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F O R E W O R D

It became already tradition that Yugoslav graph theorists meet once in a year each time in another city. First two seminars, in Belgrade 1980, and in Ljubljana 1981, were called Belgrade-Ljubljana graph theory seminars. We already had the Third Yugoslav Seminar in Kragujevac 1982. The Fourth Yugoslav Seminar on Graph Theory has been held on April 15 and 16, 1983 in Novi Sad at the Institute of Mathematics, Faculty of Sciences, University of Novi Sad. There were about 20 participants from Yugoslavia and a few from abroad. This volume contains most of the papers presented at the seminar and a few others including the papers sent by colleagues from abroad for this occassion. We are very thankful for such contributions. The papers have been refereed and revised.

For technical preparation of the manuscript we are very thankful to Milan Vujošević, Stevan Vaderna and Dragan Acketa.

Novi Sad, February 15, 1984 Editors

C O N T E N T S

Proceedings of the Fourth Yugoslav Seminar on Graph Theory, Novi Sad 1983

GAMES "DO CONNECT" AND "DON'T CONNECT" ON K-GRAPHS

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ABSTRACT

We generalize and solve two well-known games on graphs, for the case of hypergraphs of a special type.

0. PRELIMINARIES

A *k-set* is a set of cardinality k.

A *k-graph* G on V is an ordered pair (V,E), where V is a finite set and F is a family of distinct k-subsets of V. The elements of V and the sets of E are the *vertices* and the *k-edges* of G respectively.

We define an equivalence relation ∿ on vertices of a k-graph G: two vertices x and y of G are in relation ∿ if and only if there exists a sequence $e_1, e_2, \ldots, e_s$ of k-edges, such that

$$x \in e_1, \quad y \in e_s, \quad e_i \cap e_{i+1} \neq \emptyset \quad \text{for} \quad 1 \leq i \leq s-1.$$

The clases of $\sim$ are the *connected components* of G. In most cases they will be called just "components".

Given $m,n \in N$, "$\mathrm{rest}_m(n)$" will denote the remainder of n, when divided by m.

1. INTRODUCTION

We introduce two games on k-graphs, which are generalizations of the well-known ([1],[3]) corresponding games on 2-graphs (that is, simple non-oriented graphs).

The initial position consists of the set V of n isolated vertices. Two players, A (the first) and B (the second), alternatively choose a k-subset of V and create the corresponding k-edge. Each k-subset may be chosen at most once. The game ends after the move which makes a k-graph on V with just one connected component. The player who makes this last move is the winner in the game "DO CONNECT" and the loser in the game "DON'T CONNECT".

Using some auxiliary results, we give the solutions for these two games. The general solutions should be modified for $k = 2$, $k = 3$ and for some relatively small values of n.

2. SOME AUXILIARY RESULTS

E. Lucas [2] has proved a theorem, which has the following special case:

(L) Let $n,k \in N$ and let $a_t \ldots a_2 a_1$, respectively $b_i \ldots$ $\ldots b_2 b_1$, be the binary expansions of n, respectively k. The binomial coefficient $\binom{n}{k}$ is odd if and only if

$$b_j = 1 \Rightarrow a_j = 1 \qquad \text{for } 1 \le j \le i$$

We derive several necessary consequences from (L).

Let $i \in N$ be such that $2^{i-1} \le k < 2^i$. Then:

(1) Given $k \in N$, the parity of $\binom{n}{k}$ is (uniquely) determined by $\mathrm{rest}_{2^i}(n)$;

(2) $\mathrm{rest}_{2^i}(n) < k$ implies that $\binom{n}{k}$ is even;

(3) $\mathrm{rest}_{2^i}(n) = k$ implies that $\binom{n}{k}$ is odd;

(4) $\mathrm{rest}_{2^i}(n) = 2^i - 1$ implies that $\binom{n}{k}$ is odd;

(5) Given $k \in N, k \ge 4$, there does not exist a maximal sequence of exactly $k-2$ consequtive remainders y from the set $\{0,1,\ldots,2^i-1\}$, satisfying the property that all the numbers $\binom{y}{k}$ are of the same parity.

Proofs of the consequences

(1) According to (L), when k is given, then the parity of $\binom{n}{k}$ depends solely on the binary word $a_i \ldots a_2 a_1$. This word is just the binary expansion of $\mathrm{rest}_{2^i}(n)$.

(2) Follows from (1) and from $\binom{y}{k} = 0$ for $y < k$.

(3) Follows from (1) and from $\binom{k}{k} = 1$.

(4) Follows from (1) and (L), because the binary expansion of 2^i-1 consists of i 1's.

(5) We primarily note that the sequence and the claim would be "empty" for $k = 2$. We give separate proofs for $\begin{Bmatrix} y \\ k \end{Bmatrix}$ even and for $\begin{Bmatrix} y \\ k \end{Bmatrix}$ odd:

a) The numbers $\begin{Bmatrix} y \\ k \end{Bmatrix}$ are even

Using (1) - (4) and $k \geq 2^{i-1}$, we easily find that the only possibility for appearance of the described sequence with the numbers $\begin{Bmatrix} y \\ k \end{Bmatrix}$ even, is the following one:

$$k = 2^{i-1} \quad \text{and} \quad y \in [2^{i-1}+1, 2^i-2] .$$

However, as the binary expansion of 2^{i-1} is $1\underbrace{0\ldots0}_{i-1}$, (L) gives that $\begin{Bmatrix} y \\ 2^{i-1} \end{Bmatrix}$ is odd for all the remainders y from $[2^{i-1}, 2^i-1]$. Thus the only possibility for a counterexample fails, provided that $k > 2$.

b) The numbers $\begin{Bmatrix} y \\ k \end{Bmatrix}$ are odd

Lemma. Let the last $s+1$ *digits of the binary expansion of* k *be* $b_{s+1}=1$, $b_s=b_{s-1}=\ldots=b_1=0$, *for some nonnegative integer* s. *Then the length of any maximal sequence of consecutive remainders* y *from* $\{0,1,\ldots,2^i-1\}$, *which satisfy that* $\begin{Bmatrix} y \\ k \end{Bmatrix}$ *is odd, equals* 2^s.

Proof of the lemma. According to (L), all the remainders y, which satisfy that $\begin{Bmatrix} y \\ k \end{Bmatrix}$ is odd, have a binary expansion of the form $c_i\ldots c_2c_1$, where $c_j=1$ if $b_j=1$ and $c_j \in \{0,1\}$ if $b_j=0$, for $1 \leq j \leq i$. All the remainders from $[2^i-2^s, 2^i-1]$ satisfy the last condition and it is easy to see that such remainders always appear in blocks of size 2^s.Q.E.D.

Observe that the definition of s gives that k is of the form $q \cdot 2^s$ for some odd q. On the other hand, the lemma says that a counterexample to (5) for $\binom{y}{k}$ odd would give $k-2 = 2^s$. This would imply that $\frac{k}{k-2}$ is an odd integer, which is true only for $k=3$. Thus the counterexample does not exist for $k > 3$.

3. THE SOLUTIONS

Theorem 1. Let $k,n \in N$ *(*$k \geq 3$ *and* $n \geq 3k$*) and* $\lceil \log_2 k \rceil = i$. *If* $x \in \{0,1,\ldots,2^i-1\}$ *is such that* $x+k \equiv n \pmod{2^i}$, *then the player* A *has a winning strategy in the game "DO CONNECT" if and only if the number* $\binom{x}{k}$ *is odd, otherwise the player* B *has a winning strategy.*

Proof. We primarily observe that one move can reduce the number of (connected) components by at most $k-1$. Namely, a player can choose k vertices to create a k-edge from one up to k distinct components. This implies that any position with $k+1$ components is critical in the game "DO CONNECT". The player who reduces the number of components under $k+1$ is the loser, for his opponent makes one component in the next move.

Suppose that a position with $k+1$ components having $n_1, n_2, \ldots, n_{k+1}$ elements is produced. Then each player will try to create k-edges only within these components. The maximal number of such moves is given by

$$s = \binom{n_1}{k} + \ldots + \binom{n_{k+1}}{k} . \tag{3.1}$$

If both players play rationally, then the outcome of the game depends solely on the parity of s: If s is even, then B wins; if s is odd, then A wins.

We should decide which one of the players has a strategy, which would nesessarily lead to a position on $k+1$ components, which is convenient for him. This player can be determined by calculating x and checking parity of $\binom{x}{k}$, as described in the theorem. We denote this player by W (winner) and his opponent by L (loser).

Following (1) and (3.1), a position on q components having $n_1 n_2, \dots, n_q$ vertices is completely determined by

$$\{rest_{2^i}(n_1), \dots, rest_{2^i}(n_q)\}.$$

We do not write brackets and commas any more.

The initial position of the game may be written as $\underbrace{1\dots1}_{n}$. The position $k\underbrace{1\dots1}_{n-k}$ arises after the first move of A.

We proceed with the description of a winning strategy of W. It suffices to restrict attention to the part of the game until the moment when (≤) $k+1$ components are created. The main rule of the strategy is:

> If W leaves more than $k+1$ components after some his move, then he leaves at most one component with the remainder different from 1.

The initial position enables W to start applying this rule. It can be easily checked that W can always keep this type of position, regardless of the moves of L. Even more, W can always obey the mentioned rule in such a manner, that the number of components, after each two moves of L and W respectively, is reduced by $k-1$ (when L does not change the components) or k.

The second rule of the strategy is:

> If the number of components before some move of W is greater than $3k$, then W always applies the above decribed reduction of the number of components.

W must be more careful with this reduction in the final stage of the game. At the moment when the number of components before his move enters the interval $[2k+1,3k]$, W reduces the number of components to $2k+1$, unless L has not already done so.

If L reduces the number of components under $2k+1$, then W wins by creating the position $x\underbrace{1\ldots 1}_{k}$ in the next move (the choice of the winner was actually based on the parity of this last position).

The position on $2k+1$ components serves as a point from which W can control the end of the game. As there are no more than two remainders different from 1 in this position, W can produce after his next move anyone of the positions

$x_{k+j}\underbrace{1\ldots 1}_{k+j-1}$, $2 \le j \le k$, where x_{k+j} is the element of $\{0,1,\ldots,2^i-1\}$, which satisfies $x_{k+j} + (k+j-1) \equiv n \pmod{2^i}$.

Given such a position, for some j between 2 and k, L is forced to immediately reduce the number of components to exactly $k+1$; otherwise W would make either the position

$x\underbrace{1\ldots1}_{k}$ or just one component in the next move.

The only two possible answers of L are

$$x\underbrace{1\ldots1}_{k} \qquad \text{and} \qquad x_{k+j}\; j\underbrace{1\ldots1}_{k-1} .$$

The first answer obviously loses. Our next lemma will help us to show that W can always reduce the number of components so that the second position is also convenient for him.

Lemma. There exists some j_o, $2 \le j_o \le k$, *such that*

$$\binom{x_{k+j_o}}{k} + \binom{j_o}{k} \equiv \binom{x}{k} \pmod 2 .$$

Proof of the lemma. Suppose, on the contrary, that

$$\binom{x_{k+j}}{k} + \binom{j}{k} \not\equiv \binom{x}{k} \pmod 2, \quad \text{for all } j, \quad 2 \le j \le k .$$

Using (2) and (3), we have

$$\text{(3.2)} \qquad \begin{cases} \binom{x_{k+j}}{k} \not\equiv \binom{x}{k} \pmod 2, \text{ for } \; 2 \le j \le k-1, \\[2ex] \binom{x_{2k}}{k} \equiv \binom{x}{k} \pmod 2 \end{cases}$$

We observe that $x_{2k},\ldots,x_{k+3},x_{k+2},x$ are k cyclically consecutive remainders from the set $\{0,1,\ldots,2^i-1\}$. In fact, due to (3.2) and (2)-(4), they are consecutive in the ordinary sense, with the only possible exception for $x=0$. This implies that (3.2) contradicts (5). The contradiction is also preserved with $x=0$, because the $k-2$ remainders y with $\binom{y}{k}$ odd are consecutive in the ordinary sense. Q.E.D.

As $\mathrm{rest}_{2^i}(n)=1$ implies (by (2)) that $\binom{n}{k}$ is even, the lemma gives that there exists some j_o, such that the positions

$$x\underbrace{1\ldots1}_{k} \qquad \text{and} \qquad x_{k+j_o}\,j_o\underbrace{1\ldots1}_{k-1}$$

are convenient for the same player, that is for W.

We conclude that W should reduce a position on $2k+1$ components to the position $x_{k+j_o}\underbrace{1\ldots1}_{k+j_o-1}$ in the next move. This last position serves as a "support", which enables W to jump safely over the "gap" between $2k+1$ and $k+1$ components.

This completes the proof of the theorem for $k \geq 4$. We point out that the claim (5) is not valid for $k=3$. A thorough inspection shows that the only exception arises with $k=3$ and $\mathrm{rest}_4(n)=3$. The "support" does not exists in that case. The theorem says that B should win and this is true, but B should alter his strategy in this exceptional case. One possibility is the following:

B can make after each his move just one of the following two types of positions:

1...1 (all 1's) or 31...1 (all 1's except for one 3).

Such a strategy leads necessarily to one of the possitions $\underbrace{1\ldots1}_{7}$ and $\underbrace{31\ldots1}_{8}$. It is easy to check that the first player who reduces the position $\underbrace{1\ldots1}_{7}$ is the loser. However, this position is "even" and B "can wait". If B has left the position $3\underbrace{1\ldots1}_{8}$ after one his move, then he can leave $\underbrace{1\ldots1}_{7}$ after his next move, unless A has made $33\underbrace{1\ldots1}_{5}$ in the preceding move. This last position is again "even" and the first reduction loses. This implies that B can always win. Q.E.D.

Theorem 2. Let $k,n \in N$ $(k \geq 3$ *and* $n \geq 2k+1)$ $\lceil \log_2 k \rceil = i$. *If* $x \in \{0,1,\ldots,2^{i-1}\}$ *is such that* $x+1 \equiv n \pmod{2^i}$, *then the player* A *has a winning strategy in the game "DON'T CONNECT" on k-graphs with* n *vertices if and only if the number* $\binom{x}{k}$ *is odd, otherwise the player* B *has a winning strategy.*

Proof. We can almost completely imitate the proof of Theorem 1. The main difference is that the critical number of components in the game "DON'T CONNECT" is always two. The player W should now jump over the "gap" between k+2 and 2 components.

The case k=3 requires again a special treatment. The "support" does not exist with k=3 and $rest_4(n)=1$. The theorem gives that B should win in that case. It is true, but he must not use the general strategy. One possibility is to make always the position of type 1...1 or 31...1 until B leaves one of the positions $\underbrace{1\ldots1}_{9}$ or $3\underbrace{1\ldots1}_{10}$.

B can transform the second of these positions into the first, unless A has made the position $33\underbrace{1\ldots1}_{7}$. B waits the first reduction in any of the positions $\underbrace{1\ldots1}_{9}$ and $33\underbrace{1\ldots1}_{7}$ and makes one of the positions $0\underbrace{1\ldots1}_{5}$ and $\underbrace{1\ldots1}_{5}$. He waits the next reduction (all these positions are "even") and wins. Q.E.D.

Remark. We cannot extend the assertions of theorems 1 and 2 to the case $k=2$. Given $rest_4(n)$ equals 0,1,2,3, in this order, these theorems would give that the winners are A,A,B,B for "DO CONNECT" and A,B,B,A in the case of "DON'T CONNECT". However, the well-known results ([1],[3]) say that the actual respective winners are A,B,B,A for "DO CONNECT" and B,B,A,A for "DON'T CONNECT".

4. ADDITIONAL ANALYSIS OF CASES WITH "SMALL" n

We assume that $k \geq 3$ ($k=2$ is solved in [1],[3]).

In $n < k$, then no moves are allowed in any of these games. The outcome may be defined arbitrarily.

"DO CONNECT"

$n = k$: A wins in the first move.

$k+1 \leq n \leq 2k-1$: B wins, for A must reduce the number of components under $k+1$ in the first move.

$n=2k$: A wins, for $k\underbrace{1\ldots1}_{k}$ is an "odd" position on $k+1$ components.

$2k+1 \leq n \leq 3k-2$: A makes after his first move one of the positions $k\underbrace{1\ldots1}_{j}$, where $k+1 \leq j \leq 2k-2$

No player may create a k-edge on some k isolated vertices, because this would yield $2+j-k < k+1$ components. The only possibility left is that the first component should be augmented until the number of components is lessened to $k+1$. This implies that A is the winner for $n = k+j$ $(k+1 \leq j \leq 2k-2)$

if and only if $\binom{j}{k}$ is odd and B is the winner otherwise.

This agrees with the general solutions.

n = 3k-1: B wins, for he can make the "even" position $kk\underbrace{1\ldots1}_{k-1}$ after his first move.

n ≥ 3k: The general strategy (and Theorem 1) holds, for the number of components after the first move is not smaller than 2k+1.

We observe that the positions of type $x_{k+j}j1\ldots1$ are not always possible (for example, one cannot create a k-edge on j isolated vertices, where j < k). This makes no problems, for the winner does not use these positions. It is even an advantage, for any position $x_{k+j}\underbrace{1\ldots1}_{k+j-1}$, with the property that the corresponding position $x_{k+j}j\underbrace{1\ldots1}_{k-1}$ is impossible, may serve as the "support" for the winner.

"DON'T CONNECT"

n = k: B wins, for A makes one component in the first move.

n = k+1: A wins, for he makes the "odd" position k1 on two components.

k+2 ≤ n ≤ 2k-1: A makes after his first move one of the positions $k\underbrace{1\ldots1}_{j}$, where 2 ≤ j ≤ k-1.

As j isolated vertices cannot be joined into a k-edge, the only way to make two components is to let one of them have k+j-1 vertices and the other one just one vertex. This gives that A is the winner for $\binom{k+j-1}{k}$ odd and B is the winner otherwise, what again agrees with the general solution.

$n = 2k$: B wins, for he can make the "even" position kk after his first move.

$n \geq 2k+1$: The general strategy (and Theorem 2) holds, for the number of components after the first move is not smaller than k+2. Non-possibility of some "alternative" positions on two components again does not disturb the general solution.

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AN INDUCTIVE DEFINITION OF THE CLASS OF ALL TRIANGULATIONS WITH NO VERTEX OF DEGREE SMALLER THAN 5

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ABSTRACT

It is proved that each triangulation with no vertex of degree smaller than 5 can be obtained from the icosahedron graph by a finite number of applications of transformations of given types.

From Euler's formula it follows that every planar graph has some vertices of degreee smaller than 6. Therefore the maximal planar graphs (triangulations) with no vertex of degree smaller than 5 are in some sense doubly maximal. They have, for example, an important role in the proof of the Four color theorem [5, page 62].

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An inductive class I is defined by giving [1,2]:

- initial specifications; which define the class B of initial elements - the basis of I;
- generating specifications; which define the class R of rules (modes) of combination - any such rule applied to an appropriate sequence of elements, already in I, produces an element of I.

The inductive class I = Cn(B;R) consists exactly of the elements which can be obtained (constructed) from the basis by a finite number of applications of the generating rules. A powerfull proof technique for the properties of elements of the inductive class is the inductive generalization (structural induction): in order to show that every element from I has a certain property P, it is sufficient to establish that:

- every element of the basis has the property P;
- the generating rules preserve the property P.

In this paper we shall prove the following theorem about the structure of the class of all triangulations (of the sphere) with no vertex of degreee smaler than 5:

Theorem. The inductive class Cn(B;P1,P2,P3) with the base graph B from Fig.1 (the icosahedron graph):

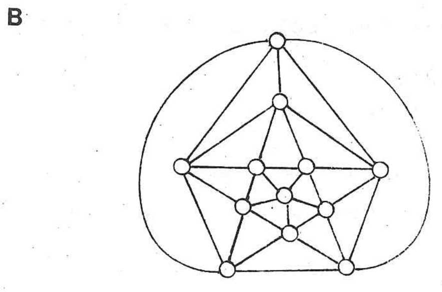

Fig.1.

and the generating rules P1,P2,P3 represented in Fig.2,

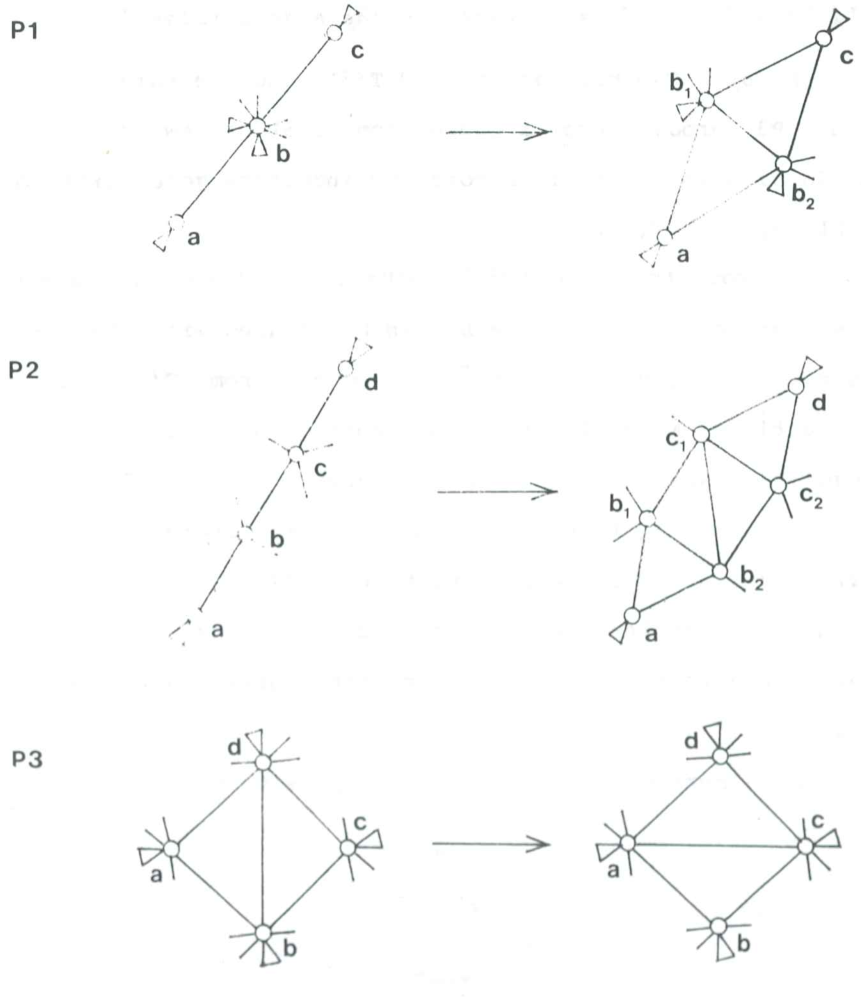

Fig.2.

is equal to the class T(5) of all triangulations (of the sphere) with no vertex of degree smaller than 5. (The small *triangles attached to the vertices in the description of the rules denote any number (zero or more) of edges. The rules should be understood as embedded in the sphere (plane)).*

Proof. The base graph $B \in T(5)$ and the rules P1, P2 and P3 produce from a graph from $T(5)$ a new graph which is also in $T(5)$. Therefore, by inductive generalization, $Cn(B;P1,P2,P3) \subseteq T(5)$.

To prove that also $T(5) \subseteq Cn(B;P1,P2,P3)$ we have to show that any graph $G \in T(5)$, $G \neq B$ can be reduced with the inverse rules $P1^-$, $P2^-$ and $P3^-$ to a graph from $T(5)$ with fewer vertices. Note also that the icosahedron is the only triangulation with all vertices of degree 5 [2, page 52].

Let $x \in V(G)$ be a vertex of maximal degree, i.e., $\deg(x) = \Delta(G)$. Because $G \neq B$ we have $\deg(x) \geq 6$. Let us say that in two triangles with a common edge the vertices, which do not belong to this edge, are opposite. There are two possibilities:

A. There exists a vertex y opposite to x with

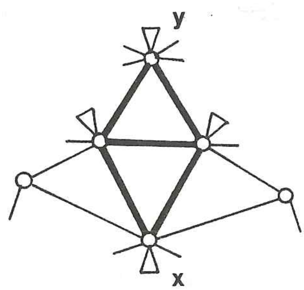

Fig.3.

deg(y) $\geq$ 6 (it can be also a neighbour of x). In this case we can apply the rule $P1^-$ (see Fig.3).

B. All vertices opposite to x are of degreee 5. Let us first show that no one of these vertices is a neighbour of x. Let us suppose the opposite (see Fig.4) the vertex y opposite to x, deg(y) = 5 is also adjacent to x. Because there are no parallel edges in G there is at least one vertex on the segment yz of "circle" around x.

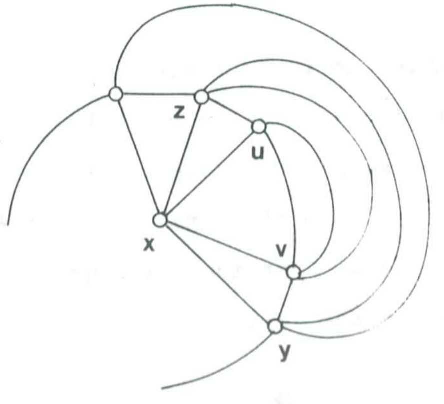

Fig.4.

Let v be the first among them from y. Because the vertex y is "saturated" there must be an edge connecting vertices z and v. But, then the vertex z is also opposite to x and by our assumption it is of degree 5. We may now repeat the same reasoning infinitely introducing a new vertex at each step. Which contradicts the finiteness of the graph G. Therefore we have the "crown" around x (see Fig.5; where black vertices denote the vertices of degree 5).

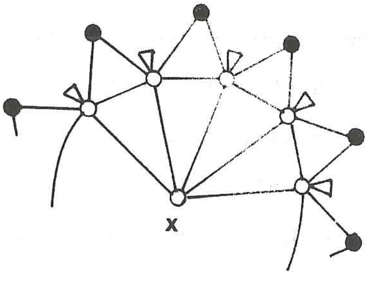

Fig.5.

We now turn our attention to the vertices on the circle around x. There are the following possibilities to be considered:

B1. There exists on the circle of x a pair of opposite vertices u and v, both of degree at least 6 (see Fig.6; the small squares denotes the vertices of degree at least 6). In this case we can apply the rule $P1^-$.

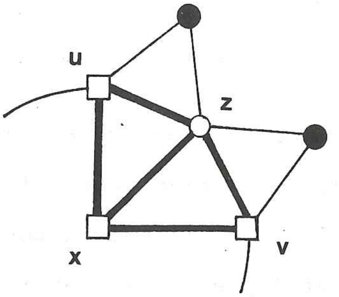

Fig.6.

This case includes as a subcase the case when deg(z) = 5. If all vertices on the circle of x are degree at least 6 we can choose for u and v any pair of opposite vertices from the circle.

B2. There exist on the circle of x a pair of consecutive vertices of degree 5 surrounded (on the circle) on both sides with vertices of degree at least 6 (see Fig.7a). We reduce this case, applying the rule $P3^-$, to the case B1.

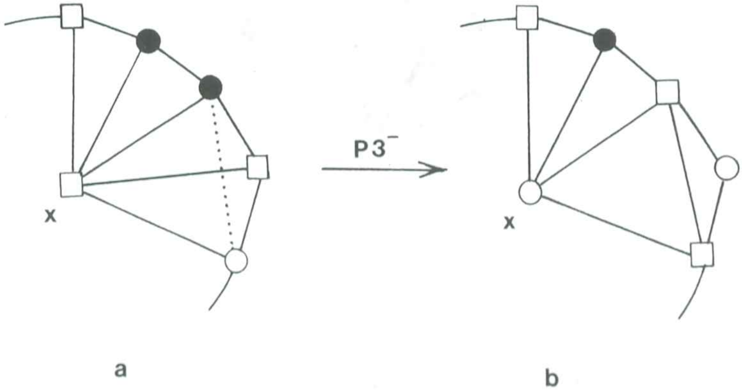

Fig.7.

B3. There exist on the circle of x at least 3 consecutive vertices of degree 5. Then we have, because there is also the crown around x, the configuration represented on Fig.8.

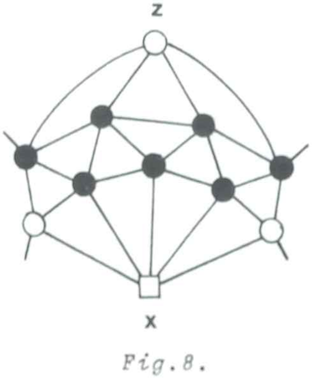

Fig.8.

which can be continued in two ways:

B3.1. The vertex z is of degree at least 6 (see Fig.9). We can apply the rule $P2^-$.

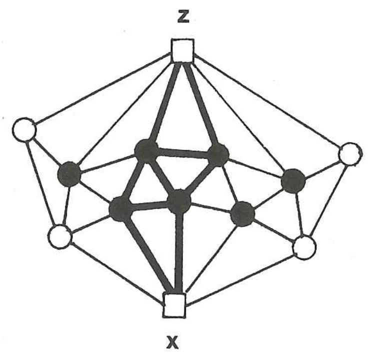

Fig.9.

B3.2. The vertex z is of degree 5. In this case we have the configuration represented on Fig.10 (or its "mirror" configuration).

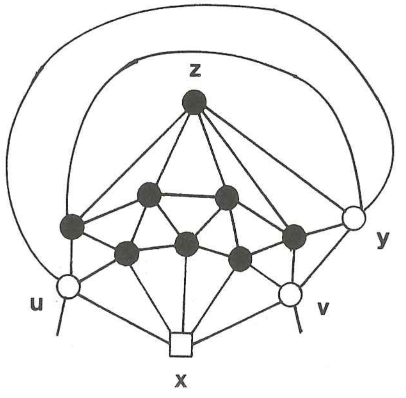

Fig.10.

If $\deg(u) = 5$ or $\deg(v) = 5$ the vertex y is opposite to x with degree at least 6 and we can apply the rule $P1^-$. Therefore we can assume in the following that $\deg(u) \geq 6$ and $\deg(v) \geq 6$. We can also assume that $\deg(x) \geq 8$; otherwise we have the situation from the case B1 or B2 on the rest of the circle. Therefore we have a configuration represented on Fig.11 which can be reduced to the case B1 by applying the rule $P3^-$ twice.

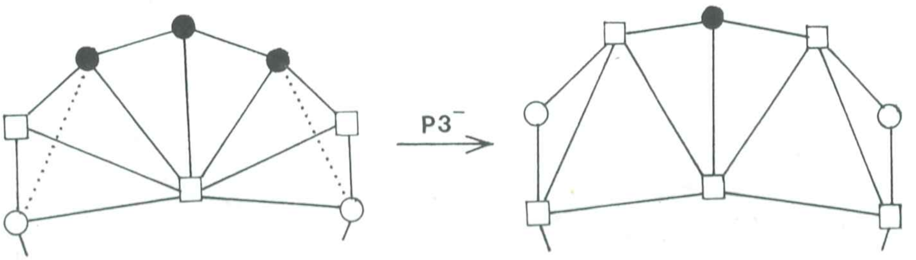

Fig.11.

This completes the proof of the theorem.

The generating rule P3 (diagonal transformation [4, page 9; 3]), although simple, has an unpleasant property - - to be bidirectional. Analyzing the proof of the theorem we can see that we used it only in two places (case B2 and case B3.2). Therefore we can replace it equivalently (with respect to its generative power) by two "augmenting" rules:

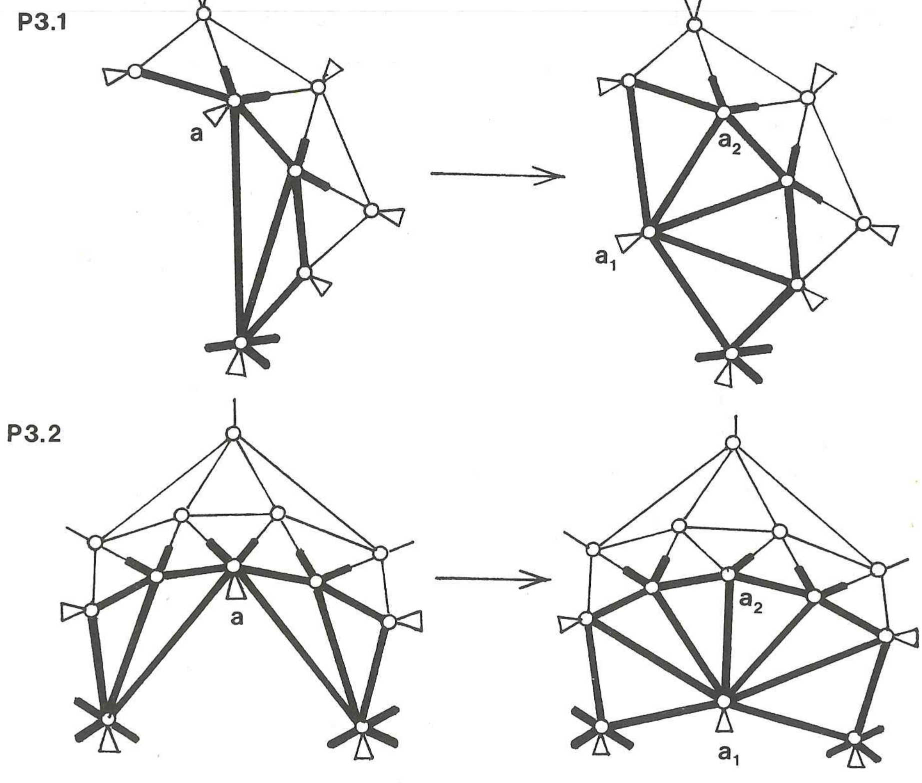

Fig.12.

where the heavy lines represent the essential parts of the rules.

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GENERAL SCHEME FOR GRAPH TRAVERSING ALGORITHMS

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ABSTRACT

Several schemata for graph (vertices and/or lines) traversing algorithms are unified into a single scheme.

In this paper, which is essentially a translation in English of the second part of the manuscript [2], an attempt is made to unify several schemata for graph traversing algorithms [1, 3, 4, 5, 9] into a single scheme. For an independent, but less general, discussion of the subject see also [7].

Let $G = (V,L)$ be a graph with the set of vertices V and the set of lines L. The set of lines is a disjoint union of the set of edges E and the set of arcs A, i.e., $L = E \cup A$.

In the following we shall use the mappings:

INIT : p $\longrightarrow$ initial vertex of arc p

TERM : p $\longrightarrow$ terminal vertex of arc p

EXT : p $\longrightarrow$ set of endpoints of line p

$$\text{ISARC} : p \longrightarrow \begin{cases} \text{true} ; & p \text{ is an arc} \\ \text{false} ; & \text{otherwise} \end{cases}$$

and the sets

$$\text{LINESTAR}(G,v) = \{p \in L \mid v \in \text{EXT}(p)\}$$

and

$$\text{OUTLINESTAR}(G,v) = \{p \in E \mid v \in \text{EXT}(p)\} \cup \{p \in A \mid v = \text{INIT}(p)\}$$

We extend the mapping EXT to subsets by defining for the subset L' of the set of lines L

$$\text{EXT}(L') = \bigcup_{p \in L'} \text{EXT}(p)$$

In the description of the scheme we shall also need the notion of the queue - an abstract data structure corresponding to explicitely or implicitely ordered multiset.

A queue of objects of type T which is explicitely ordered with components from an ordered type S is described in pascal like notation by

type Q = *queue of* T *key* v ;

where v : S is a selector from T.

Similary, a type declaration of the form

type Q = *queue of* T ;

describes an implicitely ordered queue where the order of objects in the queue is determined by the sequence of operations on the queue during program execution.

Assuming

var q : Q ; a : ↑T ; t : boolean;

we can describe the effects of the following procedures and functions:

CREATE(q)	open empty queue q
ADD(q,a)	add object a to the explicitely ordered queue q
ADDMIN(q,a)	add object a at the head of the implicitely ordered queue q
ADDMAX(q,a)	add object a to the end of the implicitely ordered queue q
t := EMPTY(q)	t = true iff the queue q is empty
a := MINFROM(q)	assigns the pointer to the first object from the queue q to a and removes the object from the queue

The explicitely ordered queues can be efficiently implemented by heaps; or by lists in the case of implicitly ordered queues. Staks and (ordinary) queues are special cases of general implicitely ordered queues:

stak (LIFO - last in first out)

```
        PUSH(q,a)   =   ADDMIN(q,a)
        POP(q)      =   MINFROM(q)
        TOP(q)      =   MIN(q)
```

(ordinary) queue (FIFO - first in first out)

```
        PUT(q,a)    =   ADDMAX(q,a)
        GET(q)      =   MINFROM(q)
```

To obtain further improvements the problem specifics should be considered.

Now we are ready to write down a general scheme for graph traversing algorithms:

```
procedure TOUR(G : graph; v0:vertex) ;
var candidates: queue of record
                    lin: line ;
                    val: values ;
                end key val ;
    test, visited, component: vertexset ;
    p: line; v: vertex ;

procedure  USEVERTEX  ...
procedure  USELINE  ...
function   ALLSEEN: boolean; ...
procedure  ADDINIT ...
procedure  ADDNEW ...
```

```
begin
   CREATE(candidates) ;
   component := [v0] + EXT(%%%STAR(G,v0)) ;
   USEVERTEX(v0) ;
   visited := [v0] ;
   ADDINIT(candidates, %%%STAR(G,v0)) ;
   while not ALLSEEN do begin
      p := MINFROM(candidates)↑.lin ;
      test := EXT(p) - visited ;
      USELINE(p) ;
      if test  <> [ ] then begin
        v := SELECT(test) ;
        USEVERTEX(v) ;
        ADDNEW(candidates, %%%STAR(G,v)) ;
        component := component + EXT(%%%STAR(G,v)) ;
        visited := visited + [v]
      end
   end
end   {TOUR} ;
```

In pascal the set-theoretic symbols {, }, ∈, ∪, ∩, - are replaced respectively by [,], in, +, *, -. The function SELECT has for its value element from a given nonempty set.

The procedures USEVERTEX and USELINE describe the problem specifics - actions to be performed in visited vertex/line. One of these two procedures can be empty.

The prefix %%% selects the type of traversing of lines:

%%% = OUTLINE - traversing of all lines/vertices lying on any path starting at v0

%%% = LINE - traversing of all lines/vertices lying on any chain starting at v0

In the scheme for the procedure ADDNEW we shall use the construct

<< condition : string >>

which indicates conditional compiling of the string.

```
procedure  ADDNEW (var candidates: cand; lines: lineset);
var   p: line ;
      q: ....
      test: vertexset ;
  begin
    for  p  in lines  do begin
      << %%% = OUTLINE :
         if ISARC(p)  then  test := INIT(p) * visited
                      else  test := EXT(p) - visited
      >>
      << %%% = LINE : test := EXT(p) - visited ; >>
      if test <> [ ]  then begin
        q.lin := p ;
        << &&& = : q.val := value(p) ;>>
        ADD&&&(candidates,q)
      end
  end
end  {ADDNEW} ;
```

The condition test <> [ ] in the procedure ADDNEW can be strengthened by specific problem constraints to improve the efficiency of the algorithm.

The procedure ADDINIT is a simplified version of the procedure ADDNEW.

The suffix &&& selects the order of traversing:

&&& = MIN - FIFO queue - stack: depth first search

&&& = MAX - LIFO queue : breadth first search

&&& = - priority queue.

In the special case when

value(p) = random

we obtain the random search strategy [8]

The condition ALLSEEN determines the type of traversing:

```
                  EMPTY(candidates)      - traversing lines
                /
ALLSEEN  =
                \
                  visited = component    - traversing vertices
```

where variable component represents:

- %%% = LINE : (weakly) connected component containing the vertex v0 of the graph G
- %%% = OUTLINE : set of vertices reachable from v0.

Note also

EMPTY(candidates) ⟹ visited = component

In a strongly connected graph (or in a weakly connected graph, if %%% = LINE) we have:

EMPTY(candidates) ⟹ visited = V

In this case we can remove from TOUR all the statements containing the variable component.

EXAMPLE: MINIMUM SPANNING TREE

Any traversing of vertices of an undirected connected graph determines a spanning tree in it. If we take for a value of the key of a queue element the value (nonnegative number) of the corresponding edge the so obtained procedure corresponds to the Prim's algorithm for minimum spanning tree.

In this case the procedure USEVERTEX is empty and the procedure USELINE consists of a single statement

```
if test <> [ ] then ADDLINE(T,p) ;
```

which is, because of "distributivity", transfered inside the conditional statement.

```
procedure MINSPANTREE(G: graph; var T: graph) ;
  . . .
begin
   CREATEGRAPH(T) ; CREATE(candidates) ;
   v := select(V) ; visited := [v] ;
   ADDINIT(candidates, LINESTAR(G,v)) ;
   while visited <> V do begin
      p := MINFROM(candidates)↑.lin ;
      test := EXT(p) - visited ;
      if test <> [ ] then begin
         ADDLINE(T,p) ;
         v := SELECT(test)
         ADDNEW(candidates, LINESTAR(G,v)) ;
         visited := visited + [v]
      end
   end
end {MINSPANTREE} ;
```

The procedure ADDINIT rest unchanged; while in the refinement of the procedure ADDNEW it is worth-while to consider that among the edges connecting a nonvisited vertex with the already visited vertices the edge with the minimal value will enter the minimum spanning tree.

For this purpose we introduce an array father which for a given vertex contains the edge connecting it to the nearest already visited vertex.

Because every edge from the queue candidates has exactly one endpoint in the set visited the condition test <> [ ] is always fulfilled and therefore it can be omited.

```
procedure  MINSPANTREE(G: graph; var T: graph) ;
   ......
begin
   CREATEGRAPH(T); CREATE(candidates);
   v := SELECT(V); visited := [v]
   for  u  in  V  do  father[u] := undefined ;
   for  p  in  LINESTAR(G,v) do begin
      u := SELECT (EXT(p) - [v]) ;
      father[u] := p ;
      ADD(candidates, (p,value(p)))
   end;
   while visited  <>  V  do begin
      p := MINFROM(candidates)↑.lin ;
      v := SELECT(EXT(p) - visited) ;
      ADDLINE(T,p) ;
      for  q  in LINESTAR(G,v) do begin
         u := SELECT(EXT(q) - [v]) ;
         if not (u  in  visited) then
            if father[u] = undefined then begin
               ADD(candidates, (q,value(q))) ;
```

```
            father u   := q
         end else if value(q) < value(father[u]) then
         begin
            DELETE(candidates, father[u]) ;
            ADD(candidates, (q,value(q)) ;
            father[u] := q
         end
      end;
      visited := visited + [v]
   end
end   {MINSPANTREE}  ;
```

In the implementation of this procedure we can also consider that one endpoint (vertex v) of the edge father[v] is already known. Therefore the edge is uniquely determined by its second endpoint. Also, at the end of the execution the array father and the graph T contains the same edges.

In [6] an implementation of the queue is described which allows us to implement the procedure MINSPANTREE in the time O(|E|).

If we introduce in the procedure MINSPANTREE an additional array d

d[v] = value of the shortest path from v0 to v

and we take for values of edges-candidates the adjusted values

d[SELECT(EXT(p) * visited)] + value(p)

we obtain Dijkstra´s shortest paths algorithm.

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A FORMULA INVOLVING THE NUMBER OF 1-FACTORS IN A GRAPH

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ABSTRACT

A formula connecting the number of 1-factors in some subgraphs of a graph is proved.

A set of independent edges that cover all vertices of a graph is called a 1-factor of that graph. The number of 1-factors of a graph G is denoted by $K(G)$, with $K(G) = 1$ if G has no vertices.

Proposition 1. If the edge u of a graph G with an even number of vertices joins the vertices x and y, then

$$K(G-x-y) \cdot K(G-u) = \sum_{Z} (K(G-Z))^2 \quad , \tag{1}$$

where Z is a circuit of G and the summation on the r.h.s. of (1) runs over all even circuits of G containing the edge u.

Proof. Let A and B be the sets of 1-factors of G which contain and which do not contain u. Then

$$|A| = K(G-x-y) \quad , \qquad |B| = K(G-u) \quad .$$

If $a_i \in A$ and $b_j \in B$, then $c_{ij} = a_i \cup b_j$ contains an even circuit Z passing through u. Both $a_i' = a_i \setminus Z$ and $b_j' = b_j \setminus Z$ are 1-factors of $G-Z$. The number of c_{ij}'s containing Z is equal to the number of ordered pairs (a_i', b_j'), i.e. $(K(G-Z))^2$. Since the total number of c_{ij}'s is equal to $|A||B|$, we get (1).

This completes the proof.

Formula (1) has been proved in [1] for hexagonal animals and its validity is now extended to all graphs with an even number of vertices.

Let now G be a graph with an odd number of vertices. Subdividing the edge u with the new vertex z we obtain from G the graph $G(u/z)$. Let us look for a number of 1-factors in $G(u/z)$. There are two possibilities represented in the Fig.1.

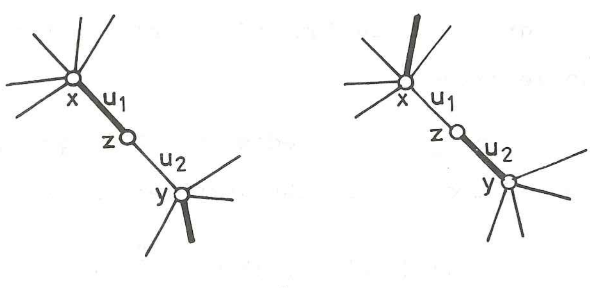

Fig.1.

It is evident that

$$(2)\qquad \begin{aligned} K(G(u/z)-x-z) &= K(G-x) = K(G(u/z)-u_2) \quad , \\ K(G(u/z)-y-z) &= K(G-y) = K(G(u/z)-u_1) \quad . \end{aligned}$$

By (1) it follows also

$$(3)\qquad K(G(u/z)-x-z)\cdot K(G(u/z)-u_1) = \sum_Z (K(G(u/z)-Z))^2 \quad ,$$

where Z runs over all even circuits of G(u/z) containing the edge u_1; and from (3) considering (2) we finally obtain the following proposition.

Proposition 2. If the edge u of a graph G with an odd number of vertices joins the vertices x and y, then

$$(4)\qquad K(G-x)\cdot K(G-y) = \sum_Z (K(G-Z))^2 \quad ,$$

where Z runs over all odd circuits of G containing the edge u.

Because graphs with odd number of vertices have no 1-factor we can combine (1) and (4) in the following theorem.

Theorem. If the edge u *of a graph* G *joins the vertices* x, y, *then*

$$K(G-x)\cdot K(G-y) + K(G-x-y)\cdot K(G-u) = \sum_Z (K(G-Z))^2 \quad ,$$

where Z *runs over all circuits of* G *containing the edge* u.

ACKNOWLEDGEMENT

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DISCUSSING GRAPH THEORY WITH A COMPUTER IV, KNOWLEDGE ORGANIZATION AND EXAMPLES OF THEOREM PROVING

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ABSTRACT

First experiences in theorem proving by the use of the interactive programming system "Graph" for the classification and extension of knowledge in the field of graph theory are described in this paper. Knowledge organization and some examples of theorem proving are described. Further experiments are necessary to make final conclusion about real effectiveness of the system.

1. INTRODUCTION

The interactive programming system "Graph" for the classification and extension of the knowledge in the field of graph theory, announced in [1] has recently been implemented at University of Belgrade, Faculty of Electrical Engineering. The main topic concerning this system are described in [2] - [5] and also in this paper. For additional literature on the system "Graph" see the list of references in [4] . See also [6] , [7] .

The system graph consists of the following three main blocks: BIBLI, ALGOR, THEOR. The block BIBLI represents a computerized graph theory bibliography [5] , ALGOR is a set of graph theoretical algorithms for solving problems on particular graphs [2] and THEOR is a system for manipulation with sentences, which represent assertions in graph theory, including theorem proving [4] .

First experiences with the ALGOR part have been described in [3] and in this paper we describe first experiences with the THEOR part. First we shall describe the work with "Graph" in general.

The system "Graph" is certainly very helpful to a researcher in the field of graph theory and its applications. However, that does not mean that "Graph" replaces completely human efforts. To get a result the researcher has to work hard and at some time "Graph" perhaps helps something, facilitating or accelerating the process of getting a new result.

The user can develop his own bibliography for a part of graph theory he is interested in. A typical command for storing bibliographical data is the following one:

STORE BOOK: "HARRARY F., GRAPH THEORY, ADDISON-WESLEY, 1969."

By a command such as:

"FIND PAPERS OF HARRARY F. ABOUT GRAPH ENUMERATION FROM 1970-1980"

the user can find the papers stored in the data base which fulfill certain requirements.

Hence, by the BIBLI part the user can produce several bibliographies or just find the papers he needs at the moment. More details about that part of the system can be found in [5] .

As described in [2] , [3] by the ALGOR part user can define a particular graph and by special commands he can ask the system to perform several tasks on that graph (e.g. to find chromatic number or the complement of this graph, to check whether it is planar, eulerian, hamiltonian, to calculate the graph spectrum etc.). An important tool in this part is the interactive graphics (graphical display with a light pen) what enables visual interaction between the user and the system.

Main features of the THEOR part are described in [4] and we shall assume that the reader is aquainted with this paper.

Quite formally, any graph theory statement can be proved by means of the system "Graph". Since the system allows the user to declare some subgoals as true, the user can say that the main goal is true and we have done. The system will consider the goal as proved. (Of course, for such a theorem proving the system "Graph" is not necessary).

To speak more seriously, quite interesting theorems can be proved if the user directs the proof in the right way. Crucial points are the branching of a proof according to some criterion (case analysis) and the introduction of a transition subgoal (forward chaining). On the other hand, completely automatic work (sending the goal or a subgoal to a resolution prover) is not much promising.

Is then the theorem prover of the system "Graph" helpful at all? This paper tries to answer this question with more details.

Generally, one of reasons why it is helpful is that in graph theory there are very long proofs (consisting of simple steps) with many branchings (case analysis). Without computer one can simply forget to consider all cases (as it occurred in some published papers).

Another reason is that the user can interrupt the proof at the moment and consider a subgoal on a model i.e. on a concrete graph which can be created or defined by the user in the ALGOR part. So, instead of checking whether a subgoal is true by a long and slow proof procedure, a graphical algorithm is invoked which does the job by a quick computation. Of course, in this way a counterexample could be found and not the proof of the validity of the subgoal. However, if some test on random graphs or graphs on which the user suspects that they could be counterexamples, gives positive results, the user is more convinced that the proof exists and will continue his efforts.

Third, it happens very often in proving theorems in graph theory that a subgoal is a statement about a concrete graph. As known, many theorems in graph theory hold for all but a finite number of graphs. These exceptional graphs give rise to the mentioned type of subgoals and again graphical algorithms from ALGOR can solve the problem.

The system "Graph" offers other possibilities to support indirectly a theorem proving process. Considering a subgoal the user could like to look at known theorems which are similar to the subgoal. The words occurring in the sentence which corresponds to the subgoal can be the basis for forming key phrases in the following command to the system:

FIND THEOREMS ABOUT "key phrases" .

Also the user can consult the bibliography with the same key phrases.

If the user is not satisfied by the search with key phrases formed on the basis of a sentence S, he can ask the system:

GENERATE SENTENCES EQUIVALENT TO S.

and form new key phrases on the basis of the sentences generated.

The system "Graph" is a good mean in teaching graph theory. Students themselves can speak with the system and learn. Together with the obvious suggestivity of the pictures of graphs on the screen and existing possibilities of modifying them with a light pen, the students can get definitions of graph theory notions, theorems connecting several notions, they can pose and check their own conjectures on concrete graphs using graph theoretic algorithm or even try to prove them by the theorem prover. They can get information about the literature.

Finally, system "Graph" is a nice toy. The messages of the system are designed so that it seems as if "Graph" is a human being. Therefore we shall refer to the system "Graph" with the pronoune "he" instead of "it". He can react in the style "I proved sentence S" or "The requested graph is not in the memory and I cannot perform the task". System "Graph" is generally polite and helpful: if you do not know how to tell him your requirement he would guide you to do that or would provide you with additional information.

If you get acquainted well with several possibilities of "Graph" and get used to him and if you are patient and persistent then he will be helpful to you in many occasions. He will become a good collaborator of yours.

Section 2 describes the way in which the knowledge of graph theory is stored in the system "Graph". Section 3 explains how some theorems are or can be proved using system "Graph". Concluding remarks are given in Section 4.

2. SOME DETAILS ABOUT THE KNOWLEDGE ORGANIZATION IN THE SYSTEM "GRAPH"

The sentence which we want to prove using system "Graph" should be typed in English (or more precisely in language GTCL (Graph Theoretical Computer Language), which represents a formalized subset of English) [4] . The sentence should be understood by the system and that means that the definitions of all necessary notions should be previously told to the system, i.e. the system should be previously taught the corresponding part of graph theory.

As explained elsewhere [1] , [4] the theory of graphs is formalized within the system "Graph" by the so-called arithmetical graph theory (AGT). The system translates the input English sentence into a formula of AGT and the theorem prover works with this formula. If necessary, formulas created in the theorem prover can be translated into GTCL.

The AGT is an extension of formal arithmetics and covers all parts of (pure) graph theory (directed and undirected graphs, multigraphs, etc). A part of the theory of finite, undirected graphs without loops or multiple lines can be covered by basic notions, definitions, axioms and lemmas given below. (Note that several other ways of developing AGT are, of course, possible).

GTCL sentences containing basic notions are followed by their translations into AGT, i.e. basic predicates of AGT. Basic predicates are denoted in the sequel by B1, B2, ...

B1. (Point) X and (point) Y are joined by (line) U ;

S1(X,Y,U) .

B2. Graph has N points;

Q1(N) .

B3. Graph has M lines;

Q2(M) .

Several versions of a sentence (with the same meaning) can be given to the system as separate definitions. For example, the last sentence can be reformulated in one of the following forms: "There are M lines", "Graph contains M lines". So, GTCL could be made richer in style, but that could make the work slower.

Definitions of graph theory notions are given in the following form. Only the definiens of the GTCL - definition is given. The whole definition is given as a formula of AGT (definens and definiendum connected by an equivalence). Definitions are labelled by $D_1, D_2, \ldots$.

Let us introduce the following predicates:

D1. X is a point;

$Q3(X) \Leftrightarrow 1 \le X \wedge (\forall N)(Q1(N) \Rightarrow X \le N)$.

D2. U is a line;

$Q4(U) \Leftrightarrow 1 \le U \wedge (\forall M)(Q2(M) \Rightarrow U \le M)$.

The basic predicates satisfy the following axioms:

A1. $\neg S1(X,X,U)$,

A2. $S1(X,Y,U) \Leftrightarrow S1(Y,X,U)$,

A3. $S1(X,Y,U) \wedge S1(X,Y,V) \Rightarrow U = V$,

A4. $(\exists N)(N \ge 1 \wedge Q1(N)) \wedge (Q1(N1) \wedge Q1(N2) \Rightarrow N1 = N2)$,

A5. $(\exists M)(M \ge 1 \wedge Q2(M)) \wedge (Q2(M1) \wedge Q2(M2) \Rightarrow M1 = M2)$,

A6. $S1(X,Y,U) \wedge S1(X1,Y1,U) \Rightarrow (X=X1 \wedge Y=Y1) \vee (X=Y1 \wedge Y=X1)$,

A7. $Q3(X) \wedge Q3(Y) \wedge S1(X,Y,U) \Rightarrow Q4(U)$,

A8. $Q4(U) \Rightarrow (\exists X)(\exists Y)(Q3(X) \wedge Q3(Y) \wedge S1(X,Y,U))$.

It is always understood in the sequel that a variable X satisfies Q3(X) and that a line variable U satisfies Q4(U). As explained in [4], the system "Graph" generates these formulas whenever a point or line variable occurs and the system considers it as useful. Hence we have a specific axiom scheme which simplifies the definitions which follow.

D3. Points X and Y are adjacent;

$R1(X,Y) \Leftrightarrow (\exists U) S1(X,Y,U)$.

D4. X and U are incident;

$R2(X,U) \Leftrightarrow (\exists Y) S1(X,Y,U)$.

D5. Lines U and V are adjacent;

$R3(U,V) \Leftrightarrow (\exists X)(R2(X,U) \wedge R2(X,V))$.

D6. X is isolated;

$Q5(X) \Leftrightarrow (\forall Y)\neg R1(X,Y)$.

D7. Graph is complete;

$P1 \Leftrightarrow (\forall X)(\forall Y)(X \neq Y \Rightarrow R1(X,Y))$.

D8. Graph is totally disconnected;

$P2 \Leftrightarrow (\forall X)(\forall Y)\neg R1(X,Y)$.

D9. Graph is trivial;

$P3 \Leftrightarrow Q1(1)$.

D10. Graph has triangle;

$P4 \Leftrightarrow (\exists X1)(\exists X2)(\exists X3)(R1(X1,X2) \wedge R1(X1,X3) \wedge R1(X2,X3))$.

Few interesting graph theory theorems can be formulated in terms of notions introduced so far. Nevertheless, when these notions are used later it is useful to have some lemmas containing them so that the system is not forced to use only axioms to achieve a goal in the proving process. Lemmas will be numbered by L1,L2, ... It is convenient, among other things, to declaire as lemmas the symmetry of predicates R1(X,Y) and R2(U,V):

L1. $R1(X,Y) \Rightarrow R1(Y,X)$,

L2. $R2(U,V) \Rightarrow R2(V,U)$,

L3. $P3 \Leftrightarrow (\forall X)(\forall Y) X = Y$.

Connectivity and metric properties of graphs can be described by introducing a new basic predicate:

B4. X and Y are joined by a walk of lenght K;

$S2(X,Y,K)$,

which satisfies the following axioms:

A9. $S2(X,Y,0) \Leftrightarrow X = Y$.

A10. $S2(X,Y,K+1) \Leftrightarrow (\exists Z)(S2(X,Z,K) \wedge (\exists U) S1(Z,Y,U))$.

Of course, the predicate $S2(X,Y,K)$ could be treated as a non-basic predicate with a recursive definition consisting of the two formulas just declared as axioms. However, we want to avoid recursive definitions and, at least formally, this can be done in the above manner.

The following notions can then easily be introduced:

D11. X and Y are joined by a walk;

$R4(X,Y) \Leftrightarrow (\exists K) S2(X,Y,K)$.

D12. Graph is connected;

$P5 \Leftrightarrow (\forall X)(\forall X)(\forall Y) R4(X,Y)$.

D13. X and Y are at distance K;

$S3(X,Y,K) \Leftrightarrow S2(X,Y,K) \wedge (\forall L)(L<K \Rightarrow \neg S2(X,Y,L))$.

D14. Graph is of diameter K;

$Q6(K) \Leftrightarrow P5 \wedge (\exists X)(\exists Y) S3(X,Y,K) \wedge (\forall X1)(\forall Y1)(\forall L)(S3(X1,Y1,L) \Rightarrow L \leq K)$.

D15. X is of excentricity K;

$R5(X,K) \Leftrightarrow (\exists Y) S3(X,Y,K) \wedge (\forall Y1)(\forall L)(S3(X,Y1,L) \Rightarrow L \leq K)$.

D16. Graph is of radius K;

$Q7(K) \Leftrightarrow P5 \wedge (\exists X R5(X,K) \wedge (\forall Y)(\forall Y)(\forall L)(R5(Y,L) \Rightarrow L \geq K)$.

D17. X is a central point;

$Q8(X) \Leftrightarrow (\forall K)(R5(X,K) \Rightarrow Q7(K))$.

The following lemmas are useful:

L4. $S2(X,Y,1) \Leftrightarrow R1(X,Y)$,

L5. $S2(X,Y,K+1) \Leftrightarrow (\exists Z)(S2(X,Z,K) \wedge R1(Z,Y))$,

L6. $S2(X,Y,K) \Leftrightarrow S2(Y,X,K)$,

L7. $R4(X,Y) \Leftrightarrow R4(Y,X)$,

L8. $R4(X,Y) \wedge R4(Y,Z) \Rightarrow R4(X,Z)$,

L9. S3(X,Y,K)<=>S3(Y,X,K),

L10. S3(X,Y,1)<=>R1(X,Y).

We want to point out the importance of these and other lemmas given in this section for the effectivness of the theorem prover. It is hopeless to expect from the system to infer everything from axioms. Among other things, it is desirable to declare as lemmas those theorems which require the induction in their proofs (although the theorem prover can handle the induction). The proposed lemmas represent, in fact, a beginner´s knowledge of graph theory. Beside these lemmas, which form a permanent part of the file of lemmas, the user can add any other graph theory theorem to this file.

The next group of notions is related to the point degree. We need more basic predicates:

B5. X and Y are joined by L of lines labelled by at most V;

T1(X,Y,V,L),

with axioms:

A11. T1(X,Y,1,1)<=>S1(X,Y,1),

A12. T1(X,Y,V+1,K+1)<=>(T1(X,Y,V,K)∧S1(X,Y,V+1)) ∨

(T1(X,Y,V,K+1)∧¬S(X,Y,V+1)),

A13. V<L=>¬T1(X,Y,V,L).

Next we define:

D18. X and Y are joined by L lines;

S4(X,Y,L)<=>(∀M)(O2(M)=>T1(X,Y,M,L)).

Another basic predicate:

B6. X is adjacent to K of points labelled by at most Y ;

S5(X,Y,K),

with axioms

A14. $S5(X,1,K) \Leftrightarrow S4(X,1,K)$,

A15. $S5(X,Y+1,K) \Leftrightarrow (\exists L)(L \leq K \wedge S5(X,Y,K-L) \wedge S4(X,Y+1,L))$.

Finally we define the degree of a point:

D19. X has degree K;

$R6(X,K) \Leftrightarrow (\forall N)(Q1(N) \Rightarrow S5(X,N,K))$.

The following definitions are now straightforward:

D20. Graph is regular of degree K;

$Q9(K) \Leftrightarrow (\forall X) R6(X,K)$.

D21. Graph is regular;

$P6 \Leftrightarrow (\exists K) Q9(K)$.

D22. Graph is a circuit;

$P7 \Leftrightarrow P4 \wedge Q9(2)$.

The following and some other lemmas are used in practical theorem proving rather than the above axioms:

L11. $Q1(N) \wedge R6(X,K) \Rightarrow 0 \leq K \wedge K \leq N-1$,

L12. $R6(X,K) \wedge K>0 \Rightarrow (\exists Y) R1(X,Y)$,

L13. $R6(X,0) \Leftrightarrow Q1(X)$,

L14. $(\exists X) R6(X,0) \Rightarrow \neg P5$,

L15. $Q1(N) \wedge (\exists X) R6(X,N-1) \Rightarrow P5$

We can treat graph operations within AGT. Definitions of graph operations will be denoted by OP1, OP2, ... For example, the complement of a graph can be introduced in the following way.

OP1. X and Y are adjacent in complement of G;

$R1GA1(X,Y) \Leftrightarrow X \neq Y \wedge \neg R1G(X,Y)$.

OP2. X and Y are adjacent in line graph of G;

$R1GA2(X,Y) \Leftrightarrow (\exists U)(\exists V)(X=U \wedge Y=V \wedge R3G(U,V))$.

Here an interesting effect of converting line variables to point variables appears.

Binary operations can be treated as well:

OP3. X and Y are adjacent in product of G_1 and G_2;

R1G1G2A3(X,Y)<=>(∃X1)(∃X2)(∃Y1)(∃Y2)(∀M)

(Q1G1(M)=>X=(X1-1)*M+Y1∧Y=(X2-1)*M+Y2∧

R1G1(X1,X2)∧R1G2(Y1,Y2)).

As explained in [4], the system will be able to understand the sentences concerning complement such as "Complement of G is connected" and will translate it as P5GA1, etc.

Useful lemmas:

L16. Q1G(N)=>Q1GA1(N),

L17. Q1G(N)∧Q2G(M)=>Q2GA1(N*(N-1)/2-M),

L18. R6G(X,K)∧Q1G(N)=>R6GA1(X,N-1-K),

L19. Q2G(M)=>Q1GA2(M),

L20. Q1G1(N1)∧Q1G2(N2)=>Q1G1G2A3(N1*N2).

Introducing subgraphs will enable to define a number of further graph theoretic notions. Subgraphs will be treated as graphs obtained by some operations from the original graph. The following three basic predicates are necessary:

B7. X and Y are adjacent in the induced subgraph number L;

S6(X,Y,L).

B8. X and Y are adjacent in the spanning subgraph number L;

S7(X,Y,L)

B9. X and Y are adjacent in the spanning subgraph number L of the induced subgraph number K;

S8(X,Y,K,L).

These predicates satisfy axioms which involve some arithmetic functions and which are rather complex. They are not reproduced here since they are used neither by the man nor by the computer in practical theorem proving. However, the system uses some lemmas (e.g., L21-L25). Next three definitions introduce subgraphs.

OP4. X and Y are adjacent in an induced subgraph of G;

R1GA4(X,Y)<=>(∃L)S6G(X,Y,L),

OP5. X and Y are adjacent in a spanning subgraph of G;

R1GA5(X,Y)<=>(∃L)S7G(X,Y,L),

OP6. X and Y are adjacent in a subgraph of G;

R1GA6(X,Y)<=>(∃K)(∃L)S8G(X,Y,K,L),

Now we have a series of definitions.

D23. Graph G has an induced circuit of length K;

Q10G(K)<=>Q1G<u>A4</u>(K)∧P7G<u>A4</u>.

In this definition the statements Q1G<u>A4</u>(K) and P7G<u>A4</u> are related to the same induced subgraph and this is indicated by underlining the operation symbol. This is an extralogical rule but it enables a more flexible treatment of definitions. Of course, the implementation of such a rule is quite easy.

D24. Graph G is hamiltonian;

P8G<=>(∀M)(Q1G(M)=>Q2G<u>A5</u>(M))∧P7G<u>A5</u>.

D25. Graph G has a circuit of length K;

Q11G(K)<=>Q1G<u>A6</u>(K)∧P7G<u>A6</u>.

D26. Graph G has girth K;

Q12G(K)<=>Q10G(K)∧(∀L)(Q10G(L)=>K≤L).

D27. Graph G is a forest;

P9G<=>¬(∃K)Q11G(K).

D28. Graph G is a tree;

P10G<=>P5G∧P9G.

D29. Graph G has a K-matching;

Q13G(K)<=>Q1GA6(K)∧Q9GA6(1).

Some useful lemmas:

L21. Q1G(N)∧Q1GA4(N1)=>N≥N1,

L22. Q1G(N)∧Q1GA5(N1)=>N = N1

L23. Q2G(M)∧Q2GA6(M1)=>M≥M1.

L24. P1G=>P1GA4.

L25. P10∧Q1(N)∧N≥2=> (∃X)R6(X,1).

Finally, we shall introduce the isomorphism of graphs. Suppose that all permutations of the points of a graph are ordered (labelled). The k-th relabelling of a graph G is the graph obtained from G by applying the k-th permutation to labels of its vertices.

B10. X and Y are adjacent in relabelling number K;

S9(X,Y,K).

Axioms are again omitted.

OP7. X and Y in a relabelling of G are adjacent.

R1GA7(X,Y)<=> (∃K)S9(X,Y,K).

D29. Graphs G_1 and G_2 are isomorphic;

P11G1/G2<=> (∃K)(∀X)(∀Y)(R1G1(X,Y)<=>S9G2(X,Y,K))

Since isomorphic graphs have the same global properties and the same values of (numerical) graph invariants we have a series of lemmas such as:

L26. P11G1/G2∧P1G1=>P1G2,

L27. P11G1/G2∧P5G1=>P5G2,

L28. P11G1/G2∧Q1G1(N1)∧Q1G2(N2)=>N1=N2,

L29. P11G1/G2∧Q2G1(M1)∧Q2G2(M2)=>M1=M2,

L30. P11G1/G2∧Q6G1(K1)∧Q6G2(K2)=>K1=K2.

In fact we have here a lemma scheme and the system alone generates them using the definition file and the types of predicates and variables in definitions. This lemma scheme is another example of domain specific features of the theorem prover of the system "Graph".

We shall not go on in this direction of considering more than one graph in one time. However, the notion of relabelling of graphs could be used to introduce further notions concerning one graph.

D30. Graph G is bipartite;

P12<=>(∃K)(∃L)(∀X)(∀Y)((X≤L∧Y≤L)∨(X>L∧Y>L)=>¬S9(X,Y,K)).

In order to make the language (proper GTCL) and the manipulation with formulas more flexible we introduce constants 01,02,..., and functions F1,F2,... by the following definitions.

F1. The number of points is equal to N;

01 = N<=>Q1(N).

F2. The number of lines is equal to M;

02 = M<=>Q2(M).

F3. The distance of X from Y is equal to K;

F1(X,Y) = K<=>S3(X,Y,K).

F4. The diameter is equal to K;

03 = K<=>Q6(K).

F5. The excentricity of X is equal to K;

F2(X) = K<=>R5(X,K).

F6. The radius is equal to K;

04 = K<=>Q7(K).

F7. Degree of X is equal to K;

F3(X) = K<=>R6(X,K).

3. EXAMPLES OF THEOREM PROVING

The Appendix of [4] contains the protocol of an interactive proof of the following theorem.

Theorem 1. If the graph G is not connected then the complement of G is connected.

As a subgoal in the proof of Theorem 1. the following lemma appears.

Lemma 1. If graph G is not connected and if points X and Y are joined by a walk in graph G then there exists Z such that X and Z are not adjacent and Y and Z are not adjacent.

Lemma 1. has been proved by a resolution based theorem prover (c.f.[6]) which is incorporated in the interactive prover. The resolution prover has used also the lemmas expressing the symmetry and transitivity of the relation "to be joined by a walk", i.e. R4(X,Y). The symmetry of this relation acts as a built-in theorem.

Before an attempt of the resolution proof begins the system consults the user about the usefulness of some relevant definitions and lemmas for the proof. The user should specify binary predicates for which the system should assume symmetry property. Also, the user could include induction as additional principle of inference. Except for this the proof procedure is fully automatized.

Commands in the interactive theorem proving are usually executed in a few seconds while the resolution prover could work several hours. Lemma 1. has been proved in about ten minutes (on a 32K machine under a nonoptimized overlay scheme).

This automated theorem prover has proved or can easily prove such simple theorems as the following ones.

Proposition 1. If points X and Y are isolated then X and Y are not adjacent.

Proposition 2. If X and Y are at distance 2, then there exists point Z such that X and Z are adjacent and Z and Y are adjacent.

Proposition 3. If graph is connected, then it is not true that there exists point X such that X is isolated, or graph is trivial.

A natural deduction (interactive) proof of Proposition 3. can easily be achieved practically only by instantiating of definitions. The proof tree has no branching and is reproduced below.

1. P5=> (∃X)Q5(X)VP3
2. P5=> (∃X)(∀Y)¬R1(X,Y)VQ1(1)
3. P5=> (∀X)(∃Y)R1(X,Y)VQ1(1)
4. (∀X1)(∀Y1)R4(X1,Y1)=>(∀X)(∃Y)R1(X,Y)VQ1(1)
5. (∀X1)(∀Y1)(∃K)S2(X1,Y1,K)=>(∀X)(∃Y)R1(X,Y)VQ1(1)
6. (∀X1)(∀Y1)(∃K)((K=0∧S2(X1,Y1,K))V(K>0∧S2(X1,Y1,K)))=> (∀X)(∃Y)R1(X,Y)VQ1(1)

7. (∀X1)(∀Y1)(∃K)((K=0∧X1=Y1)V(K>0∧(∃Z)(S2(X1,Z,K-1)∧ R1(Z,Y1))))=>(∀X)(∃Y)R1(X,Y)VQ1(1)

8. (∀X1)(∀Y1)(∃K)(∃Z)((K=0∧X1=Y1)V(K>0∧S2(X1,Z,K-1)∧ R1(Z,Y1))))=>(∀X)(∃Y)R1(X,Y)VQ1(1)

9. (∀X1)(∀Y1)(∃K)(∃Z)(X1=Y1VR1(Z,Y1))=>(∀X)(∃Y)R1(X,Y)V Q1(1)

10. (∀X1)(∀Y1)(∃Z)(X1=Y1VR1(Z,Y1))=>(∀X)(∃Y)R1(X,Y)VQ1(1)

11. (∀Y1)((∀X1)Y1=X1V(∃Z)R1(Y1,Z))=>(∀X)(∃Y)R1(X,Y)V (∀X2)(∀Y2)(X2=Y2).

Each step in this proof should be initiated by the user. However, with the improvements which are in progress the system can almost all steps do alone. Little intervention of man is necessary in step 5-6 (case analysis), 8-9 (introducing a transition subgoal) and 10-11 (using a lemma), although such steps (at least those in the form of this example) will be automatized in the future.

Next example is more complicated.

Theorem 2. If graph G has 6 vertices then graph G has a triangle or complement of G has a triangle.

A possible proof tree is given below. "Sons" of a subgoal are given two characters to the right and below their "father".

1. Q1(6)=>P4VP4A1
  2. Q1(6)=>(∃X)(F3(X)≥3VF3A1(X)≥ 3)
    3. (∃X)(Q1(6)=>F3(X)≥3VF3A1(X)≥3)
      4. (∀X)(Q1(6)=F3(X)≥3VF3A1(x)≥3)
        5. Q1(6)=>F3(X)≥3VF3A1(X)≥3)

6. $Q1(6) \wedge F3(X)<3 \Rightarrow F3A1(X) \geq 3$

7. $Q1(6) \wedge F3(X)<3 \Rightarrow 6-1-F3A1(X)<3$

true by lemma L18

8. $6-1-F3A1(X)<3 \Rightarrow F3A1(X) \geq 3$

true by arithmetical manipulation

9. $(\exists X)(F3(X) \geq 3 \vee F3A1(X) \geq 3) \Rightarrow P4 \vee P4A1$

10. $(\exists X)(F3(X) \geq 3) \vee (\exists Y)(F3A1(Y) \geq 3) \Rightarrow P4 \vee P4A1$

11. $(\exists X)(F3(X) \geq 3) \Rightarrow P4 \vee P4A1$

12. $F3(X) \geq 3 \Rightarrow P4 \vee P4A1$

this subgoal will be treated below

13. $(\exists Y)(F3A1(Y) \geq 3) \Rightarrow P4 \vee P4A1$

14. $F3A1(Y) \geq 3 \Rightarrow P4 \vee P4A1$

the procedure is similar as in subgoal 12.

The first step - spliting the goal 1 into subgoals 2 and 9 - is crucial and must be done by humane.

The subgoal 12 becomes the goal 1 in the next proof tree.

1. $F3(X) \geq 3 \Rightarrow P4 \vee P4A1$

2. $F3(X) \geq 3 \Rightarrow (\exists Y1)(\exists Y2)(\exists Y3)(Y1 \neq Y2 \wedge Y1 \neq Y3 \wedge Y2 \neq Y3 \wedge R1(X,Y1) \wedge R1(X,Y2) \wedge R1(X,Y3))$ lemma

3. $(\exists Y1)(\exists Y2)(\exists Y3)(Y1 \neq Y2 \wedge Y1 \neq Y3 \wedge Y2 \neq Y3 \wedge R1(X,Y1) \wedge R1(X,Y2) \wedge R1(X,Y3)) \Rightarrow P4 \vee P4A1$

4. $Y1 \neq Y2 \wedge Y1 \neq Y3 \wedge Y2 \neq Y3 \wedge R1(X,Y1) \wedge R1(X,Y2) \wedge R1(X,Y3) \Rightarrow P4 \vee P4A1$

5. $R1(X,Y1) \wedge R1(X,Y2) \wedge R1(X,Y3) \Rightarrow P4 \vee P4A1$

6. $R1(Y1,Y2) \wedge R1(X,Y1) \wedge R1(X,Y2) \wedge R1(X,Y3) \Rightarrow P4 \vee P4A1$

7. $R1(Y1,Y2) \wedge R1(X,Y1) \wedge R1(X,Y2) \Rightarrow P4$ true

8. $\neg R1(Y1,Y2) \wedge R1(X,Y1) \wedge R1(X,Y2) \wedge R1(X,Y3) \Rightarrow P4 \vee P4A1$

9. R1(Y1,Y3)$\wedge\neg$R1(Y1,Y2)$\wedge$R1(X,Y1)$\wedge$R1(X,Y2)$\wedge$R1(X,Y3) =>P4VP4A1

10. R1(Y1,Y3)$\wedge$R1(X,Y1)$\wedge$R1(X,Y3)=>P4 true

11. $\neg$R1(Y1,Y3)$\wedge\neg$R1(Y1,Y2)$\wedge$R1(X,Y1)$\wedge$R1(X,Y2) R1(X,Y3)=>P4VP4A1

12. R1(Y2,Y3)$\wedge\neg$R1(Y1,Y3)$\wedge\neg$R1(Y1,Y2)$\wedge$R1(X,Y1) $\wedge$R1(X,Y2)$\wedge$R1(X,Y3)=>P4VP4A1

13. R1(Y2,Y3)$\wedge$R1(X,Y2)$\wedge$R1(X,Y3)=>P4 true

14. $\neg$R1(Y2,Y3)$\wedge\neg$R1(Y1,Y3)$\wedge\neg$R1(Y1,Y2)$\wedge$R1(X,Y1) $\wedge$R1(X,Y2)$\wedge$R1(X,Y3)=>P4VP4A1

15. $\neg$R1(Y2,Y3)$\wedge\neg$R1(Y1,Y3)$\wedge\neg$R1(Y1,Y2)=>P4A1

16. Y1$\neq$Y2$\wedge$Y1$\neq$Y3$\wedge$Y2$\neq$Y3$\wedge\neg$R1(Y1,Y2)$\wedge$ $\neg$R1(Y1,Y3)$\wedge\neg$R1(Y2,Y3)=>P4A1 true

Case analyses in the proof should be done by the humane but there are some hopes that the computer processes them in the next future.

Theorem 3. There exist X,Y *such that* X$\neq$Y *and degree of* X *is equal to degree of* Y.

We shall prove this theorem by contradiction, i.e. we shall prove the following proposition from which Theorem 3 follows.

Proposition 4. If for all X,Y *if* X$\neq$Y *then degree of* X *is different from degree of* Y, *then graph is connected and graph is not connected.*

1. ($\forall$X)($\forall$Y)(X$\neq$Y=>F3(X)$\neq$F3(Y))=>P5$\wedge\neg$P5

2. ($\forall$X)($\forall$Y)(X$\neq$Y=>F3(X)$\neq$F3(Y))=>P5

3. (∀X)(∀Y)(X≠Y=>F3(X)≠F3(Y))=>(∃X1)F3(X1)=01-1

4. (∀X)(∀Y)(X≠Y=>F3(X)≠F3(Y))=>(∀K)(∃X1)F3(X1)=K

arithmetic lemma

5. (∃X1)F3(X1)=01-1=>P5

graph theoretic lemma

6. (∀X)(∀Y)(X≠Y=>F3(X)≠F3(Y))=> ¬P5

7. (∀X)(∀Y)(X≠Y=> F3(X)≠F3(Y))=>(∃X1)F3(X1)=0

8. (∀X)(∀Y)(X≠Y=>F3(X)≠F3(Y))=>(∀K)(∃X1)F3(X1)=K

the same as 4.

9. (∃X1)F3(X1)=0=>¬P5

graph theoretic lemma

The proof of the next theorem is only roughly outlined. The idea is to point out characteristic features implied by the isomorphism relation and not to deliver all technical details.

Theorem 4. If line graph of G *is isomorphic to graph* G, *then graph* G *is regular of degree 2.*

In the proof we use the predicate Q33(L) with the meaning: the sum of vertex degrees is equal to L. We use also the following lemma Q2(M) => Q33(2*M).

1. P11GA2/G=>Q9G(2)

2. P11GA2/G=>Q33G(2*01G)∧¬(∃X)F3G(X)=0∧¬(∃Y)F3G(Y)=1

3. P11GA2/G=>Q33G(2*01G)

4. P11GA2/G=>01G=02G

5. P11GA2/G=>01GA2=01G

lemma scheme

6. 01GA2=01G=>01G=02G

by L19

7. 01G=02G=>Q33G(2*01G,

by the above lemma

8. P11GA2/G=>¬(∃X) F3G(X)=0

separate proof by contradiction

9. P11GA2/G=>¬(∃Y)F3G(Y)=0

the proof is similar as in 8.

10. Q33G(2*01G)∧¬(∃X)F3G(X)=0∧¬(∃X)F3G(Y)=1=>Q9G(2)

11. Q33G(2*01G)∧¬(∃X)F3G(X)=0∧¬(∃Y)F3G(Y)=1=>

(∀Z)F3G(Z)=2

12. Q33G(2*01G)∧¬(∃X)F3G(X)<2=>(∀Z)F3G(Z)=2

arithmetical lemma

The first step is again very hard.

The subgoal 8 is equivalent to subgoal 1 in the next proof tree and this equivalence is a creative step in the proof. In the sequel we use the predicate Q34(N) with the meaning "a component of the graph is a path of length N" and some lemmas involving it.

1. P11GA2/G∧(∃X)F3G(X)=0=> (∃N)Q1G(N)∧¬(∃N)Q1G(N)

2. (∃N)Q1G(N)∧¬(∃N)Q1G(N)

3. (∃N)Q1G(N)

axiom

4. ¬(∃N)Q1G(N)

5. P11GA2/G∧(∃X)F3G(X)=0=>¬(∃N)Q1G(N)

6. P11GA2/G∧(∃X)F3G(X)=0⇒ P11GA2/G∧(∃Y)F3GA2(Y)=0

lemma scheme

7. P11GA2/G∧(∃Y)F3GA2(Y)=0=>¬(∃N)Q1G(N)

8. P11GA2/G∧Q34G(0)=>¬(∃N)Q1G(N)

9. P11GA2/G∧Q34G(0)∧(Q34G(N)=>Q34G(N+1))

=>¬(∃N)Q1G(N)

10. P11GA2/G$\wedge$Q34G(0)$\wedge$(Q34G(N)=>Q34G(N+1))=>(∀N)Q34G(N)

11. Q34G(0)$\wedge$(Q34G(N)=>Q34G(N+1))=>(∀N)Q34(N)

axiom scheme

12. (∀N)Q34G(N)=>¬(∃N)Q1G(N)

13. (∀N)Q34G(N)=>(∀X1)Q3G(X1)

lemma

14. (∀X1)Q3G(X1)=>¬(∃N)Q1G(N)

lemma

Theorems 2-4 can be proved by the use of system "Graph" in the way described above or along similar lines. Crucial steps should be done by man. However, a user which is well aquainted with the capabilities of the system could take profit of the interactive work with the system.

We believe that the above examples are typical for at least one part of graph theory and that they show both to user and to designers of the prover what the problems are.

4. CONCLUDING REMARKS

Examples from Section 3 show that it is possible to complete a proof of graph theory theorems of considerable complexity by the use of the system "Graph". In fact, with some efforts, one can construct proofs of far less trivial theorems in a similar way. However, in all these examples the user did know in advance a (non-formal) proof of theorems considered. Therefore, there does not exist so far an experience about real usefulness of the system in Theorem proving process. To explain the real help the system can provide the user, we must wait until a new theorem has been proved (if ever). We expect these experiences along the following two directions.

1. The system "Graph" has been used for some time in other parts (graph theory algorithms) for scientific research in graph theory. Now, the theorem prover will also be used and probably will help in proving new theorems. This will be supported by further development of knowledge stored in the system as well as by further (slight) improvements of the system "Graph".

2. Some experiments will be organized in which a known theorem will be given for proof to people, not very well aquainted with graph theory (say, student or mathematician working in other areas of mathematics).

Examples of Section 3 suggest also which modifications of system "Graph" should be performed to make the system more effective. As announced [4], the system should do alone some things which now are possible only with some intervention of the user. In fact, a new module - an overdirector of the proof tree - is being implemented. The overdirector would do some simple steps (e.g. deleting superflous brackets or quantifiers, applying tautologies etc.) but sometimes also more serious action (e.g. instantiating a definition or using a lemma).

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ROOT SYSTEMS, FORBIDDEN SUBGRAPHS, AND SPECTRAL CHARACTERIZATIONS OF LINE GRAPHS

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ABSTRACT

Using root systems and forbidden subgraphs we prove that the spectrum of a graph determines whether or not it is a regular connected graph except for 17 cases. Several known theorems follow from this result.

1. INTRODUCTION AND PRELIMINARY RESULTS

In this paper we shall investigate some of the relationships between a graph, G, and the spectrum of its 0-1 adjacency matrix, A(G). The amount of research concerning such relationships has grown enormously in the past ten years, and

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this is reflected in the recent book [5] on the subject. In this paper we shall use the terminology and notation of that reference. One of the basic problems in this area is the recovery of graphic (topological) information about G from its spectrum. Sometimes the graph itself is determined (up to isomorphism), and in this case we say that the graph is characterized by its spectrum. Many papers have been written concerning the characterization of graphs by their spectra and the construction of cospectral families of graphs. Regular line graphs have been of particular interest since it has been known for several years that, with only a finite number of exceptions, a graph that is cospectral with a given regular line graph is also a regular line graph. Thus the properties of the root graph come into play, and this often determines whether or not a graph is a regular line graph. The exceptions occur when the root graph is either one of the 3-connected regular graphs on 8 vertices or one of the connected semi-regular bipartite graphs on $6 + 3$ vertices.

Several tools have been used in the study of spectral properties of line graphs. A major one involves the theory of real root systems; they were exploited in the important paper of Cameron, et. al., [2], where the question of whether the spectrum determines if a graph is a line graph or not was reduced to a finite but rather large problem. Many new results were obtained, some by extensive computer searches. In fact by such a computer search it was seen that there could be only 17 exceptional cases for the regular line graphs described in the last paragraph [1, Table 1.5].

In Section 2 of this paper we shall investigate the root systems that give rise to line graphs. These have been classified previously [2], and have been derived by a variety of sometimes delicate algebraic arguments. We shall show that these arguments may be replaced by several short, straightforward, self-contained combinatorial ones, and an improved understanding of the imbeddings of one root system into another results. These will be applied in Section 3 to regular line graphs.

Another tool that has been useful is Seidel switching. We shall see that this is a natural tool to use when looking for cospectral graphs. In fact this will produce cospectral graphs that are not line graphs in all 17 of the exceptional cases.

A final tool is the construction of forbidden subgraphs. Although much of the work in this area has been subsumed by root system arguments, we shall see that it is still useful and a necessary key for the completion of the proofs.

We first some well known theorems in this area. We denote the smallest eigenvalue of G by $\lambda(G)$.

Theorem 1.1. [5, p.94] . *Let* G *be a graph with eigenvalues* $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. *Then* G *is regular if and only if* $n\lambda_1 = \sum_{i=1}^{n} \lambda_i^2$. *In that case* λ_1 *is equal to the degree of* G.

Theorem 1.2. [5, p.94]. *Let* G *be a regular graph with eigenvalues* $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. *Then the number of connected components is equal to the mutiplicity of* λ_1.

Theorem 1.3. [5, p.169] *For any graph* G, *we have* $\lambda(L(G)) \geq -2$. *The bound is attained if and only if* G *contains an even circuit or a connected component with two odd circuits.*

Theorem 1.4. Regular line graphs with least eigenvalue greater than -2 are characterized by their spectra.

Proof. From Theorem 1.3, the multiplicity of -2 being equal to zero implies that G is a tree or unicyclic with an odd cycle. Thus the only regular graphs G are odd cycles and complete graphs, both of which are characterized by their spectra by Theorems 1.1 and 1.2 (see also [5,p.72]).

Thus Theorem 1.1 tells us that the spectrum determines whether or not a graph is regular, and Theorem 1.2 tells us that when it is regular we can determine if it is connected. Theorem 1.4 tells us that the spectrum determines whether or not the graph is a regular line graph if $\lambda(G) > -2$. Hence from Theorem 1.3 we can focus our attention on graphs with $\lambda(G) = -2$. In order to do this, we must consider systems of lines in R^n.

2. STAR-CLOSED SYSTEMS OF LINES IN R^n FROM A COMBINATORIAL VIEWPOINT

In this section we wish to investigate the existence of sets of lines passing through the origin in R^n whose pairwise angles are 60°, or 90°. We wish them to be star-closed (as defined in [2]), i.e., given two lines in the set that meet 60°, the unique line that meets both of them at 60° is also in the set. The set of three lines is then called a *star*. If a system of lines always contains the third line of a star whenever the first two are present, the system is then called *star-closed*. We shall study the vectors that lie on these lines; the following lemma is obvious:

Lemma 2.1. Let S *be the set of all vectors of length* $\sqrt{2}$ *that lie on the lines of a star-closed system. Then these vectors satisfy the following properties:*

(P1) $x \cdot y = 0$ *or* ± 1 *for all* $x, y \in S$, $x \neq \pm y$,

(P2) $x \cdot x = 2$

(P3) $x \cdot y = -1$ *implies* $x + y \in S$, *and*

(P4) $x \in S$ *implies* $-x \in S$.

A system of lines that can be partitioned so that they lie in complementary orthogonal subspaces is called *decomposable*. The *dimension* of a system of lines is the dimension of the smallest real vector space containing them. Some examples of star-closed system and their relationship to real root systems are given in [2,16] and can be described as follows: let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of R^n. In each of the following cases the set of lines joining each point to the origin will form a star-closed set of lines:

(i) $A_{n-1} : \{e_i - e_j | 1 \leq i < j \leq n\}$

(ii) $D_n : \{e_i \pm e_j | 1 \leq i < j \leq n\}$

(iii) E_8: $D_8 \cup \{\pm\frac{1}{2}e_1 \pm \frac{1}{2}e_2 \pm \ldots \pm \frac{1}{2}e_8\}$ where the number of positive coefficients of the e_i's is even.

(iv) E_7: $\{x \in E_8 | x\cdot y = 0\}$ for a given vector $y \in E_8$

(v) E_6: $\{x \in E_8 | x\cdot y = x\cdot z = 0\}$ for given vectors $y, z \in E_8$ such that $y\cdot z = -1$.

Notice that all these defining vectors have length $\sqrt{2}$, E_7 is a subset of E_8 orthogonal to a line in E_8, and E_6 is a subset orthogonal to a star in E_8. The number of lines in A_n is $\frac{1}{2}n(n+1)$, in D_n is $n(n-1)$, in E_8 is 120, in E_7 is 63, and in E_6 is 36. For our purposes linear transformations will be used to imbed one system of lines (and their vectors of length $\sqrt{2}$) into another. In most cases they will be defined by their action on an orthonormal basis. As an example define $\phi(e_i) = e_i$, $i = 1,\ldots,7$, $\phi(e_8) = -e_8$. Thus ϕ multiplies the last coordinate by -1, preserves inner products and, by looking at the range of E_8 as defined above, gives a second description of E_8 as $D_8 \cup \{\pm\frac{1}{2}e_1 \pm \ldots \pm \frac{1}{2}e_8\}$ where the number of positive coefficients of the e_i's is odd. A less trivial example is given by using $j_n = e_1 + \ldots + e_n$ and defining $\phi : R^8 \to R^9$ by $\phi(e_i) = e_i - \frac{1}{6}j_9 + \frac{1}{2}e_9, i = 1,\ldots,8$. One easily verifies that the unique extension of this mapping to a linear transformation takes R^8 onto the hyperplane of R^9 orthogonal to j_9, and that the image of the lines in the second definition of E_8 consists of the lines through the vestors of A_8 plus all those through vectors of the form $-\frac{1}{3}j_9 + e_k + e_l + e_m, 1 \leq k < \ell < m \leq 9$. This is a useful alternative definition of E_8 and is also given in [2]. It also shows that A_8 can be imbedded into E_8. We shall see that this is essentially the only way to imbed A_8 into E_8.

As a final example, consider the vectors in $\{\pm\frac{1}{2}e_1 \pm \pm \frac{1}{2}e_2 \pm \dots \pm \frac{1}{2}e_6 \pm \frac{\sqrt{2}}{2}e_7\} \cup A_5$ where three of the first six coordinates are positive, and let ϕ be defined by $\phi(e_i) = = e_i$, $i = 1,\dots,6$, and $\phi(e_7) = \frac{\sqrt{2}}{2}(e_7 - e_8)$. This defines a linear transformation from R^7 into R^8 which preserves inner products, and the image of the given set is precisely the set of elements of E_8 orthogonal to the star consisting of the lines through $e_7 + e_8$, $\frac{1}{2}j_8 - e_7 - e_8$, $-\frac{1}{2}j_8$. Thus we have an alternate representation of E_6, and this will be useful later.

The basic object we wish to study is the *one-line extension* of a star-closed set. In the other words, we shall take a star-closed set, add a new line through the origin, that meets all the previous lines at 60° or 90°, and then star-close the system, i.e., add the third line of a star whenever the other two lines of it already appear in the system. If it meets all lines at 90° then a decomposable system results. Since we want to construct indecomposable systems, we shall always add a new line in such a way that it meets some line in the original system at 60°. Not only shall we see that the examples of A_n, D_n, E_8, E_7 and E_6, are essentially the only indecomposable ones, but we shall also see how the lower-dimensional examples can be imbedded into the higher-dimensional ones.

Let us start by considering A_n. We add a new line, star-close, and then consider the vectors along the new lines with length $\sqrt{2}$. We wish to consider the coordinates of the vectors with respect to the original orthonormal basis $\{e_1,\dots,e_{n+1}\}$. Adding a new line and star-closing will increase the dimension of the system by at most one, and hence we do not need to increase the size of the basis to describe the vectors along the new lines. Using our indecomposibility assumption, we may say that coordinates of a new vector with respect to $\{e_1,\dots,e_{n+1}\}$ are not all equal. By property (P1) from Lemma 2.1, any two unequal coordinates differ by 1, and hence there is a real number u such that each coordinate is equal to u or $u+1$. Now if the i^{th} coordinate is equal to u and the j^{th} is equal to $u+1$, the inner product with $e_i - e_j$ is -1 and hence by (P3) the sum is also in the system. This new vector is identical with the old one except that the i^{th} and j^{th} coordinates have been interchanged. Thus once one vector with t coordinates equal to $u+1$ and the remaining coordinates equal to u is in the system, then all such vectors with exactly t coordinates equal to $u+1$ are in the system.

Lemma 2.2. Suppose that A_n *is extended by a single line, star-closed, and that* V *is the set of vectors of length* $\sqrt{2}$ *along the resulting lines. Then*

(i) *there exists a real number* u *and a positive integer* t *such that* V *contains all vectors with* t *coordinates equal to* $u+1$ *and the remaining* $n+1-t$ *coordinates equal to* u,

(ii) $(n + 1)u^2 + 2tu + t - 2 = 0$

(iii) $t > 2$ *implies* $n \le \frac{t^2 - t + 2}{t - 2}$ *and*

(iv) $n \ge 2t - 1$.

Proof. The conclusion (i) has been explained and (ii) follows directly from (P2) of Lemma 1. Since the equation (ii) has real roots, it follows that $t^2 - (n + 1)(t - 2) \ge 0$ and hence (iii) follows. From (P4) we may assume $t \le \frac{n + 1}{2}$, and hence (iv) follows.

Putting properties (iii) and (iv) together we conclude that $1 \le t \le 4$. In fact all these values of t are realizable for appropriate values of n.

Case 1. $t = 4$. Conclusions (iii) and (iv) of Lemma 2 imply $n = 7$, and (i) implies $u = -\frac{1}{2}$. Thus we have all vectors with four coordinates equal to $\frac{1}{2}$ and four equal to $-\frac{1}{2}$, so that we have all vectors in E_8 orthogonal to the line through $\frac{1}{2}j_8$. Thus we have extended A_7 to E_7. Note that this extension of A_7 to E_7 in R^7 is in fact unique.

Case 2. $t = 3$. We now have $5 \le n \le 8$. For $n = 8$ we have $u = -\frac{1}{3}$ which gives us the alternate representation of E_8 mentioned previously. For $n = 5,6,7$, observe that two vectors which never have the value $u + 1$ in the same coordinate will have an inner product -1 by equation (ii) of Lemma 2.2. Hence their sum is in the system by Lemma 2.1, and we have $\binom{n+1}{6}$ new vectors in addition to the $\binom{n+1}{3}$ that have been added already. This gives a total of 36 lines for $n = 5$ and 63 lines for $n = 6$. For $n = 7$ the 8 vectors with one coordidinate equal to $3u + 2$ and the rest equal to $3u + 1$ are included by star-closing a vector of the first type that was added with one of the second type to give a total of 120 lines.

In each case we have imbedded A_n into E_{n+1}, as can be verified by considering the following maps: (i) $\phi(e_i) = e_i - \frac{1}{6}j_6 + \frac{1}{6(2u+1)}(e_7 - e_8)$, $i = 1,\ldots,6$; (ii) $\phi(e_i) = e_i - \frac{2u+1}{2(7u+3)}j_7 + \frac{1}{2(7u+3)}e_8$, $i = 1,2,\ldots,7$; (iii) $\phi(e_i) = e_i - \frac{2u+1}{2(8u+3)}j_8$, $i = 1,\ldots,8$. In each case this determines a linear transformation from R^n to R^8 that preserves inner products and imbeds A_n into E_{n+1}.

Case 3. $t = 2$. In this case define $\phi : R^{n+1} \to R^{n+1}$ to be linear with $\phi(e_i) = e_i + \frac{u}{2}j_{n+1}$. We then have ϕ fixing A_n and the image for the new vectors is $D_{n+1} - A_n$. Since ϕ preserves inner products, the addition of the new line creates an imbedding of A_n into D_{n+1}. Notice that vectors of this type appeared in the last case $(n = 7)$ so that in fact we have $A_7 \subset D_8 \subset E_8$.

Case 4. $t = 1$. Define $\phi : R^{n+1} \to R^{n+2}$ by $\phi(e_i) = e_i + \frac{u^2}{u-1}j_{n+2} + \frac{u}{u-1}e_{n+2}$. This is an inner product preserving linear transformation that fixes A_n and takes the new vectors into $A_{n+1} - A_n$.

Gathering the various cases together, we get the following result:

Theorem 2.1. Extending the system of lines A_n *by a single line gives precisely the following inclusions:*

(i) $A_n \subset A_{n+1}, n = 2,3...$

(ii) $A_n \subset D_{n+1}, n = 2,3...$

(iii) $A_5 \subset E_6, A_6 \subset E_7, A_7 \subset E_8, A_8 \subset E_8$, *and*

(iv) $A_7 \subset E_7$.

Now, in an analogous manner, let us extend D_n by a single line. Start by adding a new vector e_{n+1} so that $\{e_1,...,e_{n+1}\}$ in an orthonormal basis for R^{n+1}. Since D_n contains A_{n-1}, we may argue as we did for Lemma 2.2 that the first n coordinates of a vector along the line are $u + 1$ or u for some real number u. In addition since $e_i + e_j$ is in D_n, property (P1) implies that $u = 0$, $u = -\frac{1}{2}$ or $u = -1$.

Suppose $u = 0$. Then, in order to get a vector that is not already in the system, we must assume that the final coordinate is non-zero. This implies by (P2) that there is exactly one coordinate equal to 1 and hence the final coordinate is either 1 or -1. Using the vectors $e_i + e_j$ and properties (P3) and (P4) of Lemma 2.1, we get all the lines D_{n+1}.

When $u = -1$, we use property (P4) of Lemma 2.1 to put us in the case where $u = 0$.

Finally, suppose $u = -\frac{1}{2}$. Then the first n coordinates are $\pm\frac{1}{2}$ while the last coordinate is $\pm\frac{1}{2}\sqrt{8-n}$. Thus $n \leq 8$ and if $n = 8$ the extension lies in R^8, and clearly consists of all vectors with an even number of positive and an even number of negative coefficients, or of all vectors with an odd number of positive and an odd number of negative coefficients. In either case we have E_8. Since A_3 and D_3 are isomorphic, the only new case arises when $n = 4,5,6$, or 7. Now suppose we are in one of these cases and we are given a vector with the first n coordinates equal to $\pm\frac{1}{2}$. By (P4) we may assume the last coordinate is negative. For any two coordinates i and j with $1 \leq i < j \leq n$, one of the four vectors $\pm e_i \pm e_j$ has inner product -1 with the given vector and hence by (P3) there is a vector in the system identical with the original one except that the i^{th} and j^{th} coordinates have changed sign. Thus if the given vector has t of the first n coordinates positive, then any other vector of the same form whose number of positive coordinates has the same parity as t also appears in the system. Note, however, that two vectors with different parities cannot be in the system by (P1). For $n = 7$, this gives all the vectors in E_8. For $n = 6$, let $\phi(e_i) = e_i$, $i = 1,\dots,6$ and $\phi(e_7) = \frac{\sqrt{2}}{2}(e_7 + e_8)$. For $n = 5$, let $\phi(e_i) = e_i$, $i = 1,\dots,5$ and let $\phi(e_6) = \frac{1}{\sqrt{3}}(e_6 + e_7 + e_8)$. In each case we get $D_n \subset E_{n+1}$. For $n = 4$, there are 8 new vectors and these are easily identified with D_5. Collecting these results we get the following theorem:

Theorem 2.2. Extending D_n *by a single line yields precisely the following inclusions:*

(i) $D_n \subset D_{n+1}, n = 2,3,\dots,$ or

(ii) $D_5 \subset E_6, D_6 \subset E_7, D_7 \subset E_8, D_8 \subset E_8$.

Corollary 2.1. The star-closed systems of line E_n are maximal in R^n for $n = 6,7,8$ and the star-closed systems A_n and D_n are maximal for all other n.

Proof. Suppose we add a line to E_8; since $D_8 \subset E_8$, adding this line to D_8 produces an 8-dimensional extension of D_8, which by Theorem 2.2 must be E_8. But D_8 can be extended to E_8 in two ways, and any new line from one meets any new line from the other at an angle not equal to 60° or 90°. Hence E_8 must be maximal in R^8. For E_7, the result is even easier, for $A_7 \subset E_7$ and the extension of E_7 is unique. The system E_6 contains D_5 which in turn can be extended to E_6 in two ways. As with E_8, these extensions are mutually exclusive.

Theorem 2.3. The only one-line extension of E_6 is E_7; the only one-line extension of E_7 is E_8. The system E_8 cannot be further extended.

Proof. An extension of E_8 would be nine-dimensional by Corollary 2.1 and hence would be an extension of $D_8 \subset E_8$. By Theorem 2.2 this can only be D_9. Since D_9 has fewer elements than E_8, the extension of E_8 would imply the existence of a nine-dimensional extension of D_9 which is impossible. An extension of E_7 would be an eight-dimensional extension of A_7 and hence would contain A_8, D_8 or E_8. Since A_8 and D_8 both have fewer elements than E_7, and E_8 is maximal in R^8, we must have extended E_7 to E_8. Finally we consider an extension of E_6. As we saw before, we may represent E_6 by the lines through the following points: $\{\pm\frac{1}{2}e_1 \pm \frac{1}{2}e_2 \pm \ldots \pm \frac{1}{2}e_6 \pm \pm \frac{\sqrt{2}}{2}e_7\} \cup A_5 \cup \{\pm\sqrt{2}e_7\}$ where three of the first six coefficients are positive. Add a new line and form a star-closed set.

We first show that there is a line contained in the hyperplane of vectors with seventh coordinate equal to zero. Suppose z is a vector of length $\sqrt{2}$ on a line in the system and is not orthogonal to E_6. Let z_7 be the coefficient of e_7 for z. Assuming it is not zero, we have $z_7 = \pm \frac{\sqrt{2}}{2}$ and of the first six coefficients, $t \leq 3$ are equal to $u + 1$ while the rest are equal to u. Let x be a vector given in our definition of E_6 that has a coefficient of $-\frac{1}{2}$ whenever z has a coefficient of $u + 1$, and a coefficient of $-z_7$ for e_7. Then $z \cdot x = -\frac{1}{2}(t + 1) = 0$ or ± 1, and hence $t = 1$. Thus $z \cdot x = -1$ so that $x + z$ is in the system and has a coefficient of zero for e_7. Now any such vector in the system has its first six coefficients equal to v or $v + 1$, and, by the same reasoning as was used for the vector z, we get that $t = 2$, and hence we get $\binom{6}{2} = 15$ new vectors for a total (so far) of 51. Further, since this is an extension of A_5 with $t = 2$, we get that D_6 is contained in the extension, and hence the extension contains D_7 or E_7. If the extension contains D_7, then it properly contains D_7 which is not possible in a seven-dimensional space. Hence the extension contains E_7, and thus the extension is E_7.

3. REGULAR LINE GRAPHS WITH COSPECTRAL MATES

Suppose A is the adjacency matrix of a graph G with least eigenvalue $\lambda(G) \geq -2$. Then the matrix $\frac{1}{2}A + I$ is positive semidefinite, symmetric with 1's on the diagonal, and has the remaining entries equal to 0 or $\frac{1}{2}$. Hence this matrix can be interpreted as the Gram matrix of a set of unit vectors having mutual angles of 60° or 90°; the lines through these vectors meet at the origin at the same angles.

Conversely, given a set of lines passing through the origin of R^n and meeting at angles of 60° or 90°, unit vectors along the lines can be used to form a Gram matrix. This matrix is then symmetric, has 1's on the diagonal, and has 0 or $\pm \frac{1}{2}$ as entries so that in general we do not get a graph. Since there are two unit vectors on each line, an adroit choice can give rise to vectors that meet at angles of 60° or 90° so that a graph does arise. If vectors of length $\sqrt{2}$ are chosen instead and BB^T is the resulting Gram matrix, then $BB^T - 2I$ is the adjacency matrix of the corresponding graph. We then say that the set of lines *represents* the graph. Hence the study of graphs with $\lambda(G) \geq -2$ involves the star-closed lines described in the last section. Since the only maximal indecomposable sets of lines in R^n, as described in Corollary 2.1, are A_n and D_n whenever $n \neq 6,7$ or 8, and since $A_n \subset D_{n+1}$, the following proposition is clear:

Proposition 3.1. Let G be a connected graph with $\lambda(G) = -2$. Then G can be represented by lines in either D_n or E_8. In the latter case, the lines lie in a subspace of R^8 of dimension 6,7 or 8.

The *cocktail party graph* $CP(n)$ is the graph on $2n$ vertices which is regular with degree $2n - 2$. In other words, it is the graph obtained from K_{2n} by deleting a 1-factor. Notice that CP(n) can be represented in R^{n+1} by the 2n vectors $e_{n+1} \pm e_i, i = 1,\ldots,n$. For $n = 0$, it is the graph with no vertices.

Suppose G is a graph with n vertices, and $(a_1,\ldots,a_n)$ is an n-tuple of nonnegative integers. Then the *generalized line graph* $L(G;a_1,\ldots,a_n)$ is obtained by taking the line graph L(G) and adjoining n disjoint cocktail party graphs $CP(a_i), i = 1,\ldots,n$. A vertex in $CP(a_i)$ is adjacent to one in L(G) if and only if the vertex in L(G) corresponds to an edge of G with vertex i as an end point.

If $L(G;a_1,\ldots,a_n)$ is a generalized line graph and $m = \sum_{i=1}^{n} (1 + a_i)$, then the graph may be represented in R^m by taking the orthonormal basis $\{e_{i,j} | 1 \le i \le m, 0 \le j \le a_i\}$ and the vectors $\{e_{i,0} + e_{j,0} | \{i,j\}$ an edge of $G\} \cup \{e_{i,0} \pm \pm e_{i,k} | 1 \le k \le a_i, 1 \le i \le n\}$. Hence any generalized line graph can be represented by vectors in D_m, and the least eigenvalue of a generalized line graph is bounded from below by -2. In fact the singularity of the Grammian will insure that this bound is attained if the number of vertices is greater than $\sum_{i=1}^{n} (1 + a_i)$.

Proposition 3.2 [2]. A graph can be represented by the root system D_n if and only if it is a generalized line graph.

Proposition 3.3 [2]. A regular generalized line graph is either a line graph or a cocktail party graph.

Proposition 3.4. If G is a regular connected graph with least eigenvalue equal to -2, then

(i) G is a line graph,

(ii) G is a cocktail party graph, or

(iii) G is represented by an indecomposable set of lines in R^n, $6 \leq n \leq 8$.

Proof. The result follows from Proposition 3.3 and Corollary 2.1.

Proposition 3.5. If G is not a line graph but is cospectral with a regular, connected line graph, then G can be represented by a set of lines in R^n, $6 \leq n \leq 8$.

Proof. Cocktail party graphs are characterized by their spectra by Theorem 1.1. Hence the result follows immediately from Proposition 3.4.

A graph G with n vertices that is cospectral to a regular connected line graph but is not itself a line graph is called an *exceptional graph*. If the eigenvalue -2 has multiplicity p, then the matrix $I + \frac{1}{2}A$ is a Gram matrix of unit vectors and has rank r where $r = n - p$. We wish to focus on the eigenvalues of G greater than -2; this motivates the following definition: the eigenvalues of a graph G greater than $\lambda(G)$ are called the *principal eigenvalues* of G.

Proposition 3.6 [1]. The number of principal eigenvalues of an exceptional graph is either 6,7 or 8.

Proof. The rank of the associated Gram matrix is the dimension of the smallest real vector space containing the lines representing the graph. By Proposition 3.5 this dimension is equal to 6,7 or 8.

The complement of graph G is denoted by $\overline{G}$. Also, the direct sum of the graphs G and H is denoted $G \cup H$.

Proposition 3.7. A regular connected line graph can have an exceptional cospectral mate only if it is the line graph of one of the following 30 graphs:

(i) $K_{m,n}, 2 \le m \le n, 7 \le m+n \le 9$,

(ii) $K_n, n = 6,7,8$,

(iii) $CP(n), n = 3,4$,

(iv) $\overline{C}_n, n = 6,7,8$,

(v) $\overline{C_m \cup C_n}, \{m,n\} = \{3,4\},\{3,5\}$, or $\{4,4\}$

(vi) G or $\overline{G}$ where G is regular, connected, and cubic with 8 vertices,

(vii) the semiregular bipartite graph with parameters $(m,n,r_1,r_2) = (6,3,2,4)$.

Proof. As proven in [9], if G has m edges and n vertices, then the multiplicity of -2 as an eigenvalue of $L(G)$ is $m - n + 1$ if G is bipartite and $m - n$ otherwise. Hence the number of principal eigenvalues of $L(G)$ is $n - 1$ or n respectively. From Theorem 1.1 and Theorem 1.2 we see that $L(G)$ is regular and connected and hence G is connected and either regular or semiregular bipartite. From Proposition 3.6 we see that the number of vertices G is at least 6 and at most 9. The only such graphs are those in the conclusion.

Given a subset X of the vertex set of a graph G, we form a new graph G_X by letting two vertices in X or in the complement of X be adjacent if they are adjacent in G while a vertex in X and one in the complement are adjacent G_X if and only if they are not adjacent in G. A graph H is called *Seidel switching equivalent* to G if H is isomorphic to G_X for some $X \subseteq V(G)$. The necessary and sufficient condition on X to switch a regular graph into one of the same degree is easy to see.

Proposition 3.8. Suppose G is regular with n vertices and degree r. Then G_X is regular of degree r if and only if X induces a regular subgraph of degree k and $|X| = n - 2(r - k)$.

Proposition 3.9. If G and G_X are regular of the same degree, then G and G_X are cospectral.

Proof. If x is an eigenvector of $A(G)$ whose corresponding eigenvalue is not the degree, and y is defined by letting $y_i = x_i$ for $i \in X$ and $y_i = -x_i$; otherwise, then y is an eigenvector $A(G_X)$ with the same eigenvalue.

We now wish to use Seidel switching to produce cospectral regular line graphs. In other words, we wish to switch with respect to a set of lines in the root graph that induce a regular graph in the line graph; thus these edges induce a regular or semiregular bipartite subgraph of G. In particular, if we use the value $k = 0$ from Proposition 3.8, then the edges in G are disjoint. If we use $k = 1$, then the edges in the root graph consist of copies of $K_{1,2}$.

Proposition 3.10. Suppose G is regular and contains a 1-factor and a 4-cycle that intersect in a single edge. Then there exists a graph H such that H and $L(G)$ are cospectral and hence $L(G)$ is not characterized by its spectrum.

Proof. Let X be a subset of the 1-factor with cardinality $n - 2r$ that contains the edge of the circuit and apply Proposition 3.8 with $k = 0$. Then switching $L(G)$ with respect to X produces a cospectral graph that contains a copy of $K_{1,3}$ and hence is not even a line graph, much less isomorphic to $L(G)$.

Proposition 3.11. The line graphs of the following 17 graphs are cospectral with an exceptional graph:

(i) $K_{4,4}, K_{3,6}$

(ii) $CP(4)$,

(iii) K_8

(iv) $\overline{C}_8$

(v) $\overline{C_m \cup C_n}$, $\{m,n\} = \{3,4\},\{4,4\}$,

(vi)-a G where G is regular, connected, and cubic on 8 vertices (four graphs in all),

(vi)-b $\overline{G}$ where G is a regular, connected graph on 8 vertices (five graphs in all), and

(vii) the semiregular bipartite graph with parameters $(m,n,r_1,r_2) = (6,3,2,4)$.

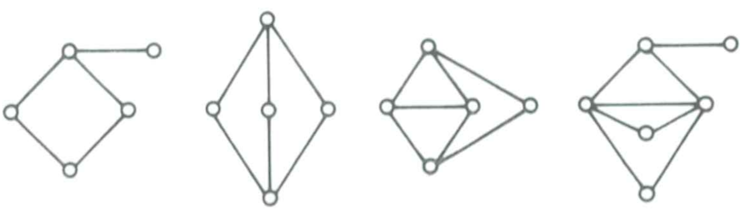
Proof. In each of the cases (i)-(vi), there is 1-factor and a 4-cycle satisfying the hypotesis of Proposition 3.10.

4. FORBIDDEN SUBGRAPHS

In the last section we showed that 17 of the 30 possible regular line graphs possessed cospectral mates. In this section we show that the remaining 13 graphs from Proposition 3.7 are indeed characterized by their spectra. In this way we will know the complete story as far as spectral characterizations of regular line graphs is concerned.

The basic technique used in these characterizations is to try to construct an exceptional graph and to deduce that a subgraph occurs which is impossible. This method, first used by A.J. Hoffman, is essentially a consequence of the bounds of the Rayleigh quotient (see [5, p. 171] for further details). The tools used for this technique are given by the following propositions.

Proposition 4.1. Let G be a graph with $\lambda(G) = -2$. Then none of the graphs in Fig.4.1 can be a subgraph of G.



F_1 F_2 F_3 F_4

Fig.4.1.

If v is a vertex of G, let G_v denote this subgraph induced by the vertices adjacent to v. If x is a vertex in G_v, let $d(x)$ and $D(x)$ denote the degrees of x in G_v and G respectively.

Proposition 4.2. [6]. Let G be a graph with $\lambda(G) = -2$, and let x, y and z be three vertices adjacent to v but mutually nonadjacent. Then

$$d(x) + d(x) + d(z) \geq D(x) + D(y) + D(z) + D(v) - |V(G)| - 2.$$

Proof. Let X, Y and Z be the sets of vertices adjacent to x, y and z respectively but not equal to v. Then $1 + d(x) + |X| = D(x)$, with similar equations holding for y and z. Further, X, Y and Z are pairwise disjoint, for any vertex in two of the sets would induce a subgraph in Fig.4.1. Since $|X| + |Y| + |Z| + d(v) + 1 \leq |V(G)|$, we have the desired result.

Corollary 4.3. If G is a regular graph with degree r and $\lambda(G) = -2$ and x, y and z are three vertices adjacent to v but mutually nonadjacent, then either G is a cubic graph or G satisfies

$$d(x) + d(y) + d(z) > 4r - |V(G)| - 2.$$

Proof. Since $D(x) = D(y) = D(z) = r$, all that must be shown is that the inequality of Proposition 4.2 is strict. If equality is attained, then any u adjacent to v not equal to x, y or z must be adjacent to every further vertex adjacent to v. This forces the existence of a forbidden subgraph from Fig.4.1 with five vertices. If $r = 3$, then no such u exists, but the graph is a cubic graph.

Proposition 4.4. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the eigenvalues of a graph G. Then the average number of triangles containing a given vertex is $\bar{t} = \frac{1}{2n} \sum_{i=1}^{n} \lambda_i^3$.

Proof. The proof is trivial; note that $\bar{t}$ is also equal to the average number of edges in G_v.

Let $E(G)$ denote the edge set of G. The spectrum of a graph is displayed by letting the exponent of a real number be its multiplicity as an eigenvalue.

Proposition 4.5. $L(\overline{C_6})$ is characterized by its spectrum.

Proof. Since $L(\overline{C_6})$ has $4,2,1^2,-1^2,-2^3$ as its spectrum, Proposition 4.4 implies that $\bar{t} < 3$. Thus there exists a vertex v with $|E(G_v)| \leq 2$. It is not possible for $|E(G_v)|$ to contain one or zero-elements because of Proposition 4.2. So consider a vertex v with $|E(G_v)| = 2$. The edges of G_v must be independent by Corollary 4.3. Now none of the remaining four vertices can be adjacent to three of the original five vertices, again because of forbidden subgraphs. Thus these four vertices form a cycle of length four, and the remaining edges can only be added in two ways. One yields $L(\overline{C_6})$ and the other yields $L(K_{3,3})$ (whose spectrum is $4,1^4,-2^4$).

Let S_5 denote the cubic graph on 8 vertices formed by taking two copies of the graph on 4 vertices with 5 edges and adding two edges to produce a regular graph.

Proposition 4.6. The graph $L(S_5)$ is characterized by its spectrum.

Proof. Suppose G has the spectrum of S_5, i.e., $4, 1+\sqrt{5}, 2, 0^4, 1-\sqrt{5}, -2^4$. Then by Proposition 4.4, the average number of edges in G_v is 3. Let us first suppose that every G_v has three edges. Then each G_v is $K_{1,3}, K_3 \cup K_1$ or P_3, the path with three edges. If G_v is $K_{1,3}$ and u is one of the vertices of degree 1, then G_u has fewer than three edges, a contradiction of our assumption. If G_v is $K_3 \cup K_1$ for one vertex v, then it is G_v for every vertex v. Thus each vertex is in exactly one complete graph with four vertices, and G is covered by three copies of K_4. The only graph with this property is the line graph of a semiregular bipartite graph and has $4, 1+\sqrt{2}^2, 0^3, 1-\sqrt{2}^2, -2^4$ as its spectrum. If G_v is P_3 for every v, then there is only one completion of the graph, and its spectrum is $4, 1+\sqrt{3}^2, 0^3, 1-\sqrt{3}^2, -2^4$. Thus there is a vertex v such that G_v has fewer than three edges. It can not have one or zero edges nor can it have two edges with a common vertex because this would contradict Corollary 4.3. Thus G_v has two independent edges.

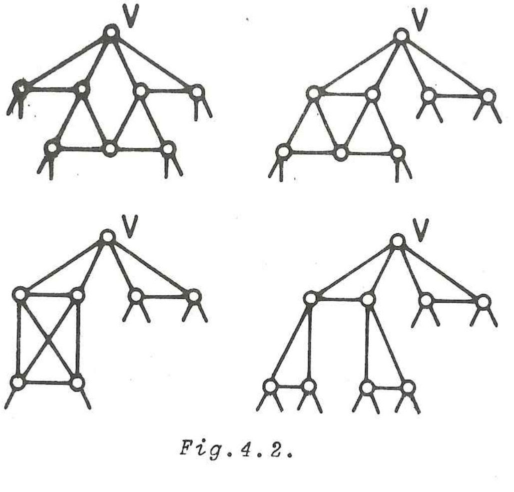


Fig. 4.2.

Possible extensions of the graph are given in Fig.4.2.

It is easy now to complete the graph in each of the cases. The only graph with the given spectrum, which is obtained in this way, is $L(S_5)$.

Details of the completion are left to the reader. Proving that a graph constructed does not have the spectrum given can be done by counting the numbers of triangles, quadrilaterals and pentagons. The completion can be done, of course, by hand and pencil. However, a practical tool for performing such extensions of graphs is the programming system "Graph", implemented at University of Belgrade, where interactive graphic (light pen) and other facilities enable a suitable performing and recording of several graph extensions as well as checking whether the graph constructed has the spectrum given.

The same remarques hold for the next proposition.

Proposition 4.7. Graphs $L(\overline{C_7})$ and $L(\overline{C_4 \cup C_3})$ are characterized by their spectra.

Proof. Suppose G is a regular graph on $n = 14$ vertices of degree $r = 6$. If the diameter of G is greater than

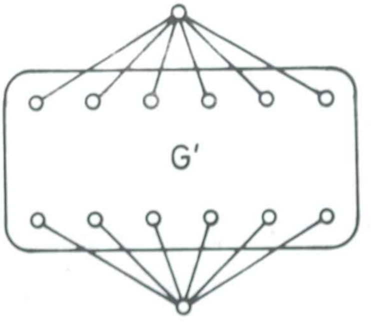


Fig.4.3.

2, the graph G looks like in Fig.4.3. The subgraph G' is regular of degree $r' = 5$ and has $n' = 12$ vertices. It must be only a line graph of a semiregular bipartite graph. The parameters n_1, n_2, r_1, r_2 of this semiregular bipartite graph are obtained from equations $n_1r_1 = n_2r_2 = 12$, $r_1 + r_2 = 7$. We have $(n_1, n_2, r_1, r_2) = (12,2,1,6)$ or $(4,3,3,4)$. In the first case $G' = 2K_6$ and by interlacing theorem $\lambda_2 \geq 5$ for G holds. This is in contradiction with spectra of both $L(\overline{C_7})$ and $L(\overline{C_4 \cup C_3})$. In the second case $G' = L(K_{4,3})$ which has the spectrum $5, 2^2, 1^3, -2^6$. By interlacing G cannot be cospectral with $L(\overline{C_7})$. The only way of expending $L(K_{4,3})$ according to Fig.4.3 so that the least eigenvalue does not drop below -2 is the way in which we get $L(\overline{C_4 \cup C_3})$.

Suppose now that diameter of G is equal to 2. For both $L(\overline{C_7})$ and $L(\overline{C_4 \cup C_3})$ we have $\bar{t} < 8$ and hence there is at least one vertex v of G for which G_v has less than 8 edges. For $x, y, z \in V(G_v)$ and non adjacent we have $|E(G_v)| \geq$ $\geq d(x) + d(y) + d(z) \geq 4r - n - 2 = 8$, where Proposition 4.2 is used. Hence 3 nonadjacent vertices do not exist, i.e. $\overline{G}_v$ contains no triangles and $|E(\overline{G}_v)| \geq 8$. Further, C_5 is out and $\overline{G}_v$ is bipartite. Hence, two disjoint cliques of G_v cover vertices of G_v. The only possibilities for G_v are $E_1 = K_4 \cup K_2$, $E_2 = 2K_3$ and $E_3 = 2K_3 + x$ (two triangles joined by an edge).

Graphs E_1 and E_2 lead quickly to a contradiction.

Considering E_3, let us notice that there exist at least seven vertices v with $G_v = E_3$. Therefore there exist adjacent vertices v and u such that $G_v = G_u = E_3$. Partial graphs on Fig.4.4 correspond to the posible situations in this case.

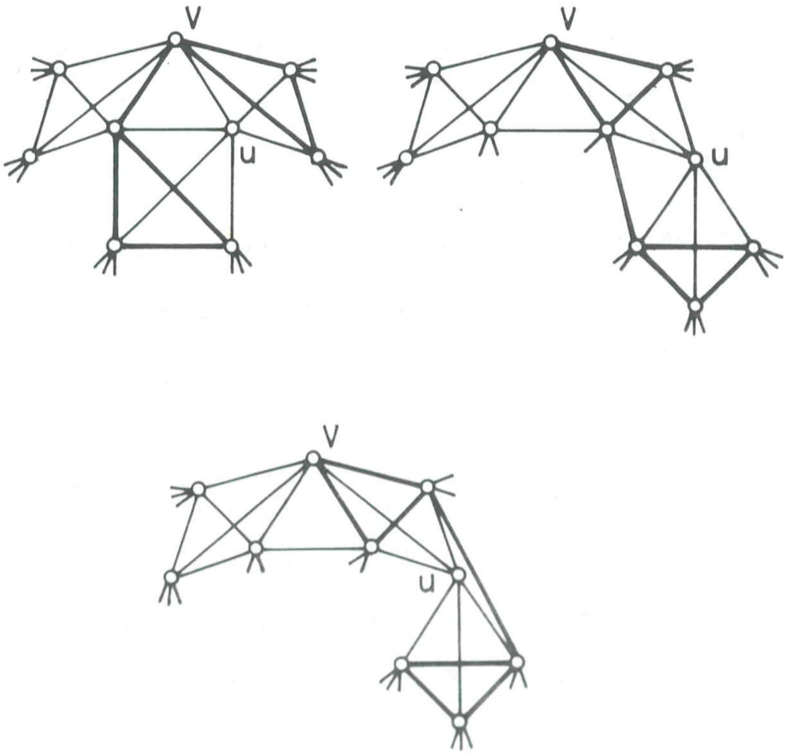


Fig.4.4.

Completion of the graph yields only $L(\overline{C_7})$ or $L(\overline{C_3 \cup C_4})$.

Proposition 4.8. The 13 graphs from Proposition 3.7, not contained in Proposition 3.11, are characterized by their spectra.

Proof. For 9 of these 13 graphs it has been already proved in the literature that they are characterized by their spectra: $L(K_{5,4})$, $L(K_{4,3})$, $L(K_{7,2})$, $L(K_{6,2})$, $L(K_{5,2})$ (see [7]), $L(K_{5,3})$ (see [6]), $L(K_7)$, $L(K_6)$, (see [11]) and $L(CP(3))$ (see [13]).

The remaining graphs $L(\overline{C_6})$, $L(S_5)$, $L(\overline{C_7})$ and $L(\overline{C_4 \cup C_3})$ are characterized by their spectra according to Propositions 4.5. - 4.7.

5. CHARACTERIZING REGULAR LINE GRAPHS BY THEIR SPECTRA

We now have the tools to prove one of our main theorems.

Theorem 5.1. The spectrum of a graph G *determines whether or not it is a regular connected line graph except for 17 cases. In these cases* G *has the spectrum of* $L(H)$ *where* H *is one of the 3-connected regular graphs on 8 vertices or* H *is a connected semiregular bipartite graph on 6+3 vertices.*

This theorem is a reformulation and generalization of a theorem from [1]. It is a generalization in the sense that the extensive computer searches used there are avoided completely and many details are sharpened.

Proof of Theorem 5.1. If H is a 3-connected regular graph on 8 vertices or a connected semiregular bipartite graph on 6 + 3 vertices, then $L(H)$ is one of the 17 graphs from Proposition 3.11. By Proposition 3.11, in these 17 cases it is not possible to tell whether the graph is a line graph or a cospectral mate obtained by Seidel switching. But in all other cases we see from Proposition 3.7 and Proposition 4.8 that we can recognize a graph as being a regular connected line graph.

This theorem was first announced in [4]. It is still possible, of course, for two nonisomorphic cospectral regular line graphs to arise from nonisomorphic root graphs (see [1] for more details about these possibilities).

Theorem 5.1 generalizes a large number of earlier results. The propositions below give a flavour of some of these previous results.

Proposition 5.2. (Hoffman and Ray-Chaudhuri [15]). Let G be a regular connected graph with degree greater than 16 and $\lambda(G) = -2$. Then either $G = CP(n)$ for some n or $G = L(H)$ for some H.

Proposition 5.3. (A.J. Hoffman [11]). The line graph of a complete graph K_n has no cospectral mates unless $n = 8$.

Proposition 5.4. (S.S. Shrikhande [17], M. Doob [6,7], D.M. Cvetković [3], F.C. Bussemaker, D.M. Cvetković, J.J. Seidel [1]). The line graph of the complete bipartite graph $K_{m,n}$ with $m \geq n$ has no exceptional cospectral mates unless $m = n = 4$ or $m = 6$ and $n = 3$.

Proposition 5.5. (A.J. Hoffman, D.K. Ray-Chadhuri [14]). The line graph of the flag graph of a symmetric balanced incomplete block design with parameters (v,k,λ) has no exceptional cospectral mates unless $v = 4$, $k = 3$ and $\lambda = 2$.

Proposition 5.6. (A.J. Hoffman, B.A. Jamil [13]). The line graph of the complete tripartite graph $K_{n,n,n}$ is characterized by its spectrum.

The results of Propositions 5.3. - 5.5 in fact contain more information. When the graphs are not characterized by their spectra, all of the cospectral mates are displayed. Similar results for Theorem 5.1 will be the subject of a future paper. In fact, using some recent result of Z. Radosavljević [18] all 68 exceptional graphs can be constructed in a natural way without the aid of computer.* Proving that no further graphs exist remains to be done.

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A CONSTRUCTION OF THE 68 CONNECTED, REGULAR GRAPHS, NON-ISOMORPHIC BUT COSPECTRAL TO LINE GRAPHS

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ABSTRACT

The graphs described in the title are constructed by means of the Seidel switching, starting from all factorizations of regular graphs on 8 vertices into two regular factors.

There are exactly 187 regular, connected graphs with the least eigenvalue -2 which are neither line graphs nor cocktail-party graphs (shortly, exceptional graphs). They have been constructed in [1] using a mixture of mathematical reasoning and computer search. Exactly 68 of them are cospectral to some line graphs. These line graphs, which are 17 in number, are line graphs of some regular graphs on 8 vertices (15 graphs) or of some semiregular bipartite graphs on 9 vertices (2 graphs). Several theorems characterizing regular line graphs by their spectra follow from these results [1], [2].

This paper is a part of efforts [2], [3] to prove all these results without the use of computer.

The following was proved without computer [1]. An exceptional graph of degree d on n vertices with $n \leq 2d + 4$ is switching equivalent to the line graph of a graph on 8 vertices. If an exceptional graph is cospectral to a line graph then we do have $n \leq 2d + 4$ (in fact, $n = 2d+4$).

Hence, in order to construct the 68 exceptional graphs cospectral to some line graphs one should switch line graphs of all 8 vertex graphs in all possible ways. However, it is proved by the computer search that it is sufficient to start with regular graphs on 8 vertices and semiregular bipartite graphs on 9 vertices.

It is proved [1] without computer search that line graphs of regular graphs on 8 vertices are switched into exceptional graphs only in the following way. Let H be a regular graph on 8 vertices. Let $F_1 \cup F_2$ be a bipartition of its edge set. $L(H)$ is converted after switching with respect to F_1 into a regular graph (of the same degree) if and only if F_1 (or F_2) is a regular factor of H.

So we should know all factorizations of regular graphs on 8 vertices into two regular factors. Two factorizations of a graph are called equivalent if there exists an automorphism of the graph which maps one factorization into another. Obviously, equivalent factorizations give rise to isomorphic graphs after switching. Therefore, it is sufficient to consider only non-equivalent factorizations of regular graphs on 8 vertices and such factorizations have been determined in [3] without computer.

Starting from results of [3] we construct 63 out of the 68 graphs mentioned above. The difficulties in constructing them are the following. Given a graph H and its factorization $F_1 \cup F_2$, if we switch $L(H)$ w.r.t. F_1 we could get again $L(H)$. Also, different (non-equivalent) factorizations can give rise still to isomorphic graphs.

All interesting regular graphs on 8 vertices are given in Fig.1 together with a labeling of their edge sets. (The case of K_8 has been treated in [5]. Exactly three exceptional graphs appear (Chung graphs or graphs no.161 - no.163 from [1]).

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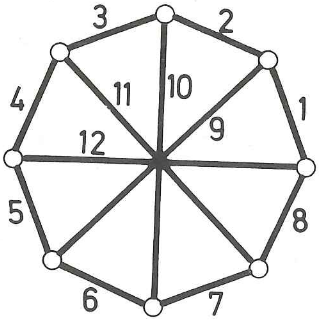

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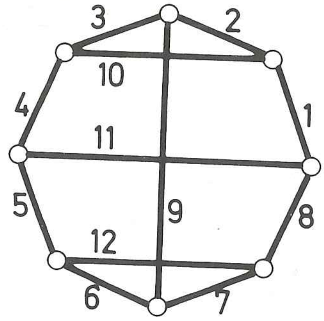

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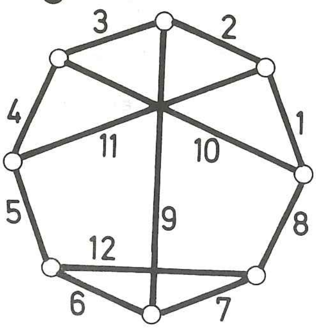

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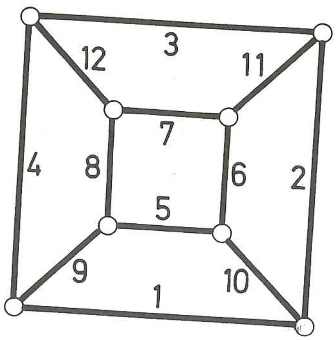
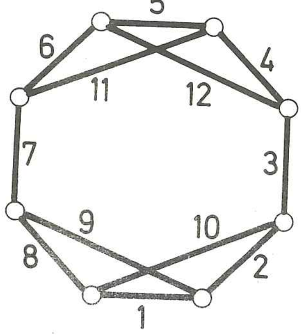
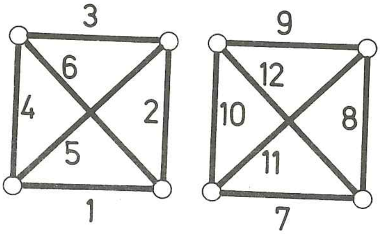


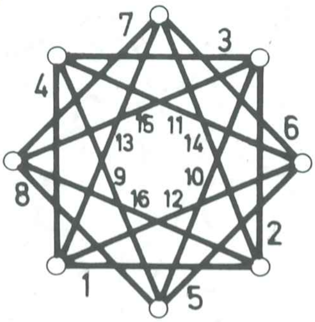
S_5



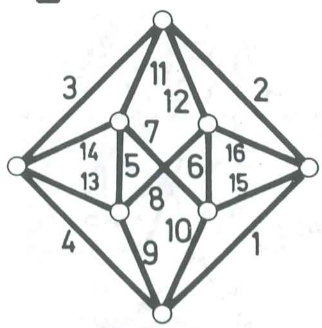
$2K_4$



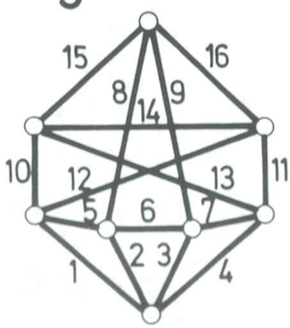
$\bar{S}_1$



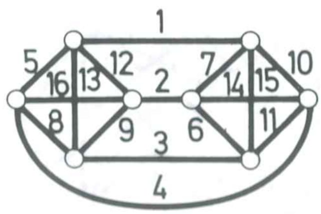
$\bar{S}_2$



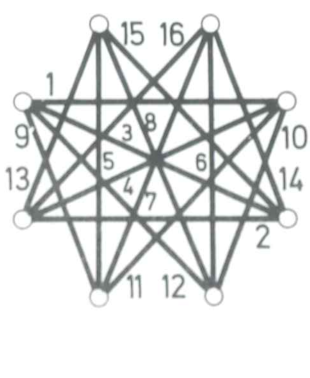
$\bar{S}_3$



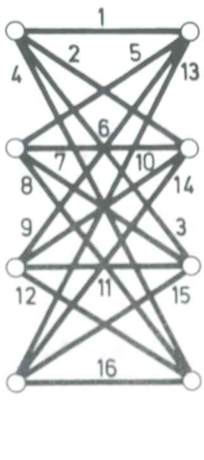
$\bar{S}_4$



$\bar{S}_5$



$\overline{2K_4}$



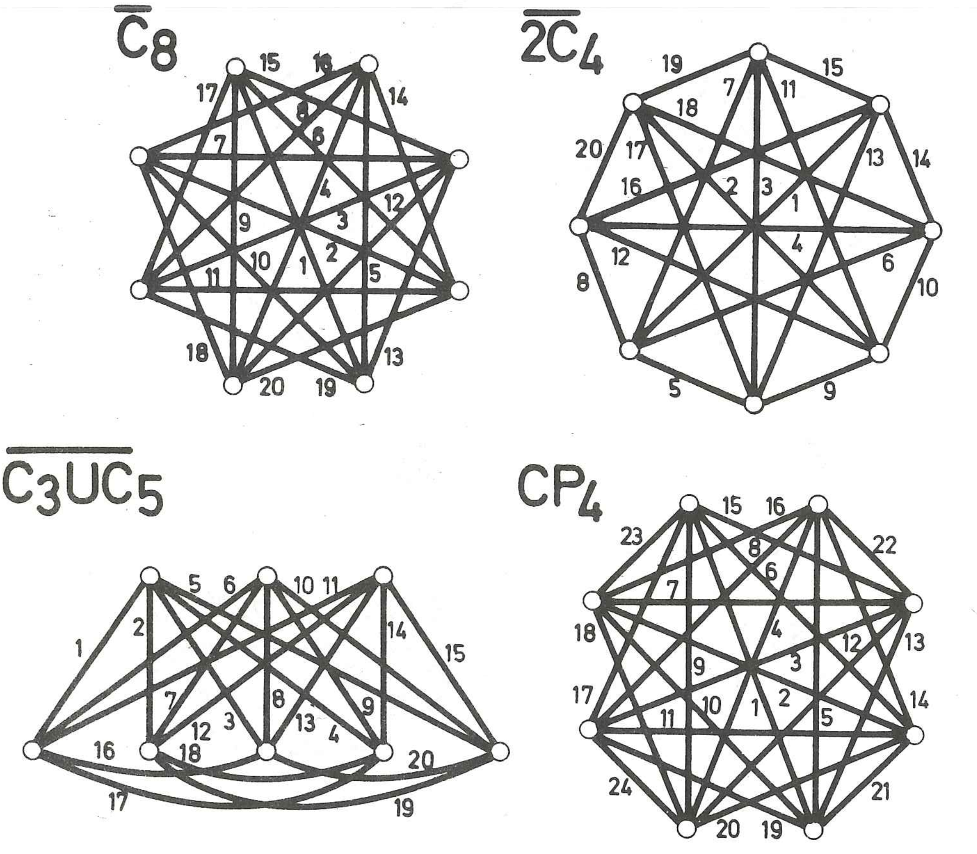


Fig.1.

The construction of exceptional graphs is given in the following tables.

Tables

S_1:

1 3 5 7	–		(1 4 3 6 5 8 7 2)(10 12)
9 10 11 12	–		(1 7 5 3)(2 8 6 4)
3 7 9 12	13 2 6 9 10		

S_2:

9 10 11 12	–		(1 5)(2 6)(3 7)(4 8)(9 11)
1 4 9 12	10 5 8 11 12	2	
2 4 6 8	11 1 4 10 11	3	

S_3:

9 10 11 12	12 1 2 3 10		

S_4:

9 10 11 12	–		(9 11)(10 12)
2 4 5 7	9 1 5 9 10		

S_5:

9 10 11 12	–				(9 12)(10 11)
1 3 5 7	–				(2 4)(3 7)(6 8)(9 11)(10 12)

$2K_4$:

5 6 11 12	–				(5 7 8 11)(6 9 10 12)

$\bar{S}_1$:

2 4 6 8	37	2 7 14 15	10	8	
10 11 13 16	–				(10 13)(11 16)
4 6 10 16	35	4 5 9 12	8	8	
1 2 3 4 5 6 7 8	–				(1 3)(2 4)(5 7)(6 8)
2 4 6 7 9 10 12 16	36	2 4 5 7	5		
2 4 6 8 9 12 14 15	37	1 6 9 15	10	8	(4 15)(5 14)
1 3 4 5 6 7 10 16	35	1 2 3 4	8	8	(3 15)(6 12)(9 11)(10 14)

$\bar{S}_2$:

1 3 5 6	49	5 7 9 10	10	6*	
1 3 7 8	49	5 7 9 10	10	6	(1 4)(2 11)(3 6)(5 8)(9 13)(10 14)(15 16)
10 11 13 16	50	5 7 9 10	10	6*	
4 5 12 15	46	5 6 7 8	10	6*	
2 8 10 14	47	5 6 7 8	8	6	
9 10 11 12 13 14 15 16	-				(1 3)(2 4)(5 6)(7 8)
1 4 5 6 11 12 13 15	54	3 5 11 13	9		
2 4 5 6 9 12 14 15	51	3 5 11 13	10	8*	
1 2 7 8 10 12 13 14	53	4 5 9 14	7	9	
1 3 5 6 9 12 14 15	51	4 5 9 14	10	8	(1 15)(2 16)(3 11)(4 5)(7 10)(8 13)(9 14)
1 2 3 5 6 8 10 14	47	6 8 9 10	8	6	(4 15)(5 14)(11 12)
1 3 4 5 7 8 12 15	46	1 2 3 4	10	6	(4 13)(6 9)(10 14)(11 15)
2 4 5 8 10 12 14 15	48	5 8 11 12	7	7	
3 4 5 7 9 12 15 16	52	5 7 9 10	10	8*	
2 4 6 7 9 11 13 16	48	6 7 11 12	7	7	(3 4)(5 10)(6 12)(9 11)

$\bar{S}_3$:

1 7 8 14	38	2 4 6 7	7	7*	
1 6 13 16	40	7 9 11 16	9		
1 4 6 7 8 10 14 16	39	2 4 6 7	8	8	
1 3 5 6 11 13 15 16	41	1 7 8 14	8	10	
1 2 6 9 10 11 13 16	42	11 13 15 16	7	7*	

$\bar{S}_4$:

1 2 3 4	-				(1 3)(2 4)(13 16)(14 15)
5 6 9 10	-				(1 3)(2 4)(7 8)(11 12)(13 15)(14 16)
6 8 10 12	44	3 4 7 12	12	8	
1 2 8 11	45	7 11 14 15	8	8	
1 2 3 4 7 8 11 12	-				(1 3)(2 4)(7 8)(11 12)(13 16)(14 15)
1 2 3 4 6 8 10 12	44	2 3 6 9	12	8	(4 13)(6 9)(11 15)(12 14)
1 2 6 8 10 11 13 16	43	8 12 13 16	8	12	
1 2 5 6 8 9 10 11	45	2 3 6 9	8	8	(3 16)(6 13)

$\bar{S}_5$:

1 2 5 6	59	3 6 11 13	12	4	
3 4 5 6	59	4 8 9 14	12	4	(2 16 5 13)(3 15 6 12)(4 8 7 14)
9 10 13 14	59	3 6 11 13	12	4	(2 15 12 5)(3 6 13 16)(4 9 7 10)(8 14)
3 6 11 13	60	4 5 6 14	10	4	
9 10 11 12 13 14 15 16	–				(5 6)(9 13)(10 14)(11 15)(12 16)
3 4 5 6 11 12 13 14	–				(3 6)(4 5)(11 13)(12 14)
3 4 6 8 9 11 13 14	60	4 5 11 13	10	4	(2 4 11 9 5 3)(6 7 8 12 13 10)(15 16)
1 2 7 8 9 10 13 14	59	3 7 9 14	12	4	(2 16 15 6 12 13)(3 5)(4 8 9 7 14 10)
1 2 4 5 6 8 9 14	60	1 2 3 4	10	4	(2 5 16)(4 15 9)(6 10 7)(8 13 12)

$\overline{2K_4}$:

1 6 11 16	69	1 2 9 10	16	0	
1 2 5 6 11 12 15 16	–				(3 7)(4 8)(9 10)(11 16)(12 15)(13 14)
1 2 6 7 11 12 13 16	69	1 2 5 6	16	0	(2 16)(5 14)(9 13)(10 15)

$\bar{C}_8$:

1 2 3 4	134	2 8 11 16	23	32	
1 4 7 11	133	5 11 14 19	21	32	
2 4 15 19	127	1 4 6 11	21	34	
4 10 11 15	126	13 15 17 19	20	33	
5 7 9 11	134	4 8 11 20	23	32	(2 15 9 16)(3 4 12 11)(5 6)(7 17 18 8)(13 14)(19 20)
5 7 17 20	124	14 16 18 20	19	34	
15 16 19 20	129	5 11 14 19	23	36	
5 6 7 8 9 10 11 12	-				(1 3)(2 4)(5 9)(6 10)(7 11)(8 12)
13 14 15 16 17 18 19 20	134	3 7 10 19	23	32	(2 9)(3 11)(4 12)(5 6)(7 17)(8 18)(13 14)(19 20)
3 5 10 14 15 17 18 20	121	4 5 10 18	17	36	
5 7 9 11 14 15 18 19	129	2 8 11 16	23	36	(1 18)(3 6 16 20)(4 17 7 14)(5 9 15 11)(10 13)(12 19)
2 4 5 7 9 11 15 19	127	1 4 5 9	21	34	(5 19)(6 15)(12 14)(16 18)

8 10 13 14 15 17 18 20	124	10 12 13 18	19	34	(1 2 20 7 9 10 16 3)(4 6) (5 12 8 18 15 13 11 14)
2 4 13 14 15 17 18 19	127	5 7 14 16	21	34	(1 4 13 9 10 7 17 2)(3 5 8 15 20 19 11 6)(12 16 18 14)
6 7 8 9 13 14 18 19	131	7 9 15 18	18	33*	
1 3 4 7 10 14 17 20	121	1 7 10 15	17	36	(1 2 5 13 9 11 6 3)(4 20 16 19) (7 10 14 15 12 8 17 18)
2 4 5 10 11 12 15 17	125	4 5 10 18	20	35	
3 4 6 10 13 14 17 18	132	10 12 13 18	17	32	
3 5 9 10 11 15 16 20	128	3 11 12 20	18	33*	
1 2 5 7 8 12 17 20	123	5 11 14 19	21	38	
1 2 3 4 9 11 13 16	124	1 9 12 13	19	34	(1 9)(2 10)(3 7)(4 19)(5 15) (6 17)(8 11)(16 20)
1 2 4 7 8 9 11 13	126	1 7 12 13	20	33	(1 17)(3 7)(4 13 20 12) (5 11 19 10)(6 16 18 15)(9 14)
2 8 9 13 14 15 18 19	128	13 15 17 19	18	33	(2 20)(3 5 4 19)(6 15 14 13) (7 9)(8 16 10 18)(12 17)
4 6 8 10 11 13 15 18	121	10 12 13 18	17	36	(1 5)(3 13)(4 19 16 20)(6 9) (7 18 12 15)(8 17 10 14)
1 2 3 5 12 16 17 20	128	1 5 8 17	18	33	(2 20)(3 19 4 5)(6 13 14 15) (7 9)(8 18 10 16)(12 17)

2 4 5 11 13 15 17 18	122	4 5 12 13	19	36	
2 8 9 10 12 13 14 17	128	2 5 10 14	18	33	(1 9)(2 11)(3 4)(5 18)(6 15)(7 17)(12 20)(13 14)(16 19)
7 8 9 10 13 14 17 20	131	9 11 17 20	18	33	(1 11)(2 10)(3 17)(6 13)(7 15)(12 20)(14 16)
1 2 3 4 5 7 9 11	-				(1 4)(2 3)(5 9)(7 11)
6 8 10 12 14 15 18 19	129	8 10 16 19	23	36	(1 18)(3 20 16 6)(4 14 7 17)(5 11 15 9)(10 13)(12 19)
3 4 6 10 14 15 18 19	130	4 5 10 18	21	36	
3 5 6 9 10 11 12 16	126	1 5 8 17	20	33	(1 3 14 8 9 7 17 2)(4 5 10 13 20 19 11 12)(6 16 18 15)
1 2 3 4 14 15 18 19	129	1 9 12 13	23	36	(1 18)(3 20)(5 9)(6 16)(10 13)(11 15)(12 19)(14 17)
1 4 7 11 13 14 17 18	130	1 3 13 17	21	36	(2 18)(3 4)(5 10)(6 14)(8 15)(9 12)(13 16)(17 19)
1 4 7 10 11 12 14 17	123	1 7 10 15	21	38	(1 7 18 2 9 3 20 11)(4 15)(6 14 8 13 19 16 10 12)
3 4 5 6 7 9 10 11	133	3 7 10 19	21	32	(2 12)(3 20 5 18 6 17 4 19)(7 15 11 13 8 14 10 16)
5 6 7 8 10 12 17 20	124	6 8 14 17	19	34	(4 6)(5 15)(8 11)(12 13)(14 18)(17 19)

$\overline{2C}_4$:

1 2 3 4					(3 4) (5 9) (6 10) (7 11) (8 12) (13 17) (14 18) (15 19) (16 20)
2 4 5 15	109	2 3 11 17	21	36	
5 10 15 20	112	5 7 17 19	25	32	
5 11 14 20	111	1 3 5 15	21	32	
6 8 9 11 14 16 17 19	-				(6 11) (7 13) (8 19) (9 14) (10 20) (16 17)
6 7 9 12 14 15 17 29	110	2 3 6 16	25	40	
2 3 6 8 11 14 16 17	109	9 12 13 16	21	36	(3 18 17 13 7 11 19 14) (4 8 20 12 6 5 16 15) (9 10)
6 8 9 11 14 15 17 20	111	6 8 14 16	21	32	(2 12) (3 20) (4 19 5 18) (6 17) (7 15 11 16) (8 14 10 13)
2 4 5 7 9 14 16 19	108	2 3 9 19	17	36	
5 7 9 12 14 16 18 19	111	5 7 9 11	21	32	(2 18 4) (5 12 19) (7 15 11) (8 10 13)
2 4 5 7 9 15 16 18	108	5 7 9 11	17	36	(4 5) (6 16) (7 11) (15 17)
1 2 3 4 8 9 14 19	112	9 12 17 20	25	32	(2 5 14) (3 7 17) (4 12 8) (6 19 9)

1 2 3 4 6 9 15 20	111	1 4 6 16	21	32	(2 12) (4 18 5 19) (7 16 11 15) (8 13 10 14)
6 7 9 12 13 16 18 19	-				(6 19 8) (7 20 18) (9 14 16) (10 12 13)
6 7 11 12 13 16 17 18	112	6 7 18 19	25	32	(2 18 15 8 6 17) (3 5 12 9 11 16) (4 14) (7 19) (10 13)
2 3 5 8 12 14 15 18	109	1 3 5 15	21	36	(3 5) (4 14) (6 13) (7 8) (9 10) (11 17) (12 16) (15 20) (18 19)

$\overline{C_3 \cup C_5}$:

3 6 15 18	113	1 10 13 18	20	32*
1 5 7 9 12 14 16 20	115	3 4 8 9	22	36
2 4 6 7 14 15 16 20	114	3 4 8 9	19	33
2 4 6 9 12 15 16 20	120	2 3 7 8	17	
1 4 7 8 13 15 17 19	117	7 8 19 20	19	31
1 5 7 8 13 14 17 19	118	7 8 12 13	21	
2 4 6 8 13 15 17 19	119	8 10 13 15	22	34
1 2 8 9 13 15 17 19	116	8 10 13 15	20	32*

CP(4):

5 7 9 11	159	2 5 16 21	44	104	
21 22 23 24	158	1 11 17 21	40	104	
8 13 14 15 18 19 20 23	160	10 13 16 22	36	96	
4 11 13 14 15 18 19 23	157	10 13 16 22	36	100	
2 4 6 8 13 15 18 19	156	13 15 17 19	36	102	
6 8 13 15 16 18 19 20	157	13 15 17 19	36	100	(1 3 13 17 11 22)(2 20 12 4 14 7)(5 23 15 24 6 16)(8 10)(9 21)
8 9 11 13 15 16 18 21	155	6 17 19 21	36	104	
2 4 8 13 15 19 20 23	157	13 15 17 19	36	100	(3 18 7 5)(4 6 8 16)(11 12 14 13)(15 17 23 21)(19 22 24 20)
6 13 14 15 16 18 19 24	158	13 15 17 19	40	104	(3 17)(4 19)(5 21)(6 16)(7 18)(8 23)(10 14)(12 13)(15 20)(22 24)
6 10 14 15 18 19 22 24	-				(2 7)(3 11)(5 9)(6 20 22 13)(8 24 23 10)(12 14 21 15)(16 18 17 19)
13 14 15 16 17 18 19 20	159	3 5 8 13	44	104	(2 18)(3 9)(5 12)(6 16)(7 24)(10 20)(11 19)(13 23)

1 3 13 14 16 17 18 20	158	10 13 15 23	40	104	(1 5 3 2 19 24)(4 18 11 6 17 13)(7 10 8 20 14 16)(9 23 22 12 21 15)
6 8 13 15 16 18 21 24	153	1 2 21 23	44	112	
1 8 11 13 15 16 18 20	156	14 16 18 20	36	102	(1 14 10)(2 11 9)(3 16 18)(4 22 15 23 19 8)(5 21 7)(20 24)
8 10 12 13 14 15 16 17 20 21 23 24	159	10 18 20 21	44	104	(1 3)(2 15)(4 20)(5 24)(6 14)(7 11)(8 21)(9 12)(13 19)(16 17)(22 23)
2 3 5 8 9 10 13 14 15 20 23 24	-				(2 5 11)(3 9 7)(8 17 21 23 16 12)(10 14 19)(13 20)(15 18 24)
8 10 12 13 14 15 16 17 20 21 23 24	159	13 14 21 22	44	104	(1 3)(2 15)(4 20)(5 24)(6 14)(7 11)(8 21)(9 12)(13 19)(16 17)(22 23)
6 8 10 12 13 14 15 16 19 20 23 24	158	13 15 17 19	40	104	(3 7 22 15)(4 16 5 23)(6 21 8 19)(9 11)(10 13 14 12)(17 18 24 20)
8 9 11 12 13 14 15 16 18 19 21 23	158	13 14 21 22	40	104	(1 4 9 8 2 21 11 16)(3 18 7 22 24 15 20 17)(5 14 6 13 19 10 23 12)
4 7 8 9 11 13 14 15 18 19 21 23	158	13 14 21 22	40	104	(1 5 14 4 11 23 2 19 10 21 9 6)(3 17 7)(8 13 16 12)(15 18)(20 24 22)

5 7 8 9 11 13 14 15 18 19 20 23	159	13 15 17 19	44	104	(1 3 8 17 24 20 22 23 21 14 7 2)(4 19 13 18 5 6 15 12 9 10 11 16)
2 5 8 9 10 11 12 13 14 15 17 18	155	13 15 17 19	36	104	(1 13 11)(2 10 14)(3 18 19)(4 20 8)(5 23 16 24 6 15)(21 22)
2 5 8 9 11 12 13 15 16 21 23 24	156	8 12 22 24	36	102	(1 2)(3 15)(4 18)(5 8)(6 24)(7 23)(9 14)(10 11)(16 19)(17 20)(21 22)
4 5 7 11 13 15 16 17 20 21 23 24	156	13 15 17 19	36	102	(1 14 12 2 11 13)(3 17 7)(4 19 20 23 22 24)(5 6 18)(8 15)(9 10)
2 4 5 7 8 13 15 17 20 21 23 24	155	13 15 17 19	36	104	(1 23 9 5 2 6 12 24)(3 19 17 18 4 8 7 20)(10 15 14 22 13 16 11 21)
1 3 6 7 8 9 13 14 16 18 21 24	154	13 14 21 22	36	108	

For each of the graphs a table is given. To each nonequivalent factorization of a graph into two regular factors there corresponds a row in the table. A row contains the following information (some data are omitted):

- a regular factor (determining a factorization);

- the identification number, referring to [1], of the exceptional graph if it is obtained, or dash "-" if the graph obtained after switching is isomorphic to the starting line graph;

- to prove that the obtained graph is exceptional, a set of 4 edges is given which gives rise to a subgraph $K_{1,3}$ which is forbidden for line graphs;

- the number of 4-cocliques (independent sets of 4 vertices) contained in the obtained graph (5-cocliques do not exist and the number of 3-cocliques is determined by the spectrum in a regular graph);

- the number of 4-cliques (complete subgraphs on 4 vertices) contained in the obtained graph (the number of 3-cliques is determined by the spectrum);

- if nonequivalent factorizations give rise to isomorphic graphs, an isomorphism is given which maps the graph into its isomorphic mate which is most highly placed in the table; however, if the graph obtained by switching is again the starting line graph, we have an isomorphism which maps the original line graph on the obtained graph (isomorphisms are given as permutations in the cyclic form where fixed points are omitted).

By the number of 4-cocliques and 4-cliques we can almost always distinguish between non-isomorphic graphs. The cases in which these numbers are not sufficient are marked in the tables by asterisks. This happens with the graphs $\bar{S}_2$, $\bar{S}_3$, $\bar{C}_8$ and $\overline{C_3 \cup C_5}$. The corresponding non-isomorphic exceptional graphs can be distinguished by analysing the incidence between vertices and 4-cocliques.

Graph $\bar{S}_2$: the exceptional graph 49 has exactly one vertex (8) belonging to four 4-cocliques, the graph 50 has two such vertices (5,6) and the graph 46 has none. The graph 51 has two vertices (14, 15) belonging to only one 4-coclique and the graph 52 has no such vertices.

Graph $\bar{S}_3$: every vertex of the graph 38 belongs to a 4-coclique and in the graph 42 the vertex 14 does not.

Graph $\bar{S}_8$: in the graph 128 there is only one vertex (15) which belongs to exactly two 4-cocliques, while in the graph 131 there are three such vertices (15, 17, 20).

Graph $\overline{C_3 \cup C_5}$: in the graph 113 the vertex 5 belongs to six 4-cocliques and in the graph 116 there is no such vertex.

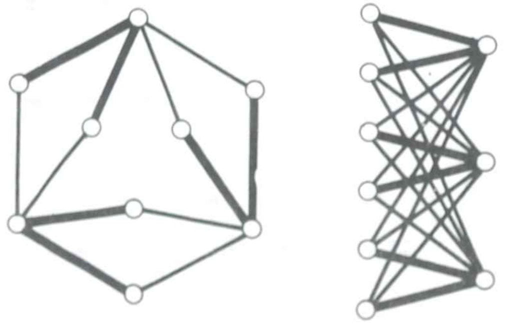


Fig.2.

Semiregular bipartite graphs, interesting for our purposes, are given in Fig.2 together with the factorizations which produce the exceptional graphs 6 and 70, respectively. The detailes are left to the reader. Other semiregular bipartite graphs on 9 vertices do not give rise to exceptional graphs (see [1]).

Tables in this paper are produced partly by the use of a computer. In fact the interactive programming system "Graph" [4] and computer facilities of Technological University, Eindhoven, The Netherlands, have been used. However, the reader can check for himself the correctness of the data; this can be done easily in the principle, although always with some effort of routine kind.

For example, 4-cocliques can be enumerated by counting sets of 4 edges of certain structure in the starting graph H on 8 vertices with a given factorization. These 4 edges are either independent and belong to the same factor or form a quadrangle, two nonadjacent of them belonging to a factor and the remaining two to another one.

Similarly, to count 4-cliques we have to find the number of sets of 4 edges in H which fulfill one of the following:

- all edges have a common vertex and belong to the same factor,

- 3 edges belong to a factor and form $K_{1,3}$ or K_3, the fourth edge being adjacent to none of these 3 and belonging to another factor,

- the edges are partitioned into 2 pairs, where each pair belongs to a different factor, edges from one pair are non-adjacent to edges from another pair and each pair forms $K_{1,2}$.

In this way the presented tables give the construction of the 68 connected regular graphs, non-isomorphic but cospectral to some line graphs and provide a proof that there are no more such graphs which can be obtained by switching regular line graphs.

AKNOWLEDGEMENT

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ON FREE PRODUCTS WITH AMALGAMATION OF FREE GROUPS

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ABSTRACT

In this paper is constructed an example of a group N admitting two different non-trivial factorizations as a free product with amalgamation of free groups. If

$$N = N_1 *_{N_{12}} N_2 = N_3 *_{N_{34}} N_4$$

then it will be denoted by $(r_1,r_2,r_3,r_4;d_1,d_2,d_3,d_4)$ where r_i denotes the rank of N_i, and d_i the index of amalgamated subgroup in the group N_i. The example obtained in this paper is with parameters (111,166,56,111;3,2,4,2) and (183,274,92,183; 3,2,4,2).

1. INTRODUCTION

We are interested in examples of groups N admitting two different non-trivial factorizations as a free product with amalgamation of free groups, say,

$$N = N_1 *_{N_{12}} N_2 = N_3 *_{N_{34}} N_4 \tag{1.1}$$

The rank of N_i will be denoted by r_i, while d_i will denote the index in N_i of the amalgamated subgroup N_{12} (if $i = 1,2$) or N_{34} (if $i = 3,4$). This information about factorizations (1.1) will be recorded in an 8-tuple

$$(r_1, r_2, r_3, r_4; d_1, d_2, d_3, d_4)$$

to which we shall refer as its set of parameters.

Two such examples were exhibited in [2] with respective parameters:

(1.2) $\quad (613,613,103,103;3,3,8,8)$, and

(1.3) $\quad (613,919,52,52;3,2,8,8)$.

In both examples the ranks r_i are large and it would be of interest to construct simpler examples. In the present article we construct two additional examples with parameters

(1.4) $\quad (111,166,56,111;3,2,4,2)$, and

(1.5) $\quad (183,274,92,183;3,2,4,2)$.

While the ranks r_i in these examples are still large, the indices d_i are rather small. In order to construct such a group we follow the method developed in [1] and [2]. We first construct a group G which acts faithfully on two trees Γ_3 and Γ_4 of valence 3 and 4, respectively. Moreover, both actions are ω-transitive. (For the definition of ω-transitivity and other undefined terms we refer the reader to our papers [1] and [2].) Then we exhibit an epimorphism $\phi : G \to PSL_2(11)$ and show that $N = \ker \phi$ admits factorizations (1.1) with parameters (1.4). Another epimorphism $\psi : G \to PSL_2(13)$ gives an $N = \ker\psi$ with parameters (1.5).

At the end we give a presentation of $PSL_2(13)$ in terms of two generators x, ξ of order 3 which is symmetrical in the sense that there is an involutory automorphism interchanging x and ξ.

2. CONSTRUCTION OF THE GROUP G

We start with a diagram

$$\begin{array}{ccccc} G_1 & \longleftarrow & G_{41} & \longrightarrow & G_4 \\ \uparrow & & & & \uparrow \\ G_{12} & & & & G_{34} \\ \downarrow & & & & \downarrow \\ G_2 & \longleftarrow & G_{23} & \longrightarrow & G_3 \end{array}$$

where

$$G_1 = \langle x,a,b \; : \; x^3=a^2=b^2=(ab)^2=1, xax^{-1}=b, xbx^{-1}=ab\rangle ,$$

$$G_{12} = \langle x \; : \; x^3=1\rangle ,$$

$$G_2 = \langle x,\alpha,\beta \; : \; x^3=\alpha^2=\beta^2=(\alpha\beta)^2=1, \alpha x\alpha=x^{-1}, \beta x=x\beta\rangle ,$$

$$G_{23} = \langle \alpha,\beta \; : \; \alpha^2=\beta^2=(\alpha\beta)^2=1\rangle ,$$

$$G_3 = \langle \xi,\alpha,\beta \; : \; \xi^3=\alpha^2=\beta^2=(\alpha\beta)^2=1, \xi\alpha\xi^{-1}=\beta, \xi\beta\xi^{-1}=\alpha\beta\rangle ,$$

$$G_{34} = \langle \xi \; : \; \xi^3=1\rangle ,$$

$$G_4 = \langle \xi,a,b \; : \; \xi^3=a^2=b^2=(ab)^2=1, a\xi a=\xi^{-1}, b\xi=\xi b\rangle ,$$

$$G_{41} = \langle a,b \; : \; a^2=b^2=(ab)^2=1\rangle ,$$

and all the arrows are the inclusion maps.

Note that $G_1 \cong G_3 \cong A_4$, $G_2 \cong G_4 \cong D_3 \times Z_2$, $G_{12} \cong G_{34} \cong Z_3$, and $G_{23} \cong G_{41} \cong Z_2 \times Z_2$, where A_n, D_n, Z_n denote respectively the alternating group of degree n, the dihedral group of order $2n$, and the cyclic group of order n.

Next we set

$$G_W = G_1 *_{G_{12}} G_2 , \qquad G_S = G_2 *_{G_{23}} G_3 ,$$

$$G_E = G_3 *_{G_{34}} G_4 , \qquad G_N = G_4 *_{G_{41}} G_1 ,$$

$$H = G_{12} * G_{34} , \qquad V = G_{23} * G_{41} .$$

We have

(2.1) $\qquad G_W \cap G_E = V , \qquad G_S \cap G_N = H ,$ and

(2.2) $\qquad [G_W:V] = [G_E:V] = 3 , \qquad [G_S:H] = [G_N:H] = 4.$

We now form the products

(2.3) $\qquad G_H = G_W *_V G_E , \qquad G_V = G_S *_H G_N .$

The canonical map $G_H \to G_V$ is an isomorphism; we use it to identify these two groups and set

(2.4) $\qquad G_0 = G_H = G_V .$

This group is generated by $a, b, \alpha, \beta, x, \xi$ and has defining relations

$$(2.5)\quad \begin{cases} a^2=b^2=\alpha^2=\beta^2=x^2=\xi^2=(ab)^2=(\alpha\beta)^2=1; \\ xax^{-1}=b, xbx^{-1}=ab, \alpha x\alpha=x^{-1}, \beta x=x\beta; \\ \xi\alpha\xi^{-1}=\beta, \xi\beta\xi^{-1}=\alpha\beta, a\xi a=\xi^{-1}, b\xi=\xi b. \end{cases}$$

It is apparent from this presentation that G_0 has an involutive automorphism θ such that

$$(2.6)\quad \theta(a)=\alpha, \theta(b)=\beta, \theta(x)=\xi, \theta(\alpha)=a, \theta(\beta)=b, \theta(\xi)=x.$$

Finally we define G to be the semidirect product

$$(2.7)\quad G = G_0 \rtimes \langle y : y^2=1\rangle \cong G_0 \rtimes Z_2$$

where y acts on G as the automorphism θ.

It follows that G is generated by a,b,x,y and has defining relations

$$(2.8)\quad \begin{cases} a^2=b^2=y^2=x^3=(ab)^2=1, xax^{-1}=b, xbx^{-1}=ab, \\ ayxya=yx^{-1}y, \quad byxy=yxyb. \end{cases}$$

Furthermore G has the following two factorizations as free product with amalgamation:

$$(2.9)\quad G = G_W *_V P = G_N *_H M$$

where

$$(2.10)\quad P = \langle V,y\rangle = V \rtimes \langle y\rangle, \qquad M = \langle H,y\rangle = \langle x,y\rangle = \langle x\rangle * \langle y\rangle,$$

and

$$(2.11)\quad [G_W:V] = 3, \quad [P:V] = 2, \quad [G_N:H] = 4, \quad [M:H] = 2.$$

3. THE ACTION OF G ON Γ_3

Let Γ_3 be the graph whose vertex-set is the set of left cosets $G/G_W = \{uG_W : u \in G\}$ and whose edge-set is the set of left cosets $G/P = \{uP : u \in G\}$, while the incidence is defined as follows: an edge uP and a vertex uG_W are incident iff $uP \cap vG_W \neq \emptyset$. It follows that the end-points of the edge uP are uG_W and uyG_W. It follows from (2.9) and (2.11) that Γ_3 is a cubic tree, see [3].

G acts on Γ_3 by left multiplication. The edge $e = P$ has the end-points G_W and yG_W. The element y stabilizes the edge e and interchanges its end-points. Since G is transitive on G/P, it follows that the action of G on Γ_3 is 1-transitive (see [1] for the definition of s-transitivity and ω-transitivity).

Lemma 1. The action of G on Γ_3 is ω-transitive.

Proof. In view of a theorem of Tutte [5] it suffices to show that G is 6-transitive. To prove the latter, it suffices to exhibit an element of G which fixes some 5-arc $(v_0, v_1, \ldots, v_5)$ and interchanges the two neighbours of v_5 different from v_4. Using the relations $xax^{-1}=b$, $xbx^{-1}=ab$, $\alpha x\alpha = x^{-1}$, $\beta x = x\beta$, $ay = y\alpha$, $by = y\beta$ it is easy to show that

$$(a\alpha)^4(xy)^4 = (xy)^4 b\alpha a\alpha\beta b\alpha ab\beta \quad \text{and}$$

$$(a\alpha)^4(xy)^5 = (xy)^4 x^{-1} y\alpha a\alpha\beta ab\alpha\beta a\beta b .$$

Hence the element $(a\alpha)^4$ fixes the 5-arc $(v_0,\dots,v_5)$ where

$$v_i = (xy)^{i-1}G_W, \qquad 0 \le i \le 5$$

and moves the vertex $(xy)^5G_W$ adjacent to v_5.

Lemma 2. H *contains no non-trivial normal subgroup of* G.

Proof. Let H_0 be the intersection of all conjugates of H in G. We have to show that $H_0 = \{1\}$. Let $\phi : G \to \mathrm{Aut}(\Gamma_3)$ be the homomorphism induced by the action of G on Γ_3. It follows from Lemma 1 that G_0 is locally ω-transitive on Γ_3. Since $\ker \phi \le V$ and $H \cap V = \{1\}$, the restriction $\phi|_H$ is injective. Hence we may identify H with its image $\phi(H)$ in $\mathrm{Aut}(\Gamma_3)$. Since $H_0 \cap \phi(G_0) \lhd \phi(G_0)$ and $H \cap \phi(G_0)$ is not locally ω-transitive, it follows from [2, Theorem 1] that $H_0 \cap \phi(G_0) = \{1\}$. This implies that $H_0 \cap G_0 = \{1\}$ and consequently $H_0 = \{1\}$.

4. THE ACTION OF G ON Γ_4

Let Γ_4 be the graph whose vertex-set is the set of left cosets $G/G_N = \{uG_N : u \in G\}$ and whose edge-set is the set of left cosets $G/M = \{uM : u \in G\}$, while the incidence is defined as follows: an edge uM and a vertex vG_N are incident iff $uM \cap vG_N \ne \emptyset$. Thus the end-points of the edge uM are uG_N and uyG_N. It follows from (2.9) and (2.11) that Γ_4 is a tree in which every vertex has valence 4.

G acts on Γ_4 by left multiplication and by Lemma 2 this action is faithful. Hence we may consider G as a subgroup of $\mathrm{Aut}(\Gamma_4)$.

Lemma 3. The action of G *on* Γ_4 *is* ω*-transitive.*

Proof. Since G is transitive on G/M and the element y inerchanges the two end-points of the edge $e = M$, we conclude that G is 1-transitive. The normal closure of $\langle x\rangle$ in M is the subgroup

$$M^+ = \langle x,\xi\rangle = \langle x\rangle * \langle\xi\rangle \cong Z_3 * Z_3 ,$$

and $M = M^+ \rtimes \langle y\rangle \cong M^+ \rtimes Z_2$. The end-points of the edge $e = M$ are $v_- = yG_N$ and $v_+ = G_N$. For any k (≥ 1) let Ω_k be the set of vertices of Γ_4 whose distance from v_+ is k and whose distance from v_- is $k+1$. We shall denote by $M^+(k)$ the permutation group induced by M^+ on the set Ω_k and by ϕ_k the canonical epimorphism $M^+ \to M^+(k)$.

We shall denote the elements of Ω_1 by $0,1,2$; explicitly we have

$$0 = ayG_N, \qquad 1 = byG_N, \qquad 2 = abyG_N.$$

The elements of Ω_2 will be written as ij where $i,j \in \langle 0,1,2\rangle$. They are defined as follows:

$$00 = a\alpha G_N, \qquad 01 = a\beta G_N, \qquad 02 = a\alpha\beta G_N,$$

$$10 = b\alpha G_N, \qquad 11 = b\beta G_N, \qquad 12 = b\alpha\beta G_N,$$

$$20 = ab\alpha G_N, \qquad 21 = ab\beta G_N, \qquad 22 = ab\alpha\beta G_N.$$

By a straightforward computation one finds that

$$\phi_1(x) = (012), \quad \phi_1(\xi) = 1;$$

$$\phi_2(x) = (00,10,20)(01,11,21)(02,12,22)$$

$$\phi_2(\xi) = (00,02,01)(10,11,12)(20,22,21)$$

For $i = 0,1,2$ let σ_i be the three cycle on Ω_2 defined by $\sigma_i = (i0,i1,i2)$. It is easy to verify that

$$\sigma_0 = \phi_2(x\xi x^{-1}\xi), \quad \sigma_1 = \phi_2(x^{-1}\xi x^{-1}\xi x^{-1}), \quad \sigma_2 = \phi_2(\xi x^{-1}\xi x).$$

It follows that

$$M^+(2) = (Z_3 \times Z_3 \times Z_3) \rtimes Z_3 = \langle \sigma_0, \sigma_1, \sigma_2 \rangle \rtimes \langle \phi_2(x) \rangle$$

where

$$\phi_2(x)\sigma_i\phi_2(x)^{-1} = \sigma_{i+1} \qquad \text{(indices mod 3)}.$$

Hence if $z \in M^+$ fixes all elements of Ω_1 then z^3 fixes all elements of Ω_2. Since G is 1-transitive on Γ_4, we conclude that if $z \in G$ fixes a vertex v and all four of its neighbours then z^3 fixes all vertices at distance 2 from v. By an obvious induction this implies that if $z \in G$ fixes a vertex v and all its four neighbours then z^{3^k} fixes all vertices at distance $k+1$ from v.

In particular the element $(x\xi)^3$ fixes the vertex v_+ and all its four neighbours. Consequently the element

$$z_k = (x\xi)^{3^k}$$

fixes each vertex at distance k from v_+. Since $z_k \neq 1$ and the action of G on Γ_4 is faithful this implies that G is ω-transitive on Γ_4. (The argument is the same as in [2], the last paragraph of the proof of Lemma 4.)

Lemma 4. V *contains no non-trivial normal subgroup of* G.

Proof. This follows from Lemma 3 in the same manner as Lemma 2 follows from Lemma 1.

It follows from Lemma 4 that the action of G on Γ_3 is also faithful.

5. THE SUBGROUP N

By using the presentation (2.8) of G it is easy to verify that there is an epimorphism $\phi : G \to PSL_2(11)$ such that

$$\phi(a) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ \phi(b) = \begin{pmatrix} 1 & 3 \\ 3 & -1 \end{pmatrix},\ \phi(x) = \begin{pmatrix} 5 & -4 \\ -3 & -4 \end{pmatrix},\ \phi(y) = \begin{pmatrix} 2 & 1 \\ -5 & -2 \end{pmatrix}.$$

We set $N = \ker \phi$, and

$$N_1 = N \cap G_W, \quad N_2 = N \cap P, \quad N_{12} = N_1 \cap N_2,$$

$$N_3 = N \cap G_N, \quad N_4 = N \cap M, \quad N_{34} = N_3 \cap N_4.$$

If $N_0 = N \cap G_0$ then by [2, Theorem 1] we conclude that N_0 is locally ω-transitive on Γ_3 and Γ_4. Taking into account that $\phi(G_0) = \phi(G)$, we infer that

$$G/N \cong G_0/N_0 \cong G_W/N_1 \cong G_N/N_3$$
$$\cong V/N_{12} \cong M/N_{34} \cong PSL_2(11),$$

and that (1.1) is valid where the indices d_i $(1 \le i \le 4)$ are as given in (1.4). All the groups N_i $(1 \le i \le 4)$ are free by [3, Theorem 14, p.56]. Their ranks r_i can be computed in the same manner as in [2]; one obtains the values listed in (1.4).

The same argument as in [2], based on a theorem of Stebe [4], shows that G is residually finite.

6. ANOTHER SUBGROUP N

By using the presentation (2.8) of G one can easily verify that we also have an epimorphism $\psi : G \to PSL_2(13)$ such that

$$\psi(a) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ \psi(b) = \begin{pmatrix} 5 & 0 \\ 0 & -5 \end{pmatrix}, \ \psi(x) = \begin{pmatrix} 3 & 2 \\ 3 & -2 \end{pmatrix}, \ \psi(y) = \begin{pmatrix} 1 & -4 \\ -6 & -1 \end{pmatrix}.$$

As in the previous section one shows that $N = \ker \phi$ has two decompositions (1.1) where N_i $(1 \leq i \leq 4)$ are free groups and the ranks r_i and indices d_i are as indicated in (1.5).

Finally we record the follow presentation of $PSL_2(13)$:

$$(6.1) \quad PSL_2(13) = \langle x, \xi : x^3 = \xi^3 = (x\xi)^6 = (x\xi(x\xi^2)^2)^2 = (x\xi(x^2\xi)^2)^2 = 1 \rangle.$$

Indeed the matrices $x = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ and $\xi = \begin{pmatrix} 0 & 6 \\ 2 & 1 \end{pmatrix}$ satisfy these relations and generate $PSL_2(13)$. On the other hand, a coset enumeration (for which I am indebted to C. Sims) shows that the group defined by this presentation has the same order as $PSL_2(13)$.

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A SURVEY OF THE UNIFYING EFFECTS OF F-POLYNOMIALS IN COMBINATORICS AND GRAPH THEORY

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ABSTRACT

An up to date account is given of the various F-polynomials and their relation to polynomials in Combinatorics and Graph Theory. These include the well-known orthogonal polynomials found in Combinatorics and many of the familiar polynomials in Graph Theory, such as chromatic polynomials, dichromatic polynomials and characteristic polynomials.

1. INTRODUCTION

Let F be a family of connected graphs. With each element of $\alpha \in F$, let us associate an indeterminate or *weight* w_α. By an F-*cover* of G we will mean a spanning subgraph of G in which each component belongs to F. With each F-cover C of G let us associate the weight

$$w(C) = \prod_{\alpha} w_\alpha \quad ,$$

where the product is taken over all the elements α of C. Then the F-*polynomial* of G is

$$F(G;\underline{w}) = \Sigma\, w(C)$$

where the summation is taken over all the F-covers in G. $\underline{w}$ is a vector of indeterminates associated with the weights w_α. For example, if we give each member of F with r nodes a weight w_r, then the elements of $\underline{w}$ will be w_1, w_2, w_3, etc.

By considering specific families of graphs, we obtain special F-polynomials. For example, we can take F to be the family of nodes and edges only, or the family of circuits, trees, paths stars or complete graphs. By restricting the members of the family, we can obtain simplified forms of $F(G;\underline{w})$ which more easily lend themselves to detailed analyses and investigation. However, some work has been done on properties of general F-polynomials and on general F-polynomials of certain graphs (see Farrell [5]).

Many significant connections have been established between certain F-polynomials and several well-known polynomials found in Combinatorics and Graph Theory. In the material which follow, we will give an up to date account of these connections. The significance of such connections is that they serve to unify all the associated polynomials and therefore create new alternate avenues by which the polynomials can be indirectly investigated.

2. THE MATCHING POLYNOMIAL

Let F be the family of (isolated) nodes and (independent) edges. Then every F-cover of G will be a matching in G. The resulting F-polynomial is called the *matching polynomial* $M(G;\underline{w})$ of G. In this case, $\underline{w} = (w_1, w_2)$. $M(G;\underline{w})$ was formally introduced in Farrell [6], although some special forms of it existed in the chemical and physical literature several years before (see Heilmann and Leib [17], Kunz [18], Gruber and Kunz [15], Hosoya [19] and Aihara [1]).

The acyclic polynomial $\alpha(G)$, of a graph G, was introduced by Gutman [16]. It was later shown (see Farrell [7]) that $\alpha(G)$ was special case of $M(G;\underline{w})$. The relationship is given in the following theorem.

Theorem 1.

$$\alpha(G;x) = M(G;x,-1) \quad .$$

Theorem 1 shows that $\alpha(G)$ is a matching polynomial and therefore an F-polynomial.

Godsil and Gutman [14] soon reported connections between $\alpha(G)$ and some of the standard orthogonal polynomials encountered in Combinatorics. These include the Chebyshev polynomials, Hermite polynomials and Laguerre polynomials. They were shown to be acyclic polynomials and hence matching polynomials of certain graphs. The connections are given in the following theorem which is a modified version of Theorem 3 of [14], using Theorem 1 above.

Theorem 2. Let P_n, C_n, K_n *denote the path, circuit and complete graph respectively, on* n *nodes. Let* $K_{m,n}$ *denote the complete bipartite graph with bipartition* m *and* n. *Then*

$$M(C_n;2\lambda,-1) = 2T_n(\lambda) \ ,$$

$$M(P_n;2\lambda,-1) = U_n(\lambda) \ ,$$

$$M(K_n;\lambda,-1) = He_n(\lambda) \ ,$$

$$2^{n/2}M(K_n;\sqrt{2}\lambda,-1) = H_n(\lambda) \ ,$$

$$M(K_{n,n};\lambda,-1) = (-1)^n L_n(\lambda^2) \ ,$$

and

$$M(K_{m,n};\lambda,-1) = (-1)^n \lambda^{m-n} L_n^{m-n}(\lambda^2) \ ,$$

where T_n *and* U_n *are Chebyshev polynomials of the first and second kind,* He_n *and* H_n *are the two standard forms of the Hermite polynomials, while* L_n *and* L_n^k *are the Laguerre and the generalized Laguerre polynomials.*

We note that the first four equations in Theorem 2 were essentially observed in [18]. This theorem shows that the Chebyshev, Hermite and Laguerre polynomials are also F-polynomials. One significance of the result is that it has now given a graph-theoretical interpretation of these famous classical polynomials.

Another classical polynomial, the rook polynomial, was also mentioned to be related to $\alpha(G)$ in [14], although no formal relation was given. The connection between the rook polynomial and the matching polynomial is essentially given in the following theorem.

Theorem 3. *Any bipartite graph* G *with* $p = m + n$ *nodes can be regarded as a chessboard* B *with* m *rows and* n *columns such that cell* (i,j) *belongs to* B *if and only if in* G, *the* i^{th} *node of* $\bar{K}_m$ *is adjacent to the* j^{th} *node of* $\bar{K}_n$. *Then a k-matching in* G *is equivalent to the placing of* k *non-taking rooks on* B.

This theorem shows that rook polynomials are matching polynomials and are therefore F-polynomials.

The results given in this section show that the matching polynomial can be very useful for establishing interconnections between certain classical polynomials. For example, Theorem 2 and 3 can be used to establish a connection between Laguerre polynomials and rook polynomials. The future of the polynomial $M(G;\underline{w})$ seems bright, and no doubt even more useful connections are envisaged in the near future.

3. THE CIRCUIT POLYNOMIAL

Let F be the family of circuits. Then every F-cover in G will be a set of circuits wich span G. In this case, the F-polynomials is caled the *circuit polynomial* of G. If we give each circuit with r nodes a weight w_r, then we will have $\underline{w} = (w_1, w_2, w_3, \ldots, w_p)$, where p is the number of nodes in G. The circuit polynomial of G will then be denoted by $C(G;\underline{w})$. This polynomial was first mentioned in [5]. However the basic paper on circuit polynomials is [8].

In [8], it was shown that the characteristic polynomial

$\phi(G;x)$ of a graph G is a special circuit polynomial. The connection between the two polynomials is given in the following theorem (Theorem 3 of [8]).

Theorem 4.

$$\phi(G;x) = C(G;x,-1,-2,-2,\ldots,-2) \quad .$$

i.e. $\phi(G;x)$ *is obtained from* $C(G;\underline{w})$ *by putting* $w_1 = x$, $w_2 = -1$ *and* $w_r = -2$ *for* $r > 2$.

It follows from the above theorem that the characteristic polynomial of a graph is a circuit polynomial. Hence the characteristic polynomial is also an F-polynomial.

It is clear that a matching is a circuit cover in which no circuit has more than two nodes. In this case, we define a circuit with one and two nodes to be an isolated node and an independent edge respectively. Hence we have the following theorem.

Theorem 5.

$$M(G;w_1,w_2) = C(G;w_1,w_2,0,0,\ldots,0) \quad .$$

Theorems 4 and 5 can be used in order to establish interconnections between the circuit polynomial, the characteristic polynomial and all the polynomial mentioned in Section 2. For example, there is a connection between rook polynomials and characteristic polynomials (Theorem 3, 4 and 5), and between the orthogonal polynomials and characteristic polynomials (Theorem 2, 4 and 5). Such connections between these polynomials were not previously suspected.

In 1972, Clarke [4] defined a polynomial $P(G;x)$ associated with the node adjacency matrix of a graph. This polynomial was as follows:

$$P(G;x) = \det.|A(G) + xI| , \qquad (1)$$

where $A(G)$ is the adjacency matrix of the graph G with p nodes and I is the $p \times p$ identity matrix. The following result was established in Farrell and Grell [9].

Theorem 6.

$$P(G;x) = C(G;x,-1,2,-2,2,\ldots) \quad .$$

i.e. $P(G;x)$ *is obtained from* $C(G;\underline{w})$ *by putting* $w_1 = x$, $w_2 = -1$ *and* $w_r = (-1)^{r+1}2$, *for* $r > 2$.

This theorem shows that $P(G;x)$ is a circuit polynomial and therefore an F-polynomial.

It is clear from Equation (1) that

$$|A(G)| = P(G;0) \quad .$$

Hence we have the following result, which is also given in [9].

Corollary 6.1.

$$|A(G)| = C(G;0,-1,2,-2,2,\ldots) \quad .$$

This corollary shows that the determinant of the node adjacency matrix of a graph is a circuit polynomial and therefore an F-polynomial. In this case, the F-polynomial is a constant. The corollary is interesting for another reason. It establishes a connection between circuit polynomials of graphs and determinants of special matrices. We suspect that this connection could be extended to circuit polynomials and determinants of arbitrary matrices. An investigation into this would be wortwhile.

4. THE TREE POLYNOMIAL

When F is the family of trees an F-cover is a spanning forest. In this case, the F-polynomial is called the *tree polynomial* of G. This polynomial will bi denoted by $T(G;\underline{w})$. The tree polynomial was introduced in Farrell [10]. In this paper, it was shown that the polyinomial $T(G;\underline{w})$ is related to the characteristic polynomial of G. The relation is given in the following theorem.

Theorem 7. Let $T(G;\underline{w})$ *be the tree polynomial of* G. *Then* $\phi(G;x)$ *is obtained from* $T(G;\underline{w})$ *by assigning to each component* α *of a tree cover, a weight* $nx - \sum_{v_i \varepsilon \alpha} d(v_i)$, *where* n *is the number of nodes in* α *and* $d(v_i)$ *is the valency of node* v_i *(belonging to* α*) in* G.

Theorem 7 is the significant for two reasons. Firstly, it shows that the characteristic polynomial of a graphs is a special tree polynomial. We have already established that $\phi(G;x)$ is a circuit polynomial (Theorem 6). These results imply that there is some relation between the tree covers and circuit covers of a graph. Secondly, Theorem 7 shows that the characteristic polynomial of a graph depends on the tree subgraphs of the graph. It was well known (See Sachs [22]) that the characteristic polynomial is related to the circuits in the graph.

Moon [21] defined a polynomial $T(G;\theta)$ which was used to investigate the number of spanning trees in graphs. In [10] it was shown that $T(G;\theta)$ is a special tree polynomial. The relation is stated formally in the following theorem, in which $\underline{w} = (w_1, w_2, \ldots, w_p)$.

Theorem 8. *The Moon polynomial $T(G;\theta)$ of a graph G is obtained from $T(G;\underline{w})$ by putting $w_n = n(-\theta)^{1-n}$, where w_n is the weight of a component with n nodes.*

This theorem shows that $T(G;\theta)$ is a tree polynomial and therefore an F-polynomial.

5. THE SUBGRAPH POLYNOMIAL

When F is the family of all connected subgraphs of the graph G, the resulting F-polynomial is called the *subgraph polynomial* of G. It is denoted by $S(G;\underline{w})$. It was shown in Farrell [11] that the chromatic polynomial and the Tutte polynomial are subgraph polynomials. The relationships are given in the following theorems.

Theorem 9. *The polynomial obtained from $S(G;\underline{w})$ by putting $w_r = (-1)^r\lambda$, where w_r is the weight of a component with r edges, is the chromatic polynomial of G.*

Theorem 10. *The polynomial obtained from $S(G;\underline{w})$ by putting $w_{n,e} = xy^{e-n+1}$, where $w_{n,e}$ is the weight of a component with n nodes and e edges, is the dichromatic polynomial of G.*

The above theorems can be used in order to obtain relationships between chromatic polynomials and dichromatic polynomials. A relationship has already been given in Tutte [23]. The theorems show that both the chromatic and dichromatic polynomials are subgraph polynomials and are therefore F-polynomials. Theorem 9 is not surprising, since it is well known (see Whitney [24]) that the chromatic polynomial of a graph is connected with the spanning subgraphs of the graph.

Borzacchini and Pulito [3] defined a polynomial, called the <u>subgraph enumerating polynomial</u> $P(G;u,v,x)$ as follows.

$$P(G;u,v,x) = \sum_{i,j,k} C_{ijk} u^i v^j x^k \quad ,$$

where C_{ijk} is the number of spanning subgraphs of G with j edges, k connected components and i non-isolated nodes, and the sum is taken over all i, j and k. This polynomial was investigated with respect to reconstruction, and it was also shown to be related to the Tutte polynomial.

The following theorem was recently proved in Farrell [12].

Theorem 11.

$$P(G;u,v,x) = S(G;\underline{w}') \quad ,$$

where $w'_{i,j} = u^i v^j x$, *for* $i > 1$ *and* $w'_{1,0} = x$.

This theorem shows that the subgraph enumerating polynomial is a subgraph polynomial. It follows that it is also an F-polynomial. Theorem 9 was used in [12] to establish results in [3] by different techniques and also to establish interconnections between the associated polynomials.

In their quest to obtain a useful polynomial which would characterize a graph (i.e. two graphs are equal if and only if they have the same polynomial) Balasubramanian and Parathasarathy [2] defined the frame polynomial. First of all, a <u>frame</u> of a graph G is defined as a spanning subgraph of G whose components are either nodes, edges, chains (tree with nodes of valencies 1 and 2 only) or cycles. Let F be a frame in G consisting of v_i chains with i nodes $(i=1,2,\ldots,n)$, and r_j cycles with j nodes for $j=3,4,\ldots,n$. Then

$$W(F) = \prod_{i=1}^{n} p_i^{v_i} \prod_{j=3}^{n} c_j^{r_j} .$$

The <u>frame polynomial</u> of G is

$$F(G;\underline{p},\underline{c}) = \Sigma\, w(F)$$

where the summation is taken over all the frames in G, $\underline{p} = (p_1,p_2,\ldots)$ and $\underline{c} = (c_1,c_2,\ldots)$. In [2] the authors conjectured that a graph G is characterized by $F(G;\underline{p},\underline{c})$.

The folloving theorem was easily established in Farrell [13]. w_{C_i} and w_{P_i} denote the weights of a cycle and chain respectively with i nodes.

Theorem 12.

$$F(G;\underline{p},\underline{c}) = S(G;\underline{w}) \quad ,$$

where $w_{P_i} = p_i,\ w_{C_i} = c_i$ *and* $w_\alpha = 0\ \forall$ *non-frame graphs.*

Thus the frame polynomial is also a subgraph polynomial. Hence it is an F-polynomial.

Associated with $F(G;\underline{p},\underline{c})$ were several other polynomials defined in [2]. These were all shown to be specially weighted subgraph polynomials, using Theorem 10. In fact, the theorem was instrumental in obtaining interrelationships between all the polynomials associated with the frame polynomial.

The conjecture that $F(G;\underline{p},\underline{c})$ is a characterizing polynomial for G, suggests that one might be able to find a convenient subgraph polynomial which characterizes a graph. Clearly, if one considers all the possible subgraph covers and give a unique weight to each component, the resulting subgraph polynomial should characterize the graph. However, such a polynomial will be far too cumbersome to be of any practical use, with even small graphs. It will be interesting therefore to find out which subgraphs could be either not counted at all, or given the same weight, without losing the characterizing property of the polynomial. The conjecture in [2] suggests that all we need to consider are the frames.

6. DISCUSSION

We have given a survey of some of the general F-polynomials and their connections with the more popular (special) F-polynomials found in Combinatorics and Graph Theory. As we have pointed out, and have illustrated in the relevant papers, the establishment of interconnections between the polynomials has povided a view of some of the more popular polynomials, from a different prospective. In some cases, this was useful for obtaining new properties of the special polynomials. In other cases, well known properties were easier derived by working with the more general F-polynomials.

The most interesting feature of the study of the general polynomials is the unifying effect on other polynomials. We are now able to classify several polynomials according to the "parent" F-polynomials. Also, we can deduce properties of a polynomial by simply identifying its "parent".

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COVERING HEXAGONAL SYSTEMS WITH HEXAGONS*

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ABSTRACT

A special type of covering of hexagonal systems with hexagons is introduced. This concept is of some relevance in chemistry. The main results are summarized in Propositions 2 and 4. Proposition 7 offers a characterization of hexagonal system with Hamiltonian cycles.

In the present paper we shall consider graphs called hexagonal systems. Such graphs correspond to the network obtained by paving the plane with congruent regular hexagons, so that two hexagons are either disjoint or have a common edge. In addition we shall assume that the hexagonal system is simply connected, i.e. has no "holes".

Graphs of this type have been named "hexagonal animals"

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* *Part XXV of the series "Topological Properties of Benzenoid Systems".*

[1,2] or "benzenoid systems" [3]. Recently H. Sachs proposed the name "hexagonal system" as a kind of compromise [4,5].

For the theory of hexagonal systems (except their enumeration) the reader should consult a review [3], where also the application of this class of graphs in chemistry is extensively discussed.

Let H be a hexagonal system having h hexagons. These hexagons will be labelled by $S_1, S_2, \ldots, S_h$. Following [3] we will divide the vertices of H into *external* and *internal*. The external vertices of H form its *perimeter*.

For example, H_1 and H_2, on Fig.1, are two hexagonal systems. H_1 has three internal vertices (marked by heavy dots) and 24 external vertices. Hence the perimeter of H_1 is a cycle of length 24. The hexagonal system H_2 has no internal vertices; all 30 vertices of H_2 lie on its perimeter.

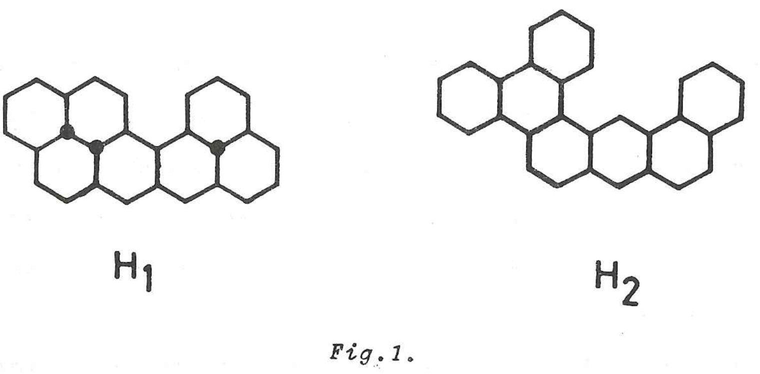

Fig.1.

If a hexagonal system has no internal vertices, it is said to be cata-condensed. The set of all cata-condenset hexagonal systems is denoted by C. If a hexagonal system has internal vertices, it is said to be peri-condensed. The set of all peri-condensed hexagonal systems is denoted by $\mathcal{P}$. For example, $H_1 \in \mathcal{P}$, $H_2 \in \mathcal{C}$.

Let $K = \{S_1, S_2, \ldots, S_k\}$, $k \geq 1$, be a collection of hexagons of a hexagonal system H. Then $H-K$ will denote the subgraph obtained by deleting from H all the vertices of all hexagons S_i, $i = 1, \ldots k$, and all the incident edges.

If G is a graph with n vertices, then a 1-factor of G is a selection of $n/2$ independent edges of G (which, of course, cover all vertices of G).

Definition. K is a *cover* of H (or K *covers* H) if the hexagons $S_1, S_2, \ldots, S_k$ are pairwise disjoint and if $H-K$ has a 1-factor. (In the case when $S_1, S_2, \ldots, S_k$ cover all the vertices of H, we assume that the empty graph $H-K$ has a 1-factor.)

If $K = \{S_1, S_2, \ldots, S_k\}$ is a cover of H, then we will also say that the hexagons $S_1, S_2, \ldots, S_k$ cover H.

The above definition is just a graph-theoretical reformulation of a concept which occurs in chemistry, within the so called *Clar aromatic sextet theory*. For more details along these lines see [3,6]. It is worth noting that the maximal cover of a hexagonal system plays an important role in chemical applications. (The definition of a maximal cover will be given later on.)

Proposition 1. If K is a cover of H and K' is a non-empty subset of K, then K' is a cover of H.

Proof. It is sufficient to prove that if $S \in K$, $|K| > 1$, then $K' = K \setminus \{S\}$ covers H.

Let F be a 1-factor of $H-K$ (which exists by hypothesis). Then a 1-factor of $H-K'$ will be obtained by making a union of F and of three independent edges of the hexagon S.□

Since three independent edges in a hexagon can be selected in two different ways, we arrive to the following result.

Corollary 1.1. If K is a cover of H and $|K| = k$, then H has at least 2^k 1-factors.

Some additional statements of this kind are obtained by similar reasoning.

Corollary 1.2. If K is a cover of H and $|K| = k$, then H has at least 2^k-1 distinct covers.

Corollary 1.3. If $K = \{S\}$ is not a cover of H, then no cover of H contains S.

Corollary 1.4. If H has no 1-factor, then H has no cover.

The reverse of Corollary 1.4 is also true.

Proposition 2. A hexagonal system has a cover if and only if it has a 1-factor.

Proof. Having in mind Corollary 1.4, it remains to demonstrate only the "if" part of Proposition 2.

We will show a stronger result, namely the following Lemma.

Lemma 2.1. Every 1-factor of a hexagonal system contains three edges which cover all the six vertices of a hexagon.

The following proof of Lemma 2.1 was suggested by B. Mohar.

Proof. Let H be a simply connected hexagonal system having n vertices, m edges and h hexagons. Let the size of the perimeter of H be p.

The edges of H can be partitioned into external (those which belong to the perimeter) and internal (those which do not belong to the perimeter). There are p external and $m-p$ internal edges. Since every internal edge belongs to two hexagons, we have $6h = 2(m-p) + p$ i.e. $m = 3h + p/2$. On the other hand, by the Euler formula, $n-m+h = 1$ and consequently

$$n - 2h - p/2 = 1 \quad . \tag{1}$$

Let F be a 1-factor of H, containing k external and $n/2-k$ internal edges of H. Suppose that Lemma 2.1 is not true for F, that is suppose that at most two edges in each hexagon belong to F. Then

$$2h \geq k + 2(n/2 - k) = n - k \geq n - p/2$$

and therefore,

$$n - 2h - p/2 \leq 0 \ .$$

This latter relation is in contradiction with eq. (1). Therefore a 1-factor of H in which at most two edges belong to each hexagon cannot exist.

Lemma 2.1 and Proposition 2 are thus proved. □

Proposition 3. If $H \in C$, then every hexagon of H covers H.

A result equivalent to the above proposition has been proved in [6]. We offer now a generalization of Proposition 3.

Proposition 4. If H possesses a Hamiltonian cycle, then every hexagon of H covers H.

Proof. Let S be a hexagon of H. Since H possesses a Hamiltonian cycle, its structure can be presented as follows (see Fig.2; the heavy line indicates the Hamiltonian cycle). After the deletion of the vertices of S, the Hamiltonian cycle of H will be decomposed into three (or less) paths. Each path will contain an even number of vertices (since all cycles of H were of even size) and therefore a 1-factor.

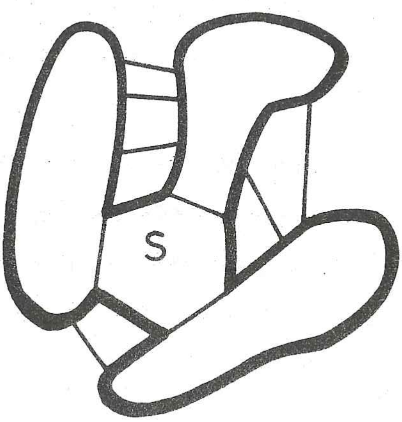

Fig.2.

Consequently H-S has a 1-factor.□

Note that the perimeter of a cata-condensed hexagonal system is a Hamiltonian cycle. Therefore Proposition 3 is a special case of Proposition 4.

* * *

Concluding this paper we would like to point at some other closely related results. A cover K is said to be maximal if K is not a proper subset of any other cover.

Proposition 5. [6] If $H \in C$ and K is a maximal cover of H, then H-K is a unique 1-factor.

Hosoya and Yamaguchi [7] observed an interesting property of cata-condensed hexagonal systems, which in the terminology of the present paper can be stated as follows. This result has been proved in [8].

Proposition 6. [7,8] If $H \in C$ and H has f 1-factors, then H has f-1 distinct covers.

Note that if $H \in C$, then $f \geq h+1$ [9].

Counterexamples show that neither Proposition 5 nor Proposition 6 can be simply extended to peri-condensed hexagonal systems. A possible progress in this direction would be the proof of the following three conjectures.

Conjecture 5.1. If $H \in P$ and K is a maximal cover of H with maximal cardinality, then H-K has a unique 1-factor.

Conjecture 6.1. If H has a Hamiltonian cycle and H has f 1-factors, then H has f-1 distinct covers.

Conjecture 6.2. If H ∈ *P* and H has f 1-factors (f > 0), then H has at least f-1 distinct covers.

* * *

Several statements in the present paper deal with hexagonal systems possessing Hamiltonian cycles. Therefore we would like to give a characterization of such systems.

Let H and H′ be two hexagonal systems having the same vertex set. If H′ can be obtained by deleting some edges from H, then we say that H is e-transformable to H′.

Proposition 7. (a) H ∈ *C* has a Hamiltonian cycle. *(b)* H ∈ *P* has a Hamiltonian cycle if and only if H is e-transformable to H′, such that H′ ∈ *C*.

Proof. Statement (a) is obvious. Let us therefore consider the case when H is peri-condensed.

Let H be e-transformable to H′, H′ ∈ *C*. Then the perimeter of H′ is the Hamiltonian cycle of H.

Let H has a Hamiltonian cycle. This cycle can be interpreted as the perimeter of another hexagonal system H′. This latter system cannot have internal vertices, i.e. H′ ∈ *C*. Obviously, H and H′ have equal vertex sets. H′ can be obtained from H by deleting some edges. Hence H is e-transformable to H′.

This proves Proposition 7.□

Corollary 7.1. If a hexagonal system with n vertices has a Hamiltonian cycle, then $n \equiv 2 \pmod 4$.

Proof. If $H \in C$, then H has $4h+2$ vertices.□

Without proof we state another related result.

Proposition 8. If H has a Hamiltonian cycle, then this cycle is unique.

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REALIZATION OF GRAPH REGULATION NUMBERS

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ABSTRACT

Let G be a graph with maximum degree n. The regulation number r(G) is the smallest number of new points which must be added to G to obtain a n-regular supergraph. Similarly the induced regulation number ir(G) is the number of new points needed to get a n-regular supergraph in which G is an induced subgraph. We show that for all integers s and t such that $0 \le s \le t$, there exists a graph G with r(G) = s and ir(G) = t.

1. INTRODUCTION

This area began with the first book on graph theory ever written [6], when König proved that for every graph G, say with maximum degree n, there is an n-regular supergraph H which contains G as an induced subgraph. Erdös and Kelly [3] specified the precise number of new points which must be added to G for this purpose. A simpler problem was solved by Akiyama, Era and Harary [1] who dropped the "induced" requirement and just regarded H as an n-regular supergraph of G.

These considerations suggest two new invariants. The *regulation number* $r(G)$ is the minimum number of additional vertices which must be added to G to construct an n-regular supergraph H. (Thus $r(G) = 0$ if and only if G has an n-regular spanning supergraph).

Similarly, the *induced regulation number* $ir(G)$ is the smallest number of new vertices needed to obtain an n-regular supergraph H containing G as an induced subgraph. Obviously, $r(G) \leq ir(G)$. Our purpose is to prove a realization result that for all integers s and t such that $0 \leq s \leq t$, there exists a graph G having $r(G) = s$ and $ir(G) = t$.

We use standard graph theoretical notation, as in [4]. Let $G = (V,E)$ be a graph with maximum degree $\Delta(G) = n$. The degree of $v \in V$ is denoted by $d(v)$, and the *deficiency* of v is the difference $n - d(v)$. Let $G' = (V',E')$ be disjoint from G. For two disjoint vertex sets V, V', we write $K(V,V')$ for the complete bipartite graph joining them. By definition, the join $G + G'$ is $G \cup G' \cup K(V,V')$; see [4, p. 21].

2. THE REALIZATION THEOREMS

We shall require just one preliminary, useful lemma. It is well-known that K_p has a factorization into hamiltonian cycles when p is odd, and a factorization into 1-factors when p is even; see, e.g. [4, p. 85]. Let $N = \{1,2,...\}$ and $N_p = \{1,2,...,p\}$. Hence we have the following observation.

<u>*Lemma 1.*</u> *Let* $V = \{v_1,...,v_p\}$ *and let* $k \in N_{p-1}$.

If p *or* k *is even, there exsists a k-regular graph* G *on* V*. If* p *and* k *are both odd, there exists a graph* G *on* V *such that* $d(v_1) = k + 1$ *and for all other vertices,* $d(v_i) = k$.

Theorem 1. *Let* $0 \leq s \leq t$ *where* $s,t \in N$*. Then there are infinitely many graphs* G *such that* $r(G) = s$ *and* $ir(G) = t$.

Proof. We first consider s positive and then handle $s = 0$. For given $0 < s \leq t$ we construct a graph G with s and t as its regulation invariants. From the construction it will be clear that there are infinitely many such graphs.

Let $n \geq t$, n even, and let $G' = (V',E')$ be any n-regular graph. Let $V" = \{v_1,\ldots,v_{n-s+1}\}$ be a set of $n-s+1$ new vertices. Then $r(G' \cup V") = s$ because K_{n+1} on $V"$ and s new points is the smallest graph such that $H = G' \cup K_{n+1}$ is n-regular.

We still must construct a graph G having not only $r(G) = s$ but also $ir(G) = t$. To do this, we have to add edges between vertices of $V"$. Note that the regulation number s will not change.

Case 1. t even.

By Lemma 1 we can add edges between vertices of $V"$ such that the resulting graph $G"$ on $V"$ is $(n - t)$-regular. Let $G = G' \cup G"$. We now show that $ir(G) = t$.

Since $G" \neq \emptyset$ and $n - d(v_i) = t$ for all $v_i \in V"$, $ir(G) \geq t$.

Let $V_t = \{w_1,\ldots,w_t\}$ be a set of t new vertices.

Since t is even and $s - 1 < t$, we conclude from Lemma 1 that we can construct an $(s - 1)$-regular graph F on V_t. Hence the graph $(F + G'') \cup G'$ is n-regular. Thus $ir(G) \leq t$ so $ir(G) = t$.

Case 2. t odd.

If s is also odd, the same construction can be used. Hence we now consider the case that s is even. Then $n-s+1$ and $s - 1$ are odd and $n - t + 1 < n - s + 1$. By Lemma 1 we can add new edges to V'' so that $d(v_1) = n - t + 1$ and $d(v_i) = n - t$ for $i = 2,\ldots,n-s+1$ in the resulting graph G''. Similarly we can construct a graph F on V_t in which $d(w_1) = s$ and $d(w_i) = s - 1$ for $i = 2,\ldots,t$. Let $G = G' \cup G''$. Obviously $ir(G) \geq t$. And since $((F + G'') \cup G') - v_1w_1$ is n-regular, $ir(G) = t$.

Finally, we consider now the case that $s = 0$. If $t = 0$ there is nothing to prove, so let $t > 0$. Now, let G' be any graph with maximum degree n, n even, in which the set V_1 of vertices of degree 1 has n elements and is independent. One example of such a graph G' is the star $K_{1,n}$. By Lemma 1, we can add joining vertices in V_1 such that in the resulting graph G every vertex in V_1 has degree $n - t$. Then $r(G) = 0$ and $ir(G) = t$.

If s and t have the same parity, we can give an alternate construction in which the graphs have arbitrarily high connectivity.

Theorem 2. *Let* $s,t \in N$, $0 \leq s \leq t$, $s \equiv t \pmod 2$, *and let* $m \in N$. *Then there are infinitely many m-connected graphs* G *such that* $r(G) = s$ *and* $ir(G) = t$.

Proof. Choose $n > m + t + 1$, n odd, and assume that m is even.

Case 1. s odd.

It is easy to construct a graph $G' = (V,E')$ of maximum degree n that is m-connected and has the following property: Every vertex of degree less than n has degree m (so that there are just two different degrees n and m), and the set V_m of these vertices is independent and contains exactly $n-s+1$ elements.

Let $k = (n - m - s)/2$ and $c = (n - m - t)/2$, and let $C_i = (V_m, E_i)$, $i = 1,\dots,k$, be pairwise edge disjoint 2-regular graphs, each a union of cycles. We define $G = G' \cup \bigcup_{i=1}^{c} C_i$. Then $r(G) \geq s$, since the deficiency of each vertex in V_m is odd, $|V_m|$ is odd, and every other vertex has deficiency 0.

On the other hand, consider $G \cup \bigcup_{i=c+1}^{k} C_i$ and K_s disjoint from G. Then $H = (K_s + V_m) \cup G$ is n-regular.

Furthermore we see that $ir(G) \geq t$. Since $s - 1$ is even, we can construct an $(s - 1)$-regular graph F with t vertices that is disjoint from G. Then $H = (F + V_m) \cup G$ is n-regular, hence $ir(G) = t$.

Case 2. s even.

It is easy to construct an m-connected graph $G = (V,E)$ with the following properties: $V = V_n \cup V_{n-t}$ where each vertex in V_i has degree i, $V_n \neq \emptyset$ and $|V_{n-t}| = n - s + 1$. Furthermore there is a vertex $v \in V_{n-t}$ such that $V_{n-t} - v$ is independent and v is adjacent to exactly $n - t$ of the vertices in $V_{n-t} - v$.

Then $r(G) = s$ and $ir(G) = t$.

3. UNSOLVED PROBLEM

What is the situation for digraphs? Beineke and Pippert [2] derived the result for digraphs (and also for oriented graphs) analogous to that of Erdös and Kelly [3] for graphs. Similarly, Harary and Karabed [5] proved for digraphs the theorem corresponding to that of Akiyama, Era and Harary [1] for graphs.

Conjecture. For all integers s and t such that $0 \leq s \leq t$, there exists a digraph D having $r(D) = s$ and $ir(D) = t$.

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A BASIS FOR CHARACTERIZATION OF HEXAGONAL ANIMALS

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ABSTRACT

A combinatorial characterization of a class of graphs known under the name of hexagonal animals is given for the first time. The characterization is based on the cyclically ordered sequence of the perimeter vertex degrees, which is shown to be the invariant describing completely the hexagonal animals.

1. INTRODUCTION

The class of graphs known in the literature as "hexagonal animals" is used for modelling the structures of molecu-

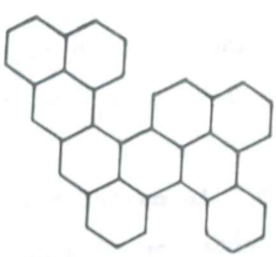

Fig.1.

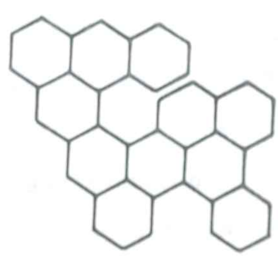

Fig.2.

les of a class of organic compounds. The atoms are represented by the vertices, and covalent bonds by the edges of the graph.

These graphs have been used and studied through a certain period of time, but they still lack a graph-theoretic characterization, despite of some endeavours made in that direction [1].

A graphical representation of a simple hexagonal animal, imbedded in the hexagonal grid, is shown in Fig.1. In the sequel the abbreviation HA shall be used for hexagonal animals.

The fundamental role in the combinatorial characterization of HA is played by the notion of the mesh of a planar graph $G(V,E)$. The set of meshes, O, of $G(V,E)$ may be defined as

$$O = \{o_i \mid o_i \text{ is a circuit in } G \wedge \text{ every edge } e_j \in E \text{ is contained in exactly two meshes of } O\}$$

For the class of two connected planar graphs, to which the HA belong, we have $|O| = n_e - n_v + 2$, where $n_e = |E|$ and $n_v = |V|$.

The following necessary condition for a graph to be a HA may be deduced from Fig.1:

Condition 1. A HA has a set of meshes with n_e-n_v+1 meshes of the length 6 and one mesh of the length n_p, which is either 6 or ≥ 10.

The mesh with the unique length n_p is called the perimeter of the HA. In a graphical representation of a HA imbedded in the hexagonal grid the perimeter is represented by the contour inclosing the drawing of the HA.

The example from Fig.2 shows that Condition 1 is not sufficient.

2. INVARIANT CHARACTERIZATION OF HEXAGONAL ANIMALS

From Fig.3 it can be seen that to every HA corresponds a unique cyclically ordered sequence of the perimeter-vertex degree-values $B = b_1, b_2, \ldots, b_{n_p}, b_i \in \{2,3\}$ which shall be called the "code B". The particular code B of the HA from Fig.3 may be written as

$$B = 22223322232233332223222323222333.$$

Group	n_s	Δk_b
22223333	4	+3
222333	3	+2
2233	2	+1
23	1	0
333	3	-1
33	2	-1
3	1	-1
222	0	+3
22	0	+2
2	0	+1

Table 1

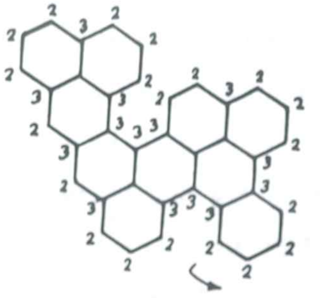

Fig. 3

That, conversely, a code B of a HA determines that unique HA, may be verified by considering a tour around a graphical representation of the perimeter when the HA is imbedded in the hexagonal grid. Whenever a vertex is passed on such a tour, a change in the direction of moving takes place. When the degree of the vertex is 2, the change is $+\pi/3$, and when the degree is 3, the change is $-\pi/3$, assuming an anticlockwise tour. Thus, the sequence of the perimeter-vertex degrees determines completely the graphical representation of a HA imbedded in the hexagonal grid, and consequently, it determines the HA too.

From Fig.2 it can be deduced that the code B of a HA must satisfy the following necessary, but not the sufficient condition.

Condition 2. The code B of a HA does not contain a subsequence with more than four adjacent elements having the value 3.

The consideration of more complex HA is needed in order to obtain a similar, but stronger, condition for the subsequences with more than four nonadjacent vertices of the value 3. Such a necessary and sufficient condition, for a code B to belong to a HA, in terms of the properties of these critical subsequences, has to exist because the code B contains the complete information on the HA to which it belongs. The great amount of the possible "shapes" the HA can take, has prevented the author to find such a direct characterization.

Instead, the invariant characterization for the class of the HA has been found in an indirect form, that includes the third condition, imposed on another cyclical code, the code H, derivable from the code B.

The code H is defined for graphs which satisfy Condition 1 and Condition 2. Its form is

$$H = h_1, h_2, \ldots, h_{n_h}, h_i = (x_i, y_i),\ x_i, y_i \in Z.$$

The set of the necessary and sufficient conditions for a graph G to be a HA consists of three conditions: Condition 1, Condition 2 and

Condition 3. Every pair of the elements, (h_i, h_k) of the code H of a HA which do not belong to two adjacent meshes of the HA satisfies the relations:

$$|x_i - x_k| > 2 \vee |y_i - y_k| > 1.$$

A procedure for the calculation of elements of H is given in part 3.

3. A PROCEDURE FOR CALCULATING THE CODE H

Given a graph G satisfying the Condition 1, its code B is readily obtainable from the information on the perimeter. If the code B satisfies Condition 2, one can proceed to the calculation of the elements of the code H using the following method.

A preliminary step includes:

a) finding, in the code B, one of the longest subsequences, B_x, of the adjacent elements having the value 2;

b) cyclical permutation of the record of B which places B_x on the beginning of the record;

c) grouping the elements of B into the first four groups of Table 1, in the order of appearance of the groups in Table 1;

d) grouping the rest of elements of B into other six groups of Table 1, using the same system of priorities as in c).

The code B record of the HA from Fig.3, obtained after the application of the preliminary step, is

B = 22 2233 22 23 2233 33 22 23 22 23 23 23 222333.

Upon the completion of the preliminary step, the calculation of the components of the code H elements, h_i, for $i = 2,3,\ldots,n_h$ is further governed by the groups of elements formed during the preliminary step. To every group containing at least one element of the value 3 corresponds one element h_i. The last three groups from Table 1, containing no elements of value 3, do not generate an element in H. On the other hand, they contribute to the values of the components, x_j and y_j, of the code element introduced by the next group in the sequence.

The calculation of the components, x_i and y_i, of the i-th code H element, h_i, for $i = 2,3,\ldots,n_h$ proceeds in three steps:

1) calculation of the coefficient k_{bi} is done according to the formula $k_{bi} = k_{i-1} + \Delta k_{bn} + \Delta k_{bg}$, where Δk_{bn} and Δk_{bg} take their values from Table 1 as follows:

Δk_{bg} belongs to the group with $n_s \neq 0$ generating h_i,

Δk_{bn} belongs to the group with $n_s = 0$, which immediately preceeds in B the group generating h_i; if such a group does not exist, $\Delta k_{bn} = 0$.

2) calculations of x_i and y_i are done according to the formulas

$$x_i = x_{i-1} + \sum_{\ell=0}^{n_s-1} \Delta x_{i\ell}, \qquad \Delta x_{i\ell} = \left|\frac{5-|k_{bi}-\ell|}{2}\right| - |k_{bi}-\ell|$$

$$y_i = y_{i-1} + \sum_{\ell=0}^{n_s-1} \Delta y_{i\ell}, \qquad \Delta y_{i\ell} = \begin{cases} \dfrac{k_{bi}-\ell}{|k_{bi}-\ell|}\,(2-|\Delta x_{i\ell}|), & \ell \neq k_{bi} \\ 0, & \ell = k_{bi} \end{cases}$$

3) calculation of the parameter k_i is done using the formula

$$k_i = k_{bi} - n_s + 1 .$$

The initial values for this calculation are arbitrarily taken as

$$h_1 = (0,0), \quad k_1 = 0.$$

4. MEANING OF THE CODE H

For the HA shown in Fig.3, the following code H is obtained by the application of the procedure which is described in part 3:

H = (0,0),(-3,1),(-4,0),(-4,2),(-6,2),(-7,-1),(-8,2),
(-6,2),(-4,2),(-2,-2)

Referring to Figs.4 and 5 it may be verified that the elements of the code H represent the coordinates of the hexagons of the hexagonal grid, as coded in Fig.4, which are occupied by the perimeter meshes when the HA is imbedded in the way shown in Fig.5.

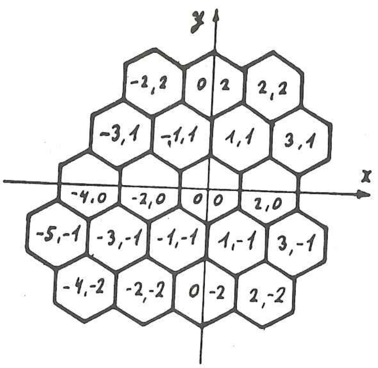

Fig.4.

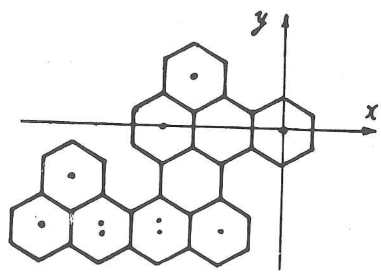

Fig.5.

5. CONCLUSION

The necessary and sufficient conditions for a graph to be a hexagonal animal are given in the terms of the graph invariants. The code B consisting of the cyclically ordered sequence of the perimeter-vertex degree-values, is shown to contain the necessary and sufficient information for the verification, whether the graph to which it belongs is, or is not, a hexagonal animal. The existence of a more direct characterization, having a single condition on the code B, instead of the Condition 2 and Condition 3, is conjectured.

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CONVEX HULLS OF HYPERGRAPH CLASSES

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ABSTRACT

In this paper a survey of results concerning convex hulls of hypergraph classes is given. Some applications are discussed and some open problems are pointed out.

1. INTRODUCTION, SPERNER-HYPERGRAPHS

Let X be a finite set of n elements and let 2^X be its power set. We call the pair (X,H) *hypergraph* where $H \subset 2^X$. The elements of X and H are called *vertices* and *edges*, resp. (X,H) is *k-uniform* (or briefly *uniform*) if all of its edges are of size k. If no edge of (X,H) contains another one we say that (X,H) is a *Sperner-hypergraph*. It is easy to see that any uniform hypergraph is a Sperner-hypergraph.

Sperner theorem [16]. *A Sperner-hypergraph on* n *vertices* has maximally

(1) $$\binom{n}{\lfloor \frac{n}{2} \rfloor}$$

edges.

The complete $\binom{n}{\lfloor\frac{n}{2}\rfloor}$-uniform hypergraph shows that there is a Sperner-hypergraph with this many edges. The non-trivial part of the above theorem is that the number of edges is at most (1). For a short proof see [13].

In order to study the possible sizes of the edges a Sperner-hypergraph let us introduce the concept of the *profile* of a hypergraph:

$$p(H) = (p_0, p_1, \dots, p_n)$$

where p_i is the number of edges of (X,H) of size i $(0 \le i \le n)$. Therefore the profile of a hypergraph is a point of the $(n+1)$-dimensional Euclidean space R^{n+1}. Let σ denote the set of profiles of all Sperner-hypergraphs. A good approximation of σ is its convex hull. The *convex hull* $\hat{\alpha}$ of a set $\alpha \subset R^{n+1}$ is

$$\hat{\alpha} = \left\{ \sum_{i=1}^{\ell} c_i A_i : A_i \in \alpha \, , \; c_i \ge 0 \quad (1 \le i \le \ell), \quad \sum_{i=1}^{\ell} c_i = 1 \right\} ,$$

that is, the set of all *convex linear combinations* of the elements of α. $A \in \alpha$ is an *extreme point of* α iff A is not a convex linear combination of elements of α different from A. It is easy to see that 1) if α has finitely many extreme points then any element A of α can be expressed as a convex linear combination of the extreme points of α; 2) α and $\hat{\alpha}$ have the same extreme points; 3) the extreme points of α uniquely determine $\hat{\alpha}$.

After these preliminaries we can formulate our first (rather trivial)

Theorem 1. [5]. *The extreme points of* σ *(=set of profiles of Sperner-hypergraphs) are*

$$Z = (0,\dots,0)$$

and

$$V_i = (\underset{o}{0},\dots,0,\underset{i}{\tbinom{n}{i}},0,\dots,\underset{n}{0}) \qquad (0 \le i \le n).$$

Proof. Z is the profile of the hypergraph without any edge while V_i is the profile of the complete i-uniform hypergraph. These are Sperner-hypergraphs, so $Z, V_i \in \sigma$ $(0 \le i \le n)$. Let us show that they are extreme points of σ. Suppose that

$$(2) \qquad V_i = \sum_{j=1}^{\ell} c_j A_j$$

where $c_j > 0$, $A_j \in \sigma$ $(1 \le j \le \ell)$, $\sum_{j=1}^{\ell} c_j = 1$. The components of A_j are non-negative, therefore their k-th$(k \ne i)$ component must be 0 by (2). Their i-th components are $\le \binom{n}{i}$, so their i-th components must be equal to $\binom{n}{i}$, again by (2). Hence $V_i = A_j$ $(1 \le i \le n)$, that is, V_i is an extreme point. It can be shown in the same way that Z is also extreme point.

We will see that any element $A \in \sigma$ is a convex linear combination of Z and V_i $(0 \le i \le n)$. This obviously implies that these are all the extreme points of σ. Indeed, let $A = (p_0,\dots,p_n) \in \sigma$. We have to find coefficients c, $c_0,\dots,c_n \ge 0$ satisfying

(3) $$c + \sum_{i=0}^{n} c_i = 1$$

and

$$A = cZ + \sum_{i=0}^{n} c_i V_i \ .$$

This latter equation is equivalent to

$$p_i = c_i \binom{n}{i} \qquad (0 \le i \le n) \ .$$

That is, c_i are unambiguously determined. We can find a $c \ge 0$ satisfying (3) iff

$$\sum_{i=0}^{n} c_i =$$

(4) $$\sum_{i=0}^{n} \frac{p_i}{\binom{n}{i}} \le 1 \ .$$

However, (4) is well known and called the LYM inequality after Lubell [13], Yamamoto [17] and Meshalkin [14]. The proof is complete.

The convex hull of σ is bordered by the trivial hyperplanes

$$p_i \ge 0 \qquad (0 \le i \le n)$$

and by the hyperplane determined by (4). Therefore Theorem (1) is only a reformulation of the LYM inequality. However this is not so for other classes of hypergraphs. In general, more non-trivial inequalities are needed. In the next section we will list classes of hypergraphs whose extreme points are determined.

2. CONVEX HULLS OF SOME CLASSES OF HYPERGRAPHS

(X,H) is a *k-Sperner-hypergraph* if it contains no $k+1$ different edges $H_1,\dots,H_{k+1} \in H$ satisfying

$$H_1 \subset \dots \subset H_{k+1} .$$

A hypergraph is 1-Sperner iff it is a Sperner-hypergraph. The set of profiles of the k-Sperner-hypergraphs will be denoted by σ_k.

Theorem 2 [6]. *The extreme points of* σ_k *are the vectors whose i-th component is either zero or* $\binom{n}{i}$ *but the number of their non-zero components is at most* k.

Using this theorem, it is easy to determine the maximum number of edges of a k-Sperner-hypergraph (X,H). The number of edges of (X,H) is nothing else but the sum $\sum_{i=0}^{n} p_i$ of the profile $p(H) = (p_0,\dots,p_n)$. It is easy to see that

$$\max_{(X,H) \in \sigma_k} \sum_{i=0}^{n} p_i$$

can be attained only for extreme points of σ_k. Hence *the maximum number of edges in a k-Sperner-hypergraph is the sum*

$$\sum_{i=\lfloor\frac{n-k+1}{2}\rfloor}^{\lfloor\frac{n+k-1}{2}\rfloor} \binom{n}{i}$$

of the k *largest binomial coefficients.* This is a well known theorem of Erdös [3].

We say that (X,H) is an *intersecting* hypergraph if any two edges have non-empty intersection. The set of profiles of intersecting hypergraphs is denoted by $\imath$.

Theorem 3 [6]. *The extreme points of* $\imath$ *are the following ones:*

$$(5)\qquad (\underset{0}{0},\dots,\underset{k}{\tbinom{n-1}{k-1}},\underset{k+1}{\tbinom{n-1}{k}},\dots,\underset{n-k}{\tbinom{n-1}{n-k-1}},\underset{n-k+1}{\tbinom{n}{n-k+1}},\dots,\underset{n}{\tbinom{n}{n}})\qquad (1\le k<n/2),$$

$$(6)\qquad (\underset{0}{0},\dots,\underset{n/2}{\tbinom{n-1}{n/2-1}},\underset{n/2+1}{\tbinom{n}{n/2+1}},\dots,\underset{n}{\tbinom{n}{n}})$$

$$(7)\qquad \left(0,\dots,0,\binom{n}{\left\lceil\frac{n+1}{2}\right\rceil},\dots,\binom{n}{n}\right)$$

and the vector obtained by replacing 1) any but the k*-th components of (5) by zero and 2) any components of (6) or (7) by zero.*

It is easy to construct hypergraphs with these profiles: take all the edges of size i with $k \le i \le n-k$ containing a fixed vertex x and all the possible edges of size $> n-k$. The rest of the proof is more complicated.

One can deduce from this theorem the maximum number of edges of an intersecting hypergraph: 2^{n-1}. However, even this deduction is longer than the original proof in [4]. A more interesting consequence of Theorem 3 is the Erdös-Ko-Rado theorem [4]: *The maximum number of edges of an intersecting k-uniform hypergraph is* $\binom{n-1}{k-1}$ *if* $k \le n/2$. Indeed, the i-th component $(k \le n/2)$ of any extreme point of ι is $\le \binom{n-1}{k-1}$. For a short direct proof of this theorem see [10].

The (up to now) deepest result of this theory is

Theorem 4 [5]. *The extreme points of* $\sigma \cap \iota$ *are*

$$Z\ , \qquad V_j \qquad (n/2 < j \le n)$$

$$W_i = (\underset{0}{0},\dots,\underset{i}{\binom{n-1}{i-1}},\dots,\underset{n}{0}) \qquad (1 \le i \le n/2)$$

$$W_{ij} = (\underset{0}{0},\dots,\underset{i}{\binom{n-1}{i-1}},\dots,\underset{j}{\binom{n-1}{j}},\dots,\underset{n}{0}) \qquad (1 \le i \le n/2,\ i+l>n)$$

The main part of the statement of this theorem is that the extreme points can have at most two non-zero components.

The Erdös-Ko-Rado theorem can be easily deduced, again. Indeed, the k-th component of any extreme point is $\le \binom{n-1}{k-1}$ if $k \le n/2$.

Let us consider the problem, what is the maximum number of edges of an intersecting Sperner hypergraph. It is sufficient to maximize the sum of the components of the extreme points in Theorem 4. Examine first the extreme points W_{ij}. $j \ge n-i+1 > n/2$ follows from the conditions $1 \le i \le n/2$, $i+j > n$. Therefore

$$\binom{n-1}{i-1}+\binom{n-1}{j} \le \binom{n-1}{j-1}+\binom{n-1}{n-i+1} = \binom{n-1}{i-1}+\binom{n-1}{i-2} = \binom{n}{i-1} \le \binom{n}{\lfloor \frac{n-2}{2} \rfloor}$$

gives an upper estimate for the sum of the components in W_{ij}. The only non-zero components of W_i and V_j are $\binom{n-1}{i-1} \leq \binom{n-1}{\lfloor\frac{n-2}{2}\rfloor}$ and $\binom{n}{j} \leq \binom{n}{\lceil\frac{n+1}{2}\rceil}$, resp. Consequently, *the complete* $\lfloor\frac{n+1}{2}\rfloor$*-uniform hypergraph has the maximum number of edges among all intersecting Sperner-hypergraphs.* This is a special case of a theorem of Milner [15].

It is worth-while to determine the nontrivial hyperplanes bordering the convex hull of $\sigma \cap \iota$. The following class of inequalities contains the inequalities corresponding to these hyperplanes:

(8) $$\sum_{1\leq i\leq n/2} (1-y_{n-i+1}) \frac{p_i}{\binom{n-1}{i-1}} + \sum_{n/2\leq j\leq n-1} y_j \frac{p_j}{\binom{n-1}{j}} \leq 1$$

for any $(p_0,\dots,p_n) \in \sigma\cap\iota$ and for any sequence $y_{\lfloor n/2\rfloor+1} \geq \dots \geq y_n \geq 0$ satisfying

(9) $$y_j \leq 1 - \frac{j}{n} \qquad (n/2 < j \leq n).$$

It is interesting to mention that some authors tried to find inequalities well characterizing the elements of $\sigma \cap \iota$. Bollobás [1] proved

$$\sum_{1\leq i\leq n/2} \frac{p_i}{\binom{n-1}{i-1}} \leq 1$$

which can be obtained from (8) by substituting $y_{\lfloor n/2\rfloor+1} = \dots = y_n = 0$, while the inequality

$$\sum_{1\le i\le n/2} \frac{p_i}{\binom{n}{i-1}} + \sum_{n/2<j\le n} \frac{p_j}{\binom{n}{j}} \le 1$$

of Greene, Katona and Kleitman [8] follows from (8) by choosing $y_j = 1-\frac{j}{n}$ $(n/2<j\le n)$. Now it is clear that these inequalities were too weak to characterize the elements of $\sigma \cap \iota$, alone. Many of them are needed.

Let us investigate a problem of somewhat different character. Suppose that the hypergraphs $(X, H_1), \ldots, (X, H_t)$ satisfy the following condition:

$$\text{(10)} \qquad \begin{gathered} G, H \subset H, \quad G \ne H, \quad G \in H_i, \quad H \in H_j, \quad i \ne j \\ \text{implies} \\ G \not\subset H \end{gathered}$$

That is, two different hypergraphs cannot contain different edges, one containing the other one. (But $H_i \cap H_j$ is not necessarily empty.) The *profile* of the sequence of the hypergraphs $(X, H_1), \ldots, (X, H_t)$ is

$$p(H_1, \ldots, H_t) = \sum_{i=1}^{t} p(H_i) .$$

Let $\sigma(t)$ denote the set of the profiles of hypergraphs satisfying (10).

Theorem 5 [6]. *The extreme points of* $\sigma(t)$ *are*

$$Z, \quad tV_i$$

if $t \ge n+1$. *Otherwise there are some additional extreme points* *with at least* t+1 *non-zero components* $p_i = \binom{n}{i}$.

A theorem of Daykin, Frankl, Greene and Hilton [2] easily follows:

$$\max_{\substack{(X,H_1),\dots,(X,H_t)\\ \text{satisfy (10)}}} \sum_{i=1}^{t} |H_i| = \max\left(t\binom{n}{\lfloor \frac{n}{2}\rfloor},\ 2^n\right).$$

3. APPLICATIONS

Let c_i $(0\le i\le n)$ be reals and suppose that $\sum_{i=0}^{n} c_i p_i$ has to be maximized for a certain class of hypergraphs. Let α be the set of profiles of these hypergraphs. If the extreme points of α are determined, our situation is very easy. We have to maximize $\sum_{i=0}^{n} c_i p_i$ only for these extreme points.

In the previous section we applied this idea only for the cases when (i) $c_0 = c_1 = \dots = c_n = 1$ and (ii) $c_i = 1$, $c_j = 0$ $(j \ne i)$. However, more complicated functions can be arised. For instance, one could ask for the maximum of the sums of the sizes of the edges in a hypergraph. That is, $c_i = i$ $(0\le i\le n)$. [11] solves this problem for Sperner-hypergraphs:

$$\text{(11)}\qquad \max_{(p_0,\dots,p_n)\in\sigma} \sum_{i=0}^{n} i p_i = \binom{n}{\lceil \frac{n}{2}\rceil}\left\lceil \frac{n}{2}\right\rceil .$$

This is an easy consequence of Theorem 1 (that is, of the LYM inequality). Indeed, the extreme point V_i gives $\binom{n}{i}i$. It is easy to verify that $\max_i \binom{n}{i} i$ is equal to the right hand side of (11).

Another application can be found in [5] where

$$\sum_{i=0}^{n} (n-i)!\, p_i \tag{12}$$

is to be maximized for intersecting Sperner-hypergraphs. We have to use Theorem 4. (12) gives more for W_{ij} than for W_i, therefore we have to check only V_j $(n/2<j\le n)$ and W_{ij} $(1\le i\le n/2, i+j>n)$ from the extreme point. If $V_j = (p_0,\dots,p_n)$ then we have trivial inequality for (12)

$$(n-j)!\binom{n}{j} \le (n-1)!+1 \quad .$$

On the other hand, if $W_{ij} = (p_0,\dots,p_n)$ then the following sequence of inequalities gives the same estimate:

$$(n-i)!\binom{n-1}{i-1}+(n-j)!\binom{n}{j} = \frac{(n-j)!}{(i-1)!} + \frac{(n-1)!(n-j)}{j!} \le$$

$$\le \frac{(n-1)!}{(i-1)!} + \frac{(n-1)!(i-1)}{(n-i+1)!} \begin{cases} = (n-1)! & \text{if } i = 1 \\ = (n-1)!+1 & \text{if } i = 2 \\ \le \dfrac{(n-1)!}{2} + \dfrac{(n-1)!}{2} & \text{if } 3\le i\le n/2 \\ (n \le 4 \text{ should be checked separately}). & \end{cases}$$

Summarizing, $(n-1)!+1$ is the maximum of (12). The hypergraph (X,H), where

$$H = \{\{x,y\}: y \in X-x\} \cup \{X-x\} \quad (x \in X \quad \text{fixed}) ,$$

gives the equality.

4. OPEN PROBLEMS

1. Any problem of the extremal hypergraphs (see e.g. [12]) can be extended in the present way. However, some of these extended questions are blocked by longstanding open problems. See e.g. the following condition for (X,H)

(13) $\qquad H_1, H_2 \in H \qquad \text{implies} \qquad |H_1 \cap H_2| \geq k .$

Let us denote by $\iota(k)$ the set of profiles of the hypergraphs satisfying (13). The extreme points of $\iota(1) = \iota$ are determined in Theorem 3. However knowing the extreme points of $\iota(k)$ would imply the determination of $\max p$ for $(p_0,\dots,p_n) \in \iota(k)$, too. This is known to be $\binom{n-k}{\ell-k}$ only for $n > (\ell+1)(k+1)$ $(k \geq 15)$ (see [4], [7]). A nice open problem of this kind from [4]: Is it true that the optimal construction of the above problem for $k = 2$, $n = 4m$ $\ell = 2m$ is the hypergraph (X,H) with

$$H = \{A: |A \cap X_1| > m\}$$

where

$$X_1 \subset X , \qquad |X_1| = 2m \ ?$$

On the other hand, one extreme point of $\iota(k)$ maximizing $\sum_{i=0}^{n} p_i$ is known [17].

2. We determined the extreme points for several cases. However it is non-trivial to make a detailed description of these convex hulls. Determine e.g. the graph of the edges (1-dimensional faces) of the convex hulls.

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SEARCH FOR MINIMAL NON-HAMILTONIAN SIMPLE 3-POLYTOPES

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ABSTRACT

All simple 3-polytopes with up to 30 vertices which have no triangular faces and no two adjacent quadrangular faces are constructed. The obtained polytopes are used to show that every simple 3-polytope having 30 or less vertices admits a Hamiltonian circuit, which was independently shown by Okamura.

1. INTRODUCTION

3-connected planar cubic graphs were extensively studied because of their close relationship with the Four Color Problem. Tait once conjectured [7] that every 3-connected planar graph possesses a Hamiltonian circuit. The verification of this conjecture (at least for cubic graphs) would have settled the Four Color Conjecture. But in 1946, Tutte disproved this

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by constructing his now famous 3-connected cubic planar non--Hamiltonian graph on 46 vertices [8]. The smallest such graph found to date is due to Bosák [1] and Barnette, as reported by Lederberg [6]. This graph has 38 vertices and is shown in Fig.1. It is also known that there is no such graph on 22 or less vertices [2,6]. The proof of this fact was obtained by computer. Lederberg examined all the 44 cubic graphs on 18 or less vertices which were possible counterexamples, while Butler had to examine 400 cyclically 4-connected graphs on 20 and 22 vertices. Note that our approach requires to consider only 50 graphs having 22 or less vertices.

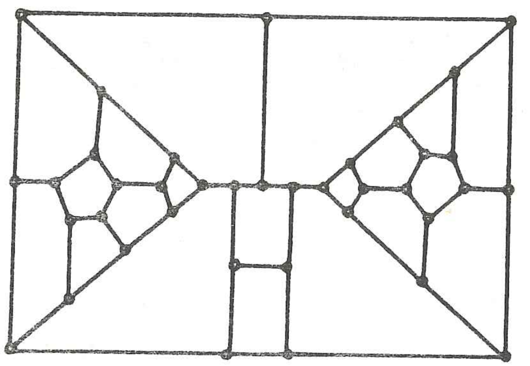

Fig.1.

In the present paper we confirm and extend the results of Lederberg and Butler and show that every simple 3-polytope with 30 or less vertices admits a Hamiltonian circuit. This result was obtained by an exhaustive computer search. We greatly reduce the number of graphs to be examined using results which are presented in Section 3.

After finishing the first version of this paper it came to the authors attention that N. Okamura [9,10] recently obtained the same result without the use of a computer.

The numbers of obtained nonisomorphic simple polytopes are also interesting. They increase very fast and it is seen that the complete list of graphs on 32 vertices is almost impossible to obtain.

2. BASIC DEFINITIONS

Wi will limit ourselves to defining only lesser known terms and those which may cause confusion. Other definitions may be found in standard textbooks, e.g. [4,5].

We note that the notion of a simple 3-polytopal graph will be used to describe a cubic planar 3-connected graph. The graphs are considered to be embedded on the 2-sphere (as a 1-skeleton of the corresponding polytope). The data about the embedding are "superfluous" since 3-connected planar graphs are (combinatorially) uniquely embeddable on the 2-sphere. The graph itself therefore uniquely determines the corresponding 3-polytope.

We recall two well-known theorems which will be used in the sequel:

(a) (Theorem of Steinitz) The graph G is a graph of a 3-polytope iff it is planar and 3-connected.

(b) The graph is 3-connected iff the removal of any two or less vertices does not produce a disconnected or trivial graph.

In the spirit of the above theorems and the unique embeddability of 3-connected graphs we will not distinguish between 3-polytopes and the corresponding polytopal graphs.

To refer to the number of edges of a face of a 3-polytope we say that a (2-dimensional) face is a *k-face* if it is a k-gonal face. Note that this is not a very standard notion and that some authors use k-face to describe a k-dimensional face. We adopt this notation because of our frequent use of it and since we do not consider other than 2-dimensional faces.

3. BASIC THEOREMS

The search for a smallest non-Hamiltonian simple 3-polytope requires some preliminary theoretical work. We need an algorithm to produce all graphs which are "of interest". It is based on Theorem 3. Next we want to reduce the class of "interesting graphs" to be as small as possible. Theorem 1 and Theorem 2 greatly reduce the number of graphs which have to be examined. Two lemmas precede the theorems.

Lemma 1. Let G *be a simple 3-polytope which has two adjacent triangular faces. Then* G *is the tetrahedron.*

Proof. Denote by e the common edge of the triangular faces. We have the situation shown in Fig.2. The vertices a and b are not adjacent provided G is not a tetrahedron. Then we obtain a disconnected graph if we remove the vertices a and b. This contradicts the 3-connectedness of G./////

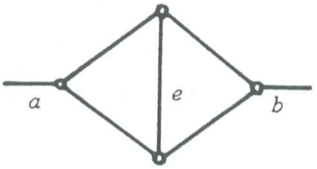

Fig.2.

Lemma 2. Let G *be a simple 3-polytope which has three mutually adjacent 4-faces,* F_1, F_2 *and* F_3. *Suppose that there is another 4-face* $F \neq F_3$ *which is adjacent to* F_1 *and* F_2. *Then* G *is the cube 3-polytope.*

Proof. Suppose that G is not equal to the 3-cube and that the assumptions of the lemma are satisfied. Then we have the situation of Fig.3 with $a \neq b$ and not adjacent. Removing the vertices a and b we obtain a disconnected graph. But this is not possible since G is 3-connected./////

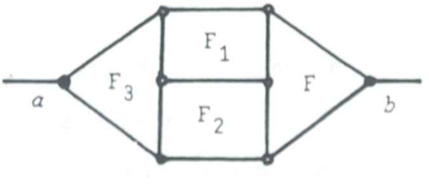

Fig.3.

Theorem 1. The smallest non-Hamiltonian simple 3-polytope does not have any triangular faces.

Proof. This is obvious, since a Hamiltonian circuit through a triangle is equivalent to that through a vertex in the graph obtained by shrunking the triangle to a single vertex. The possibility of the reduction is guaranteed by Lemma 1./////

Theorem 2 will reduce our observations to polytopes without adjacent quadrangular faces.

Theorem 2. A smallest non-Hamiltonian simple 3-polytope has no two adjacent quadrangular faces.

Proof. Let G be one of the smallest non-Hamiltonian simple 3-polytopes, i.e. every simple 3-polytope which has less vertices than G is Hamiltonian. By Theorem 1, G has no triangular faces. Suppose that G has two adjacent 4-faces, F_1 and F_2. We distinguish three cases.

Case 1. There is another 4-face which is adjacent to both, F_1 and F_2. Since G is not the 3-cube, we may because of Lemma 2 apply the reduction $P2^-$ of Theorem 3. $P2^-(G)$ has a Hamiltonian circuit which is easily seen to be extendable to a Hamiltonian circuit in G. This would contradict the non--Hamiltonicity of G.

Case 2. The faces adjacent to F_1 and F_2 are not 4-gons and the reduction of Fig.4 produces a 3-connected graph G'. This is also impossible since every Hamiltonian circuit from G' can be extended to a Hamiltonian circuit in G.

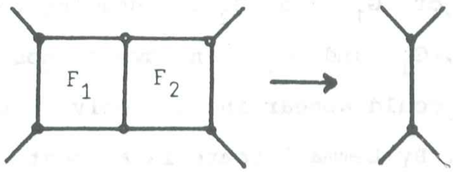

Fig.4.

Case 3. The reduction of Fig.4 produces a graph which is not 3-connected. Then we have the situation as shown in Fig.5. Take new vertex v and join the vertices v_1, v_2 and v_3 of G_1' to this vertex. Denote the obtained graph by G_1. Similarly we obtain a graph G_2 by joining vertices u_1, u_2 and u_3 of G_2' to a newly introduced vertex u.

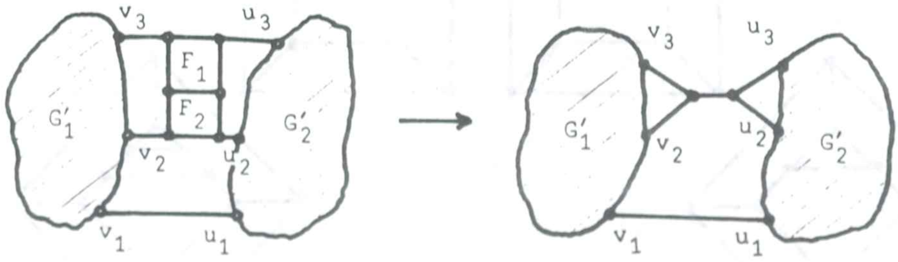

Fig.5.

Obviously, G_1 and G_2 are 3-connected planar cubic graphs. One easily verifies that any Hamiltonian circuit in G induces Hamiltonian circuits in G_1 and G_2. A short examination shows that G is non-Hamiltonian iff every Hamiltonian circuit in G_1 contains the edge vv_1 and every Hamiltonian circuit in G_2 by-passes the edge uu_1 (or symmetrically if we change the rules of G_1 and G_2). Assuming that G has no triangular faces, G_1 and G_2 can have at most one triangular face. 3-faces could appear in G_1 only if they contain the added vertex v. By Lemma 1 there is at most one such face in G_1. The same arguments apply to G_2.

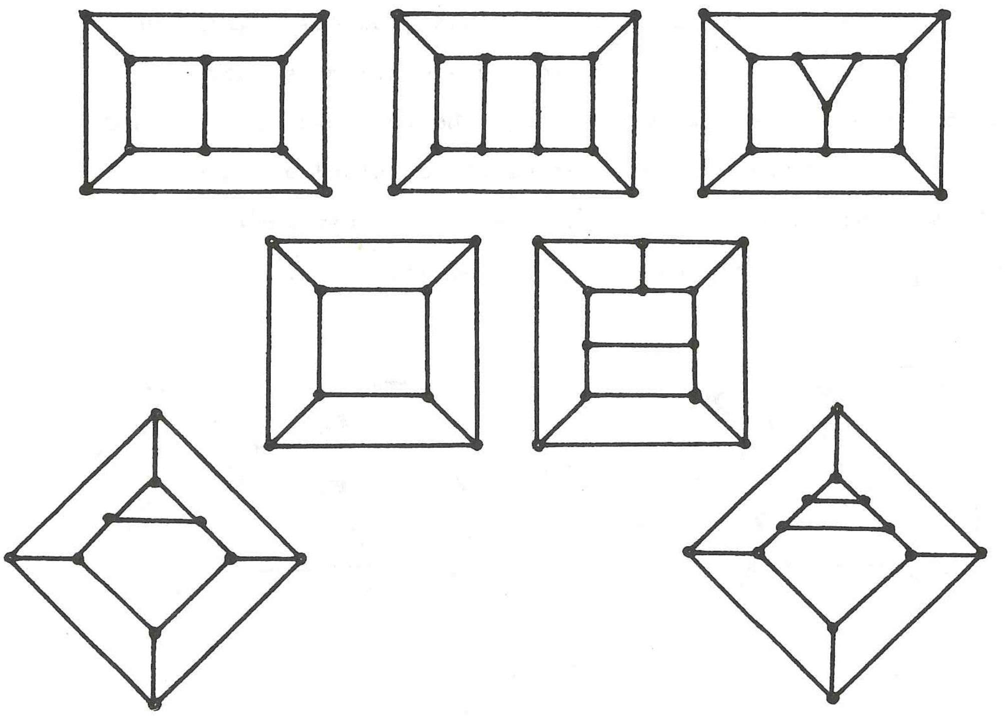

Fig.6.

Fig.6 contains all 3-connected cubic planar graphs up to 12 vertices which have at most one triangular face. Using the symmetries of these graphs one immidiately verifies that all of them have the following property. For every edge there exists a Hamiltonian circuit containing it and for every edge there exists a Hamiltonian circuit avoiding it. It was established by computer that the same is true for all such graphs having 14 vertices. There are examples on 16 vertices having an edge which lies on every Hamiltonian circuit. But the graphs up to 18 vertices admit for every edge a Hamiltonian circuit containing this edge. This was also established by computer. See also the results in [1,2].

Suppose that G_i has k_i vertices $(i = 1,2)$. Then G has $k_1 + k_2 + 4$ vertices. By the foregoing, if G does not admit a Hamiltonian circuit, G_1 has at least 16 and G_2 has at least 20 vertices. Therefore G should have at least $16 + 20 + 4 = 40$ vertices. This would not be the smallest non-Hamiltonian example since there are known examples having 38 vertices [1,2]./////

Theorem 3. Every simple 3-polytope without triangular faces is obtained from Q_3, *the cube polyhedron, by a finite number of the following generating rules:*

P1 :

P2 :

P3 :

Proof. It sufficies to show the following. Let G be a simple 3-polytope without triangular faces, which is not equal to Q_3. Then it is possible to apply on G one of the inverse rules $P1^-$, $P2^-$ or $P3^-$ to obtain a simple 3-polytope without 3-faces having less vertices than G.

Before we continue we introduce some useful concepts. We say that a 4-face is *P1-reducible* if it admits the reduction $P1^-$, i.e. $P1^-(G)$ is 3-connected and without 3-faces. Similarly, a 5-face is *P3-reducible* if it admits the reduction $P3^-$. *P2-reducibility* is defined on three mutually adjacent 4-faces. Further we say for a 4-face F that it is *separated* if there are no two 4-faces F_1 and F_2 such that F, F_1 and F_2 are mutually adjacent.

We claim the following: let F be a separated quadrangular face in G. Then F is P1-reducible.

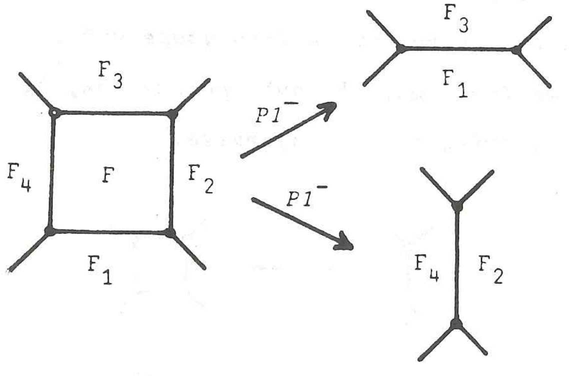

Fig.7.

That the reduction $P1^-$ is possible, i.e. $P1^-(G)$ is 3-connected, was shown by Bowen and Fisk [3]. If none of the faces adjacent to F is a 4-gon then the reduction produces a polytope without triangular faces. Otherwise suppose that F_1 is a 4-face (see Fig.7). The faces F_2 and F_3 are not 4-faces since F is separated. After the reduction $P1^-$ which preserves the faces F_1 and F_3, the obtained cubic plane graph G' has no triangular faces. Suppose that G' is not 3-connected. This is possible iff the faces F_1 and F_3 are adjacent in G. Then we have the situation shown in Fig.8. Removing the vertices a and b we disconnect G which contradicts its 3-connectedness.

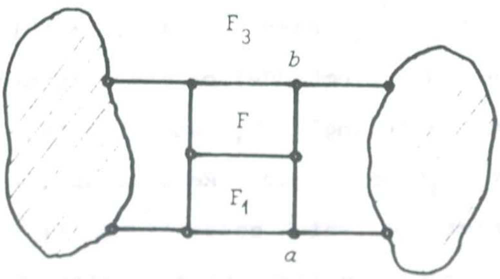

Fig.8.

Suppose that G has no separated 4-faces and that F is a 5-face of G. Then F is P3-reducible. To show this we distinguish three cases. For the notations see Fig.9.

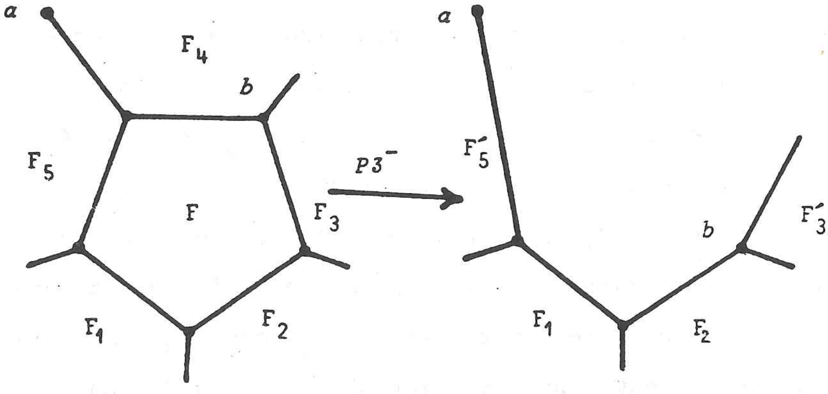

Fig.9.

Case 1. Suppose that F_3 and F_5 are adjacent faces of G. Then the face F_4 cannot be adjacent either to F_1 or to F_2 (consider G as embedded on the 2-sphere). Moreover, since F_4 is not a triangle, F_3 and F_5 cannot be 4-gons. To see this let F_4 be a 4-gon. Removal of the vertices a and b disconnects G which contradicts its 3-connectedness. From all this we conclude that the reduction of Fig.9 is possible.

Because of the symmetry of the reduction $P3^-$, the P3-reducibility is also proved if at least one of the following pairs of faces of G are adjacent: F_1 and F_3, F_1 and F_4, F_2 and F_4, F_2 and F_5, or F_3 and F_5.

Case 2. None of the above mentioned pairs of faces are adjacent and for at least one pair, both faces are larger than four. Suppose that F_3 and F_5 are 5- or more-faces. Then obviously the reduction $P3^-$ of Fig.9 works.

Case 3. All pairs (F_i,F_j), $i - j = 2$, contain a 4-gon. Then for an index k $(1 \le k \le 5)$, F_k, F_{k+1} and F_{k+2} are 4-gons ($k+1$ and $k+2$ are "modulo 5"). By the assumption, there is no separated 4-face in G. Since F_{k+1} is not separated, there is another 4-gonal face which is adjacent to F_k, F_{k+1} and F_{k+2}. By Lemma 2 G is equal to Q_3.

Finally, suppose that G has no 5-faces and that there are three mutually adjacent 4-gons. By Lemma 2 the faces surrounding these 4-gons are at least 6-gons. It is easy to see that the reduction $P2^-$ produces a 3-polytope which obviously does not contain any triangular faces.

It is known (e.g. [4]) that every 3-polytope contains a k-face where $k < 6$. Since G has no triangular faces it contains a 4-face or a 5-face. Every separated 4-face is P1-reducible; if there are no separated 4-faces then every 5-face is P3-reducible and every triple of nonseparated mutually adjacent 4-faces is P2-reducible if there are no 5-gons. Therefore G admits at least one of the reductions $P1^-$, $P2^-$ or $P3^-$ producing a simple 3-polytope without triangular faces. /////

4. COMPUTATIONAL RESULTS

Using the deduction rules of Theorem 3 it was not very difficult to write a computer program to generate all simple 3-polytopes without triangular faces having up to a certain number (i.e. 30) of vertices.

Because of simplicity we have dealt with the simplicial 3-polytopes, i.e. dual polytopes of simple 3-polytopes. The simplicial polytopes are also called the triangulations (of the 2-sphere). We adopt this notation. Note that there is a one-to-one correspondence between the simplicial and simple 3-polytopes. The advantage of considering triangulations as opposed to cubic graphs is in easier isomorphism checking, since we may conclude that two triangulations are non-isomorphic if they have different vertex-degree sequences. Moreover, when constructing an isomorphism between two triangulations we can make good use of the vertex-degrees information.

The most important thing we use in the generating routine is a kind of maximality principle which can be described as follows. Given a triangulation G' and using the (dual) reduction $P1^-$ one may decide that the reduction is to be made at a vertex v of degree four which has some maximal property. To distinguish between the two possible reductions at vertex v we use the maximizing function

$$f(v) = 100 \cdot (\deg(a) + \deg(c)) + \deg(b) + \deg(d)$$

(see Fig.10).

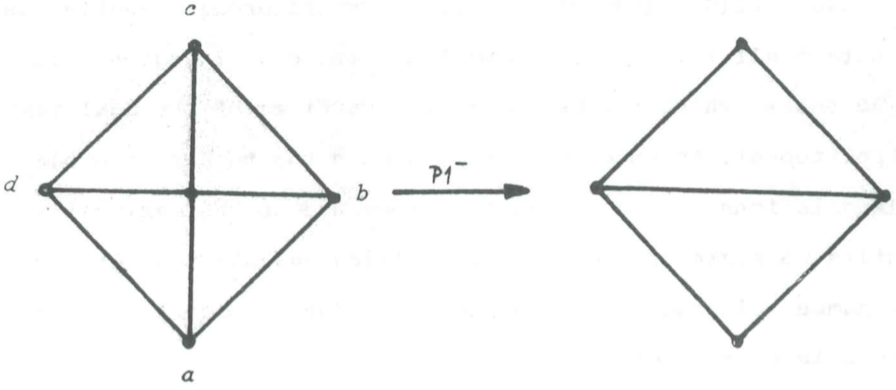

Fig.10.

The generating procedure is as follows. Take a graph G. Generate the graph G′ = P1(G). If the new vertex does not have maximal value under f(v) then we disregard the graph G′. Note that f(v) must be computed at all separated vertices of degree 4, and at every vertex possibly twice. Naturally, this is to be done iff the reduction $P1^-$ is possible.

Consider P2. It must be applied iff P2(G) does not have any separated vertex of degree 4 (at which $P1^-$ could be applied), and does not have any vertex of degree 5, since then $P3^-$ can be used. P3 must be applied iff P3(G) does not have any separated vertex of degree 4. In both cases maximality is tested. The maximality function is chosen in a similar way to the function for P1, so that it distinguishes between the five possible reductions at a vertex. We note that many tests concerning maximality can be made in a graph G without our being always obliged to do the explicit generation of P1(G), P2(G) and P3(G). The described process enables us to obtain all triangulations without vertices of degree 3 up to 30 faces (which correspond to the vertices of the dual simple polytopes). As an example of how much the number of produced triangulations is reduced as a consequence of the maximality testing we state the somewhat surprising result: the ratio of the number of produced graphs to the number of non-isomorphic graphs is on average 2:1.

The numbers of nonisomorphic graphs obtained are presented in Table 1. T(n) is the number of simple 3-polytopes

Table 1

n	T(n)	Q(n)
8	1	-
10	1	-
12	2	-
14	5	1
16	12	2
18	34	3
20	130	12
22	525	32
24	2472	123
26	-	506
28	-	2313
30	-	26933*

without triangular faces. Bowen and Fisk [3] determined $T(n)$ up to $n = 20$ and our numbers agree with their results up to this point. $Q(n)$ is the number of such polytopes having no two adjacent quadrangular faces. We also generated simple polytopes having no triangular faces on 26, 28 and 30 vertices but we did not do any isomorphism checking. We carried out isomorphism testing only on the graphs without adjacent 4-faces. The obtained numbers for $n = 26$ and $n = 28$ are given in Table 1. The graphs on 30 vertices were not checked under isomorphism. The number of obtained graphs is also given in Table 1 and is marked with an asterisk. We estimate that the number of non--isomorphic graphs is about one third of this number.

The following result is an important outcome of our computational research. We have checked the Hamiltonicity of the obtained polytopes without adjacent 4-faces. All of them turned out to have a Hamiltonian circuit. Since there is always some doubt in results obtained by computer, we formulate it in the form of:

Computational result. Every simple 3-polytopal graph having 30 or less vertices admits a Hamiltonian circuit.

In conclusion we note that all the computations were run on the DEC - 10 computer at the Computer Center of the E. Kardelj University, Ljubljana.

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SIMPLICIAL SCHEMES AND SOME COMBINATORIAL APPLICATIONS**

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ABSTRACT

A simplicial scheme is a graph with a certain additional structure. The purpose of this concept is a graph-theoretical description of pseudocomplexes. Theory can be applied only in the case when the underlying polyhedron of the pseudocomplex is a pseudomanifold. We discuss some combinatorial properties of pseudocomplexes which are obtained from simplicial schemes using combinatorial properties of the graphs on which the simplicial schemes are defined. As the first application we show that the dual graph of a simply connected combinatorial manifold is bipartite iff every simplex of codimension two is contained in an even number of top dimensional simplexes. This result is extended to a larger family of simplicial complexes. The second application concerns the colorability of vertices of triangulated n-dimensional pseudomanifolds with n+1 colors.

1. INTRODUCTION

Suppose that we are given an n-dimensional simplicial complex K. Usually we describe its structure by the incidence

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relations between the simplexes of K. Such a description is tedious and space consuming and is not very useful in many instances. We shall introduce a new way of describing a large class of simplicial complexes and pseudocomplexes which are pseudomanifolds. The description is purely graph-theoretical and enables us to use its properties to obtain some nontrivial combinatorial results. For example, the covering projections between graphs correspond to nonsingular nondegenerate simplicial maps between complexes described by the graphs. Hence we may use any graph covering tool, e.g. voltage graphs, to describe nonsingular maps between complexes.

An approach similar to ours was given before by Pezzana, Ferri, Gagliardi, et al. [Pe74, Fe79, Ga79a, Ga79b, FGG, FG81]. However, their main goal is topological - to represent PL-manifolds as simplicial pseudocomplexes, in particular as so called crystallizations. These are n-dimensional pseudocomplexes such that their set of 0-simplexes has cardinality $n+1$. Their description - a special case of ours - applies to every PL-manifold, and so it is useful for some topological purposes, e.g. computing homology groups, computing the fundamental groups, classifying PL-manifolds, etc. Our way is more combinatorial in nature. We give up the special properties of crystallizations in order to be able to describe a much larger class of pseudocomplexes. Our approach will be useful if we consider combinatorial properties of simplicial complexes and pseudocomplexes.

Throughout this paper we shall use the term *graph* for what is usually called a "multigraph". Hence we allow multiple edges but loops are forbidden. A graph is *simple* if it has no multiple edges. Given a graph G, let $V(G)$ and $E(G)$ denote its vertex-set and edge-set, respectively. Let $S(G)$ denote the set of directed edges of G. Every edge $e \in E(G)$ gives rise to two directed edges (or arcs) in oposite direction having same boundary vertices as e. We denote the elements of $S(G)$ by letters with arrows, e.g. $\vec{e}$. If the initial vertex of $\vec{e}$, $\partial_-\vec{e}$, is v and the terminal vertex, $\partial_+\vec{e}$, is u, we also write $\vec{e} = v\vec{u}$ provided there can be no confusion. With $\vec{e}^{-1}$ we denote the inverse arc of $\vec{e}$, e.g. $v\vec{u}^{-1} = u\vec{v}$.

A *walk* W in a graph G is a sequence of directed edges, $W = (\vec{e}_1\vec{e}_2\ldots\vec{e}_d)$, such that the initial vertex of $\vec{e}_{i+1}$ is the same as the terminal vertex of $\vec{e}_i$, $i = 1,2,\ldots,d-1$. Hence a walk is supposed to have an orientation. A walk W is *closed* if $\partial_-\vec{e}_1 = \partial_+\vec{e}_d$. For other terms of graph theory we refer to [Ha69].

For the terms of topology we refer to any standard text-book on topology or algebraic topology, e.g. [HW60, Ma67, RS72]. We only mention that a topological space is *simply connected* if it has trivial fundamental group.

2. BASIC DEFINITIONS OF COMBINATORIAL TOPOLOGY

It is supposed that the reader is familiar with basic notions of combinatorial topology, such as *simplex* of dimension n (also n-simplex), *i-face* or i-dimensional face of a simplex, *simplicial complex, dimension* of a simplicial complex, *codimension* of a simplex in a complex, etc.

A *pseudocomplex* K is a finite collection of topological simplexes such that

(i) if $A \in K$ and $B \subseteq A$ is a face of A, then $B \in K$,

(ii) if $A,B \in K$, then $A \cap B$ is a union of simplexes of K, and

(iii) if $|K| = \cup\{B;\ B \in K\}$, then $|K| = \coprod\{B^o;\ B \in K\}$ where $\coprod$ denotes disjoint union and B^o the interior of B.

The notion of a pseudocomplex was introduced in [HW60] in order to simplify the calculations of homology groups. A 1-dimensional pseudocomplex is a graph. A graph is simple if it is a simplicial complex

A pseudocomplex K is *homogeneous (of dimension n)* if every simplex in K is contained in an n-simplex of K. K is *strongly connected* if for any two top-dimensional simplexes A and B of K there exists a sequence of top simplexes $A = P_1, P_2, \ldots, P_{k-1}, P_k = B$ such that P_i and P_{i+1}, $i = 1,2,\ldots k-1$, have a common codimension-one face.

Suppose that

(1) K is homogenous of dimension n,

(2) K is strongly connected, and

(3) every (n-1)-simplex of K is contained in exactly two n-simplexes of K.

Then K is said to be a (pseudotriangulated) *pseudomanifold.*

In a pseudocomplex K we define also the star and the link of a simplex. If $A \in K$ then

$$\mathrm{star}(A,K) = \{B \in K;\ \exists C \in K \text{ such that } B \subseteq C \text{ and } A \subseteq C\}$$

and

$$\mathrm{link}(A,K) = \{B \in \mathrm{star}(A,K)\ ;\ B \cap A = \emptyset\}\ .$$

If K is an n-dimensional pseudocomplex we define its *dual graph* G as the 1-skeleton of the dual of K. G is connected iff K is strongly connected. If K is a pseudomanifold then the vertex-set of G is the set of n-simplexes of K, and two arbitrary vertices are joined by one edge for each common (n-1)-face of the corresponding two n-simplexes. In this case G is (n+1)-regular and connected.

3. SIMPLICIAL SCHEMES

Let G be a connected regular graph. A *presimplicial scheme* g on G is a function which assigns to every directed edge $\vec{e} \in S(G)$ a bijective map

$$g(\vec{e}) : \mathrm{star}(\partial_-\vec{e},G) \longrightarrow \mathrm{star}(\partial_+\vec{e},G)$$

such that the following conditions are fulfilled

(SS1) for every $\vec{e} \in S(G)$, $g(\vec{e})\vec{e} = \vec{e}^{-1}$, and

(SS2) for every $\vec{e} \in S(G)$, $g(\vec{e}^{-1}) = (g(\vec{e}))^{-1}$.

Let $W = (\vec{f}_1\vec{f}_2 \ldots \vec{f}_d)$ be a walk in G. An arc

$\vec{e} \in \mathrm{star}(\partial_{-}\vec{f}_1, G)$ is said to *avoid the walk* W (w.r.t. a presimplicial scheme g) if for $i = 0,1,\ldots,d-1$

$$g(\vec{f}_i) \circ g(\vec{f}_{i-1}) \circ \ldots \circ g(\vec{f}_2) \circ g(\vec{f}_1)e \neq \vec{f}_{i+1} \quad .$$

A presimplicial scheme g is called *simplicial scheme* if, in addition to (SS1) and (SS2), also the following condition is satisfied

(SS3) for every closed walk $W = (\vec{f}_1\vec{f}_2\ldots\vec{f}_d)$ and every arc $\vec{e} \in \mathrm{star}(\partial_{-}\vec{f}_1, G)$ which avoids W

(1) $$g(\vec{f}_d) \circ g(\vec{f}_{d-1}) \circ \ldots \circ g(\vec{f}_1)e = e \quad .$$

Simpicial schemas and pseudocomplexes meet in the following construction. Let G be an $(n+1)$-regular graph and g be a simplicial scheme on G. Then we construct an n-dimensional pseudocomplex $K = K(G,g)$ such that G is isomorphic with the dual graph of K. For each vertex $v \in V(G)$ take an n-simplex A_v and choose a bijective correspondence between the edges in $\mathrm{star}(v,G)$ and the $(n-1)$-faces of A_v. For every arc $\vec{e}$ of $\mathrm{star}(v,G)$ denote by $\mathrm{face}(\vec{e})$ the $(n-1)$-face of A_v corresponding to $\vec{e}$, and let $\mathrm{vert}(\vec{e})$ be the vertex of A_v opposite to $\mathrm{face}(\vec{e})$ (thus $A_v = \mathrm{vert}(\vec{e}) * \mathrm{face}(\vec{e})$, where $*$ means join).

We are now ready to define the pseudocomplex $K(G,g)$. It has n-simplexes A_v, $v \in V(G)$, and if $\vec{f}$ is an arc from v to u, then identify $\mathrm{face}(\vec{f}) \subset A_v$ and $\mathrm{face}(\vec{f}^{-1}) \subset A_u$ so that the vertices $\mathrm{vert}(\vec{e})$ and $\mathrm{vert}(g(\vec{f})\vec{e})$ are identified, for every arc $\vec{e} \in star(v,G)$, $\vec{e} \neq \vec{f}$.

It is easy to see that a vertex a of A_v is identified with a vertex b of A_u (by successive identifications of (n-1)-faces) iff in G exists a walk W from v to u, and if $\vec{e}$ is the arc of star(v,G) with $vert(\vec{e}) = a$, then $\vec{e}$ avoids W. This shows why we need the condition (SS3) suppose that there is a closed walk $W = (f_1 f_2 \ldots f_d)$ and an arc $\vec{e}$ which avoids W but (1) is not true for $\vec{e}$ on W. Then a vertex $vert(\vec{e})$ of A_v is identified with a vertex $vert(g(\vec{f}_d) \circ \ldots \circ g(\vec{f}_1)\vec{e})$ of A_v and this vertex is different from $vert(\vec{e})$. The n-simplex A_v is in this case deformed to something that is not a simplex. We remark that also a theory of complexes built from such deformed simplexes can be developed.

Similarly, the n-simplexes A_v and A_u intersect in a k-face iff there is a walk W from v to u, and there are arcs $\vec{e}_o, \vec{e}_1, \ldots, \vec{e}_k$ of star(v,G) which avoid W.

Above we have shown the folloving fact which will be of great importance when considering the properties of pseudo-complexes obtained from graphs and simplicial schemes.

<u>*3.1. Proposition.*</u> Let X and Y be n-dimensional simplexes of K(G,g). If X and Y have a common k-face, then

(1) there exists a walk $W = (\vec{f}_1 \vec{f}_2,,,\vec{f}_d)$ in G such that $X = A_{\partial_- \vec{f}_1}$ and $Y = A_{\partial_+ \vec{f}_d}$, and

(2) there exists arcs $\vec{e}_o, \vec{e}_1, \ldots, \vec{e}_k$ in $star(\partial_- \vec{f}_1, G)$ which avoid W.

Conversely, if (1) and (2) are satisfied, then X and Y have a common k-face with vertices $vert(\vec{e}_o),\dots,vert(\vec{e}_k)$. ////

A simplicial pseudocomplex is a simplicial complex iff for every simplexes A and B of K and for every two common faces of A and of B, X and Y, there exists a common face Z of A and of B such that X and Y are faces of Z. We may restrict ourselves to the case where A and B are top-dimensional simplexes and X and Y differ only by one vertex, i.e. $X = C * x$, $Y = C * y$, $Z = C * x * y$. Hence, also, X and Y are of same dimension. Let vertices v and u correspond to A and B, respectively. If C is a (k-1)-simplex, then X and Y are k-simplexes. According to Proposition 3.1, there are arcs $\vec{e}_1,\vec{e}_2,\dots,\vec{e}_k$ in star(v,G) (they determine C), and arcs $\vec{e}_x$ and $\vec{e}_y$ (they determine x and y, respectively), and there are walks P_X and P_Y from v to u such that $\vec{e}_1,\vec{e}_2,\dots,\vec{e}_k$ avoid both paths, P_X and P_Y, arc $\vec{e}_x$ avoids P_X (but not P_Y), and $\vec{e}_y$ avoids P_Y (but not P_X). $C * x * y$ is not a common face of A and B iff on every walk P from v to u at least one of the arcs $\vec{e}_x$, $\vec{e}_y$, $\vec{e}_1$, $\vec{e}_2$, ..., $\vec{e}_k$ does not avoid P. This is a sufficient and necessary condition when a pseudocomplex is a simplicial complex.

An easy consequence of Proposition 3.1 is the following characterization of simplexes of codimension two. The proof is left as an easy exercise.

Corollary 3.2. A cycle $C = (\vec{f}_1\vec{f}_2\ldots\vec{f}_d)$ in G corresponds to the star of a codimension-two simplex iff

(2) $$g(\vec{f}_i)\vec{f}_{i-1}^{-1} = \vec{f}_{i+1}\,, \qquad i = 1,2,\ldots,d$$

where $\vec{f}_{d+1}$ is $\vec{f}_1$, and $\vec{f}_o$ is $\vec{f}_d$./////

We call the reader's attention to a work of Pezzana, Ferri, Gagliardi, at.al. [Pe74, Fe79, Ga79a, Ga79b, FG81, FGG]. They define a simplicial scheme only in the case when the graph G is 1-factorable. In fact, they discuss also the case when G is not regular, using edge-coloration instead of 1-factorization. In this case, the obtained pseudocomplexes have a "boundary" (there are some simplexes of codimension one that are contained only in one top-dimensional simplex). Suppose that we have a 1-factorable graph G and choose a particular 1-factorization of G. This 1-factorization determines a simplicial scheme g on G as follows. Let $g(\vec{f})\vec{e}$ be the arc of the 1-factor to which $\vec{e}$ belongs and such that it has $\partial_+\vec{f}$ as the initial vertex. g is obviously a simplicial scheme. On every closed walk, (1) is satisfied not only for those arcs which avoid the walk but for every arc $\vec{e}$ from the star of the initial vertex of the walk. Pezzana [Pe74] has shown that every closed, connected n-dimensional PL-manifold can be pseudo-triangulated so that the set of 0-simplexes has cardinality n+1. The corresponding simplicial scheme is called a crystallization. The cited papers are mainly devoted to the theory of crystallizations.

Next we establish some combinatorial and topological properties of $K = K(G,g)$. The pseudocomplex K is homogeneous by construction, and since G is connected, it is strongly connected. Every $(n-1)$-simplex is contained in exactly two n-simplexes. Thus K is a pseudomanifold. It has some additional properties. Take a k-simplex A in K and consider $star(A,K)$. By Proposition 3.1, for any n-simplexes, P and Q, in $star(A,K)$ there is a sequence of n-simplexes which connects P and Q. That means that $star(A,K)$ is strongly connected. The subgraph of G which corresponds to $star(A,K)$ is $(n-k)$-regular and connected, and g can be restricted (in an obvious meaning) to this subgraph. Conversely, if g can be restricted to an $(n-k)$-regular connected subgraph of G, then this subgraph corresponds to a star of a k-simplex. This is also an immediate consequence of Proposition 3.1. It is easy to see that this subgraph with the induced simplicial scheme is isomorphic with the dual graph (and a simplicial scheme on it) of $link(A,K)$, if K is simplicial complex.

Let K be a pseudomanifold of dimension n. Then the dual graph G of K is $(n+1)$-regular and connected. Define a simplicial scheme g on G as follows. Every arc $\vec{e} \in S(G)$ corresponds to an $(n-1)$-simplex of K, denote it by $face(\vec{e})$. Suppose that v is a vertex in n-simplex which corresponds to $\partial_- \vec{e}$ and is opposite to $face(\vec{e})$ in this n-simplex. Let $vert(\vec{e}) := v$. If $\vec{e} \in star(\partial_- \vec{f},G)$, $\vec{e}' \in star(\partial_+ \vec{f},G)$, and $verc(\vec{e}) = vert(\vec{e}')$, then define $g(\vec{f})\vec{e} := \vec{e}'$. The obtained collection of maps $\{g(\vec{f});\ \vec{f} \in S(G)\}$ is a simplicial scheme on G. Denote a pair (G,g) obtained from K by the above construction by $G(K)$.

To classify the complexes which can be obtained from graphs and simplicial schemes we enlarge a definition of strong connectedness. Let K be a pseudocomplex of dimension n and let L be a subset of (n-1)-simplexes of K. We say that K is *strongly connected by L* if for any two n-simplexes, A and B of K there exists a sequence of n-simplexes $A = P_1, P_2, \ldots, P_k = B$ such that P_i and P_{i+1} $(i=1,2,\ldots,k-1)$ have a common (n-1)-face which belongs to L.

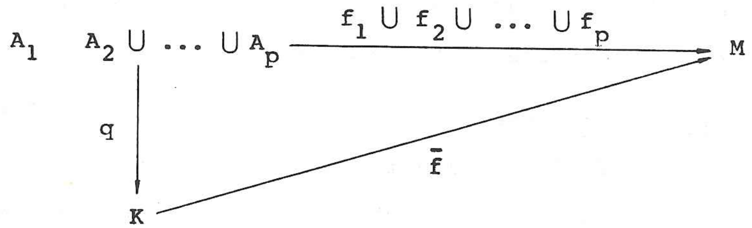
Theorem 3.3. *Let* G *be a connected (n+1)-regular graph and* g *a simplicial scheme on* G. *The pseudocomplex* $K(G,g)$ *is a pseudomanifold in which the star of every simplex* A *is strongly connected by (n-1)-simplexes which contain* A. *Moreover,* $G(K(G,g)) = (G,g)$.

Proof. We have shown before everything except that $G(K(G,g)) = (G,g)$. But this is immediate by constructions./////

Theorem 3.4. *Let* M *be a pseudomanifold of dimension* n *such that the star of every simplex* $A \in M$ *is strongly connected by (n-1)-simplexes which contain* A. *Then* $K(G(M)) = M$.

Proof. $K(G(M))$ is obtained as follows. Take a set of n-simplexes $\{A_1, A_2, \ldots, A_p\}$ such that there is a bijective correspondence $f : \{A_1, A_2, \ldots, A_p\} \to$ {set of n-simplexes of M}, and for every $i = 1,2,\ldots,p$ take a simplicial isomorphism $f_i : A_i \longrightarrow f(A_i)$. If two n-simplexes, $f(A_i)$ and $f(A_j)$ of M have a common (n-1)-face F then identify corresponding (n-1)-faces, $f_i^{-1}(F)$ and $f_j^{-1}(F)$, of A_i and A_j, respectively. As the identification map take $f_j^{-1}f_i$ restricted to $f_i^{-1}(F)$. If we make all the possible identifactions, we obtain a pseudocomplex K which is isomorphic with $K(G(M))$.

Define a simplicial map $\bar{f} : K \longrightarrow M$ such that the following diagram is commutative



where q is the natural projection onto K (K is a quotient pseudocomplex of $A_1 \cup A_2 \cup \ldots \cup A_p$). If $\bar{f}$ exists then it is unique. We shall now prove that $\bar{f}$ is well-defined. To see this it suffices to establish that if $q(A) = q(B)$, A a face of A_i, B a face of A_j, then $f_i(A) = f_j(B)$. If $q(A) = q(B)$ then there is a sequence $A_i = A_{i_1}, A_{i_2}, \ldots, A_{i_k} = A_j$ such that $f(A_{i_r})$ and $f(A_{i_{r+1}})$ have a common (n-1)-face which contains $f_i(A)$. Since the identification map between A_{i_r} and $A_{i_{r+1}}$ ($r = 1,2,\ldots,k-1$) is $f_{i_{r+1}}^{-1} f_{i_r}$, it follows that $q(f_{i_r}^{-1}(f_i(A))) = q(f_{i_{r+1}}^{-1}(f_i(A)))$. By transitivity we conclude that $q(f_i^{-1}(f_i(A))) = q(f_j^{-1}(f_i(A)))$. Since this is equal to $q(A) = q(B)$, it follows that $f_j^{-1}(f_i(A)) = B$, and $f_i(A) = f_j(B)$.

$\bar{f}$ is a simplicial map which is onto by the construction. To prove that it is also one-by-one (and thus establishing the theorem) it suffices to see that if a simplex $A \in M$ is a face of $f(A_i)$ and of $f(A_j)$ then $qf_i^{-1}(A) = qf_j^{-1}(A)$. By the assumptions of the theorem, star(A,M) is strongly connected by (n-1)-simplexes which contain A. Therefore there exists a sequence $A_i = A_{i_1}, A_{i_2}, \ldots, A_{i_k} = A_j$ such that $f(A_{i_r})$ and $f(A_{i_{r+1}})$ $(r = 1,2,\ldots,k-1)$ have a common (n-1)-face which contains A. As above we see that $qf_{i_r}^{-1}(A) = qf_{i_{r+1}}^{-1}(A)$, and consequently, $qf_i^{-1}(A) = qf_j^{-1}(A)$./////

At the end of this section we consider orientability of K(G,g). Let K be a pseudomanifold. As an *orientation* of K we refer to orientation (in the usual meaning) of |K| minus |(n-2)-skeleton of K|. If K is orientable (i.e. admits an orientation), it has exactly two different orientations. Every orientation is described as a choice of coherent orientations at every n-simplex in K. Orientation of a simplex can be described by a sequence of its vertices, and two sequences represent the same orientation iff they differ by an even permutation. Suppose that n-simplexes A and B have a common (n-1)-face, and that their vertex-sets are $\{a,x_1,x_2,\ldots,x_n\}$ and $\{b,x_1,x_2,\ldots,x_n\}$, respectively. Then the sequence (i.e. orientations) $(a,x_1,x_2,\ldots,x_n)$ and $(b,x_1,x_2,\ldots,x_n)$ are not coherent.

Let $W = (\vec{f}_1,\vec{f}_2,\ldots,\vec{f}_d)$ be a closed walk in G. Then $g(W) := g(\vec{f}_d) \circ \ldots \circ g(\vec{f}_2) \circ g(\vec{f}_1)$ can be viewed in an obvious way as a permutation of arcs of $star(\partial_- \vec{f}_1, G)$. Define $s(W,g)$ to be equal to 0, if $g(W)$ is an even permutation, and equal to 1, otherwise. Note that $s(W,g)$ is well-defined and invariant with respect to cyclic permutations of W.

Theorem 3.5. $K(G,g)$ *is orientable iff for every closed walk* W, $s(W,g)$ *is equal to the parity of the length of* W.

Proof. The proof is an easy consequence of the following fact. Let A and B be n-simplexes in $K(G,g)$ which have a common (n-1)-face F, let $v, u \in V(G)$ correspond to A and B, respectively, and let $\vec{f}$ correspond to F in G (it is supposed that $\vec{f}$ is directed from v to u). If $\vec{e}_o, \vec{e}_1, \ldots, \vec{e}_n$ are the arcs of $star(v,G)$ then for $i = 0,1,\ldots,n$, vertices $vert(\vec{e}_i)$ and $vert(g(\vec{f})\vec{e}_i)$ are equal in $K(G,g)$ provided $\vec{e}_i \neq \vec{f}$. That implies that orientations in A and in B represented by sequences $(vert(\vec{e}_o), vert(\vec{e}_1), \ldots, vert(\vec{e}_n))$ and $(vert(g(\vec{f})\vec{e}_o), vert(g(\vec{f})\vec{e}_1), \ldots, vert(g(\vec{f})\vec{e}_n)$, respectively, are not coherent. By extending this along a closed walk W we see that a sequence $(vert(g(W)\vec{e}_o, \ldots, vert(g(W)\vec{e}_n))$ represents in A the same orientation iff either the length of W is even and $g(W)$ is an even permutation, or the length of W is odd and $g(W)$ is odd permutation. The theorem is thus established./////

Suppose that a simplicial scheme g on G is obtained from a 1-factorization of G. For this case in [FGG] it is shown that $K(G,g)$ is orientable iff G is bipartite. This is also immediate by Theorem 3.5 since $g(W)$ is equal to identity on every closed walk W.

4. CATEGORY OF GRAPHS AND SIMPLICIAL SCHEMES

From a categorical point of view we consider a category G_n^S of n-regular graphs together with simplicial schemes. A morphism p between (G,g) and (H,h) is a covering projection $p : G \longrightarrow H$ which maps edges to edges, and for every directed edge $\vec{e} \in S(G)$ with initial vertex v and terminal vertex u the following diagram comutes

$$\begin{array}{ccc} \mathrm{star}(v,G) & \xrightarrow{\quad g(\vec{e}) \quad} & \mathrm{star}(u,G) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \mathrm{star}(p(v),H) & \xrightarrow{\quad h(p(\vec{e})) \quad} & \mathrm{star}(p(u),H) \end{array}$$

Let $\tilde{G}$ be a covering graph over G and let g be a simplicial scheme on G. Then there exists exactly determined simplicial scheme $\tilde{g}$ on $\tilde{G}$ such that the covering $p : \tilde{G} \to G$ is a morphism from $(\tilde{G},\tilde{g})$ onto (G,g).

Let K and L be homogeneous pseudocomplexes of dimesion n. A simplicial map $f : K \longrightarrow L$ is *nondegenarate* iff f maps every k-simplex $(0 \leq k \leq n)$ onto a k-simplex, and is *nonsingular* iff every two n-simplexes sharing a common $(n-1)$-face are mapped under f onto two different n-simplexes in L.

A technical proof of the following theorem is left as an exercise.

Theorem 4.1. *The assignment* K *which assigns a pseudocomplex* K(G,g) *with a pair* (G,g) *from* G^s_{n+1} *is a covariant functor from* G^s_{n+1} *into category of n-dimensional homogeneous pseudocomplexes and nondegenerate nonsingular maps.*/////

$$\begin{array}{ccc} (G,g) & \xrightarrow{K} & K(G,g) \\ \Big\downarrow p & & \Big\downarrow K(p) \\ (H,h) & \xrightarrow{K} & K(H,h) \end{array}$$

Theorem 4.1 is an important tool when considering nonsingular maps between pseudocomplexes. We give two applications of this. Another application can be found in |Mo|. Some nontrivial results of the same type can also be found in |Fi77|.

For a simplex A in K, let $\rho(A,K)$ denote the number of top-dimensional simplexes which contain A.

Lemma 4.2. *Let* K = K(G,g) *and* L = K(H,h) *be n-dimensional simplicial complexes, and let* q: K $\longrightarrow$ L *be a nondegenerate nonsingular map. Suppose that the link of every k-simplex in* L, $k < n-2$, *is simply connected, and that for any (n-2)-simplex* A *of* K, $\rho(A,K) = \rho(q(A),L)$. *Then* q *is a covering projection.*

Proof. First of all note that q is onto since G and H are assumed to be connected. Every nonsingular map is covering projection in the interior of n- and (n-1)-simplexes.

We have to show that q is covering on every simplex. Using induction on the dimension k we shall show that for every k-simplex A of K, $k \leq n-2$, link(A,K) is isomorphic with link$(q(A),L)$. This is true for $k = n-2$, since link of $(n-2)$-simplex A is isomorphic with a cycle $C_{\rho(A,K)}$ and link of $q(A)$ is isomorphic with $C_{\rho(q(A),L)}$.

For general k, $k < n-2$, suppose that for every $(k+1)$-simplex X, link(X,K) is isomorphic with link$(q(X),L)$. Let x be a vertex of link(A,K). By the induction hypothesis, link$(x * A,K)$ is isomorphic with link$(q(x * A), L) =$ $=$ link$(q(x) * q(A), L)$. We know that

$$\text{link}(x * A, K) = \text{link}(x, \text{link}(A,K)) \quad ,$$

and

$$\text{link}(q(x) * q(A), L) = \text{link}(q(x), \text{link}(q(A), L)) \ .$$

Therefore link$(x$, link$(A,K))$ is isomorphic with link$(q(x)$, link$(q(A),L))$. This is true for every vertex x of link(A,K), and consequently q restricted to link(A,K) is a covering projection of link(A,K) onto link$(q(A),L)$. By the assumption of the lemma, link(A,K) is simply connected, and since it is also connected (K is obtained as $K(G,g)$), it follows that q restricted to link(A,K) is an isomorphism between link(A,K) and link $(q(A),L)$. This completes the proof./////

Theorem 4.3. *Let* $L = K(H,h)$ *be simply connected n-dimensional simplicial complex such that the link of every k-simplex in* L, $k < n-2$, *is simply connected. If every $(n-2)$-simplex in* L *is contained in an even number of n-simplexes, then* H *is bipartite.*

Proof. Proof will be done by induction on the dimension n of $K(H,h)$. If $n = 1$, the only possible pseudomanifold is a cycle C_k. It has exactly one $(n-2)$-simplex, namely the empty set. The link of it is whole C_k and hence k is even by assumption. Since the dual graph of C_k is isomorphic with C_k, it is bipartite.

Let us now suppose that the theorem is true for every dimension less than n. Let $G = H \otimes K_2$ be a tensor product of H with K_2. G is a 2-fold covering graph over H, and has vertex-set $V(H) \times \{0,1\}$, two vertices, (v,i) and (u,j) being adjacent iff v is adjacent with u in H and $j \neq i$. It is known that G is connected iff H is not bipartite. Denote by g the lift of the simplicial scheme h to G.

We shall use Lemma 4.2. To be able to do this we have to prove that for any $(n-2)$-simplex A of $K(G,g)$, $\rho(A,K(G,g)) = \rho(q(A),L)$ where q is the nonsingular map determined by the covering projection $p\colon H \otimes K_2 \longrightarrow H$. This is obvious since $\rho(q(A),L)$ is even by assumption, and every closed walk of even length lifts to two walks of the same length in G.

Next we have to prove that $K(G,g)$ is a simplicial complex. We shall use the notations introduced after Proposition 3.1 where we characterise when $K(G,g)$ is a simplicial complex. We only write (v,i) and (u,j) instead of v and u, respectively. In H, the arcs $p(\vec{e}_x)$, $p(\vec{e}_1),\ldots,p(\vec{e}_k)$ avoid the walk $p(P_x)$ and $p(\vec{e}_y)$, $p(\vec{e}_1),\ldots,p(\vec{e}_k)$ avoid $p(P_y)$. Since $K(H,h)$ is simplicial complex, there is a walk P_o such that all arcs $p(\vec{e}_x)$, $p(\vec{e}_y)$, $p(\vec{e}_1), \ldots, p(\vec{e}_k)$ avoid P_o. Let P be a walk in G which is the lift of P_o and has (v,i) as the initial vertex. The arcs $\vec{e}_x, \vec{e}_y, \vec{e}_1, \ldots, \vec{e}_k$ all avoid P. We shall prove that the terminal vertex of P is (u,j) in any case, thus establishing that $K(G,g)$ is simplicial complex.

Suppose the contrary - the terminal vertex of P is $(u,1-j)$. Then it is easy to see (use the definition of $H \otimes K_2$) that the length of P has different parity as the length of P_X. Consequently, a closed walk W in H which is a concentration of $p(P_X)$ and P_o^{-1} has odd length. But W lies as a dual in $\mathrm{star}(p(C * x), K(H,h))$. This means that in the dual graph of $\mathrm{link}(p(C * x), K(H,h))$ there exists a closed walk of odd length. In other words, the dual graph of the complex $\mathrm{link}(p(C * x), K(H,h))$ is not bipartite. Note that this complex is of dimension less than n and satisfies the assumptions of the theorem. Using the induction hypothesis we obtain a contradiction with non-bipartiteness of the dual graph.

All the assumptions for the map $q\colon K(G,g) \longrightarrow K(H,h)$ of Lemma 4.2 are satisfied. Hence q is a covering projection. Now we use a fact that $K(H,h)$ is simply connected. Hence it admits only 1-fold connected covers. Since $K(G,g)$ is a 2-fold cover, it must consist of two components, hence $G = H \otimes K_2$ is not connected. This implies that H is bipartite./////

Theorem 4.3 is well-known in the case where $|K(H,h)|$ is a 2 sphere, e.g. [Fi77].

Next application concerns colorability of pseudomanifolds. By a *coloring* of an n-dimensional pseudocomplex we mean a (n+1)-coloring of the vertices of the pseudocomplex. Suppose that K(G,g) is n-dimensional. It is easy to see that K(G,g) admits a coloring iff the simplicial scheme g is induced by some 1-factorization of G. Let H be (n+1)-regular graph on two vertices, and let h be the simplicial scheme on H which is induced by the 1-factorization of H (in fact, this is the only simplicial scheme on H). The following theorem which is easily proved can be useful when considering colorable complexes.

Theorem 4.4. *Suppose that* K(G,g) *is orientable. Then* K(G,g) *admits a coloring iff* (G,g) *maps to* (H,h)./////

In other words, every orientable n-dimensional colorable pseudocomplex which can be obtained as K(G,g) can be described by its dual which is a covering graph over H. The simplicial scheme on G is exactly determined as a lift of h.

ACKNOWLEDGEMENT

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COMPARABILITY LINE GRAPHS

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ABSTRACT

We determine all graphs G such that the line graph L(G) is a comparability graph.

1. INTRODUCTION

For several graph-theoretic properties characterizations of graphs whose line graphs have the property in question have been given (cf. [2], [3]). Here we give such a characterization for the property of being a comparability graph.

Definition. A *graph* G is an ordered pair $G = (V,E)$ where V is a finite set and E is a binary relation in V: $E \subseteq V \times V$. If E is symmetric and irreflexive ($E^{-1} = E$, $E \cap I = \emptyset$) then G is *undirected*. A binary realtion $F \subseteq E$ is an *orientation* of an undirected graph G if $F \cap F^{-1} = \emptyset$

and $F \cup F^{-1} = E$. An orientation F is *transitive* if $F^2 \subseteq F$. An undirected graph is a *comparability graph* if it admits a transitive orientation. Otherwise it is an *incomparability graph*.

We shall restrict our attention to undirected graphs. More about comparability graphs can be found in [1].

Example 1. Every bipartite graph is a comparability graph since we can orient each edge from the first partitive set of vertices to the second and get an orientation F with $F^2 = \emptyset$. (It is clear that only bipartite graphs admit such an orientation as they have no odd cycles). Thus trees, even cycles, and cubes (Q_n) are comparability graphs.

Example 2. For every graph G its transitive closure is a comparability graph. To construct a transitive orientation assign different integers to vertices and orient the edges from low to high. Thus complete graphs (K_n) are comparability graphs since they are transitive closures of themselves.

Example 3. The graphs $H_1 - H_7$ shown in Fig.1 are incomparability graphs. This can be verified by trying to orient their edges transitively and eventually deducing conflicting orientations of the same edge.

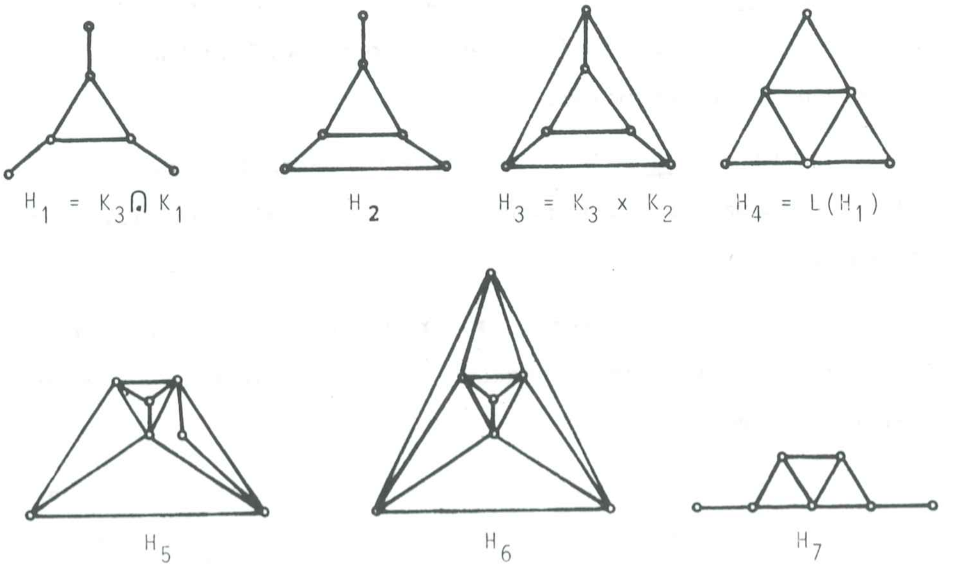

Fig.1. Some incomparability graphs

Problem. Characterize graphs G such that the line graph $L(G)$ is a comparability graph!

To solve this problem we need some auxiliary lemmas.

Lemma 1. If G is a comparability graph, then every induced subgraph H of G is a comparability graph as well.

Proof. Since the subgraph H is induced we can use the same orientation of edges as in G.

Lemma 2. (a) If H is a subgraph of G, then $L(H)$ is an induced subgraph of $L(G)$.

(b) If K is an induced subgraph of $L(G)$, then there exists a subgraph H of G such that $L(H) = K$.

Proof. (a) is obvious. (b) Let H be the subgraph of G consisting of edges of G which correspond to the vertices of K, and their endpoints.

Corollary 1. If $L(G)$ is a comparability graph and $L(H)$ an incomparability graph, then G contains no subgraph isomorphic to H.

Proof. If H were a subgraph of G, then, by Lemma 2a, $L(H)$ would be an induced subgraph of $L(G)$. But this is impossible, by Lemma 1.

Corollary 2. If $L(G)$ is a comparability graph, then G has no subgraph isomorphic to an odd cycle C_{2k+1} with $k \geq 2$, or to any of the graphs $G_1 - G_7$ shown in Fig.2.

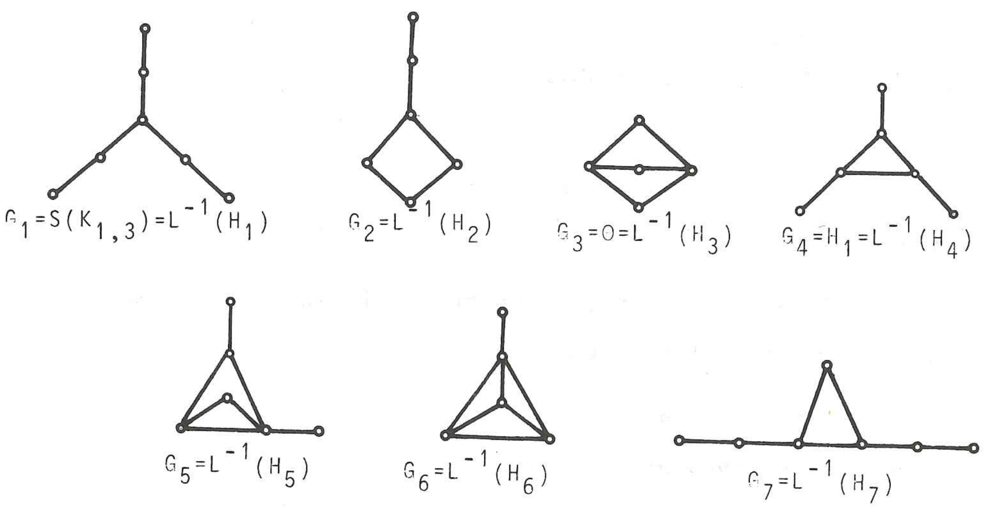

Fig.2. Inverse line graphs of graphs $H_1 - H_7$

Proof. By Corollary 1, all we have to show is that the line graphs of the said graphs are all incomparability graphs. - $L(C_{2k+1}) = C_{2k+1}$, and for $k \geq 2$ this is clearly an incomparability graph. - For the remaining graphs, note that the line graphs of G_1 - G_7 are isomorphic to the incomparability graphs H_1 - H_7 shown in Fig.1.

In sections 2 and 3, we shall prove the following result using Corollary 2.

Theorem. $L(G)$ *is a comparability graph if and only if each connected component of* G *is of one of the following types:*

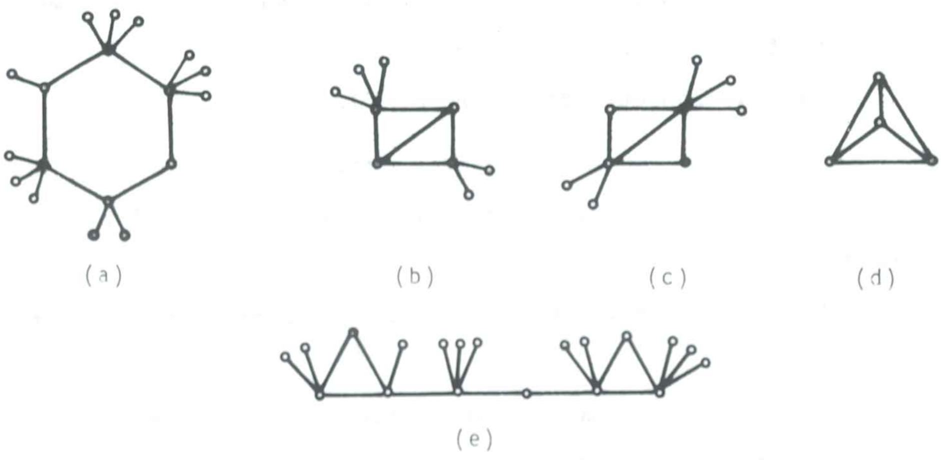

Fig.3. Some graphs of types (a) - (e)

(a) *an even cycle with zero or more pendant edges in each vertex,*

(b) K_4-e *with zero or more pendant edges in vertices of degree 2,*

(c) K_4-e *with zero or more pendant edges in vertices degree 3,*

(d) K_4,

(e) *a path with zero or more pendant edges in each vertex, possibly with one triangle over one or both end-edges.*

A graph of each of the types (a) - (e) is shown in Fig.3.

2. NECESSITY

In this section we prove the necessity part of the Theorem above: if L(G) is a comparability graph then each connected component of G is of one of the types (a) - (e).

Take a connected component of G.

<u>*Case 1.*</u> G contains an even cycle C_{2m}. Denote the set of vertices that lie on the cycle by X and the set of the vertices of G by Y.

First we consider the edges with at least one endpoint in Y.

If there exists a path of length 2 or more such that its first vertex is in X and its second and third vertices are in Y, then G contains a subgraph isomorphic to G_1, if $m \geq 3$, or to G_2, if $m = 2$. (See Fig.4. The forbidden subgarphs are shown with a dashed line). If there exists a vertex c in Y adjacent to vertices a and b from X, $a \neq b$, then G contains:

- C_{2m+1}, if $d(a,b) = 1$ (here $m \geq 2$)
- G_1, if $d(a,b) \geq 3$
- G_2, if $d(a,b) = 2$ and $m \geq 3$
- G_3, if $d(a,b) = 2$ and $m = 2$

(See Fig.5. The forbidden subgraph are shown with a dashed line). Here the distance is measured around the cycle C_{2m}.

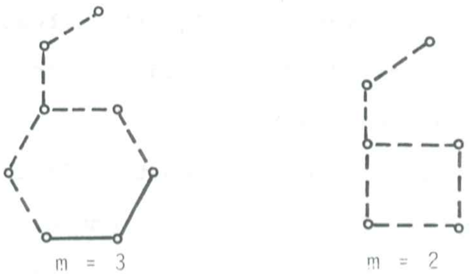

Fig.4.

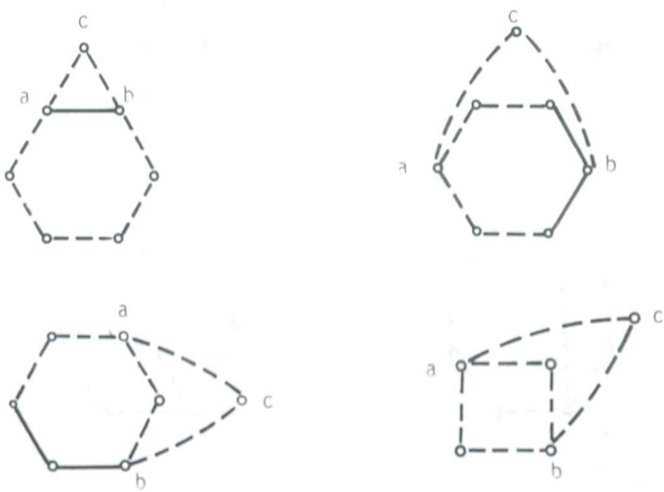

Fig.5.

Thus, by Corollary 2, the only possible edges with at least one endpoint in Y are the pendant edges in the vertices of the cycle C_{2m}.

Second, we consider edges which are not edges of the cycle C_{2m}, but have both endpoints in X, that is, the diagonals of the cycle. Take a diagonal D. If D cuts the cycle into two even cycles, then G contains G_1 if one of the new cycles has length 6 or more, or G_2, if at least one of the new cycles os of length 4. (See Fig.6. The forbidden subgraphs are shown with a dashed line).

If D splits the cycle into two odd cycles, they must be triangles and m = 2. If there are pendant edges in two consecutive vertices of C_4, then G contains G_5. If both diagonals of C_4 are present and there are pendant edges in some vertex of C_4, then G contains G_6. (See Fig.7. The forbidden subgraphs are shown with a dashed line).

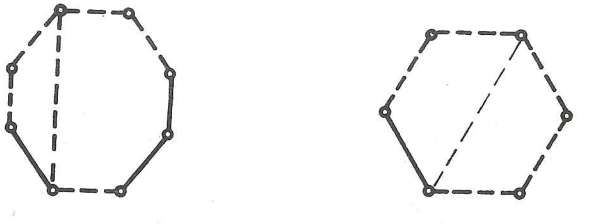

Fig.6.

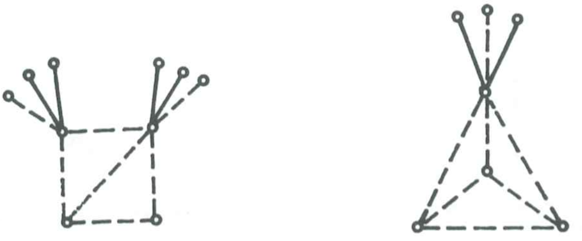

Fig.7.

Thus, by Corollary 2, if a connected component of G contains an even cycle, it is of one of the type (a) - (d).

Case 2. G contains no even cycle.

Let P be one of the longest paths in G. Denote the set of vertices that lie on P by X and the set of the rest of the vertices of G by Y.

First we consider the edges with at least one endpoint in Y. If there exists a path of length 2 or more, such that its first vertex x is X and its second and third vertices are in Y, then x must be at distance 2 or more from either endpoint of P, or P were not a longest path. But then G contains G_1. (See Fig.8. The forbidden subgraph is shown with a dashed line).

If there exists a vertex c in Y adjacent to different vertices a and b from X, then $d(a,b) \geq 2$ or P were not a longest path. But then G contains a cycle of length 4 or more. (See Fig.9). Here the distance is measured along the path P.

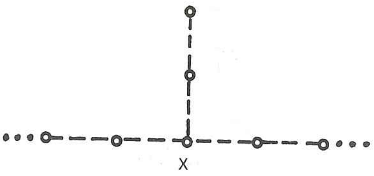

Fig.8.

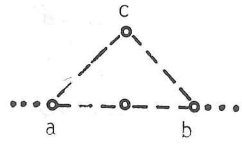

Fig.9.

Thus, by Corollary 2, the only possible edges with at least one endpoint in Y are the pendant edges in the vertices of the path P. In fact, there may not be any pendant edges in the endpoints of P, but we shall rather regard the end-edges as pendant edges attached in the next-to-end vertices of P and hence allow pendant edges in every vertex of P. Then, of course, P might cease to be a longest path.

Second, we consider edges connecting two nonconsecutive vertices a,b of P (shortcuts).

If $d(a,b) \geq 3$, then G contains a cycle of length 4 or more. (See Fig.10). If a and b are both at distance 2 or more from either endpoint of P, then G contains G_7. (See Fig.11).

There can be no pendant edges attached to the top of a triangle. For if neither of a and b is an endpoint of P, then G would contain G_4, and this is the case as P is a longest path. (See Fig.12 and 13).

Also two triangles cannot share a common edge since then G would contain C_4. (See Fig.14).

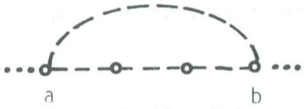

Fig.10.

Fig.11.

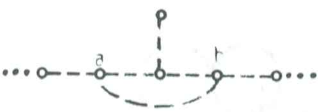

Fig.12.

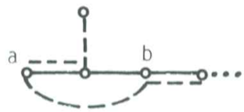

Fig.13.

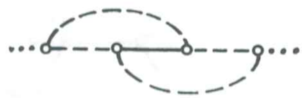

Fig.14.

Thus, by Corollary 2, if a connected component of G contains no even cycle, it is of type (e).

3. SUFFICIENCY

In this section we prove the sufficiency part of the Theorem above: if each connected component of G is of one of the types (a) - (e), then L(G) is a comparability graph. For each of the types (a) - (e) we first construct the line graph and then assign a transitive orientation to its edges.

A simple closed curve in the figures represents a complete subgraph. Its edges are assigned a transitive orientation as described in Example 2. Only the choice of the lowest and the highest vertices is shown in the figures. They are denoted by L and H respectively.

A transitive orientation of the line graph of a graph of type (a) is shown in Fig.15.

A transitive orientation of the line graph of a graph of type (b) is shown in Fig.16.

A transitive orientation of the line graph of a graph of type (c) is shown in Fig.17.

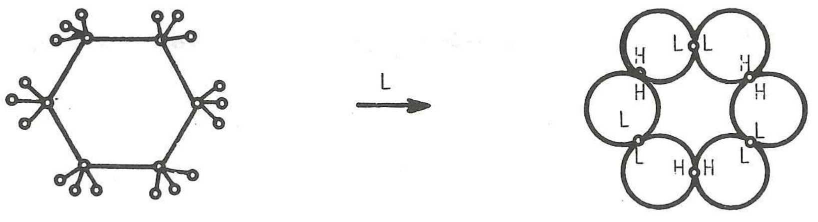

Fig.15.

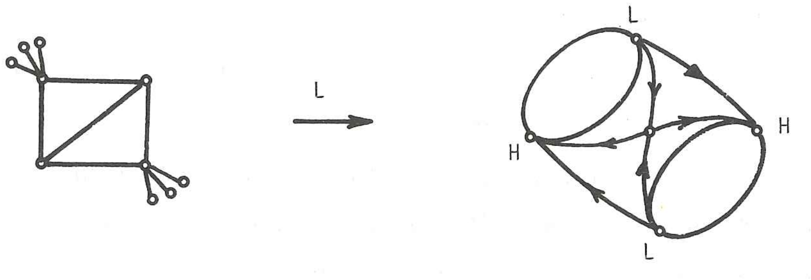

Fig.16.

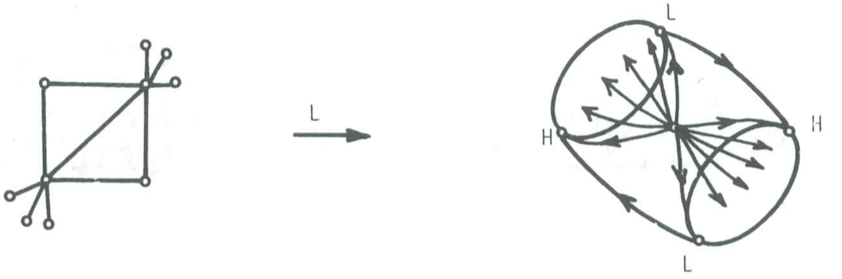

Fig.17.

A transitive orientation of the octahedron which is the line graph of the tetrahedron is shown in Fig.18.

Transitive orientations of the line graphs of some graphs of type (d) are shown in Fig.19 - 21.

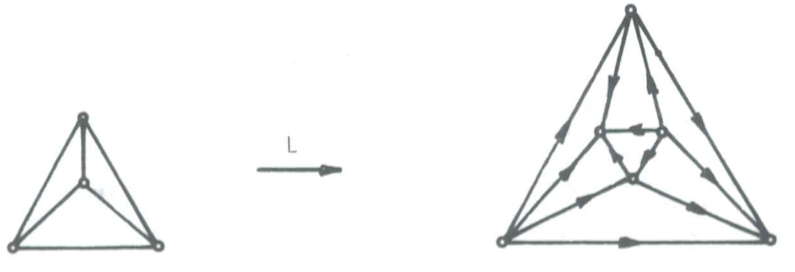

Fig.18.

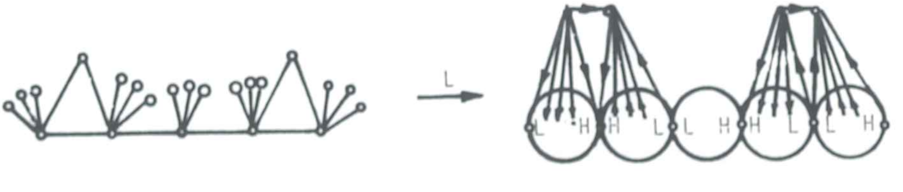

Fig.19.

Fig.20

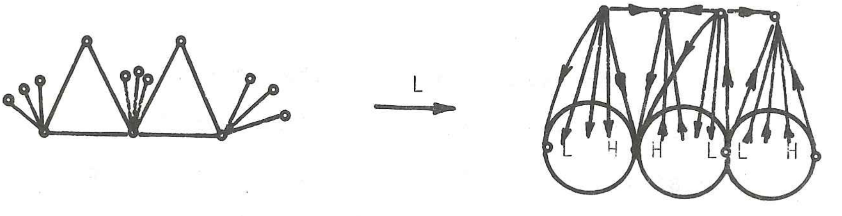

Fig.21.

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ON GRAPHS WHOSE SECOND SPREAD DOES NOT EXCEED 3/2

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ABSTRACT

The quantity $s_2 = \lambda_2^+ - \lambda_2^-$, where λ_2^+ is the second largest positive eigenvalue and λ_2^- is the second least negative eigenvalue of a graph, is called the second spread of the graph. In this paper all minimal graphs with the property $s_2 > 3/2$, are determined. In addition, all connected graphs with $s_2 \le 3/2$ are described.

1. INTRODUCTION

All considered graphs in this paper are assumed to be undirected graphs without loops or multiple edges. By eigenvalues of a graph we mean eigenvalues of its 0-1 adjacency matrix.

Throughout the paper, we consider only the graphs having at least two positive eigenvalues, the largest of which is $\lambda_1^+(G)$ and the second largest of which is $\lambda_2^+(G)$. As is easily seen, any such graph has at least two negative eigenvalues, the least of which is $\lambda_1^-(G)$ and the second least of which is $\lambda_2^-(G)$. For each such graph G, we denote by

$$s_2(G) = \lambda_2^+(G) - \lambda_2^-(G)$$

the second spread of G. For definition and properties of the first spread of matrices, one can consult [2].

In this paper we determine all minimal graphs with the property

$$s_2 = \lambda_2^+ - \lambda_2^- > 3/2 \tag{1}$$

and all graphs without isolated vertices with the property

$$s_2 = \lambda_2^+ - \lambda_2^- \leq 3/2. \tag{2}$$

We first quote the following lemma, which is an immediate consequence of the results J.H. Smith ([3]).

Lemma 1. A connected graph G has at least two positive eigenvalues if and only if it has one of the graphs displayed in Figure 1 as an induced subgraph.

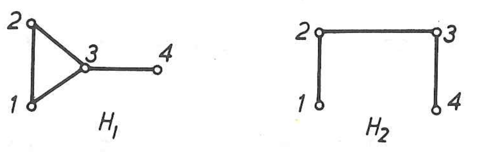

Fig.1.

Next, define an equivalence relation $\sim$ in the vertex set V(G) of G in the following way: vertices x and y are equivalent if and only if they have the same neighbours. Let $\{N_1,\dots,N_k\}$ be the corresponding quotient set and $|N_i| = n_i$ $(i=1,\dots,k)$. The subsets $N_1,\dots,N_k$ (characteristic subsets of G) have the following property: any two vertices from the same subset are not adjacent, and any two subsets are completely adjacent or completely nonadjacent in G. The induced subgraph g of G obtained by choosing an arbitrary vertex from each of the characteristic subsets is said to be canonical image of G.

Lemma 2. ([4]) *Let* G *be a finite graph and* g *its canonical image. Then the number* $p(G)$ *of the non-zero eigenvalues of* G *equals the number* $p(g)$ *of the non-zero eigenvalues of* g.

The non-zero eigenvalues of G *are determined by the equation*

$$f(\lambda) = \det(b_{ij} - \frac{\lambda}{n_i}\delta_{ij}) = 0$$

where $B = [b_{ij}]$ *is the adjacency matrix of* g *and* δ_{ij} *is the Kronecker* δ*-symbol**.

Denote by (n) any graph with n vertices and without edges. Next, denote by (m) (hatched) the complete graph K_m. The line between two circles denotes that there are all possible edges

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* *Although the proof was given for infinite graphs only, the proof for finite graphs is completely similar.*

between corresponding graphs. Next, by $P(\ell,m,n)$ and $Q(m,n)$ denote the following classes of graphs:

$P(\ell,m,n)$ $Q(m,n)$

Lemma 3. A graph G *from the class* $P(\ell,m,n)$ *satisfies (2) if and only if one of the following holds:*

1° $\ell = 2,\ m \geq 1,\ n \leq 2$;

2° $\ell = 3,\ m \geq 1,\ n = 1$;

3° $\ell = 4,\ m = n = 1$.

Proof. Let $G \in P(\ell,m,n)$. If $\ell = 1$, the canonical image g of G is K_2. Then by Lemma 2, we conclude that G has exactly two non-zero eigenvalues and does not satisfy (2).

If $\ell \geq 2$, then by Lemma 2, we have that non-zero eigenvalues of G are determined by equation

$$f(\lambda) = \frac{(-1)^{\ell}(\lambda+1)^{\ell-1}}{m\cdot n}(\lambda^3-(\ell-1)\lambda^2-m(n+\ell)\lambda+(\ell-1)mn) = 0 .$$

Hence we conclude that $\lambda_2^- = -1$ and $\lambda_2^+ \leq \frac{1}{2}$ if and only if one of 1°, 2° or 3° holds.

Lemma 4. A graph G *from the class* $Q(m,n)$ *satisfies (2) if and only if one of the following holds:*

1° $m = 1,\ n \geq 1$;

2° $m = n = 2$.

Proof. By Lemma 2 we get that the non-zero eigenvalues of $G \in Q(m,n)$ are determined by equation

$$f(\lambda) = \frac{\lambda}{m \cdot n} (\lambda^4 - (mn+m+n)\lambda^2 + mn) = 0 .$$

Hence we conclude that $\lambda_2^+ = |\lambda_2^-| \le \frac{3}{4}$ if and only if $m = 1$, $n \ge 1$ or $m = n = 2$ holds.

Next, denote by $R(n)$ $(n \ge 3)$ the graph obtained from the complete graph K_n by removing two incidence edges.

Lemma 5. A graph G *from the class* $R(n)$ *satisfies (2) for each* $n \ge 4$.

Proof. As is easily shown, the characteristic polynomial of $G \in R(n)$ is

$$p(\lambda) = (\lambda+1)^{n-3}(\lambda^3 - (n-3)\lambda^2 - (2n-5)\lambda + (n-3)) .$$

Hence we conclude that $\lambda_2^- = -1$ and $\lambda_2^+ \le \frac{1}{2}$ for all $n \ge 4$.

2. MINIMAL GRAPHS WITH $S_2(G) > \frac{3}{2}$

Recall that a graph is minimal with respect to the property P if it has the property P and none of its proper induced subgraphs has this property.

Now we determine all minimal graphs with the property (1).

Theorem 1. There are exactly 16 minimal graphs with respect to the property of having the second spread greater than 3/2 and they are displayed in Figure 2.

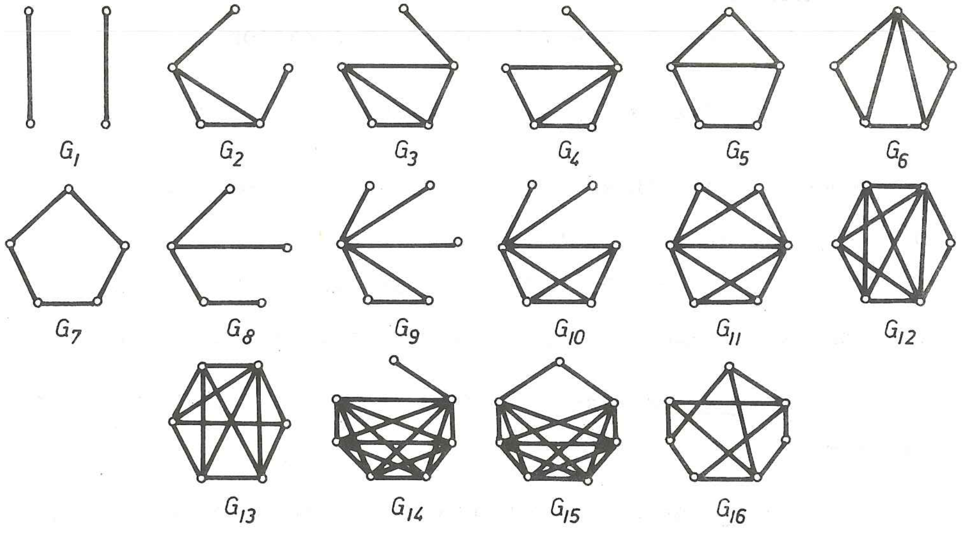


Fig.2.

Proof. As is easily seen, the graphs G_1-G_{16} in Figure 2 are minimal with respect to the property (1).

Conversely, let G be any minimal graph with respect to the property (1). We shall prove that G is one of the graphs depicted in Figure 2.

The graph G obviously has no isolated vertices. If G is a disconnected graph, then G_1 is an induced subgraph and hence equals G.

Next, let G be a connected graph. Acorrding to Lemma 1, G contains one of the labelled graphs depicted in Figure 1 as an induced subgraph. We distinguish the following two cases:

Case 1. G has H_1 as an induced subgraph.

Then, let $T_{i_1 \ldots i_k}$ $(1 \leq i_1 < \ldots < i_k \leq 4;\ 1 \leq k \leq 4)$ be the set of vertices from $V(G)\setminus V(H_1)$ which are adjacent exactly to the vertices $i_1,\ldots,i_k$ of H_1. Let T_o be the set of vertices from $V(G)\setminus V(H_1)$ which are not adjacent to any vertex of H_1. Excluding the symmetric cases, we distinguish the following subcases:

1^o $T_1 \neq \emptyset$. Then G_2 is an induced subgraph, and hence equals G.

2^o $T_4 \neq \emptyset$. Then G contains the proper induced subgraph G_1 contradicting the minimality condition.

3^o $T_{12} \neq \emptyset$. Then G is G_3.

4^o $T_{13} \neq \emptyset$. Then G is G_4.

5^o $T_{14} \neq \emptyset$. Then G is G_5.

6^o $T_{34} \neq \emptyset$. Then G contains the proper induced subgraph G_1 contradicting the minimality condition.

7^o $T_{134} \neq \emptyset$. Then G is G_6.

8^o $T_o \neq \emptyset$. Then G has a proper induced subgraph with the property (1) contradicting the minimality condition.

Let next $T_1 = T_2 = T_4 = T_{12} = T_{13} = T_{14} = T_{23} = T_{24} = T_{34} = T_{134} = T_{234} = T_o = \emptyset$.

9^o $|T_3| > 1$. Then G is G_9.

10^o $|T_3| = 1$, $T_{123} \neq \emptyset$. Then G is G_{10}.

11^o $|T_3| = 1$, $T_{124} \neq \emptyset$. Then either $G \in P(2,m,2)$ (and by Lemma 3, it satisfies (2)), or it has a proper induced subgraph with the property (1), contradiction.

12^o $|T_3| = 1$, $T_{1234} \neq \emptyset$. Then G is G_{11}.

Let next $T_3 = \emptyset$.

13^o $|T_{123}| > 2$. Then G is G_{14}.

14^o $|T_{123}| = 2$, $T_{124} \neq \emptyset$. Then G is G_{15}.

15^o $|T_{123}| = 2$, $T_{1234} \neq \emptyset$. Then G has a proper induced subgraph with the property (1) contradicting the minimality condition.

16^o $|T_{123}| = 1$, $T_{124} \neq \emptyset$. Then either $G \in P(3,m,1)$ (and by Lemma 3, it satisfies (2)), or it has proper induced subgraph with the property (1), contradiction.

17^o $|T_{123}| = 1$, $T_{1234} \neq \emptyset$. Then G is G_{12}.

Let next $T_{123} = \emptyset$.

18^o $|T_{124}| > 1$. Then either $G \in P(2,m,1)$ (and by Lemma 3, it satisfies (2)), or $G = H_3$ (and satisfies (2), too), or it has a proper induced subgraph with the property (1), contradiction.

19^o $|T_{124}| = 1$, $T_{1234} \neq \emptyset$. Then G is G_{13}.

Finally, let $T_{124} = \emptyset$. Then obviously G is G_{13}.

Case 2. G has H_2 and has no H_1 as an induced subgraph.

Let $T_{i_1 \ldots i_k}$ and T_o have the same meaning as in the first case, now with respect to H_2. Then $T_{12} = T_{23} = T_{34} = T_{123} = T_{124} = T_{134} = T_{234} = T_{1234} = \emptyset$. Excluding the symmetric cases, we distinguish the following subcases:

1^o $T_1 \neq \emptyset$. Then G contains the proper induced subgraph G_1 contradicting the minimality condition.

2^o $T_2 \neq \emptyset$. Then G is G_8.

3^o $T_{14} \neq \emptyset$. Then G is G_7.

4^o $T_o \neq \emptyset$. Then G has a proper induced subgraph with the property (1) contradicting the minimality condition.

Finally, let $T_1 = T_2 = T_3 = T_4 = T_{14} = T_o = \emptyset$.

5^o $T_{13} \neq \emptyset$, $T_{24} = \emptyset$. Then $G \in Q(1,n)$ (and by Lemma 4, it satisfies (2)), contradiction.

6^o $T_{13} \neq \emptyset$, $T_{24} \neq \emptyset$. Then G is G_{16}.

This completes the proof of Theorem 1.

3. GRAPHS WHOSE SECOND SPREAD DOES NOT EXCEED 3/2

In this section we determine all graphs without isolated vertices whose second spread does not exceed 3/2.

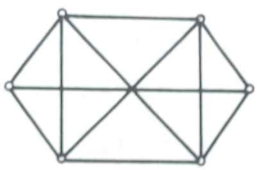
Theorem 2. *Let G be a graph without isolated vertices. The second spread of G is not greater than 3/2 if and only if one of the following holds:*

(i) $G \in P(\ell,m,n)$ ($\ell = 2$, $m \geq 1$, $n \leq 2$; $\ell = 3$, $m \geq 1$, $n = 1$ or $\ell = 4$, $m = n = 1$);

(ii) $G \in Q(m,n)$ ($m = 1$, $n \geq 1$ or $m = n = 2$);

(iii) $G \in R(n)$ ($n \geq 5$);

(iv) G is the graph H_3 from Figure 3.



H_3

Fig. 3.

Proof. Let a graph G has the second spread not greater than 3/2. To describe G, we use the method of impossible subgraphs. By the Interlacing theorem we conclude that G contains none of the graphs $G_1 - G_{16}$ from Figure 2 as an induced subgraph, because they satisfy (1).

The graph G is connected, because in the contrary case G_1 is an induced subgraph, contradiction. According to Lemma 1, G contains one of the graphs from Fig.1 as an induced subgraph. We distinguish the following two cases:

Case 1. G has H_1 as an induced subgraph.

Let $T_{i_1 \ldots i_k}$ and T_o have the same meaning as in Theorem 1.

Then $T_1 = T_2 = \emptyset$, because in the contrary case G_2 is an induced subgraph, contradiction. Similarly, we have that $T_4 = T_{12} = T_{13} = T_{14} = T_{23} = T_{24} = T_{34} = T_{134} = T_{234} = \emptyset$ (otherwise we obtain one of the graphs in Figure 2 as an induced subgraph, contradiction). Moreover, we have that $T_o = \emptyset$. Indeed, in the contrary case G has at least one of the graphs G_1, G_2, G_3, G_4 as an induced subgraph, contradiction. Thus, all these sets except eventually T_3, T_{123}, T_{124} and T_{1234} are empty. Next, we distinguish the following five cases:

1^o $T_3 = T_{123} = T_{124} = T_{1234} = \emptyset$.

Then $G = H_1 \in P(\ell,m,n)$ $(\ell = 2,\ m = n = 1)$.

2^o $T_3 \neq \emptyset$, $T_{123} = T_{124} = T_{1234} = \emptyset$. In this case we have that $|T_3| = 1$, since G has no G_1 or G_9 as an induced subgraph. Thus, $G \in P(\ell,m,n)$ $(\ell = 2,\ m = 1,\ n = 2)$.

3° $T_{123} \neq \emptyset$, $T_3 = T_{124} = T_{1234} = \emptyset$. Then the set T_{123} cannot have two nonadjacent vertices, since otherwise G_4 is an induced subgraph, contradiction. But since T_{123} has no triangles (for G has no G_{14} as an induced subgraph), we have that $|T_{123}| \leq 2$ and $G \in P(\ell,m,n)$ $(3 \leq \ell \leq 4,$ $m = n = 1)$.

4° $T_{124} \neq \emptyset$, $T_3 = T_{123} = T_{1234} = \emptyset$. Then the subgraph H of G induced by the vertex set T_{124} is either K_2 or the graph without edges. Indeed, in the contrary case G has at least one of the graphs G_4, G_{12}, G_{13} as an induced subgraph, contradiction. Wherefrom, $G = H_3$ or $G \in P(\ell,m,n)$ $(\ell = 2,\ m \geq 2,\ n = 1)$.

5° $T_{1234} \neq \emptyset$, $T_3 = T_{123} = T_{124} = \emptyset$. Then the set T_{1234} is complete, because otherwise G_{13} is an induced subgraph, contradiction. Thus, $G \in R(n)$ $(n \geq 5)$.

Now we determine the edge-relations between the sets T_3, T_{123}, T_{124}, T_{1234} in G and represent them by Table 1.

Table 1

	T_3	T_{123}	T_{124}	T_{1234}
T_3		$\emptyset$	1	$\emptyset$
T_{123}			1	$\emptyset$
T_{124}				0

If the corresponding sets are completely adjacent, completely nonadjacent, or not consistent, the symbols 1, 0 or $\emptyset$, respectively, are used. So for example, the sets T_3 and T_{123} are not consistent, since G has no G_4 or G_{11} as an induced subgraph.

Taking into account all possible combinations with the sets T_3, T_{123}, T_{124} and T_{1234}, we can distinguish the following three cases:

6^o $T \neq \emptyset$, $T_{124} \neq \emptyset$, $T_{123} = T_{1234} = \emptyset$. Then, having in view 2^o and 4^o, we conclude that $G \in P(\ell,m,n)$ ($\ell = 2$, $m \geq 2$, $n = 2$). Indeed, in the contrary case G contains G_{11} as an induced subgraph, contradiction.

7^o $T_{123} \neq \emptyset$, $T_{124} \neq \emptyset$, $T_3 = T_{1234} = \emptyset$. Then, if the subgraph induced by T_{124} is K_2, we have that G has G_{13} as an induced subgraph, contradiction. Let T_{124} consist of $n \geq 1$ isolated vertices. Then if $|T_{123}| = 2$, G_{15} will be an induced subgraph, contradiction, while if $|T_{123}| = 1$, we get that $G \in P(\ell,m,n)$ ($\ell = 3$, $m \geq 2$, $n = 1$).

8^o $T_{124} \neq \emptyset$, $T_{1234} \neq \emptyset$, $T_3 = T_{123} = \emptyset$. Then if the subgraph induced by T_{124} is K_2, we have that G has G_1 as an induced subgraph, contradiction. If T_{124} consists of $n \geq 1$ isolated vertices, then G has one of G_5, G_{12} as an induced subgraph, provided that $|T_{124}| > 1$ or $|T_{1234}| > 1$, contradiction. In the remaining case, we obviously have that $G = H_3$.

Case 2. The graph G has H_2 and has no H_1 as an induced subgraph.

Then, obviously, $T_{12} = T_{23} = T_{34} = T_{123} = T_{124} = T_{134} =$

$= T_{234} = T_{1234} = \emptyset$. Besides, $T_1 = T_4 = \emptyset$, since otherwise G contains G_1 as an induced subgraph, contradiction. In a similar way we conclude that $T_2 = T_3 = T_{14} = \emptyset$ (otherwise G contains G_8 or G_7 as an induced subgraph, contradiction). Similarly $T_o = \emptyset$, since otherwise G_8 is an induced subgraph, contradiction. Thus, all above sets except eventually T_{13} and T_{24} must be empty.

The sets T_{13} and T_{24} consist only of isolated vertices. Indeed, in the contrary case G has H_1 as an induced subgraph, contradiction. Next, each vertex from T_{13} must be adjacent to any other vertex from T_{24} (otherwise G contains the impossible subgraph G_1, contradiction).

So, we determined the structure of the considered graph. Taking into account all possible combinations and having in mind the symmetry, we distinguish the following subcases:

1^o $T_{13} = T_{24} = \emptyset$. Then $G \in Q(m,n)$ $(m = n = 1)$.

2^o $T_{13} \neq \emptyset$, $T_{24} = \emptyset$. Then $G \in Q(m,n)$ $(m = 1, n \geq 2)$.

3^o $T_{13} \neq \emptyset$, $T_{24} \neq \emptyset$. Then if $|T_{13}| = |T_{24}| = 1$, we have that $G \in Q(m,n)$ $(m = n = 2)$, while if $|T_{13}| > 1$ or $|T_{24}| > 1$, we get that G contains G_{16} as an induced subgraph, contradiction.

This, together with Lemmas 3-5, completes the proof.

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ANTIDIRECTED HAMILTONIAN CIRCUITS IN TOURNAMENTS

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ABSTRACT

In this paper we prove that every tournament T with $n = 2k \geq 14$ vertices, which contains a transitive subtournament TT_5 and does not contain a TT_6, has an antidirected Hamiltonian circuit. From this fact and Rosenfeld's result [6] it follows that every tournament T_n with $n = 2k \geq 16$ vertices has an ADH circuit.

1. INTRODUCTION

An n-tournament T_n is an oriented complete graph with n vertices. A simple path (circuit) in a diagraph is antidirected if every two adjacent edges of the path (circuit) have opposite orientations. An antidirected Hamiltonian path (circuit) - - ADH path (circuit) is a simple antidirected path (circuit) which contains all vertices.

B. Grünbaum introduced the ADH paths and circuits in [3] and proved that every tournament, except T_3^c , T_5^c and T_7^c , has an ADH path. A simpler proof with some additional results was given by M.Rosenfeld in [4] . Since a tournament T_n can have ADH circuit only if n is even and T_8 which contain a T_7^c cannot have an ADH circuit, B.Grünbaum conjectured that every tournament T_n with $n = 2k \geq 10$ has an ADH circuit. The conjecture was proved by C.Thomassen [5] for $n \geq 50$ and by M.Rosenfeld [6] for $n \geq 26$. We prove that Grünbaum's conjecture is true for $n \geq 16$.

DEFINITIONS AND NOTATIONS. TT_n is a transitive tournament with n vertices. We usually denote the vertex set of TT_n by $\{1,2,\ldots,n\}$ where $i \to j$ iff $i<j$. T_3^c and T_5^c are unique regular tournaments with three and five vertices, respectively. T_7^c is a regular tournament with seven vertices $v_0,v_1,\ldots,v_6$ where $v_i \to v_j$ if $i-j$ is a quadratic residue modulo 7. T_p+T_q is every $(p+q)$ - tournament in which p vertices span a subtournament isomorphic to T_p while the remaining q vertices span a subtournament isomorphic to T_q. If a vertex v dominates vertices $v_1,v_2,\ldots,v_k$ $(k>1)$, we write $v \to \{v_1,v_2,\ldots,v_k\}$. Similary we write $A \to B$ to express the fact that every vertex from the set A dominates every vertex from the set B. A vertex v is a starting (terminating) vertex in T_n if there exists an ADH path $v \to v_1 \leftarrow \ldots (v \leftarrow v_1 \to \ldots)$. If v is both starting and terminating vertex, then v is a double point.

We shall use the following results due to B.Grünbaum and M.Rosenfeld ([3], [4], [6]):

(A) Every tournament with an odd number of vertices, that has an ADH path, has a double point.

(B) Every tournament, except T_3^c, T_5^c and T_7^c, has an ADH path.

(C) (a) If $n=2k$, then TT_n has an ADH path starting at i $(i \neq n)$ and terminating at j except for following cases:

(i) $j=1$,

(ii) $i=1$, $j=2$ $(n>2)$,

(iii) $i=2k-1$, $j=2k$.

(b) If $n=2k+1$, then TT_n has ADH path with i and j as starting vertices if $i,j\neq 2k+1$ and $\{i,j\}=\{2k-1,2k\}$, $(n>3)$.

2. ADH-CIRCUITS

Let T_n be a tournament with $n=2k\geq 14$ vertices. Since every tournament with at least 14 vertices has a transitive subtournament TT_5 ([2]), we can consider T_n as $T_n=TT_5+T_m$, $m=2k+1\geq 9$. Denote by $\{1,2,3,4,5\}$ and $\{v_1,v_2,\ldots,v_m\}$ the vertex sets of TT_5 and T_m , respectively. Without loss of generality we may assume (by (A) and (B)) that

$$v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_{m-1} \rightarrow v_m$$

is an ADH path in T_m and at least three vertices of TT_5 dominate the vertex v_1 .

At first we shall prove some lemmas. In all these lemmas T_n is the above mentioned tournament. Furthermore we assume that T_n does not contain a TT_6 .

Lemma 1. For every vertex v_i *of* T_m *there are vertices* p *and* q *of* TT_5 *such that* $v_i \rightarrow p$ *and* $v_i \rightarrow q$.

Proof. It follows immediately from the condition $T_n \not\supset TT_6$.

Lemma 2. If the tournament T_n *has no ADH circuits and* $v_1 \leftarrow \{i,j,k\}$ $(\{i,j,k\} \subset \{1,2,3,4,5\}$, $i<j<k$, $i<3)$ *, then* $v_m \rightarrow \{1,2,3,4\}$ and $v_m \leftarrow 5$.

Proof. If $v_m \leftarrow \ell$, $\ell \in \{1,2,3,4\}$, then $x \rightarrow v_1 \leftarrow v_2 \rightarrow \ldots$ $\ldots \leftarrow v_{m-1} \rightarrow v_m \leftarrow \ell \rightarrow \ldots \leftarrow x$ is an ADH circuit in T_n, where $x = \begin{cases} i & \text{for } \ell \neq i \\ j & \text{for } \ell = i \end{cases}$ and $\ell \rightarrow \ldots \leftarrow x$ is an ADH path in TT_5 which exists by (C). This is in contradiction with conditions of the lemma. So $v_m \rightarrow \{1,2,3,4\}$ and by Lemma 1. $v_m \leftarrow 5$.

Lemma 3. If the tournament T_n *has no ADH circuits,* $v_1 \leftarrow \{3,4,5\}$ *and* $v_1 \rightarrow \{1,2\}$, *then* $v_m \rightarrow \{1,2\}$.

Proof. If $v_m \leftarrow \ell$, $\ell \in \{1,2\}$, then $3 \rightarrow v_1 \leftarrow v_2 \rightarrow \ldots \leftarrow v_{m-1} \rightarrow$ $\rightarrow v_m \leftarrow \ell \rightarrow \ldots \leftarrow 3$ is an ADH circuit in T_n where $\ell \rightarrow \ldots \leftarrow 3$ is an ADH path in TT_5 ((C)).

Lemma 4. If the tournament T_n *has no ADH circuits,* $v_1 \leftarrow \{3,4,5\}$, $v_1 \rightarrow \{1,2\}$, $v_m \leftarrow \{i,j\}$ *and* $v_m \rightarrow \{1,2,k\}$ ($\{i,j,k\} = \{3,4,5\}$) *then:*

(i) $v_{m-1} \leftarrow \{1,2,k\}$,

(ii) $v_{m-2} \rightarrow \{1,2,v_m\}$,

(iii) $v_{m-3} \leftarrow \{1,2,v_{m-1},k\}$,

(iv) $v_{m-4} \rightarrow \{1,2,v_{m-2}\}$.

Proof.

(i) For $v_{m-1} \rightarrow \ell$, $\ell \in \{1,2,k\}$ we have ADH circuit $x \rightarrow v_1 \leftarrow v_2 \rightarrow \ldots \leftarrow v_{m-1} \rightarrow \ell \leftarrow v_m \rightarrow y \leftarrow \ldots \leftarrow x$ in TT_n where $x = \begin{cases} 3 & \text{for } k \neq 3 \\ 4 & \text{for } k = 3 \end{cases}$, $y = \begin{cases} 2 & \text{for } \ell = k \\ k & \text{for } \ell \neq k \end{cases}$ and $y \leftarrow \ldots \leftarrow x$ is an ADH path in $TT_5 \setminus \{\ell\}$.

(ii) Suppose that $v_{m-2} \leftarrow \ell$, $\ell \in \{1,2\}$. Let $\{x,y,k\} =$ $= \{3,4,5\}$ and $\{\ell,\ell'\} = \{1,2\}$. By (i) $\ell \rightarrow v_{m-2} \leftarrow v_{m-3} \rightarrow \ldots \rightarrow v_1 \leftarrow y \rightarrow$ $\rightarrow v_m \leftarrow v_{m-1} \rightarrow x \leftarrow \ell' \rightarrow k \leftarrow \ell$ is an ADH circuit in T_n . If $v_{m-2} \leftarrow v_m$, then, by (i) , $x \rightarrow v_1 \leftarrow v_2 \rightarrow \ldots \rightarrow v_{m-2} \leftarrow v_m \rightarrow k \leftarrow \ell \rightarrow v_{m-1} \leftarrow 2 \rightarrow y \leftarrow x$ is an ADH circuit in T_n with $x = \begin{cases} 3 & \text{for } k \neq 3 \\ 4 & \text{for } k = 3 \end{cases}$ and $y = \begin{cases} 4 & \text{for } k = 5 \\ 5 & \text{for } k \neq 5 \end{cases}$.

(iii) If $v_{m-3} \rightarrow \ell$, $\ell \in \{1,2\}$, we have, according to (ii), ADH circuit $v_1 \leftarrow v_2 \rightarrow \ldots \leftarrow v_{m-3} \rightarrow \ell \leftarrow v_{m-2} \rightarrow v_m \leftarrow v_{m-1} \rightarrow x \leftarrow \ell' \rightarrow y \leftarrow z \rightarrow v_1$

($\{\ell,\ell'\}=\{1,2\}$, $\{x,y,z\}=\{3,4,5\}$). If $v_{m-3}\to v_{m-1}$ then, according to (i) and (ii), $v_1\leftarrow v_2\to\ldots\leftarrow v_{m-3}\to v_{m-1}\leftarrow 1\to 2\leftarrow v_{m-2}\to v_m\leftarrow x\to 5\leftarrow y\to v_1$ ($\{x,y\}=\{3,4\}$) is an ADH circuit in T_n. Finally, suppose that $v_{m-3}\to k$. If $v_{m-1}\to j$, then $k\leftarrow v_{m-3}\to v_{m-4}\leftarrow\ldots\to v_1\leftarrow i\to j\leftarrow v_{m-1}\to v_m\leftarrow v_{m-2}\to$ $\to 2\leftarrow 1\to k$ is an ADH circuit in T_n. If $v_{m-1}\leftarrow j$ (then $v_{m-1}\to i$ by (i)), we have for $v_{m-2}\leftarrow j$ ADH circuit $k\leftarrow v_{m-3}\to v_{m-4}\leftarrow\ldots\to v_1\leftarrow j\to$ $\to v_{m-2}\leftarrow v_{m-1}\to i\leftarrow 1\to 2\leftarrow v_m\to k$ and for $v_{m-2}\to j$ ADH circuit $k\leftarrow v_{m-3}\to v_{m-4}\leftarrow$ $\leftarrow\ldots\to v_1\leftarrow i\to v_m\leftarrow v_{m-2}\to j\leftarrow 2\to v_{m-1}\leftarrow 1\to k$.

(iv) Let $v_{m-4}\leftarrow\ell$, $\ell\in\{1,2\}$. Then by (iii) $\ell\to v_{m-4}\leftarrow$ $\leftarrow v_{m-5}\to\ldots\to v_1\leftarrow x\to v_m\leftarrow v_{m-1}\to v_{m-2}\leftarrow v_{m-3}\to y\leftarrow\ell'\to z\leftarrow\ell$ ($\{\ell,\ell\}=\{1,2\}$, $\{x,y,z\}=\{3,4,5\}$) is an ADH circuit in T_n. If $v_{m-4}\leftarrow v_{m-2}$, we get from (ii) and (iii) ADH circuit $v_1\leftarrow v_2\to\ldots\to v_{m-4}\leftarrow v_{m-2}\to v_m\leftarrow$ $\leftarrow v_{m-1}\to v_{m-3}\leftarrow 1\to 5\leftarrow 2\to 4\leftarrow 3\to v_1$.

Since all cases lead to cntradictions, Lemma 4. is proved.

<u>*Lemma 5*</u>. *If the tournament* T_n *has no ADH circuits,* $\{v_1,v_m\}\leftarrow\{3,4,5\}$, $\{v_1,v_m\}\to\{1,2\}$ *and* $v_1\leftarrow v_m$, *then:*

(i) $v_{m-1}\leftarrow\{2,4,5\}$,

(ii) $v_{m-2}\to\{1,v_m\}$.

<u>*Proof.*</u>

(i) For $v_{m-1}\to 2$ we have ADH circuit $2\leftarrow v_{m-1}\to v_{m-2}\to\ldots\to$ $\to v_1\leftarrow 3\to v_m\leftarrow 4\to 5\leftarrow 1\to 2$. From conditions of the lemma it follows that $v_m\to v_1\leftarrow v_2\to\ldots\leftarrow v_{m-1}$ is an ADH path in T_m and according to the dual assertion of (C) (assuming that T_n has no ADH circuits) we get $v_{m-1}\leftarrow\{4,5\}$.

(ii) Since $1\to v_{m-2}\leftarrow v_{m-3}\to\ldots\to v_1\leftarrow 5\to v_m\leftarrow 4\to v_{m-1}\leftarrow 2\to 3\leftarrow 1$ is an ADH circuit in T_n we conclude that $v_{m-2}\to 1$. Suppose that $v_{m-2}\leftarrow v_m$. According to (i) and Lemma 1., three cases are possible:

(a) $v_{m-1}\to 1$, $v_{m-1}\leftarrow 3$ (b) $v_{m-1}\to\{1,3\}$ (c) $v_{m-1}\leftarrow 1$, $v_{m-1}\to 3$. But in the first case we have ADH circuit $v_1\leftarrow v_2\to\ldots\to v_{m-2}\leftarrow v_m\to 2\leftarrow 1\to 5\leftarrow 4\to v_{m-1}\leftarrow 3\to v_1$, in the second $v_1\leftarrow v_2\to\ldots\to v_{m-2}\leftarrow v_m\to 1\leftarrow v_{m-1}\to 3\leftarrow 2\to 5\leftarrow 4\to v_1$ and in the third $v_1\leftarrow v_2\to\ldots\to v_{m-2}\leftarrow v_m\to 2\leftarrow 1\to v_{m-1}\leftarrow 4\to 5\leftarrow 3\to v_1$.

This completes the proof of Lemma 5.

Now we can prove the main result of the paper.

Theorem 1. *Every tournament* T_n *with* $n=2k\geq 14$ *vertices which does not contain a transitive subtournament* TT_6 *has an ADH circuit.*

Proof. Since $n\geq 14$, T_n has a TT_5 as a subtournament [2] . Therefore we can write $T_n=TT_5+T_m$, $m\geq 9$, as in begining of this chapter.

Let $v_1\leftarrow\{i,j,k\},(\{i,j,k\}\subset\{1,2,3,4,5\},\ i<j<k)$. There, are two characteristic cases to be examined.

Case 1. $i<3$.

Let z be a vertex of TT_5 such that $v_{m-1}\to z$ (it follows from Lemma 1.). If $z\neq 5$ we have by Lemma 2. ADH circuit

$$x\to v_1\leftarrow v_2\to\ldots\leftarrow v_{m-1}\to z\leftarrow v_m\to y\leftarrow\ldots\leftarrow x ,$$

where $x = \begin{cases} i & \text{for } z\neq i \\ j & \text{for } z=i \end{cases}$, and $y\leftarrow\ldots\leftarrow x$ is an ADH path in $TT_5\setminus\{z\}$ for suitably chosen $y\in\{1,2,3,4\}\setminus\{z\}$.

If $v_{m-1}\to 5$ and $v_{m-1}\leftarrow\{1,2,3,4\}$, we have ADH circuit

$$i\to v_1\leftarrow v_2\to\ldots\leftarrow v_{m-1}\to 5\leftarrow y\to\ldots\leftarrow i ,$$

where $y\to\ldots\leftarrow i$ is an ADH path in the transitive tournament $TT_5 = \{1,2,3,4,v_m\}$, $y\in\{1,2,3,4\}\setminus\{i\}$.

Thus case 1. is settled.

It is obvious that $v_1 \leftarrow \{3,4,5\}$ and $v_1 \rightarrow \{1,2\}$. According to Lemmas 1 and 3 we have to consider three cases:

(a) v_m is dominated by exactly one of the vertices 3,4,5.

(b) v_m is dominated by exactly two of the vertices 3,4,5.

(c) v_m is dominated by all the vertices 3,4,5.

(a) $v_m \leftarrow i$, $v_m \rightarrow \{1,2,j,k\}$, $(\{i,j,k\}=\{3,4,5\})$. Denote by z a vertex of TT_5 such that $v_{m-1} \rightarrow z$. By Lemma 3. for $z \neq i$

$$x \rightarrow v_1 \leftarrow v_2 \rightarrow \ldots \leftarrow v_{m-1} \rightarrow z \leftarrow v_m \rightarrow y \leftarrow \ldots \leftarrow x$$

is an ADH circuit in T_n , where $x = \begin{cases} 3 \text{ for } z \neq 3 \\ 4 \text{ for } z=3 \end{cases}$ and $y \leftarrow \ldots \leftarrow x$ is an ADH path in $TT_5 \setminus \{z\}$ for suitably chosen z . For $z=i$ we have ADH circuit

$$x \rightarrow v_1 \leftarrow v_2 \rightarrow \ldots \leftarrow v_{m-1} \rightarrow i \leftarrow 1 \rightarrow 2 \leftarrow v_m \rightarrow y \leftarrow x \quad ,$$

with $x = \begin{cases} 3 \text{ for } z \neq 3 \\ 4 \text{ for } z=3 \end{cases}$ and $\{i,x,y\}=\{3,4,5\}$.

(b) $v_m \leftarrow \{i,j\}$, $v_m \rightarrow \{1,2,k\}$, $(\{i,j,k\}=\{3,4,5\})$. Assume at first that $k \neq 5$. By Lemma 4. (iii) and Lemma 1. vertex v_{m-3} dominates at least one of the vertices i and j . If $v_{m-3} \rightarrow i$, then

$$i \leftarrow v_{m-3} \rightarrow v_{m-4} \leftarrow \ldots \rightarrow v_1 \leftarrow k \rightarrow j \leftarrow 1 \rightarrow 2 \leftarrow v_{m-2} \rightarrow v_m \leftarrow v_{m-1} \rightarrow i$$

is an ADH circuit in T_n for $v_{m-1} \rightarrow i$ and

$$i \leftarrow v_{m-3} \rightarrow v_{m-4} \leftarrow \ldots \rightarrow v_1 \leftarrow k \rightarrow j \leftarrow v_{m-1} \rightarrow v_m \leftarrow v_{m-2} \rightarrow 2 \leftarrow 1 \rightarrow i$$

for $v_{m-1} \leftarrow i$. (Observe that in the last case $v_{m-1} \rightarrow j$ by Lemma 4. (i)). If $v_{m-3} \rightarrow j$, we get ADH circuit

$$j \leftarrow v_{m-3} \rightarrow v_{m-4} \leftarrow \ldots \rightarrow v_1 \leftarrow i \rightarrow v_m \leftarrow v_{m-2} \rightarrow 2 \leftarrow 1 \rightarrow v_{m-1} \leftarrow k \rightarrow j \quad ,$$

where $v_{m-1} \leftarrow k$ (by Lemma 4. (i)) and $k<j$.

Examine now the case $k=5$. The ADH circuit $4 \leftarrow v_{m-3} \rightarrow$

$\to v_{m-4} \leftarrow \ldots \to v_1 \leftarrow 5 \to v_{m-1} \leftarrow 1 \to 2 \leftarrow v_{m-2} \to v_m \leftarrow 3 \to 4$ implies that

(1) $$v_{m-3} \leftarrow 4 \quad .$$

From (1) and Lemma 4, it follows that

(2) $$v_{m-3} \to 3 \quad .$$

Further, ADH circuit $3 \leftarrow v_{m-3} \to v_{m-4} \leftarrow \ldots \to v_1 \leftarrow 4 \to 5 \leftarrow 1 \to 2 \leftarrow v_{m-2} \to v_m \leftarrow v_{m-1} \to 3$ implies that

(3) $$v_{m-1} \leftarrow 3$$

and according to Lemma 4.

(4) $$v_{m-1} \to 4 \quad .$$

By Lemmas 1 and 4 v_{m-4} is dominated by at least one of vertices 3,4 and 5. Using (1), (2), (3) and (4) we have ADH circuits

$$3 \to v_{m-4} \leftarrow v_{m-5} \to \ldots \to v_1 \leftarrow 5 \to v_{m-3} \leftarrow 1 \to 2 \leftarrow v_{m-2} \to v_m \leftarrow v_{m-1} \to 4 \leftarrow 3 \quad ,$$

$$4 \to v_{m-4} \leftarrow v_{m-5} \to \ldots \to v_1 \leftarrow 5 \to v_{m-1} \leftarrow 3 \to v_m \leftarrow v_{m-2} \to 2 \leftarrow 1 \to v_{m-3} \leftarrow 4 \quad ,$$

$$5 \to v_{m-4} \leftarrow v_{m-5} \to \ldots \to v_1 \leftarrow 4 \to v_{m-3} \leftarrow 1 \to 2 \leftarrow v_{m-2} \to v_m \leftarrow 3 \to v_{m-1} \leftarrow 5 \quad ,$$

respectively.

(c) $v_m \leftarrow \{3,4,5\}$, $v_m \to \{1,2\}$. We shall examine the three cases mentioned in the proof of Lemma 5.

1) $v_{m-1} \to 1$, $v_{m-1} \leftarrow 3$. Notice at first that

(5) $$v_{m-2} \to \{3,4,5\} \quad .$$

Indeed, if $v_{m-2} \leftarrow i$, $i \in \{3,4,5\}$, then $i \to v_{m-2} \leftarrow v_{m-3} \to \ldots \to v_1 \leftarrow v_m \to 2 \leftarrow 1 \to j \leftarrow k \to v_{m-1} \leftarrow i$ $(\{i,j,k\}=\{3,4,5\}$, $j>k)$ is an ADH circuit in T_n.

The vertex v_{m-3} dominates at least one of the vertices 1,2,3,4 and 5. If it is 1 or 2, we have, using (5) and Lemma 5., ADH circuits

$$1\leftarrow v_{m-3}\rightarrow v_{m-4}\leftarrow\ldots\rightarrow v_1\leftarrow 3\rightarrow 5\leftarrow 2\rightarrow 4\leftarrow v_{m-2}\rightarrow v_m\leftarrow v_{m-1}\rightarrow 1 ,$$

$$2\leftarrow v_{m-3}\rightarrow v_{m-4}\leftarrow\ldots\rightarrow v_1\leftarrow 3\rightarrow v_{m-1}\leftarrow 4\rightarrow v_m\leftarrow v_{m-2}\rightarrow 5\leftarrow 1\rightarrow 2 ,$$

If $v_{m-3}\rightarrow i$, $i \in \{3,4,5\}$, then there is ADH circuit

$$i\leftarrow v_{m-3}\rightarrow v_{m-4}\leftarrow\ldots\rightarrow v_1\leftarrow v_m\rightarrow 1\leftarrow v_{m-2}\rightarrow j\leftarrow k\rightarrow v_{m-1}\leftarrow 2\rightarrow i ,$$

where $\{i,j,k\}=\{3,4,5\}$ and $j>k$.

2) $v_{m-3}\rightarrow\{1,3\}$. Since $2\rightarrow v_{m-2}\leftarrow v_{m-3}\rightarrow\ldots\rightarrow v_1\leftarrow 4\rightarrow v_m\leftarrow v_{m-1}\rightarrow$ $\rightarrow 3\leftarrow 1\rightarrow 5\leftarrow 2$ and $4\rightarrow v_{m-2}\leftarrow v_{m-3}\rightarrow\ldots\rightarrow v_1\leftarrow v_m\rightarrow 1\leftarrow v_{m-1}\rightarrow 3\leftarrow 2\rightarrow 5\leftarrow 4$ are ADH circuits in T_n we conclude that

$$v_{m-2}\rightarrow\{2,4\} . \tag{6}$$

Let $v_{m-3}\rightarrow i$, $i \in \{1,2,3,4,5\}$. For $i \in \{1,3\}$ we have, using (6), ADH circuit

$$i\leftarrow v_{m-3}\rightarrow v_{m-4}\leftarrow\ldots\rightarrow v_1\leftarrow k\rightarrow 5\leftarrow j\rightarrow \ell\leftarrow v_{m-2}\rightarrow v_m\leftarrow v_{m-1}\rightarrow i$$

where $k = \begin{cases} 3 \text{ for } i=1 \\ 4 \text{ for } i=3 \end{cases}$, $j = \begin{cases} 2 \text{ for } i=1 \\ 1 \text{ for } i=3 \end{cases}$, $\ell = \begin{cases} 4 \text{ for } i=1 \\ 2 \text{ for } i=3 \end{cases}$. For $i=2$ we obtain ADH circuit

$$2\leftarrow v_{m-3}\rightarrow v_{m-4}\leftarrow\ldots\rightarrow v_1\leftarrow 3\rightarrow 5\leftarrow 4\rightarrow v_m\leftarrow v_{m-1}\rightarrow 1\leftarrow v_{m-2}\rightarrow 2 .$$

Finally, for $i \in \{4,5\}$ we get ADH circuit

$$i\leftarrow v_{m-3}\rightarrow v_{m-4}\leftarrow\ldots\rightarrow v_1\leftarrow j\rightarrow v_m\leftarrow v_{m-2}\rightarrow 1\leftarrow v_{m-1}\rightarrow 3\leftarrow 2\rightarrow i$$

where $\{i,j\}=\{4,5\}$.

3) $v_{m-1}\leftarrow 1$, $v_{m-1}\rightarrow 3$. In this case

$$v_{m-2}\rightarrow\{2,3\} . \tag{7}$$

Indeed, for $v_{m-2}\leftarrow 2$ we have ADH circuit $2\rightarrow v_{m-2}\leftarrow v_{m-3}\rightarrow\ldots\leftarrow v_1\leftarrow 4\rightarrow$ $\rightarrow v_m\leftarrow v_{m-1}\rightarrow 3\leftarrow 1\rightarrow 5\leftarrow 2$ and for $v_{m-2}\leftarrow 3$ $3\rightarrow v_{m-2}\leftarrow v_{m-3}\rightarrow\ldots\rightarrow v_1\leftarrow v_m\rightarrow 2\leftarrow 1\rightarrow$ $\rightarrow v_{m-1}\leftarrow 4\rightarrow 5\leftarrow 3$, by Lemma 5.

Let $v_{m-3}\rightarrow i$, $i \in \{1,2,3,4,5\}$. If $i \in \{1,2,3\}$, we get, using (7) and Lemma 5, ADH circuit

$$i \leftarrow v_{m-3} \rightarrow v_{m-4} \leftarrow \ldots \rightarrow v_1 \leftarrow 5 \rightarrow v_m \leftarrow 4 \rightarrow v_{m-1} \leftarrow j \rightarrow k \leftarrow v_{m-2} \rightarrow i$$

where $\{i,j,k\}=\{1,2,3\}$, $j<k$. For $i \in \{4,5\}$

$$i \leftarrow v_{m-3} \rightarrow v_{m-4} \leftarrow \ldots \rightarrow v_1 \leftarrow j \rightarrow v_m \leftarrow v_{m-1} \rightarrow 3 \leftarrow v_{m-2} \rightarrow 2 \leftarrow 1 \rightarrow i$$

with $\{i,j\}=\{4,5\}$ is an ADH circuit in T_n .

In all cases we have obtained contradictions which prove Theorem 1.

Theorem 2. Every tournament T_n *with* $n = 2k \geq 16$ *vertices has an ADH circuit.*

Proof. It is proved in [1] that every tournament T_n with $n \geq 16$ vertices has a transitive subtournament TT_5 . If T_n does not contain TT_6, the proof follows from Theorem 1. The case $T_n \supset TT_6$ was proved by M. Rosenfeld [6] .

3. REMARKS

Three types of tournaments, T_{10}, T_{12} and T_{14}, should be checked to prove Grünbaum's conjecture. Since every T_{14} contains TT_5 [2], the conjecture is valid for n=14 if $T_{14} \not\supset T_6$. This is an immediate consequence of Theorem 1. According to [6], the only case of interest for $n=14$ is $T_{14} = TT_7 + T_7^{cc}$. In our opinion, techniques similar to those used in the proof of Theorem 1. can be applied to this case as well. For tournaments T_{10} and T_{12} the problem remains open.

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THE DIAGONAL CONSTRUCTION AND GRAPH EMBEDDINGS

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ABSTRACT

The purpose of this note is to show that quadrilateral embeddings are at least as important as triangular embeddings. Namely, the study of cellular graph embeddings can be reduced to the study of quadrilateral embeddings of bipartite graphs. We show this by exhibiting an easily computable one-to-one correspondence between cellular graph embeddings and quadrilateral embeddings of bipartite graphs. Finally, the diagonal construction applied to the orientable embedding of the n-cube graph is extensively studied.

1. INTRODUCTION

Triangulations are important in topology. In topological graph theory where the embeddings of graphs into surfaces are studied, they are important, too: by virtue of Euler´s equation any triangular embeddings is also minimal embedding (i.e. it is a genus embedding if it is orientable). However, we believe that the importance of triangular embeddings of graphs was overemphasized in the past. Not all graphs admit triangular embeddings. Take for instance bipartite graphs: since they possess no odd cycles the quadrilateral embeddings are the best one can hope for.

2. THE PURPOSE

It is the purpose of this note to show that quadrilateral embeddings are at least as important as triangular embeddings. Namely, the study of cellular graph embeddings can be reduced to the study of quadrilateral embeddings of bipartite graphs! We show this by exhibiting an easily computable one-to-one correspondence between cellular graph embeddings and quadrilateral embeddings of bipartite graphs.

We start with an arbitrary cellular embedding of some graph G into some orientable or non-orientable surface S. First, we construct the *web graph* $W(G)$ determined by this embedding. The embedding of G into S induces also a cellular embedding of $W(G)$ into S. Its dual $B(G)$ on S is a quadrilaterally embedded bipartite graph.

The reverse process, which we call the *diagonal construction*, produces - from a quadrilateral embedding of some bipartite graph B into the surface S - dual embeddings on S of graphs $D_o(B)$ and $D_1(B)$ in such a way that

$$W(D_o(B)) = W(D_1(B)) = B^*, \text{ the dual of } B.$$

Finally, the diagonal construction applied to the orientable embedding of the n-cube graph is extensively studied.

3. NOTATION AND TERMINOLOGY

It is assumed that the reader is familiar with the fundamental results of the theory of graph embeddings in particular with the combinatorial tools that describe cellular embeddings of graphs into surfaces. All embeddings in this note are cellular! The survey paper [12] by Stahl is highly recommended, since we adopt his point of view. However, the terminology and notations differ slightly. For terms not defined here the reader is reffered to any standard textbook on graph theory, e.g. Harary [3].

4. THE WEB CONSTRUCTION

Let G be a graph and let (P,s) be any of its (generalized) embedding schemes. We call the triple $(G,P,s) = S$ an *embedding* of graph G into the surface S. This abuse of language is suggested by Theorem 4 of Stahl [12]. Compare also with Ringel [10].

Given an embedding (G,P,s) we construct the *web graph* $W(G)$ together with its embedding into the same surface S as follows: let $W(G)$ be a spanning subgraph of the line graph $L(G)$; two edges e and f of G are adjacent as vertices of $W(G)$ iff $f = (e)P_v$ for some vertex v of G.

Figure 1 represents an embedding of K_5, the complete graph on 5 vertices, into S_1, the torus, together with the corresponding web graph $W(K_5)$.

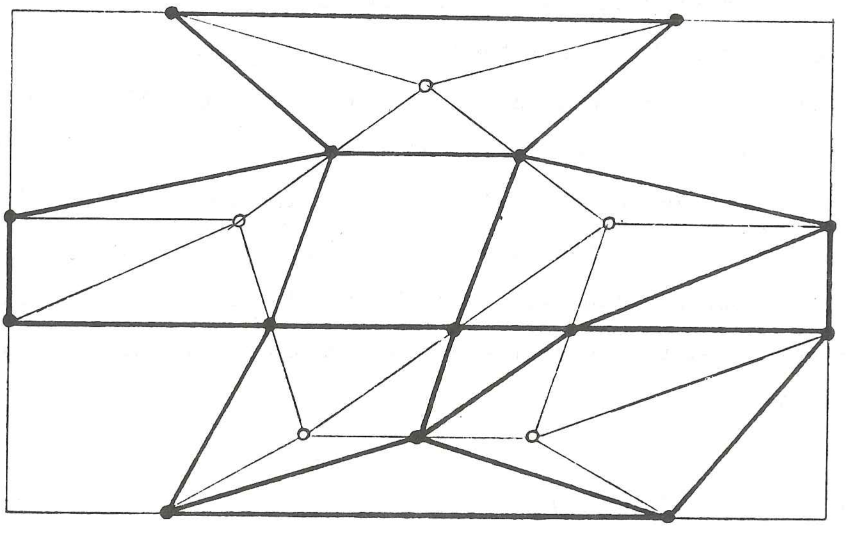


Fig.1. Embedding of K_5 and its web $W(K_5)$ on the torus

If G is a simplicial graph with no vertices of degree less than 3, then W(G) is a simplicial graph, too. We leave to the reader to fill in the detailes of the definition of W(G) if G has multiple edges, loops, or vertices of degree one or two. See Figure 2 for help.

We omit the lengthy combinatorial description of the induced embedding of the web graph.

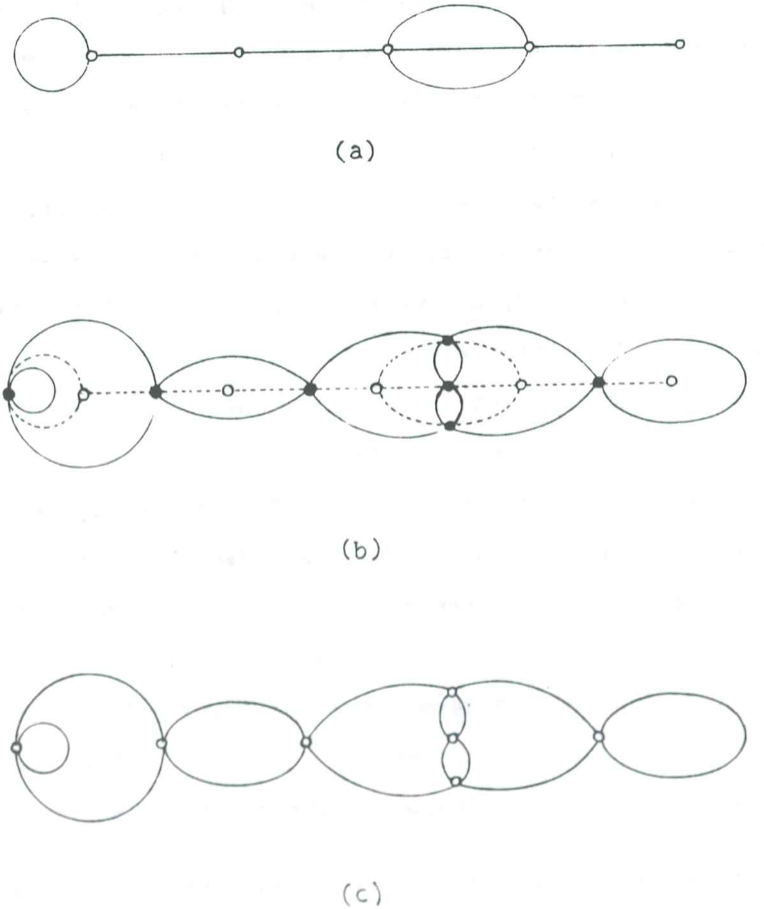


Fig.2. (a) An example of a planar graph G with loops, multiple edges, and vertices of degree one and two. (b) The web construction applied to G. (c) The web graph $W(G)$.

5. SOME PROPERTIES OF THE WEB GRAPH

5.1. Although $W(G)$ was defined using the triple (G,P,s), it turns out that it is independent of s. This can be seen if we consider the line graph $L(G)$. Recall that the Krausz partition K of $L(G)$ is the family of complete subgraphs of $L(G)$ induced by the stars at vertices of G. (See R.L. Hemminger and L.W. Beineke in [4, page 278]). Every edge of $L(G)$ is in exactly one member of the Krausz partition K and every vertex is in exactly two members of K. Now we change $L(G)$ into $W(G)$ by supstituting each complete graph from the Krausz partition with an arbitrary Hamiltonian cycle. Again, pay attention to the difficulties when G possesses vertices of degree less than 3. To set a specific web induced by say, (G,P,s), we have to choose the Hamiltonian cycles induced by P_v. This proves that $W(G)$ is independent of s. Even more, if we change P by replacing P_v by its inverse $Inv(P_v)$ at some vertices v then $W(G)$ does not change.

This discussion suggests the following definition: a *cycle Krausz partition* of a graph H is a collection C of subgraphs of H with the following three properties:

(i) Each member of C is a cycle (1 and 2-cycles allowed)

(ii) Every edge of H is in exactly one member of C.

(iii) Every vertex of H is in exactly two members of C.

5.2. Let H be a (connected) graph, then the following statements are equivalent:

(i) H is a web graph (of some graph).

(ii) H has cycle Krausz partition.

(iii) H is (connected) regular of degree 4.

We saw that (i) implies (ii) in 5.1. Now, (ii) implies (i). The cycle Krausz partition C of H can be augmented into a (usual) Krausz partition K of some L(G) by changing cycles into complete graphs. Now, apply again 5.1 to see that H = W(G) for a suitable embedding of G. For instance take the embedding (G,P,1), where P_v is induced by the corresponding (oriented) cycle of C. Here 1 denotes the constant map. It is trivial to see that (iii) follows from (ii). The converse is a trivial corollary to the Petersen´s Theorem, e.g. [3, Theorem 9.9].

5.3. Let G^* be the dual graph of graph G on the surface S = (G,P,s). Then $W(G^*) = W(G)$.

We only give a sketch of the proof. Consider the embedding of W(G) induced by (G,P,s). The regions of this embedding fall into two classes. The regions of the first class correspond to the regions of (G,P,s). The regions of the second class correspond to the local rotations P_v. Passing from G to its dual G^* we only interchange the regions and local rotations. Therefore, $W(G^*) = W(G)$. Notice, that the regions of the second class are bounded precisely by the cycles of the cycle Krausz partition of W(G) induced by P. The remaining regions correspond to the cycle Krausz partition of $W(G^*)=W(G)$

determined by the dual embedding (G^*,P^*,s^*). Since the cycles of the cycle Krausz partition are edge-disjoint we immediately obtain the following:

5.4. Let $W^*(G)$ be the dual of $W(G)$ on S, where the embedding of $W(G)$ is induced by (G,P,s). Then $W^*(G)$ is a bipartite graph, quadrilaterally embedded into S.

5.5. As a by-product of the preceding discussion we get:

(i) Every cellular embedding of a cubic graph G determines the same web graph $W(G)$, namely the line graph $L(G)$.

(ii) Every regular graph of degree 4 admits an embedding with a bipartite dual. (The embedding can be chosen to be orientable).

5.6. Lastly, it might be noted that the quadrilaterally embedded bipartite graph $W^*(G)$ corresponding to G and G^* can be nicely pictured directly from G and G^*: given a cellular embedding of G on S (with vertices represented by solid dots, and edges by solid curves, say - see Fig.3), construct G^* in the usual way (with vertices represented by open dots and edges by dashed curves); now for each edge e of G, draw "squiggly" curves connecting the endpoints of e with the vertices of G^* immediately on either side of e. The result will be a bipartite graph $B = W^*(G)$ with vertices those of G and G^* and with edges the "squiggly" curves.

Its (quadrilateral) faces contain as diagonals the pairs of corresponding edges of G and G^*.

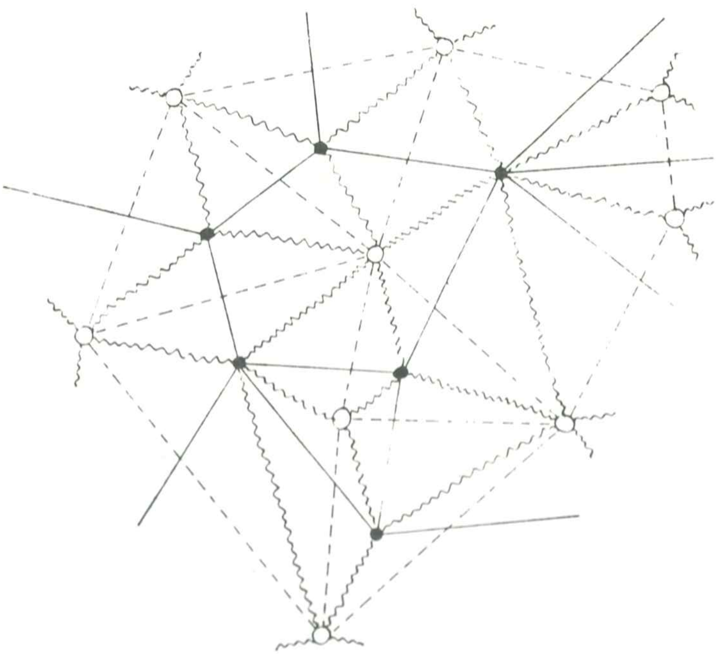


Fig.3. Construction of $B = W^*(G)$ directly from G and its dual G^*

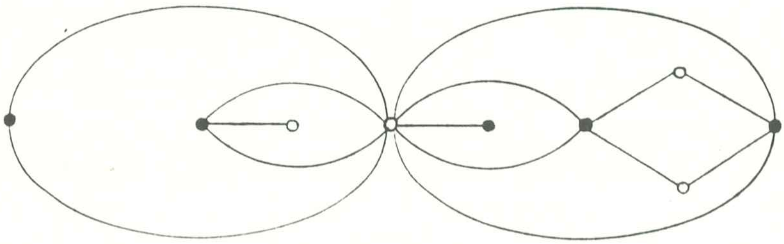
6. EXAMPLE

Take for instance the complete bipartite graph $K_{m,n}$. Its line graph $L(K_{m,n})$ is isomorphic to the Cartesian product the complete graphs: $L(K_{m,n}) = K_m \times K_n$. The Krausz partition is given by both projections. It consists of m copies of K_n and n copies of K_m. Now, we may choose spanning cycles C_n in K_n and C_m in K_m in such a way that $C_m \times C_n$ is a spanning subgraph of $L(K_{m,n})$. The graph $C_m \times C_n$ intersects each element of the Krausz partition in a Hamiltonian cycle. This proves that $C_m \times C_n = W(K_{m,n})$ for some embedding of $K_{m,n}$.

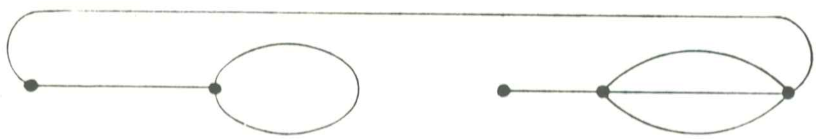
7. THE DIAGONAL CONSTRUCTION

We start with an arbitrary connected bipartite graph B, quadrilaterally embedded into some surface S. Let V_0 and V_1 be the bipartition of its vertex set V. Now we define two graphs G_0 and G_1 (together with their cellular embedding into S). Let $V(G_i) = V_i$, and $E(G_i)$ = (diagonals of quadrilateral faces of the embedding of B with their endpoints in V_i), $i = 0, 1$.

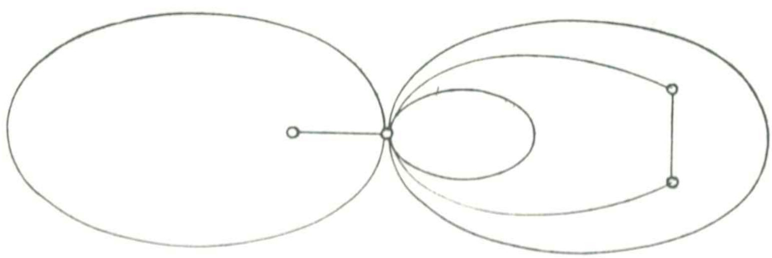
This construction $G_i = D_i(B)$, $i = 0, 1$ is called the *diagonal construction*. Notice, that the embedding of both graphs G_i is specified at the same time. Again we omit its lengthy combinatorial description. The result of the diagonal construction on the graph of Fig.4a can be seen on Fig.4b and 4c.



(a)



(b)



(c)

Fig.4. (a) The diagonal construction applied to $W^*(G)$, where G is the graph of Fig.2a. One of the two graphs obtained in this way (b) is G, and the other is its dual G^*, (c).

8. SOME PROPERTIES OF THE DIAGONAL CONSTRUCTION

8.1. The embeddings of graphs G_o and G_1 as above are cellular.

8.2. Even more! G_o and G_1 are duals on S.

8.3. It is not difficult to see that the dual of B on S is precisely the web of G_o (and/or G_1).

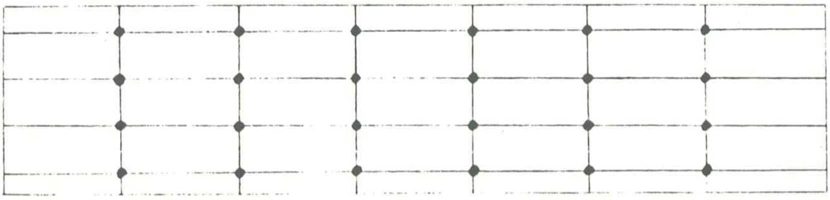
Moreover, if G and G^* are dually embedded graphs on a surface S, and if B is the dual of $W(G)$ $(= W(G^*))$, then $D_o(B) = G$ and $D_1(B) = G^*$ (although the subscript labels are arbitrary here).

This means that the study of arbitrary cellular graph embeddings can be reduced via web construction and its dual to the study of quadrilateral embeddings of bipartite graphs. No informations is lost during this process: it can be completely recovered by applying the diagonal construction.

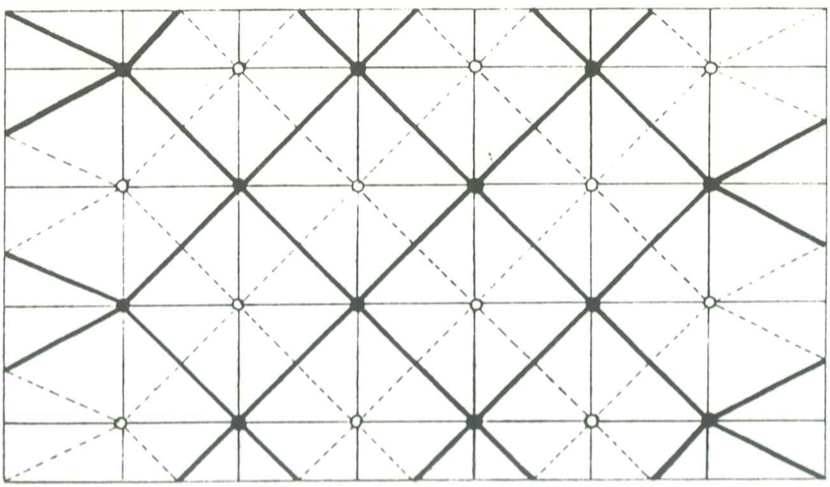
9. EXAMPLE

Take for instance the product of two even cycles $C_{2m} \times C_{2n}$. This is a bipartite graph, quadrilaterally embeddable into the torus, see A.T. White [16]. Figure 5 is showing the case $C_4 \times C_6$. If we apply the diagonal construction to $B = C_{2m} \times C_{2n}$ embedded into the torus we obtain duals $G_i = D_i(B)$, i = 0, 1. It is easy to see that G_o and G_1 are isomorphic and their union is precisely the tensor product of the cycles C_{2m} and C_{2n}: $G_o \cup G_1 = C_{2m} \otimes C_{2n}$.

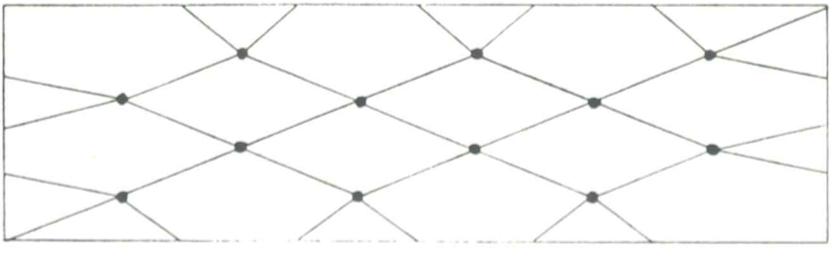
Since C_{2m} and C_{2n} are both bipartite, their tensor



(a)



(b)



(c)

Fig.5. (a) Cartesian product of cycles $C_4 \times C_6$ quadrilaterally embedded into the torus. Since this is a bipartite graph, the diagonal construction applies (b), producing two copies of the diamond graph $D_{4,6}$ (c).

product is disconnected. It consists of two identical copies of a graph $D_{m,n}$ which we call the *diamond graph*. $D_{m,n}$ consists of 2mn vertices and is regular of degree 4. This construction proves that $D_{m,n}$ admits a quadrilateral embedding into the torus which is of course self-dual. Applying the web construction to the quadrilateral embedding of $C_m \times C_n$ into the torus we get that $W(C_m \times C_n) = D_{m,n}$.

10. THE N-CUBE GRAPH

Let Q_n denote the n-cube graph, the 1-skeleton of the n-dimensional cube. It is a bipartite graph on 2^n vertices, which is regular of degree n, therefore it contains $n2^{(n-1)}$ edges. G. Ringel computed its genus long ago [8]. For some recent generalizations see Pisanski [7].

Since Q_n is a Cayley graph of the 2-group Z_2^n it is possible to label the vertices of Q_n by elements of Z_2^n, that can be regarded as binary n-tuples.

Let $H = \{h_1, h_2, \ldots, h_n\}$ denote the set of generators for the group Z_2^n, where

$$pr_j(h_i) = \delta_{ij}, \quad i,j = 1,\ldots,n$$

(The function $pr_j\colon Z_2^n \longrightarrow Z_2$ means projection on j-th component, and δ_{ij} is the Kronecker symbol).

An edge $e = uv$ of Q_n is labeled by h_i if and only if u and v differ only in the i-th component.

Define the bipartition of the vertex set $V(Q_n)$ into V_i, with $\mathrm{card}(V_i) = 2^{(n-1)}$, for $i = 0,1$ as follows:

> the vertex $u = (u_1, u_2, \ldots, u_n)$ of Q_n belongs to V_i iff $u_1 + u_2 + \ldots + u_n = i \pmod{2}$.

Now, we construct a quadrilateral orientable embedding of Q_n into the surface of genus $1 + (n - 4)2^{(n-3)}$ that will be used later on. For each vertex v define P_v, the cyclic permutation of the edges incident with v. We choose once and for all the cyclic permutation p of the set of generators H. Assume $p = (h_1, h_2, \ldots, h_n)$.

This means that $p(h_i) = h_{i+1}$ (and $p(h_n) = h_1$). Since there are exactly n edges incident with v and each edge is colored by one of the generators of H we may define the P_v in terms of cyclic permutations of H. (This argument can be used for all Cayley color graphs as was first discovered by A.T. White [14]). Of course we choose P_v to be induced by the same p for all vertices v. To complete the definition of the embedding (Q_n, P, s) we only need to specify the 2-coloring s of edges of the n-cube. In our case we choose s to be the constant -1 mapping. We leave to the reader the simple argument showing that (Q_n, P, s) is a quadrilateral embedding. It is orientable as can be proved by Theorem 5 of [11].

The conditions for the diagonal construction are fulfilled. The diagonal graphs $D_i(Q_n)$, $i = 0,1$ are isomorphic to each other. Here is the proof!

Define $f: Z_2^n \longrightarrow Z_2^n$ as $f(a) = a + h_1$, $a \in Z_2^n$. We have $f(a) + f(b) = a + b$, for all a and b.

Since a and b are adjacent in Q_n iff $(a + b)$ is an element of H, we see that f is an automorphism of the graph Q_n. (Note, that f is not a group homomorphism!). Furthermore, f interchanges the sets V_o and V_1. Concerning the embedding (Q_n, P, s), f carries the boundary of faces into the boundary of faces. Therefore, it can be extended to the homeomorphism of the surface onto itself preserving the embedding.

The preceding discussion enables us to use a simpler notation $D(Q_n)$ for both diagonal graphs and to conclude that $D(Q_n)$ possesses a self-dual embedding.

The rest of this section is dedicated to the identification of the graph $D(Q_n)$. This task is not too difficult. Consider the graph $D_o(Q_n)$ defined on the vertex set V_o. The edges of $D_o(Q_n)$ can be labeled by the elements of the group Z_2^n. Let K denote the set of n labels used. It is determined as follows: the edges are diagonals of quadrilaterals. As we know from above the edges of a quadrilateral are labeled by two generators h_i and h_{i+1}. (Non-adjacent edges of a quadrilateral are labeled by the same generator). This labeling induces the labeling of diagonals. We label the diagonal by the element $k_i = h_i + h_{i+1}$. (Obviously, we have to define k_n to be $h_n + h_1$).

Let $\langle K\rangle$ denote the subgroup generated by K. The group $\langle K\rangle$ is identified later on. Let $E = \{e_1, \ldots, e_{n-1}\}$

denote the set of standard generators for the group Z_2^{n-1}, i.e. $pr_j(e_i) = \delta_{ij}$, $i,j = 1,\ldots,(n-1)$, and let e_n be their sum $e_n = e_1 + \ldots + e_{n-1}$.
Since Z_2^{n-1} can be viewed as a vector space over the field Z_2 the set E can be viewed as its basis. Now we define a homomorphism $F\colon Z_2^{n-1} \longrightarrow Z_2^n$ by specifying it on the basis: $F(e_i) = k_i$, for $i = 1,\ldots,(n-1)$. Clearly, $F(e_n) = k_n$.

The reader may verify that $\mathrm{Ker}\, F = 0$ and $\mathrm{Im}\, F = \langle K\rangle$. Hence, F is an isomorphism between Z_2^{n-1} and $\langle K\rangle$. The group $\langle K\rangle$ is a subgroup of Z_2^n of index 2. It consists of all binary n-tuples with 0 check-sum mod 2. Therefore the vertices of V_o are labeled precisely by the elements of $\langle K\rangle$. Note, that $D_o(Q_n)$ represents the Cayley color graph for $\langle K\rangle$ if we leave its vertices labeled by elements of $\langle K\rangle$ and its edges labeled by generators from K, as we described above. Changing the labels by applying the inverse of F to the existing labels, the very same graph $D_o(Q_n)$ now serves as a Cayley color graph of the group Z_2^{n-1}, generated by the set $E \cup \{e_n\}$.

Since the first $(n-1)$ generators define Q_{n-1} it is obvious that the last one, i.e. $e_n = (1,1,\ldots,1)$, defines additional 2^{n-2} edges, joining antipodal vertices of Q_{n-1}.

Let Q_n^+ denote the *augmented n-cube graph*, i.e. the Q_n plus all edges joining antipodal vertices. We proved that $D(Q_n) = Q_{n-1}^+$.

11. SOME PROPERTIES OF THE AUGMENTED CUBE

11.1. It is easy to see that the augmented cube Q_n^+ is bipartite iff n is odd.

11.2. For $n \geq 3$ the girth of Q_n^+ is 4. The length of any odd cycle in Q_n^+ is at least $n+1$. Obviously, odd cycles exists only for n even: $n = 2m$. We could say that Q_{2m}^+ is "almost bipartite" since it has no "short" cycles of odd length.

This property of the augmented cubes suggests the introduction of some new graph invariants, e.g. *even (odd) girth* of graph G, i.e. the length of its shortest even (odd) cycle.

11.3. The augmented cube graph Q_n^+ admits a self-dual orientable embedding which is $(n+1)$-gonal as follows from the diagonal construction of section 10 applied to Q_{n+1}. Combining this fact with 11.2 we deduce that any embedding of Q_n^+ into an orientable surface of genus less than $1 + (n-3)2^{(n-2)}$ (this is the genus of Q_{n+1}), must exhibit at least one face bounded by a cycle of even length!

We may speak about *even (odd) embeddings* in the case of 2-cell embeddings with all faces even (odd). We may introduce in the obvious way the notion of *even (odd) genus* of a graph G. We may even talk about *even (odd) non-orientable genus*. In the case of bipartite graphs the even genus coincides with the (usual) genus and the odd genus is infinite (doesn´t exist).

11.4. It is possible to find a quadrilateral embedding for Q_n^+ which is orientable for odd n and non-orientable for even n, $n \geq 3$. Therefore, the orientable genus of Q_{2m+1}^+, $m \geq 1$, is $\gamma(Q_{2m+1}^+) = 1 + 2(m-1)4^{m-1}$ while the non-orientable genus of the graph Q_{2m}^+, $m \geq 2$, is $\tilde{\gamma}(Q_{2m}^+) = 2 + (m-1)4^m$.

The upper bound for the non-orientable genus of Q_{2m+1}^+, $m \geq 1$, $\tilde{\gamma}(Q_{2m+1}^+) \leq 3 + (m-1)4^m$ is obtained using the fact $\tilde{\gamma}(G) \leq 2\cdot\gamma(G) + 1$. The upper bound for the orientable genus of Q_{2m}^+, $m \geq 2$, $\gamma(Q_{2m}^+) \leq 1 + (m-1)4^{m-1}$ is obtained by exhibiting an ad hoc embedding of Q_{2m}^+ with the property that each vertex is incident with $2m-1$ quadrilaterals and 2 octagons.

11.5. Remark. We saw above that the augmented cube graph admits a self-dual orientable embedding. This fact is not new. It follows from a more general result of Stahl [13, Lemma 3]. According to Stahl this lemma was proved independently by Bouchet in 1975. Bouchet´s result, in turn, generalizes those of White and Pengelley. Unfortunately, these results were inaccessible to the authors.

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A NOTE ON INFINITE GENERALIZED LINE GRAPHS

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ABSTRACT

It is proved in [4] that a connected infinite graph G is a generalized line graph if and only if its specially defined least eigenvalue $\lambda(G)$ is ≥ -2. Here we offer another shorter proof of this fact. Besides, we describe the automorphism group of an infinite generalized line graph.

1. INTRODUCTION

Throughout the paper, a graph G is a countable connected infinite graph, without loops or multiple edges, whose vertex set is $V(G) = N$ (the set of natural numbers).

If P, Q are arbitrary two finite or infinite graphs, then $P \subseteq Q$ means that P is an induced subgraph of Q and $P \subset Q$ means that P is a subgraph of Q (in a wider sense).

If G is a connected infinite graph, then its least "limiting eigenvalue" $\lambda(G)$ (see [4]) can be defined by

$$\lambda(G) = \operatorname{Inf}\{\lambda(F_n) \mid F_n \subseteq G\} ,$$

where the infimum is taken over all finite induced subgraphs F_n of G with n vertices $(n = 2,3,\ldots)$. In the general case we have that $\lambda(G) \geq -\infty$. The question when $\lambda(G)$ is finite remains open [4].

Throughout the paper, the abbreviations "GLG" and "GLIG" will denote a finite generalized line graph and an infinite generalized line graph, respectively. If $G = L(H;\bar{k})$ is any GLIG with a (finite or infinite) root graph H, and the corresponding sequence of cardinals $\bar{k} = (k_\alpha)$ $(k_\alpha \leq \aleph_0)$, then $C(\bar{k}) = \bigcup_\alpha CP(k_\alpha)$ is the underlying union of cocktail party graphs of G (induced subgraph of G). The root graph H is obviously connected, and if it is finite then at least one between cocktail party graphs $CP(k_\alpha)$ in G must be infinite.

If next, $G_i = L(H_i;\bar{k}_i)$ $(i \in N)$ is a sequence of GLGs, then for brevity, the induced subgraphs $C(\bar{k}_i) = \bigcup_\alpha CP(k_{i\alpha})$ of G_i are denoted by C_i.

In this paper we prove that a connected infinite graph G is a GLIG if and only if it has the property $\lambda(G) \geq -2$. The proposed proof is shorter than this in [4], which is based on infinite root systems, and needs some recent results of D. Cvetković, M. Doob and S. Simić ([1] or [2]). In the last section, we consider the automorphism group of a GLIG in the sense of definition given in [3], and find all GLIGs for which this group is non-trivial.

2. GRAPHS WITH THE PROPERTY $\lambda(G) \geq -2$

First, let G be any connected infinite graph and G_i ($i \in N$) be a strictly increasing (in the sense of induced subgraphs) sequence of its finite connected induced subgraphs. We call G_i a <u>good</u> sequence if all of G_i are GLG, and $G_i = L(H_i;\bar{k}_i)$ where

a) $\bigcup G_i = G$

b) both H_i (as subgraphs) and $C_i = C(\bar{k}_i)$ (as induced subgraphs) increase in i.

<u>*Lemma.*</u> *G is a GLIG if and only if it has a good sequence of its induced subgraphs.*

<u>*Proof.*</u> If $G = L(H;\bar{k})$ is a GLIG, it obviously has the mentioned property. The proof is slightly different depending on whether the root graph H is finite or infinite.

Conversely, assuming that G_i is a good sequence for G, and denoting $\bigcup H_i = H$, $\bigcup C(\bar{k}_i) = C(\bar{k})$, we easily have that $G = L(H;\bar{k})$, q.e.d.

<u>*Theorem 1.*</u> *If two connected GLIGs are isomorphic, then their root graphs and their underlying CP graphs are isomorphic, too.*

<u>*Proof.*</u> Let $G = L(H;\bar{k})$ and $G' = L(H';\bar{k}')$ be such two GLIGs, and $\omega : G \to G'$ be the corresponding isomorphism. Next, let $G_i = L(H_i;\bar{k}_i)$ be a good sequence of connected induced subgraphs of G. Then, obviously, $\omega(G_i)$ is a sequence of connected GLG which are induced subgraphs of the graphs G'. Let $\omega(G_i) = L(M_i;\bar{\ell}_i)$. Then by theorem 2.6 of [2], for $|G_i| \geq 5$

we have that H_i, M_i as well as $C(\bar{k}_i)$, $C(\bar{\ell}_i)$ are isomorphic. Consequently, we easily find that $\omega(G_i)$ is a good sequence in G'. Therefore, we have that $H' = \bigcup \omega(H_i)$, and $C(\bar{k}') = \bigcup \omega(C(\bar{k}_i))$, whence we get that H and H' as well as $C(\bar{k})$ and $C(\bar{k}')$ are isomorphic.

We now prove the main theorem.

Theorem 2. A connected infinite graph G *is a* GLIG *if and only if* $\lambda(G) \geq -2$.

Proof. Assuming, first, that G is a GLIG and $G_i = L(H_i;\bar{k}_i)$ a good sequence of subgraphs in G, we get that $\lambda(G_i) \geq -2$ for each i. Whence, obviously,

$$\lambda(G) = \lim_{i\to\infty} \lambda(G_i) > -2 .$$

Next, assume that $\lambda(G) \geq -2$ and prove that G must be a GLIG. Note that $\lambda(G) \geq -2$ is equivalent to the condition $\lambda(F_n) \geq -2$ valid for all finite connected $F_n \subseteq G$ on n vertices $(n \in N)$.

First, as in [2], consider the next equivalence relation $\sim$ in the set $V(G)$: $x \sim y$ if and only if they are not adjacent and have the same neighbours.

We want to prove that there are no three equivalent vertices in G. Indeed, in the contrary case, there is a connected induced subgraph $F_n \subseteq G$ which contains these vertices, and these vertices are equivalent in F_n. But since, for a sufficiently large n, F_n must be a GLG (for $\lambda(F_n) \geq -2$), we get a contradiction.

Consider, next, the pairs of equivalent vertices in G (if such pairs exist). If a,b and c,d are two such pairs, then it is easily seen that there is either none or all possible edges joining these pairs. This fact immediately gives that all pairs of equivalent vertices in G (if such pairs exist) are concentrated into a (finite or infinite) sequence of independent cocktail party graphs $CP(k_i)$, each of which is either finite or infinite.

Now choose an arbitrary strictly increasing sequence G_i of connected induced subgraphs of G, such that $\bigcup G_i = G$. We can suppose that all G_i are GLGs, since they must be such at least for $|G_i| \geq 37$. Additionally, one can suppose that in each G_i two vertices are equivalent if and only if they are equivalent in G. Indeed, in the contrary case, we can do following:

1) for any two equivalent vertices a,b from G_i which are not equivalent in G, remove from G_i (and also from all other $G_k, k > i$, in which a,b are equvalent) exactly one of these vertices;

2) for any vertex a from G_i which is without equivalent vertex in G_i, but with equivalent vertex b from G, add b to G_i and to all other $G_k, k > i$. Then the subgraphs $\tilde{G}_i$ so obtained are connected, we have that $\tilde{G}_i \subseteq \tilde{G}_{i+1}$, $\bigcup \tilde{G}_i = G$, and as is easily seen, all $\tilde{G}_i$ have the mentioned property.

In such a way we have that two vertices a,b from G_i are equivalent in G_i if and only if they are equivalent in G, and consequently if and only if they are equivalent in G_{i+1}. From this it is clear that if $G_i = L(H_i;\bar{k}_i)$ and $G_{i+1} = L(H_{i+1};\bar{k}_{i+1})$, then there exists an $\bar{H} \subset H_{i+1}$ and a $C(\bar{k}) \subseteq C(\bar{k}_{i+1})$ such that $G_i = L(\bar{H};\bar{k})$. But then, by Theorem 2.6 [2] (for $|G_i| \geq 5$), we have that H_i and $G(\bar{k}_i)$ must be isomorphic to $\bar{H}$ and $C(\bar{k})$, respectively. So we get that $H_i \subset H_{i+1}$ and $C(\bar{k}_i) \subseteq C(\bar{k}_{i+1})$. In such a way, we have proved that the sequence G_i is good, and Lemma completes the proof.

3. THE AUTOMORPHISM GROUP OF A GLIG

In this section we consider the automorphism group (c.f. [3]) of an infinite generalized line graph.

If G is an arbitrary connected infinite graph, consider all permutations $P = [p_{ij}]$ of G (each p_{ij} is 0 or 1) satisfying the relation $AP = PA$, where $A = [a_{ij}]$ is the adjacency matrix of G defined by $a_{ij} = a^{i+j-2}$ if i,j are adjacent, and $a_{ij} = 0$ in the contrary case (a is a fixed positive constant less than 1). Each permutation P with this property saves the adjacency relation in G, but the converse is not true. The group of all such permutations on G is denoted by $\Gamma(G)$ and called the automorphism group of G.

In [3], we proved that $|\Gamma(G)| \leq 2$, and described the graphs with the non-trivial automorphism group. In this section we describe GLIGs with the non-trivial automorphism group. If $\Gamma(G)$ is non-trivial, then ([3]) a special relabelling on $V(G)$ must be performed.

Theorem 3. The infinite two-way path is the only GLIG *with a non-trivial automorphism group.*

Proof. As is proved in [3], a graph with non-trivial automorphism group is bipartite and, hence, it contains no triangles. If $G = L(H;\bar{k})$ is such a GLIG, then obviously the degrees of all vertices in H must be at most 2, thus H is two-way or one-way infinite path. Additionally, each k_i is ≤ 1 and we have that $k_i = 0$ for each vertex i of degree 2.

Assuming, first, that H is an one-way infinite path with the end-point i, we easily get that the non-trivial automorphism $\omega \in \Gamma(G)$ satisfies $\omega(i) = i$, in the both possi-blle cases $k_i = 0$ or 1. But then (Lemma 2 [3]) ω must be identity, contradiction. The rest of the proof is then trivial.□

As a trivial consequence of Theorem 3, we remark that if a GLIG and its root graph have non-trivial automorphism groups, then these groups are isomorphic.

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ON INFINITE GRAPHS WHOSE SPECTRUM IS UNIFORMLY BOUNDED BY $\sqrt{2+\sqrt{5}}$

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ABSTRACT

In this paper, all connected infinite graphs whose spectrum $\sigma(G;a)$ is uniformly bounded by $\varepsilon = \sqrt{2+\sqrt{5}}$, i.e. whose spectrum lies in the interval $[-\varepsilon,\varepsilon]$ for each value of the parameter $a \in (0,1)$, are found.

1. INTRODUCTION

Throughout the paper, by an infinite graph G we mean a connected countable (undirected) infinite graph without loops or multiple edges, with the vertex set $V(G) = N$ (set of natural numbers).

The adjacency matrix $A(G) = [a_{ij}]$ of G is an infinite $N\times N$ matrix, where $a_{ij} = a^{i+j-2}$ if v_i, v_j are adjacent, and $a_{ij}=0$ otherwise, and <u>a</u> is a fixed positive constant $(0<a<1)$. Hence, the whole graph G is labelled and weighted, and the weight at the vertex v_i is $w_i = a^{i-1}$ [8]. Then we briefly say that G has $(0-a)$ adjacency matrix.

Matrix $A(G)$ is the matrix of a symmetric Hilbert-Schmidt operator A in a separable Hilbert space H, with respect to a fixed orthonormal basis $\{e_i\}$ in H.

The spectrum $\sigma(G)$ of G is the spectrum of this operator; it consists of the zero and of a sequence λ_n of real eigenvalues of finite multiplicities, which is finite (for graphs with finite spectrum) or infinite. If it is infinite, then $\lambda_n \to 0$ $(n\to\infty)$.

In each case,

$$\sigma(G) = \{\lambda_1, \lambda_2, \ldots\} \cup \{0\} .$$

In the general case, spectrum $\sigma(G)$ of G depends on the constant $\underline{a}$ $(0<a<1)$ and on the labellings of the vertex set $V(G)$. To stress it, we sometimes write $\sigma(G) = \sigma(G;a)$.

The maximal and the minimal eigenvalue of G are denoted by $r(G;a)$ and $\lambda(G;a)$ respectively. It is known that $r(G;a) = \|A(G)\|$, and the whole spectrum $\sigma(G)$ lies in the interval $[-r(G;a), r(G;a)]$.

Next, we need the following results (Proposition 1 and Lemmas 2-3), which have been proved in [10].

Proposition 1.

1^o *The function* $r(G;a)$ $(0<a<1)$ *is strictly increasing in* a;

2^o *The limit*

$$r(G) = \lim_{a\to 1} r(G;a) = \sup\{r(G;a) \mid 0<a<1\} \leq +\infty$$

exists for any G;

3^o *We have*

$$r(G) = \sup\{r(H_n;1) \mid H_n \subseteq G\} ,$$

where the supremum is taken over all finite induced subgraphs $H_n \subseteq G$ $(n \in N)$ *on* n *vertices.*

It is immediate from 3^o that the invariant $r(G)$ of G coincides with the spectral radius of G considered by B.Mohar in [6]. It is proved in [6] that $r(G)$ is finite iff G has bounded degrees, i.e. $\sup \deg(i) < +\infty$.

The next lemma is obvious.

<u>Lemma 1</u>. *For any connected infinite graph* G, *there always exists a sequence of connected induced subgraphs* $G_{n_i} \subseteq G$ *such that* $G_{n_i} \subseteq G_{n_{i+1}}$ *and* $\bigcup G_{n_i} = G$.

<u>Lemma 2</u>. *For each monotonically increasing sequence* $\{G_{n_i}\}$ *of connected induced subgraph* G_{n_i} *of* G *satisfying* $\bigcup G_{n_i} = G$, *we have that*

$$\lim_{i\to\infty} r(G_{n_i};1) = r(G) \quad .$$

<u>Lemma 3</u>. *For each* $a \in (0,1)$, *we have*

$$r(G;a) < r(G) \quad .$$

For any finite induced subgraph $G_n \subseteq G$, *we have*

$$r(G_n;1) < r(G) \quad .$$

Now, for any $r>0$, denote by $M(r)$ the class of all connected infinite graphs whose spectrum is uniformly bounded by r, that is $\sigma(G;a) \subseteq [-r,r]$ (for any $a \in (0,1)$), or equivalently $r(G) \le r$.

Graph G is called a graph with uniformly bounded spectrum if $G \in M(r)$ for some $r>0$. By [6] we then have that a graph $G \in M(r)$ for some $r>1$ if and only if it has bounded degrees d_i of vertices, thus $d_i \le K$ for each $i \in N$.

The problem that we consider in this paper reads: Describe the class $M(\varepsilon)$ for $\varepsilon = \sqrt{2+\sqrt{5}} \approx 2,06$.

The similar problem for finite graphs is considered and solved by J.H.Smith [7] and D.Cvetković, I.Gutman [3] for r=2 and D.Cvetković, M.Doob, I.Gutman [1] for r=ε. Here, we shall use their solutions, and some results of A.J.Hoffman.

A general property of the class $M(r)$, for any $r>1$, is given by the following theorem.

Theorem 1. Each connected infinite graph G with the uniformly bounded spectrum has, for any $a \in (0,1)$, *an infinite spectrum.*

Proof. Indeed, assume that $r(G) \leq K$ for some $K>0$. Then the valencies d_i satisfy $d_i \leq d = K^2$ for each $i \in N$. Hence, obviously, each characteristic subset N_p of G ($p = 1, 2, \ldots$) can have at most $\underline{d}$ elements, thus since $V(G) = \bigcup N_p$, there exists an infinite number of characteristic subsets in G (see [9]). But this means that G has an infinite type, which is by a result of [9], equivalent to the claim that $\sigma(G)$ is infinite (for any $0<a<1$).

2. ON THE CLASS $M(\varepsilon)$

1. We first consider the class $M(2)$. The main result concerning this class is following.

Theorem 2. Infinite graphs P_∞, P_∞^+, Z_∞ (Fig.1) *are the unique connected infinite graphs with the property* $r(G) \leq 2$.

Proof. By Proposition 1 (3^o) and Lemma 3, we have that $r(G) \leq 2$ if and only if $r(G_n;1) \leq 2$ for each induced subgraph $G_n \subseteq G$, and arbitrary $n = 2,3,\ldots$.

Let next, G be any infinite connected graph with the property $r(G) \leq 2$. Then, by Lemma 1, there is a sequence of connected induced subgraphs $G_{n_1}, G_{n_2}, \ldots$ in G such that $G_{n_i} \subset G_{n_{i+1}}$

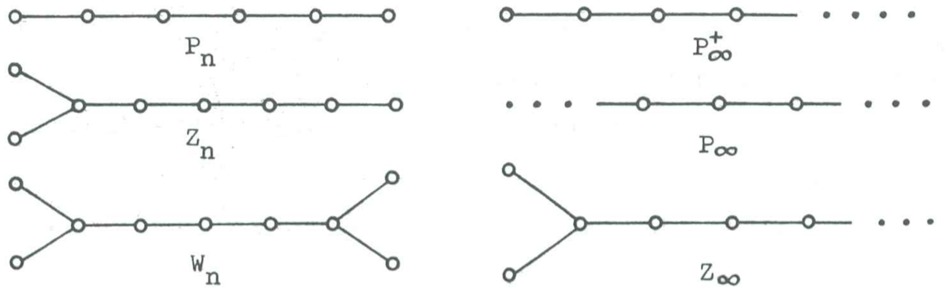

Fig.1.

and $\bigcup G_{n_i} = G$. Consequently, $r(G_{n_i})<2$, and according to [7] or [3] , for $n_i \geq 10$, each of subgraphs G_{n_i} must be one of the graphs P_{n_i}, Z_{n_i}, W_{n_i} (Fig.1).

We prove that $G_{n_i} \neq W_{n_i}$. Indeed, in the contrary case, each of the cases $G_{n_{i+1}} = P_{n_{i+1}}$, $Z_{n_{i+1}}$, $W_{n_{i+1}}$ would be impossible. Therefore, each G_{n_i} $(n_i \geq 10)$ must be one of the graphs P_{n_i}, Z_{n_i} . But if $G_{n_i} = Z_{n_i}$ for some $i = i_o$, then easily $G_{n_k} = Z_{n_k}$ $(k \geq i+1)$, and immediately, $G = \bigcup G_{n_i} = Z_\infty$. If, finally, each $G_{n_i} = P_{n_i}$, we obviously have that $G = \bigcup G_{n_i} = P_\infty$ or $G = P_\infty^+$.

Conversely, if G is any of graphs P_∞, P_∞^+, Z_∞ , then by Proposition $1(3^o)$ $r(G) \leq 2$, which completes the proof. □

Next, since any graph $G \in M(2)$ obviously contains an infinite path, and since

$$\sigma(P_n) = \{2 \cos \frac{k\pi}{n+1} \mid k=1,\ldots,n\} ,$$

we have that $r(P_n) \to 2$ $(n \to \infty)$, whence $r(G) = 2$.

Hence, we have

<u>*Corollary*</u>. *Any infinite connected graph* G *with the property* $r(G)\leq 2$ *is bipartite, and its spectral radius* $r(G)=2$.

<u>*Remark*</u>. Example of the graph Z_∞ shows that for connected infinite graphs the equality $r(G) = r(G')$ can hold, with G' as a proper induced subgraph of G, even if $G' = G-v$ $(v \in V(G))$.

2. Now, we can consider the class $M(\varepsilon)$, more precisely we describe the set $M(\varepsilon)\setminus M(2)$.

In the paper [1], D.Cvetković, M.Doob and I.Gutman described the class $M(\sqrt{2+\sqrt{5}})$ for finite graphs, in fact, they found all finite graphs G_n with the property $2<r(G_n;1)\leq \varepsilon = \sqrt{2+\sqrt{5}}$. Following this, we do similarly for infinite connected graphs, finding all connected infinite graphs with the property $2<r(G)<\varepsilon$, or by Lemma 3, with the equivalent property $2<r(G_n;1)\leq\varepsilon$, for any finite induced subgraph $G_n\subseteq G$. The key-lemma is again Proposition 1 (3^o).

In the sequel, we again explicitly use the corresponding solution from [1], together with two classes of finite graphs $T(a,b,c)$, $S(a,b,c)$ $(a,b,c\geq 1)$ (Fig.2), appearing in [4] and [1].

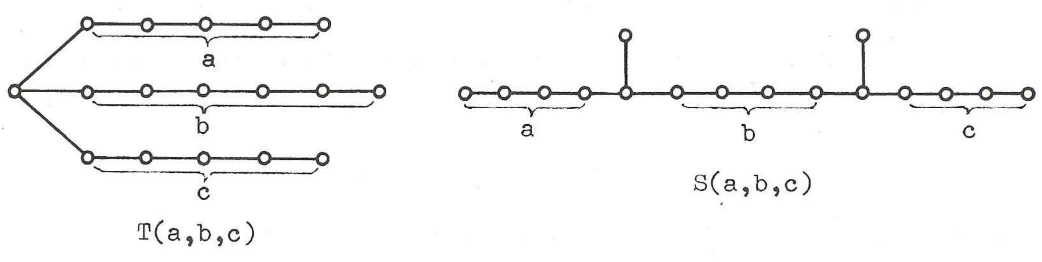

Fig.2.

We shall also use the infinite graphs $T(a,\infty,\infty)$, $S(\infty,h,\infty)$ and $S(a,h,\infty)$ $(a,h\geq 1)$, defined in an obvious way.

We note another interesting fact.

There is a closed connection between our problem and the problem considered by A.J.Hoffman [4]. Namely, by Proposition 1, the values $r(G)$ of spectral radii of infinite graphs are exactly the limit points of spectral radii of finite graphs, investigated in the mentioned paper of A.J.Hoffman. Moreover, some of his results are immediately applicable to our situation.

The main auxiliary results are the following.

Lemma 4. ([4]). *The spectral radii of graphs* $T(1,n,n)$, $S(n,h,n)$, $S(a,h,n)$ $(a,h\geq 1)$ *converge as* $n\to\infty$.

Proposition 2. *If* $G = T(1,\infty,\infty)$, *then* $r(G)=\varepsilon=\sqrt{2+\sqrt{5}}$.

Proof. By Proposition 1, we have

$$r(G) = \lim_{n\to\infty} r(T(1,n,n)) ,$$

and by a Hoffman's result [4], this limit equals ε.

Proposition 3. *If* $G=T(2,2,\infty)$, *then* $r(G) = \varepsilon$.

The proof is similar to the proof of Proposition 2.

Proposition 4. *If* $G=T(1,h,\infty)$ $(h\geq 2)$, *then* $2<r(G)<\varepsilon$.

Proof. Indeed, $r(G)>2$ and $r(G)\leq r(T(1,\infty,\infty))=\varepsilon$.

Now, $r(T(1,h,\infty))$ is the maximal root of a characteristic equation $F_{h,n}(\lambda)=0$, and assuming that it converges to ε as $n\to\infty$ we obtain that $\lambda=\varepsilon$ is the root of a limit equation. But, by straightforward calculations, we obtain a contradiction. We omit the details about the last equation.

Proposition 5. *If* $G=S(\infty,h,\infty)$ $(h\geq 1)$, *then* $r(G)>\varepsilon$.

Proof (partial). Obviously, $r(S(\infty,h,\infty))\geq r(T(1,\infty,\infty)) = \varepsilon$. The proof that $r(G)\neq\varepsilon$ is similar to the corresponding one in the previous proposition, and we omit the details.

Proposition 6. If $G=S(a,h,\infty)$ $(a,h\geq 1)$, *then* $r(G)>\varepsilon$.

Proof. Obviously, $r(S(a,h,n))\geq r(S(1,h,n))$. However, A.J.Hoffman [4] proved that

$$r(S(1,h,n))>r(T(1,n,n)) \quad ,$$

for n sufficiently large, whence as $n\to\infty$, we obtain $r(S(1,h,\infty))\geq\varepsilon$.

We omit the proof that $r(G)\neq\varepsilon$.

Theorem 3. Graphs $T(1,\infty,\infty)$, $T(2,2,\infty)$ *and* $T(1,h,\infty)$ $(h\geq 2)$ (Fig.3) *are the only connected infinite graphs with* $2<r(G)\leq\varepsilon$.

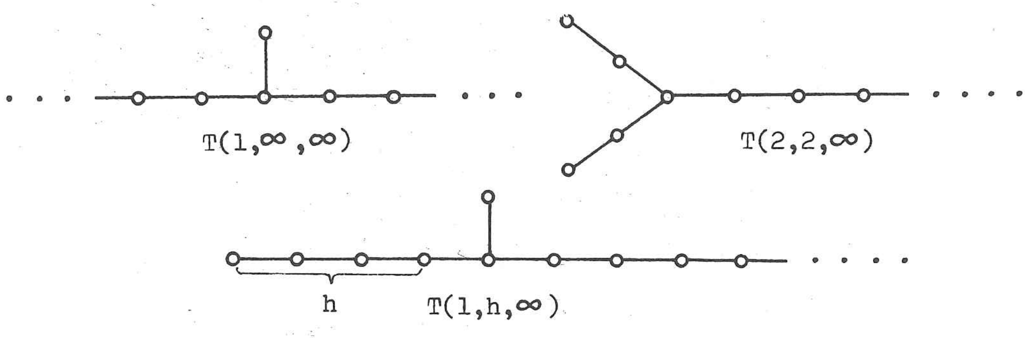

Fig.3.

Proof. By Propositions 2,3 and 4, all the mentioned infinite graphs have the property $r(G)\in(2,\varepsilon]$.

Conversely, let G be any connected infinite graph with $2<r(G)\leq\varepsilon$, and $\{G_{n_i}\}$ be a fixed increasing sequence of induced subgraphs of G such that $\bigcup G_{n_i}=G$. Then by Lemmas 2 and 3, for i sufficiently large, we have

$$2<r(G_{n_i};1)<\varepsilon \quad ,$$

whence, by the main result in [1] , each G_{n_i} for sufficiently large i , must be one of graphs $T(1,h,n)$ $(h\geq 2)$, $T(2,2,n)$, $S(a,b,n)$ $(a,b\geq 1)$. Hence, one easily concludes that all candidates for G are the graphs $T(1,\infty,\infty)$, $T(2,2,\infty)$, $T(1,h,\infty)$ $(h\geq 2)$, as well as the graphs $S(\infty,h,\infty)$ $(h\geq 1)$ and $S(a,h,\infty)$ $(h\geq 1)$.

However, Propositions 5 and 6 exclude the last two graphs, which completes the proof.

3. ON THE CLASS M(r) CONTAINING ONLY BIPARTITE GRAPHS

We observe that class $M(r)$ with $r=\sqrt{2+\sqrt{5}}$ contains only bipartite infinite graphs, and pose the following question: Find maximal $r>0$ such that the class $M(r)$ contains only bipartite graphs.

Theorem 4. $r=3/\sqrt{2}$ *is the maximal* r *such that the class* $M(r)$ *contains only bipartite graphs.*

Proof. We immediately have that the desired value $r=r_o$ satisfies

(1) $$r_o = \inf\{r(G) \mid G \text{ infinite and not bipartite}\} .$$

Now, use the following characterization of non-bipartite graphs: G is non-bipartite if and only if it contains an odd cycle as an induced subgraph.

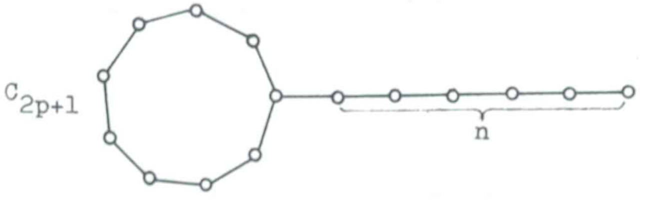

Fig.4.

Since all the considered $r(G)$ in (1) can be assumed - finite, we immediatelly conclude that each G from (1) contains an odd cycle and a sequence of paths joined with the vertices of cycle, whose lengths tend to infinity.

Consider now the following finite graph T_n : (Fig.4), and denote $r_n = r(T_n)$.

We have that $r_n \leq r_{n+1}$, and for each G from (1),

$$r(G) \geq \lim_{n\to\infty} r(T_n) = \rho \quad .$$

But now, one can establish that ρ is the greatest root of the equation

$$P_{2p+1}(\theta+P_{2p-1}) = (P_{2p}+1)^2 \quad ,$$

where $\theta = (\lambda+\sqrt{\lambda^2-4})/2$, and $P_m = (\theta^m-1/\theta^m)/2$ $(m\in N)$ (see also [4]) .

By a simplification, the last equation reduces to

$$\theta^{2p-1}(\theta^2-2) = 1 \quad ,$$

whence if $\rho=\theta(p)$, we immediately have that $\theta(p)>\theta(p+1)$, $\theta(p)\to\sqrt{2}$ $(p\to\infty)$, the corresponding value $\lambda(p)>\lambda(p+1)$, and $\lambda(p)\to 3/\sqrt{2}$ $(p\to\infty)$. Therefore, for each G from (1) we have $r(G)\geq 3/\sqrt{2}$; moreover $r_o=3/\sqrt{2}$ (≈ 2.12) .

Finally, we find at least one bipartite infinite graph G with $r(G)=3/\sqrt{2}$. Consider the graph $T=T(\infty,\infty,\infty)$ from the next figure.

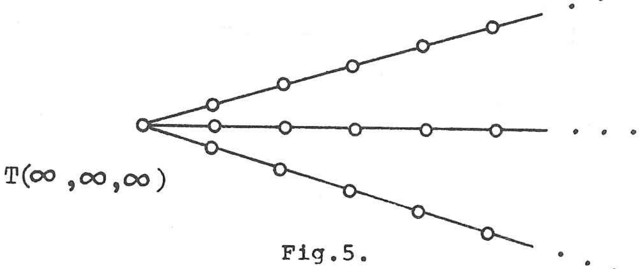

Fig.5.

By considering the corresponding finite graph T(n,n,n) and putting $n \to \infty$, we immediately have that $r(G)=3/\sqrt{2}$, which completes the proof.

Remark. It is interesting to note that in similar theorems for the finite case, as a rule, some exceptional graphs exist. Such exceptions, which represent both difficulty and challenge, usually do not exist in the infinite case.

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AN ANALYSIS OF SOME PARTIZAN GRAPH GAMES

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ABSTRACT

A class of partizan graph games is defined which are neither normal nor misère. For the progressively finite game digraphs the winning strategies are determined. Some take-away games are treated in this way and solved completely.

1. INTRODUCTION

The games we shall consider are two-person games without chance moves (such as rolling dice or shuffling cards) and with complete information (i.e. both players know what is going on). All definitions are taken from [1] and [2]. Two players we shall call A and B (First and Second). There are several, usually finitely many, posiitons, and often a particular starting position. There are clearly defined rules that specify the moves that either player can make from a given position to its options.

A and B move alternately, in the game as a whole. A play of a given game come to an end when some player is unable to move under the rules of the game. In the normal play convention a player unable to move loses. In the misère play convention a player unable to move wins.

Games in which from any position exactly the same moves are available to either player are called impartial. Games in which the two player may have different options are called partizan.

Digraphs are natural representation of such games. The vertices represent the positions in the game and the edges represent the moves. There is a directed edge from vertex u to v iff the game can be transformed from position u to v by a move permissible under the rules of the game. In the game digraph of a partizan game not all edges are permissible moves for each player.

2. THE RESULTS

As the game digraph we shall consider a digraph satisfying some special conditions.

Let $G = (V,E)$ be a digraph and let E_A and E_B be two subsets of E such that $E_A \cup E_B = E$. It is not necessary for E_A and E_B to be disjoint. It is sometimes suitable to consider two digraphs $G_A = (V,E_A)$ and $G_B = (V,E_B)$ with the same set of vertices V. We call elements of E_A and E_B A-arcs and B-arcs respectively. We can consider G as game digraph in which A-arcs represent the permissible moves for the first player (player A) while B-arcs represent the permissible moves for the second player (player B).

Let $T \subset V$ be the set of terminal vertices of G, and T_A and T_B the sets of terminal vertices of G_A and G_B respectively.

Suppose that for some $J_A \subset V$ and $J_B \subset V$, the following conditions are satisfied:

(a) $T \subset J_A \cup J_B$,

(b) $T_A = T_B = T$,

(c) For any nonterminal vertex $u \in V \setminus J_B$, there is a $v \in J_A$ such that $uv \in E_A$,

(d) For any $uv \in E_A$ such that $u \in J_B$, $v \notin J_A$,

(e) For any nonterminal vertex $u \in V \setminus J_A$, there is a $v \in J_B$ such that $uv \in E_B$,

(f) For any $uv \in E_B$ such that $u \in J_A$, $v \notin J_B$.

Each directed path from the starting vertex to a closing vertex in which there are neither two adjacent arc from $E_A \setminus E_B$ nor two adjacent arcs from $E_B \setminus E_A$ represents one complete play of the game. This path consists of arcs representing the moves of two players alternately. We shall say that the first player A is the winner if the terminal vertex of this path is in $J_A \setminus J_B$, and that the second player B is the winner if the terminal vertex is in $J_B \setminus J_A$, independently of the fact which player brings the game to that closing position. That is why we say that this game is neither normal nor misere (nnnm). If the terminal vertex is in $J_A \cap J_B$, the play is draw.

Theorem 1. Let the game digraph of a nnnm *game satisfies the conditiones* (a)-(f). *Then:*

(i) If the starting vertex is not in J_B, *then the second player* B *has not a winning strategy.*

(ii) If the starting vertex is in J_B, *then the first player* A *has not a winning strategy.*

Proof. (i) Suppose that starting vertex $u \notin J_B$. If it is a terminal vertex, it must be in $J_A \setminus J_B$, and the first player A is the winner. If u is a nonterminal vertex, then according to (c), A can by his first move select a vertex $v \in J_A$. If v is a terminal, the game ends in a position which is not in $J_B \setminus J_A$, so either A is the winner or the play is draw. If v is not terminal, then according to (f), the second player B must select some vertex $w \in V \setminus J_B$. If w is a terminal vertex, then $w \in J_A \setminus J_B$ and A is the winner. If w is not terminal, then the play continues in such a way that whenever A takes turn, he can always select a vertex in J_A, while the second player B taking turn in a nonterminal position must select some vertex in $V \setminus J_B$. So, if the play ends, it obviously ends in a position belonging to J_A, i.e. B can not be the winner.

(ii) Suppose that the starting vertex $u \in J_B$. If u is a terminal vertex, then $u \notin J_A \setminus J_B$ and A is not the winner. If u is not terminal, then according to (d), A must select some vertex $v \in V \setminus J_A$. If v is terminal, then $v \in J_B \setminus J_A$, and B is the winner. If v is not terminal, then the second player B continues playing as the first player in the case (i).

So, in this case, A has not a winning strategy.

Theorem 2. Let the game digraph of a nnnm *game satisfying the conditions* (a)-(f) *and the additional condition*

(g) $T \cap J_A \cap J_B = \emptyset$

is progressively finite. Then:

(i) If the starting vertex is not in J_B*, then the first player* A *has a winning strategy, and* A *can win by always selecting vertices in* J_A.

(ii) If the starting vertex is in J_B*, then the second player* B *has a winning strategy, and* B *can win by always selecting vertices in* J_B.

Proof. (i) Each directed path representing one complete play must be finite, because the game digraph is progressively finite. So, each play must end. According to (g), the play can not be draw. Now, it follows from Theorem 1, that A can win by always selecting vertices in J_A.

The proof of (ii) is similar.

3. THE ANALYSIS OF SOME TAKE - AWAY GAMES

1. (n,m) - E - game.

Let two positive integers n and m are given, where n is odd. A pile of n sticks is given and player A and B take turns, each taking any number a of sticks from the pile, where $1 \leq a \leq m$. The play ends when all the sticks are taken away from the pile. The player who has an even number of sticks at the and of the play wins.

Since the finite quantity of sticks will eventually be exhausted, and exactly one of the players will have an even number of sticks, it is obvious that the game allows no draw. We shall call this game (n,m) - E - game.

The game digraph of this game is $G = (V,E)$, where V is the set of all threedimensional vectors $(n-a-b,a,b)$ and a and b are integers such that $a \geq 0$, $b \geq 0$, $a+b \leq n$. An A-arc from $(n-a_1-b_1, a_1,b_1)$ to $(n-a_2-b_2,a_2,b_2)$ exists iff $b_1=b_2$ and $1 \leq a_1-a_2 \leq m$. A B-arc from $(n-a_1-b_1, a_1,b_1)$ to $(n-a_2-b_2, a_2,b_2)$ exists iff $a_1=a_2$ and $1 \leq b_1-b_2 \leq m$. The starting vertex $(n,0,0)$ is the only vertex with zero in-degree. The set T of terminal vertices is the set of all the vectors $(0,a,b)$ such that $a+b=n$. This digraph is progressively finite because it is a finite acyclic digraph.

Each play ends in a terminal vertex $(0,a,b)$. A wins if a is even, B wins if a is odd. We shall analyse two cases separately:

(i) m even

It can be checked that the game digraph of (n,m)-E-game satisfies all conditions (a)-(g), where

$$J_A = \{(n-a-b,a,b) \mid (n-a-b \equiv 1 \pmod{m+2} \wedge b \equiv 0 \pmod 2) \vee$$
$$\vee (n-a-b \equiv m+1 \pmod{m+2} \wedge b \equiv 1 \pmod 2) \vee$$
$$\vee (n-a-b \equiv 0 \pmod{m+2} \wedge b \equiv 1 \pmod 2)\}$$

and

$$J_B = \{(n-a-b,a,b) \mid (n-a-b \equiv 1 \pmod{m+2} \wedge b \equiv 0 \pmod 2) \vee$$
$$\vee (n-a-b \equiv m+1 \pmod{m+2} \wedge b \equiv 1 \pmod 2) \vee$$
$$\vee (n-a-b \equiv 0 \pmod{m+2} \wedge b \equiv 0 \pmod 2)\}.$$

Since n is odd, the starting vertex $(n,0,0)$ is in J_B iff $n \equiv 1 (\mathrm{mod}\,(m+2))$. So, we have the following statement:

Theorem 3. Let m *is an even integer. Then:*

If $n \not\equiv 1 (\mathrm{mod}\,(m+2))$, *then in* (n,m)-E-game *the first player* A *has a winning strategy, and* A *can win by always transforming the instantaneous position into a position in* J_B. *If* $n \equiv 1 (\mathrm{mod}\,(m+2))$, *then the second player* B *can win by always transforming the instantaneous position into a position in* J_B.

(ii) m odd

It can be checked that the game digraph of (n,m)-E--game satisfies all conditions (a)-(g), where

$$\begin{aligned} J_A = \{ (n-a-b,a,b) \mid & (n-a-b \equiv 1 (\mathrm{mod}\,(2m+2)) \wedge b \equiv 0 (\mathrm{mod}\ 2)) \vee \\ & \vee (n-a-b \equiv 0 (\mathrm{mod}\,(2m+2)) \wedge b \equiv 1 (\mathrm{mod}\ 2)) \vee \\ & \vee (n-a-b \equiv m+2 (\mathrm{mod}\,(2m+2)) \wedge b \equiv 1 (\mathrm{mod}\ 2)) \vee \\ & \vee (n-a-b \equiv m+1 (\mathrm{mod}\,(2m+2)) \wedge b \equiv 0 (\mathrm{mod}\ 2)) \} \end{aligned}$$

and

$$\begin{aligned} J_B = \{ (n-a-b,a,b) \mid & (n-a-b \equiv 1 (\mathrm{mod}\,(2m+2)) \wedge b \equiv 0 (\mathrm{mod}\ 2)) \vee \\ & \vee (n-a-b \equiv 0 (\mathrm{mod}\,(2m+2)) \wedge b \equiv 0 (\mathrm{mod}\ 2)) \vee \\ & \vee (n-a-b \equiv m+2 (\mathrm{mod}\,(2m+2)) \wedge b \equiv 1 (\mathrm{mod}\ 2)) \vee \\ & \vee (n-a-b \equiv m+1 (\mathrm{mod}\,(2m+2)) \wedge b \equiv 1 (\mathrm{mod}\ 2)) \} . \end{aligned}$$

Since n is odd, the starting vertex $(n,0,0)$ is in J_B iff $n \equiv 1 (\mathrm{mod}\,(2m+2))$. So, we have the following statement:

Theorem 4. Let m *is an odd integer. Then:*

If $n \not\equiv 1 (\mathrm{mod}\,(2m+2))$, *then in* (n,m)-E-game *the first player* A *has a winning strategy, and* A *can win by always transforming the instantaneous position into a position in* J_A.

If $n \equiv 1 \pmod{2m+2}$, *then the second player* B *can win by always transforming the instantaneous position into a position in* J_B.

2. (n,2)-3-game

A pile of n sticks is given, where n is an integer such that $n \equiv 0 \pmod 3$. Two player A and B take turns, each taking one or two sticks from the pile. The play ends when all the sticks are taken away from the pile. Let a and b respectively are the numbers of sticks which the first player A and the second player B have at the end of the play. If $a \equiv b \equiv 0 \pmod 3$, then A is the winner, otherwise B wins. We shall call this game (n,2)-3-game.

The game digraph of this game is $G = (V,E)$, where V is the same set is in (n,m)-E-game. An A-arc from $(n-a_1-b_1, a_1, b_1)$ to $(n-a_2-b_2, a_2, b_2)$ exists iff $b_1 = b_2$ and $1 \le a_1-a_2 \le 2$. A B-arc from $(n-a_1-b_1, a_1, b_1)$ to $(n-a_2-b_2, a_2, b_2)$ exists iff $a_1 = a_2$ and $1 \le b_1-b_2 \le 2$. The starting vertex $(n,0,0)$ is the only vertex with zero in - degree. The set T of terminal vertices is the set of all vectors $(0,a,b)$ such that $a+b = n$. This digraph is also a finite acyclic digraph.

Each play ends in a terminal vertex $(0,a,b)$. A wins if $a \equiv b \equiv 0 \pmod 3$, otherwise B wins.

It can be checked that the game digraph of (n,2)-3-game satisfies all conditions (a)-(g), where

$$J_A = \{(n-a-b,a,b) \mid (n-a-b \equiv 1 \pmod 2 \wedge b-a \equiv 2 \pmod 3) \vee \vee (n-a-b \equiv 0 \pmod 4 \wedge b-a \equiv 0 \pmod 3)\}$$

and

$$J_B = \{(n-a-b,a,b) \mid (n-a-b \equiv 1 \pmod 2 \wedge b-a \equiv 2 \pmod 3) \vee$$
$$\vee (n-a-b \equiv 0 \pmod 2 \wedge b-a \equiv 1 \pmod 3) \vee$$
$$\vee (n-a-b \equiv 1 \pmod 2 \wedge b-a \equiv 0 \pmod 3) \vee$$
$$\vee (n-a-b \equiv 0 \pmod 4 \wedge b-a \equiv 2 \pmod 3)\} .$$

Since $n \equiv 0 \pmod 3$, the starting vertex $(n,0,0)$ is in J_B iff $n \equiv 1 \pmod 2$, i.e. iff $n \equiv 3 \pmod 6$. So, we have the following statement:

Theorem 5. If $n \equiv 0 \pmod 6$, *then in* $(n,2)$-3-game *the first player* A *has a winning strategy, and* A *can win by always transforming the instaneous position into a position in* J_A.

If $n \equiv 3 \pmod 6$, *then the second player* B *has a winning strategy, and* B *can win by always transforming the instaneous position into a position in* J_B.

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