\pi} v^2 (u^*)^{p-2}\, d\theta \leqslant C_p \int_0^{2\pi} (u^*)^p\, d\theta, \qquad p < 2, \tag{7.4}$$

where $C_p = 1 + 2(2-p)/p$. To obtain (7.2) assume that $\int_0^{2\pi} (u^*)^p\, d\theta = 1$. Then, by Jensen's inequality,

$$\begin{aligned}\left\{\int_0^{2\pi} |v|^p (u^*)^{-p} (u^*)^p\, d\theta\right\}^{2/p} &\leqslant \int_0^{2\pi} \{|v|^p (u^*)^{-p}\}^{2/p} (u^*)^p\, d\theta \\ &= \int_0^{2\pi} |v|^2 (u^*)^{p-2}\, d\theta;\end{aligned}$$

now (7.2) follows from (7.4).

Proof of (7.3). We can suppose that $0 < |E_\lambda| < 2\pi$. Let

$$F(z) = \sup_{\zeta \in A} |u(\zeta z)|, \ z \in \mathbb{C}.$$

The set $F_\lambda = \{z \in \mathbb{C} : F(z) \leqslant \lambda\} \cap \overline{\mathbb{D}}$ is nonempty and simply connected because the function $\sup\{F(z), |z|/\lambda\}$ is subharmonic in $\mathbb{C}$. Also, F_λ contains the set $H_\lambda = \bigcup\{e^{i\theta} A: \theta \in E_\lambda\}$. Let $\Gamma_\lambda = (\partial H_\lambda) \setminus E_\lambda$. By Cauchy's integral formula, we have

$$\frac{1}{2\pi i} \int_{\partial H_\lambda} \frac{f(z)^2}{z}\, dz = f(0)^2 = 0,$$

whence

$$-\int_{E_\lambda} f(e^{i\theta})^2\, d\theta = \int_{\Gamma_\lambda} f(e^{i\theta})^2\, (d\theta + i\, dr/r).$$

Taking the real parts we get

$$\int_{E_\lambda} (v^2 - u^2)\, d\theta = \int_{\Gamma_\lambda} (u^2 - v^2)\, d\theta - \int_{\Gamma_\lambda} 2uv\, dr/r.$$

Since $|dr/rd\theta| \leqslant 1$ on Γ_λ and $2|uv| \leqslant u^2 + v^2$, we conclude that

$$\int_{E_\lambda} v^2\, d\theta \leqslant \int_{E_\lambda} u^2\, d\theta + \int_{\Gamma_\lambda} 2u^2\, d\theta$$

Now the desired result follows from the inequality

$$\int_{\Gamma_\lambda} u^2\,d\theta \leqslant \int_{\Gamma_\lambda} \lambda^2\,d\theta = \int_{G_\lambda} \lambda^2\,d\theta = \lambda^2 m(\lambda).$$

This completes the proof of the theorem. □

Remark. The above proof is a slight modification of Koosis' proof (see Garnett [22]). The only difference is in that Koosis uses (7.3) to show that

$$|\{\theta\colon |v(\theta)| > \lambda\}| \leqslant m(\lambda) + \frac{2}{\lambda^2}\int_0^\lambda s\,m(s)\,ds.$$

7.3 "Smooth" Cesàro means

In contrast to the case $1 < p < \infty$, the sequence $\{z^n\}$ is not a Shauder basis in H^p for $p \in (0,1]$. Hardy and Littlewood proved that this sequence is a (C,α) basis in H^p for $\alpha > 1/p - 1$ $(p \leqslant 1)$ (for a proof, see [67]). Instead, we shall construct the "smooth" Cesàro basis, which has an advantage in that it is "universal", i.e., independent of p (Theorem 7.3.4). Before, we state the theorem of Hardy and Littlewood.

The Cesàro means of order $\alpha > -1$ of an analytic function f are defined by

$$\sigma_n^\alpha f(z) = \frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)} \sum_{k=0}^{n} \frac{\Gamma(\alpha+n+1-k)}{\Gamma(n+1-k)} \widehat{f}(k) z^k,$$

where Γ is the Euler gamma function. In particular

$$\sigma_n^1 f(z) = \sum_{k=0}^{n} \Big(1 - \frac{k}{n+1}\Big) \widehat{f}(k) z^k.$$

Define the maximal operator $\sigma_{\max}$ by

$$(\sigma_{\max}^\alpha f)(\zeta) = \sup_n |\sigma_n^\alpha f(\zeta)| \qquad (\zeta \in \mathbb{T}).$$

It should be noted that the nontangential maximal function $M_* f$ is dominated by a constant multiple of $\sigma_{\max}^\alpha f$; in the case $\alpha = 1$, this follows from the inequality

$$|f(\zeta z)| \leqslant \frac{|1-z|^2}{(1-|z|)^2} (\sigma_{\max}^1 f)(\zeta),$$

and this follows from the formula

$$f(\zeta z) = (1-z)^2 \sum_{n=0}^{\infty} (n+1)(\sigma_n^1 f)(\zeta)\, z^n.$$

7.3.1 Theorem *If* $0 < p \leqslant 1$, $\alpha > 1/p - 1$, *and* $f \in H^p$, *then* $\|\sigma_{\max}^\alpha f\|_p \leqslant C_p \|f\|_p$, $\lim_{n\to\infty} \|f - \sigma_n^\alpha f\|_p = 0$, *and* $\lim_{n\to\infty} \sigma_n^\alpha f(e^{i\theta}) = f(e^{i\theta})$ a.e.

In the case $1 < p < \infty$, there is a much deeper result, due to Carleson[12] and Hunt[31], which states that the above relations hold for the partial sums of $f \in h^p$ (see p. 41).

The polynomials W_n

Let ψ be a complex-valued C^∞-function with compact support in $\mathbb{R}$. Define the trigonometric polynomials W_n, $n \geqslant 1$, by

$$W_n(e^{it}) = W_n^\psi(e^{it}) = \sum_{|k|<\infty} \psi\Big(\frac{k}{n}\Big) e^{ikt}.$$

7.3.2 Lemma *For every positive integer N there holds*

$$|W_n(e^{i\theta})| \leqslant C_N \min\{n, |\theta|^{-N} n^{1-N}\} \qquad (|\theta| < \pi),$$

where C_N depends only of N and ψ.

Proof. Suppose that $\operatorname{supp}\psi$ is contained in, say, the interval $[-2, 2]$. Then

$$\begin{aligned}
(1 - e^{i\theta})^N W_n(e^{i\theta}) &= \sum_{k=-\infty}^{\infty} \psi\left(\frac{k}{n}\right) (1 - e^{i\theta})^N e^{ik\theta} \\
&= \sum_{k=-\infty}^{\infty} \psi\left(\frac{k}{n}\right) \sum_{m=0}^{N} \binom{N}{m} (-1)^m e^{i(m+k)\theta} \\
&= \sum_{m=0}^{N} (-1)^m \binom{N}{m} \sum_{k=-\infty}^{\infty} \psi\left(\frac{k}{n}\right) e^{i(m+k)\theta} \\
&= \sum_{m=0}^{N} (-1)^m \binom{N}{m} \sum_{k=-\infty}^{\infty} \psi\left(\frac{k-m}{n}\right) e^{ik\theta} \\
&= \sum_{k=-\infty}^{\infty} \left(\sum_{m=0}^{N} (-1)^m \binom{N}{m} \psi\left(\frac{k-m}{n}\right)\right) e^{ik\theta} \\
&= \sum_{k=-2n}^{N+2n} \left(\sum_{m=0}^{N} (-1)^m \binom{N}{m} \psi\left(\frac{k-m}{n}\right)\right) e^{ik\theta}.
\end{aligned}$$

The inner sum in the last expression, denote it by $S_{k,N}$, is equal to the symmetric difference of order N of the function $\psi(x/n)$. By Lagrange's theorem, for every k there exists $\xi_{m,N}$ such that $S_{k,n} = n^{-N}\psi^{(N)}(\xi_{m,N})$. Since the derivative $\psi^{(N)}$ is bounded, we see that $|W_n(e^{i\theta})| \leqslant C|\theta|^{-N}n^{1-N}$, for $-\pi < \theta < \pi$, with C depending only on N and ψ. On the other hand, from the definition of W_n it follows that $|W_n(e^{i\theta})| \leqslant Kn$, where $K = 2\max|\psi|$. This concludes the proof. □

The operator $W_{\max}$

For a function $f \in H(\mathbb{D})$, we define the maximal function $W_{\max} f$ by

$$(W_{\max} f)(\zeta) = \sup_n |W_n * f(\zeta)| \qquad (\zeta \in \mathbb{T}).$$

7.3.3 Lemma *If $0 < q \leqslant 1$, then*

$$(W_{\max} f)^q \leqslant C_q \mathcal{M}(|f|^q), \qquad f \in H^q. \tag{7.5}$$

Proof. Let $\operatorname{supp}(\psi) \subset [-2, 2]$. Then

$$\begin{aligned} |W_n * f(\zeta)| &\leqslant \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(r\zeta e^{it}) \sum_{k=-2n}^{2n} r^{-k} \psi(k/n) e^{-ikt} \, dt \right| \\ &\leqslant \frac{r^{-2n}}{2\pi} \int_{-\pi}^{\pi} \left| f(r\zeta e^{it}) \right| \left| \sum_{-2n}^{2n} r^{2n-k} \psi(k/n) e^{(2n-k)it} \right| dt. \end{aligned}$$

For fixed ζ, n put

$$g(z) = f(z\zeta) \sum_{k=-2n}^{2n} \psi(k/n)\, z^{2n-k}$$

and rewrite the preceding inequality as $|W_n * f(\zeta)| \leqslant r^{-2n} M_1(r, g)$. The function g is analytic and hence $M_1(r, g) \leqslant (1 - r^2)^{1-1/q} M_q(1, g)$ (see Corollary 5.1.2). Putting $r = 1 - 1/(n+1)$, we get

$$\begin{aligned} |W_n * f(\zeta)|^q &\leqslant C\, n^{1-q} \int_{-\pi}^{\pi} |g(e^{i\theta})|^q \, d\theta \\ &= C\, n^{1-q} \int_{-\pi}^{\pi} |f(\zeta e^{i\theta})|^q \, |W_n(e^{i\theta})|^q \, d\theta. \end{aligned}$$

Hence, by Lemma 7.3.2, we get

$$\begin{aligned} |W_n * f(e^{it})|^q &\leqslant C n^{1-q} \int_{|\theta|<1/n} |f(\theta + t)|^q \, n^q \, d\theta \\ &\quad + C n^{1-q} \int_{1/n<|\theta|<\pi} |f(\theta + t)|^q \, n^{(1-N)q} |\theta|^{-Nq} \, d\theta \\ &= C n \int_{|\theta|<1/n} |f(\theta + t)|^q \, d\theta \\ &\quad + C n^{1-Nq} \int_{1/n}^{\pi} \{ |f(t + \theta)|^q + |f(t - \theta)|^q \} \, \theta^{-Nq} \, d\theta. \end{aligned}$$

The first summand in the latter sum is dominated by $C\, M(|f|^q)(e^{it})$. Write the integral in the second summand in the form

$$J = \int_{1/n}^{\pi} \theta^{-Nq} \, dF(\theta) = F(\theta) \Big|_{1/n}^{\pi} + Nq \int_{1/n}^{\pi} F(\theta) \theta^{-Nq-1} \, d\theta,$$

where

$$F(\theta) = \int_0^\theta \left(|f(t+x)|^q + |f(t-x)|^q\right) dx.$$

From this we obtain

$$\begin{aligned} J &\leqslant F(\pi) + Nq \sup_{0<\theta<\pi} \frac{F(\theta)}{\theta} \int_{1/n}^{\pi} \theta^{-Nq}\, d\theta \\ &\leqslant F(\pi) + \frac{Nq}{Nq-1} \sup_\theta \frac{F(\theta)}{\theta} n^{Nq-1} \\ &\leqslant C n^{Nq-1} \mathcal{M}(|f|^q)(e^{it}), \end{aligned}$$

where we choose N so that $Nq > 1$. This completes the proof. □

The "W-maximal" theorem

From the preceding lemma, by the maximal theorem and the theorem on convergence almost everywhere (Theorem 2.7.2), we get the following.

7.3.4 Theorem *For every $p \in (0, \infty]$ we have*

$$\|W_{\max} f\|_p \leqslant C_p \|f\|_p \qquad (f \in H^p). \tag{7.6}$$

If $\psi(0) = 1$ and $p < \infty$, then

$$\lim_{n\to\infty} \|f - W_n * f\|_p = 0, \tag{7.7}$$

$$\lim_{n\to\infty} W_n * f(e^{i\theta}) = f(e^{i\theta}) \quad \text{almost everywhere} \tag{7.8}$$

for every $f \in H^p$.

Proof. Let $0 < q < p$ and $s = p/q$. Then $(W_{\max} f)^p \leqslant C\left\{\mathcal{M}(|f|^q)\right\}^s$, because of (7.5), so we get (7.6) from the maximal theorem 2.3.1, applied to L^s. The relation (7.7) holds if f is a polynomial; that (7.7) holds for all $f \in H^p$, follows from $\|W_n * f\|_p \leqslant C\|f\|_p$ and Theorem 1.2.2. Finally, (7.8) is proved by using Theorem 2.7.2. □

7.3.5 Exercise Let $\sigma = s + it$ be a complex number and $p > 0$. Then

$$\left\| \sum_{k=n}^{4n} k^\sigma a_k z^k \right\|_p \asymp n^s \left\| \sum_{k=n}^{4n} a_k z^k \right\|_p.$$

This can be proved by taking $\psi(x) = x^\sigma \varphi(x)$, where φ is a C^∞-function such that $\operatorname{supp}(\varphi) \subset (0, \infty)$ and $\varphi(x) = 1$ for $x \in [1, 4]$.

Assume there exist constants C_1 and C_2, independent of f, such that

$$\|Tf\|_{p,\infty} \leqslant C_1\|f\|_p, \qquad f \in H^p, \tag{7.9}$$
$$\|Tf\|_{q,\infty} \leqslant C_2\|f\|_q, \qquad f \in H^q.$$

Then for every $s \in (p,q)$ there exists a constant C independent of f such that

$$\|Tf\|_s \leqslant C\|f\|_s, \qquad f \in H^s.$$

Observe that the case $q = \infty$ is now excluded. In that case the things lie much deeper, as one can see in [7, Ch. 5].

Proof. The idea is the same as in the case of the classical Marcinkiewicz's theorem. The unique obstacle is in that we cannot use the old decomposition of f because g_λ need not be in $H^p(\mathbb{T})$. Fortunately, we have the decomposition $f = g_\lambda + h_\lambda$, where

$$\|g_\lambda\|_p^p \leqslant A\int_{|f|>\lambda} |f|^p\,d\sigma,$$
$$\|h_\lambda\|_q^q \leqslant A\int_{|f|\leqslant\lambda} |f|^q\,d\sigma + A\lambda^{2q}\int_{|f|>\lambda} |f|^{-q}\,d\sigma,$$

where A=const, and $d\sigma$ is the normalized measure on $\mathbb{T}$ (Lemma 7.4.2 below). Assuming that T is subadditive and $C_1 = C_2 = 1$, we have

$$\begin{aligned}\mu(Tf,\lambda) &\leqslant \mu(Tg_\lambda,\lambda/2) + \mu(Th_\lambda,\lambda/2)\\ &\leqslant A(2/\lambda)^p\int_{|f|>\lambda}|f|^p\,d\sigma + A(2/\lambda)^q\int_{|f|\leqslant\lambda}|f|^q\,d\sigma\\ &\quad + A(2\lambda)^q\int_{|f|>\lambda}|f|^{-q}\,d\sigma = I_1(\lambda)+I_2(\lambda)+I_3(\lambda).\end{aligned}$$

Now we multiply this by $s\lambda^{s-1}$ and integrate these three summands from $\lambda = 0$ to ∞. For instance, we have

$$s\int_0^\infty I_3(\lambda)\lambda^{s-1}\,d\lambda = A2^q\int_{\mathbb{T}}|f|^{-q}\,d\sigma\int_0^{|f|}\lambda^q\lambda^{s-1}\,d\lambda = \frac{A2^q}{q+s}\int_{\mathbb{T}}|f|^{-q}|f|^{q+s}\,d\sigma.$$

In the case of I_1 and I_2 we proceed similarly. □

The following lemma is essentially due to Bourgain [9]. The following presentation is taken from Kislyakov/Xu [43].

7.4.2 Lemma *If $f \in H^p$ $(0 < p < \infty)$ and $\lambda > 0$, then there are functions $h \in H^\infty$ and $g \in H^p$ such that*

$$|h_*| \leqslant C\lambda \min\left(\frac{|f_*|}{\lambda}, \frac{\lambda}{|f_*|}\right) \quad \text{and}$$

$$\|g\|_p^p \leqslant C \int_{\zeta \in \mathbb{T},\ |f_*(\zeta)| > \lambda} |f_*(\zeta)|^p \,|d\zeta|,$$

where C depends only on p.

Proof. Let $\lambda > 0$ and define the functions α on $\mathbb{T}$, and F on $\mathbb{D}$ by

$$\alpha = \max\left(1, \left(\frac{|f_*|}{\lambda}\right)^{p/2}\right) \quad \text{and} \quad F = \frac{1}{P[\alpha] + i\widetilde{P}[\alpha]}.$$

Since $P[\alpha] \geqslant 1$ in $\mathbb{D}$, we have $0 < |F| \leqslant 1$ in $\mathbb{D}$. Therefore the function

$$G = 1 - (1 - F^{4/p})^{2/p}$$

is well defined, analytic and bounded in $\mathbb{D}$. We claim that the functions $h = Gf$ and $g = (1-G)f$ satisfy the desired conditions.

Since $|F_*| \leqslant 1/\alpha$ and $|G| \leqslant C|F|^{4/p}$ (by Schwarz' lemma), we have

$$|h_*| \leqslant C|f_*| \min\left(1, \left(\frac{|f_*|}{\lambda}\right)^{-2}\right),$$

and this gives the desired estimate for h. On the other hand,

$$\begin{aligned} |g_*| &\leqslant C|1 - F_*|^{2/p}\,|f_*| \\ &\leqslant C\left(\frac{\alpha - 1}{\alpha} + \frac{|H(\alpha - 1)|}{\alpha}\right)^{2/p} |f_*| \\ &\leqslant C(1 - 1/\alpha)^{2/p}\,|f_*| + C\lambda\,|H(\alpha - 1)|^{2/p}, \end{aligned}$$

where H is the Hilbert operator. Since $\alpha = 1$ on the set $\{|f_*| < \lambda\}$, and H is bounded on L^2, we see that

$$\begin{aligned} \|g\|_p^p &\leqslant C\int_{\mathbb{T}} (1 - 1/\alpha)^2 |f_*|^p + C\lambda^p \int_{\mathbb{T}} |H(\alpha - 1)|^2 \\ &\leqslant C\int_{|f_*|>\lambda} |f_*|^p + C\lambda^p \int_{\mathbb{T}} (\alpha - 1)^2 \\ &\leqslant C\int_{|f_*|>\lambda} |f_*|^p. \end{aligned}$$

This completes the proof. □

Application to Taylor coefficients

As an example, consider the operator

$$(Tf)(n) = (n+1) \sup_{0\leqslant k\leqslant n} |\widehat{f}(k)|, \qquad f \in H^p,\ n \geqslant 0.$$

By Corollary 5.1.3, for every $p \in (0,1)$ we have

$$(Tf)(n) \leqslant C_p\,(n+1)^{1/p}\|f\|_p, \qquad f \in H^p.$$

Define the measure μ on $\Omega = \{0,1,2,\ldots\}$ by $\mu(\{n\}) = (n+1)^{-2}$. Then, arguing as in the proof of Theorem 2.2.5 we find that T satisfies (7.9) for every $p \in (0,1)$. Hence, by Theorem 7.4.1, T maps H^p into $L^p(\Omega,\mu)$, for every $p \in (0,1)$. Thus we have proved:

7.4.3 Theorem [57] *If $f \in H^p$, $0 < p < 1$, then* $\sum_{n=0}^{\infty} (n+1)^{p-2} \sup_{k\leqslant n} |\widehat{f}(k)|^p < \infty$.

As a corollary we have part (a) of the following theorem of Hardy and Littlewood.

7.4.4 Theorem (a) *If $f \in H^p$, $0 < p < 1$, then $|\widehat{f}(n)| = o(n^{1/p-1})$.*

(b) *Assertion* (a) *is optimal in the following sense: If $|\widehat{f}(n)| = O(\psi_n)$ for every $f \in H^p$, where $\psi_n > 0$, then there is a constant $c > 0$ such that $\psi_n \geqslant cn^{1/p-1}$.*

Proof. To prove (b) assume that $|\widehat{f}(n)| = O(\psi_n)$. Consider the space ℓ^∞_ψ of all scalar sequences $\{a_n\}$ for which $\|\{a_n\}\|_\psi := \sup_n \psi_n^{-1}|a_n| < \infty$. By the hypothesis we have $H^p \subset \ell^\infty_\psi$. To prove that the inclusion operator is continuous assume that (1) $f_j \to f$ in H^p, and (2) $\{\widehat{f_j}(n)\} \to \{a_n\}$ in ℓ^∞_ψ, as $j \to \infty$. It follows from assertion (a) and (1) that $\widehat{f_j}(n) \to \widehat{f}(n)$, $j \to \infty$, for every n. On the other hand, (2) implies that $\widehat{f_j}(n) \to a_n$ for every n. Hence $a_n = \widehat{f}(n)$ for every n, and therefore the closed graph theorem tells us that the inclusion is continuous. In particular, we have $\sup_n \psi_n^{-1}|\widehat{f_r}(n)| \leqslant C_p\|f_r\|_p$, $0 < r < 1$, where $f_r(z) = (1-rz)^{-2/p}$ and C_p depends only on p. Since

$$\widehat{f_r}(n) = \binom{-2/p}{n}(-1)^n r^n \geqslant cn^{2/p-1} \quad \text{and} \quad \|f_r\|_p = (1-r^2)^{-1/p},$$

we conclude that $\psi_n^{-1}n^{2/p-1}r^n \leqslant C(1-r)^{-1/p}$, where C is independent of r and n. Now the desired result is obtained by taking $r = 1 - 1/n$. □

L^q-integrability of $M_p(r,f)$

We have proved that if $f \in H^p$, $p < \infty$, then

$$M_q(r,f) = O\left((1-r)^{1/q-1/p}\right) \quad (r \to 1) \quad \text{for} \quad q > p, \tag{7.10}$$

see (5.3). Then, using the fact that the polynomials are dense in H^p, we can prove that (7.10) remains valid if we replace "O" by "o". This is further improved by inequality (7.11) below because $M_p(r, f)$ increases with r.

7.4.5 Theorem (Hardy/Littlewood) *If $f \in H^p$, $0 < p < \infty$ and $\infty \geqslant q > p$, then*

$$\int_0^1 (1-r)^{-p/q} M_q^p(r, f)\, dr \leqslant C\|f\|_p^p, \tag{7.11}$$

where C depends only on p, q.

From (5.3) and (7.11) we get the following:

7.4.6 Corollary *If $f \in H^p$, $0 < p < \infty$, $\infty \geqslant q > p$ and $s \geqslant p$, then*

$$\int_0^1 (1-r)^{s\alpha-1} M_q^s(r, f)\, dr \leqslant C\|f\|_p^s,$$

where $\alpha = 1/p - 1/q$ (> 0) and C is independent of f.

In the case $s = q$, this can be written in the form:

7.4.7 Corollary *If $f \in H^p$, $0 < p < \infty$ and $q > p$, then*

$$\int_{\mathbb{D}} |f(z)|^q (1-|z|^2)^{q/p-2}\, dA(z) \leqslant C_{p,q}\, \|f\|_p^q.$$

In connection with this corollary see 5.3.5.

Proof of Theorem 7.4.5. We can suppose that $q < \infty$ because

$$M_\infty(r, f) \leqslant (1-r)^{-1/q} M_q(\sqrt{r}, f).$$

Define the (quasilinear) operator $T\colon H^p \mapsto C(0,1)$ by

$$(Tf)(r) = (1-r)^{-1/q} M_q(r, f), \qquad 0 < r < 1.$$

We will prove that T maps H^s into $L^{s,\infty}(0,1)$ for $s = p$ and $s = q$. By Theorem 7.4.1, this will imply that T maps H^s into $L^s(0,1)$ for $p < s < q$; this will conclude the proof.

In the case $s = p$ we use inequality (5.3); we get

$$Tf(r) \leqslant A(1-r)^{-1/p}\|f\|_p, \tag{7.12}$$

where A is a positive constant. If $\|f\|_p = 1$, then it follows from (7.12) that

$$\big|\{r \in (0,1)\colon Tf(r) > \lambda\}\big| \leqslant \big|\{r \in (0,1)\colon A(1-r)^{-1/p} > \lambda\}\big| = \min(1, (A/\lambda)^p),$$

which proves that T is of weak type (p, p). In the case $s = q$ the desired conclusion follows from the inequality $M_q(r, f) \leqslant \|f\|_q$. The proof is complete. □

Remark. For a more elementary proof, see [18].

7.4.8 Exercise If f is harmonic in $\mathbb{D}$, then (7.11) holds for $q > p > 1$.

7.5 On the Hardy/Littlewood inequality

As a consequence of Theorems 7.4.3 and 2.2.4 we have another result of Hardy and Littlewood.

7.5.1 Theorem *If $f \in H^p$, $0 < p < 2$, then*

$$K := \sum_{n=0}^{\infty} (n+1)^{p-2} |\widehat{f}(n)|^p \leqslant C_p \|f\|_p^p. \tag{7.13}$$

If a function $f \in H(\mathbb{D})$ satisfies the condition $K < \infty$, for some $p > 2$, then $f \in H^p$ and $\|f\|_p^p \leqslant C_p K$.

In the case $1 < p < 2$, this theorem is, of course, weaker than Theorem 2.2.4. The following proof contains, however, an improvement of (7.13) in other direction.

Proof. In the case $q = 2 > p$, inequality (7.11) can be written as

$$\int_0^1 (1-r)^{-p/2} \left(\sum_{n=0}^{\infty} |\widehat{f}(n)|^2 r^{2n} \right)^{p/2} dr \leqslant C_p \|f\|_p^p. \tag{7.14}$$

Let $f \in H^p$, $0 < p < 2$. We use (7.14) and Lemma 7.5.4 below to get

$$\sum_{n=0}^{\infty} 2^{-n(1-p)} \left(2^{-n} \sum_{k \in I_n} |\widehat{f}(k)|^2 \right)^{p/2} \leqslant C_p \|f\|_p^p. \tag{7.15}$$

Since $\operatorname{card}(I_n) = 2^n$ for $n \geqslant 1$, and the function $t \mapsto t^{p/2}$, $t > 0$, is concave, we have

$$\left(2^{-n} \sum_{k \in I_n} |\widehat{f}(k)|^2 \right)^{p/2} \geqslant 2^{-n} \sum_{k \in I_n} |\widehat{f}(k)|^p,$$

and this together with the previous relation implies (7.13).

To discuss the case $p > 2$ we consider the space $X_p \subset H(\mathbb{D})$ defined by condition (7.13). It is easily seen that $(X_p)^* = X_q$, $1/p + 1/q = 1$, with respect to the bilinear form

$$(f, g) = \sum_{n=0}^{\infty} \widehat{f}(n) \widehat{g}(n)$$

Since $(H^p)^* = H^q$, $1 < p < \infty$, with respect to the same form, we deduce from $H^p \subset X_p$, $p < 2$, that $X_q \subset H^q$, $q > 2$. The proof is complete. □

Inequality (7.15) can be improved by combining Theorem 7.4.5 with the Hausdorff/Young theorem. For example, in the case $p = 1$ we get:

7.5.2 Theorem [57] *If $f \in H^1$, then*

$$\sum_{n=0}^{\infty} \left(2^{-n} \sum_{k \in I_n} |\widehat{f}(k)|^q \right)^{1/q} < \infty, \qquad 0 < q < \infty. \tag{7.16}$$

Note that the quantity in (7.16) increases with q. In the case $q = 1$, (7.16) reduces to

$$\sum_{n=0}^{\infty} \frac{|\widehat{f}(n)|}{n+1} < \infty.$$

Addendum: Littlewood's conjecture

In connection with the so called Littlewood's conjecture,

$$\int_0^{2\pi} \left| \sum_{k=1}^{n} e^{i\lambda_k\theta} \right| d\theta \geqslant c\log n,$$

the following extension of Theorem 7.5.1 ($p = 1$) was proved in [65].

7.5.3 Theorem *If $\{\lambda_n\}_1^\infty$ is a strictly increasing sequence of positive integers, and $f \in H^1$ is such that $\operatorname{supp}\widehat{f} \subset \{\lambda_n : n \geqslant 1\}$, then $\sum_{n=1}^\infty n^{-1}|\widehat{f}(\lambda_n)| \leqslant C\|f\|_1$, where C is independent of $\{\lambda_n\}$ and f.*

L^p-integrability of power series with positive coefficients

7.5.4 Lemma *Let $\alpha > -1$, $0 < q < \infty$, and $I_n = \{k : 2^n \leqslant k < 2^{n+1}\}$ for $n \geqslant 1$, $I_0 = \{0,1\}$. If $\{a_n\}$ is a sequence of nonnegative numbers such that the series $G(r) = \sum_{n=0}^\infty a_n r^n$ converges for every $r \in (0,1)$, then the following three conditions are equivalent and the corresponding quantities are "proportional":*

$$\int_0^1 (1-r)^\alpha G(r)^q \, dr < \infty,$$

$$\sum_{n=0}^{\infty} 2^{-(\alpha+1)n} \Big(\sum_{k\in I_n} a_k \Big)^q < \infty,$$

$$\sum_{n=0}^{\infty} (n+1)^{-\alpha-2} \Big(\sum_{k=0}^{n} a_k \Big)^q < \infty.$$

In the case of the function $G(r) = \sup_{n\geqslant 0} a_n r^n$ the expressions $\sum_{k\in I_n} a_k$, $\sum_{k=0}^n a_k$ should be replaced by $\sup_{k\in I_n} a_k$, $\sup_{0\leqslant k\leqslant n} a_k$, respectively.

This lemma is an immediate consequence of its special case:

7.5.5 Lemma *Let $\{a_k\}$ be a sequence of nonnegative real numbers such that the series $F(r) = \sum_{k=0}^\infty a_k r^{2^k}$ converges for every $r \in (0,1)$. Let $\alpha > -1$ and $0 < q < \infty$. Then the conditions*

$$A := \int_0^1 (1-r)^\alpha F(r)^q \, dr < \infty \quad \text{and} \quad B := \sum_{k=0}^{\infty} 2^{-(\alpha+1)} a_k^q < \infty$$

are equivalent. There holds the inequality $B/C \leqslant A \leqslant CB$, where C depends only on α, q. The same holds for the function $F(r) = \sup_k a_k r^{2^k}$.

Proof. The proof of the inequality $A \geqslant B/C$ is very simple (see, e.g., the proof of Theorem 11.1.1, p. 165)(*). To prove the reverse inequality, observe first that the case $q \leqslant 1$ is trivial; we integrate the inequality

$$(1-r)^\alpha F(r)^q \leqslant (1-r)^\alpha \sum_{k=0}^{\infty} a_k^q r^{q2^k}.$$

In the case $q > 1$ we can use Jensen's inequality (as in [58]), or proceed as follows.

Write the inequality $A \leqslant CB$ as $\|T(\{a_n\})(r)\|_{L^q(\mu)} \leqslant C_q \|\{a_n\}\|_{\ell^q}$, where

$$T(\{a_n\})(r) = (1-r)^\beta \sum_{n=0}^{\infty} 2^{\beta n} a_n r^{2^n}, \qquad \beta > 0,$$

and $d\mu(r) = dr/(1-r)$. It is easily verified that T maps ℓ^q to $L^q(\mu)$ for $q = 1, \infty$, so we can apply the Riesz/Thorin theorem. □

Miscellaneous

7.5.6 If $\alpha > 0$ and let $\{a_n\}$ be a nonnegative sequence, then the following conditions are equivalent:

$$\sum_{n=0}^{\infty} a_n r^n = O\Big((1-r)^{-\alpha}\Big) \quad (r \to 1^-), \quad \text{and} \quad \sum_{k \in I_n} a_k = O(2^{\alpha n}) \quad (n \to \infty).$$

The equivalence remains true if we replace "O" by "o."

7.5.7 If $f \in H^p$, $1 < p < 2$, and $\{\lambda_n\}_1^\infty$ is a strictly increasing sequence of positive integers, then $\sum_{n=1}^{\infty} n^{p-2} |\widehat{f}(\lambda_n)|^p < \infty$.

7.5.8 [59] Lemma 7.5.4 can be generalized in the following way:

For an analytic function f let $\Delta_n f(z) = \sum_{k \in I_n} \widehat{f}(k) z^k$. Let $1 < p < \infty$, $0 < q < \infty$, $\alpha > 0$. Then

$$\int_0^1 (1-r)^{q\alpha - 1} M_p^q(r, f)\, dr < \infty \quad \textit{if and only if} \quad \sum_{n=0}^{\infty} \left(2^{-\alpha n} \|\Delta_n f\|_p\right)^q < \infty.$$

In the case $q = \infty$ there holds the equivalence

$$M_p(r, f) = O(1-r)^{-\alpha} \iff \|\Delta_n f\|_p = O(2^{\alpha n}).$$

This can be deduced from Lemma 7.5.5 in the following way. We have

$$M_p(r, f) \leqslant \sum_{n=0}^{\infty} M_p(r, \Delta_n f), \quad 1 \leqslant p \leqslant \infty,$$

and, by Riesz' projection theorem, $M_p(r, f) \geqslant c_p \sup_{n \geqslant 0} M_p(r, \Delta_n f)$, $1 < p < \infty$. It remains to apply 4.1.4.

(*)It is, however, difficult to prove that this inequality remains hold if we drop the hypothesis $a_n \geqslant 0$; see Theorems 11.4.1 and 11.4.2.

7.5.9 *Let $1 < p < \infty$, $f \in H(\mathbb{D})$, and let the sequence $\widehat{f}(n)$ be decreasing. Then, $f \in H^p$ if and only if*

$$\sum_{n=0}^{\infty}(n+1)^{p-2}|\widehat{f}(n)|^p < \infty.$$

This can be deduced from 7.5.8 and the Littlewood/Paley theorem.

7.5.10 Assertion 7.5.8 does not hold for $p \in (0,1) \cup \{\infty\}$. Then we can use Theorem 7.3.4 to replace $\Delta_n f$ by $\Gamma_n * f$, where Γ_n is a sequence of polynomials Γ_n, $n \geqslant 0$, satisfying, for all $p > 0$,

$$\begin{aligned} \operatorname{supp}(\widehat{\Gamma}_n) &\subset [2^{n-1}, 2^{n+2}], \quad n \geqslant 1, \\ f &= \sum_{n=0}^{\infty} \Gamma_n * f, \quad f \in H(\mathbb{D}), \\ \|\Gamma_n * f\|_p &\leqslant C_p \|f\|_p, \quad f \in H^p, \\ \|\Gamma_n\|_p &\asymp 2^{n(1-1/p)}. \end{aligned}$$

A more complicated polynomials can be constructed by means of Theorem 7.3.1 (cf. [69]). Such polynomials play a central role in calculating multipliers for various spaces of analytic functions (cf. [33, 70]).

7.6 On the dual of H^1

A function $\phi \in L^1(\mathbb{T})$ is called a function of bounded mean oscillation if

$$\sup_{I \subset \mathbb{T}} \frac{1}{|I|} \int_I |\phi(e^{it}) - \phi_I| \, dt = \|\phi\|_* < \infty, \quad \text{where} \quad \phi_I = \frac{1}{|I|} \int_I \phi(e^{it}) \, dt,$$

and the supremum is taken over all subarcs of $\mathbb{T}$.

The class BMO $= \{\phi \colon \|\phi\|_* < \infty\}$ is normed by $\|\phi\|_{\mathrm{BMO}} = \|\phi\|_{L^1} + \|\phi\|_*$. The intersection of BMO with $H^1(\mathbb{T})$ is denoted by BMOA.

The famous theorem of Fefferman states that BMOA is isomorphic to the dual of $H^1(\mathbb{T})$. This means the following:

A function $\phi \in H^1(\mathbb{T})$ belongs to BMOA *if and only if*

$$\sup_{\|Q\|_1 \leqslant 1} \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(e^{it}) Q(e^{-it}) \, dt \right| = \|\phi\|'_* < \infty, \tag{7.17}$$

where the supremum is taken over the set of all polynomials in H^1. Moreover, the norm $\|\cdot\|'_*$ is equivalent to the original norm in BMOA.

The proof of Fefferman's theorem as well as of various other properties of BMOA can be found in Garnett [22, Ch. VI]. Here we can accept (7.17) as a definition of BMOA. Also, we can treat BMOA as a space of analytic functions on $\mathbb{D}$, via the Poisson integral. Then Theorem 7.5.2 leads to the following.

7.6.1 Theorem [57] *Let $\{a_n\}_0^\infty$ be a sequence of complex numbers satisfying*

$$\sup_{n\geqslant 0} 2^n\Big(2^{-n}\sum_{k\in I_n}|a_k|^q\Big)^{1/q} < \infty, \quad \text{for some } q>1. \tag{7.18}$$

Then the function $f(z)=\sum_0^\infty a_n z^n$ belongs to BMOA.

Note that (7.18) is satisfied if $a_n = O(1/n)$.

7.6.2 Remark It is interesting that condition (7.18) is sufficient for the validity of the implication

$$\lim_{r\to 1^-}\sum_{n=0}^{\infty} a_n r^n \text{ exists} \implies \sum_{n=0}^{\infty} a_n \text{ converges;}$$

see Theorem 11.2.2 and Corollary 11.2.3.

Miscellaneous

7.6.3 The space BMOA is closely related to two important spaces of functions—the space H^∞ and the Bloch space $\mathfrak{B}=\{f\in H(\mathbb{D})\colon \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)|<\infty\}$. It is clear from (7.17) that $\|f\|_*' \leqslant \|f\|_\infty$ so $H^\infty \subset$ BMOA. This inclusion is proper because the function $f(z)=\sum_{n=1}^\infty z^n/(n+1)$ is in BMOA but not in H^∞.

The inclusion BMOA $\subset \mathfrak{B}$ can be deduced from the easily proved relations

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\phi(e^{it})Q(e^{-it})\,dt = f(0)Q(0) + 2\int_{\mathbf{D}} f'(z)Q'(\bar z)\log\frac{1}{|z|}\,dA(z),$$

and

$$\|Q\|_1 \leqslant |Q(0)| + C\int_{\mathbf{D}}|Q'(z)|\,dA(z),$$

where C is an absolute constant. The inclusion is proper. Namely, the function $f(z)=\sum z^{2^n}$ is in $\mathfrak{B}$. On the other hand, since $H^1\subset H^p$ for $1<p<\infty$, and $(H^p)^* = H^{p'}$ (Theorem 6.3.2), we see that BMOA $\subset H^p$ for every $p<\infty$ and therefore f is not in BMOA because f is not in H^2.

7.6.4 Fefferman's duality theorem can be expressed in terms of multipliers, namely:

$$\text{BMOA} = (H^1, H^\infty) := \{g\in H(\mathbb{D})\colon g*f\in H^\infty \text{ for all } f\in H^1\}.$$

It was proved in [61] that $(H^1,\text{BMOA})=\mathfrak{B}$.

7.6.5 If $f\in H^1$ and $g\in$ BMOA, then $f(e^{it})g(e^{-it})$ need not be integrable on $[-\pi,\pi]$. Example:

$$f(z) = z^2\frac{1-z}{1+z}\Big(\log\frac{1-z}{1+z}\Big)^{-2}, \quad g(z)=\log\frac{1-z}{1+z}.$$

8 Bergman spaces: Atomic decomposition

In this chapter we consider the Coifman/Rochberg theorem on the atomic decomposition of Bergman spaces (Theorems 8.3.1, 8.3.4) as well as some applications, due to Kalton [37], to the theory of vector-valued analytic functions (Theorems 8.4.2, 8.4.3). Some technical simplifications in the proof of the Coifman/Rochberg theorem are made. Kalton's results are formulated in a more precise form (Theorems 8.4.3, 8.4.5).

Further topics in the theory of Bergman spaces can be found in the book of Hedenmalm, Korenblum and Zhu [28].

8.1 Bergman spaces

Let $0 < p < \infty$. The Bergman space A^p is the subspace of $L^p(\mathbb{D}, dA)$ consisting of analytic functions. The quasinorm can be written as

$$\|f\|^p = \|f\|^p_{A^p} = 2\int_0^1 I_p(r,f)\, r\,dr, \quad \text{where} \quad I_p(r,f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^p\, d\theta.$$

8.1.1 Proposition *For $p \in (0,\infty)$ there hold the following:*
(a) *For every $z \in \mathbb{D}$ the functional $f \mapsto f(z)$ is continuous on A^p;*
(b) *The space A^p is complete;*
(c) *If $f \in A^p$, and $f_\varrho(z) = f(\varrho z)$, then $f_\varrho \in A^p$ and $\|f - f_\varrho\| \to 0$ $(\varrho \to 1)$;*
(d) *The set of all polynomials is a dense subset A^p.*

Proof. The function $|f|^p$ is subharmonic, which means that

$$|f(z)|^p \leqslant \frac{1}{R^2}\int_{|w-z|<R} |f(w)|^p\, dA(w)$$

whenever $R \leqslant 1 - |z|$. It follows that

$$|f(z)|^p \leqslant \frac{1}{(1-|z|)^2}\int_{\mathbb{D}} |f|^p dA, \tag{8.1}$$

which proves (a).

(b) If $\{f_n\}$ is a Cauchy sequence in $A^p(\mathbb{D})$, then, because of (8.1), it converges uniformly on compact subsets to a function f analytic in $\mathbb{D}$. On the other hand, since L^p is complete, $\{f_n\}$ converges in the L^p metric to some function $g \in L^p$.

And since $f_n \to f$ in $L^p(K)$, where K is any compact subset of D, we have $f = g$ almost everywhere.

(c) Let $f \in A^p$, $0 < p < 1$. Then

$$\|f - f_\varrho\|^p = 2\int_0^1 I_p(r, f - f_\varrho)\, r dr.$$

Since $I_p(r, f - f_\varrho) \leqslant I_p(r, f) + I_p(r, f_\varrho)$ and $I_p(r, f_\varrho) = I_p(r\varrho, f) \leqslant I_p(r, f)$, the function $r \mapsto 2I_p(r, f)$ is an integrable dominant so we can apply the dominated convergence theorem. The case $p \geqslant 1$ is discussed in a similar way.

(d) Let $f \in A^p$, $\varepsilon > 0$ and $0 < \varrho < 1$. The Taylor series of f_ϱ converges uniformly on $\mathbb{D}$, which implies $\|f_\varrho - s_n\| \leqslant \varepsilon$ for n large enough, where s_n are the partial sums of the Taylor series of f_ρ. Now (d) can easily be deduced from (c). □

Exercises

8.1.2 The assertion (c) of Proposition 8.1.1 holds and for an arbitrary function $f \in L^p(\mathbb{D})$.

8.1.3 The inclusion $A^q \subset A^p$ is compact for $q > p$.

Remark. It is known that A^p is isomorphic to ℓ^p (see Theorem 1.5.1). And since every operator from ℓ^q to ℓ^p, $q > p$, is compact (see Theorem 1.5.6), every operator from A^q u A^p, $q > p$, is compact.

8.1.4 There holds the inequality $|f(z)| \leqslant (1 - |z|^2)^{-2/p}\|f\|_{A^p}$. See Lemma 5.1.1.

8.1.5 For every $p > 0$ and every polynomial $f(z) = \sum_{k=m}^{n} a_k z^k$ there holds the inequality

$$\frac{1}{n+1}\|f\|_{H^p} \leqslant \|f\|_{A^p} \leqslant \frac{1}{m+1}\|f\|_{H^p}.$$

8.2 Reproductive kernels

Let

$$K_p(w, z) = \frac{1}{(1 - \bar{w}z)^{2/p+1}} \qquad (z, w \in \mathbb{D}).$$

Let $d\mu_p$ denote the measure defined over $\mathbb{D}$ by $d\mu_p(w) = c_p(1 - |w|^2)^{2/p-1} dA(w)$, where

$$\frac{1}{c_p} = \int_{\mathbb{D}} (1 - |w|^2)^{2/p-1} dA(w) = \frac{p}{2}.$$

The following theorem shows that da K_p has the reproducing property.

8.2.1 Theorem *Let $0 < p \leqslant 1$ and $f \in A^p$. Then for every $z \in \mathbb{D}$ the function $w \mapsto f(w)K_p(w,z)$ belongs to $L^1(\mathbb{D}, d\mu_p)$ and there holds the formula*

$$f(z) = \int_{\mathbb{D}} K_p(w,z) f(w)\, d\mu_p(w). \tag{8.2}$$

Proof. If $\|f\|_{A^p} = 1$ and $z \in \mathbb{D}$, then, according to (8.1),

$$|f(w)| = |f(w)|^p |f(w)|^{1-p} \leqslant |f(w)|^p (1-|w|)^{-2/p+2},$$

which implies that the function $w \mapsto f(w)K_p(w,z)$ is in $L^1(\mathbb{D}, d\mu_p)$. It follows that the functional

$$\Lambda_z f = \int_{\mathbb{D}} K_p(w,z) f(w)\, d\mu_p(w)$$

is well defined and bounded on A^p. Since the functional $f \mapsto f(z)$ is bounded (Proposition 8.1.1(a)) and the polynomials are dense in A^p, we see that the proof of (8.2) reduces to the proof that $\Lambda_z f = f(z)$ for $f(z) = z^n$, $n \geqslant 0$. However this follows from the formulas

$$\frac{1}{(1-z)^{\alpha+1}} = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+k+1)}{\Gamma(\alpha+1)\Gamma(k+1)} z^k,$$

$$\int_{\mathbb{D}} (1-|w|^2)^{\alpha-1} |w|^{2k} dA(w) = \frac{\Gamma(\alpha+1)\Gamma(k+1)}{\Gamma(\alpha+k+1)} \int_{\mathbb{D}} (1-|w|^2)^{\alpha-1} dA(w),$$

which are true for $\alpha > 0$, and the formula $\int_{\mathbb{D}} w^m \bar{w}^n\, dA(w) = 0$ $(m \neq n)$. □

Remark. A more interesting proof of (8.2) can be found in [63].

8.2.2 Lemma *Let $f \in A^p$, $0 < p \leqslant 1$ and let*

$$g(z) = \int_{\mathbb{D}} |K_p(w,z)|\, |f(w)|\, d\mu_p(w) \qquad (z \in \mathbb{D}).$$

Then $g \in L^p(\mathbb{D}, dA)$ and there holds the inequality $\|g\|_{L^p} \leqslant C_p \|f\|_{A^p}$.

Proof. Let $\mathcal{E}$ be a partition of the unit disk into disjoint sets E with the following properties:

$$\frac{1}{C} \leqslant \frac{\operatorname{diam}(E)}{1-|w|} \leqslant C \qquad (w \in E),$$

$$\frac{1}{C} \leqslant \frac{\operatorname{area}(E)}{(1-|w|)^2} \leqslant C \qquad (w \in E),$$

$$\frac{1}{C} \leqslant \frac{1-|\zeta|}{1-|w|} \leqslant C \qquad (\zeta, w \in E),$$

where C is an absolute constant. Such a family consists of the sets

$$\left\{ z\colon \frac{1}{2^{k+1}} < 1-|z| \leqslant \frac{1}{2^k},\ \frac{2\pi j}{2^{k+2}} \leqslant \arg z < \frac{2\pi(j+1)}{2^{k+2}} \right\},$$

where $k = 0, 1, 2, \ldots, 0 \leqslant j < 2^{k+2}$. Let $\mathcal{E} = \{E_n : n \geqslant 1\}$. Then

$$g(z) \leqslant C \sum_{n=1}^{\infty} (1 - |w_n|)^{2/p-1} \int_{E_n} |f(w) K_p(w, z)| \, dA(w)$$
$$\leqslant C \sum_{n=1}^{\infty} (1 - |w_n|)^{2/p+1} \sup_{w \in E_n} |f(w) K_p(w, z)|,$$

where $\{w_n\}$ is an arbitrary sequence such that $w_n \in E_n$ and C is a constant depending only on p. It follows that

$$g(z)^p \leqslant C \sum_{n=1}^{\infty} (1 - |w_n|)^{2+p} \sup_{w \in E_n} |f(w) K_p(w, z)|^p.$$

For a fixed z the function

$$F(w) = |f(w) K_p(w, z)|^p = \left| \frac{f(w)}{(1 - w\bar{z})^{2/p+1}} \right|^p$$

is subharmonic in $\mathbb{D}$ because the function $f(w)/(1 - w\bar{z})^{2/p+1}$ is analytic with respect to w. Therefore

$$F(w) \leqslant \frac{1}{A(D_w)} \int_{D_w} F(\zeta) \, dA(\zeta),$$

where $D_w \subset \mathbb{D}$ is an arbitrary disk centered at w. From this and the properties of the family $\mathcal{E}$ we get

$$\sup_{E_n} F \leqslant C \frac{1}{(1 - |w_n|)^2} \int_{B_n} \frac{|f(w)|^p}{|1 - \bar{w}z|^{2+p}} \, dA(w),$$

where B_n is the union of those $E \in \mathcal{E}$ for which the set $\overline{E} \cap \overline{E}_n$ is nonempty. Combining these with the inequality

$$\int_{\mathbb{D}} \frac{1}{|1 - \bar{w}z|^{2+p}} \, dA(z) \leqslant \frac{C}{(1 - |w|)^p}$$

(see Lemma 8.2.3 below), we get

$$\int_{\mathbb{D}} g(z)^p \, dA(z) \leqslant C \sum_{n=1}^{\infty} \int_{B_n} |f(w)|^p \, dA(w).$$

Now our result follows from the easily checked fact that each B_n contains at most N members of the family $\mathcal{E}$, with N being independent of n. □

8.2.3 Lemma *Let*

$$I_s(a) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |1 - \bar{a}e^{i\theta}|^{-s} \, d\theta, \quad J_s(a) = \int_{\mathbb{D}} |1 - \bar{a}w|^{-s} \, dA(w).$$

There hold the relations(*):

$$I_s(a) \asymp \begin{cases} (1-|a|)^{-s+1} & \text{for } s>1 \\ \log \frac{2}{1-|a|} & \text{for } s=1 \\ 1 & \text{for } s<1 \end{cases} \qquad (a \in \mathbb{D}),$$

$$J_s(a) \asymp \begin{cases} (1-|a|)^{-s+2} & \text{for } s>2 \\ \log \frac{2}{1-|a|} & \text{for } s=2 \\ 1 & \text{for } s<2 \end{cases} \qquad (a \in \mathbb{D}).$$

Proof. Since $J_s(a) = 2\int_0^1 I_s(ra)\,dr$, it suffices to discuss the case of I_s. Let $\rho = |a| < 1$. We have

$$I_s(a) = \frac{1}{\pi}\int_0^\pi \left[(1-r)^2 + 4r\sin^2(\theta/2)\right]^{-s/2} d\theta$$
$$\asymp \frac{1}{\pi}\int_0^\pi (1-r+\theta)^{-s} d\theta. \qquad \square$$

8.3 The Coifman/Rochberg theorem

It is the idea of Coifman and Rochberg [15] to represent a member of A^p as a sum of "atoms" by replacing the integral in (8.2) by a Riemannian sum over a sufficiently fine partition of the disk. They proved the "decomposition theorem" for every $p > 0$ and for a class of domains in $\mathbb{C}^n$, in particular on balls. Here we consider the case $p < 1$ because this case, as was shown by Kalton [37], is of fundamental importance in the theory of vector-valued analytic functions.

8.3.1 Theorem (on atomic decomposition) *Let* $0 < p \leqslant 1$. *Then*

(a) *There exists a sequence* $\{w_n\}$ *in* $\mathbb{D}$ *and a constant* C *such that for every* $f \in A^p$ *there exists a sequence* $\{a_n\} \subset \ell^p$ *with the properties*

$$f(z) = \sum_{n=1}^{\infty} a_n \frac{1-|w_n|^2}{(1-\bar{w}_n z)^{2/p+1}} \tag{8.3}$$

$$\sum_{n=1}^{\infty} |a_n|^p \leqslant C \int_{\mathbb{D}} |f|^p\, dA.$$

(b) *The function* f *of the form* (8.3), *where* $\{a_n\} \in \ell^p$, *belongs to* A^p *and there holds* $\|f\|_{A^p} \leqslant C\|\{a_n\}\|_p$.

For the proof we need a lemma.

(*) "$A \asymp B$ " means that the ratio A/B lies between two positive constants.

8.3.2 Lemma *There exists a constant C such that*

$$|K_p(w,z) - K_p(\zeta,z)| \leqslant C\frac{|w-\zeta|}{\operatorname{diam} E}|K_p(w,z)|$$

for all $E \in \mathcal{E}$, $w, \zeta \in E$ and $z \in \mathbb{D}$.

Proof. By the mean value theorem, we have

$$\begin{aligned}|K_p(w,z) - K_p(\zeta,z)| &\leqslant (2/p+1)|w-\zeta| \sup_{a\in E} |1-\bar{a}z|^{-2/p-2} \\ &\leqslant (2/p+1)\frac{|w-\zeta|}{1-|a|} \sup_{a\in E} |K_p(a,z)| \\ &\leqslant C_p\frac{|w-\zeta|}{\operatorname{diam} E} \sup_{a\in E} |K_p(a,z)|.\end{aligned}$$

On the other hand,

$$\left|1-\frac{1-\bar{w}z}{1-\bar{a}z}\right| = \left|\frac{(w-a)z}{1-\bar{a}z}\right| \leqslant \frac{|w-a|}{1-|a|} \leqslant C.$$

Hence $|1-\bar{w}z| \leqslant C|1-\bar{a}z|$ and hence $|K_p(a,z)| \leqslant C|K_p(w,z)|$ for $a, w \in E$. The result follows. □

Proof of Theorem 8.3.1. Assertion (b) follows from the boundedness of the sequence $\|\psi_k\|_{A^p}$, where

$$\psi_k(z) = \frac{1-|w_k|^2}{(1-\bar{w}_k z)^{1+2/p}}.$$

Let us prove (a). Let $\varepsilon > 0$. Dividing each $E \in \mathcal{E}$ into N subsets, where N is a sufficiently large integer independent of E, we can represent $\mathbb{D}$ as a disjoint union $D_1 \cup D_2 \cup \ldots$, where $D_1, D_2, \ldots$ are subsets of $\mathbb{D}$ with the properties:

$$\frac{\varepsilon}{C_1} \leqslant \frac{\operatorname{diam}(D_n)}{1-|w|} \leqslant C_1\varepsilon \quad\text{and}\quad \frac{\varepsilon^2}{C_1} \leqslant \frac{\operatorname{area}(D_n)}{(1-|w|)^2} \leqslant C_1\varepsilon^2 \quad (w \in D_n).$$

Let $\{w_n\}$ be a sequence such that $w_n \in D_n$. Define the operator T by

$$Tf(z) = \sum_{n=1}^{\infty} b_n(1-|w_n|^2)K_p(w_n,z) \quad (z \in \mathbb{D}),$$

where

$$b_n = \frac{1}{1-|w_n|^2}\int_{D_n} f(w)\,d\mu_p(w).$$

Proceeding as in the proof of Lemma 8.2.2, we can prove that T maps A^p into A^p. In order to conclude the proof it suffices to prove that T is an isomorphism for ε small enough and that

$$\sum_{n=1}^{\infty} |b_n|^p \leqslant C\int_{\mathbb{D}} |f|^p dA.$$

The proof of the latter is similar to that of Lemma 8.2.2. To prove the rest we start from the relation

$$f(z) - Tf(z) = \sum_{n=1}^{\infty} \int_{\mathbf{D}} (K_p(w,z) - K_p(w_n,z)) f(w)\, d\mu_p(w).$$

From this, by Lemma 8.3.2, we get

$$\begin{aligned} |TF(z) - f(z)| &\leqslant C\varepsilon \sum_{n=1}^{\infty} \int_{D_n} |f(w)|\, |K_p(z,w)|\, d\mu_p(w) \\ &= C\varepsilon \int_{\mathbf{D}} |f(w)|\, |K_p(z,w)|\, d\mu_p(w). \end{aligned}$$

Now Lemma 8.2.2 shows that $||Tf - f|| \leqslant C_p\varepsilon ||f||$. Finally we take $\varepsilon = 1/2C_p$ and apply Proposition 1.2.1. □

It is of importance that the sequence w_n in (8.3) can be chosen from the annulus $\{w : r < |w| < 1\}$, where $r < 1$ is fixed.

8.3.3 Theorem (Kalton [38]) *Let $0 < r < 1$ and $\psi \in A^p$. Then ψ can be represented as*

$$\psi(z) = \sum_{k=0}^{\infty} \alpha_k \frac{1 - |\zeta_k|^2}{(1 - \zeta_k z)^{\frac{2}{p}+1}}, \quad \text{where} \quad r \leqslant |\zeta_k| < 1, \ \Big(\sum_{k=0}^{\infty} |\alpha_k|^p\Big)^{1/p} \leqslant C||\psi||_{A^p},$$

and C depends only on r and p.

In contrast to Theorem 8.3.1, here we do not assert $\{\zeta_k\}$ is independent of ψ (though this is probably true).

Proof. If Γ is an arbitrary set, then the space $\ell^p(\Gamma)$ consists of those functions $\omega \mapsto a_\omega$ for which

$$||a|| := \Big(\sum_{\omega\in\Gamma} |a_\omega|^p\Big)^{1/p} < \infty.$$

Let $\Gamma = \{w : r \leqslant |w| < 1\}$. Define the operator $S : \ell^p(\Gamma) \mapsto A^p$ by

$$S(\{a_w\})(z) = \sum_{w\in\Gamma} a_w \frac{1 - |w|^2}{(1 - \bar{w}z)^{2/p+1}}.$$

Let $E = S(\ell^p)$ and Λ an arbitrary (not necessarily continuous) linear functional on A^p such that $\Lambda(E) = 0$. We want to prove $\Lambda = 0$ on A^p, which implies that S is *onto*; this proves the desired result, according to the open mapping theorem.

Let F be the linear hull of the vectors $\varphi_k(z) = 1/(1 - \bar{w}_k z)^{2/p+1}$, $|w_k| < r$, where $\{w_k\}$ is the sequence from Theorem 8.3.1. If $f \in A^p$, then $f = g + h$, where $g \in E$, $h \in F$ and $||g|| + ||h|| \leqslant C||f||$. Since $\dim(F) < \infty$, we see that Λ is bounded on F, i.e., there exists a constant C_1 such that $|\Lambda h| \leqslant C_1||h||$, which implies that $|\Lambda f| = |\Lambda h| \leqslant C_2\|f\|$, and this means that Λ is bounded on A^p. Take $\phi(w) = \Lambda(\psi_w)$, where $\psi_w(z) = 1/(1-\bar{w}z)^{2/p+1}$. The function ϕ is antianalytic and $\phi(w) = 0$ for $r \leqslant |w| < 1$ because $\Lambda(E) = 0$. It follows that $\phi(w) = 0$ for every $w \in \mathbb{D}$ and in particular $\Lambda\varphi_k = 0$ for every k. By Theorem 8.3.1, this implies $\Lambda = 0$ on A^p, which was to be proved. □

The case $p > 1$

If $1 < p < \infty$, then the formulation of the Coifman/Rochberg theorem is similar to that of Theorem 8.3.1; we only have to replace (8.3) by

$$f(z) = \sum_{n=1}^{\infty} a_n \frac{(1-|w_n|^2)^{2/q}}{(1-\bar{w}_n z)^2},$$

where $1/p + 1/q = 1$ (see [99]).

The case of weighted spaces

The **weighted Bergman space** A^p_β, $0 < p < \infty$, $\beta > -1$, consists of those $f \in H(\mathbb{D})$ for which

$$\|f\|_{p,\beta} := \left\{ \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\beta \, dA(z) \right\}^{1/p} < \infty.$$

8.3.4 Theorem *Let $0 < p \leqslant 1$, $\beta > -1$ and $\gamma > 0$.*

(a) *There exists a sequence $\{w_n\}$ in $\mathbb{D}$ and a constant C such that every $f \in A^p_\beta$ can be represented as*

$$f(z) = \sum_{n=1}^{\infty} a_n \frac{(1-|w_n|^2)^\gamma}{(1-\bar{w}_n z)^{\gamma+(\beta+2)/p}} \tag{8.4}$$

with $\|\{a_n\}\|_{\ell^p} \leqslant C\|f\|_{p,\beta}$.

(b) *Every function f of the form (8.4) with $\{a_n\} \in \ell^p$ belongs to A^p_β and $\|f\|_{p,\beta} \leqslant C\|\{a_n\}\|_{\ell^p}$.*

This theorem is proved in the same way as its particular case, Theorem 8.3.1 ($\beta = 0$, $\gamma = 1$); the key is in the formula

$$f(z) = c_s \int_{\mathbb{D}} \frac{f(w)(1-|w|^2)^{s-2}}{(1-\bar{w}z)^s} \, dA(w) \qquad (s = \gamma + (\beta+2)/p),$$

which holds for $f \in A^p_\beta$.

Envelops of Hardy spaces

We have defined the q-Banach envelope of a quasi-Banach space (see p. 4). Now we can give a nontrivial example.

8.3.5 Theorem *If $0 < p < q \leqslant 1$, then the q-Banach envelope of the Hardy space H^p is equal to A^q_β, $\beta = q/p - 2$.*

Proof. The space $X = H^p$ is embedded into the q-Banach space $Y = A^q_\beta$ (see 7.4.7). On the other hand, every $f \in Y$ can be represented as $f = \sum_{n=1}^\infty f_n$, where

$$f_n = a_n \frac{(1-|w_n|^2)^\gamma}{(1-\bar{w}_n z)^{\gamma+1/p}}$$

and $(\sum |a_n|^q)^{1/q} \leqslant C\|f\|_Y$. Lemma 8.2.3 shows that

$$\int_{-\pi}^{\pi} \frac{1}{|1-\bar{w}_n e^{i\theta}|^{\gamma p+1}}\, d\theta \leqslant \frac{C}{(1-|w_n|)^{\gamma p}},$$

which implies

$$\sum_{n=1}^\infty \|f_n\|_X^q \leqslant C \sum_{n=1}^\infty |a_n|^q \leqslant C\|f\|_Y.$$

Now the result follows from Proposition 1.2.5. □

8.4 Coefficients of vector-valued analytic functions

A function $F : \Omega \mapsto X$, where Ω is a domain in $\mathbb{C}$, is said to be analytic if every point in Ω admits a neighborhood in which f can be expanded into a power series with X-valued coefficients. In the case where Ω is the unit disk, it turns out that the analyticity implies the existence of vectors $\widehat{F}(n)$ such that $F(z) = \sum_{n=0}^\infty \widehat{F}(n)\, z^n$, $|z| < 1$, with uniform convergence on compact subsets. The vectors $\widehat{F}(n)$, uniquely determined by F, are called the Taylor coefficients and, as one expects, satisfy the condition $\limsup_{n\to\infty} \|\widehat{F}(n)\|^{1/n} \leqslant 1$. On the other hand, if $\{f_n\}$ is a sequence of vectors in X, then the condition

$$\limsup_{n\to\infty} \|f_n\|^{1/n} \leqslant 1 \tag{8.5}$$

is necessary and sufficient for the series $\sum_{n=0}^\infty f_n\, z^n$ to converge for every $z \in \mathbb{D}$. In the case of convergence, the sum of that series is analytic in $\mathbb{D}$. Therefore the set of the functions $F : \mathbb{D} \mapsto X$ analytic in $\mathbb{D}$ can be identified with the set of the formal power series satisfying (8.5), i.e., with the set of the power series converging in $\mathbb{D}$. We will denote this set by $H(\mathbb{D}, X)$.

Derivatives The derivative of a function $f \in H(\mathbb{D}, X)$ is defined by means of power series, $F'(z) = \sum_{n=1}^\infty n\widehat{F}(n)\, z^{n-1}$, or by the formula

$$F'(z) = \lim_{w\to z} \frac{F(w) - F(z)}{w - z}. \tag{8.6}$$

It should be noted, however, that the existence of the limit (8.6) for an arbitrary function F does not guarantee that F is analytic.

Hadamard product

The Hadamard product of a (scalar) function $\psi \in H(\mathbb{D})$ and a function $F \in H(\mathbb{D}, X)$ is defined by

$$(\psi * F)(z) = \sum_{n=0}^{\infty} \widehat{\psi}(n)\widehat{F}(n)\, z^n.$$

The proof that $\psi * F$ belongs to $H(\mathbb{D}, X)$ is straightforward. There holds

$$z\,F'(z) = (\psi * F)(z), \quad \text{where} \quad \psi(z) = \frac{z}{(1-z)^2}.$$

8.4.1 Proposition *Let $F \in H(\mathbb{D}, X)$, where X is a p-Banach space, and let the series $\sum_{k=1}^{\infty} \psi_k(z) = \psi(z)$, where $\psi_j \in H(\mathbb{D})$, converge uniformly on compact subsets. Then*

$$(\psi * F)(z) = \sum_{k=1}^{\infty} (\psi_k * F)(z) \qquad (|z| < 1).$$

Proof. Let $|z| = r < 1$, $F_N(z) = \sum_{j=1}^{N} (\psi_j * F)(z)$, and $R_N(z) = \sum_{j=N+1}^{\infty} \psi_j(z)$. Then

$$\|F_N(z) - \psi * F(z)\|^p = \|R_N * F(z)\|^p \leqslant \sum_{j=0}^{\infty} A_N(j), \tag{8.7}$$

where $A_N(j) = |\widehat{R}_N(j)|^p\, \|\widehat{F}(j)\|^p\, |z|^{jp}$. The sequence $R_N(z)$ is uniformly bounded on compact subsets and therefore for every $\rho < 1$ there exists a constant $C = C(\rho)$ such that $|\widehat{R}_N(j)| \leqslant C/\rho^j$ for all N and j. From this and the inequality $\|\widehat{F}(j)\| \leqslant K(\rho)/\rho^j$, we get $A_N(j) \leqslant M(\rho)(r/\rho^2)^{jp}$ where $M(\rho) = C(\rho)^p K(\rho)^p$. Thus by taking $\rho^2 = \sqrt{r}$, we see that the sequence A_N has a summable majorant. On the other hand, from the hypothesis $R_N(z) \to 0$, uniformly on compact subsets, it follows $A_N(j) \to 0$ $(N \to \infty)$, for every j. Therefore we can apply the dominated convergence theorem:

$$\lim_{N\to\infty} \sum_{j=0}^{\infty} A_N(j) = \sum_{j=0}^{\infty} \lim_{N\to\infty} A_N(j) = 0.$$

Now the desired assertion follows from (8.7). □

Inequalities for the coefficients

Let $H^\infty(X)$ denote the set of all bounded functions $F \in H(\mathbb{D}, X)$; the quasinorm is given by $\|F\|_{\infty,X} = \sup_{|z|<1} \|F(z)\|_X$. If $F \in H^\infty(X)$, where X is a Banach space, then one can use the Hahn/Banach theorem to deduce the inequality $\|\widehat{F}(n)\| \leqslant \|F\|_{\infty,X}$ from the corresponding inequality for scalar-valued functions. However if X is p-Banach, then this inequality does not hold, and proving that

$$\|\widehat{F}(n)\| \leqslant C_p(n+1)^{1/p-1}\|F\|_{\infty,X}$$

is quite nontrivial. Following Kalton [37], we will prove the latter by means of the Coifman/Rochberg theorem. We first consider a weak variant of Schwarz lemma.

8.4.2 Theorem *Let $F \in H^\infty(X)$, where X p-Banach space. Then there holds the inequality*

$$\|F'(z)\|_X \leqslant C_p (1-|z|^2)^{-1} \|F\|_{\infty,X} \qquad (z \in \mathbb{D}),$$

where C_p is a constant depending only on p.

Proof. Let ψ be a scalar-valued analytic function belonging to $L^p(\mathbb{D})$. According to Theorem 8.3.1, there holds the formula

$$\psi(w) = \sum_{k=1}^{\infty} \alpha_k \frac{1-|z_k|^2}{(1-z_k w)^{2/p+1}},$$

where $\{z_k\}$ is a sequence in $\mathbb{D}$ and $\{\alpha_k\}$ is a sequence of complex numbers such that $\|\{\alpha_k\}\|_p \leqslant C_p \|\psi\|_{L^p(\mathbb{D})}$, where C_p is independent of ψ (z_k are independent of ψ as well). The series converges uniformly on compact subsets so we can apply Proposition 8.4.1; we get

$$\psi * G(w) = \sum_{k=1}^{\infty} \alpha_k (1-|z_k|^2) D^{2/p} G(w z_k) \qquad (w \in \mathbb{D}), \tag{8.8}$$

where $G \in H(\mathbb{D}, X)$ and

$$D^s G(z) = G(0) + \sum_{j=1}^{\infty} \frac{(s+1)(s+2)\ldots(s+j)}{j!} \widehat{G}(j)\, z^j.$$

Now choose G so that $D^{2/p}G = F$ and put $\psi(w) = w$ in (8.8). We get

$$\frac{\widehat{F}(1)}{2/p+1} w = \sum \alpha_k (1-|z_k|^2) F(z_k w).$$

Hence

$$\begin{aligned} \|F'(0)\|_X^p\, |w| &\leqslant (1+2/p)^p \sum |\alpha_k|^p (1-|z_k|^2)^p\, \|F(z_k w)\|^p \\ &\leqslant (1+2/p)^p \sum |\alpha_k|^p\, \|F\|_{\infty,X}^p \leqslant C_p \|F\|_{\infty,X}^p. \end{aligned}$$

This proves the result for $z = 0$. If $z \in \mathbb{D}$ is arbitrary, then we apply this special case to the function $F_1(w) = F(z + (1-|z|)w)$; the derivative can be found by the formula (8.6): $F_1'(0) = (1-|z|)F'(z)$. We need however to show that the function $F_1(w)$ is analytic in $\mathbb{D}$; this is true because

$$\begin{aligned} F_1(w) &= \sum_{n=0}^{\infty} \widehat{F}(n) \sum_{j=0}^{n} \binom{n}{j} z^{n-j} (1-|z|)^j\, w^j \\ &= \sum_{j=0}^{\infty} \left(\sum_{n=j}^{\infty} \widehat{F}(n) \binom{n}{j} z^{n-j} (1-|z|)^j \right) w^j, \end{aligned}$$

where we have used 1.1.5. □

8.4.3 Theorem *Let $F \in H^\infty(X)$, where X is a p-Banach space, $0 < p \leqslant 1$. Then there exists a constant C_p depending only on p such that*

$$\|\widehat{F}(n)\|_X \leqslant C_p\, n^{1/p-1}\|F\|_{\infty,X} \qquad (n \geqslant 1). \tag{8.9}$$

Before proving the theorem consider a simple example. Let $f \in H^p$, $0 < p < 1$. Define the function $F : \mathbb{D} \mapsto H^p$ by $F(w)(z) = f(zw)$ $(z, w \in \mathbb{D})$. The function F belongs to $H(\mathbb{D}, H^p)$ because $F(w) = \sum_{n=0}^{\infty} \widehat{f}(n)e_n\, w^n$, $e_n(z) = z^n$, and the series converges for $|w| < 1$. The function F is bounded; moreover $\|F\|_\infty = \|f\|_{H^p}$. An application of Theorem 8.4.3 gives $|\widehat{f}(n)| \leqslant C_p\,(n+1)^{1/p-1}\|f\|_{H^p}$ because $\|e_n\|_{H^p} = 1$. Thus, in this special case, Theorem 8.4.3 reduces to the well known result of Hardy and Littlewood (Corollary 5.1.3).

Proof of Theorem. In the case $n = 1$ the assertion is contained in Theorem 8.4.2 because $F'(0) = \widehat{F}(1)$. Let $n \geqslant 2$. We use the atomic decomposition again. In this situation we choose G in (8.8) so that $D^{2/p}G = F'$. Then we take $\psi(w) = w^n$ in (8.8) and get

$$A_n\widehat{F}(n+1)w^n = \sum \alpha_k(1-|z_k|^2)F'(z_k w),$$

where

$$A_n = \frac{(n+1)!}{(2/p+1)\ldots(2/p+n)}.$$

Since A_n behaves as $n^{1-2/p}$, it follows that

$$n^{p-2}\|\widehat{F}(n+1)\|^p \leqslant C\sum |\alpha_k|^p(1-|z_k|^2)^p\|F'(z_k)\|^p.$$

From this and Lemma 8.4.2 it follows

$$n^{p-2}\|\widehat{F}(n+1)\|^p \leqslant \sum |\alpha_k|^p\|F\|^p_{\infty,X}.$$

Finally since

$$\sum |\alpha_k|^p \leqslant C\int_{\mathbb{D}} |z|^{np}\, dA(z) = \frac{2c}{np+2}$$

we get (8.9). □

Vector-valued Hardy spaces

Theorem 8.4.3 can be used for its own improvement. For a quasi-Banach space X let $H^p(X)$ $(0 < p < \infty)$ denote the space of analytic functions $F : \mathbb{D} \mapsto X$ such that

$$\|F\|_{p,X} = \sup_{0<r<1}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \|F(re^{i\theta})\|^p\, d\theta\right\}^{1/p} < \infty.$$

So $H^p(X)$ is a "vector" Hardy space.

8.4.4 Theorem *Let $F \in H^p(X)$, where X is a p-Banach space ($0 < p \leqslant 1$). Then $\|\widehat{F}(n)\|_X \leqslant C_p n^{1/p-1}\|F\|_{p,X}$, for $n \geqslant 1$.*

Proof. Let Y be the completion of the space of those continuous functions $g : \mathbb{T} \mapsto X$ for which the function $\|g\|$ belongs to $L^p(\mathbb{T})$. The space Y is p-Banach. For a fixed r, $0 < r < 1$, define the function $G : \mathbb{D} \mapsto Y$ by $G(z)(e^{it}) = F(rze^{it})$. Then $G \in H^\infty(Y)$ and $\|G\|_{\infty,Y} \leqslant \|F\|_{p,X}$. Since $\widehat{G}(n)(e^{it}) = r^n \widehat{F}(n)e^{int}$, we can apply Theorem 8.4.3 to get $r^n\|\widehat{F}(n)\|_X \leqslant C_p n^{1/p-1}\|F\|_{p,X}$, where C_p is independent of r. This concludes the proof. □

Inequalities for a Hadamard product

The Hadamard product of a function $Q \in L^1(\mathbb{T})$ and a function $F \in H(\mathbb{D}, X)$ is defined by

$$Q * F(z) = \sum_{k=-\infty}^{\infty} \widehat{Q}(k)\widehat{F}(k)\, z^k.$$

Since $|\widehat{Q}(n)| \leqslant \|Q\|_1$, the function $Q * F(z)$ belongs to $H(\mathbb{D}, X)$.

8.4.5 Theorem *Let $F \in H^\infty(X)$, where X is p-Banach, and let $Q \in L^1(\mathbb{T})$ be such that $\operatorname{supp}(\widehat{Q})$ is contained in $(-\infty, n]$ for some positive integer n. Then there exists a constant C_p such that $\|Q * F\|_{\infty,X} \leqslant C_p n^{1/p-1}\|Q\|_p\|F\|_{\infty,X}$.*

Note that $Q * F(z) = \sum_{k=0}^{n} \widehat{Q}(k)\widehat{F}(k)\, z^k$. Note that in the special case $Q(e^{i\theta}) = e^{int}$ we have Theorem 8.4.3; on the other hand, we will deduce Theorem 8.4.5 from Theorem 8.4.3 by means of Theorem 8.4.4.

Proof. Let $a_k = \widehat{Q}(n-k)$ for $k \geqslant 0$ and $A(z) = \sum_{k=0}^{\infty} a_k z^k$. Then $Q * F(w) = \sum_{k=0}^{n} a_{n-k}\widehat{F}(k)w^k$. Thus $Q * F(w)$ is the n-th coefficient in the Taylor expansion of the function $A(z)F(zw)$, which implies, by Theorem 8.4.4,

$$\|Q * F(w)\|_X^p \leqslant Cn^{1-p} \sup_{0<r<1} \int_0^{2\pi} |A(re^{it})|^p\, \|F(re^{it}w)\|^p\, dt.$$

Hence

$$\|Q * F\|_{\infty,X}^p \leqslant Cn^{1-p} \left(\sup_{0<r<1} \int_0^{2\pi} |A(re^{it})|^p\, dt\right) \|F\|_{\infty,X}^p.$$

Since the function $|A(z)|^p$ is subharmonic the last integral increases with r and therefore the supremum is equal to

$$\int_0^{2\pi} |A(e^{it})|^p\, dt = \int_0^{2\pi} |Q(e^{it})|^p\, dt.$$

The result follows. □

As an immediate consequence of Theorem 8.4.5 and Lemma 7.3.2 we have:

8.4.6 Theorem *Let $F \in H^\infty(X)$, where X is a quasi-Banach space. Let*

$$W_n(e^{it}) = W_n^\psi(e^{it}) = \sum_{|k|<\infty} \psi\Big(\frac{k}{n}\Big)e^{ikt},$$

where ψ is a complex-valued C^∞-function with compact support in $\mathbb{R}$. Then there holds the inequality $\|W_n * F\|_{\infty,X} \leqslant C\|F\|_{\infty,X}$, $n \geqslant 1$, where C is independent of n and F.

Miscellaneous

8.4.7 If a sequence $F_n \in H(\mathbb{D}, X)$ converges uniformly on compact subsets of $\mathbb{D}$, then its limit belongs to $H(\mathbb{D}, X)$.

8.4.8 If the function $\phi : \mathbb{D} \mapsto \mathbb{D}$ is analytic and $F \in H(\mathbb{D}, X)$, then the composition $F \circ \phi$ belongs to $H(\mathbb{D}, X)$ and there holds the standard formula for the derivative.

8.4.9 If X is quasi-Banach, then $H^p(X)$ is complete for every $p \in (0, \infty]$.

8.4.10 (Kalton [38]) If X is quasi-Banach, then for every $\rho \in (0, 1)$ there exists a constant $C = C(\rho)$ such that $\|F(0)\| \leqslant C \sup_{\rho<|z|<1} \|F(z)\|$.

8.4.11 A function $P : X \mapsto [-\infty, \infty)$ is said to be plurisubharmonic if it is upper semicontinuous and

$$P(x) \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} P(x + ye^{i\theta})\, d\theta$$

for all $x, y \in X$. The space X is said to be PL-convex if its quasinorm is plurisubharmonic. Since convex functions are plurisubharmonic, Banach spaces are PL-convex. Some of quasi-Banach spaces are PL-convex, e.g., L^p $(p > 0)$, and some are not.

A quasi-Banach space X is PL-convex iff $\|f(0)\| \leqslant \max_{|\zeta|=1}\|f(\zeta)\|$ for every continuous function $f\colon \overline{\mathbb{D}} \mapsto X$ analytic in $\mathbb{D}$.

Further information can be found in [68, 16, 38, 3, 74].

8.4.12 Let X be quasi-Banach space and $\sigma = s+it$ an arbitrary complex number. Then

$$\Big\|\sum_{k=n}^{4n} k^\sigma a_k z^k\Big\|_{\infty,X} \asymp n^s \Big\|\sum_{k=n}^{4n} a_k z^k\Big\|_{\infty,X} \qquad (\{a_k\} \subset X).$$

This generalizes 7.3.5.

9 Subharmonic behavior

The basic topological properties of the spaces $hA^p = h(\mathbb{D}) \cap L^p$ and h^p, $p < 1$, can be deduced from the inequality

$$|g(0)|^p \leqslant C_p \int_{\mathbb{D}} |g|^p \, dA, \quad g \text{ harmonic,} \tag{†}$$

essentially due to Hardy and Littlewood. In this chapter we give two proofs of this inequality (Section 9.1) as well as some generalizations (Section 9.3). Using (†) we prove the Hardy and Littlewood's theorem that hA^p is "self-conjugate" for every $p > 0$ (Theorem 9.1.2). In Section 9.2 we apply (†) to prove four estimates, due to Hardy and Littlewood, concerning h^p (Theorems 9.2.1 and 9.2.2).

9.1 Subharmonic behavior of $|u|^p$, $p < 1$, and Bergman spaces

The space $hA^p = h(\mathbb{D}) \cap L^p(\mathbb{D}, dA)$, $0 < p < \infty$, (quasi)normed in the obvious way, is called the harmonic Bergman space. That this space is complete for $p \geqslant 1$ can be proved by appealing to the subharmonicity of $|g|^p$, $g \in h(\mathbb{D})$ (see the proof of Proposition 8.1.1). However, as we already noted (see 4.1.5), if $0 < p < 1$ and $g \in h(\mathbb{D})$, then $|g|^p$ need not be subharmonic. Nevertheless, as the following result shows, the functional $g \mapsto g(0)$ is continuous on hA^p, for every $p > 0$, and this can be used to prove that *hA^p is complete for every p.*

9.1.1 Theorem *If $u = |g|$, where g is harmonic in $\mathbb{D}$, and $0 < p < 1$, then there holds the inequality*

$$u(0)^p \leqslant C_p \int_{\mathbb{D}} u^p \, dA, \tag{9.1}$$

where C_p is constant depending only on p. More generally, this inequality remains valid if we assume that $u \geqslant 0$ is an arbitrary subharmonic function in $\mathbb{D}$.

Recall that we have used this theorem in proving the important Theorem 7.1.8.

Of course, if $p \geqslant 1$, then (9.1) holds with $C_p = 1$; if $u = |f|$, f analytic, then $C_p = 1$ for every $p > 0$.

In the case where $u = |g|$, g harmonic, Theorem 9.1.1 is a formal consequence of the following theorem of Hardy and Littlewood [26]:

9.1.2 Theorem (Hardy/Littlewood) *If $g = \operatorname{Re} f$, where f is analytic in $\mathbb{D}$, then*

$$\int_{\mathbb{D}} |f|^p \, dA \leqslant C_p \Big(|\operatorname{Im} f(0)|^p + \int_{\mathbb{D}} |g|^p \, dA \Big). \tag{9.2}$$

Indeed, since the function $|f|^p$ is subharmonic for every $p > 0$, and since $|g(0)|^p = |f(0)|^p$, we see that (9.2) implies (9.1). However, it seems more natural first to prove Theorem 9.1.1 and then deduce Theorem 9.1.2.

Proof of Theorem 9.1.1

First proof.(*) If we apply (9.1) to the disk of radius $1-|z|$ centered at $z \in \mathbb{D}$, we get

$$u(z)^p \leqslant C_p(1-|z|)^{-2} \int_{\mathbb{D}} u^p \, dA. \tag{9.3}$$

Hence, it is natural to consider the maximum of the function $F(z) = (1-|z|)^2 u(z)^p$, which exists under the condition that u is bounded on $\mathbb{D}$. We shall assume that u is bounded(†) and that

$$\int_{\mathbb{D}} u^p \, dA = 1. \tag{9.4}$$

Then the function F is above semicontinuous on the closed disk and equal to zero on the boundary. Consequently, F attains its maximum at a point $a \in \mathbb{D}$. Now we apply the sub-mean-value property of u on the disk D_a of radius $(1-|a|)/2$ centered at a. It follows that

$$u(a) \leqslant 4(1-|a|)^{-2} \int_{D_a} u \, dA.$$

Writing $u = u^p u^{p-1}$, we see that, in view of (9.4), $(1-|a|)^2 u(a) \leqslant 4(M_a)^{1-p}$, where M_a is the supremum of u on D_a. Assume we have found a constant K_p such that $M_a \leqslant K_p u(a)$; then $(1-|a|)^2 u(a) \leqslant 4(K_p)^{1-p} u(a)^{1-p}$, i.e., after multiplying by $u(a)^{p-1}$,

$$u(0)^p = F(0) \leqslant F(a) \leqslant 4(K_p)^{1-p},$$

which implies (9.1).

In order to prove $M_a \leqslant K_p u(a)$, let $z \in D_a$. Then $|z| - |a| \leqslant (1-|a|)/2$, whence $1-|a| \leqslant 2(1-|z|)$. By the definition of a, there holds the inequality $F(z) \leqslant F(a)$, i.e.,

$$u(z)^p \leqslant \left(\frac{1-|a|}{1-|z|}\right)^2 u(a)^p.$$

From these two inequalities it follows that $M_a \leqslant 4^{1/p} u(a)$, which completes the proof. □

Remark. Let $u : \mathbb{D} \mapsto \mathbb{R}$ be a Borel function that satisfies the condition

$$u(a)^p \leqslant \frac{C}{r^2} \int_{D_r(a)} u^p \, dA$$

(*)In the case of the modulus of a harmonic function, the proof is given by Kuran [47] and Fefferman and Stein [21]. Fefferman and Stein's proof can be found in Koosis [46] and Garnett [22]. Here we present two proofs from [75].

(†)If u is not bounded, then we apply (9.1) to the functions $u(\rho z)$.

for some $p > 0$, provided that $D_r(a) = \{z : |z - a| < r\} \subset \mathbb{D}$. Then this condition is satisfied for every $p > 0$.

Second proof. This proof is applicable only to the case where $u = |g|$, g harmonic. This time we start from the inequality

$$|\nabla g(a)| \leqslant \frac{K}{r} \sup\{|g(z)| : |z - a| < r\}, \tag{9.5}$$

which is valid whenever the disk $D_r(a) = \{z : |z - a| < r\}$ is contained in $\mathbb{D}$. (The constant K is independent of r and a.) From Lagrange's theorem and inequality (9.5) it follows that

$$u(a) \leqslant u(z) + K(t/r) \sup_{D_s(a)} u \ (s = t + r)$$

provided $z \in D_t(a)$ and $D_s(a) \subset \mathbb{D}$. The point a is chosen in the same way as in the first proof (with other hypothesis for u). Then, we choose t and r so that $s = (1 - |a|)/2$. Then $u(w)^p \leqslant 4u(a)^p$ for $w \in D_s(a)$ and therefore

$$u(a) \leqslant u(z) + K_p(t/r)u(a), \qquad z \in D_t(a),$$

where $K_p = 4^{1/p}$. Next we choose t and r so that $K_p(t/r) = 1/2$, and get the inequality $u(a)^p \leqslant 2^p u(z)^p$, $z \in D_t(a)$. Integrating this inequality in $z \in D_t(a)$, we get

$$t^2 u(a)^p \leqslant 2^p \int_{D_t(a)} u^p \, dA \leqslant 2^p.$$

Since $t = c_p(1 - |a|)$, we see that $u(0)^p \leqslant (1 - |a|)^2 u(a)^p \leqslant 2^p / c_p^2$, which concludes the proof. □

Proof of Theorem 9.1.2.

Let $p < 1$. We start from the inequality

$$|f'(0)|^p \leqslant C \int_{\varepsilon \mathbf{D}} |g(w)|^p \, dA(w) \qquad (\varepsilon = 1/2), \tag{9.6}$$

which follows from (9.5) and (9.1); write this as

$$|f'(0)|^p \leqslant C \int_{\varepsilon \mathbf{D}} |g(w)|^p \, d\tau(w) \qquad (\varepsilon = 1/2),$$

where

$$d\tau(w) = (1 - |w|^2)^{-2} dA(w).$$

The measure $d\tau$ invariant with respect to conformal automorphisms, which means that

$$\int_{\mathbf{D}} G \circ \varphi_a \, d\tau = \int_{\mathbf{D}} G \, d\tau$$

for every positive Borel function G, where

$$\varphi_a(z) = \frac{a-z}{1-\bar{a}z} \qquad (|a|<1).$$

The automorphism φ_a has the property $\varphi_a(\varphi_a(z)) \equiv z$. Hence, applying (9.6) to the function $f \circ \varphi_a$, we get

$$|f'(a)|^p(1-|a|^2)^p \leqslant C\int_{\varepsilon\mathbf{D}} |g(\varphi_a(w)|^p\, d\tau(w) = C\int_{H_\varepsilon(a)} |f(z)|^p\, d\tau(z), \tag{9.7}$$

where $H_\varepsilon(a) = \varphi_a(\varepsilon\mathbb{D}) = \{z : |\varphi_a(z)| < \varepsilon\}$. Integrating inequality (9.7) and changing the order of integration, we obtain

$$\int_{\mathbf{D}} |f'(a)|^p(1-|a|^2)^p\, dA(a) \leqslant C\int_{\mathbf{D}} |f(z)|^p\, d\tau(z)\int_{H_\varepsilon(z)} dA(a). \tag{9.8}$$

The set $H_\varepsilon(z)$ is a hyperbolic disk centered at z. The euclidean area of $H_\varepsilon(z)$ is "proportional" to $(1-|z|^2)^2$, so (9.8) implies

$$\int_{\mathbf{D}} |f'(a)|^p(1-|a|^2)^p\, dA(a) \leqslant C_p\int_{\mathbf{D}} |g(a)|^p\, dA(a). \tag{9.9}$$

Now it is enough to prove the inequality

$$\int_{\mathbf{D}} |f|^p\, dA \leqslant C_p\Big(|f(0)|^p + \int_{\mathbf{D}} |f'(a)|^p(1-|a|^2)^p\, dA(a)\Big); \tag{9.10}$$

indeed, (9.2) is implied by (9.9), (9.10) and (9.1). In order to prove (9.10), we start from

$$\begin{aligned}\int_{\mathbf{D}} |f|^p\, dA = \int_0^1 M_p^p(r,f)r\, dr &\leqslant 2\sum_{n=0}^{\infty} 2^{-n}M_p^p(r_n,f)\\ &= 2|f(0)|^p + \sum_{n=0}^{\infty} 2^{-n}\big(M_p^p(r_{n+1},f) - M_p^p(r_n,f)\big),\end{aligned}$$

where $r_n = 1-2^{-n}$. Hence, by Proposition 7.1.6,

$$\int_{\mathbf{D}} |f|^p\, dA \leqslant 2|f(0)|^p + C_p\sum_{n=0}^{\infty} 2^{-n}2^{-np}M_p^p(r_{n+1},f'),$$

from which (9.10) is obtained immediately.

In the case $p \geqslant 1$ we can proceed in a similar way; we only have to modify the proof of (9.10) (see [61]). Also, we can proceed as in the proof of Riesz' theorem 6.2.6, i.e., if $1 \leqslant p \leqslant 2$ apply the preceding result to the function f^2; etc. □

9.1.3 Remark As Theorem 9.1.2 shows, the space hA^p is "self-conjugate" for every $p > 0$, in contrast to the case of h^p, which is self-conjugate only for $p > 1$. Using Riesz' theorem on conjugate functions, we deduced that the sequence $e_n(re^{i\theta}) = r^{|n|}e^{in\theta}$ is a Shauder basis of h^p (Theorem 6.3.1), which implies that $\{e_n\}$ is a basis of hA^p for $p > 1$. However, this sequence is not a basis of hA^p for $p \leqslant 1$. More precisely, for every $p \leqslant 1$ there exists a function $f \in A^p$ the Taylor series of which does not converge in A^p.

9.1.4 Remark The space hA^p is equal to the direct sum of its subspaces A^p and $\bar{A}_0^p = \{\bar{f} : f \in A^p,\ f(0) = 0\}$, for every $p > 0$. This can be used to show that hA^p and A^p are isomorphic for every $p > 0$ (see 6.3.3).

9.2 The space h^p, $p < 1$

Recall that the harmonic Hardy space h^p, $0 < p < \infty$, is defined by the requirement

$$\|u\|_p := \sup_{r<1} M_p(r,u) < \infty.$$

In contrast to the case of Bergman spaces, the space h^p for $p < 1$ differs very much from its analytic analogue; for information we refer to [89]. However, there are similarities; for example:

9.2.1 Theorem *For $u \in h^p$ $(0 < p < 1)$ we have:*

$$|u(z)| \leqslant C_p(1-|z|)^{-1/p}\|u\|_p \quad (z \in \mathbb{D}), \tag{9.11}$$

$$|\widehat{u}(n)| \leqslant C_p(|n|+1)^{1/p-1}\|u\|_p \quad (n \in \mathbb{Z}). \tag{9.12}$$

Proof. By (9.3) we have

$$|u(re^{i\theta})|^p \leqslant C_p(1-r)^{-2}\int_{2r-1<|w|<1}|u(w)|^p\,dA(w) \leqslant 2C_p\int_{2r-1}^{1} M_p^p(\rho,u)\rho\,d\rho$$

(for $1/2 < r < 1$), which implies (9.11). Hence

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}|u(re^{i\theta})|\,d\theta &= \frac{1}{2\pi}\int_{-\pi}^{\pi}|u(re^{i\theta})|^p\,|u(re^{i\theta})|^{1-p}\,d\theta\\ &\leqslant \{M_\infty(r,u)\}^{1-p}\,\|u\|_p^p \leqslant (C_p)^{1-p}(1-r)^{-1/p+1}\,\|u\|_p^p.\end{aligned}$$

This gives (9.12) via the inequality

$$r^{|n|}|\widehat{u}(n)| \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi}|u(re^{i\theta})|\,d\theta, \quad r = 1-1/(|n|+1).$$

The proof is complete. □

In a similar way one proves the following estimates.

9.2.2 Theorem (Hardy/Littlewood [26]) *If* $u = \operatorname{Re} f$, $f \in H(\mathbb{D})$ *and* $u \in h^p$ $(0 < p \leqslant 1)$, *then*

$$M_p(r, f') \leqslant C_p(1-r)^{-1}\|u\|_p, \quad M_p(r,f) \leqslant C_p\Big(\log\frac{2}{1-r}\Big)^{1/p}\|u\|_p \qquad (0<r<1).$$

Two open problems

As far I know, the following two problems are still unsolved.

PROBLEM A. Whether there exists a function $u \in h^p$ $(p < 1)$ such that

$$|\widehat{u}(n)| \geqslant c(|n|+1)^{1/p-1} \qquad (c = \text{const} > 0,\ n \in \mathbb{Z})\,? \tag{9.13}$$

PROBLEM B. Whether there exists a function $f \in H(\mathbb{D})$ such that $u = \operatorname{Re} f \in h^p$ and

$$M_p^p(r,f) \geqslant c \log\frac{2}{1-r} \qquad (0<r<1)\,? \tag{9.14}$$

Hardy and Littlewood [26] proved that the answer is positive provided that $p = 1/N$, $N = 2, 3, \ldots$; their example is

$$u(z) = D^N P(z) = \frac{\partial^N P}{\partial \theta^N}(re^{i\theta}),$$

where P is the Poisson kernel. It is clear that u satisfies condition (9.13) for $p = 1/N$. That condition (9.14) is satisfied follows from the estimate (e.g., [89])

$$|D^N P(z)| \leqslant C\frac{1-|z|^2}{|1-z|^{N+1}}.$$

It should be noted that solving Problem A leads to solution of B. Namely, *condition* (9.13) *implies condition* (9.14). Indeed, if (9.13) is satisfied, then $|\widehat{f}(n)| \geqslant c(n+1)^{1/p-1}$, so the conclusion follows from the inequality

$$M_p^p(r,f) \geqslant c\sum_{n=0}^{\infty}(n+1)^{p-2}|\widehat{f}(n)|^p\, r^{np}$$

(see Theorem 7.5.1).

9.2.3 Exercise (Hardy/Littlewood) Let $u = \operatorname{Re} f$, $f \in H(\mathbb{D})$, $\alpha > 0$, and $0 < p < \infty$. Then the following three conditions are equivalent:

$$M_p(r,f) = O(1-r)^{-\alpha}; \quad M_p(r,u) = O(1-r)^{-\alpha}; \quad M_p(r,f') = O(1-r)^{-\alpha-1}.$$

9.3 Subharmonic behavior of smooth functions

In this section we present some results that are closely related to Theorem 9.1.1. Unless otherwise stated, we consider real-valued functions defined on a proper subdomain G of $\mathbb{C}$. For the proofs of slightly more general results we refer to [75, 77]. For generalizations to the case of the so called $\mathcal{M}$-harmonic functions, see [73, 34].

The class $SH(G)$. For a constant $K \geqslant 1$, let $SH_K(G)$ denote the class of non-negative, continuous functions u on G such that

$$u(z) \leqslant Kr^{-2} \int_{D_r(z)} u\, dm$$

whenever $D_r(z) := \{w\colon |w - z| < r\} \subset G$. We put $SH(G) = \bigcup_{K \geqslant 1} SH_K(G)$.[‡]

As remarked on page 138, the first proof of Theorem 9.1.1 gives the following result:

9.3.1 Proposition *Let $p > 0$. If $u \in SH_K(G)$, then $u^p \in SH_C(G)$, where C is a constant depending only on K and p.*

Classes of C^1 functions

The class $HC^1(G)$ The second proof of Theorem 9.1.1 (p. 139) was based on the property

$$|\nabla f(z)| \leqslant Kr^{-1} \sup_{D_r(z)} |f|, \qquad D_r(z) \subset G, \tag{9.15}$$

where $K \geqslant 1$ is a constant independent of $D_r(z) \subset G$. We denote by $HC^1_K(G)$ the class of all locally Lipschitz functions f on G satisfying (9.15). Note that (9.15) is implied by

$$|\nabla f(z)| \leqslant K|f(z)|/\delta_G(z), \quad \delta_G(z) = \operatorname{dist}(z, \partial G), \tag{9.16}$$

which is a restriction on the growth of f and is therefore stronger than (9.15).

9.3.2 Example (a) It is a simple but important fact that condition (9.16) is satisfied if f is a positive function harmonic in G.

(b) If $f = |g'|$, where g is a univalent function in $G = \mathbb{D}$, then the proof of Theorem 4.3.5 shows that

$$|\nabla f(z)| = |g''(z)| \leqslant \frac{3f(z)}{1 - |z|^2}.$$

The class $OC^1_K(G)$ is the subclass of $HC^1_{2K}(G)$ consisting of those f such that

$$|\nabla f(z)| \leqslant Kr^{-1}\mathcal{O}f(z,r), \quad D_r(z) \subset G,$$

where $\mathcal{O}f(z,r)$ is the oscillation of f on $D_r(z)$,

$$\mathcal{O}f(z,r) = \sup\{\, |f(w) - f(z)| \,:\, w \in D_r(z)\,\}.$$

9.3.3 Example If a function $f\colon G \mapsto \mathbb{R}$ is convex or harmonic, then $f \in OC^1_K(G)$ for some K independent of f. In particular the function $f(x+iy) = e^x$ is in $HC^1(G)$, where G is the right half-plane, but f does not satisfy (9.16).

[‡]In defining other classes we proceed in a similar way.

9.3.4 Example As a further example, we have that $|f| \in OC^1(G)$, if f is analytic in G. (See Lemma 10.2.4.)

9.3.6 Corollary *Let $p > 0$. A function $f \in C^1(G)$ belongs to $HC^1(G)$ if and only if there is a constant K such that*

$$|\nabla f(z)|^p \leqslant K r^{-2-p} \int_{D_r(z)} |f|^p \, dm, \quad 0 < r < \delta_G(z).$$

Let

$$\mathcal{O}_p f(z,r) = \left\{ \frac{1}{|D_r(z)|} \int_{D_r(z)} |f(w) - f(z)|^p \, dm(w) \right\}^{1/p},$$

the L^p-oscillation over $D_r(z)$.

9.3.7 Corollary *Let $p > 0$. A function f belongs to $OC^1(G)$ if and only if $|\nabla f(z)| \leqslant K r^{-1} \mathcal{O}_p f(z,r)$, $0 < r < \delta_G(z)$, for some constant K.*

This is deduced from the preceding corollary by considering the functions $f -$ const.

Proof of Theorem 9.3.5. The proof of (a) is the same as the second proof of Theorem 9.1.1, p. 139. It remains to prove that $f \in OC^1(G)$ implies $|\nabla f| \in SH(G)$. Let $f \in OC^1_K(G)$. By Theorem 9.3.1, it suffices to prove that, for some q, the function $|\nabla f|^q$ belongs to $SH(G)$. This can be reduced to proving that

$$|\nabla f(0)|^q \leqslant C \int_{\mathbb{D}} |\nabla f|^q dm$$

provided that $\mathbb{D} \subset G$. By Corollary 9.3.7, we have

$$|\nabla f(0)| \leqslant K \int_{\mathbb{D}} |f(z) - f(0)| \, dm(z).$$

On the other hand,

$$|f(z) - f(0)| \leqslant |z| \int_0^1 |\nabla f(rz)| \, dr,$$

whence

$$\int_{\mathbb{D}} |f(z) - f(0)| \, dm(z) \leqslant \int_0^1 dr \int_{\mathbb{D}} |\nabla f(rz)| \, |z| \, dm(z).$$

Hence, by the change $z = w/r$ and Fubini's theorem,

$$|\nabla f(0)| \leqslant K \int_{\mathbb{D}} |\nabla f(w)| \, dm(w) \int_{|w|}^1 r^{-3} |w| \, dr \leqslant K \int_{\mathbb{D}} |\nabla f(w)| |w|^{-1} dm(w).$$

Now the required inequality is proved by Hölder's inequality with the indices $q = 3$ and $q' = 3/2$, using the fact that the function $w \mapsto |w|^{-1}$ belongs to the space $L^{3/2}(\mathbb{D}, dm)$. □

Classes of C^2-functions

The class $HC^2(G)$ consists of those $f \in C^2(G)$ for which

$$|\Delta f(z)| \leqslant Kr^{-1} \sup_{D_r(z)} |\nabla f| + K_0 r^{-2} \sup_{D_r(z)} |f| \tag{9.17}$$

for some constants K, K_0. The condition

$$|\Delta f(z)| \leqslant K|\nabla f(z)|\delta_G(z)^{-1} + K_0|f(z)|\delta_G(z)^{-2}$$

implies (9.17) with the same values of K and K_0.

The class $OC^2(G)$ consists of those $f \in C^2(G)$ such that

$$|\Delta f(z)| \leqslant Kr^{-1} \sup_{D_r(z)} |\nabla f| \tag{9.18}$$

for some constant K. In particular, $OC^2(G)$ contains every function f for which

$$|\Delta f(z)| \leqslant K|\nabla f(z)|/\delta_G(z), \qquad z \in G.$$

By Lagrange's theorem, f belongs to $OC^2(G)$ iff

$$|\Delta f(z)| \leqslant Kr^{-1} \sup_{D_r(z)} |\nabla f| + K_0 r^{-2} \mathcal{O}(f, r) \quad \text{for some } K \text{ and } K_0.$$

9.3.8 Example Condition (9.18) is satisfied if f is a harmonic function on an arbitrary domain G. Let f be an eigenfunction of Δ, i.e., $\Delta f \equiv \lambda f$ for some constant λ. Assuming that the closure of $r\mathbb{D} = D_r(0)$ is contained in G, we have

$$2\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d}{dr} f(re^{i\theta})\, d\theta = r^{-1} \int_{r\mathbf{D}} \Delta f\, dm.$$

Hence

$$r\Delta f(0) = 2\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d}{dr} f(re^{i\theta})\, d\theta - r^{-1} \int_{r\mathbf{D}} (f - f(0))\, dm,$$

and hence

$$|\Delta f(0)| \leqslant 2r^{-1} \sup_{r\mathbf{D}} |\nabla f| + r|\lambda| \sup_{r\mathbf{D}} |\nabla f|.$$

Applying this to the function $w \mapsto f(w+z)$ we conclude that $f \in OC^1(G)$ provided G is bounded. On the other hand, if $f(x+iy) = \sin x$, and G is the right half-plane, then $\Delta f = -f$ but f is not in $OC^2(G)$.

9.3.9 Theorem *The following inclusions hold:* (a) $HC^2(G) \subset HC^1(G)$; *and* (b) $OC^2(G) \subset OC^1(G)$.

9.3.10 Corollary *A function $f \in C^2(G)$ belongs to $HC^2(G)$ if and only if there is a constant K such that $|\Delta f(z)| \leqslant Kr^{-2} \sup_{D_r(z)} |f|$.*

The proof of Theorem 9.3.9 is based on the following lemmas.

9.3.11 Lemma *If $f\colon D_r(z) \to \mathbb{R}$ is a C^2-function, then*

$$|\nabla f(z)| \leqslant 2r^{-1} \sup_{D_r(z)} |f| + (2/3)\, r \sup_{D_r(z)} |\Delta f|. \tag{9.19}$$

This inequality is a consequence of the formula

$$\frac{\partial f}{\partial z}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-i\theta}\, d\theta - \frac{1}{4\pi} \int_{\mathbb{D}} \Delta f(w)(1 - |w|^2) w^{-1}\, dm(w),$$

where f is a C^2-function defined in a neighborhood of the closed unit disk. The verification of this formula can be reduced to the case of $f(z) = z^m \bar{z}^n$, where m, n are nonnegative integers.

9.3.12 Lemma *Let F_1, F_2 and F_3 be nonnegative, continuous functions on G such that, for some constant K,*

$$F_1(z)/K \leqslant r^{-1} \sup_{D_r(z)} F_2 + r \sup_{D_r(z)} F_3 \qquad \text{and} \tag{9.20}$$

$$F_3(z)/K \leqslant r^{-1} \sup_{D_r(z)} F_1 + r^{-2} \sup_{D_r(z)} F_2 \tag{9.21}$$

whenever $D_r(z) \subset G$. Then there is a constant $C = C(K)$ such that

$$F_1(z) \leqslant Cr^{-1} \sup_{D_r(z)} F_2. \tag{9.22}$$

Proof. By translations the proof of (9.22) reduces to the case $z = 0$. Let the closure of $D_\varepsilon(0)$ be contained in G and $F_2 \leqslant 1$ on $D_\varepsilon(0)$. (In the general case we consider the functions F_i/A, where A is chosen so that $F_2(z) \leqslant A$ for all $z \in D_\varepsilon(0)$.) Choose $z \in D_\varepsilon(0)$ so that $F_1(w)(\varepsilon - |w|) \leqslant F_1(z)(\varepsilon - |z|)$ for all $w \in D_\varepsilon(0)$. This implies that $F_1(w) \leqslant 2F_1(z)$ for $w \in D_\delta(z)$, where $\delta = (\varepsilon - |z|)/2$. Now we use the hypotheses to find w in $D_\delta(z)$ so that $F_1(z)/K \leqslant r^{-1} + (Kr/t)F_1(w) + Krt^{-2}$ for all r, $t > 0$ such that $r + t = \delta$, which implies $F_1(z)/K \leqslant r^{-1} + (2Kr/t)F_1(z) + Krt^{-2}$. Now choose r, t so that $r + t = \delta$ and $2Kr/t = 1/2K$, which implies that $r = c_1(\varepsilon - |z|)$, $t = c_2(\varepsilon - |z|)$ for some $c_i = c_i(K)$, to obtain $F_1(z)/K \leqslant F_1(z)/2K + K_1(\varepsilon - |z|)^{-1}$, where $K_1 = c_1^{-1} + c_1 c_2^{-2}$. Hence $F_1(0)\varepsilon \leqslant F_1(z)(\varepsilon - |z|) \leqslant 2KK_1$, and this concludes the proof. □

Proof of Theorem 9.3.9. Let f satisfy (9.17). We may assume that $K \geqslant 2$ and $K_0 \geqslant 2$. Define the functions $F_1(z) = |\nabla f(z)|$, $F_2(z) = |f(z)|$, $F_3(z) =$

$|\Delta f(z)|$. Then (9.20) is satisfied because of (9.19), and (9.21) is satisfied because of (9.17). Hence $f \in HC^1(G)$, by Lemma 9.3.12. This proves assertion (a).

To prove (b) let $f \in OC^2(G)$. Applying (a), together with its proof, to the functions $f - c$ we find a constant K_1 independent of z, r, c so that $|\nabla f(z)| \leqslant K_1 r^{-1} \sup_{D_r(z)} |f - c|$. Finally we take $c = f(z)$ to finish the proof. □

Vector-valued functions The preceding notions and results can be extended to vector-valued functions. For example, if $f = (f_1, f_2)$ is a function from G to $\mathbb{R}^2$, then we replace $|\nabla f(z)|$ by $\|Df(z)\|$, where $Df(z) \colon \mathbb{R}^2 \mapsto \mathbb{R}^2$ is the derivative of f at z. The Laplacian of f is defined as $\Delta f = (\Delta f_1, \Delta f_2)$. Applying Lemma 9.3.11 to the functions $af_1 + bf_2$, $a^2 + b^2 = 1$, we find that

$$\|Df(z)\| \leqslant 2r^{-1} \sup_{D_r(z)} |f| + (2/3)\, r \sup_{D_r(z)} |\Delta f|.$$

It turns out that Theorem 9.3.9 remains valid. As an application we note a sufficient condition for a C^3-function to be in $OC^2(G)$.

9.3.13 Theorem *A real valued C^3-function f belongs to $OC^2(G)$ if there are constants K_1 and K_2 such that*

$$|\nabla(\Delta f)(z)| \leqslant K_1 r^{-1} \sup_{D_r(z)} \nabla_2(f) + K_2 r^{-2} \sup_{D_r(z)} |\nabla f|. \tag{9.23}$$

Here

$$\nabla_2(f) = \left\{ \left|\frac{\partial^2 f}{\partial x^2}\right|^2 + \left|\frac{\partial^2 f}{\partial y^2}\right|^2 + \left|\frac{\partial^2 f}{\partial x \partial y}\right|^2 \right\}^{1/2}.$$

Proof. Suppose that a real-valued function f satisfies (9.23). Since $\nabla(\Delta f) = \Delta(\nabla f)$ and $\|D(\Delta f)\| \asymp \nabla_2(f)$, we see that condition (9.23) means that $\nabla f \in HC^2(G)$. Therefore (9.23) implies $\nabla f \in HC^1(G)$, by Theorem 9.3.9, which means $\|D(\nabla f)(z)\| \leqslant Kr^{-1} \sup_{D_r(z)} |\nabla f|$ for some constant K. Since obviously $|\Delta f| \leqslant \mathrm{const} \cdot \|D(\nabla f)\|$, it follows that $f \in OC^2(G)$. □

Remark. Condition (9.23) implies the formally stronger condition

$$|\nabla(\Delta f)(z)| \leqslant Kr^{-2} \sup_{D_r(z)} |\nabla f|.$$

9.3.14 Corollary *A C^4-function $f \colon G \to \mathbb{R}$ belongs to $OC^2(G)$ if so does Δf. Consequently a C^∞-function f belongs to $OC^2(G)$ if so does $\Delta^k f$ for some integer k. In particular every polyharmonic function of finite order belongs to OC^2.*

A function f is **polyharmonic** of order k, where k is a positive integer, if $\Delta^k f \equiv 0$. For the theory of polyharmonic functions we refer to [5].

Proof. Let $\Delta f \in OC^2$. Then $\Delta f \in HC^1$, by Theorem 9.3.9, which means that $|\nabla(\Delta f)(z)| \leqslant Kr^{-1} \sup_{D_r(z)} |\Delta f|$. Now the desired conclusion follows from Theorem 9.3.13. □

10 Lipschitz spaces

In this chapter we are concerned with some results which relate the modulus of smoothness of n-th order of a function $g \in C(\mathbb{T})$ with the n-th tangential derivative of the Poisson integral of g. In Section 10.1 we consider the case $n = 1$ (Theorem 10.1.1); as an application we prove Privalov's theorem on conjugate functions (Theorem 10.1.3 and 10.1.4). In Section 10.2 we use Section 10.1 to prove some results on the Lipschitz condition for the modulus of an analytic function. The case $n \geqslant 2$ is considered in the next two sections.

10.1 Lipschitz spaces of first order

The space $\Lambda_\alpha(K)$ If K is a *bounded* subset of $\mathbb{C}$, then, by definition, $\Lambda_\alpha(K)$ $(0 < \alpha \leqslant 1)$ is the set of all complex-valued functions g on K such that

$$|g(z) - g(w)| \leqslant C\,|z - w|^\alpha \qquad (z,\, w \in K),$$

where C is a constant independent of z, w.

The space $\mathrm{Lip}(\omega, K)$ More generally, let ω be a **majorant**, i.e., a continuous increasing function on $[0, t_0]$, where t_0 is large enough, such that $\omega(0) = 0$ and that $\omega(t)/t$ decreases $(t > 0)$. Then the space $\mathrm{Lip}(\omega, K)$ is defined by the requirement

$$|g(z) - g(w)| \leqslant C\,\omega(|z - w|). \tag{10.1}$$

The norm is given by $C_g + |g(a)|$, where $C_g = C$ $(\geqslant 0)$ is the smallest constant satisfying (10.1) and a is any fixed point from K; with this norm the space $\mathrm{Lip}(\omega, K)$ is Banach. Since $\max_K |g| \leqslant |g(a)| + C_g\,\omega(\operatorname{diam} K)$, we see that the inclusion $\mathrm{Lip}(\omega, K) \subset L^\infty(K)$ is continuous.

Lipschitz condition and the tangential derivative

In the case $K = \mathbb{T}$ condition (10.1) is equivalent to $|g(e^{it}) - g(e^{i\theta})| \leqslant C\,\omega(|t - \theta|)$, t, $\theta \in \mathbb{R}$ (the value of C need not be the same). In particular, this means that the "classical" Lipschitz space $\Lambda_1(\mathbb{T})$ consists of absolutely continuous functions $g \in C(\mathbb{T})$ with the bounded derivative $(d/d\theta)g(e^{i\theta})$, and that therefore is isomorphic to $L^\infty(\mathbb{T})$. Since

$$\int_{-\pi}^{\pi} P(r, \theta - t)\,\frac{d}{dt} g(e^{it})\,dt = \frac{\partial}{\partial\theta}\int_{-\pi}^{\pi} P(r, \theta - t) g(e^{it})\,dt,$$

we can conclude that g belongs to $\Lambda_1(\mathbb{T})$ iff $D(P[g])$ belongs to h^∞, where D is the operator of "tangential" differentiation, $(Du)(re^{i\theta}) := \partial u(re^{i\theta})/\partial\theta$. This can be generalized in the following way.

10.1.1 Theorem *Let ω be a majorant such that*

$$\frac{\omega(t)}{t^\alpha} \quad \textit{is almost increasing for some } \alpha > 0. \tag{10.2}$$

Then a function u, harmonic in $\mathbb{D}$, is equal to the Poisson integral of some function $g \in \operatorname{Lip}(\omega, \mathbb{T})$ iff there exists a constant C such that

$$|Du(z)| \leqslant C\,\frac{\omega(1-|z|)}{1-|z|} \qquad \textit{for all } z \in \mathbb{D}. \tag{10.3}$$

A real function φ, defined on some interval, is called **almost increasing** if there exists a constant C such that $x < y$ implies $\varphi(x) < C\varphi(y)$. (An almost decreasing function is defined similarly.)

Let $A(\mathbb{D})$ denote the **disk-algebra**, i.e., the set of functions that are analytic in $\mathbb{D}$ and have a continuous extension to the boundary. Since $Df(re^{i\theta}) = ire^{i\theta} f'(re^{i\theta})$, $f \in A(\mathbb{D})$, we have the following consequence.

10.1.2 Corollary *Let ω be as in the theorem and let $f \in A(\mathbb{D})$. The boundary function f_* belongs to $\operatorname{Lip}(\omega, \mathbb{T})$ iff*

$$|f'(z)| \leqslant C\,\frac{\omega(1-|z|)}{1-|z|} \qquad (z \in \mathbb{D}).$$

This last condition implies $f \in A(\mathbb{D})$.

Theorem 10.1.1 is an immediate consequence of the following assertions which will be proved later on in a more general situation (see Propositions 10.4.4 and 10.4.5); the direct proof is rather simple.

If $u = P[g]$, $g \in C(\mathbb{T})$, then $M_\infty(r, Du) \leqslant C(1-r)^{-1}\omega(g, 1-r)$, $0 < r < 1$, where $\omega(g,t)$ is the modulus of continuity of g.

On the other hand, if $u \in h(\mathbb{D})$ and $\int_0^1 M_\infty(r, Du)\,dr < \infty$, then $u = P[g]$ for some $g \in C(\mathbb{T})$ satisfying

$$\omega(g,t) \leqslant C \int_{1-t}^{1} M_\infty(r, Du)\,dr, \qquad 0 < t < 1.$$

Privalov's theorem on conjugate functions

The condition $g \in \Lambda_1(\mathbb{T})$ does not imply that the radial derivative $\partial P[g]/\partial r$ is bounded (in contrast to the tangential derivative) because the functions $\partial u/\partial\theta$ and $r\partial u/\partial r$ represent an arbitrary pair of harmonic conjugates. However, under an additional hypothesis on the majorant we have the following theorem of Privalov:

10.1.3 Theorem *Let (10.2) be satisfied and suppose that*

$$\omega(t)/t^\beta \quad \textit{is almost decreasing for some } \beta < 1. \tag{10.4}$$

If $u = P[g]$ and $g \in \operatorname{Lip}(\omega, \mathbb{T})$, then the conjugate function $\tilde{u}$ has a continuous extension to $\overline{\mathbb{D}}$ and its boundary function belongs to $\operatorname{Lip}(\omega, \mathbb{T})$.

In other words,

If conditions (10.2) *and* (10.4) *are satisfied, then the Hilbert operator maps* $\mathrm{Lip}(\omega, \mathbb{T})$ *to* $\mathrm{Lip}(\omega, \mathbb{T})$.

Proof of Theorem 10.1.3. Let u be real-valued. The function $v = Du$ is harmonic and therefore $|\nabla v(z)| \leqslant 2R^{-1} \sup\{ |v(w)| : |w - z| < R\}$, where $R = (1 - |z|)/2$. Using the hypotheses and Theorem 10.1.1, we get

$$|\nabla v(z)| \leqslant C \frac{\omega(1 - |z|)}{(1 - |z|)^2}.$$

Since $|\nabla v| \geqslant |\partial v/\partial\theta|/r = |\partial^2 u/\partial\theta^2|/r$ and $\partial^2 u/\partial\theta^2 = -r^2\, \partial^2 u/\partial r^2$, it follows that

$$\left|\frac{\partial^2 u}{\partial r^2}(re^{i\theta})\right| \leqslant C \frac{\omega(1 - r)}{(1 - r)^2}, \qquad 0 < r < 1,$$

where C is independent of $re^{i\theta}$. Integrating this from $r = 0$ and using (10.4) we get

$$\left|\frac{\partial u}{\partial r}(re^{i\theta})\right| \leqslant C \frac{\omega(1 - r)}{1 - r},$$

which together with (10.3) gives

$$|\nabla \widetilde{u}(z)| = |\nabla u(z)| \leqslant C \frac{\omega(1 - |z|)}{1 - |z|}. \tag{10.5}$$

Finally, (10.2) and (10.5) imply $\widetilde{u} \in \mathrm{Lip}(\omega, \mathbb{D})$ (see Lemma 10.1.6 below). □

According to the above proof, we have the following theorem on extension of a Lipschitz condition from $\mathbb{T}$ to $\mathbb{D}$.

10.1.4 Theorem (a) *Let* ω *satisfy* (10.2) *and* (10.4). *Then the function* g *belongs to* $\mathrm{Lip}(\omega, \mathbb{T})$ *iff the Poisson integral of* g *belongs to* $\mathrm{Lip}(\omega, \mathbb{D})$.

(b) *If* $f \in A(\mathbb{D})$, *then* (10.2) *is sufficient for the validity of the implication*

$$f_* \in \mathrm{Lip}(\omega, \mathbb{T}) \;\Rightarrow\; f \in \mathrm{Lip}(\omega, \mathbb{D}).$$

10.1.5 Remark There are cases where a Lipschitz condition on the circle extends to the disk with saving the corresponding Lipschitz constant. For example, if $f \in A(\mathbb{D})$, $0 < \alpha \leqslant 1$ and $|f(\zeta) - f(\eta)| \leqslant |\zeta - \eta|^\alpha$, $\zeta, \eta \in \mathbb{T}$, then $|f(z) - f(w)| \leqslant |z - w|^\alpha$, $z, w \in \mathbb{D}$. The latter is equivalent to

$$|f(z) - f(w)| \leqslant \frac{|z - w|}{|1 - \bar{w}z|^{1-\alpha}} \qquad (z,\, w \in \mathbb{D}),$$

and this implies $|f'(z)| \leqslant (1 - |z|^2)^{\alpha - 1}$, $z \in \mathbb{D}$.

10.1.6 Lemma *If a* C^1*-function* $u\colon \mathbb{D} \mapsto \mathbb{C}$ *satisfies* (10.5) *and* ω *satisfies the (Dini) condition*

$$\omega_1(x) = \int_0^x \frac{\omega(t)}{t}\, dt < \infty, \qquad x > 0, \tag{10.6}$$

then $u \in \mathrm{Lip}(\omega_1, \mathbb{D})$.

Proof (cf. [84, Lemma 6.4.8]). Let

$$|\nabla u(z)| \leqslant \frac{\omega(1-|z|)}{1-|z|}, \qquad z \in \mathbb{D}.$$

Let $|a| \leqslant |b| \leqslant 1$. By Lagrange's theorem,

$$|u(a) - u(b)| \leqslant \frac{\omega(1-|c|)}{1-|c|}\,|a-b|,$$

where $c = (1-\lambda)a + \lambda b$ for some $\lambda \in (0,1)$. Since $|c| \leqslant |b|$ and $\omega(t)/t$ decreases, we see that

$$\frac{\omega(1-|c|)}{1-|c|} \leqslant \frac{\omega(1-|b|)}{1-|b|},$$

hence $|u(a) - u(b)| \leqslant \omega(|a-b|) \leqslant \omega_1(|a-b|)$, under the condition $|a-b| \leqslant 1-|b|$.

If $1-|b| \leqslant |a-b| \leqslant 1-|a|$, then $|u(a)-u(b)| \leqslant |u(a)-u(b')| + |u(b')-u(b)|$, where $b' = (1-\delta)b/|b|$, $\delta = |a-b|$. Using Lagrange's theorem as above we get

$$|u(a) - u(b')| \leqslant \frac{\omega(1-|b'|)}{1-|b'|}\,|a-b'| = \frac{\omega(\delta)}{\delta}|a-b'| \leqslant \omega(\delta) \leqslant \omega_1(\delta).$$

In the case of $|u(b') - u(b)|$, we have

$$|u(b') - u(b)| \leqslant \int_{|b'|}^{|b|} \frac{\omega(1-t)}{1-t}\,dt \leqslant \int_{1-\delta}^{1} \frac{\omega(1-t)}{1-t}\,dt = \omega_1(\delta).$$

Finally, if $\delta > 1-|a|$, we use the inequality $|u(a)-u(b)| \leqslant |u(a)-u(a')| + |u(a') - u(b')| + |u(b')-u(b)|$, where $a' = (1-\delta)a/|a|$, and then proceed in a similar way as above. □

10.1.7 Remark If ω satisfies (10.6), then ω_1 is a concave majorant. Let

$$\omega_2(x) = x\int_x^1 \frac{\omega(t)}{t^2}\,dt, \qquad 0 < x < 1.$$

This function is not increasing but has the properties: ω_2 is concave, $\omega_2(t)/t$ is decreasing and $\omega_2(0+) = 0$. It follows that ω_2 is a majorant near 0. We have

$$\int_0^x \frac{\omega_2(t)}{t}\,dt = \omega_1(x) + \omega_2(x), \qquad 0 < x < 1.$$

Then Lemma 10.1.6 and the proof of Privalov's theorem yield:

Suppose that ω satisfies the Dini condition. (a) *If $u = P[g]$ and $g \in \mathrm{Lip}(\omega, \mathbb{T})$, then the conjugate function $\widetilde{u}$ has a continuous extension to $\overline{\mathbb{D}}$, and, moreover, u and $\widetilde{u}$ belong to* $\mathrm{Lip}(\omega_3, \mathbb{D})$, *where* $\omega_3(x) = \omega_1(x) + \omega_2(x)$.

(b) *If $f \in A(\mathbb{D})$ and $f_* \in \mathrm{Lip}(\omega, \mathbb{T})$, then $f \in \mathrm{Lip}(\omega_1, \mathbb{D})$.*

10.1.8 Remark A majorant ω is said to be **regular** if there exists a constant C such that

$$\int_0^x \frac{\omega(t)}{t}\,dt + x\int_x^1 \frac{\omega(t)}{t^2}\,dt \leqslant C\omega(x) \qquad (0 < x < 1).$$

It is easily verified that condition (10.2)&(10.4) implies that ω is regular. The converse is true as well. Moreover, ω satisfies (10.2) iff $\omega_1(x) \leqslant C\omega(x)$ for some constant C; and ω satisfies (10.4) iff $\omega_2(x) \leqslant C\omega(x)$ for some constant C.

10.2 Lipschitz condition for the modulus

A function $f \in H(\mathbb{D})$ satisfies the condition $|f(z) - f(w)| \leqslant |z - w|$ in $\mathbb{D}$ iff $|f'| \leqslant 1$ in $\mathbb{D}$. On the other hand, the corresponding Lipschitz condition for $|f|$ is satisfied iff $\big|\nabla|f|\big| \leqslant 1$. Since $\big|\nabla|f|\big| = |f'|$, we conclude that there holds the relation

$$f \in \Lambda_1(\mathbb{D}) \iff |f| \in \Lambda_1(\mathbb{D}), \qquad f \in H(\mathbb{D}).$$

This is the simplest case of the following theorem.

10.2.1 Theorem *Let ω satisfy the Dini condition* (10.6). *If $f \in H(\mathbb{D})$ and $|f| \in$ Lip$(\omega, \mathbb{D})$, then $f \in$ Lip$(\omega_1, \mathbb{D})$.*

The theorem states, in particular, that if $|f| \in \operatorname{Lip}(\omega, \mathbb{D})$, and ω satisfies (10.6), then $f \in A(\mathbb{D})$. On the other hand, there exists a function $f \in H(\mathbb{D}) \setminus A(\mathbb{D})$ such that the function $|f|$ has a continuous extension to the closed unit disk. To show this, we use the known fact that there exists a bounded analytic function $u + iv$ such that u is continuous on $\overline{\mathbb{D}}$, while v has no continuous extension to $\overline{\mathbb{D}}$. Then there are a point $\zeta \in \mathbb{T}$, two sequences $\{z_n\} \subset \mathbb{D}$ and $\{w_n\} \subset \mathbb{D}$ tending to ζ, and two points $a, b \in \mathbb{C}$ $(a \neq b)$ such that $v(z_n) \to a$ and $v(w_n) \to b$. We can assume that $e^{ia} \neq e^{ib}$ since otherwise we can consider the function $(u + iv)/\lambda$ for a suitable $\lambda > 0$. Then the desired function is $f = \exp(u + iv)$.

As a consequence of the above theorem we have the following result of Dyakonov.

10.2.2 Theorem [20] *Let ω satisfy* (10.2). *A function $f \in H(\mathbb{D})$ belongs to the space* Lip$(\omega, \mathbb{D})$ *iff $|f|$ belongs to the same space.*

Before stating another theorem of Dyakonov recall that the function $|f|$ is subharmonic and consequently $P[|f_*|](z) - |f(z)| \geqslant 0$, for all $z \in \mathbb{D}$. We also know that

$$P[|f_*|](z) - |f(z)| \to 0 \quad \text{as} \quad |z| \to 1, \tag{$\dagger$}$$

so it is a natural question how fast the convergence in (†) can be. It turns out that this is closely related to Lipschitz condition for f, i.e., to growth of the first derivative.

10.2.3 Theorem (Dyakonov [20]) *Let ω satisfy* (10.2) *and* (10.4), *$f \in A(\mathbb{D})$. Then the following conditions are equivalent:*

(i) $|f_*| \in \mathrm{Lip}(\omega, \mathbb{T})$ *and* $|f(e^{i\theta})| - |f(re^{i\theta})| \leqslant C\omega(1-r)$ *for some constant* C;

(ii) $|f_*| \in \mathrm{Lip}(\omega, \mathbb{T})$ *and* $P[|f_*|](z) - |f(z)| \leqslant C\omega(1-|z|)$ *for some constant* C;

(iii) $f \in \mathrm{Lip}(\omega, \mathbb{D})$.

Dyakonov's proofs are based on theorems on pseudoanalytic continuation and theorems on division by inner functions. Here we present the proof from [78]. The key is in connection between the modulus of the first derivative and the oscillation of the modulus of the function.

10.2.4 Lemma *Let* $D_z = \{w : |w-z| \leqslant 1-|z|\}$, $f \in A(\mathbb{D})$. *Then*

$$\frac{1}{2}(1-|z|)|f'(z)| \leqslant \sup_{w\in D_z} \big(|f(w)| - |f(z)|\big) \qquad (z \in \mathbb{D}).$$

Proof. Let $M_z = \sup\{|f(w)|\colon w \in D_z\}$. If $z = 0$ and $M_0 = 1$, then Schwarz' lemma gives $|f'(0)| \leqslant 1 - |f(0)|^2 \leqslant 2(1-|f(0)|)$, which is the required inequality in a special case. In the general case we apply this special case to the function $F(\zeta) = f(z + \zeta(1-|z|))/M_z$, $\zeta \in \mathbb{D}$. □

Proof of Theorem 10.2.1 Assuming $|f| \in \mathrm{Lip}(\omega, \mathbb{D})$, we have

$$|f(w)| - |f(z)| \leqslant C\omega(|w-z|) \leqslant C\omega(1-|z|)$$

for every $z \in \mathbb{D}$ and $w \in D_z$. Taking the supremum over $w \in D_z$ and using Lemma 10.2.4, we get $|f'(z)|(1-|z|) \leqslant 2C\omega(1-|z|)$. Now the result follows from Lemma 10.1.6. □

Proof of Theorem 10.2.3. [78] The implication (iii) ⇒ (i) is trivial. Consider the implication (i) ⇒ (ii). Assuming (i), let $h(z) = P[|f_*|](z)$, $f \in \mathrm{Lip}(\omega, \mathbb{D}) =: X$ and $|f_*| \in \mathrm{Lip}(\omega, \mathbb{T})$. Then $h \in X$, by Theorem 10.1.4, and $|f| \in X$ because $f \in X$. Hence

$$\begin{aligned} h(z) - |f(z)| &= h(z) - |f(z/|z|)| + |f(z/|z|)| - |f(z)| \\ &= h(z) - h(z/|z|) + |f(z/|z|)| - |f(z)| \\ &\leqslant C\omega(1-|z|) \end{aligned}$$

for $z \in \mathbb{D} \setminus \{0\}$, which implies (ii).

In order to prove that (ii) implies (iii), we start from the inequality

$$|f(w)| - |f(z)| \leqslant h(w) - |f(z)| = h(w) - h(z) + h(z) - |f(z)|,$$

valid for $z \in \mathbb{D}$ and $w \in D_z$. From the hypothesis $|f_*| \in \mathrm{Lip}(\omega, \mathbb{T})$ and Theorem 10.1.4 it follows that

$$h(w) - h(z) \leqslant C\omega(|w-z|) \leqslant C\omega(1-|z|), \qquad w \in D_z.$$

By (ii), we have $h(z) - |f(z)| \leqslant C\omega(1-|z|)$, so we get

$$|f(w)| - |f(z)| \leqslant C\omega(1-|z|), \qquad w \in D_z.$$

The proof is now concluded as in the case of Theorem 10.2.2. □

10.3 Lipschitz spaces of higher order

Moduli of smoothness If h is a complex-valued function defined on $\mathbb{R}$, then $\Delta_t^n h$ (n is a positive integer, $t \in \mathbb{R}$) denotes the n-th symmetric difference with step t:

$$\Delta_t^1 h(\theta) = h(\theta + t) - h(\theta) \quad (\theta \in \mathbb{R}), \quad \text{and} \quad \Delta_t^n h = \Delta_t^1 \Delta_t^n h \quad (n \geqslant 2).$$

In particular, $\Delta_t^2 h(\theta) = h(\theta + 2t) - 2h(\theta + t) + h(\theta)$. In the general case there holds the formula

$$\Delta_t^n h(\theta) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} h(\theta + kt).$$

If g is a function on the unit circle, then $\Delta_t^n g$ is defined by $\Delta_t^n g(e^{i\theta}) = \Delta_t^n h(\theta)$, where $h(\theta) = g(e^{i\theta})$. For fixed n and t, Δ_t^n is a linear operator which preserves $C(\mathbb{T})$; we have $\|\Delta_t^n g\| \leqslant 2^n \|g\|$, where $\|\cdot\| = \|\cdot\|_\infty$ denotes the max-norm in $C(\mathbb{T})$. The modulus of smoothness of order n is defined by $\omega_n(g,t) = \sup\{\|\Delta_s^n g\| : |s| < t\}$, $t > 0$, $g \in C(\mathbb{T})$.

Lipschitz spaces Let ϕ be a positive function on $(0, \pi]$ and let n be an integer. The Lipschitz space $\mathrm{Lip}_n(\phi)$ consists of those functions $g \in C(\mathbb{T})$ for which

$$\omega_n(g,t) = O(\phi(t)), \qquad t \to 0;$$

the norm is defined by

$$\|g\|_{\phi,n} = |\widehat{g}(0)| + \sup_{0<t\leqslant\pi} \frac{\omega_n(g,t)}{\phi(t)}.$$

The analogous space defined by the "little oh" condition will be denoted by $\mathrm{lip}_n(\phi)$.

Zygmund classes In the case where $n = 2$ and $\omega(t) = t$ the spaces $\mathrm{Lip}_n(\phi)$ and $\mathrm{lip}_n(\phi)$ are known as the Zygmund classes (cf. [100, II.§§3,4]) and are denoted by Λ_* and λ_*, respectively. A function $g \in \lambda_*$ is called a **smooth function** because the existence of the left derivative at a point implies the existence of the right as well as that they are equal. The set of points where g is differentiable may be of measure zero although must be everywhere dense and of the power of the continuum. See [100, p. 43–48]. The **Weierstrass function**

$$g(\theta) = \sum_{n=1}^{\infty} b^{-n\alpha} \cos(b^n \theta),$$

where $b = 2, 3, \ldots$ and $\alpha > 0$, belongs to Λ_* for $\alpha = 1$, and to $\Lambda_\alpha(\mathbb{T})$ for $\alpha < 1$. It is known that for $\alpha \leqslant 1$ the function g is nowhere differentiable (Weierstrass, Hardy).

The spaces $h_{\infty,n}(\psi)$ Let $h_\infty(\psi)$ denote the class of the functions $u \in h(\mathbb{D})$ for which $u(z) = O(\psi(1/(1-r)))$, $r = |z| \to 1^-$, where $\psi(x)$, $x \geqslant 1$, is a positive nonincreasing function that grows slower than some power of x. The latter means that, for some $\beta > 0$, the function $\psi(x)/x^\beta$ is almost decreasing. Such functions are, for example $\psi(x) = x^\alpha(\log(2x))^\gamma$, where $\alpha > 0$ and $\beta \in \mathbb{R}$, or $\alpha = 0$ and $\beta > 0$. In the simplest(*) case, when $\psi(x) \equiv 1$, we have $h_\infty(\psi) = C(\mathbb{T})$, and then, as we have seen, the Poisson integral acts as an isometrical isomorphism from $C(\mathbb{T})$ onto $hC(\mathbb{D}) = C(\overline{\mathbb{D}}) \cap h(\mathbb{D})$ (see Theorem 3.1.7). Hence every subclass of $C(\mathbb{T})$ can be regarded as a subclass of $hC(\mathbb{D})$, and conversely.

Following the paper [71], we shall show that any $h_\infty(\psi)$ is isomorphic to some Lipschitz space via the tangential derivative of sufficiently large order.

Under the above hypothesis on ψ, we define

$$\|u\|_\psi = |u(0)| + \sup_{0<r<1} \frac{M(r,u)}{\psi(1/(1-r))},$$

where $M(r,u) = M_\infty(r,u) = \max\{|u(z)| : |z| = r\}$. Let

$$h_{\infty,n}(\psi) = \{u \in h(\mathbb{D}) : \|D^n u\|_\psi < \infty\}, \quad n = 0,1,2,\ldots,$$

where

$$(D^n u)(re^{i\theta}) = \frac{\partial^n u}{\partial \theta^n}(re^{i\theta}) = \sum_{j=-\infty}^{\infty} (ij)^n \hat{u}(j) r^{|j|} e^{ij\theta}, \qquad re^{i\theta} \in \mathbb{D}.$$

The subspace of $h_{\infty,n}(\psi)$ defined by $M(r, D^n u) = o(\psi(1/(1-r)))$, $r \to 1^-$, will be denoted by $h_{o,n}(\psi)$.

Conditions for majorants From now on we shall assume that ψ is a positive almost increasing function on $[1,\infty)$ satisfying the condition:

$$\text{there exists a constant } C < \infty \text{ such that } \psi(2x) \leqslant C\psi(x),\ x \geqslant 1. \qquad (U)$$

This is equivalent to the existence of a positive constant α such that

$$\frac{\psi(x)}{x^\alpha} \text{ is almost decreasing } (x \geqslant 1). \qquad (U_\alpha)$$

Concerning the function ϕ, we shall assume that it positive and almost increasing on $(0,1]$ and that there exists $\beta > 0$ such that

$$\frac{\phi(t)}{t^\beta} \text{ is almost decreasing } (0 < t < 1). \qquad (U_\beta^0)$$

(*)with respect to ψ

10.4 Growth of derivatives

The following theorem is classical and is due Hardy and Littlewood, Zygmund, and Privalov (see [18, Ch. 2, §§1,2], [100], and [67]).

10.4.1 Theorem *Let $0 < a \leqslant n$ ($n = 1, 2, \dots$) and $u \in h(\mathbb{D})$. Then the following two conditions are equivalent:*

$$u \in hC(\mathbb{D}) \quad \& \quad \omega_n(u_*, t) = O(t^a)\,; \tag{10.7}$$

$$D^n u(z) = O((1 - |z|)^{a-n}). \tag{10.8}$$

If in addition $a < n$ and $u = \operatorname{Re} f$, $f \in H(\mathbb{D})$, then the condition $f^{(n)}(z) = O((1 - |z|)^{a-n})$ is equivalent to each of the preceding.

Recall that $hC(\mathbb{D}) = C(\overline{\mathbb{D}}) \cap h(\mathbb{D})$. In this section we extend Theorem 10.4.1 in the following way.

10.4.2 Theorem *If $\phi(t)/t^n$ almost decreases and $\phi(t)/t^b$ almost increases, for some $b > 0$, then the Poisson integral acts as an isomorphism of $\operatorname{Lip}_n(\phi)$ onto $h_{\infty,n}(\psi)$, where $\psi(x) = x^n \phi(1/x)$, $x \geqslant 1$.*

Remark. The analogous assertion for the pair $\operatorname{lip}_n(\phi)$, $h_{o,n}(\psi)$ is also valid.

This theorem is contained in the following.

10.4.3 Theorem *Let n be a positive integer and let ψ and ϕ satisfy conditions (U) and (U_n^0). Then the following assertions are equivalent:*

(a) $\operatorname{Lip}_n(\phi) = h_{\infty,n}(\psi)$;

(b) *There are constants $\alpha < n$ and C such that ψ satisfies (U_α) and*

$$\phi(t)/C \leqslant t^n \psi(1/t) \leqslant C\phi(t), \quad 0 < t < 1.$$

Condition (U_n^0) is natural because the modulus of smoothness has the property that $\omega_n(g, t)/t^n$ almost decreases (Lemma 10.4.7). However, the theorem cannot be applied to the case

$$\phi(t) = \frac{1}{(1 + \log(1/t))^b} \qquad (b > 0),$$

and we do not know whether the space $\operatorname{Lip}_n(\phi)$ is isomorphic to some of the spaces $h_\infty(\psi)$.

Proof of Theorem 10.4.3

The implication (b)$\Rightarrow$(a) is a consequence of the following two propositions, which will be proved later on.

10.4.4 Proposition *If $u \in hC(\mathbb{D})$, then*

$$M(r, D^n u) \leqslant C(1-r)^{-n}\omega_n(u_*, 1-r), \quad 0 < r < 1, \tag{10.9}$$

where $C < \infty$ depends only on n $(n = 1, 2, \dots)$.

For the proof see page 160.

10.4.5 Proposition *If $u \in h(\mathbb{D})$ and*

$$\int_0^1 (1-r)^{n-1} M(r, D^n u)\, dr < \infty, \tag{10.10}$$

then $u \in hC(\mathbb{D})$ and

$$\omega_n(u_*, t) \leqslant C \int_{1-t}^1 (1-r)^{n-1} M(r, D^n u)\, dr, \quad 0 < t < 1, \tag{10.11}$$

where C depends only on n.

For the proof see page 162.

Let n be a fixed integer. Condition (U_α) from the preceding section can be written as

$$\psi(y) \leqslant C(y/x)^\alpha \psi(x), \quad y \geqslant x \geqslant 1. \tag{U_α}$$

Hence (U_α), $\alpha < n$, implies

$$\int_x^\infty \psi(y) y^{-n-1}\, dy \leqslant C x^{-n} \psi(x), \quad x \geqslant 1, \tag{A_n}$$

which is part of the following lemma.[(†)]

10.4.6 Lemma *The function ψ satisfies (A_n) iff there exists $\alpha < n$ such that ψ satisfies (U_α).*

Proof. We need to prove that (A_n) implies (U_α) for some $\alpha < n$. Let ψ satisfy (A_n), and let

$$F(x) = \int_x^\infty \psi(y) y^{-n-1}\, dy, \quad x \geqslant 1.$$

It is easy to see that $cF(x) \leqslant x^{-n}\psi(x) \leqslant CF(x)$, $x \geqslant 1$, and hence it suffices to find $b > 0$ such that $x^b F(x)$ is nonincreasing for $x \geqslant 1$. Choose b so that $F(x) \leqslant (1/b)\psi(x)x^{-n}$, $x \geqslant 1$, which can be written as $F(x) \leqslant -(1/b)xF'(x)$. This implies that the derivative of $x^b F(x)$ is $\leqslant 0$, which concludes the proof. □

Now the implication (b)⇒(a) of Theorem 10.4.3 is obtained immediately from Propositions 10.4.4, 10.4.5, Lemma 10.4.6 and the formula

$$\int_{1/t}^\infty \psi(y) y^{-n-1}\, dy = \int_{1-r}^1 (1-r)^{n-1} \psi(1/(1-r))\, dr, \quad 0 < t < 1.$$

For the proof of the implication (a)⇒(b) we need some more lemmas.

(†)Such assertions are often encountered in the theory of regularly varying functions (see [88]).

10.4.7 Lemma *If $g \in C(\mathbb{T})$, then the function $\omega_n(t)/t^n$ is almost decreasing for $t > 0$.*

Proof. The lemma is easily deduced from the inequality

$$\omega_n(g, 2t) \leqslant 2^n \omega_n(g, t), \quad t > 0, \tag{10.12}$$

while (10.12) can be proved by means of the formula

$$\Delta_t^n g(e^{i\theta}) = \sum_{|j|<\infty} \hat{g}(j)(e^{ijt} - 1)^n e^{ij\theta} \tag{10.13}$$

(g is a trigonometric polynomial), from which one gets

$$\Delta_{2t}^n g(e^{i\theta}) = \sum_j (e^{ijt} + 1)^n \hat{g}(j)(e^{ijt} - 1)^n e^{ij\theta} = \sum_{k=0}^{n} \binom{n}{k} \Delta_t^n g(e^{i(\theta+kt)}). \qquad \square$$

10.4.8 Lemma *If $g \in C(\mathbb{T})$ and $g_k(e^{i\theta}) = g(e^{ik\theta})$, where $k = 1, 2, \ldots$, then*

$$|\hat{g}(0)| + \omega_n(g_k, \pi/k) \geqslant \|g\|_\infty.$$

Proof. From (10.13) it follows that

$$(-1)^n \left[g(e^{i\theta}) - \hat{g}(0) \right] = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\Delta_t^n g)(e^{i\theta})\, dt.$$

Hence

$$\|g - \hat{g}(0)\|_\infty \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} \|\Delta_t^n\|_\infty\, dt \leqslant \omega_n(g, \pi) = \omega_n(g_k, \pi/k). \qquad \square$$

10.4.9 Lemma *Let $X = \mathrm{Lip}_n(\phi)$ or $h_{\infty,n}(\psi)$. Then there hold the assertions:* (i) *If a sequence $\{u_m\} \subset X$ converges in norm to u, then $u_m(z) \to u(z)$ uniformly on compact subsets of $\mathbb{D}$.* (ii) *X is complete.*

Proof. Let $X = \mathrm{Lip}_n(\phi)$. According to Lemmas 10.4.8 ($k = 1$) and 10.4.7,

$$\|u\|_X \geqslant |u(0)| + \omega_n(u_*, \pi)/\phi(\pi) \geqslant c\|u_*\|_\infty = c\|u\|_\infty, \qquad u \in X.$$

This shows that X is continuously embedded into $C(\mathbb{T}) = hC(\mathbb{D})$, which implies (i). The same fact can be used to prove the completeness of X.

In the case when $X = h_{\infty,n}(\psi)$ the proof is even simpler. $\square$

10.4.10 Lemma *Let ϕ and ψ satisfy the conditions (U) and (U_n^0). Let $u_k(z) = z^k$, where $|z| \leqslant 1$ and $k = 1, 2, \ldots$ Then*

$$\|D^n u_k\|_\psi \asymp \frac{k^n}{\psi(k)} \qquad (k \geqslant 1), \tag{10.14}$$

$$\|u\|_{\phi,n} \asymp \frac{1}{\phi(1/k)} \qquad (k \geqslant 1). \tag{10.15}$$

Proof. The proof of (10.14) is straightforward. In order to prove (10.15) we use the equality $(\Delta_t^n u_{k*}(w)) = w^k(e^{ikt}-1)^n$, $w \in \mathbb{T}$. Hence

$$\omega_n(u_{k*}, t) = 2^n \sup\{|\sin(ks/2)|^n : 0 < s \leqslant t\}, \qquad t > 0,$$

and therefore

$$\|u_k\|_{\phi,n} = 2^n \sup\{|\sin(ks/2)|^n/\phi(t) : s \leqslant t \leqslant 1, 0 < s \leqslant 1\},$$

Since $\phi(t) \geqslant \phi(s)/C$ for $0 < s \leqslant t \leqslant 1$, we have

$$\|u_k\|_{\phi,n} \leqslant C \sup\{|\sin(ks/2)|^n/\phi(s) : 0 < s \leqslant 1\},$$

where C is independent of k. If $1/k \leqslant s \leqslant 1$, then $|\sin(ks/2)|^n/\phi(s) \leqslant C/\phi(1/k)$ because the function $1/\phi$ is almost decreasing. If $0 < s \leqslant 1/k$, then

$$|\sin(ks/2)|^n/\phi(s) \leqslant 2^{-n}k^n s^n/\phi(s) \leqslant C\, k^n(1/k)^n/\phi(1/k)$$

because $s^n/\phi(s)$ is almost increasing. Thus $\|u\|_{\phi,n} \leqslant C/\phi(1/k)$. The proof of the reverse inequality is simpler. □

Now we ready to prove the the implication (a)⇒(b) in Theorem 10.4.3. Let $\mathrm{Lip}_n(\phi) = h_{\infty,n}(\psi)$. It follows from Lemma 10.4.9 and the closed graph theorem that $\|D^n u\|_\psi \asymp \|u_k\|_{\phi,n}$, $k \geqslant 1$, where u_k is as in Lemma 10.4.10. Hence, by Lemma 10.4.10, $\phi(1/k) \asymp (1/k)^n\psi(k)$, $k \geqslant 1$, which yields

$$\phi(t) \asymp t^n\psi(1/t), \qquad 0 < t \leqslant 1, \tag{10.16}$$

and this is part of (b).

In order to prove that (a) implies (U_α) for for some $\alpha < n$, we consider the functions

$$U_k(z) = k^{-n}\psi(k)z^k + \sum_{j=2}^{\infty}(jk)^{-n}\big(\psi(jk) - \psi((j-1)k)\big)z^{jk}, \qquad z \in \mathbb{D}.$$

Assume, as we may, that ψ is nondecreasing. Then

$$\begin{aligned}
M(r, D^n U_k) &\leqslant \psi(k)r^k + \sum_{j=2}^{\infty}\big(\,\psi(jk) - \psi((j-1)k)\,\big)\, r^{jk} \\
&\leqslant \psi(k)r^k + \sum_{j=2}^{\infty}\ \sum_{p=(j-1)k}^{jk-1}\big(\,\psi(p+1) - \psi(p)\,\big)\, r^{p+1} \\
&= \psi(k)r^k + \sum_{j=k}^{\infty}\big(\,\psi(p+1) - \psi(p)\,\big)\, r^{p+1} \\
&= (1-r)\sum_{p=k}^{\infty}\psi(p)\, r^p \leqslant C\,\psi\big(1/(1-r)\big).
\end{aligned}$$

It follows that $\{U_k\}$ is a norm bounded sequence in $h_{\infty,n}(\psi)$. Now we use the inclusion $h_{\infty,n}(\psi) \subset \mathrm{Lip}_n(\phi)$ to conclude that the functions U_k are continuous on the closed disk and

$$\omega_n(U_{k*}, t) \leqslant C\phi(t), \qquad 0 < t < 1, \tag{10.17}$$

where C is independent of t, k.

On the other hand, by Lemmas 10.4.7 and 10.4.8,

$$\begin{aligned} C\omega_n(U_{k*}, 1/k) \geqslant \omega_n(U_{k*}, \pi/k) &\geqslant \|U_{k*}\|_\infty \\ &= k^{-n}\psi(k) + k^{-n}\sum_{j=2}^{\infty} j^{-n}\big(\psi(jk) - \psi((j-1)k)\big) \\ &= k^{-n}\sum_{j=1}^{\infty}\big(j^{-n} - (j+1)^{-n}\big)\psi(jk). \end{aligned}$$

Hence, by (10.17), (10.16) and (U),

$$k^{-n}\sum_{j=1}^{\infty} j^{-n-1}\psi((j+1)k) \leqslant C\phi(1/k) \leqslant Ck^{-n}\psi(k)$$

and therefore

$$\int_k^\infty y^{-n-1}\psi(y)\,dy = k^{-n}\int_1^\infty y^{-n-1}\psi(yk)\,dy \leqslant Ck^{-n}\psi(k), \qquad k = 1, 2, \ldots.$$

It is easily verified that this implies (A_n). Thus ψ satisfies (U_α) for some $\alpha < n$ (Lemma 10.4.6), and this concludes the proof of Theorem 10.4.3.

Proof of Proposition 10.4.4

In proving Proposition 10.4.4 and Proposition 10.4.5 we shall use the inequalities

$$\begin{aligned} M(r, D^{n+1}f) &\leqslant C(1-r)^{-1}M((1+r)/2, D^n u) \\ M(r, f^{(n+1)}) &\leqslant C(1-r)^{-1}M((1+r)/2, D^n u), \end{aligned} \tag{10.18}$$

where $u = \operatorname{Re} f$, $f \in H(\mathbb{D})$, and $n \geqslant 0$. Equivalently: if $D^n u$ is bounded in $\mathbb{D}$, then

$$|D^{n+1}f(z)| \leqslant C(1-|z|)^{-1}\,\|D^n u\|_\infty, \quad |f^{(n+1)}(z)| \leqslant C(1-|z|)^{-1}\,\|D^n u\|_\infty,$$

where C is independent of f and z. The proof is left to the reader as an exercise.

In proving Proposition 10.4.4 we may assume that u is real-valued and harmonic in a neighborhood of the closed disk. For fixed $r < 1$ let $h(\theta) = u_r(\theta) = u(re^{i\theta})$. Then

$$(\Delta_t^n h)(\theta) = \int_{tE} h^{(n)}(\theta + x_1 + \cdots + x_n)\,dx_1 \ldots dx_n, \tag{10.19}$$

where tE is the n-dimensional cube $[0,t]^n$. Hence

$$h^{(n)}(\theta) = (D^n u)(re^{i\theta})t^n$$
$$= (\Delta_t^n)(\theta) - \int_{tE} \left(h^{(n)}(\theta + x_1 + \cdots + x_n) - h^{(n)}(\theta)\right) dx_1 \ldots dx_n.$$

Since

$$|h^{(n)}(\theta+x) - h^{(n)}(\theta)| = \left| \int_0^x h^{(n+1)}(\theta+y)\, dy \right| \leqslant M(r, D^{n+1})x, \qquad x = x_1 + \ldots x_n,$$

we get

$$M(r, D^{n+1}u) \leqslant \|\Delta_t^n u_r\|_\infty + \int_{tE} M(r, D^{n+1}u)(x_1 + \cdots + x_n)\, dx_1 \ldots dx_n$$
$$= \|\Delta_t^n u_r\|_\infty + (n/2)M(r, D^{n+1})t^{n+1}, \quad 0 < r < 1,\ t > 0.$$

The function $\Delta_t^n u$ defined by $(\Delta_t^n u)(re^{i\theta}) = (\Delta_t^n u_r)(\theta)$ is harmonic on the closed disk and therefore $\|\Delta_t^n u_r\|_\infty \leqslant \|\Delta_t^n u_*\| \leqslant \omega_n(u_*, t)$, $t > 0$. These inequalities together with (10.18) yield

$$M(r, D^n u) \leqslant t^{-n}\omega_n(u_*, t) + Kt(1-r)^{-1}M((1+r)/2, D^n u) \tag{10.20}$$

($t > 0$, $0 < r < 1$,) where K depends only on n. Let $A(r) = (1-r)^{-n}M(r, D^n u)$, $0 < r < 1$. It follows from (10.20) that

$$A(r) \leqslant t^{-n}(1-r)^n\omega(t) + 2^n Kt(1-r)^{-1}A((1+r)/2),$$

where $\omega(t) = \omega(u_*, t)$. Choose an integer m so that $2^n K \leqslant (1/4)2^m$ and take $t = a(1-r)$, $a = 2^{-m}$. Then we have $A(r) \leqslant a^{-m}\omega(1-r) + (1/4)A((1+r)/2)$, $0 < r < 1$. Integrating this inequality from ρ (< 1) to 1, and introducing appropriate substitutions, we get

$$\int_\rho^1 A(r)\, dr \leqslant a^{-m}\int_0^{1-\rho} \omega(t)\, dt + (1/2)\int_{(1+\rho)/2}^1 A(r)\, dr$$
$$\leqslant a^{-m}\int_0^{1-\rho} \omega(t)\, dt + (1/2)\int_\rho^1 A(r)\, dr.$$

Hence, since the integral $\int_\rho^1 A(r)\, dr$ is finite,

$$(1/2)\int_\rho^1 A(r)\, dr \leqslant a^{-m}\int_0^{1-\rho} \omega(t)\, dt.$$

Now (10.9) follows from the inequalities

$$\int_0^{1-\rho} \omega(t)\, dt \leqslant (1-\rho)\,\omega(1-\rho), \quad M(r, D^{n+1}u)(1-\rho)^{n+1} \leqslant (n+1)\int_\rho^1 A(r)\, dr,$$

which are valid because the functions ω and M are increasing. Thus the proof of Proposition 10.4.4 is finished.

Proof of Proposition 10.4.5

Let $u = \operatorname{Re} f$, where f is analytic u $\mathbb{D}$. Then

$$f(z) = \sum_{k=0}^{n} \frac{f^{(k)}(rz)}{k!} z^k (1-r)^k + \frac{1}{n!} \int_r^1 (1-s)^n z^{n+1} f^{(n+1)}(sz)\, ds \tag{10.21}$$

($z \in \mathbb{D},\ 0 < r < 1$). Denoting the sum by $f_{r,n}$ we have

$$|f(z) - f_{r,n}(z)| \leqslant \frac{1}{n!} \int_r^1 (1-s)^n M(s, f^{(n+1)})\, ds.$$

From this and (10.18) it follows that (10.10) implies $\|f - f_{r,n}\|_\infty \to 0$ $(r \to 1^-)$. Since the functions $f_{r,n}$ $(r < 1)$ are continuous on the closed disk, we see that (10.10) implies the continuity of f, and consequently of u, on the closed disk.

In order to prove (10.11) let $u_r(\theta) = u(re^{i\theta})$, $0 < r \leqslant 1$. Then (10.11) is equivalent to

$$\|\Delta_t^n u_1\|_\infty \leqslant C \int_{1-t}^1 (1-s)^{n-1} M(s, D^n u)\, ds, \qquad 0 < t < 1. \tag{10.22}$$

Let $r = 1 - 2t$, $0 < t < 1/4$. Then $\|\Delta_t^n u_1\| \leqslant \|\Delta_t^n (u_1 - u_r)\| + \|\Delta_t^n u_r\|$. It follows from (10.19) and the "increasing" property of $M(r, D^n u)$ that

$$\|\Delta_t^n u_r\| \leqslant t^n M(r, D^n u) \leqslant n \int_{1-t}^1 (1-s)^{n-1} M(s, D^n u)\, ds,$$

and therefore we have to prove that $\|\Delta_t^n (u_1 - u_r)\|$ is dominated by the right-hand side of (10.22). Since $\|\Delta_t^n (u_1 - u_r)\| \leqslant \|\Delta_t^n (f_1 - f_r)\|$, it is enough to prove that

$$\|\Delta_t^n (f_1 - f_r)\| \leqslant C \int_{1-t}^1 (1-s)^{n-1} M(s, D^n u)\, ds.$$

To prove this write (10.21) in the form

$$f_1(\theta) - f_r(\theta) = H(\theta) + \sum_{k=1}^{n} h_k(\theta)(1-r)^k / k!, \quad \text{where}$$

$$H(\theta) = \frac{1}{n!} \int_r^1 (1-s)^n e^{i(n+1)\theta} f^{(n+1)}(se^{i\theta})\, ds, \quad \text{and} \quad h_k(\theta) = f^{(k)}(re^{i\theta}) e^{ik\theta}.$$

We have

$$\begin{aligned}
\|\Delta_t^n H\| \leqslant 2^n \|H\| &\leqslant \frac{2^n}{n!} \int_r^1 (1-s)^n M(s, f^{(n+1)})\, ds \\
&\leqslant C \int_r^1 (1-s)^{n-1} M((1+s)/2, D^n u)\, ds \\
&= 2^n C \int_{1-t}^1 (1-s)^{n-1} M(s, D^n u)\, ds,
\end{aligned}$$

where we have applied (10.18). In order to estimate $\|\Delta_t^n h_k\|$, let $m = n - k + 1$ $(1 \leqslant k \leqslant n)$ and observe that (10.19) implies

$$\|\Delta_t^n h_k\| = \|\Delta_t^{k-1}\Delta_t^m h_k\| \leqslant 2^{k-1}\|\Delta_t^m h_k\| \leqslant 2^{k-1}t^m\|h_k^{(m)}\|.$$

From this and the inequality $\|h_k^{(m)}\| \leqslant C(1-r)^{-1}M((1+r)/2, D^n u)$ (see (10.18)) it follows that

$$\|\Delta_t^n h_k\| \leqslant Ct^{n-k}M(1-t, D^n u) \leqslant Ct^{-k}\int_{1-t}^{1}(1-s)^{n-1}M(s, D^n u)\,ds,$$

where C is independent of t. Combining all the above results yields (10.11) for $0 < t < 1/4$. If $t > 1/4$, we can apply Lemma 10.4.7 to reduce (10.11) to the case $0 < t < 1/4$, and this completes the proof. □

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10.4.11 (conjugate functions) If $\psi(x)$, $x \geqslant 1$, satisfies (U) (p. 155) and there exists a constant $\beta > 0$ such that

$$\psi(x)/x^\beta \qquad \text{almost increases} \tag{10.23}$$

then the space $h_\infty(\psi)$ is "self-conjugate", i.e., there holds $u \in h_\infty(\psi) \Rightarrow \widetilde{u} \in h_\infty(\psi)$, or, what is the same, the Riesz projector $(R_+u)(z) = \sum_{n=0}^{\infty}\widehat{u}(n)z^n$ acts from $h_\infty(\psi)$ to $h_\infty(\psi)$. From this and Theorem 10.4.3 it follows that the same holds for $\mathrm{Lip}_n(\phi)$ provided there exist constants $\beta < n$ and $\alpha > 0$ such that $\phi(t)/t^\alpha$ is almost increasing and $\phi(t)/t^\beta$ is almost decreasing. It was proved in [91] that

if (U) holds, then (10.23) *is necessary for $h_\infty(\psi)$ to be self-conjugate.*

10.4.12 If $a < n$, then the derivative $D^n u$ in Theorem 10.4.1 may be replaced by each of the derivatives $\partial^n u/\partial^j r\partial^{n-j}\theta$ $(j = 0, 1, \ldots, n)$, see 10.4.11. If $0 < a < n$, then Theorem 10.4.1 remains true when "O" is replaced by "o".

10.4.13 A real function $g \in C(\mathbb{T})$ belongs to Λ_* iff the Cauchy integral

$$f(z) := \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{u(e^{i\theta})}{1 - ze^{-it}}\,d\theta,$$

satisfies

$$|f''(z)| \leqslant \frac{C}{1-|z|} \qquad (|z| < 1).$$

This condition implies that $f \in A(\mathbb{D})$ and that the boundary function f_* belongs to Λ_*.

11 Lacunary series

In the first section we consider Paley's theorems on lacunary series in H^p (Theorems 11.1.1 and 11.1.3). The rest of the chapter is devoted to the proof of a generalized version of the Gurarij/Matsaev theorem on $L^p(0,1)$-integrability of lacunary power series (see Theorems 11.4.2 and 11.3.1). Since the proof is based on the ideas from Karamata's proof of Littlewood's tauberian theorem, we included a short discussion of Karamata's ideas (Section 11.2).

11.1 Lacunary series in H^p

A sequence $\{n_k\}_{k\geqslant 1}$ of positive real numbers is called lacunary if it satisfies the condition

$$\inf_{k\geqslant 1} \frac{n_{k+1}}{n_k} > 1.$$

The corresponding series $\sum a_k x^{n_k}$ is then called a lacunary series.

11.1.1 Theorem (Paley) *Let n_k be a lacunary sequence of positive integers. If $f \in H^1$, then*

$$\|f\|_1 \geqslant c\Big(\sum_{k=1}^{\infty} |\widehat{f}(n_k)|^2\Big)^{1/2},$$

where $c > 0$ is a constant independent of f.

This theorem does not extend to the case of $L^1(\mathbb{T})$, and this can be seen from 6.1.4.

Theorem 11.1.1 will be deduced from the following result.

11.1.2 Theorem (Hardy/Littlewood) *Let f be analytic in $\mathbb{D}$. Then there hold the assertions:*

(a) *If $f \in H^p$, $0 < p \leqslant 2$, then*

$$K := \int_0^1 M_p^2(r, f')(1-r)\,dr < \infty \tag{11.1}$$

and there exists a constant C_p such that $K \leqslant C_p\|f\|_p^2$.

(b) *If $2 \leqslant p < \infty$, then (11.1) imply $f \in H^p$ and $\|f\|_p^2 \leqslant C_p(K + |f(0)|^2)$.*

Proof. (a) Let $0 < p \leqslant 2$. In view of Lemma 5.1.7, it is enough to discuss the case where f has no zeros in $\mathbb{D}$. Further, we can assume that $\|f\|_p = 1$. Let

$g = f^{p/2}$. Then $\|f\|_p^p = \|g\|_2^2$ and

$$M_p^p(r, f') = (2/p)^p \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(re^{i\theta})|^{2-p} |g'(re^{i\theta})|^p d\theta.$$

Hence, by Hölder's inequality with the indices $2/(2-p)$, $2/p$, we get

$$M_p^p(r, f') \leqslant (2/p)^p M_2^{2-p}(r, g) M_2^p(r, g').$$

Since $M_2(r, g) \leqslant \|g\|_2 = 1$, we see that $M_p^2(r, f') \leqslant (2/p)^2 M_2^2(r, g')$, and this reduces the proof to the easily proved inequality

$$\int_0^1 M_2^2(r, g')(1-r)dr \leqslant C\|g\|_2^2.$$

(b) As remarked in the proof of Lemma 6.2.4, the function $|f|^p$ is of class C^2 so we can apply Green's formula to prove that the Hardy/Stein identity,

$$\|f\|_p^p = |f(0)|^p + \frac{p^2}{2} \int_{\mathbf{D}} |f|^{p-2} |f'|^2 \log \frac{1}{|z|} \, dA,$$

holds for every $f \in H^p$. Let $g(z) = f(\rho z)$, $0 < \rho < 1$. By Hölder's inequality we get

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |g(re^{i\theta})^{p-2} |g'(re^{i\theta})|^2 \, d\theta \leqslant M_p^{p-2}(r, g) M_p^2(r, g') \leqslant \|g\|_p^{p-2} M_p^2(r, g').$$

Hence

$$\|g\|_p^p \leqslant |g(0)|^2 \|g\|_p^{p-2} + p^2 \|g\|_p^{p-2} \int_0^1 M_p^2(r, g') r \log \frac{1}{r} \, dr.$$

Multiplying this inequality by $\|g\|_p^{2-p}$ and then letting ρ tend to 1, we get the desired result. □

Proof of Theorem 11.1.1. Let $\lambda = \inf_{k \geqslant 1} \frac{n_{k+1}}{n_k}$. From Theorem 11.1.2 it follows that

$$\|f\|_1 \geqslant c_1 \sum_{m=1}^{\infty} \int_{r_m}^{r_{m+1}} M_1^2(r, f')(1-r) \, dr,$$

where $r_m = 1 - \lambda^{-m}$, c_1=const> 0. For each m, the block $I_m = [\lambda^m, \lambda^{m+1})$ contains at most one member of the sequence $\{n_k\}$. Therefore we can suppose that $n_m \in I_m$ for all m. Since $M_1(r, f') \geqslant n|\widehat{f}(n)| r^{n-1}$, we have

$$\int_{r_m}^{r_{m+1}} M_1^2(r, f')(1-r) \, dr \geqslant n_m^2 |\widehat{f}(n_m)|^2 \int_{r_m}^{r_{m+1}} r^{2n_m - 2}(1-r) \, dr.$$

Now the proof is easily completed. □

11.1.3 Theorem (Paley) *Let the series* $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$, *where* $\{n_k\}$ *is a lacunary sequence, converge in* $\mathbb{D}$. *Then* $f \in H^p$ $(0 < p < \infty)$ *iff* $\sum_{k=1}^{\infty} |a_k|^2 < \infty$. *There exists a constant* $C = C_p > 0$ *such that*

$$C^{-1}\|\{a_k\}\|_2 \leqslant \|f\|_p \leqslant C\|\{a_k\}\|_2. \tag{11.2}$$

Proof. In the case $1 \leqslant p < 2$, the result is an immediate consequence of Paley's theorem. Let $0 < p < 1$. Let f be analytic in a neighborhood of the closed disk. Then, by means of the Cauchy/Schwarz inequality, we get

$$\|f\|_1 = \int |f|^{p/2}\,|f|^{1-p/2} \leqslant \|f\|_p^{p/2}\|f\|_{2-p}^{(2-p)/2} \leqslant \|f\|_p^{p/2}\|f\|_2^{(2-p)/2}.$$

Since $\|f\|_1 \geqslant c\|f\|_2$, we see that $c\|f\|_2^{p/2} \leqslant \|f\|_p^{p/2}$. If f is arbitrary, then we apply this inequality to the functions f_ρ $(\rho \to 1)$ and this completes the proof in the case $0 < p < 2$.

Let $2 < p < \infty$ and $q = p/(p-1)$. t follows from Paley's theorem that the operator P, $(Pf)(z) = \sum \widehat{f}(n_k)z^{n_k}$, is bounded from H^q to H^2. The dual operator P^* is formally equal to P, and since $(H^q)^* = H^p$ (Theorem 6.3.2), we have $\|Pf\|_p \leqslant C_p\|f\|_2$ for $f \in H^2$. Hence $\|f\|_p \leqslant C_p\|f\|_2$ if $Pf = f$.

In the case $p > 2$, the theorem can also be deduced from Theorem 11.1.2(b) by using 7.5.5. □

Miscellaneous

11.1.4 Inequality (11.2), known as Paley's inequality, holds for every $p \in (0, \infty)$ under the more general assumption that f is an arbitrary trigonometric polynomial with lacunary coefficients.

On the other hand, in the case $p = \infty$, there holds **Sidon's theorem** [100, Theorem VI.(6.1)]:

11.1.5 Theorem *A trigonometric series with lacunary coefficients is the Fourier series of a bounded function iff the sequence of coefficients is absolutely summable.*

For further properties of lacunary series see Zygmund [100, Ch. V, §§6,7].

11.2 Karamata's theorem and Littlewood's theorem

Recall Abel's theorem:

Let

$$f(z) := \sum_{k=0}^{\infty} A_k\, z^k = (1-z)\sum_{n=0}^{\infty} s_n\, z^n \quad (|z| < 1,\ z \in \mathbb{C}), \tag{11.3}$$

where $\{A_k\}$ $(k \geqslant 0)$ *is a sequence of vectors in a Banach space* $(X, |\cdot|)$, *and* $s_n = \sum_{k=0}^{n} A_k$. *If there exists the limit* $\lim_{n\to\infty} s_n =: s \in X$, *then* $\lim_{r\to 1^-} f(r) = s$.

In other words, *convergence of a series implies its summability by Abel's method;* the converse is not true. Tauber proved that if the "tauberian" condition $\lim_{n\to\infty} n A_n = 0$ is satisfied, then the existence of $\lim_{r\to 1} f(r) =: l$ implies $\lim_{n\to\infty} s_n = l$. Littlewood [54] improved Tauber's theorem by replacing Tauber's condition by (*)

$$\sup_{k\geqslant 0} (k+1)|A_k| < \infty. \tag{11.4}$$

The original proof of Littlewood is very complicated. Karamata [39] found a simple approach to Littlewood's theorem, which enabled him to improve and generalize some more tauberian theorems, mainly due to Hardy and Littlewood [39, 41, 40].

Karamata's theorem

Condition (11.4) and boundedness of the function $f(r) = (1-r)\sum_{k=0}^{\infty} s_k r^k$ on $(0,1)$ imply boundedness of the sequence s_n (Lemma 11.2.4).

11.2.1 Theorem (Karamata) *If the sequence* s_n *is bounded and the condition* $\lim\limits_{r\to 1^-} (1-r)\sum\limits_{n=0}^{\infty} s_n r^n = S$ *is satisfied, then*

$$\lim_{r\to 1^-} (1-r)\sum_{k=0}^{\infty} s_k\, \psi(r^k) r^k = S\int_0^1 \psi(t)\,dt, \tag{11.5}$$

where ψ *is an arbitrary Riemann*(†) *integrable function on* $[0,1]$.

Proof of Littlewood's theorem. In order to deduce Littlewood's theorem from Karamata's theorem, i.e., to prove that the conditions (11.4) and

$$\lim_{r\to 1^-} \sum_{k=0}^{\infty} A_k r^k = S \tag{11.6}$$

imply convergence of the series $\sum A_k$, we choose ψ so that $r\psi(r) = 1/(\lambda-1)$ for $e^{-\lambda} \leqslant r < e^{-1}$, $\psi(r) = 0$ otherwise, where $\lambda > 1$. Then from (11.5) we get, by taking $r = e^{-1/n}$,

$$\lim_{n\to\infty} \frac{1}{(\lambda-1)n} \sum_{k=n+1}^{[\lambda n]} s_k = S. \tag{11.7}$$

On the other hand,

$$\left(\frac{1}{(\lambda-1)n}\sum_{k=n+1}^{[\lambda n]} s_k\right) - s_n = \left(\frac{[\lambda n]-n}{(\lambda-1)n} - 1\right) s_n + \frac{1}{(\lambda-1)n}\sum_{k=n+1}^{[\lambda n]} (s_k - s_n).$$

(*)See Theorem 11.2.2.

(†)"Riemann" cannot be replaced by "Lebesgue".

From this and condition (11.4) it follows that

$$\limsup_{n\to\infty}\left|\left(\frac{1}{(\lambda-1)n}\sum_{k=n+1}^{[\lambda n]} s_k\right)-s_n\right| \leqslant \limsup_{n\to\infty}\max_{n<k\leqslant[\lambda n]}|s_k-s_n| \leqslant C\log\lambda.$$

Now the result follows from (11.7) because $\lambda > 1$ is arbitrary. □

Proof of Karamata's theorem. Observe that the expression

$$(1-r)\sum_{k=0}^{\infty}\psi(r^k)\,r^k=\sum_{k=0}^{\infty}\psi(r^k)(r^k-r^{k+1})$$

is a Riemannian sum of the function $\psi(r)$. The diameter of the underlying partition is equal to $1-r$ and therefore

$$\lim_{r\to 1^-}(1-r)\sum_{k=0}^{\infty}\psi(r^k)\,r^k=\int_0^1\psi(t)\,dt.$$

It follows that condition (11.6) can be written as

$$\lim_{r\to 1^-}(1-r)\sum_{k=0}^{\infty}(s_k-S)r^k=0.$$

Replacing r by $r^{1+\alpha}, \alpha > 0$, we find that

$$\lim_{r\to 1^-}(1-r)\sum_{k=0}^{\infty}(s_k-S)r^{\alpha k}r^k=0,$$

and this implies

$$\lim_{r\to 1^-}(1-r)\sum_{k=0}^{\infty}(s_k-S)P(r^k)\,r^k=0,$$

where P is an arbitrary polynomial. Hence

$$\begin{aligned}
&\limsup_{r\to 1^-}\left|(1-r)\sum_{k=0}^{\infty}(s_k-S)\psi(r^k)\,r^k\right|\\
&\leqslant \limsup_{r\to 1^-}\left|(1-r)\sum_{k=0}^{\infty}(s_k-S)\big(\psi(r^k)-P(r^k)\big)r^k\right|\\
&\leqslant \limsup_{r\to 1^-}(1-r)\sum_{k=0}^{\infty}\big|\psi(r^k)-P(r^k)\big|r^k\\
&= M\int_0^1\big|\psi(t)-P(t)\big|\,dt,
\end{aligned}$$

where $M = \sup_{n\geqslant 0}|s_n|$. This concludes the proof because the polynomials are dense in $L^1(0,1)$. □

On Littlewood's theorem

The proof of Littlewood's theorem shows that condition $A_n = O(1/n)$ can be weakened. Here we state a special case of a very general result of Karamata [40].

11.2.2 Theorem *Let the function* (11.3) *have property* (11.6). *The series* $\sum_0^\infty A_n$ *converges if there exists a function* $\delta(\lambda)$, $\lambda > 1$, *such that* $\lim_{\lambda\to 1^+} \delta(\lambda) = 0$, *and*

$$\limsup_{n\to\infty} \max_{n<k\leqslant[\lambda n]} \Big|\sum_{j=n}^{k} A_j\Big| \leqslant \delta(\lambda) \quad \textit{for all } \lambda > 1. \tag{11.8}$$

11.2.3 Corollary *With the above hypotheses, the series* $\sum_0^\infty A_n$ *converges if*

$$\Big\{\frac{1}{n}\sum_{k=n+1}^{2n} |A_k|^q\Big\}^{1/q} \leqslant \frac{C}{n} \quad \textit{for some } q > 1, \tag{11.9}$$

where C *is independent of* n.

It should be observed that condition (11.8) is necessary for convergence of an arbitrary series and is weaker than the condition

$$\limsup_{n\to\infty} \sum_{j=n+1}^{[\lambda n]} |A_j| \leqslant \delta(\lambda),$$

while the latter is weaker than (11.9); namely, if (11.9) holds, then by Hölder's inequality,

$$\sum_{j=n+1}^{[\lambda n]} |A_j| \leqslant \Big(\sum_{j=n+1}^{[\lambda n]} |A_j|^q\Big)^{1/q}([\lambda n] - n)^{1-1/q} \leqslant C(\lambda-1)^{1-1/q},$$

so we can take $\delta(\lambda) = C(\lambda-1)^{1-1/q}$.

Theorem 11.2.2 is an immediate consequence of the above proof of Littlewood's theorem and the following lemma.

11.2.4 Lemma *If the function* f *is bounded and*

$$\sup_{n\geqslant 0} \max_{n\leqslant j\leqslant 2n} \Big|\sum_{k=n}^{j} A_k\Big| < \infty, \tag{11.10}$$

then the sequence s_n *bounded.*

We leave the proof of this lemma to the interested reader. We only note that one can start from the inequality

$$|f(r_n) - s_{2^n}| \leqslant \Big|\sum_{j=0}^{2^n+1} A_j(1-r_n^j)\Big| + \Big|\sum_{j=2^n}^{\infty} A_j r_n^j\Big|, \tag{11.11}$$

where $r_n = 1 - 2^{-n}$. If (11.10) is strengthened to $\sup_{n\geqslant 0} \sum_{j=n}^{2n} |A_j| < \infty$, then the proof becomes shorter; namely, then it easy to show that the right-hand side of (11.11) is bounded.

Miscellaneous

11.2.5 The "uniform" version of **Dirichlet/Jordan test** says:

The Fourier series of a 2π-periodic continuous *function $f(\theta)$ of bounded variation converges uniformly on $\mathbb{R}$.*

Let $A_k(\theta) = a_k \cos k\theta + b_k \sin k\theta$ ($\theta \in \mathbb{R}$, $k \geqslant 1$), where a_k, b_k are the Fourier coefficients of $f(\theta)$. From Theorem 3.1.6 it follows that

$$\lim_{r\to 1^-} \left\| \frac{a_0}{2} + \sum_{k=1}^{\infty} A_k\, r^k \ - f \right\|_{\infty} = 0$$

If in addition f is of bounded variation on $[0, 2\pi]$, then $\|A_k\|_\infty = O(1/k)$, so the conclusion of the test can be obtained by Littlewood's theorem.

11.2.6 Equality (11.5) holds if the sequence s_n is real and bounded from above. In his proof Karamata appeals to the following: If ψ is a Riemann integrable function, then for every $\varepsilon > 0$ there are polynomials $P(r)$ and $Q(r)$ such that $P(r) < \psi(r) < Q(r)$ and $\int_0^1 \left(Q(r) - P(r)\right) dr < \varepsilon$.

11.2.7 Let the sequence $s_n = \sum_{k=0}^{n} A_k$ be bounded and let condition (11.6) be satisfied. If φ is an absolutely continuous function on the segment $[0, 1]$ and $\varphi(0) = 0$, then

$$\lim_{r\to 1^-} \sum_{k=0}^{\infty} A_k\, \varphi(r^k) = \varphi(1) \lim_{r\to 1^-} f(r).$$

11.3 Lacunary series in $C[0, 1]$

We continue to denote by $\{A_k\}$ a sequence of vectors in a Banach space X. We consider the series

$$\mathcal{L}(r) = \sum_{k=0}^{\infty} A_k\, r^{\lambda_k}, \tag{11.12}$$

where λ_k is a *lacunary* sequence, i.e., a sequence satisfying

$$\inf_{k\geqslant 0} \frac{\lambda_{k+1}}{\lambda_k} = q > 1. \tag{11.13}$$

The following theorem is taken from Gurarij/Matsaev [24].

11.3.1 Theorem *If there exists the limit $S := \lim_{r\to 1^-} \mathcal{L}(r)$, and is finite, then the series $\sum_{k=0}^{\infty} A_k$ converges.*

Proof. First we prove that the hypotheses imply

$$\sup_{k\geqslant 0}|A_k| \leqslant C\,M, \tag{11.14}$$

where $M = \sup_{0<r<1}|\mathcal{L}(r)|$ and C is independent of the sequence $\{A_k\}$. Let

$$P(r) = p_1 r + p_2 r^2 + \cdots + p_n r^n.$$

Then

$$\sum_{k=0}^{\infty} A_k\, P(r^{\lambda_k}) = p_1\mathcal{L}(r) + \cdots + p_n\mathcal{L}(r^n),$$

whence

$$\left|\sum_{k=0}^{\infty} A_k\, P(r^{\lambda_k})\right| \leqslant (|p_1| + \cdots + |p_n|)\, M.$$

Choose P so that $P(1/2) = 1$, $|P(r)| \leqslant \delta r$ for $0 < r < 2^{-q}$, and $|P(r)| \leqslant \delta(1-r)$ for $2^{-1/q} < r < 1$, where δ is small enough; for example, we can take $P(r) = \{4r(1-r)\}^N$, N large enough. Next, if $A_k \to 0$, choose ν so that $|A_\nu| = \sup_{k\geqslant 0}|A_k|$. Then

$$\begin{aligned}|A_\nu|\,|P(r^{\lambda_\nu})| &\leqslant (|p_1| + \cdots + |p_n|)M + \sum_{k\neq\nu}|A_k|\,|P(r^{\lambda_k})| \\ &\leqslant (|p_1| + \cdots + |p_n|)M + |A_\nu|\sum_{k\neq\nu}|P(r^{\lambda_k})|.\end{aligned}$$

Let $r = 2^{-1/\lambda_\nu}$. We have

$$\begin{aligned}\sum_{k\neq\nu}|P(r^{\lambda_k})| &\leqslant \delta\sum_{k<\nu}(1 - 2^{-\lambda_k/\lambda_\nu}) + \delta\sum_{k>\nu}2^{-\lambda_k/\lambda_\nu} \\ &\leqslant \delta\sum_{k<\nu}\lambda_k/\lambda_\nu + \delta\sum_{k>\nu}2^{-q^k} \leqslant \delta\Big(\sum_{k=0}^{\nu-1}q^{k-\nu} + \sum_{k=1}^{\infty}2^{-q^k}\Big).\end{aligned}$$

Now choose δ so that the latter is less than $1/2$. We get $(1/2)|A_\nu| \leqslant (|p_1| + \ldots|p_n|)\,M$, which is just the desired assertion in the case that $A_k \to 0$. In the general case, we consider the function $\mathcal{L}(\rho r)$ for $0 < \rho < 1$; since the constant $C = |p_1| + \ldots|p_n|$ is independent of $\{A_k\}$, we see, by the preceding, that $(1/2)|A_k|\rho^{2^k} \leqslant C\sup_r|\mathcal{L}(\rho r)| \leqslant C\,M$ for all ρ and k, which completes the proof of inequality (11.14).

Further, let us prove that $|s_n| \leqslant C_1 M$, where $s_n = \sum_{k=0}^{n} A_k$, and C_1 is an absolute constant. Putting $r_n = e^{-1/\lambda_n}$ we get

$$\begin{aligned}|\mathcal{L}(r_n) - s_n| &= \left|\sum_{k=0}^{n} A_k\,(r_n^{\lambda_k} - 1) + \sum_{k=n+1}^{\infty} A_k\, r_n^{\lambda_k}\right| \\ &\leqslant \sum_{k=0}^{n} |A_k|\lambda_k(1-r_n) + \sum_{k=n+1}^{\infty} |A_k| r_n^{\lambda_k} \\ &\leqslant CM\sum_{k=0}^{n} \lambda_k/\lambda_n + CM\sum_{k=n+1}^{\infty} e^{-\lambda_k/\lambda_n},\end{aligned}$$

whence, by condition (11.13), $|\mathcal{L}(r_n) - s_n| \leqslant$ const. M.

Finally, let $C_X[0,1]$ be the space of continuous functions from $[0,1]$ to X and let Y be the subspace of $C_X[0,1]$ consisting of the functions g of the form $g = \mathcal{L}$ with the property that there exists the limit $\lim_{r\to 1} g(r)$. By what we have proved, the formula $\Lambda_n(g) = s_n$ defines a bounded sequence of linear operators from Y to X. On the other hand, because of uniform continuity of functions $g \in Y$, we have $\lim_{\rho\to 1^-} \|g - g_\rho\|_\infty = 0$. Since $g_\rho \in Y$ and $\lim_{n\to\infty} \Lambda_n(g_\rho)$ exists, we see that $\lim_{n\to\infty} \Lambda_n(g)$ exists. □

11.4 L^p-integrability of lacunary series on $(0,1)$

With the above notation, we denote by $F(x,y)$ $(0 \leqslant x \leqslant 1,\ y \geqslant 0)$ a nonnegative real function that satisfies the condition

$$\lambda^a\mu^c\,F(x,y) \leqslant F(\lambda x, \mu y) \leqslant \lambda^b\mu^d\,F(x,y) \quad (0 < \lambda,\ \mu < 1), \tag{11.15}$$

where a, b, c, d are positive constants $(a \geqslant b,\ c \geqslant d)$. We write $F \in \Delta(a,b;\,c,d)$ or $F \in \Delta$ according to whether the values of a,b,c,d are or are not important. The simplest example is $F(x,y) = x^\alpha y^p$ $(\alpha > 0, p > 0)$. Further examples can be found by considering the functions $F(x,y) = x^\alpha\Phi(x^\beta y)$, where $\Phi : [0,\infty) \mapsto [0,\infty)$ is a function for which there are positive constants γ and δ such that $\Phi(t)/t^\gamma$ decreases and $\Phi(t)/t^\delta$ increases.

11.4.1 Theorem *If $F \in \Delta$, and $\mathcal{L}$ is given by* (11.12), *then the following conditions are equivalent:*

$$\int_0^1 (1-r)^{-1}F(1-r, |\mathcal{L}(r)|)\,dr < \infty; \tag{11.16}$$

$$\sum_{n=0}^{\infty} F\Big(\frac{1}{\lambda_n}, |A_n|\Big) < \infty. \tag{11.17}$$

In the case where $F(x,y) = xy^p$, this reduces to the following theorem of Gurarij and Matsaev [24]:

11.4.2 Theorem *For every $p \in (0, \infty)$, the following conditions are equivalent:*

$$\int_0^1 |\mathcal{L}(r)|^p \, dr < \infty; \qquad \sum_{n=0}^{\infty} (1/\lambda_n)|A_n|^p < \infty.$$

We shall deduce Theorem 11.4.1 from a weaker result, namely:

11.4.3 Proposition *There holds the inequality*

$$\int_0^1 (1-r)^{-1} F(1-r, |\mathcal{L}(r)|) \, dr \geqslant c_0 \sup_{n \geqslant 0} F(1/\lambda_n, |A_n|), \tag{11.18}$$

where c_0 is a positive constant.

Proof. Let $F \in \Delta(a, b; c, d)$. The integral in (11.18) is $\geqslant$

$$\int_0^1 (1-r)^{-1} F(1-r, |\mathcal{L}(r)| r^{1/d}) \, dr/r,$$

and since the series $\mathcal{L}(r) r^{1/d}$ is lacunary, it follows that in (11.18) we can write $r^{-1}\, dr$ instead of dr. Next, assuming that $\lambda_{k+1}/\lambda_k \geqslant q > 1$ for every k, let

$$P(r) = \frac{[r(1-r)]^N}{[\rho(1-\rho)]^N},$$

where N is an integer and ρ satisfies the condition $2^{-q} < \rho < 1/2 < 1 - \rho < 2^{-1/q}$. Choose ε so small that

$$2^{-q(1-\varepsilon)} < \rho < 2^{-(1+\varepsilon)}, \quad 2^{-(1-\varepsilon)} < 1 - \rho < 2^{-(1+\varepsilon)/q}.$$

For every $\delta > 0$ we can choose N so that

$$\begin{aligned} P(r) &\geqslant 1, && \text{for } 2^{-(1+\varepsilon)} < r < 2^{-(1-\varepsilon)}, \\ 0 < P(r) &\leqslant \delta r, && \text{for } 0 < r < 2^{-q(1-\varepsilon)}, \\ 0 < P(r) &\leqslant \delta(1-r), && \text{for } 2^{-(1+\varepsilon)/q} < r < 1. \end{aligned}$$

It is easily checked that

$$\begin{aligned} \text{Int} := &\int_0^1 (1-r)^{-1} F\Big(1-r, \Big|\sum_0^{\infty} A_k P(r^{\lambda_k})\Big|\Big) \frac{dr}{r} \\ &\leqslant C \int_0^1 (1-r)^{-1} F(1-r, |\mathcal{L}(r)|) \frac{dr}{r}, \end{aligned}$$

where C depends only of P and F. Therefore to prove (11.18) it is enough to find a polynomial P so that $\text{Int} \geqslant c_0 \sup_{n \geqslant 0} F(1/\lambda_n, |A_n|)$. Suppose that $A_n \to 0$

and take ν so that $F(1/\lambda_\nu, |A_\nu|) \geqslant F(1/\lambda_n, |A_n|)$ for all n. We shall prove the (formally) stronger inequality

$$\mathrm{Int}_\nu(F) \geqslant c_0 \sup_{n\geqslant 0} F(1/\lambda_n, |A_n|), \tag{11.19}$$

where

$$\mathrm{Int}_\nu(F) = \int_{J_\nu} (1-r)^{-1} F\Big(1-r, \Big|\sum_0^\infty A_k P(r^{\lambda_k})\Big|\Big) \frac{dr}{r},$$
$$J_\nu = \big(2^{-(1+\varepsilon)/\lambda_\nu}, 2^{-(1-\varepsilon)/\lambda_\nu}\big).$$

There hold the implication

$$\begin{aligned} k > \nu,\ r \in J_\nu &\implies r^{\lambda_k} \leqslant 2^{-q(1-\varepsilon)}, \\ k < \nu,\ r \in J_\nu &\implies r^{\lambda_k} \geqslant 2^{-(1+\varepsilon)/q}. \end{aligned} \tag{11.20}$$

Now we pass to the proof of (11.19). Assume first that $F \in \Delta(a,b;\,1,d)$. This means that the function $y \mapsto F(x,y)/y$ $(y > 0)$ decreases, which implies that $F(x, y_1 + y_2) \leqslant F(x,y_1) + F(x,y_2)$, whence

$$\begin{aligned} \mathrm{Int}_\nu(F) \geqslant{}& \int_{J_\nu} (1-r)^{-1} F\big(1-r,\, P(r^{\lambda_\nu})|A_\nu|\big) \frac{dr}{r} \\ &- \sum_{k\neq\nu} \int_{J_\nu} (1-r)^{-1} F\big(1-r,\, P(r^{\lambda_k})|A_k|\big) \frac{dr}{r}. \end{aligned}$$

By the relations (11.20) and the properties of P, for δ small enough, one can prove that the subtrahend is small with respect to $F(1/\lambda_\nu, |A_\nu|)$, while the minuend is $\geqslant K\,F(1/\lambda_\nu, |A_\nu|)$, where K is independent of δ, and this concludes the proof in the special case. (Details are omitted.)

On the other hand, applying Minkowski's inequality to the preceding inequality, we get, for $p > 1$,

$$\begin{aligned} \big\{\mathrm{Int}_\nu(F^p)\big\}^{1/p} \geqslant{}& \Big\{\int_{J_\nu} (1-r)^{-1} F^p\big(1-r,\, P(r^{\lambda_\nu})|A_\nu|\big) \frac{dr}{r}\Big\}^{1/p} \\ &- \sum_{k\neq\nu} \Big\{\int_{J_\nu} (1-r)^{-1} F^p\big(1-r,\, P(r^{\lambda_k})|A_k|\big) \frac{dr}{r}\Big\}^{1/p}, \end{aligned}$$

which is enough to complete the proof because an arbitrary function of class Δ can be represented as F^p, where $F \in \Delta(a,b;\,1,d)$. □

In order to deduce Theorem 11.4.1 from Proposition 11.4.3 we need two technical lemmas.

11.4.4 Lemma *If* $\alpha > 0$, $\beta > 0$, *then*

$$\int_r^1 (1-\rho)^{\alpha-1}(\rho - r)^{\beta-1} d\rho \leqslant C_{\alpha,\beta}(1-r)^{\alpha+\beta-1} \quad (0 < r < 1). \tag{11.21}$$

Proof. Let $\alpha < 1$ and $\beta < 1$.[‡] We split the interval $(0,1)$ by the point $\rho = \sqrt{r}$. If $r < \rho < \sqrt{r}$, then $(1-\rho)^{\alpha-1}(\rho-r)^{\beta-1} \leqslant (1-\sqrt{r})^{\alpha-1}(\rho-r)^{\beta-1}$. If $\sqrt{r} < \rho < 1$, then $(1-\rho)^{\alpha-1}(\rho-r)^{\beta-1} \leqslant (1-\rho)^{\alpha-1}(\sqrt{r}-r)^{\beta-1}$. The result follows by integration of these inequalities over the corresponding intervals. □

11.4.5 Lemma *If G is a positive measurable function on $(0,1)$ and $\gamma > 0$, then*

$$\int_0^1 (1-r)^{\gamma-1}G(r)\,dr \geqslant c_0 \int_0^1 (1-\rho)^{\gamma/2-1}d\rho \int_0^\rho (\rho-r)^{\gamma/2-1}G(r)\,dr,$$

where c_0 is a positive constant depending only on γ.

Proof. We can apply Fubini's theorem and appeal to inequality (11.21). □

Proof of Theorem 11.4.1. Let F satisfy condition (11.15) and assume there holds (11.16). Put $G(r) = (1-r)^{-b}F(1-r, |\mathcal{L}(r)|)$. According to Lemma 11.4.5 we have

$$\int_0^1 (1-r)^{-1}F(1-r, |\mathcal{L}(r)|)\,dr \geqslant c_0 \int_0^1 (1-\rho)^{b/2-1}d\rho \int_0^\rho (\rho-r)^{b/2-1}G(r)\,dr.$$

On the other hand, the inner integral equals

$$\begin{aligned}&\rho^{b/2}\int_0^1 (1-r)^{b/2-1}G(\rho r)\,dr\\ &= \rho^{b/2}\int_0^1 (1-r)^{b/2-1}(1-\rho r)^{-b}F(1-\rho r, |\mathcal{L}(\rho r)|)\,dr, \qquad \text{which is}\\ &\geqslant \rho^{b/2}\int_0^1 (1-r)^{-b/2-1}F(1-r, |\mathcal{L}(\rho r)|)\,dr.\end{aligned}$$

(Here we used the fact that $x^{-b}F(x,y)$ increases with x, which follows from (11.15).) Further, the function $x^{-b/2}F(x,y)$ belongs to $\Delta(a-b/2, b/2; c, d)$ so we can apply Proposition 11.4.3; we get

$$\int_0^1 (1-r)^{-b/2-1}F(1-r, |\mathcal{L}(\rho r)|)\,dr \geqslant c\,\lambda_n^{b/2}\,F(1/\lambda_n, |A_n|\rho^{\lambda_n}).$$

From these inequalities and the estimates

$$(1-\rho)^{-\gamma} \geqslant c_0 \sum_{n=0}^\infty \lambda_n^\gamma \rho^{\lambda_n}, \quad \text{where} \quad \gamma = 1-b/2 > 0,$$

we get

$$\int_0^1 (1-r)^{-1}F(1-r, |\mathcal{L}(r)|)\,dr \geqslant c_0 \int_0^1 d\rho \sum_{n=0}^\infty \lambda_n^{1-b/2}\rho^{\lambda_n+b/2}\lambda_n^{b/2}\,F(1/\lambda_n, |A_n|\rho^{\lambda_n}),$$

whence, by condition (11.15), we see that (11.17) holds. The proof of the implication (11.17) $\Longrightarrow$ (11.16) is much simpler and we omit it. The reader could see the papers [60] and [58]. □

[‡]Only this case will be used in the proof of the theorem.

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Index of notation

(A_n), 172
A^p, Bergman space, 11
A^p_β, weighted Bergman space, 141

$BV[a,b]$, 42
$\mathfrak{B}$, Bloch space, 133
BMO, 132

$C_0^2(D)$, 66
$\mathfrak{c}_0$, 11

$D_r(z)$, 156
$Du = \partial u/\partial\theta$, 163
$\Delta(a,b;\,c,d)$, 189
Δ_t^n, symmetric difference, 169
$d\mu_p$, 135
$dA = dm/\pi$, 83
dm, Lebesgue measure in $\mathbb{C}$, 36

$E_\varepsilon(z)$, 69
$\mathcal{E}$, 136

$\widetilde{f}$, conjugate function, 93, 94
f_*, boundary function, 48

H, Hilbert operator, 94
$H(\mathbb{D}, X)$, all X-valued analytic functions, 143
$H(\Omega)$, all analytic functions, 36
HC^1, 156
$HC^2(G)$, 158
$H^\infty(X)$, 144
H^p, Hardy space, 73
$\underline{H^p}(\mathbb{T})$, 102
$\overline{H^p}(\mathbb{T})$, 102
$h(\Omega)$, all harmonic functions, 36
hA^p, harmonic Bergman space, 150
$hC(\mathbb{D}) = C(\overline{\mathbb{D}}) \cap h(\mathbb{D})$, 39
h^1, 40
h^p, harmonic Hardy space, 46
$h_\infty(\psi)$, 170
$h_{\infty,n}(\psi)$, 170

$I(r,u)$, mean values, 58
$I(u)$, 69
$I_p(r,f)$, 134

$J_f(z)$, 106
$J_p(r,f)$, 60

$K_p(w,z)$, 135

$L^{p,\infty}$, weak Lebesgue space, 17
Λ_*, Zygmund class, 169
$\Lambda_\alpha(K)$, Lipschitz space, 162
$\mathrm{Lip}(\omega, K)$, Lipschitz space, 162
$\mathrm{Lip}_n(\phi)$, Lipschitz space, 169
λ_*, little Zygmund class, 169
$\mathcal{L}(r)$, 187
$\mathcal{L}_0$, all measurable functions, 26

$M(\mathbb{T})$, Borel measures, 39
M_*u, nontangential maximal function, 115
$M_p(r,f)$, integral mean, 46
$M_{\mathrm{rad}}u$, radial maximal function, 113
$\mu(f,\lambda)$, distribution function, 17
$\mathcal{M}$, maximal function, 21

$N_q(f)$, 5
$\mathcal{N}(\mathbb{D})$, Nevanlinna class, 10
$\mathcal{N}^+(\mathbb{D})$, Smirnov class, 10
$\mathfrak{N}_0(f)$, quasinorm in $\mathcal{L}_0$, 26

$OC^2(G)$, 159
$\mathcal{O}f(z,r)$, 157
$\mathcal{O}_pf(z,r)$, 157
$\omega_n(g,t)$, modulus of smoothness, 169

$P(z) = P(r,\theta)$, Poisson kernel, 37
PS, Poisson/Stieltjes integral, 42
$P[\cdot]$, Poisson integral, 38
P_nf, 103
$\widetilde{P}$, conjugate Poisson kernel, 93

R_+, Riesz projector, 98
$r_j(t)$, Rademacher functions, 24

SH, 156
$\sigma^\alpha_{\max}f$, 120
$\sigma^\alpha_n f$, 120

(U), 171
(U_α), 171
(U^0_β), 171

W_n (polynomials), 120
$W_{\max}$, 121

Subject index

absolute continuity of f_*, 79
approximation by inner

Banach envelope, 5
Blaschke condition, 81
Blaschke product, 81
Bourgain's lemma, 124

complemented subspaces of ℓ^p, 12
conformal mappings, 80
convexity of mean values, 58, 59

decomposition of H^p functions, 75
density of polynomials in H^p, 76
density of rational functions in L^p, 23
Dini condition, 165
Dirichlet problem, 38
Dirichlet/Jordan test, 186
disk-algebra, 163

F-norm, 9
Formula
    Cauchy integral, 79
    Green, 36, 67
    Jensen, 67, 81
    Parseval, 102
    Riesz representation, 68
Function
    almost increasing, 163
    atomic, 85
    conjugate, 94, 98
    harmonic conjugate, 93
    inner, 85, 90
    inner, singular, 86
    logarithmically convex, 59
    main maximal, 21
    nontangential maximal, 115
    of bonded mean oscillation, 132
    outer, 86, 87
    polyharmonic, 161
    Rademacher, 24
    radial maximal, 113
    semicontinuous, 53
    subharmonic, 53
    superharmonic, 53
    Weierstrass, 170
[illegible] property, [illegible]

Inequality
    Chebyshev, 17
    Hardy, 80, 84
    Hardy/Littlewood, 128
    Harnack, 41
    Heinz, 107
    Hilbert, 84
    isoperimetric, 83, 84
    Khintchine, 24
    Littlewood/Paley, 49, 70
    Paley, 26, 182
    Riesz/Fejer, 84
    Riesz/Zygmund, 97
integral means, 46
    of univalent functions, 60
interpolation for Hardy spaces, 123
isometry L^p to h^p, 48
isomorphism A^p to ℓ^p, 11

lacunary series, 130, 182, 187
    L^p-integrability, 190
Lebesgue point, 22, 23
Lebesgue set, 22, 23
Littlewood's conjecture, 129

majorant, 162
maximum principle, 55
    Smirnov, 77
mean value property, 36
moduli of smoothness, 169
multiplier, 11
multipliers, 32

Nevanlinna class, 10

Operator
    bilinear, 8
    from ℓ^p to ℓ^q, 12
    Hilbert, 94, 98, 164
    invertible, 5, 7

of strong type, 18
of weak type, 18
quasilinear, 18
[illegible], 18
of f_*, 48, 79
of a Borel measure, 39
of $\log|f_*|$, 77
of the conjugate function, 95
Poisson kernel, 37, 48
conjugate, 93
Poisson/Stieltjes integral, 42
Principle
Banach, 34
Littlewood subordination, 63
maximum, 55, 77
maximum modulus, 37
uniform boundedness, 8

q-Banach envelope, 5
of the space H^p, 142
quasinorm, 2

Radial limits
of h^p functions, 48
of H^p-functions, 76
of conjugate function, 94
of the Poisson integral, 43, 44
reproductive kernels, 136
Riesz measure, 66
Riesz projector, 98

Shauder basis, 8, 101
Smirnov class, 10
Space
$M(\mathbb{T})$, 39
$\mathcal{L}_0$, 26
$hC(\mathbb{D})$, 39
h^1, 40
$h_{\infty,n}(\psi)$, 170
$h_{o,n}(\psi)$, 170
F-space, 9
$h_{\infty,n}(\psi)$, 170
p-Banach, 4
Bergman, 11, 134
Bergman harmonic, 150
Bergman, weighted, 141
BMO, 132
Hardy, 73
Hardy harmonic, 46
weak Lebesgue, 17
Zygmund, 169
subordination, 63
symmetric difference, Δ_t^n, 169

Theorem
on coefficients in BMO, 132
on radial and nontangential limits, 46
Abel, 183
Ahern, 64
Aleksandrov, 102
Aoki/Rolewicz, 3
Banach/Steinhauss, 8
Beurling approximation, 88
Bieberbach, 61
Burkholder, Gundy
and Silverstein, 117
Carathéodory convergence, 111
Choquet, 97
closed graph, 9
Coifman/Rochberg, 138
complex maximal, 115
Dyakonov, 167
Fatou on nontangential limits, 45
Fatou on radial limits, 44
Fefferman, 132
Fefferman/Stein, 116
Gurarij/Matsaev, 187, 190
Hadamard's three circles, 59
Hardy, 59
Hardy/Littlewood, 127, 180
on h^p, $p < 1$, 154, 155
on Cesaro means, 120
on coefficients, 20, 126
on conjugate functions, 156
on conjugates, 151
Harnack, 42
Hausdorff/Young, 16
Hurwitz, 41
inner-outer factorization, 86
Kalton, 103

Karamata, 183, 187
Koebe, 61
Kolmogorov, 117
Kolmogorov/Smirnov, 64
Konyagin, 105
Lebesgue points, 22
Lindelöf, 46
Liouville, 60
Littlewood tauberian, 183, 185
Littlewood/Paley, 49, 70
local integrability, 56
Marcinkiewicz interpolation, 18, 19, 123
maximal, 21, 120, 123
McGehee, Pigno and Smith, 129
Mori, 107
Nikishin, 27
Nikishin/Stein, 31
on a.e. convergence, 34
on coefficients in $F \in H^\infty(X)$, 146
on coefficients in H^1, 128
on quasiconformal
harmonic homeomorphisms, 107
on subharmonic behavior, 150
on the dual of H^p, 101
open mapping, 7
Paley
on Fourier coefficients, 20
on lacunary series, 180
Prawitz, 60
Privalov on conjugates, 164, 166
Privalov/Plessner, 94, 96
radial maximal, 114
Riesz factorization, 82
Riesz on conjugate functions, 98
Riesz projection, 98
Riesz representation, 87
Riesz the brothers, 79
Riesz/Herglotz, 40–42
Riesz/Thorin, 13, 15
Rogosinski, 66
Sidon, 182
Stephenson on composition, 91
subharmonic maximal, 114
Tauber, 183
uniqueness, 35, 77
Weierstrass approximation, 38

weak type, 18