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PREFACE

The title of this monograph could read *modification of the rheonomic systems dynamics* or *revision of the rheonomic systems* as these titles more closely point to the subject of this study. The monograph changes the essential formulae, equations and principles of analytical mechanics of rheonomic systems which is evident from the comparison below:

Widespread and adopted in analytical mechanics	Contained in this monograph
Rheonomic constraints	
$f_\mu(\mathbf{r}_1,\dots,\mathbf{r}_N;t)=0$ or $\mathbf{r}_\nu=\mathbf{r}_\nu(q^1,\dots,q^n;t)$	$f_\mu(\mathbf{r}_1,\dots,\mathbf{r}_N;\tau(t))=0$ or $\mathbf{r}_\nu=\mathbf{r}_\nu(q^0,q^1,\dots,q^n),\ q^0=\tau(t)$
Generalized momenta	
$p_\alpha=g_{\alpha\beta}\dot q^\beta+\overline{g}_{\alpha 0}$ $p_0=-H,\ q^0=t$	$p_\alpha=g_{\alpha\beta}\dot q^\beta+g_{\alpha 0}\dot q^0$ $p_0=g_{0\beta}\dot q^\beta+g_{00}\dot q^0$
and consequences	
$\dot q^\alpha=g^{\alpha\beta}(p_\beta-\overline{g}_{\beta 0})$ $\lvert g_{\alpha\beta}\rvert\neq 0,\ \alpha=1,\dots,n$	$\dot q^i=g^{ij}p_j$ $\lvert g_{ij}\rvert\neq 0,\ j=0,1,\dots,n$
Kinetic energy	
$T=\frac{1}{2}g_{\alpha\beta}\dot q^\alpha\dot q^\beta+\overline{g}_{0\alpha}\dot q^\alpha+\frac{1}{2}\overline{g}_{00}$	$T=\frac{1}{2}g_{ij}\dot q^i\dot q^j$ $=\frac{1}{2}g_{\alpha\beta}\dot q^\alpha\dot q^\beta+g_{\alpha 0}\dot q^\alpha\dot q^0+\frac{1}{2}g_{00}\dot q^0\dot q^0$
$\overline{T}=\frac{1}{2}g^{\alpha\beta}(p_\beta-\overline{g}_{\beta 0})(p_\beta-\overline{g}_{\beta 0})-T_0$	$T=\frac{1}{2}g^{ij}p_ip_j$ $=\frac{1}{2}g^{00}p_0p_0+g^{\alpha 0}p_\alpha p_0+\frac{1}{2}g^{\alpha\beta}p_\alpha p_\beta$
Lagrangian	
$L=T-\Pi$	$\mathcal{L}=T-V=T-(\Pi+P)$ where P is rheonomic potential

Hamiltonian

$$\underbrace{H \stackrel{\text{def}}{=} p_\alpha \dot{q}^\alpha - L}_{\Downarrow}$$
$$H = T_2 - T_0 + \Pi$$
$(\alpha = 1, \dots, n)$
not invariant

$$H \stackrel{\text{def}}{=} E = \tfrac{1}{2} g^{ij} p_i p_j + V$$
$$= \underbrace{T + V = T_2 + T_1 + T_0 + \Pi + P}_{\Downarrow}$$
$$E = p_i \dot{q}^i - \mathcal{L} \quad (i = 0, 1, \dots, n)$$
invariant

Legendre-Hamiltonian transformations

$$p_\alpha = \frac{\partial L}{\partial \dot{q}^\alpha} \wedge \dot{q}^\alpha = \frac{\partial H}{\partial p_\alpha}$$

not necessary because the relations
$p_i = g_{ij} \dot{q}^j \leftrightarrow \dot{q}^i = g^{ij} p_j$
are sufficient

Lagrange differential equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\alpha} - \frac{\partial L}{\partial q^\alpha} = 0$$
$\alpha = 1, \dots, n$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha} - \frac{\partial \mathcal{L}}{\partial q^\alpha}$$
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^0} - \frac{\partial \mathcal{L}}{\partial q^0}$$

Hamiltonian equations

$$\dot{q}^\alpha = \frac{\partial H}{\partial p_\alpha}$$
$$\dot{p}_\alpha = -\frac{\partial H}{\partial q^\alpha}$$
$\alpha = 1, \dots, n$

$$\dot{q}^i = \frac{\partial E}{\partial p_i}$$
$$\dot{p}_i = -\frac{\partial E}{\partial q^i}$$
$i = 0, 1, \dots, n$

Coupled differential equations of motion and differential equations of perturbation

$$\dot{q}^\alpha = \frac{\partial H}{\partial p_\alpha}, \quad \dot{p}_\alpha = -\frac{\partial H}{\partial q^\alpha},$$
$$\dot{\xi}^\alpha = \frac{\partial^2 H}{\partial q^\beta \partial p_\alpha}\xi^\beta + \frac{\partial^2 H}{\partial p_\alpha \partial p_\beta}\eta_\beta,$$
$$\dot{\eta}_\alpha = \frac{\partial^2 H}{\partial q^\alpha \partial^\beta}\eta_\beta - \frac{\partial^2 H}{\partial q^\alpha \partial q^\beta}$$

$$\dot{q}^i = \frac{\partial \mathcal{H}}{\partial \eta_i}, \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial \xi^i},$$
$$\dot{\xi}^i = \frac{\partial \mathcal{H}}{\partial p_i}, \quad \dot{\eta}_i = -\frac{\partial \mathcal{H}}{\partial q^i}$$

Generalized differential equations of motion

$$\frac{\partial Z}{\partial a^p} + c^\mu_p \frac{\partial Z}{\partial a^\mu} = 0$$
$$2Z = g_{ij}(a^i - Q^i)(a^j - Q^j)$$

for holonomic system

$$\frac{\partial Z}{\partial a^i} = 0 \ (i = 0, 1, \dots, n)$$

Appell's equations

$$\frac{\partial S^*}{\partial \ddot{q}^\alpha} = Q_\alpha$$

$$S^* = \tfrac{1}{2}\sum_{s=1}^{n}\sum_{k=1}^{n} A_{sk}\ddot{q}_s\ddot{q}_k + \sum_{s=1}^{n}\sum_{k=1}^{n}\sum_{r=1}^{n}[s,k;r]\dot{q}_s\dot{q}_k\ddot{q}_r + \sum_{r=1}^{n}\sum_{s=1}^{n}\left(\frac{\partial A_{sr}}{\partial t}+\frac{\partial B_r}{\partial q_s}-\frac{\partial B_s}{\partial q_r}\right)\ddot{q}_r\dot{q}_s + \sum_{r=1}^{n}\left(\frac{\partial B_r}{\partial t}-\frac{\partial T_0}{\partial q_r}\right)\ddot{q}_r$$

$$\frac{\partial S}{\partial a^i} = Q_i$$

$$2S = g_{ij}a^i a^j$$

D'Alembert-Lagrange's principle

$$\left(Q_\alpha - g_{\alpha j}\frac{D\dot{q}^j}{dt}\right)\delta q^\alpha = 0$$

$$\left(Q_i - g_{ij}\frac{D\dot{q}^j}{dt}\right)\delta q^i = 0$$

$$\left(Q_\alpha - g_{\alpha j}\frac{D\dot{q}^j}{dt}\right)\delta q^\alpha + \left(Q_0 - g_{0j}\frac{D\dot{q}^j}{dt}\right)\delta q^0 = 0$$

Conditions for equilibrium

$$Q_\alpha = 0$$

$$Q_\alpha = 0$$
$$Q_0 + R_0 = 0$$

Principle of least action

$\emptyset$

$$\delta\int_{t_0}^{t_1}(T-P)\,dt = 0$$

Hamilton's principle

$$\delta\int_{t_0}^{t_1} L\,dt = 0$$

$$\delta\int_{t_0}^{t_1} \mathcal{L}\,dt = 0$$

or

$$\delta\int_{t_0}^{t_1} p_\alpha\,dq^\alpha - H\,dt = 0$$
$(\alpha = 1,\dots,n)$

$$\delta\int_{t_0}^{t_1} p_i\,dq^i - E\,dt = 0$$
$(i = 0,1,\dots,n)$

Gauss principle

$$\frac{\partial Z}{\partial \dot{q}^\alpha}\delta\ddot{q}^\alpha = 0$$

$$\delta Z = \frac{\partial Z}{\partial a^i}\delta a^i = 0$$

$$a^\alpha = D\dot{q}^\alpha/dt = \ddot{q}^\alpha + \Gamma^\alpha_{ij}\dot{q}^i\dot{q}^j$$

When a comparison is made with the standpoints adopted in analytical mechanics the major changes entered here appear both substantial and formal and they may not be easily accepted due to the nature of inertia. They are substantial because the cause of change in constraints is now introduced into the description of motion and they are formal because the relations and expressions for rheonomic systems are reduced to homogeneous harmonic forms of relations and expressions for scleronomic systems. Namely, it is not only the matter of improving the forms of some analytical expressions, but also improving the entire description of the motion of rheonomic systems and their constraints. The term "modification" would imply a comparative listing of the original texts with those adopted that needed correction, as I am doing in this foreword but this approach I did not consider a useful one. That is why I opted for a more purposeful title *The dynamics of rheonomic systems*. It may happen that such a title has already been used somewhere, but I think that the subject matter practiced here has not been covered anywhere since that probably would have been known. From the classics to modern scientists there have been attempts to solve this problem, but as shown in the comparative analysis, the search for solutions took another route that led to "another mechanics". The most widespread approach was the one which introduced time as the $(n+1)$-st coordinate $q^{n+1} = t$ and, corresponding to this, the $(n+1)$-st momentum $p_{n+1} = -H$ as the negative Hamilton's function. Corresponding to them is the $(n+1)$-st differential equation

$$\frac{d}{dt}\frac{\partial \overline{L}}{\partial t'} - \frac{\partial \overline{L}}{\partial t} = 0$$

where $\overline{L}$ is the transformed Langrange's function of L. However, it has been proved that this equation is a dependent one and that it only formally fills the gap in the enlarged configurational space.

Debates at scientific conferences induced me to provide some details, which may have been left out of research papers or to prove some results in two or three ways comparatively. For the same reason, several simple and clear examples have been included. I recognized this need at the above mentioned conferences and where the reader is concerned this will make it easier for him to compare the widespread viewpoints from the analytical mechanics of rheonomic systems with the ones offered here. I hope that such a book will be useful to the advanced students studying analytical mechanics.

Some results appearing in this monograph have already been published in my papers [11–20] which I explained and proved at the sessions of the Department for Mechanics of the Mathematical Institute in Belgrade, the Seminar for Analytical Mechanics on the Faculty of Mathematics in Belgrade, the Institute of Physics on the Faculty of Mathematics, the Yugoslav Congresses for Theoretical and Applied Mechanics, the XVII Congress of the International Union for Mechanics in Grenoble, the Department for Mechanics in the Computer Centre of the Academy of Sciences of USSR, the Institute dealing with problems in mechanics of the same Academy, Moscow, and the Mathematical Institute of the Georgian Academy of Science, Tbilisi, USSR.

SURVEY OF ELEMENTARY MODIFICATIONS

The concept of *rheonomic system* first appeared in the book of L. Boltzmann "Vorlesungen über die Prinzipe der Mechanik", Part II, Leipzig, 1922, meaning a system of material particles whose motion is limited by the constraints changing with time. C. Lanczos in his book "The Variational Principles of Mechanics"[1] (1962) states that Boltzmann introduced the terms "*rheonomic*" (*ρέος* — flow, flowing, *νόμος* — law, decree) and "scleronomic" (*σκληρός* — hard, strong). But, apart from the terms, which are not even today used often and uniformly, the impact of time-dependent constraint upon the transformation of differential equations of motions and principles were also considered earlier by e.g. Ostrogradski[2] (1848) and Sophus Lie[3] (1877). As regards the way in which variable (rheonomic) constraints are described in the theory of mechanics we find them in the classics, for instance A. M. Ljapunov[4] (1893). In numerous papers on system mechanics, however, the term "rheonomic constraints" is often replaced by the term "non-stationary constraints". Scarce are the books in which the general term "constraints" is covered that these constraints, beside coordinates and velocities also depend on time t.

Even more important than the title itself for these dynamic systems is the knowledge gained in classical mechanics which ought to be modified. To this end, it is necessary not only to clearly define the rheonomic constraints but to make their geometrical and dynamic meanings clearer. It is well known that the constraints are the source of the forces which act upon material particles in the system and limit their motion. As for the constraints which themselves undergo changes, there must be a cause of their change. Related to this fact, which was either overlooked or bypassed in analytical mechanics are the basic and general viewpoints in dynamics which are reflected upon the subsequently derived relations on the system motion. Because of that, non-uniformity, non-equivalence and non-invariance have appeared in the analytical description of the rheonomic systems motion. Differences were noticed among Newton's, Lagrange's and Hamilton's mechanics though

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[1] C. Lanczos, *The Variational Principles of Mechanics*, sec. edition, Toronto, 1962.

[2] М. Б. Остроградский, *Mémoire sur les équations différentielles relative aux problèmes des isoperimètres*, St. Peterburg, 1848.

[3] S. Lie, *Die Störungstheorie und die Berührungstransformationen*, Arch. for Math. T. 2, s. 2, Kristiania 1877, 129-156.

[4] А. М. Ляпунов, *Курс теоретической механики*, Технологический институт, Харьков, 1893.

they all described the same natural phenomena. They were not of mere formal nature. When the motions of systems with invariable (scleronomic) constraints are described the harmony, uniformity and invariance are absolute in the choice of the methods of description and the choice of coordinate systems. This harmony is disturbed when talking of the rheonomic systems for the very reason that the cause of change of the constraints is overlooked. An attempt was made in this study to overcome this deficiency and make the principles of mechanics invariant both for scleronomic and rheonomic systems. The causative attributes related to the change of constraints are named here: "The changing force of constraint", "the changing power of constraints", and "the potential of rheonomic constraints" depending on the context in which the said term is used. The true meaning of these phenomena can be noticed from the description of motion of a material particle over a variable surface, this matter being devoted much attention in the beginning of this paper.

1. Limiting the motion of a point

The area in which a particle of mass m moves is in the general case a three dimensional space E^3 with an orthogonal vector base $\mathbf{e} = \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$:

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{1.1}$$

satisfying

$$\frac{d\mathbf{e}_i}{dt} = 0 \qquad (i, j = 1, 2, 3) \tag{1.2}$$

where time t (tempus) is considered homogeneous. If the set of all real numbers is denoted by $\mathbf{R}$, then we can always write $t \in \mathbf{R}$. The values of t form an axis which is named the time axis, denoted by T. Then time has one dimension i.e.

$$[\dim t] = T. \tag{1.3}$$

The common origin O of base vectors $\mathbf{e}_i$ may belong to any inertial basis so that when related to the arbitrarily chosen pole O the position of the particle can be determined by the position vector

$$\mathbf{r} = y^1\mathbf{e}_1 + y^2\mathbf{e}_2 + y^3\mathbf{e}_3 = y^i\mathbf{e}_i = \mathbf{r}(y^1, y^2, y^3) \tag{1.4}$$

where y^i $(i = 1, 2, 3)$ are Descartes rectilinear coordinates. The motion of a point M in space E^3 may be limited by constraints of the form

$$f(y^1, y^2, y^3, \tau) = 0, \qquad f \in C^1. \tag{1.5}$$

In the case when parameter τ is constant, the relation (1.5) i.e. $f(y^1, y^2, y^3, \tau_0) = 0$ represents a surface $S_2 \subset E^3$, whose dimension equals the number of coordinates of a point reduced by the number of constraints i.e.

$$\dim S_2 = 2. \tag{1.6}$$

However, if $\tau = \tau(t)$ is a variable parameter, the considered surface will change depending on the parameter $\tau = \tau(t)$. Hence, the dimension of the variable surface is

$$\dim S_{2+1}(t) = 2 + 1. \tag{1.7}$$

where an additional dimension defines the parameter τ as a function of time t. Using the relation (1.5) it is possible to determine one of the coordinates y^i as a function of the other two y^α $(\alpha = 1, 2)$ and of parameter τ, provided that $\partial f/\partial y^i \neq 0$. The parameter τ is free until we chose its dependence on time, i.e. $\tau = \tau(t)$.

It means that the constraint $f(y^1, y^2, y^3; \tau) = 0$ permits a solution in a neighbourhood of a stationary non-singular point $M(y_0^1, y_0^2, y_0^3) \in E^3$ if

$$\frac{\partial f}{\partial \tau} = 0. \tag{1.8}$$

Really, the solution $y^3 = \varphi(y^1, y^2, \tau)$ exists in a neighbourhood of point M_0 if there exists the differential

$$df = \frac{\partial f}{\partial y^1}dy^1 + \frac{\partial f}{\partial y^2}dy^2 + \frac{\partial f}{\partial y^3}dy^3 + \frac{\partial f}{\partial \tau}d\tau = 0,$$

hence it follows that

$$\frac{\partial \varphi}{\partial y^1} = -\frac{\partial f}{\partial y^1} : \frac{\partial f}{\partial y^3}; \quad \frac{\partial \varphi}{\partial y^2} = -\frac{\partial f}{\partial y^2} : \frac{\partial f}{\partial y^3}; \quad \frac{\partial \varphi}{\partial \tau} = -\frac{\partial f}{\partial \tau} : \frac{\partial f}{\partial y^3}.$$

Therefore a change in the function $y^3 = \varphi(y^1, y^2, \tau)$ occurs even if the coordinates y_0^1 and y_0^2 are fixed, only if $\partial f/\partial \tau \neq 0$. The constraint $f(y^1, y^2, y^3, y^0) = 0$, $y^0 = \tau(t)$, is non-singular for the point $M_0(y_0^1, y_0^2, y_0^3)$ if the gradient of the function f for point M_0 differs from zero, i.e.

$$\operatorname{grad} f(M_0) = \frac{\partial f(M_0)}{\partial y^1}\mathrm{e}_1 + \frac{\partial f(M_0)}{\partial y^2}\mathrm{e}_2 + \frac{\partial f(M_0)}{\partial y^3}\mathrm{e}_3 \neq 0. \tag{1.9}$$

The following simple examples ought to make clearer the nature and character of parameter τ, which satisfies the dimensional equation for the constraint.

1. A heavy particle moves over a horizontal smooth plane $y^3 = \varphi(t)$. The function $\varphi(t)$ shows that the plane itself moves vertically during time. It can be seen that parameter τ is a function of time and the constraint (1.5) is in the form:

$$f(y^3, \tau(t)) = 0. \tag{1.10}$$

Hence it clearly follows that parameter τ has physical dimension of coordinate y^3 i.e. $[\dim \tau] = L$. The parametric form of this constraint: $y^3 = \tau(t)$ should be distinguished from the motion of a free point according to the law $y^3 = y^3(t)$ in the sense that the constraint (1.10) limits the motion of the point by the force named the constraint reaction.

The same horizontal plane may be considered in some other rectilinear coordinate system z^1, z^2, z^3 for which we have linear homogeneous transformation $y^i = c^i_k z^k$, $c^i_k \neq 0$. If $y^3 = c^3_1 z^1 + c^3_2 z^2 + c^3_3 z^3$ is substituted into (1.10), the rheonomic constraint equation is $f = c_i z^i - \tau(t) = f(z^1, z^2, z^3, \tau) = 0$, or in parametric form

$$z^3(z^1, z^2, \tau) = b^3_1 z^1 + b^3_2 z^2 + b^3_0 \tau, \tag{1.11}$$

where it is obvious that $b^3_1 = -c^3_1/c^3_3$, $b^3_2 = -c^3_2/c^3_3$, $b^3_0 = -1/c^3_3$.

2. A point moves over the plane

$$a(t)y^1 + b(t)y^2 + cy^3 = 0, \qquad c \in \mathbf{R}, \tag{1.12}$$

which rotates around point $M_0(0,0,0)$. The equation of the constraint may be written in the form (1.5) i.e.

$$f = \tau(t)y^1 + b(\tau)y^2 + cy^3 = 0 \tag{1.13}$$

where for τ the coefficient $a(t) = \tau(t) \to t = t(\tau)$ is chosen. Here, as in (1.10) or (1.12), the constraint (1.12) can be written in parametric form

$$y^3 = y^3(\tau, y^1, y^2) = -\frac{\tau}{c}y^1 - \frac{b}{c}y^2$$

or

$$y^3 = y^3(y^0, y^1, y^2) \tag{1.14}$$

where it is established that $\tau = y^0$. Here τ is not a coordinate y^1, y^2 or y^3, as chosen in the relation (1.10), but a symbol for that function of the coordinates with zero index: $y^0 = \tau(t) = y^0(t)$.

Let us assume further that in (1.13) the coefficient $b = 0$ and $a = \operatorname{tg}\omega t$, where ω is the frequency of rotation of the plane $y^3 = -(\operatorname{tg}\omega t/c)y^1$ about y^2 axis. It is suitable to choose ωt for the function τ, i.e. $\omega t = \tau = y^0$. Thus the equation of that surface can be written in the form

$$f(y^0, y^1, y^2) = y^3 + (\operatorname{tg} y^0/c)y^1 = 0. \tag{1.15}$$

Time t itself could have been chosen for τ such that $\tau = t = y^0(t)$, but one should pay attention that the dimensional equation of constraints is satisfied. In this case the relation (1.15) in the general expression is not changed because

$$f(y^0, y^1, y^3) = y^3 + (\operatorname{tg}\omega y^0/c)y^1 = 0. \tag{1.16}$$

The only change is in the dimension of function τ, that is y^0, depending on the selection of geometrical or kinematic coefficients. In case (1.10) the dimension of parameter τ was the length dimension and in cases (1.14) and (1.15) the same coordinate stood for angle. The given examples for the motion of a point over two-dimensional movable planes S_2 shows that the motion of a point over those planes is defined by one "coordinate" more, $\dim S_{2+1} = 2+1$, as is generalized in the relation (1.7). Still more obvious is the meaning of that one dimension more if the motion of a point is observed on a straight line, which can be determined by two movable planes

(1.17)
$$\begin{cases} f_1(y^1, y^2, y^3; y^0) = 0 \qquad y^0 = \tau(t) \\ f_2(y^1, y^2, y^3; y^0) = 0 \end{cases}$$

or one movable plane, let it be $f_1 = 0$ and one stationary plane $f_3(y^1, y^2, y^3) = 0$.

Movements of dy^i about a non-singular point $M \in f_1 \cap f_2$ will satisfy the constraints

(1.18)
$$\begin{aligned} df_1 &= \frac{\partial f_1}{\partial y^1} dy^1 + \frac{\partial f_1}{\partial y^2} dy^2 + \frac{\partial f_1}{\partial y^3} dy^3 1 + \frac{\partial f_1}{\partial y^0} dy^0 = \frac{\partial f_1}{\partial y^i} dy^i = 0 \\ df_2 &= \frac{\partial f_2}{\partial y^i} y^i \qquad (i = 0, 1, 2, 3). \end{aligned}$$

For non-singular two-dimensional matrix

$$\left\{ \frac{\partial f_\sigma}{\partial y^i} \right\}_2^2$$

it is always possible to find two coordinates y^i using the third one and the time coordinate y^0. Such an equation for a moving line, or a more generalized line if the relations (1.17) represent surfaces, when given in parametric form will be, for instance,

(1.19)
$$\begin{cases} y^2 = y^2(y^0, y^1) \\ y^3 = y^3(y^0, y^1). \end{cases}$$

Pursuant to this it can be said that the dimension of a rheonomic straight-line (or a more generalized line) $p = f_1 \cap f_2$ equals two, i.e.

$$\dim p = 1 + 1.$$

An example of such a constraint is the relation (1.15) together with another constraint: $y^2 =$ const. or $y^2 = \tau(t)$.

Limiting the displacement of a point using three rheonomic constraints. In classical mechanics it is known that the number of degrees of freedom of a material particle on retaining constraints is equal to $n = 3 - k$ where k is the number of constraints. Therefore, if there are three constraints

$$f_\sigma(y^1, y^2, y^3, \tau) = 0 \qquad (\sigma = 1,2,3) \tag{1.20}$$

it would follow that the number of degrees of freedom of so limited motion equals zero. That is correct if the constraints are scleronomic ones

$$\psi_\sigma(y^1, y^2, y^3) = 0. \tag{1.21}$$

In this case the material particle remains in the fixed intersection of three surfaces (1.21) in a region. In order to have displacement in the neighbourhood of a common point the following three equations should be satisfied

$$\frac{\partial \psi_\sigma}{\partial y^i} dy^i = 0, \qquad (\sigma = 1,2,3). \tag{1.22}$$

It is obvious that these homogeneous equations are satisfied for $dy^1 = dy^2 = dy^3 = 0$, and there is no displacement. For other possible solutions for $dy^i \neq 0$ the determinant

$$\left| \frac{\partial \psi_\sigma}{\partial y^i} \right|_3^3 \tag{1.23}$$

of the system (1.22), should equal zero, but this is not the case because it is assumed that the constraints (1.21) are mutually independent. However, the common point of the three independent rheonomic constraints (1.20) is not fixed but moves into E^3 in the course of time $t \in T$, if the following three equations

$$\frac{\partial f_\sigma}{\partial y^i} dy^i = -\frac{\partial f_\sigma}{\partial y^0} dy^0, \qquad (\sigma, i = 1,2,3), \tag{1.24}$$

permit the solution for dy^i other than zero. And this is precisely possible when $|\partial f_\sigma / \partial y^i|_3^3 \neq 0$. Using three independent non-homogeneous algebraic equations (1.24) with $3+1$ magnitudes dy^i and dy^0 it is possible to determine the motions of dy^i dependent on dy^0. Furthermore, the intersection itself $p(t) = f_1 \cap f_2 \cap f_3$ is a function of time if at least one of the constants change with time. Therefore the point M, at which three rheonomic constraints act, can move as caused by the limiting constraints. In other words, it is the additional dimension denoted by τ or $y^0 = y^0(t)$, which extends the space in which the point moves by that one dimension. *One should not overlook the fact that the possibility of motion is conditioned by constraints and as such it represents by itself a constraint acting on the particle.*

Similarly to the notion of retaining rheonomic constraints (1.20) the unilateral constraint will be recorded with $f(y^1, y^2, y^3; y^0) \geq 0$, $y^0 = \tau(t)$.

Transformation of constraint equations. By making a good choice of the coordinate system, the holonomic constraints (1.5) can be reduced to simple coordinate forms as shown on following pages. The rectilinear coordinate systems y^i are convenient for movable planes and straight lines. The changing of constraints makes curved surfaces in the course of time so it is more convenient to introduce curvilinear coordinates x_1, x_2, x_3, for such constraints which are oriented by coordinate vectors $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3$. It is necessary to determine the inertiality of the vector base, introduced in this way, which can be done by comparing them with the inertial basis arranged by the relations (1.1) and (1.2). That is always possible if there exists interdependence of coordinates

$$y^i = y^i(x^1, x^2, x^3) \quad \text{and} \quad x^i = x^i(y^1, y^2, y^3). \tag{1.25}$$

The position vector of the point may in such a case be written in the form

$$y^i \mathbf{e}_i = y^i \frac{\partial \mathbf{r}}{\partial y^i} = y^i \frac{\partial x^k}{\partial y^i} \frac{\partial \mathbf{r}}{\partial x^k} = y^i \frac{\partial x^k}{\partial y^i} \mathbf{g}_k,$$

where obviously for arbitrary and independent coordinates y^i,

$$\mathbf{g}_k = \frac{\partial \mathbf{r}}{\partial x^k} = \mathbf{g}_k(x^1, x^2, x^3) \tag{1.26}$$

and

$$\mathbf{e}_i = \frac{\partial x^k}{\partial y^i} \mathbf{g}_k \iff \mathbf{g}_i = \frac{\partial y^k}{\partial x^i} \mathbf{e}_k \tag{1.27}$$

under condition

$$\left| \frac{\partial y^i}{\partial x^k} \right| \neq 0. \tag{1.28}$$

Pursuant to the derived relations, the vector derivative (1.27) in time t may be represented in the following way

$$\frac{d\mathbf{e}_i}{dt} = \frac{\partial}{\partial y^j}\left(\frac{\partial x^k}{\partial y^i}\right) \frac{\partial y^j}{\partial x^l} \frac{dx^l}{dt} \mathbf{g}_k + \frac{\partial x^k}{\partial y^i} \frac{d\mathbf{g}_k}{dt}.$$

Partial derivatives beside g_k may be further transformed as follows:

$$\frac{\partial}{\partial y^j}\left(\frac{\partial x^k}{\partial y^i}\right) \frac{\partial y^j}{\partial x^l} = \frac{\partial^2 x^k}{\partial y^j \partial y^i} \frac{\partial y^j}{\partial x^l} = \begin{Bmatrix} k \\ \mu\ \nu \end{Bmatrix} \frac{\partial x^\mu}{\partial y^i} \frac{\partial x^\nu}{\partial y^j} \frac{\partial y^j}{\partial x^l} = -\begin{Bmatrix} k \\ \mu\ l \end{Bmatrix} \frac{\partial x^\mu}{\partial y^i},$$

since

$$\frac{\partial^2 x^k}{\partial y^i \partial y^j} = -\begin{Bmatrix} k \\ m\ \nu \end{Bmatrix} \frac{\partial x^m}{\partial y^i} \frac{\partial x^\nu}{\partial y^j},$$

where $\left\{ \begin{matrix} k \\ m\,\nu \end{matrix} \right\}$ are Christoffel symbols of the second kind. Therefore we obtain that

$$\frac{d\mathbf{e}_i}{dt} = \left(\frac{d\mathbf{g}_m}{dt} - \left\{ \begin{matrix} k \\ m\,l \end{matrix} \right\} \frac{dx^l}{dt} \mathbf{g}_k \right) \frac{\partial x^m}{\partial y^i}.$$

Composing this with $\partial y^i/\partial x^s$ it follows

$$\frac{d\mathbf{g}_s}{dt} - \left\{ \begin{matrix} k \\ s\,l \end{matrix} \right\} \frac{dx^l}{dt} \mathbf{g}_k = \frac{D\mathbf{g}_s}{dt} = \frac{\partial y^i}{\partial x^s} \frac{d\mathbf{e}_i}{dt},$$

which means that for the inertiality of the vector base $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3$ of a curvilinear coordinate system, the following will be necessary

$$\frac{D\mathbf{g}_s}{dt} = 0 \qquad (s = 1,2,3). \tag{1.29}$$

If the relations (1.25) are substituted into relations (1.5) an equation of the retaining rheonomic constraint is obtained in the curvilinear coordinate system x^i with the vector base $\mathbf{g}_i$, i.e.

$$f(x^1, x^2, x^3; \tau) = 0. \tag{1.30}$$

This equation is equivalent to equation (1.5) of the rheonomic constraint for each $x^i \in E$ for which (1.28) is valid and for which the gradient of the function f is different from zero. The rheonomic constraint equation (1.30) has considerably simpler forms for curved surfaces than equation (1.5). When a convenient coordinate system x_1, x_2, x_3 is chosen the constraint equation (1.30) may be reduced to a simple form of a movable or variable coordinate surface $x = x(t)$.

Example. An equation of the sphere whose radius changes with time in a coordinate system y^i is $\delta_{ij} y^i y^j = r^2(t)$, while with respect to the spherical coordinate system $x^1 = \varphi$, $x^2 = \theta$, $x^3 = r$, this equation of the constraint has a much simpler form namely $x^3 = x^0 = r(t)$, where the zero index shows that for the "rheonomic coordinate" $x^0 = r(t)$ just the radius of the sphere changing with time was chosen. It should not be forgotten that, when passing from one coordinate system to the other, i.e. when the rheonomic constraint equations (1.5) are transformed into (1.30) then the form of the constraint equations is changed while the constraint as a mechanical object limiting the motion of a point remains as it is, because objectively it is independent of any coordinate system and its transformations. Thus the equations of rheonomic constraints in curvilinear coordinate systems should also be distinguished, for example

$$f(x^k, x^0(t)) = x^k - x^0(t) = 0 \qquad \text{for } k = 1 \text{ or } k = 1,2, \tag{1.31}$$

from the finite equations of motion $x^i = x^i(t)$, regardless of the fact that they are identical in form. The dimension of the space in which the limited motion of a point

(1.7) is observed does not change transformation (1.25). When the constraint is active (1.30) the number of possible displacements is the same as (1.7), $2+1$, and when the point moves on a variable curved line the dimension of the space is $1+1$.

In the cases when one rheonomic constraint has the form (1.31), for instance $x^3 = x^0(t)$, the equations of the pointwise transformation may be given parametric form of the rheonomic constraint equations

$$y^i = y^i(x^1, x^2, x^0(t)), \qquad (i = 1,2,3). \tag{1.32}$$

When two independent rheonomic constraints are active

$$f_1(x^1, x^2, x^3, x^0) = 0 \quad \text{and} \quad f_2(x^1, x^2, x^3, x^0) = 0$$

then two coordinates can be expressed by means of the third known function of $x^0(t)$, which is called "rheonomic coordinate", and in this case the parametric equations of the rheonomic constraint are

$$y^i = y^i(x^0, x^1), \qquad (i = 1,2,3). \tag{1.33}$$

The relations (1.32) are parametric equations of a variable surface (a cylinder for instance: $y^1 = x^0(t)\cos x^1$, $y^2 = x^0(t)\sin x^1$, $y^3 = x^3$), while the equations (1.33) are parametric equations of a curved line (for instance of a circle: $y^1 = x^0(t)\cos x^1$, $y^2 = x^0(t)\sin x^1$, $y^3 = x^3 = \text{const.}$). Then the vector equations

$$\mathbf{r} = \mathbf{r}(x^0, x^1, x^2)q \qquad \mathbf{r} = \mathbf{r}(x^0, x^1) \tag{1.34}$$

can be considered as parametric equations of the retaining rheonomic constraints which limit the motion of a material point.

By substituting (1.23) or (1.33) into expression (1.4) equations (1.34) are obtained. With such rheonomic constraints (1.32) it is easily noted from (1.34) that the coordinate vector

$$\mathbf{g}_0 = \frac{\partial \mathbf{r}}{\partial x^0} \tag{1.35}$$

is a function of two variables x^1, x^2 and a rheonomic coordinate, while in the case when the point moves on a variable line, (1.33) is a function of one independent coordinate and one rheonomic coordinate i.e. $\mathbf{g}_0 = \mathbf{g}_0(x^0(t), x^1)$. A statement can be made after a synthesis of the study of the rheonomic coordinate properties that: when the motion of a particle is limited by double sided rheonomic constraints, the dimension of the space is enlarged by one; that the added coordinate function known in the general case has the kinematic properties given to it when chosen. The parametric constraint equations, let us repeat, $x^i - x^0(t) = 0$ differ by quality from the definite equations of motion $x^i = x^i(t)$, even when condition that time t is chosen for the $n+1$-th coordinate x_0.

2. Motion of a material particle over the rheonomic surface

The essence of generalizing the description of the motion of systems of particles on multi-dimensional manifolds lies in the description of the motion of a particle over a smooth rheonomic (variable or movable) surface. For this reason and due to modification of classical standpoints about rheonomic systems in mechanics it is in the interest of methodology to carry out a detailed analysis of the motion of a point of mass m which is constrained by surface. The resultant force acting on the particle is $\mathbf{F}$. The equation for a rheonomic constraint, a variable surface in this case, may be described by means of

a) general relation

$$f(y^1, y^2, y^3, \tau(t)) = 0 \tag{2.1}$$

related to Cartesian inertial coordinate system y^1, y^2, y^3 (for example, a variable sphere $f = \delta_{ij} y^i y^j - 4a^2t^4 = 0$);

b) by equation

$$f(x^1, x^2, x^3, \tau) = 0 \tag{2.2}$$

with respect to a curvilinear coordinate system (for example, a sphere: related to polar spherical system $x^1 = \rho$, $x^2 = \varphi$, $x^3 = \theta$ the sphere being given by the equation $f = \rho - 2at^2 = 0$), or

c) in parametric form

$$y^i = y^i(x^1, x^2, \tau) \qquad (i = 1, 2, 3) \tag{2.3}$$

(an example of the observed sphere of a variable radius is $y^1 = 2at^2 \sin\varphi \cos\theta$, $y^2 = 2at^2 \sin\varphi \sin\theta$, $y^3 = 2at^2 \cos\varphi$). In mechanics the equations (2.3) are often written in the form

$$\mathbf{r} = \mathbf{r}(x^0, x^1, x^2), \qquad x^0 = \tau(t), \tag{2.4}$$

in order to determine analytically the coordinate vectors (1.26) including the vector (1.35).

According to the fundamental law of dynamics the differential equation

$$\frac{d}{dt}(m\mathbf{v}) = \mathbf{F} \tag{2.5}$$

describes the motion of each point of mass m, where $\mathbf{v} = d\mathbf{r}/dt$ is the velocity of motion of each point, t is time, and $\mathbf{F}$ resultant vector of all the forces acting on each point. However, this differential equation itself does not help to determine the motion without additional conditions and an analysis of all the factors influencing the motion, which figure in the differential equation of motion (2.5).

Let us take for the moment that the mass is a function of time t, namely

$$m = m_0 + \int_0^t \mu(t)\,dt, \qquad \mu = \frac{dm}{dt}. \tag{2.6}$$

The velocity $\mathbf{v} = d\mathbf{r}/dt$ of a particle must be consistent with the constraint limiting the motion and therefore it is necessary to determine the condition of consistency. This can be achieved in two ways: a) by dynamic method of constraint elimination and b) by the method of satisfying kinematic conditions of constraints. Let us analyze both methods.

a. Dynamic elimination of constraints. The motion is limited by the objects of constraints. The rheonomic constraints act towards a change in motion and the force is, as seen in (2.5), the cause of a change in motion. Therefore, it is logical that the constraints can be thought as replaced by a force which is called the reaction of constraints. Judging from the physical nature of the reaction of constraints it is known that this vector of force may be divided into two components. One is $\mathbf{R}_\tau$ which lies in the plane tangent to the surface at the point which touches the particle and the other is $\mathbf{R}_N$ normal to that plane i.e.

$$\mathbf{R} = \mathbf{R}_\tau + \mathbf{R}_N. \tag{2.7}$$

Since the force $\mathbf{R}$ replaces the time-change constraint it can be logically assumed that the vector of force $\mathbf{R}$ also depends on time.

By replacing, as comes to mind, the constraint (2.1) by the force (2.6) the differential equation of the motion of a point (2.5) now becomes more determinate

$$\frac{d}{dt}(m\mathbf{v}) = \mathbf{F} + \mathbf{R} \tag{2.8}$$

where the vector $\mathbf{F}$ represents the resultant, namely the main vector of active and known forces, appearing in a general case as a function of the position $\mathbf{r}$, the velocity $\mathbf{v}$ and time t. The vector of velocity $\mathbf{v}$ which figures in equation (2.9) equals the time derivative of $\mathbf{r}$:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dy^i}{dt}\mathbf{e}_i = \dot{y}^i\mathbf{e}_i, \qquad (i = 1, 2, 3) \tag{2.9}$$

or due to (1.26) and (1.27)

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{\partial \mathbf{r}}{\partial x^i}\dot{x}^i = \dot{x}^i\mathbf{g}_i, \qquad \mathbf{g}_i = \frac{\partial \mathbf{r}}{\partial x^i}. \tag{2.10}$$

Therefore by eliminating, in thoughts, the surface which limits the motion of a material particle and by replacing the reaction of the constraint (2.6), the motion of a point may be considered as the motion of a free point in E^3 which should be preceded by the choice of a coordinate system. If we first substitute the expression for velocity (2.9) into the differential equation of motion (2.8) it will become

$$\frac{d}{dt}(m\dot{y}^i \mathbf{e}_i) = \mathbf{F} + \mathbf{R}. \tag{2.11}$$

Scalar multiplication of this relation by coordinate vectors $\mathbf{e}_k$ while equation (1.2) should be considered, gives three scalar differential equations for the motion of a material particle over the surface namely

$$\frac{d}{dt}\left(\partial_{ik}\frac{d\dot{y}^i}{dt}\right) = Y_k + R_k \qquad (k = 1, 2, 3) \tag{2.12}$$

where

$$\partial_{ik} = m\,\mathbf{e}_i \cdot \mathbf{e}_k, \tag{2.13}$$

$$Y_k = \mathbf{F} \cdot \mathbf{e}_k, \qquad R_k = \mathbf{R} \cdot \mathbf{e}_k. \tag{2.14}$$

When the motion $y^k = y^k(t)$ is known from the law of mass and force Y^k, then the reactions $R_k = R_k(t)$ as a function of time can be determined from (2.12). It is also explicit that the motion $y^k = y^k(t)$ can be determined if the forces Y_k, R_k and mass m are known. It is also possible to combine the above. And when by entering more than three known magnitudes which figure in equations (2.12) the other three magnitudes are determined. Thus, if one knows, for example, the forces Y_i $(i = 1, 2, 3)$, R_s $(s = 1, 2)$ and the final equation of motion, as well the mass m, one can determine from the differential equations of motion (2.12): $y^1 = y^1(t)$, $y^2 = y^2(t)$, and consequently the reaction R_3 as a function of time because of

$$R_3 = \frac{d}{dt}(m\dot{y}_3) - Y_3 = R_3(t). \tag{2.15}$$

The assumption that the vector coordinates of the reactions of the constraints R_1 and R_2 are known is probable for flat surfaces. However, for the rheonomic curved surfaces such an assumption can hardly be realized. Therefore in order to facilitate the determination of motion and reactions of the constraints it is convenient to transform the differential equation of the motion of a point (2.8) with respect to the curvilinear coordinate system x^1, x^2, x^3 with the basis (1.26). The scalar multiplication of the equation (2.8) by the coordinate vectors $\mathbf{g}_k$, gives

$$\frac{d}{dt}(m\mathbf{v}) \cdot \mathbf{g}_k = (\mathbf{F} + \mathbf{R}) \cdot \mathbf{g}_k, \tag{2.16}$$

where

$$\frac{d}{dt}(m\mathbf{v}) \cdot \mathbf{g}_k = \frac{d}{dt}(m\mathbf{v} \cdot \mathbf{g}_k) - m\mathbf{v} \cdot \frac{d\mathbf{g}_k}{dt}.$$

If we consider the expression for velocity (2.10) and the relation (1.29) it comes out that

$$\frac{d}{dt}(m\mathbf{v})\cdot\mathbf{g}_k = \frac{d}{dt}(g_{ik}\dot{x}^i) - g_{ij}\left\{\begin{matrix} j \\ k\ l \end{matrix}\right\}\dot{x}^i\frac{dx^l}{dt}$$
$$= \frac{D(g_{ik}\dot{x}^i)}{dt} = g_{ik}\frac{D\dot{x}^i}{dt}$$

where

$$g_{ik} = m\,\mathbf{g}_i\cdot\mathbf{g}_k \tag{2.17}$$

and $Dg_{ik}/dt = 0$ for $m = \text{const.}$

If the projections of the vector of force $\mathbf{F}$ are marked off on the coordinate direction x^k with the letter X_k, i.e. if it is considered that covariant coordinates of the vector of force $\mathbf{F}$ and reaction $\mathbf{R}$ are determined by the following inner products

$$X_k = \mathbf{F}\cdot\mathbf{g}_k, \qquad R_k = \mathbf{R}\cdot\mathbf{g}_k \tag{2.18}$$

it follows that from the equation (2.14) three differential equations of the motion of a point can be obtained in covariant form

$$g_{ik}\frac{D\dot{x}^i}{dt} = X_k + R_k, \qquad (k = 1,2,3) \tag{2.19}$$

where

$$\frac{D\dot{x}^i}{dt} = \frac{d\dot{x}^i}{dt} + \left\{\begin{matrix} i \\ j\ k \end{matrix}\right\}\dot{x}^j\frac{dx^k}{dt} \qquad (i = 1,2,3) \tag{2.19a}$$

are the vector coordinates for acceleration of the observed particle.

The form of these differential equations of motion is more general than the form of differential equations (2.2) as much as the tensor of inertia

$$g_{ik} = g_{ik}(x^1, x^2, x^3) = \mathfrak{z}_{jl}\frac{\partial y^j}{\partial x^i}\frac{\partial y^l}{\partial x^k} \tag{2.20}$$

is more general than the tensor (2.13) and by as much as the curvilinear coordinates offer more possibilities to describe motion in Euclidian space E^3. The equations (2.12) are consequences of equations (2.19) because in the case where $x^i = y^i$ the form of equations (2.12) follows from equations (2.19) and then because $g_{ij} = \mathfrak{z}_{ij}$ all Christoffel's symbols equal zero, $\left\{\begin{matrix} i \\ j\ k \end{matrix}\right\} = 0$ and so $D\dot{y}^i = d\dot{y}^i$. Namely both equations (2.19) and (2.12) describe the motion of a particle as it was a free point, over a variable surface on the assumption that the point is acted upon by the equivalent force (2.18) instead of by the constraint (2.2). But in order to provide a solution for this problem, some additional conditions have to be met as seen in example (2.15). It is necessary to know all the coordinates of the active forces X_k, the nature of the constraints, the coefficients of possible friction, and the law of the change of constraints.

Example. A heavy particle of mass $m=$ const. moves over a vertical circular cylinder whose radius changes according to the law $r = kt^2$, $k > 0$. The cylinder is smooth.

The requirement of smoothness of the cylinder surface makes this problem more definite, because this condition also determines the direction of the constraint reaction. If the motion is described in the cylindrical coordinate system $x^1 = r$, $x^2 = \theta$, $x^3 = z$, then the coordinate vectors are $\mathbf{g}_1 = \mathbf{g}_r$, $\mathbf{g}_2 = \mathbf{g}_\theta$, $\mathbf{g}_3 = \mathbf{g}_z$. With respect to such a coordinate system in which the z-axis is vertical, the reaction of the constraint has only the component in the direction of the normal to the surface i.e. $\mathbf{R} = R\mathbf{g}_r$, and the active force only in the direction of the vertical line $\mathbf{F} = -mg\mathbf{g}_z$. The inertia tensor in this example is

$$g_{ij} = \left\{\begin{matrix} 1 & 0 & 0 \\ 0 & r^2(t) & 0 \\ 0 & 0 & 1 \end{matrix}\right\} \tag{2.21}$$

so the differential equations of motion (2.19) are

$$m\frac{D\dot{r}}{dt} = R, \quad mr^2\frac{D\dot{\theta}}{dt} = 0, \quad m\frac{D\dot{z}}{dt} = -mg, \tag{2.22}$$

hence it simply comes out that:

$$\dot{\theta} = \dot{\theta}_0, \quad \dot{z} = \dot{z}_0 - mgt, \quad \text{and} \quad R = m(2k + 4\dot{\theta}_0/t - k\dot{\theta}_0 t^2). \tag{2.23}$$

The above example clearly shows that, by applying the method of constraint substitution by their reactions, it is not possible to obtain from the differential equations (2.19) three definite equations of motion $x^i = x^i(t)$ and three vector coordinates of the reaction of constraint R_i without any additional conditions. Such a condition here was the given rheonomic coordinate $x^0 = r(t) = kt^2$.

b. Kinematic conditions for the constraints. If the holonomic constraints are given in any analytical form they should satisfy the conditions of point velocity on a surface. If the surface is in the form of (1.5) the condition for the point velocity on the variable surface will be

$$\frac{\partial f}{\partial y^i}\dot{y}^i + \frac{\partial f}{\partial y^0}\dot{y}^0 = 0 \qquad (i = 1,2,3) \tag{2.24}$$

or if (2.9) and (1.9) are considered then

$$\operatorname{grad} f \cdot \mathbf{v} = -\frac{\partial f}{\partial y^0}\dot{y}^0. \tag{2.25}$$

Should the transformation (1.25) be substituted into (2.24) it shows that the condition for velocity of motion of a particle over the described variable surface in a curvilinear coordinate system is

$$\frac{\partial f}{\partial x^i}\dot{x}^i + \frac{\partial f}{\partial x^0}\dot{x}^0 = 0, \tag{2.26}$$

this being the derivative in time of the function (2.2) which describes the variable surface.

From the equation (2.24) or (2.26) it is possible to define one coordinate of the velocity vector as a function of the other two and of the derivative $\dot{\tau}(t)$ providing that the corresponding partial derivative of the function f on the corresponding coordinate is not zero. Since the indices $i = 1, 2, 3$ can be arbitrarily chosen, nothing is lost from the generality if it is assumed that $\partial f/\partial y^3 \neq 0$ holds in equation (2.24) and in the equivalent equation (2.26) $\partial f/\partial x^3 \neq 0$, which is always possible for the functions $f \in C^1$. Then it is possible to determine

$$\dot{y}^3 = -\left(\frac{\partial f}{\partial y^0}\dot{y}^0 + \frac{\partial f}{\partial y^1}\dot{y}^1 + \frac{\partial f}{\partial y^2}\dot{y}^2\right) : \frac{\partial f}{\partial y^3} \neq 0 \tag{2.27}$$

or

$$\dot{x}^3 = -\left(\frac{\partial f}{\partial x^0}\dot{x}^0 + \frac{\partial f}{\partial x^1}\dot{x}^1 + \frac{\partial f}{\partial x^2}\dot{x}^2\right) : \frac{\partial f}{\partial x^3}. \tag{2.28}$$

If (2.27) is substituted into (2.9) or (2.28) into (2.10) then the velocity vector of point motion over the surface $f = 0$ is obtained

$$\mathrm{v} = \dot{y}^1\left(\mathrm{e}_1 - \frac{\partial f}{\partial y^1}\mathrm{e}_3\right) + \dot{y}^2\left(\mathrm{e}_2 - \frac{\partial f}{\partial y^2}\mathrm{e}_3\right) - \dot{y}^0\frac{\partial f}{\partial y^0}\mathrm{e}_3, \tag{2.29}$$

or with respect to curvilinear system

$$\mathrm{v} = \dot{x}^1\left(\mathrm{g}_1 - \frac{\partial f}{\partial x^1}\mathrm{g}_3\right) + \dot{x}^2\left(\mathrm{g}_2 - \frac{\partial f}{\partial x^2}\mathrm{g}_3\right) - \dot{x}^0\frac{\partial f}{\partial x^0}\mathrm{g}_3. \tag{2.30}$$

If a common notation $q = (q^0, q^1, q^2)$ is introduced for independent coordinates y^0, y^1, y^2 and independent curvilinear coordinates x^0, x^1, x^2 where for the function $q^0(t)$ we will keep the name of generalized rheonomic coordinate or simply rheonomic coordinate, the expressions for velocity (2.29) or (2.30) may be written in the form

$$\mathrm{v} = \dot{q}^i \mathrm{g}_i \qquad (i = 0, 1, 2) \tag{2.31}$$

where g_i are base vectors which are in the cases (2.29) and (2.30) equal to

$$\mathrm{g}_i = \left(\mathfrak{z}_i - \frac{\partial f}{\partial x^i}\mathfrak{z}_3\right) = \mathrm{g}_i(q^0, q^1, q^2); \qquad \mathfrak{z}_0 = 0 \tag{2.32}$$

wherefrom it becomes clear that these base vectors are functions of q^1, q^2 and time t through the influence of the rheonomic coordinate $q^0 = r(t)$, and that their structure depends on the structure of the surface $f = 0$. The Greek letters α, β, γ for the indices will denote the independent coordinates and the corresponding vector and tensor coordinates, and will take the values of natural numbers $1, 2, \ldots, n$.

If the equation of the rheonomic constraint is determined by parametric relation (2.4) which is in this notation written as

$$\mathbf{r} = \mathbf{r}(q^0, q^1, q^2), \tag{2.33}$$

then the velocity of motion of the point limited by the rheonomic surface (2.33) will obviously be

$$\mathbf{v} = \frac{\partial \mathbf{r}}{\partial q^i}\dot{q}^i = \frac{\partial \mathbf{r}}{\partial q^0}\dot{q}^0 + \frac{\partial \mathbf{r}}{\partial q^1}\dot{q}^1 + \frac{\partial \mathbf{r}}{\partial q^2}\dot{q}^2 = \frac{\partial \mathbf{r}}{\partial q^0}\dot{q}^0 + \frac{\partial \mathbf{r}}{\partial q^\alpha}\dot{q}^\alpha. \tag{2.34}$$

A comparison with (2.31) and (2.32) shows that the base vectors may be written as partial derivatives of the position vectors (2.33) on generalized coordinates

$$\mathbf{g}_i = \frac{\partial \mathbf{r}}{\partial q^i}. \tag{2.34a}$$

The structure of these base vectors is considerably more complex than the structure of coordinate vectors (1.26) and in general they do not satisfy the inertial relation (1.29). The vectors $\mathbf{g}_1$ and $\mathbf{g}_2$ from (2.34a) as seen in (2.34) correspond to the coordinates q^1 and q^2 and lie in the plane tangent to the surface at the observed point $M(q^1, q^2)$ at any moment of the surface change. The vector $\mathbf{g}_0$ however, does not lie in the plane tangent to the surface except its origin which coincides with the intersection of the tangent plane and the surface. If q are generalized coordinates x^i or y^i from (1.32) it follows that the vector $\mathbf{g}_0$ for the coordinate surface $x^k = x^0(t)$ is proportional to the vector $\mathbf{\ni}_k$, i.e. $\mathbf{g}_0 = \kappa \mathbf{\ni}_k$. For the example of the motion of a point on a circular variable cylinder $f = r - x^0(t) = 0$, for which $\partial f/\partial x^1 \neq 0$ holds, it follows from the relation (2.32) that

$$\mathbf{g}_0 = \left(\mathbf{\ni}_0 - \frac{\partial f}{\partial x^0}\mathbf{\ni}_1\right) = \mathbf{\ni}_1 = \mathbf{\ni}_r, \qquad (\mathbf{\ni}_0 = 0).$$

The same thing follows from (2.34a) because parametric equation (2.33) in the cylinder coordinate system for a circular variable cylinder is $\mathbf{r} = r(t)\mathbf{\ni}_r + x\mathbf{\ni}_z = x^0\mathbf{\ni}_r + z\mathbf{\ni}_z$. For the example of the constraint (1.10) $\mathbf{g}_0 = \mathbf{\ni}_3$; for the constraint (1.15) $\mathbf{g}_0 = (scy^0/c)\mathbf{e}_3$. For the example (2.3) $\partial f/\partial x^1 \neq 0$ and it follows that $\mathbf{g}_0 = \mathbf{\ni}_1 = \mathbf{\ni}_r$ if it is chosen $x^0 = 2t$.

Real variable surfaces which limit the motion of a point in general are not smooth and as such oppose the motion of a point by a resisting or friction force. Those resistant forces $\mathbf{R}_t$ are most often determined experimentally and are included in the active forces. However the surface described by the relations (2.1) or

(2.33) is smooth and therefore a priori recognized as frictionless i.e. that $\mathbf{R}_t = 0$ and as written in (2.7) the reaction of the constraint $\mathbf{R}$ has the direction of the normal to the surface at a given point, $\mathbf{R} = \mathbf{R}_N$. The direction of that force is determined by the gradient of the observed surface. Therefore, the vector $\mathbf{R}$ of the surface reaction (2.1) or (2.2) is

$$\mathbf{R} = \lambda \operatorname{grad} f \tag{2.35}$$

where λ is a multiplier to be determined. For this reason the equation of the motion of point (2.9) over the surface (2.1) or (2.2) becomes more determinate namely

$$\frac{d}{dt}(m\mathbf{v}) = \mathbf{F} + \lambda \operatorname{grad} f, \quad f(\mathbf{r}, t) = 0. \tag{2.36}$$

The coordinate form of these equations is reduced to the equations (2.19) on the understanding that here

$$R_k = \lambda \operatorname{grad} f \cdot \mathbf{\vartheta}_k = \lambda \frac{\partial f}{\partial x^k},$$

so there follow three differential equations of the motion of point

$$\begin{aligned} &g_{ik}\frac{D\dot{x}^i}{dt} = F_k + \lambda \frac{\partial f}{\partial x^k}, \qquad (i,k = 1,2,3) \\ &f(x^1, x^2, x^3) = 0. \end{aligned} \tag{2.37}$$

With a given force $\mathbf{F}$ there are enough equations to determine the motion $x^i = x^i(t)$ and the multiplier λ since it is necessary that the constraint equation satisfies the condition of acceleration

$$\begin{aligned} &\frac{\partial f}{\partial x^i}\left(\frac{D\dot{x}^i}{dt} - \left\{ {i \atop j\,k} \right\}\dot{x}^j\dot{x}^k\right) + \frac{\partial^2 f}{\partial x^i \partial x^j}\dot{x}^i\dot{x}^j \\ &= -\left(\frac{\partial f}{\partial x^0}\ddot{x}^0 + \frac{\partial^2 f}{\partial x^j \partial x^0}\dot{x}^j\dot{x}^0 + \frac{\partial^2 f}{\partial x^0 \partial x^0}\dot{x}^0\dot{x}^0\right) \end{aligned} \tag{2.37a}$$

When using the parametric equations of the constraint (2.34a), the differential equation of the motion of a point (2.36) will be reduced to the covariant coordinate form after the scalar multiplication by the base vectors $\mathbf{g}_\alpha$ ($\alpha = 1,2$) and the vector $\mathbf{g}_0$. As the accepted standpoints are deviated from and for the purpose of better clarity some more details are used to prove this statement. To this end we can multiply the differential equation (2.36) by the vector $\mathbf{g}_1$ and it follows

$$\mathbf{g}_1 \cdot \frac{d}{dt}(m\mathbf{v}) = \mathbf{g}_1 \cdot \mathbf{F} + \lambda \mathbf{g}_1 \cdot \operatorname{grad} f. \tag{2.38}$$

Due to the orthogonality of the vectors $\mathbf{g}_1$ and grad f it follows that $\mathbf{g}_i \cdot \text{grad}\, f = 0$. The inner product of the vector of force $\mathbf{F}$ and vector $\mathbf{g}_1$ represents the covariant coordinate of the generalized force

$$Q_1 = \mathbf{g}_1 \cdot \mathbf{F} = Q_1(q^0, q^1, q^2). \tag{2.39}$$

It is indispensable to comment that if force $\mathbf{F}$ depends on time t, apart from the constraints, then the generalized force (2.39) may be written in the form

$$Q_1 = Q_1(t, q^0, q^1, q^2) \tag{2.39a}$$

since the rheonomic coordinate q^0 is introduced during the transformation of rheonomic constraints. Naturally, the very expression for the force (2.39a) may be reduced to the form (2.39) because of the relation $q^0 = q^0(t) \longleftrightarrow t = t(q^0)$, but one should distinguish the dependence of force upon time apart from the constraints and the dependence which results from the rheonomic constraints.

In the same way the left-hand side of the equation (2.38) can be transformed. If the expressions (2.34) and (2.35) are considered, this side of the equation for $m =$ const. may be reduced to the analytical form by the following procedure:

$$\begin{aligned} \mathbf{g}_1 \frac{d}{dt}(m\mathbf{v}) &= \frac{\partial \mathbf{r}}{\partial q^1} \cdot \frac{d}{dt}\left(m \frac{\partial \mathbf{r}}{\partial q^i} \dot{q}^i\right) \\ &= m \frac{\partial \mathbf{r}}{\partial q^1} \cdot \frac{\partial \mathbf{r}}{\partial q^i} \frac{d\dot{q}^i}{dt} + m \frac{\partial \mathbf{r}}{\partial q^1} \cdot \frac{\partial^2 \mathbf{r}}{\partial q^j \partial q^i} \dot{q}^i \dot{q}^j . \end{aligned} \tag{2.40}$$

Analogous to the formula (2.13) let

$$m \frac{\partial \mathbf{r}}{\partial q^1} \cdot \frac{\partial \mathbf{r}}{\partial q^i} = a_{1i}. \tag{2.41}$$

The inner products

$$\frac{\partial \mathbf{r}}{\partial q^1} \cdot \frac{\partial^2 \mathbf{r}}{\partial q^j \partial q^i} = \Gamma_{1,ji} = a_{1k} \Gamma^k_{ji} \tag{2.42}$$

are obviously functions of coordinates q^0, q^1, and q^2. If (2.42) and (2.41) are returned into (2.40) we get

$$\mathbf{g}_1 \cdot \frac{d}{dt}(m\mathbf{v}) = a_{1i} \frac{d\dot{q}^i}{dt} + a_{1k} \Gamma^k_{ji} \dot{q}^j \dot{q}^i .$$

By substituting this expression, and the expressions from (2.39) and (2.38) into the equation (2.37) we get the differential equations of motion in scalar form

$$a_{1k}\left(\frac{d\dot{q}^k}{dt} + \Gamma^k_{ij} \dot{q}^i \dot{q}^j\right) = Q_1 \tag{2.43}$$

or shorter

$$a_{1k}\frac{D\dot{q}^k}{dt} = Q_1, \tag{2.44}$$

where the notations are obvious.

In the same way the vector equation (2.36) is projected on the direction of vector $\mathbf{g}_2$ and the following is obtained

$$a_{2k}\frac{D\dot{q}^2}{dt} = Q_2. \tag{2.45}$$

Let us multiply the vector equations (2.36) by the vector $\mathbf{g}_0 = \partial\mathbf{r}/\partial q^0$:

$$\mathbf{g}_0 \cdot \frac{d}{dt}(m\mathbf{v}) = \mathbf{g}_0 \cdot \mathbf{F} + \lambda\mathbf{g}_0 \cdot \operatorname{grad} f. \tag{2.46}$$

In this case the term $\mathbf{g}_0 \cdot \operatorname{grad} f$ does not vanish.

If we denote

$$Q_0 = \mathbf{g} \cdot \mathbf{F} = \mathbf{F} \cdot \frac{\partial\mathbf{r}}{\partial q^0} \tag{2.47}$$

and

$$R_0 = \lambda\mathbf{g}_0 \cdot \operatorname{grad} f \tag{2.48}$$

then from (2.46) we obtain the third differential equation of the motion of a point on a moving surface namely

$$a_{0i}\frac{D\dot{q}^i}{dt} = Q_0 + R_0 \tag{2.49}$$

in which an unknown force R_0 appears as a new unknown factor which does not figure in the first two differential equations of the motion (2.44) and (2.45). In a concise form these three differential equations may be written as a system

$$a_{ij}\frac{D\dot{q}^j}{dt} = Q_i + R_i, \qquad (i, j = 0, 1, 2;\ R_1 = R_2 = 0), \tag{2.50}$$

where

$$a_{ij} = m\frac{\partial\mathbf{r}}{\partial q^i} \cdot \frac{\partial\mathbf{r}}{\partial q^j} = a_{ji}(q^0, q^1, q^2) \tag{2.51}$$

are covariant coordinates of the inertia tensor and

$$\frac{D\dot{q}^j}{dt} = \frac{d\dot{q}^j}{dt} + \Gamma^j_{ik}\dot{q}^j\frac{dq^k}{dt} \tag{2.52}$$

are covariant coordinates of the acceleration vector of the point. Prior to any analysis of the equation (2.49) or of the magnitude R_0 it should be noted that the system of differential equations (2.50) describes the same motion as the system of equations (2.37) together with the equation of the surface.

In a curvilinear coordinate system x^i the equation for the rheonomic surface can be described by the equation $f = x^3 - x^0(t) = 0$ for instance. Then the system of differential equations (2.37) is reduced to

$$\text{(2.53)} \qquad \mathfrak{s}_{1i}\frac{D\dot{q}^i}{dt} = F_1, \quad \mathfrak{s}_{2i}\frac{D\dot{q}^i}{dt} = F_2, \quad \mathfrak{s}_{3i}\frac{D\dot{q}^i}{dt} = F_3 + \lambda,$$

and the conditions of acceleration to

$$\text{(2.53a)} \qquad \frac{D\dot{x}^3}{dt} = \Gamma^3_{jk}\dot{x}^j\dot{x}^k - \frac{\partial f}{\partial x^0}\ddot{x}^0 = \ddot{x}^0 + \Gamma^3_{jk}\dot{x}^j\dot{x}^k$$

since

$$\frac{\partial f}{\partial x^1} = \frac{\partial f}{\partial x^2} = 0; \quad \frac{\partial f}{\partial x^3} = 1.$$

Written in this way, differential equations (2.37) are fully similar with the differential equations (2.50) or one to one correspondently with (2.44), (2.45) and (2.49). They are identical if curvilinear coordinates x are chosen for generalized coordinates q and the known function of time for the zero rheonomic coordinate $q^0 = x^3(t) = \tau(t)$, which is possible. In such a case, as it can be seen, the generalized force R_0 equals the multiplier λ of the rheonomic constraint. Though the differential equations of the motion (2.37) and (2.50) describe the same motion and are identical in some of the examples, from the mathematical point of view we speak of two geometries: Euclidian E^3 in which motion is described by means of relations (2.37) and the variable interior geometry on $(2+1)$-dimensional manifolds M_{2+1} by which the same motion is described by means of differential covariant equations (2.50). For the case of invariable constraints the equation (2.49) is nullified and the motion of a point over a surface is described with only the two equations (2.44) and (2.45) in the spaces M_2, but in the number and structure of equations (2.37) in E^3 no change occurs. The essential difference between the equations (2.37) and (2.50) lies in the inertia tensors (2.51) and (2.17). This further involves a difference among the Christoffel symbols $\left\{ {}_{j}{}^{i}{}_{k} \right\}$ in equations (2.37) and the coefficients of connection in equations (2.50). In order to estimate this difference it is necessary to identify the coefficients of connection Γ^i_{jk}, which are introduced here with the relation (2.42). And the identity relation implies that:

1) the covariant derivatives $\nabla_k a_{ji}$ of the tensor a_{ji} on coordinates q^k equal zero;

2) the coefficients of connection Γ^k_{ij} are Christoffel symbols of the second kind over tensor (2.51) i.e. that the relations

$$\text{(2.54)} \qquad \nabla_k a_{ji} = 0$$

and

$$\Gamma^k_{ij} = a^{kl}\Gamma_{ij,l} = \frac{1}{2}a^{kl}\left(\frac{\partial a_{jl}}{\partial q^i} + \frac{\partial a_{il}}{\partial q^j} - \frac{\partial a_{ij}}{\partial q^l}\right) \tag{2.55}$$

hold.

If we start from the known relations in affine geometry in which the factor of a normal $\mathbf{n}$, $|\mathbf{n}| = 1$, to the surface figures, i.e. from the relations

$$\frac{\partial^2 \mathbf{r}}{\partial q^\alpha \partial q^\beta} = \Gamma^\gamma_{\alpha\beta}\mathbf{g}_\gamma + b_{\alpha\beta}\mathbf{n} \tag{2.56}$$

where $b_{\alpha\beta}$ is the second metric tensor of the surface, the equation (2.36) if (2.34) is kept in mind, may be written in the following developed form

$$\begin{aligned} &m[\mathbf{g}_\alpha \ddot{q}^\alpha + (\Gamma^\gamma_{\alpha\beta}\mathbf{g}_\gamma + b_{\alpha\beta}\mathbf{n})\dot{q}^\alpha \dot{q}^\beta] \\ &\quad + \left[\mathbf{g}_0 \ddot{q}^0 + 2\frac{\partial^2 \mathbf{r}}{\partial q^0 \partial q^\alpha}\dot{q}^0 \dot{q}^\alpha + \frac{\partial^2 \mathbf{r}}{\partial q^0 \partial q^0}\dot{q}^0 \dot{q}^0\right] \\ &= \mathbf{F} + \lambda \operatorname{grad} f = \mathbf{F} + R_N \mathbf{n}. \end{aligned} \tag{2.57}$$

For a scleronomic surface the second square bracket in the preceding equation vanishes and the constraint reaction normal to the surface is obtained:

$$R_N = m b_{\alpha\beta}\dot{q}^\alpha \dot{q}^\beta - F_N,$$

where $F_N = \mathbf{F} \cdot \mathbf{n}$. However, for the variable (rheonomic) surface the expression for the constraint reaction R_N is considerably more complex, namely

$$R_N = m\left[b_{\alpha\beta}\dot{q}^\alpha \dot{q}^\beta + \mathbf{g} \cdot \mathbf{n}\ddot{q}^0 + \left(2\frac{\partial^2 \mathbf{r}}{\partial q^0 \partial q^\alpha}\dot{q}^0 \dot{q}^\alpha + \frac{\partial^2 \mathbf{r}}{\partial q^0 \partial q^0}\dot{q}^0 \dot{q}^0\right)\mathbf{n}\right] - F_N. \tag{2.58}$$

Therefore, the reaction R_N of the rheonomic constraint depends on the acceleration of the change in the constraint $\ddot{q} = 0$, because in general $\mathbf{g}_0 \cdot \mathbf{n} \neq 0$. This more clearly separates the notions of the constraint reaction R_N from the "reaction of the constraint change" R_0.

If the reaction of an smooth rheonomic surface is written in a vector form

$$\mathbf{R} = \lambda \operatorname{grad} f = R_N \mathbf{n}$$

it follows that

$$R_N = \mathbf{R} \cdot \mathbf{n},$$

and

$$R_0 = \mathbf{R} \cdot \mathbf{g}_0 = R_N \mathbf{n} \cdot \mathbf{g}_0 = R_N g_0 \cos(\mathbf{n}, \mathbf{g}_0). \tag{2.59}$$

If the above is multiplied by the vector $\mathbf{g}_0$ the equation (2.57) shows that R_0 depends on the complex sum of different terms including all other derivatives of generalized coordinates. The covariant equation (2.49) gives this in a concise covariant form.

Example. A heavy particle of mass $m =$ const. moves on a steep plane making angle α with a horizontal line and moves translationally at a constant velocity v on a horizontal plane. If a coordinate system is chosen so that y^3 is the vertical axis, the coordinate $y^1 = x^1 = q^1$ is independent and y^2 makes an angle α with the steep plane, then the motion is limited by the rheonomic constraint

$$f = \frac{y_3}{y_2 - vt} - \operatorname{tg}\alpha = 0.$$

If we choose that $q^0 = vt$, the parametric equation of the rheonomic constraint is

$$\mathbf{r} = q^0\mathbf{e}_2 + q^1\mathbf{e}_1 + q^2(\cos\alpha\mathbf{e}_2 + \sin\alpha\mathbf{e}_3).$$

The coordinates of the inertia tensor (2.51) are constants

$$a_{ij} = \left\{\begin{matrix} m & 0 & m\cos\alpha \\ 0 & m & 0 \\ m\cos\alpha & 0 & m \end{matrix}\right\}.$$

The generalized forces are $Q_0 = Q_1 = 0$, $Q_2 = -mg\cos\alpha$.

Since all coefficients a_{ij} are constant, the differential equations of motion (2.50) are easy to write, since $D\dot{q}^i = d\dot{q}^i$, and it follows

$$\begin{aligned} &m(\ddot{q}^0 + \cos\alpha\ddot{q}^2) = R_0, \\ &m\ddot{q}^1 = 0, \\ &m(\ddot{q}^0\cos\alpha + \ddot{q}^2) = -mg\sin\alpha. \end{aligned} \tag{2.59a}$$

Then we have

$$R_0 = m\ddot{q}^2\cos\alpha = -mg\sin\alpha\cos\alpha.$$

For the case where time is chosen for the rheonomic coordinate, i.e. $q^0 = t$, from the constraint equation it would follow that

$$R_0 = -mgv\sin\alpha\cos\alpha.$$

Kinetic energy. When describing the motion of a point by means of analytical expressions it is necessary to first opt either for a) dynamic method of constraint elimination (2.6)–(2.20), or for b) kinematic conditions of constraint (2.25)–(2.50).

The kinetic energy in the motion of a material particle with inertia is a scalar invariant

$$T = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\mathbf{v}\cdot\mathbf{v} > 0, \qquad \text{for every } v \neq 0 \tag{2.60}$$

As such, it neither changes the physical substance nor the mathematical form on the occasion of various transpositions from one coordinate system into another. However, a question arises: if the velocity vector is described by various expressions and is subjected to the conditions such as (2.9), (2.10), (2.29), (2.30) or (2.34) would the analytical expressions for the form (2.0) not differ for the same motion of a point over the rheonomic surface? In method a) motion is considered as motion of a free material particle whose velocity is given in the expressions (2.9) or (2.10). By substituting (2.9) into the expression (2.60) it follows, as already known,

$$T = \tfrac{1}{2}e_{ij}\dot{y}^i\dot{y}^j; \quad e_{ij} = m\delta_{ij}, \qquad (i,j = 1,2,3), \tag{2.61}$$

and, if the curvilinear coordinate system x^i is chosen in which the velocity vector of the point is determined by the expression (2.10), then we have

$$T = \tfrac{1}{2}\vartheta_{kl}\dot{x}^k\dot{x}^l, \tag{2.62}$$

where, as in (2.17):

$$\vartheta_{kl} = e_{ij}\frac{\partial y^i}{\partial x^k}\frac{\partial y^j}{\partial x^l} = \vartheta_{kl}(x^1, x^2, x^3). \tag{2.63}$$

In such a procedure for describing motion of a particle over a variable surface, expressions for kinetic energy (2.61) and (2.62) are the same as for the motion over an invariable (scleronomic) surface.

When describing the motion on expanded configurational manifolds M_{2+1} i.e. by method b) corresponding velocities (2.34) or adequately (2.27) and (2.28) should be substituted into the expression for kinetic energy (2.60) or (2.61) and (2.62) accordingly. By substituting (2.34) into (2.60) we get

$$T = \tfrac{1}{2}a_{ij}\dot{q}^i\dot{q}^j \qquad (i,j = 0,1,2), \tag{2.64}$$

or in a developed form

$$T = \tfrac{1}{2}a_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta + a_{0\alpha}\dot{q}^0\dot{q}^\alpha + \tfrac{1}{2}a_{00}\dot{q}^0\dot{q}^0, \quad (\alpha,\beta = 1,2) \tag{2.65}$$

where the inertia tensor a_{ij} is determined by formula (2.51). All these formulas for kinetic energy are homogeneous quadratic forms. They should be so written even if we chose $q^0 = t$. Only for the purpose of final calculation of kinetic energy, if necessary, the velocity values will be introduced into formula (2.65) and with them $\dot{q}^0 = 1$. In other words, doing different mathematical operations in the expression for kinetic energy, which is a function of the coordinates q^0, q^1, q^2 and the respective velocity vector coordinates $\dot{q}^0, \dot{q}^1, \dot{q}^2$ it will be necessary to retain the form (2.65) regardless of the accepted and widespread expression in mechanics

$$T = \tfrac{1}{2}a_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta + a_j\dot{q}^j + \tfrac{1}{2}a_0, \tag{2.66}$$

when it is emphasized that $a_0 = a_0(q^1, a^2, t)$ is a function of coordinates and time. Then, independently of the definition of inertia coefficients (2.51), the coefficient a_0 is attributed the energy dimension E i.e.

$$[\dim a] = [\dim E] = \mathrm{ML^2T^{-2}}. \tag{2.67}$$

The weakness of expressions (2.66) and (2.67) will become clearer when the link between generalized velocities and momentum covector coordinates is determined (3.30).

Differential equations of motion in Lagrange form. The differential equations of motion of a point over a surface (2.37), as differential equations of the form (2.50), can be written in the form known as Lagrange equations of the first kind and Lagrange equations of motion of the second kind. Indeed, differential equations (2.37) in more developed forms are

$$\mathfrak{z}_{ik}\left(\frac{d\dot{x}^k}{dt} + \left\{ {k \atop j\,l} \right\}\dot{x}^j\dot{x}^l\right) = F_k + \lambda\frac{\partial f}{\partial x^k}$$

or

$$\frac{d}{dt}(\mathfrak{z}_{ik}\dot{x}^k) - \frac{\partial \mathfrak{z}_{ik}}{\partial x^j}\dot{x}^k\dot{x}^j + [jl,k]\dot{x}^j\dot{x}^l = F_k + \lambda\frac{\partial f}{\partial x^k}.$$

If the indices are developed:

$$[jl,k] = \frac{1}{2}\left(\frac{\partial \mathfrak{z}_{lk}}{\partial x^j} + \frac{\partial \mathfrak{z}_{jk}}{\partial x^l} - \frac{\partial \mathfrak{z}_{jl}}{\partial x^k}\right),$$

after a permitted substitution of addition indices, we get

$$\frac{d}{dt}(\mathfrak{z}_{ik}\dot{x}^i) - \frac{1}{2}\frac{\partial \mathfrak{z}_{ij}}{\partial x^k}\dot{x}^i\dot{x}^j = F_k + \lambda\frac{\partial f}{\partial x^k}.$$

On the other side, based on (2.62), it can be seen that

$$\frac{\partial T}{\partial \dot{x}^k} = \mathfrak{z}_{jk}\dot{x}^j, \quad \frac{\partial T}{\partial x^k} = \frac{1}{2}\frac{\partial \mathfrak{z}_{ij}}{\partial x^k}\dot{x}^i\dot{x}^j \tag{2.68}$$

and the preceding relations are then reduced to three differential equations of motion of a particle over a variable surface

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}^k} - \frac{\partial T}{\partial x^k} = F_k + \lambda\frac{\partial f}{\partial x^k} \qquad (k = 1,2,3) \tag{2.69}$$

which are, together with constraint equation $f(x^1, x^2, x^3, t) = 0$, equivalent to equations (2.37).

Analogous to the above transformation of differential equations from one form into another the covariant differential equations of motion (2.50), with regard to (2.64) and (2.65), can be written in the form

$$\frac{d}{dt}(a_{ik}\dot{q}^k) - \frac{1}{2}\frac{\partial a_{jk}}{\partial q^i}\dot{q}^j\dot{q}^k = Q_i + R_i \qquad (i, j, k = 0, 1, 2).$$

However, if

$$\frac{\partial T}{\partial \dot{q}^i} = a_{ik}\dot{q}^k, \quad \frac{\partial T}{\partial q^i} = \frac{1}{2}\frac{\partial a_{jk}}{\partial q^i}\dot{q}^j\dot{q}^k$$

then Lagrange's differential equations of motion of the second kind will follow

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^i} - \frac{\partial T}{\partial q^i} = Q_i \tag{2.70}$$

or

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\alpha} - \frac{\partial T}{\partial q^\alpha} = Q_\alpha \qquad (\alpha = 1, 2) \tag{2.70a}$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^0} - \frac{\partial T}{\partial q^0} = Q_0 + R_0, \tag{2.70b}$$

which are equivalent to equations (2.64). This differential form of motion equations enables a clearer analysis of the law of change of energy of the particle moving over a rheonomic surface, which will be useful for an understandable generalization later.

The law of change and conservation of energy. A parallel description in two ways of motion of a particle over a variable surface contributes considerably to clarify the need for modification of laws of energy of point motion constrained by a rheonomic constraint.

a) It is first assumed that active forces F_k have a potential Π and

$$F_k = -\frac{\partial \Pi}{\partial x^k} \qquad (k = 1, 2, 3) \tag{2.71}$$

where $\Pi = \Pi(x^1, x^2, x^3, t)$ is a function of position x and time t. Multiplying each equation (2.69) by corresponding differential $dx^k = \dot{x}^k\, dt$ and summing by index k it follows that

$$\dot{x}^k d\left(\frac{\partial T}{\partial \dot{x}^k}\right) - \frac{\partial T}{\partial x^k}dx^k = -\frac{\partial \Pi}{\partial x^k}dx^k + \lambda\frac{\partial f}{\partial x^k}dx^k.$$

But, as

$$d\left(\frac{\partial T}{\partial \dot{x}^k}\dot{x}^k\right) - \frac{\partial T}{\partial x^k}d\dot{x}^k - \frac{\partial T}{\partial x^k}dx^k = d\left(\frac{\partial T}{\partial \dot{x}^k}\dot{x}^k - T\right) = dT,$$

it follows that

$$dT = -\frac{\partial \Pi}{\partial x^k} dx^k + \lambda \frac{\partial f}{\partial x^k} dx^k. \tag{2.72}$$

If we take condition (2.26) into consideration then

$$dT + d\Pi = -\frac{\partial \Pi}{\partial t} dt + \lambda \frac{\partial f}{\partial t} dt$$

or

$$T + \Pi = -\int \left(\frac{\partial \Pi}{\partial t} - \lambda \frac{\partial f}{\partial t} \right) dt. \tag{2.73}$$

This relation represents the law of change of energy for the motion of a point over a rheonomic surface $f(x^1, x^2, x^3, \tau(t)) = 0$. It is evident that this law (2.73) has a somewhat simpler form if the potential energy does not explicitly depend on time t in which case

$$T + \Pi = \int \lambda \frac{\partial f}{\partial t} dt. \tag{2.73a}$$

Since we also call the differential equation (2.72) by the name "law of change of energy", which it surely is, the relation (2.73), may be called the "law of energy in integral form" or more simple the "law or integral of energy". This is different from the notion of the "first integral of differential equations of motion (2.69)" or the notion "law of energy conservation". The relation (2.74) will be the first integral of differential equations of motion if one of the following conditions is satisfied:

$$\begin{aligned} &1) \quad \lambda = \frac{\partial \Pi}{\partial t} : \frac{\partial f}{\partial t}, \\ &2) \quad \frac{\partial \Pi}{\partial t} = 0 \wedge \frac{\partial f}{\partial t} = 0, \qquad \text{or} \\ &3) \quad \frac{\partial \Pi}{\partial t} = 0 \wedge \lambda \frac{\partial f}{\partial t} = R(t) \end{aligned} \tag{2.74}$$

when $R(t)$ is an integrable function, i.e. when there is a function $P(t)$ for which

$$R(\tau) = -\frac{\partial P}{\partial \tau}. \tag{2.75}$$

In this case integral (2.73) could be reduced to

$$T + \Pi + P = C = \text{const.}, \tag{2.76}$$

where

$$P(t) = -\int R\, dt. \tag{2.77}$$

In general, the derivative $\partial f/\partial t$ is a function of three position coordinates of a particle and time t. However, since one dependent coordinate can be expressed, from a constraint equation, as a function of the remaining two and one rheonomic coordinate $\tau(t)$, then only two unknowns will appear in two independent differential equations and they can be determined dependent on time. By substituting into a third equation, in which partial derivative $\partial f/\partial t$ appears with a constraint multiplier, the constraint reaction at the end appears in the form of a function of the rheonomic coordinate, because the multiplier can be determined from the conditions of acceleration. However, in order to determine the first integral (2.77) it is necessary that R is a priori a known function of the rheonomic coordinate and that the natural potential does not generaly depend on time. The same result is obtained by means of differential equations (2.70).

b) If the differential equations of motion (2.70) are combined with differentials $dq^i = \dot{q}^i\, dt$ $(i = 0, 1, 2)$ it follows that

$$\dot{q}^i\, d\left(\frac{\partial T}{\partial \dot{q}^i}\right) - \frac{\partial T}{\partial q^i}\, dq^i = (Q_i + R_i)\, dq^i. \tag{2.78}$$

For a potential system of force with the natural potential $\Pi = \Pi(q^0, q^1, q^2)$ and generalized forces

$$Q_i = -\frac{\partial \Pi}{\partial q^i}, \tag{2.79}$$

for which the following conditions are satisfied

$$\frac{\partial Q_i}{\partial q^j} = \frac{\partial Q_j}{\partial q^i} \qquad (i, j = 0, 1, 2) \tag{2.80}$$

the expression $(\partial\Pi/\partial q^i)\, dq^i$ will represent the total differential of potential Π. This means that

$$Q_i\, dq^i = -d\Pi. \tag{2.81}$$

On the other hand, for the accepted formula of kinetic energy (2.64) or (2.65) the kinetic energy differential is

$$dT = \frac{\partial T}{\partial q^i} dq^i + \frac{\partial T}{\partial \dot{q}^i} d\dot{q}^i, \tag{2.82}$$

so

$$\dot{q}^i\, d\left(\frac{\partial T}{\partial \dot{q}^i}\right) = d\left(\dot{q}^i \frac{\partial T}{\partial \dot{q}^i}\right) - \frac{\partial T}{\partial \dot{q}^i} d\dot{q}^i = d(2T) - \frac{\partial T}{\partial \dot{q}^i} d\dot{q}^i.$$

Because of this and relation (2.81) differential expression (2.78) is reduced to

$$d(T + \Pi) = R_0\, dq^0, \tag{2.83}$$

or

$$T + \Pi = \int R_0\, dq^0 + C, \tag{2.84}$$

which coincides with integral relation (2.73). On the assumption that R_0 is an integrable function of the rheonomic coordinate q^0, which has its own rheonomic potential

$$P(q^0) \stackrel{\text{def}}{=} -\int R_0(q^0)\, dq^0, \tag{2.85}$$

and also

$$R_0 = -\frac{\partial P}{\partial q^0} \tag{2.86}$$

the integral of energy for the motion of a point over a rheonomic surface there will exist the integral

$$T + \Pi + P = C \tag{2.87}$$

or

$$T + V = C, \tag{2.88}$$

where V is the potential

$$V = \Pi + P \tag{2.89}$$

consisting of natural potential Π and the potential P resulting from the change of constraint [16].

Example. Let the first differential equation of the system (2.59a) is multiplied by differential dq^0, the second by dq^1, and the third equation by dq^3; adding one obtains

$$\begin{aligned} &m(\dot{q}^0\, d\dot{q}^0 + \dot{q}^0 \cos\alpha\, d\dot{q}^2 + \dot{q}^2 \cos\alpha\, d\dot{q}^0 + \dot{q}^1\, d\dot{q}^1 + \dot{q}^2\, d\dot{q}^2) \\ &\quad = -mg \sin\alpha\, dq^2 + R_0\, dq^0. \end{aligned}$$

On the other hand kinetic energy $T = \frac{1}{2} a_{ij} \dot{q}^i \dot{q}^j$, since the tensor a_{ij} (given before (2.59a)) is determined, is

$$T = (m/2)(\dot{q}^0 \dot{q}^0 + \dot{q}^1 \dot{q}^1 + \dot{q}^2 \dot{q}^2 + 2\dot{q}^0 \dot{q}^2 \cos\alpha).$$

Hence it follows that

$$dT = -mg \sin\alpha\, dq^2 + R_0\, dq^0.$$

If it is known that $Q_2 = -\partial\Pi/\partial q^2 = -mg\sin\alpha = \text{const.}$, then it results that $\Pi = mgq^2\sin\alpha$ and integral (2.84) is obtained. Since in this case $R_0 = \text{const.}$, the law of energy is

$$E = T + \Pi + mvgt\sin\alpha\cos\alpha = C = \text{const.} \tag{2.90}$$

This agrees with integral (2.73), as in this example

$$\frac{\partial\Pi}{\partial t} = \frac{\partial}{\partial t}(mg\xi\sin\alpha) = 0, \qquad \frac{\partial f}{\partial t} = \frac{v\,\mathrm{tg}\,\alpha}{y_2 - vt},$$
$$\lambda = \frac{mg(y_2 - vt)}{1 + \mathrm{tg}^2\alpha}, \qquad \text{so} \quad \lambda\frac{\partial f}{\partial t} = mgv\sin\alpha\cos\alpha.$$

Conditions for the absolute rest of a point on the rheonomic surface. For any holonomic constraint $f(\mathbf{r}, t) = 0$ the conditions of rest of a particle are: velocity of the observed particle equals zero and the sum of all forces acting on that point equals zero. For equations (2.36), therefore, the conditions of rest are as follows:

$$\mathbf{F} + \lambda\,\mathrm{grad}\,f = 0 \quad \text{and} \quad \mathbf{v} = 0. \tag{2.91}$$

But, as $\mathbf{v} = 0$, another condition results from the relation (2.25):

$$\frac{\partial f}{\partial \tau}\dot{\tau} = 0 \quad \text{i.e.} \quad \frac{\partial f}{\partial t} = 0. \tag{2.92}$$

This condition does not show, as it is often understood, that a constraint must not depend on time if a point is to rest in equilibrium, but only that for any time t, a partial derivative of a constraint in time or on a rheonomic coordinate should be zero at the point of rest.

In coordinate form conditions (2.91), as seen from the equations (2.37) or (2.69), are

$$F_k + \lambda\frac{\partial f_k}{\partial x^k} = 0 \qquad (k = 1, 2, 3) \tag{2.93}$$

which together with (2.92) represent the conditions of a point resting on a rheonomic surface $f(x^1, x^2, x^3, \tau(t)) = 0$.

With respect to independent generalized coordinates the conditions of a point at rest, while a variable surface is active and has parametric equations $\mathbf{r} = \mathbf{r}(q^0, q^1, q^2)$ are reduced to three equations

$$Q_1 = 0, \qquad Q_2 = 0, \tag{2.94}$$
$$Q^0 + R_0 = 0, \tag{2.95}$$

which is obvious if the motion is described with the aid of differential equations (2.50).

For a scleronomic retaining constraint-surface, two equations (2.94) are sufficient to determine the conditions for point equilibrium. The same two conditions are also sufficient for equilibrium on a variable surface when observed with respect to that surface, i.e. if change or motion of the constraint itself is not considered. Therefore, one should distinguish two notions of equilibrium for a point on a rheonomic surface namely

1) relative equilibrium determined by: either equations (2.93), provided that all velocity coordinates are $\dot{x}^1 = \dot{x}^2 = \dot{x}^3 = 0$ or equations (2.94) provided that generalized velocities are $\dot{q}^1 = \dot{q}^2 = 0$;

2) equilibrium determined by: either equations (2.92) and (2.93) provided that $\dot{x}^1 = \dot{x}^2 = \dot{x}^3 = 0$, or equations (2.94) and (2.95) provided that generalized velocities are $\dot{q}^1 = \dot{q}^2 = 0$.

Example. A particle of mass G on a variable ellipsoid. Depending whether variable ellipsoid is determined by equation

$$f(y,t) = c^2(t)(y_1^2 + y_2^2) + a^2(t)y_3^2 - a^2(t)b^2(t) = 0, \tag{2.96}$$

or relations (2.3), i.e.

$$y_1 = a(t)\cos\theta\sin\varphi, \quad y_2 = a(t)\sin\theta\cos\varphi, \quad y_3 = a(t)\cos\varphi \tag{2.97}$$

the problem should be solved with appropriate equations (2.92) and (2.93) for the case (2.96), namely with the aid of equations (2.94) and (2.95) for the case of constraints in form of (2.97). In either case, the simplest solution is to select time t for a rheonomic coordinate.

Let the half-axis c be vertical and axis y_3 be directed upwards. Then it follows from equation (2.93) for $x = y$ that

$$y_1 = 0, \quad y_2 = 0, \quad y_3 = \frac{G}{2a^2\lambda}, \tag{2.98}$$

and from condition (2.92)

$$c(y_1^2 + y_2^2 + a^2)\dot{c} + a(y_3^2 - c^2)\dot{a} = 0, \tag{2.99}$$

where $\dot{c} = dc/dt = \partial c/\partial t$. By substituting (2.98) into (2.99) we get

$$\lambda = \pm\frac{G}{2a^2c} \implies y_3 = \pm c.$$

Therefore, the points $(0, 0, \pm c)$ are equilibrium positions on an ellipsoid provided condition (2.99) is satisfied, namely that the ellipsoid axis under the influence of force of gravity is not changing i.e. that

$$\frac{dc}{dt} = 0.$$

In the case of parametric constraints (2.93) let the generalized coordinates be $q^1 = \varphi$ and $q^2 = \theta$, $q^0 = t$. As the force vector is $\mathbf{F} = -G\mathbf{e}_3 = Y^3\mathbf{e}_3$, it follows that $Q_0 = Y_3(\partial y/\partial\theta) = -G\dot{c}\cos\varphi$, $Q_1 = Gc(t)\sin\varphi$, $Q_2 = Y_3(\partial y^3/\partial\theta) = 0$. By substituting into conditions (2.94) and (2.95) it comes out that in equilibrium: $\sin\varphi = 0$ and $R_0 = G\dot{c}\cos\varphi_{|\varphi=k\pi}$. As shown, equilibrium positions $\varphi = k\pi$ $(k = 0, 1, 2, \dots)$ are possible provided that a heavy particle is acted upon by power $R_0 = G\dot{c}$ at points $2k\pi$ and $R_0 = -G\dot{c}$ at points $((2k+1)\pi)$ or if $\dot{c} = 0$, which agrees with the above result. It is convenient to note here that the power of a rheonomic constraint $R_0(t) = |G\dot{c}|$ differs from the reaction of the point on the surface.

3. Rheonomic systems

Main concepts. The concept of a rheonomic system means a non-empty set of particles whose motion is limited by a non-empty set of rheonomic constraints. Therefore, there is at least one particle whose motion is limited by at least one rheonomic constraint. With this definition, even the motion of one particle over a rheonomic surface represents a rheonomic system. If there is a notion which defines a system more precisely, then such a more definite notion will be used; for example, the motion of a point over a smooth retaining surface, or the motion of a point on a retaining curved line or a pendulum varying in length etc. It is stated by relation (1.7) that the position of a point on a rheonomic surface may be defined with the use of 2 + 1 parameters. That is why it can be said that the motion of a point over a surface is the motion in 2 + 1 dimensional space. However, the notion of a 2 + 1 dimensional space is much more general than the concept of motion of a point over a variable surface. This may be a 2 + 1 dimensional manifold (several particles whose motion is limited by a definite number of rheonomic constraints). However, there is no arbitrariness in understanding the systems cited. Just as the rheonomic system has a strict definition so should other notions characteristic for rheonomic systems be as strictly defined. A rheonomic system is composed of a set of N particles M_ν, mass m_ν $(\nu = 1, 2, \dots, N)$, whose motion is limited by non-empty set of k rheonomic constraints

$$f_\mu(\mathbf{r}_1, \dots, \mathbf{r}_N, \mathbf{v}_1, \dots, \mathbf{v}_N, t) = 0, \quad (\mu = 1, 2, \dots, k;\ k < 3N)$$

where $\mathbf{r}_\nu$ are position vectors of points M_ν and $\mathbf{v}_\nu$ their velocities.

If we assume that constraints do not depend on velocities or that the constraints may be reduced by integral calculation to a form of the so-called holonomic constraints

$$f_\mu(\mathbf{r}_1, \dots, \mathbf{r}_N, t) = 0 \tag{3.1}$$

then the system is more precisely termed as a holonomic system but, if it is wished to emphasize that constraints on it do depend on time then one says a holonomic rheonomic system. Also related to the motion of these systems is the following text unless otherwise indicated. It is always possible to introduce such a function

$$\tau = \tau(t) \longleftrightarrow t = t(\tau)$$

that constraints (3.1) can be written depending on the function $\tau(t)$, i.e.

$$f_\mu(\mathbf{r}_1, \dots, \mathbf{r}_N, \tau) = 0. \tag{3.2}$$

An analysis of the above subject matter in which function τ was introduced as $\tau = y^0(t)$, $\tau = x^0(t)$, $\tau = q^0(t)$ directs one to introduce the name of a rheonomic coordinate for function τ. Corresponding to it is the zero or $n+1$-st index q^0 or q^{n+1}. It appears as such for rheonomic systems only. In this sense and with respect to a rectilinear coordinate system, same as in (1.4), the holonomic rheonomic retaining constraints (3.2) can be recorded in the form

$$f_\mu(y_1^1, y_1^2, y_1^3, \dots, y_N^1, y_N^2, y_N^3; y^0) = 0 \tag{3.3}$$

or after transformations (1.25),

$$f_\mu(x_1^1, x_1^2, x_1^3, \dots, x_N^1, x_N^2, x_N^3; x^0) = 0 \tag{3.4}$$

If a simpler single-index notation of coordinates is wanted then the following can be written

$$x_\nu^1 = x^{3\nu-2}, \quad x_\nu^2 = x^{3\nu-1}, \quad x_\nu^3 = x^{3\nu} \tag{3.5}$$

When denoting coordinates in this way, constraint equations (3.4) can be more simply written

$$f_\mu(x^0, x^1, x^2, \dots, x^{3N}) = 0, \quad f_\mu \in C^1; \tag{3.6}$$

$$\operatorname{rank}\left\{\frac{\partial f_\mu}{\partial x^i}\right\} = k, \qquad k < 3N, \quad \mu = 1, 2, \dots, k; \ i = 0, 1, \dots, 3N. \tag{3.7}$$

In this way a rheonomic system of N material particles M_ν of mass m_ν whose motion is limited by a set of k holonomic rheonomic constraints (3.3) is reduced, analogous to the motion of a point over a rheonomic surface (2.2), to the study of motion of a representative point on a $(3N - k + 1)$-dimensional manifold M_{n+1}, where $n = 3N - k$. On every point of this manifold, a set of constraints (3.6) enables motion of a system of particles with velocities $\dot{x}^0, \dot{x}^1, \dots, \dot{x}^{3N}$, for which the determinant of the matrix (3.7) is not zero. Conditions for velocities are obtained from time derivatives of constraints (3.4) in time, i.e.

$$\frac{\partial f_\mu}{\partial x^i}\dot{x}^i = \frac{\partial f_\mu}{\partial x^j}\dot{x}^j + \frac{\partial f_\mu}{\partial x^0}\dot{x}^0, \quad (j = 1, 2, \dots, 3N), \tag{3.7a}$$

which can be rewritten as

$$\frac{\partial f_\mu}{\partial x^1}\dot{x}^1 + \dots + \frac{\partial f_\mu}{\partial x^k}\dot{x}^k = -\frac{\partial f_\mu}{\partial x^\alpha}\dot{x}^\alpha - \frac{\partial f_\mu}{\partial x^0}\dot{x}^0$$

where indices α have values $k+1, k+2, \dots, 3N$. As regards the condition that

$$\det\left\{\frac{\partial f_\mu}{\partial x^\sigma}\right\} \neq 0 \qquad (\mu, \sigma = 1, 2, \dots, k) \tag{3.8}$$

the k velocity coordinates $\dot{x}^\sigma$ are determined as a linear combination of the other $3N - k + 1$ velocity coordinates $\dot{x}^\alpha$ and $\dot{x}^0$.

In the neighborhood of point x for which condition (3.8) is satisfied, the k coordinates x^σ can be obtained from the set (3.6) as function of the remaining coordinates:

$$x^\sigma = x^\sigma(x^0, x^{k+1}, \dots, x^{3N}). \tag{3.9}$$

In the same way, the set (3.4) when $\det\{\partial f_\mu / \partial y^\sigma\} \neq 0$ can serve to determine the following

$$y^\sigma = y^\sigma(y^0, y^{k+1}, \dots, y^{3N}), \tag{3.10}$$

where $y^{k+1}, \dots, y^{3N}$ are independent rectilinear Cartesian coordinates. Due to transformations (1.25) it follows that

$$y^\sigma = y^\sigma(x^0, x^{k+1}, \dots, x^{3N}). \tag{3.11}$$

If some more general notations $q^0, q^1, \dots, q^n$ are introduced for rectilinear coordinates $y^0, y^{k+1}, \dots, y^{3N}$ and for curvilinear coordinates $x^0, x^{k+1}, \dots, x^{3N}$, relations (3.10) and (3.11) can be written in the form

$$y^\sigma = y^\sigma(q^0, q^1, \dots, q^n), \qquad q^0 = \tau(t), \tag{3.12}$$

where $n = 3N - k$. A set of dependent coordinates y^μ or x^μ may be determined as a function of a rheonomic coordinate or they are constant. In this way, the rheonomic manifold is given by a set of constraints of the form

$$x^\sigma = x^\sigma(q^0) \quad \text{or} \quad x^k = \text{const.}, \qquad \text{i.e.} \tag{3.13}$$

$$\begin{cases} x^1 = \text{const.} \\ \dots \\ x^i = x^i(q^0), \quad q^0 = q^0(t) \\ \dots \\ x^k = \text{const}, \end{cases} \tag{3.14}$$

where at least one dependent coordinate is a function of time t.

The set of parametric constraints (3.13) and definite equations of motion

$$q^\alpha = q^\alpha(t) \tag{3.15}$$

on configurational manifold $M_n \subset E^{3N}$ define a motion of a rheonomic system of points in Euclidian space E^{3N}. The sets (3.11) or (3.10) are parametric forms of rheonomic constraints (3.3) or (3.4) and they may be written in vector form:

$$\mathbf{r}_\nu = \mathbf{r}_\nu(q^0, q^1, \dots, q^n), \qquad q^0 = \tau(t). \tag{3.16}$$

Indeed, a position vector of any ν-th point will be $\mathbf{r}_\nu = y^1_\nu \mathbf{e}_1 + y^2_\nu \mathbf{e}_2 + y^3_\nu \mathbf{e}_3$. With the use of indices as in (3.5) these vectors can be written as

$$\mathbf{r}_\nu = y^{3\nu-2}\mathbf{e}_1 + y^{3\nu-1}\mathbf{e}_2 + y^{3\nu}\mathbf{e}_3. \tag{3.16a}$$

The relations (3.12) are followed by the written forms (3.13) which, analogous to parametric equations of surface (2.33), represent here parametric equations of a rheonomic $n+1$ dimensional manifold, named an expanded configurational manifold or a rheonomic configurational manifold.

Velocity vectors of particles of a rheonomic system at any point of an expanded configurational manifold are

$$\mathbf{v}_\nu = \frac{d\mathbf{r}_\nu}{dt} = \frac{\partial \mathbf{r}_\nu}{\partial q^1}\dot{q}^1 + \dots + \frac{\partial \mathbf{r}_\nu}{\partial q^n}\dot{q}^n + \frac{\partial \mathbf{r}}{\partial q^0}\dot{q}^0,$$

or written in a more concise form

$$\mathbf{v}_\nu = \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha}\dot{q}^\alpha + \frac{\partial \mathbf{r}_\nu}{\partial q^0}\dot{q}^0 = \frac{\partial \mathbf{r}_\nu}{\partial q^i}\dot{q}^i, \quad (\alpha = 1, 2, \dots, n;\ i = 0, 1, \dots, n). \tag{3.17}$$

Therefore, velocity vectors of particles (3.17) on an M_{n+1} rheonomic manifold are decomposed into $n+1$ components made up of generalized velocities $\dot{q}^0, \dot{q}^1, \dots, \dot{q}^n$ and the base or coordinate vectors corresponding to them.

Since the position vector of the ν-th point is $\mathbf{r}_\nu = y^s_\nu \mathbf{e}_s \longrightarrow \dfrac{\partial \mathbf{r}_\nu}{\partial q^\alpha} = \dfrac{\partial y^s_\nu}{\partial q^\alpha}\mathbf{e}_s$, and velocity vector is $\mathbf{v}_\nu = \dot{y}^s_\nu \mathbf{e}_s$ from (3.17) it follows that

$$\dot{y}^j = \frac{\partial y^j}{\partial q^\alpha}\dot{q}^\alpha + \frac{\partial y^j}{\partial q^0}\dot{q}^0 = \frac{\partial y^j}{\partial q^i}\dot{q}^i \qquad \begin{aligned} j &= 1, \dots, 3N \\ i &= 0, 1, \dots, n \\ s &= 1, 2, 3 \end{aligned} \tag{3.18}$$

where it can be seen that the rank of the matrix of transition (from one coordinates to the others) $\{\partial y^j / \partial q^i\}$ equals the number of dimensions of a rheonomic manifold M_{n+1}

$$\operatorname{rank}\left\{\frac{\partial y^j}{\partial q^i}\right\} = n + 1. \tag{3.18a}$$

The remaining $3N - (n+1) = k$ coordinates of velocity vectors are determined from the set (3.14).

Momentum. In rheonomic mechanics, the question of momenta has great importance. The more so, since the definition of this essential notion is not uniform in analytical mechanics. The basic (Newton's) definition of momentum $\mathbf{p}_\nu$ of a ν-particle of mass m_ν, moving at velocity $\mathbf{v}_\nu$ is:

$$\mathbf{p}_\nu \stackrel{\text{def}}{=} m_\nu \mathbf{v}_\nu . \tag{3.19}$$

For the motion of a point limited by rheonomic constraints, the momentum of each ν-th point with respect to velocity (3.17) is

$$\mathbf{p}_\nu = m_\nu \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha} \dot{q}^\alpha + m_\nu \frac{\partial \mathbf{r}_\nu}{\partial q^0} \dot{q}^0 \tag{3.20}$$

or in a more concise form (3.21)

$$\mathbf{p}_\nu = m_\nu \frac{\partial \mathbf{r}_\nu}{\partial q^i} \dot{q}^i . \qquad (i = 0, 1, \dots, n). \tag{3.21}$$

Since momenta of material particles are related to these points they cannot simply be mathematically reduced, by simple vector additions to one point. Not for the momentum $\mathbf{p}$ of such a system can it be said that it represents a vector sum of momenta of the particles in this system. To avoid the difficulty of vector addition, the scalar product of momentum $\mathbf{p}$ of each point with the corresponding coordinate vectors $\partial \mathbf{r}_\nu / \partial q^i$, gives projections of all point momenta on each corresponding coordinate q^i. The sum of all these images $\mathbf{p}_\nu \cdot (\partial \mathbf{r}_\nu / \partial q^i)$, which essentially represent projections of momentum vector $\mathbf{p}$ on coordinate directions, make up the system momentum covector. Therefore, if each scalar product of each momentum vector (3.20) with the corresponding vectors $\partial \mathbf{r}_\nu / \partial q^i$ is made and then all these products are added, the j-th coordinate of the system momentum is obtained

$$p_j \stackrel{\text{def}}{=} \sum_{\nu=1}^{N} \mathbf{p}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^j} \tag{3.22}$$

$$= \sum_{\nu=1}^{N} m_\nu \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha} \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^j} \dot{q}^\alpha + \sum_{\nu=1}^{N} m_\nu \frac{\partial \mathbf{r}_\nu}{\partial q^0} \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^j} \dot{q}^0, \quad \text{or,}$$

$$p_j = \sum_{\nu=1}^{N} m_\nu \frac{\partial \mathbf{r}_\nu}{\partial q^i} \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^j} \dot{q}^i . \quad (j = 0, 1, \dots, n) \tag{3.23}$$

If analogous to inertia tensor (2.51), the corresponding inertia tensor is denoted on $(n+1)$ dimensional manifold

$$a_{ij} = \sum_{\nu=1}^{N} m_\nu \frac{\partial \mathbf{r}_\nu}{\partial q^i} \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^j} = a_{ji}(q^0, q^1, \dots, q^n) \tag{3.24}$$

the momentum covector coordinates (3.22) can be written in a more concise form as

(3.25) $$p_j = a_{\alpha j}\dot{q}^\alpha + a_{0j}\dot{q}^0$$

or as (3.23)

(3.26) $$p_j = a_{ij}\dot{q}^i. \qquad (i,j = 0,1,\dots,n)$$

The tensor (3.24) is covariant, of the second order, symmetrical, and its determinant is

(3.27) $$\det(a_{ij}) \neq 0, \qquad (i,j = 0,1,\dots,n)$$

which is not difficult to prove if one has (3.18a) in mind. Therefore, system momentum covector coordinates are linear combinations of inertia tensors on configurational and generalized velocities $\dot{q}^i$ in that system. With regard to (3.27) contravariant coordinates a^{jk} of inertia tensor (3.24) can be determined, too. By composing the momentum covector (3.26) with contravariant tensor a^{jk} $(k = 0,1,\dots,n)$ generalized velocities are obtained as functions of momenta. Indeed,

(3.28) $$a^{jk}p_j = a^{jk}a_{ij}\dot{q}^i = \delta^k_i\dot{q}^i = \dot{q}^k,$$

or

(3.29) $$\dot{q}^k = a^{k\alpha}p_\alpha + a^{k0}p_0.$$

If time is chosen for rheonomic coordinate q^0, $q^0 = t$, which is possible, the corresponding momentum p_0 is

(3.30)* $$p_0 = a_{0\alpha}\dot{q}^\alpha + a_{00}$$

which can be seen from formula (3.25) since in this case $\dot{q}^0 = 1$.

In the case of holonomic scleronomic constraints from formula (3.24) it follows that all coordinates of inertia tensor $a_{0\alpha}$ and a_{00}, which correspond to rheonomic coordinate q^0 are zero. The physical dimensions of the momentum vector are MLT^{-1}. This comes out from its definition (3.19) since

$$[\dim p] = [\dim m]\cdot[\dim \mathbf{v}] = MLT^{-1}.$$

*In classical analytical mechanics the time momentum coordinate p_0 is given the value of the negative of Hamilton's function H, i.e. $p_0 = -H$, which cannot be made harmonious with momentum definition (3.22) neither by its physical meaning nor the form of expression (3.30).

The momentum covector dimension, however, is not always the same. If the generalized coordinate has the dimension of lenght, $[\dim q^k] = \mathrm{L}$, then the dimension of the respective coordinate p_k of the momentum covector will equal the momentum vector dimension since in this case, due to (3.22):

$$[\dim p_k] = [\dim p] \cdot [\dim r] : [\dim q^k] = \mathrm{MLT}^{-1}.$$

If the generalized coordinate q^j is angle, the corresponding coordinate p_j of the momentum covector has the dimension of impulse momentum $\mathrm{ML^2T^{-1}}$, as in this case

$$[\dim p_j] = \mathrm{MLT}^{-1} \cdot \mathrm{L} = \mathrm{ML^2T^{-1}}.$$

The physical dimension of the momentum covector coordinate p_0 depends on the dimension of the chosen rheonomic coordinate q^0 because then

$$[\dim p_0] = \mathrm{ML^2T^{-1}} : [\dim q^0]. \tag{3.30}$$

If time t is chosen for the rheonomic coordinate, then

$$[\dim p_0] = \mathrm{ML^2T^{-2}},$$

which coincides with energy dimensions. But even with this statement one should bear in mind that p_0 is only one of the momentum covector coordinates (3.26).

Kinetic energy. According to the definition from classical mechanics the kinetic energy T of a system of points is equal to the sum of kinetic energies of all particles which means

$$T = \frac{1}{2}\sum_{\nu=1}^{N} m_\nu \mathbf{v}_\nu^2 = \frac{1}{2}\sum_{\nu=1}^{N} m_\nu \mathbf{v}_\nu \cdot \mathbf{v}_\nu.$$

Velocities of motion of particles limited by holonomic rheonomic constraints are written as formulas (3.17) and substitution into the preceding expression for kinetic energy of the system of particles gives

$$T = \frac{1}{2}\sum_{\nu=1}^{N} m_\nu \frac{\partial \mathbf{r}_\nu}{\partial q^i} \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^j}\dot q^i \dot q^j,$$

namely, with regard to (3.24),

$$T = \tfrac{1}{2}a_{ij}\dot q^i \dot q^j, \qquad (i,j = 0,1,\dots,n). \tag{3.31}$$

It is obvious that kinetic energy (3.31) of a rheonomic holonomic system represents a homogeneous quadratic form of generalized velocities $\dot q^0, \dot q^1, \dots, \dot q^n$, which may be developed into the form

$$T = \tfrac{1}{2}a_{\alpha\beta}\dot q^\alpha \dot q^\beta + a_{0\alpha}\dot q^0 \dot q^\alpha + \tfrac{1}{2}a_{00}\dot q^0 \dot q^0 \quad (\alpha,\beta = 1,2,\dots,n). \tag{3.32}$$

The form accepted in literature is

$$T = \tfrac{1}{2}a_{\alpha\beta}\dot{q}^{\alpha}\dot{q}^{\beta} + a_{0\alpha}\dot{q}^{\alpha} + \tfrac{1}{2}a_{00}. \tag{3.33}$$

This is a consequence which comes out from the expression (3.32) where the rheonomic coordinate q^0 is time t. In this case $\dot{q}^0 = 1$ and formula (3.33) comes out of (3.32). Still in this case, kinetic energy should be written in the form of (3.31) or (3.32). If relations (3.28) are taken into consideration, it is not difficult to express kinetic energy as a homogeneous quadratic form of generalized momenta p_i. By substituting (3.28) into (3.31) it follows that

$$T = \tfrac{1}{2}a_{ij}a^{ik}p_k a^{jl}p_l = \tfrac{1}{2}\delta^k_j a^{jl}p_k p_l = \tfrac{1}{2}a^{kl}p_k p_l \tag{3.34}$$

or

$$T = \tfrac{1}{2}a^{\alpha\beta}p_\alpha p_\beta + a^{0\alpha}p_0 p_\alpha + \tfrac{1}{2}a^{00}p_0 p_0. \tag{3.35}$$

For rheonomic coordinate $q^0 = t$ based on (3.28) the relation is found to be

$$a^{0i}p_i = a^{0\alpha}p_\alpha + a^{00}p_0 = 1. \tag{3.36}$$

In order to use this relation, kinetic energy can be written in the form

$$T = \tfrac{1}{2}a^{\alpha\beta}p_\alpha p_\beta + \tfrac{1}{2}(a^{0\alpha}p_\alpha + \underbrace{a^{0\alpha}p_\alpha + a^{00}p_0})p_0,$$

and in this case

$$T = \tfrac{1}{2}a^{\alpha\beta}p_\alpha p_\beta + \tfrac{1}{2}a^{0\alpha}p_0 p_\alpha + \tfrac{1}{2}p_0. \tag{3.37}$$

If this expression is used for kinetic energy, then equation (3.36) should be joined to it. Also the relation $\dot{q}^0 = 1$ should be joined to expression (3.33). In order to avoid these additional conditions it is easier to use expressions of homogeneous quadratic forms (3.31) and (3.34) for kinetic energy.

Conditions of point acceleration; system acceleration covector. When the motion of a particle over a rheonomic surface was described, the question of acceleration was considered in relations from (2.9) through (2.60); and it was not by chance that it was connected with forces and the constraint reaction. Constraints are, inter alia, a source of forces. Rheonomic constraints, in particular as forces are direct and reflexive causes of system point acceleration. Because of that, three methods for the analysis of acceleration of constrained points, can be distinguished:

a) dynamic method of constraint elimination (as in 2.6),

b) kinematic conditions of constraint (same as in 2.37a), and

c) acceleration of point systems on manifolds (as in 2.52).

a) *Dynamic method of constraint elimination* understands the substitution of constraints by vectors which should be determined by means of some dynamic conditions. In this case, accelerations of system points are observed as accelerations of free points which are subject to additional conditions. This method is as simple as it is inapplicable to solving differential equations of motions since an excessive number of unknown magnitudes of constraint reactions appears in them.

b) *Kinematic conditions of constraints limit the velocities*, as in (3.7), and the accelerations of systems points. Time derivatives of relation (3.7) are

$$\frac{\partial f_\mu}{\partial x^{i'}}\ddot{x}^{i'} + \frac{\partial^2 f_\mu}{\partial x^{j'}\partial x^{i'}}\dot{x}^{i'}\dot{x}^{j'} = 0, \qquad (i', j' = 0, 1, \dots, 3N). \tag{3.38}$$

On the other hand, the vector coordinates of point acceleration, as seen in (2.19a), in E^{3N} are

$$\frac{D\dot{x}^i}{dt} = \frac{d\dot{x}^i}{dt} + \Gamma^i_{jk}\dot{x}^j\frac{dx^k}{dt}, \qquad (i, j = 1, \dots, 3N). \tag{3.39}$$

By substituting ordinary second derivatives $\ddot{x}^i = d\dot{x}^i/dt$ into equations (3.38) the k conditions are obtained, which are to satisfy the point acceleration as follows:

$$\frac{\partial f_\mu}{\partial x^i}\left(\frac{D\dot{x}^i}{dt}\right) + \left(\frac{\partial^2 f_\mu}{\partial x^j \partial x^k} - \frac{\partial f_\mu}{\partial x^i}\Gamma^i_{jk}\right)\dot{x}^j\dot{x}^k = 0. \tag{3.40}$$

As in the case of conditions for velocities (3.7a), because of (3.7) and (3.8) respectively, it can be seen that k vector coordinates of acceleration $D\dot{x}^\mu/dt$ are dependent of $3N-k$ independent acceleration vector coordinates $D\dot{x}^\alpha/dt$ and $D\dot{x}^0/dt$, $(\alpha = 1, \dots, 3N-k)$.

c) *The acceleration of particles on rheonomic manifolds* considerably differs from the acceleration of points in system (3.39), which is characteristic for methods a) and b). Accelerations (3.39) are the acceleration vector coordinates of particles in the curvilinear coordinate system, which are, by means of indices (3.5), extended to all $3N$ system coordinates, and there are $3N$ in number, just as many as there are vector coordinates of all system point positions. Such accelerations are only subject to conditions (3.40) which establishes dependence among the acceleration coordinates. It is possible to solve the problem how to get freedom from such dependence with equations (3.40) in which derivatives of holonomic constraints appear in form (3.6), but similarly this problem can be solved in a more elegant way using parametric forms of constraints (3.12) and (3.16) where the generalized point velocities (3.17) on the observed configurational manifolds are determined. Starting with just (3.17) and the definitions, the acceleration vectors $\mathbf{a}_\nu$ or the ν-th particle in a system, i.e.

$$\mathbf{a}_\nu = \frac{d\mathbf{v}_\nu}{dt} = \frac{d}{dt}\left(\frac{\partial \mathbf{r}_\nu}{\partial q^i}\dot{q}^i\right), \quad (i = 0, 1, \dots, n)$$

the following is obtained

$$\mathbf{a}_\nu = \frac{\partial \mathbf{r}_\nu}{\partial q^i}\frac{d\dot{q}^i}{dt} + \frac{\partial^2 \mathbf{r}_\nu}{\partial q^i \partial q^j}\dot{q}^i\frac{dq^j}{dt}. \tag{3.41}$$

Vectors $\dfrac{\partial^2 \mathbf{r}_\nu}{\partial q^i \partial q^j}$ can be decomposed into $n+1$ components along base vectors $\dfrac{\partial \mathbf{r}_\nu}{\partial q^\alpha}$, corresponding to relations (2.42), i.e.

$$\frac{\partial^2 \mathbf{r}_\nu}{\partial q^i \partial q^j} = \Gamma^k_{ij}\frac{\partial \mathbf{r}_\nu}{\partial q^k} + b_{(\nu)ij}\mathbf{n}_\nu \qquad (j,k = 0,1,\dots,n) \tag{3.42}$$

where Γ^k_{ij}, for the moment, are some coefficients of connection which are to be determined, and $b_{(\nu)ij}$ is the second basic tensor. By returning to (3.41) we have

$$\mathbf{a}_\nu = \left(\frac{d\dot{q}^k}{dt} + \Gamma^k_{ij}\dot{q}^i\frac{dq^j}{dt}\right)\frac{\partial \mathbf{r}_\nu}{\partial q^k} + b_{(\nu)ij}\dot{q}^i\dot{q}^j\mathbf{n}_\nu, \tag{3.43}$$

which can be written in a shorter way

$$\mathbf{a}_\nu = \frac{D\dot{q}^k}{dt}\frac{\partial \mathbf{r}_\nu}{\partial q^k} + b_{(\nu)ij}\dot{q}^i\dot{q}^j\mathbf{n}_\nu = a^k\frac{\partial \mathbf{r}_\nu}{\partial q^k} + a_{(\nu)N}\mathbf{n}_\nu, \tag{3.43a}$$

when acceleration vector coordinates are known to be equal to

$$a^k = \frac{D\dot{q}^k}{dt} = \frac{d\dot{q}^k}{dt} + \Gamma^k_{ij}\dot{q}^i\frac{dq^j}{dt}. \tag{3.44}$$

Although the coefficients of connection are still unknown, it can be seen from (3.43) that the acceleration vector of the ν-th point has $n+1$ independent coordinates on as many dimensional manifold M_{n+1}. It is convenient to determine the coefficients of connection Γ^k_{ij} in configurational space with the help of inertia tensor (3.24). For this reason, one should multiply each vector (3.43) by scalar mass m and then find the scalar product with vector $\partial\mathbf{r}_\nu/\partial q^j$, i.e.

$$m_\nu\mathbf{a}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^j} = m_\nu\frac{\partial \mathbf{r}_\nu}{\partial q^k}\cdot\frac{\partial \mathbf{r}_\nu}{\partial q^j}a^k.$$

If these two relations are summed on ν the j-th coordinate of the system acceleration covector is obtained

$$a_j = \sum_{\nu=1}^{N} m_\nu\frac{\partial \mathbf{r}_\nu}{\partial q^j}\cdot\mathbf{a}_\nu = \sum_{\nu=1}^{N} m_\nu\frac{\partial \mathbf{r}_\nu}{\partial q^j}\cdot\frac{\partial \mathbf{r}_\nu}{\partial q^k}a^k \tag{3.45}$$

or, with respect to (3.24),

$$a_j = a_{jk}a^k. \tag{3.46}$$

Contrary to vector coordinates of acceleration which, as seen in (3.43) or (3.44) is of kinematic nature, the acceleration covector $(a_0, a_1, \ldots, a_n)$ of a holonomic rheonomic system has kinetic characteristic since the inertia tensor, as it can be seen in (3.24) and (3.45), contains masses of system points. Because of that, covariant acceleration vector coordinates in configurational space qualitatively differ from: contravariant coordinates of that vector physical dimension.

If inertia tensor (3.24) of an extended configurational space is used, then coefficients of connection introduced in relations (3.42) can be determined. The partial derivative of tensor (3.24) by coordinate q^k is

$$\frac{\partial a_{ij}}{\partial q^k} = \sum_{\nu}^{N} m_\nu \left(\frac{\partial^2 \mathbf{r}_\nu}{\partial q^k \partial q^i} \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^j} + \frac{\partial \mathbf{r}_\nu}{\partial q^i} \cdot \frac{\partial^2 \mathbf{r}_\nu}{\partial q^k \partial q^j} \right). \tag{3.47}$$

When indices i, j, k change places cyclicaly (by permutation) the relations obtained are

$$\begin{aligned} \frac{\partial a_{jk}}{\partial q^i} &= \sum_{\nu}^{N} m_\nu \left(\frac{\partial^2 \mathbf{r}_\nu}{\partial q^i \partial q^j} \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^k} + \frac{\partial \mathbf{r}_\nu}{\partial q^j} \cdot \frac{\partial^2 \mathbf{r}_\nu}{\partial q^i \partial q^k} \right), \\ \frac{\partial a_{ki}}{\partial q^j} &= \sum_{\nu}^{N} m_\nu \left(\frac{\partial^2 \mathbf{r}_\nu}{\partial q^j \partial q^k} \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^i} + \frac{\partial \mathbf{r}_\nu}{\partial q^k} \cdot \frac{\partial^2 \mathbf{r}_\nu}{\partial q^j \partial q^i} \right). \end{aligned} \tag{3.48}$$

If (3.47) is subtracted from the sum of relations (3.48) then we get

$$\frac{\partial a_{jk}}{\partial q^i} + \frac{\partial a_{ki}}{\partial q^j} - \frac{\partial a_{ij}}{\partial q^k} = 2 \sum_{\nu=1}^{N} m_\nu \frac{\partial^2 \mathbf{r}_\nu}{\partial q^i \partial q^j} \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^k}.$$

On the other hand, if the relations (3.42) are multiplied by vectors $\partial \mathbf{r}_\nu / \partial q^k$ and corresponding masses m_ν and then everything is summed by indices ν, then a sum in the following form is obtained

$$\sum_{\nu=1}^{N} m_\nu \frac{\partial^2 \mathbf{r}_\nu}{\partial q^i \partial q^j} \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^k} = \sum_{\nu=1}^{N} m_\nu \frac{\partial \mathbf{r}_\nu}{\partial q^l} \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^k} \Gamma^l_{ij},$$

and with respect to (3.24),

$$\sum_{\nu=1}^{N} m_\nu \frac{\partial^2 \mathbf{r}_\nu}{\partial q^i \partial q^j} \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^k} = a_{lk} \Gamma^l_{ij} = \Gamma_{ij,k}.$$

These last symbols, as it can be seen in (3.47), are

$$\Gamma_{ij,k} = \frac{1}{2}\left(\frac{\partial a_{jk}}{\partial q^i} + \frac{\partial a_{ki}}{\partial q^j} - \frac{\partial a_{ij}}{\partial q^k}\right) \tag{3.49}$$

and are known as Christoffel symbols of the first kind. Thence, the coefficients of connection Γ^l_{ij} would be $\Gamma^l_{ij} = a^{lk}\Gamma_{ij,k}$ Christoffel symbols of the second kind over the inertia tensor of $n+1$ dimensional configurational rheonomic manifolds. In this way, vector (3.44) and covector (3.46) of acceleration on multidimensional manifolds M_{n+1} are fully determined.

Acceleration in surrounding space. A more complete determination of covectors and vectors of acceleration becomes clear in differential equations of motions, which will appear below, but, here it is possible to formulate a somewhat clearer meaning of acceleration vector coordinates (3.39) in E^{3N}, and a considerably smaller number of vector coordinates (3.44) upon manifolds M_{n+1}. A general information is given by the k relations (3.40) containing k dependent acceleration coordinates $a^\sigma = D\dot{x}^\sigma/dt$ and $3N-k$ independent ones $a^\alpha = D\dot{x}^\alpha/dt$ ($\sigma = 1,\dots,k$; $\alpha = k+1,\dots,3N$). For rheonomic constraints required by the coordinate form (3.13) i.e.

$$f^\sigma = x^\sigma - x^\sigma(t) = 0 \qquad (\sigma = 1,2,\dots,k) \tag{3.50}$$

it comes out from (3.40) that the dependent acceleration coordinates

$$\frac{D\dot{x}^\sigma}{dt} = \frac{d\dot{x}^\sigma}{dt} + \Gamma^\sigma_{ij}\dot{x}^i\frac{dx^j}{dt} \quad (i,j = 1,2,\dots,3N) \tag{3.51}$$

are determined as a function of velocities $\dot{x}^1,\dots,\dot{x}^{3N}$ in each point in space $x^1,\dots,x^{3N}$. The other $3N-k$ acceleration vector coordinates have the forms of (3.39) and (3.44) respectively show this for the case when rheonomic constraints are required in parametric form (3.16). Indeed, if the coordinates y^i ($i = 1,\dots,3N$) of the ν-th point are indexed as in (3.5)

$$y^1_\nu = y^{3\nu-2}, \quad y^2_\nu = y^{3\nu-1}, \quad y^3_\nu = y^{3\nu} \qquad (\nu = 1,\dots,N)$$

and we will then position vectors may be written as vector functions of coordinates $y^1, y^2, \dots, y^{3N}$, i.e.

$$\mathbf{r}_\nu = \mathbf{r}_\nu(x^1,\dots,x^{3N}) \tag{3.52}$$

or due to transformation (1.25),

$$\mathbf{r}_\nu = \mathbf{r}_\nu(x^1,\dots,x^{3N}). \tag{3.52a}$$

A choice of curvilinear coordinates can be done so that rheonomic constraints are given the form of (3.13) or (3.14), namely more general (3.50). Let k of the coordinates so chosen be denoted by $x^1 = x^1(t), \ldots, x^k = x^k(t)$, and let the other $3N - k$ independent coordinates be denoted by q^α $(\alpha = k+1, \ldots, 3N)$. Thus

$$\mathbf{r}_\nu = \mathbf{r}_\nu(x^1, \ldots, x^k; q^{k+1}, \ldots, q^{3N}) \tag{3.53}$$

which corresponds to (3.16). The system point velocities are therefore

$$\mathbf{v}_\nu = \frac{\partial \mathbf{r}_\nu}{\partial x^\sigma}\dot{x}^\sigma + \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha}\dot{x}^\alpha, \quad (\sigma = 1, \ldots, k;\ \alpha = k+1, \ldots, 3N). \tag{3.54}$$

Let vectors $\frac{\partial \mathbf{r}_\nu}{\partial x^\sigma} = \mathfrak{z}_\sigma$ make the base of space $E^k \subset E^{3N}$ and vectors $\frac{\partial \mathbf{r}_\nu}{\partial q^\alpha}$ the base of the manifold $M_n \subset E^{3N}$ $(n = 3N - k)$. When introducing rheonomic coordinate q^0 as in (3.13) the expression for velocities (3.54) is reduced to (3.17), because

$$\frac{\partial \mathbf{r}_\nu}{\partial x^\sigma}\dot{x}^\sigma = \frac{\partial \mathbf{r}_\nu}{\partial x^\sigma}\frac{\partial x^\sigma}{\partial q^0}\dot{q}^0 = \frac{\partial \mathbf{r}_\nu}{\partial q^0}\dot{q}^0.$$

Acceleration of system points, based on (3.54), will be

$$\begin{aligned}\mathbf{a}_\nu &= \frac{\partial^2 \mathbf{r}_\nu}{\partial x^\sigma \partial x^\kappa}\dot{x}^\sigma \dot{x}^\kappa + \frac{\partial \mathbf{r}_\nu}{\partial x^\sigma}\ddot{x}^\sigma + \frac{\partial^2 \mathbf{r}_\nu}{\partial x^\sigma \partial q^\alpha}\dot{x}^\sigma \dot{q}^\alpha \\ &+ \frac{\partial^2 \mathbf{r}}{\partial x^\sigma \partial q^\alpha}\dot{x}^\sigma \dot{q}^\alpha + \frac{\partial^2 \mathbf{r}_\nu}{\partial q^\alpha \partial q^\beta}\dot{q}^\alpha \dot{q}^\beta + \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha}\ddot{q}^\alpha.\end{aligned} \tag{3.55}$$

Scalar multiplication of this relation by vectors $m_\nu \frac{\partial \mathbf{r}_\nu}{\partial x^k}$ and summation by index ν, analogously to the procedures given for (3.44) to (3.46), will result in

$$\sum_{\nu=1}^{N} m_\nu \frac{\partial \mathbf{r}_\nu}{\partial x^k} \cdot \mathbf{a}_\nu = \sum_{\nu=1}^{N} m_\nu \frac{\partial \mathbf{r}_\nu}{\partial x^k} \cdot \frac{\partial^2 \mathbf{r}_\nu}{\partial x^i \partial x^j}\dot{x}^i \dot{x}^j + \sum_{\nu=1}^{N} m_\nu \frac{\partial \mathbf{r}_\nu}{\partial x^i} \cdot \frac{\partial \mathbf{r}_\nu}{\partial x^k}\ddot{x}^i.$$

Furthermore if the relations (3.45) and (3.48) are considered, then one obtains

$$a_k = g_{ki}a^i = g_{ki}\frac{D\dot{x}^i}{dt} \quad (i, j, k = 1, 2, \ldots, 3N) \tag{3.56}$$

or

$$a_\mu = \mathfrak{z}_{\mu\sigma}\frac{D\dot{x}^\sigma}{dt} + a_{\mu\beta}\frac{D\dot{q}^\beta}{dt}, \tag{3.57}$$

where:

$$\mathfrak{z}_{\mu\sigma} = \sum_{\nu=1}^{N} m_\nu \frac{\partial \mathbf{r}_\nu}{\partial x^\mu} \cdot \frac{\partial \mathbf{r}_\nu}{\partial x^\sigma}, \qquad (\mu, \sigma = 1, \ldots, k) \tag{3.59}$$

(3.60) $$э_{\mu\beta} = \sum_{\nu=1}^{N} m_\nu \frac{\partial \mathbf{r}_\nu}{\partial x^\mu} \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^\beta} = a_{\mu\beta},$$

(3.61) $$a_{\alpha\beta} = \sum_{\nu=1}^{N} m_\nu \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha} \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^\beta}, \qquad (\alpha,\beta = k+1,\dots,3N)$$

In this way the main tensor of subspace E^k and M_n is separated:

(3.62) $$g_{ij} = \begin{Bmatrix} э_{\mu\sigma} & a_{\mu\beta} \\ a_{\beta\mu} & a_{\alpha\beta} \end{Bmatrix},$$

which enables a clear survey of acceleration distribution in the space E^{3N}. By composing relation (3.57) with a contravariant tensor (3.59) i.e. $э^{\mu\kappa}$ $(\kappa = 1,\dots,k)$ it results that

(3.63) $$\frac{D\dot{x}^\kappa}{dt} = э^{\sigma\kappa} a_\sigma = \frac{d\dot{x}^\kappa}{dt} + \Gamma^\kappa_{ij}\dot{x}^i\dot{x}^j.$$

In a similar way we obtain

(3.64) $$\frac{D\dot{q}^\gamma}{dt} = a^{\alpha\gamma} a_\alpha = \frac{d\dot{q}^\gamma}{dt} + \Gamma^\gamma_{ij}\dot{x}^i\dot{x}^j.$$

The acceleration vector of the ν-th particle $\mathbf{a}_\nu$ can be, therefore, decomposed into components along covectors

(3.65) $$э_{\nu\sigma} = \frac{\partial \mathbf{r}_\nu}{\partial x^\sigma}, \qquad \mathbf{g}_{\nu\alpha} = \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha}$$

as follows

(3.66) $$\mathbf{a}_\nu = a^\sigma э_{\nu\sigma} + a^\alpha \mathbf{g}_{\nu\alpha}.$$

A combination using relations (3.63) and (3.64) is also possible. Thus, if (3.63) is substituted into (3.66) the acceleration vector of the ν-th point will be

(3.67) $$\mathbf{a}_\nu = a_\kappa э^\kappa_\nu + a^\alpha \mathbf{g}_{\nu\alpha},$$

where the vectors are

(3.68) $$э^\kappa_\nu = э^{\sigma\kappa} э_{\nu\sigma}.$$

Accelerations (3.64) are equivalent to acceleration vector (3.44). To prove this is simple in the case of rheonomic constraints (3.50). Acceleration vector coordinates (3.44) and (3.64) are equal if following hold:

(3.69)
$$\Gamma^{\alpha}_{ij}\dot{x}^i\dot{x}^j = \Gamma^{\alpha}_{\beta'\gamma'}\dot{q}^{\beta'}\dot{q}^{\gamma'}$$
$$(i,j=1,\dots,3N;\ \alpha=1,\dots,n;\ \alpha',\gamma'=0,1,\dots,n)$$

In a chosen coordinate system $x^1,\dots,x^k;\ x^{k+1}=q^{k+1},\ \dots,\ x^{3N}=q^{3N}$, with respect to which the constraints are described by equations $x^{\kappa}=x^{\kappa}(q^0)$ $(\kappa=1,\dots,k)$, the first time derivatives of these coordinates are

$$\dot{x}^i = \frac{\partial x^i}{\partial q^{\beta'}}\dot{q}^{\beta'}$$

where $\partial x^i/\partial q^{\beta}=\delta^i_{\beta}$ are Kronecker symbols. When returning to the left hand side of relation (3.69) i.e.

(3.70)
$$\Gamma^{\alpha}_{ij}\dot{x}^i\dot{x}^j = \Gamma^{\alpha}_{ij}\frac{\partial x^i}{\partial q^{\beta'}}\frac{\partial x^j}{\partial q^{\gamma'}}\dot{q}^{\beta'}\dot{q}^{\gamma'} = \Gamma^{\alpha}_{ij}\delta^i_{\beta'}\delta^j_{\gamma'}\dot{q}^{\beta'}\dot{q}^{\gamma'} = \Gamma^{\alpha}_{\beta'\gamma'}\dot{q}^{\beta'}\dot{q}^{\gamma'}$$

this proves the equality of acceleration vector coordinates (3.44) and (3.64).

Acceleration function. Under the notion of "acceleration function" we understand the function

(3.71)
$$\mathcal{A} \overset{\text{def}}{=} \frac{1}{2}\sum_{\nu=1}^{N} m_{\nu}\mathbf{a}_{\nu}\cdot\mathbf{a}_{\nu},$$

which is in analytical mechanics known under the names: "energy of acceleration", "Gibbs function", "Appell's function" or "Gibbs-Appell's function". As the physical dimensions of this function

(3.72)
$$[\dim\mathcal{A}] = \mathrm{ML^2T^{-4}}$$

are not equal to dimensions of energy E, $[\dim\mathcal{A}]\neq[\dim E]$, and the final analytical expressions partly differ from the expressions accepted for Gibbs or Appell's function in generalized Lagrange coordinates, this function (3.71) is here named "acceleration function". The accelerations of points, as it can be seen in (3.66), are distributed in the surrounding space E^{k+n} and the acceleration function therefore belongs to that space. In order to subdivide the acceleration functions into subspaces, expressions for point acceleration (3.67) should be substituted into formula (3.71),

(3.73)
$$\mathcal{A} = \frac{1}{2}\sum_{\nu=1}^{N} m_{\nu}(a_{\kappa}\ni^{\kappa}_{\nu} + a^{\alpha}\mathbf{g}_{\nu\alpha})(a_{\sigma}\ni^{\sigma}_{\nu} + a^{\beta}\mathbf{g}_{\nu\beta})$$

where $\sigma,\kappa=1,\dots,k$ and $\alpha,\beta=k+1,\dots,3N$.

With reference to (3.68) the function $\mathcal{A}$ is reduced to

$$\mathcal{A} = \frac{1}{2}\sum_{\nu=1}^{N} m_\nu \text{э}_\nu^\kappa \cdot \text{э}_\nu^\sigma a_\kappa a_\sigma + \frac{1}{2}\sum_{\nu=1}^{N} m_\nu \mathbf{g}_{\nu\alpha} \cdot \mathbf{g}_{\nu\beta} a^\alpha a^\beta. \tag{3.74}$$

If linear transformation (3.68) is used once again it results in

$$\sum_{\nu=1}^{N} m_\nu \text{э}_\nu^\kappa \cdot \text{э}_\nu^\sigma = \sum_{\nu=1}^{N} m_\nu \text{э}^{\kappa\chi} \text{э}_{\nu\chi} \cdot \text{э}^{\sigma\mu} \text{э}_{\nu\mu} = \text{э}_{\chi\mu} \text{э}^{\kappa\chi} \text{э}^{\sigma\mu} = \delta_\mu^\kappa \text{э}^{\sigma\mu} = \text{э}^{\sigma\kappa}$$

where

$$\text{э}_{\chi\mu} = \sum_{\nu=1}^{N} m_\nu \frac{\partial \mathbf{r}_\nu}{\partial x^\chi} \cdot \frac{\partial \mathbf{r}_\nu}{\partial x^\mu} = \sum_{\nu=1}^{N} m_\nu \text{э}_{\nu\chi} \cdot \text{э}_{\nu\mu}. \tag{3.75}$$

If the inertia tensor of the n-dimensional rheonomic manifold is also considered

$$a_{\alpha\beta} = \sum_{\nu=1}^{N} m_\nu \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha} \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^\beta} \tag{3.76}$$

the acceleration function is obtained

$$\mathcal{A} = \tfrac{1}{2}\text{э}^{\sigma\kappa} a_\sigma a_\kappa + \tfrac{1}{2} a_{\alpha\beta} a^\alpha a^\beta \tag{3.77}$$

as a sum of two homogeneous quadratic forms:

$$\mathcal{A}_E = \tfrac{1}{2}\text{э}^{\sigma\mu} a_\sigma a_\mu \tag{3.78}$$

and

$$\mathcal{A}_M = \tfrac{1}{2} a_{\alpha\beta} a^\alpha a^\beta. \tag{3.79}$$

On $(n+1)$-dimensional rheonomic manifolds, where the point acceleration vectors are as in (3.43), the acceleration function (3.71) gets the homogeneous quadratic form of $n+1$ vector coordinates of acceleration

$$\mathcal{A} = \sum_{\nu=1}^{N} m_\nu \frac{\partial \mathbf{r}_\nu}{\partial q^i} \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^j} a^i a^j = \tfrac{1}{2} a_{ij} a^i a^j, \qquad (i, j = 0, 1, \dots, n) \tag{3.80}$$

where, as seen in (2.24), the inertia tensor $a_{ij} = a_{ji}(q^0, q^1, \dots, q^n)$ is a function of rheonomic and generalized coordinates.

4. Nonholonomic rheonomic constraints

In the classics of analytical mechanics there are often comparative relations of holonomic and non-holonomic constraints although they essentially differ one from another. The degree of constraints is determined according to the degree of velocity coordinates which appear in constraint equations. In this agreed classification the most widespread class is the class of linear constraints which are most often analytically reduced to the form of

$$\varphi_\mu = b_{\mu\alpha}\dot{q}^\alpha + b_{\mu 0} = 0 \qquad (\mu = 1, \dots, l;\ \alpha = 1, \dots, n) \tag{4.1}$$

where the coefficients $b_{\mu\alpha}$ and $b_{\mu 0}$ are functions

$$b_{\mu\alpha} = b_{\mu\alpha}(q^1, \dots, q^n; t) \tag{4.2}$$

$$b_{\mu 0} = b_{\mu 0}(q^1, \dots, q^n; t) \tag{4.3}$$

which eliminate the possibility of integrating the relations (4.1). If relations (4.1) are nonhomogeneous, i.e. if $b_{\mu 0} \neq 0$ then the nonholonomic constraints are in general rheonomic ones independent of whether coefficients (4.2) and (4.3) depend on time. This is because in such a case there is a time differential next to coefficients $b_{\mu 0}$ and it follows that

$$b_{\mu\alpha}\, dq^\alpha + \overset{\bullet}{b}_{\mu 0}\, dt = 0 \qquad (\alpha = 1, \dots, n) \tag{4.4}$$

By introducing rheonomic coordinates $q^0 = q^0(t) \longrightarrow t = t(q^0)$ nonhomogeneous constraints (4.4) are reduced to homogeneous form

$$b_{\mu\alpha}\, dq^\alpha + \overset{\bullet}{b}_{\mu 0} \frac{dt}{dq^0}\, dq^0 = 0$$

or

$$b_{\mu i}\, dq^i = 0 \qquad (i = 0, 1, \dots, n) \tag{4.5}$$

where $b_{\mu 0} = \overset{\bullet}{b}_{\mu 0}(dt/dq^0) = b_{\mu 0}(q^0, q^1, \dots, q^n)$ now depend on rheonomic and generalized coordinates same as all coefficients (4.2) i.e.

$$b_{\mu i} = b_{\mu i}(q^0, q^1, \dots, q^n). \tag{4.6}$$

Rheonomity of constraints (4.5) ensues from whether they contain either a rheonomic coordinate q^0 or its differential form.

If each relation (4.5) is divided by differential dt it becomes out

$$b_{\mu i}\dot{q}^i = b_{\mu\alpha}\dot{q}^\alpha + b_{\mu 0}\dot{q}^0 = 0 \tag{4.7}$$

or

$$b_{\mu i}\dot{q}^\alpha = -b_{\mu 0}\dot{q}^0,$$

hence it can be seen that if $\det|b_{\mu\alpha}| \neq 0$, l of the generalized velocities $\dot{q}^\mu$ can be determined using the remaining $n + 1 - l$ independent velocities q^σ $(\sigma = 0, l + 1, \dots, n)$.

PRINCIPLES OF MECHANICS

It is understood that principles, as fundamental and most general standpoints in mechanics, equally apply to all mechanical systems regardless of the character and functional dependence of the constraints. In systems with scleronomic constraints the mathematical harmony of analytical mechanics is really established by the principles of mechanics, the variational ones in particular. Disharmony appears in systems with rheonomic constraints which are the subject of study here. Without commenting in more detail the standpoints of classical analytical dynamics, there is an effort in the ensuing text to show the equivalent nature of the principles of mechanics in the sphere where they apply as well as the invariance with respect to various transformations most often used in mechanics.

5. Equivalence and invariance of principles

We say that two principles are equivalent if they have the same natural values for the same atributes of motion, regardless of their formulation. And we say that a principle is invariant if neither its content nor mathematical form changes when transforming coordinates.

Galilean principle of relativity for equilibrium of dynamical particles can be written in the form

$$\mathbf{F}_\nu + \mathbf{R}_\nu = 0, \qquad \mathbf{v}_\nu = 0, \tag{5.1}$$

where $\mathbf{F}_\nu$ and $\mathbf{R}_\nu$ are forces and $\mathbf{v}_\nu$ is the velocity of ν-th particle. In inertial coordinate systems, using the equations (5.1), we can find equilibrium of a system. The constraints (3.1) are additional ones. The velocities of the particles should satisfy

$$\sum_\nu \mathrm{grad}_\nu f_\mu \cdot \mathbf{v}_\nu + \frac{\partial f_\mu}{\partial t} = 0 \textit{ implies } \left.\frac{\partial f_\mu}{\partial t}\right|_{\mathbf{r}=\mathbf{r}_0} = 0. \tag{5.2}$$

The relations (5.1) and (5.2) together should be equivalent to the relation (the principle of virtual displacements):

$$\sum_\nu (\mathbf{F}_\nu + \mathbf{R}_\nu) \cdot \delta\mathbf{r}_\nu = 0. \tag{5.3}$$

It is really so in the mechanics of scleronomic systems, but it is not so in the systems with rheonomic constraints. Here we shall remove the difference. Furthermore, the scalar invariant (5.3) should satisfy

$$\sum_{i=1}^{3N}(Y_i + R_i^y)\,\delta y^i = \sum_{i=1}^{3N}(X_i + R_i^x)\,\delta x^i = \sum_{j=0}^{3N-k} Q_j\,\delta q^j = 0.$$

But the relations $\sum_{i=1}^{3N}(Y_i + R_i^y)\,\delta y^i = 0$ and $\sum_{\alpha=1}^{3N-k} Q_\alpha\,\delta q^\alpha = 0$ are used in mechanics, although it can be proved that

$$\sum_{i=1}^{3N}(Y_i + R_i^y)\,\delta y^i \neq \sum_{\alpha=1}^{3N-k} Q_\alpha\,\delta q^\alpha.$$

Similarly, Galilean principle of relativity and the principle of least action are equivalent in the mechanics of scleronomic systems, while in the mechanics of rheonomic systems they are not usually considered equivalent. For scleronomic system the principle of least action can be written in one of the forms

$$\delta\int_1^2 2T\,dt = \delta\int_1^2 a_{\alpha\beta}\dot{q}^\alpha\,dq^\beta = \delta\int_1^2 p_\beta\,dq^\beta = 0, \qquad (\alpha,\beta = 1,\dots,n). \tag{5.4}$$

while for rheonomic systems we have

$$\int 2T\,dt \neq \int a_{\alpha\beta}\dot{q}^\alpha\,d\dot{q}^\beta \neq \int p_\beta\,dq^\beta.$$

We shall show that it is possible to unify that and to write for rheonomic systems the following

$$\delta\int_1^2 2T\,dt = \delta\int_1^2 a_{ij}\dot{q}^i\,dq^j = \delta\int_1^2 p_j\,dq^j = 0, \qquad (i,j = 0,1,\dots,n).$$

The difference will be only in the dimension of the configuration space.

Similarly, the principle of stationary action for scleronomic systems can be written in one of the forms

$$\delta\int_1^2 (T-\Pi)\,dt = \delta\int_1^2 \left(\tfrac{1}{2}a_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta - \Pi\right)dt = \delta\int_1^2 (p_\alpha\,dq^\alpha - H\,dt) = 0,$$

while for rheonomic systems we have

$$\int (T+\Pi)\,dt \neq \int \left(\tfrac{1}{2}a_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta + \Pi\right)dt \neq \int (p_\alpha\,dq^\alpha - H\,dt),$$

where H is Hamiltonian and Π is the natural potential of forces. We shall also show that for rheonomic systems the principle can be written in one of the following invariant forms

(5.5)
$$\delta\int_1^2 \mathcal{L}\,dt = \delta\int_1^2 (T-V)\,dt = \delta\int_1^2 \left(\tfrac{1}{2}a_{ij}\dot{q}^i\dot{q}^j - V\right)dt = \delta\int_1^2 (p_i\,dq^i - E\,dt) = 0$$

where $E = T + \Pi + P$ is mechanical energy of the system and $V = \Pi + P$ is the potential of the forces.

The relations (5.5) are equivalent to the modified relation expressing other related principles of the mechanics. We shall demonstrate later that they are also invariant.

6. D'Alembert's principle

To determine the motion of a body with respect to various coordinate systems, inertial and non-inertial ones, an important place is given to D'Alembert's principle in which the inertia force $\mathbf{I} \stackrel{\text{def}}{=} -m\mathbf{a}$ has an eminent place and which states:

Each system of forces which act upon a body (particle) is balanced by the force of inertia of that body.

Therefore, if on some ν-th particle M_ν there acts the resultant force $\mathbf{F}_\nu$, the resultant $\mathbf{R}_\nu$ of rheonomic holonomic constraints (3.2) and the force of inertia

(6.1)
$$\mathbf{I}_\nu = -m_\nu\mathbf{a}_\nu,$$

the principle asserts that the following equations must be satisfied

(6.2)
$$\mathbf{F}_\nu + \mathbf{R}_\nu + \mathbf{I}_\nu = 0, \qquad (\nu = 1,\dots,N)$$

a) These are basic equations of dynamics which, if (6.1) is taken into consideration, are most often written in the form of (6.3) where it is understood that the constraints are, in mind, "removed" and replaced by forces $\mathbf{R}_\nu$. However, equations (6.3) do not solve the problem of motion without additional conditions. It is necessary to know motion $\mathbf{r}_\nu = \mathbf{r}_\nu(t)$ and forces $\mathbf{F}_\nu$ in advance in order to determine the constraint equations or to know the forces and sufficient conditions for constraint equations in order to determine motion. And preliminary determination of rheonomic constraint reactions at any point by the direction and by the modulus, without establishing a connection with motion is not always possible.

b) To simplify this task the constraint reactions are resolved into two components. One in the direction of the gradient at the given point on a geometrical constraint and the other normal to the constraint gradient, which is most often called friction force and is included into the composition of force $\mathbf{F}_\nu$. In the case of smooth constraints the remaining reaction lies in the direction of the constraint gradient $f_\mu(\mathbf{r}_1,\dots,\mathbf{r}_N;\tau)=0$, i.e.

$$\mathbf{R}_\nu=\sum_{\mu=1}^{k}\lambda_\mu \operatorname{grad}_{\mathbf{r}_\nu} f_\mu. \tag{6.4}$$

As the friction force is most often determined experimentally according to the laws of friction, mistakes will not be made if all holonomic rheonomic constraints $f_\mu(\mathbf{r}_1,\dots,\mathbf{r}_N;\tau)=0$ are considered smooth and the reaction force has the form of (6.4) In this case together with differential equations of motion

$$m_\nu \mathbf{a}_\nu=\mathbf{F}_\nu+\sum_{\mu=1}^{k}\lambda_\mu \operatorname{grad}_{\mathbf{r}_\nu}\varphi_\mu \tag{6.5}$$

there appear the constraint reaction equations

$$f_\mu(\mathbf{r}_1,\dots,\mathbf{r}_N;\tau)=0 \tag{6.6}$$

which satisfy the conditions of velocity and acceleration. Since these equations for holonomic rheonomic constraints are most frequently required in the form (3.3), (3.6) or parametric form (3.16), then also the differential equations of motion can be presented in scalar form on the same coordinate system. If the constraints are required in Cartesian coordinate system $(y,\mathbf{e})$, as in (3.3), the differential equations of motion (6.5), if indices as in (3.5) are chosen, will be

$$m_i\ddot{y}_i=Y_i+\sum_{\mu=1}^{k}\lambda_\mu\frac{\partial f_\mu}{\partial y^i}\quad (i=1,\dots,3N) \tag{6.7}$$

which, together with k of the constraints (3.3) i.e.

$$f_\mu(y^0,y^1,\dots,y^{3N})=0 \tag{6.8}$$

give a solution to the problem of motion of a system of points. There is a better possibility in a curvilinear coordinate system $(x,\mathbf{g})$ to reduce constraint equations to a more simple form

$$f_\mu(x^0,x^1,\dots,x^{3N})=0 \qquad (\mu=1,\dots,k) \tag{6.9}$$

In such coordinate systems, analogous to (2.19), if indices and notations are as in (3.5), the differential equations of motion will be reduced to the form

$$g_{ij}\frac{D\dot{x}^j}{dt}=F_i+\sum_{\mu=1}^{k}\lambda_\mu\frac{\partial f_\mu}{\partial x^i},\qquad (i,j=1,\dots,3N), \tag{6.10}$$

where $D\dot{x}^j/dt$ are acceleration coordinates for each particle M_ν, each of them derived from formulas (3.39) and the inertia tensor g_{ij} is derived from formulas (2.17) or (2.20), the notations for points being taken from (3.5). The acceleration coordinates from these differential equations (6.10) should satisfy k conditions of constraint acceleration (3.40), i.e.

$$\begin{aligned}(6.11)\qquad \frac{\partial f_\mu}{\partial x^i}g^{ij}\left(F_j+\sum_{\sigma=1}^{k}\lambda_\sigma\frac{\partial f_\sigma}{\partial x^j}\right)+\left(\frac{\partial^2 f_\mu}{\partial x^j\partial x^k}-\frac{\partial f_\mu}{\partial x^i}\Gamma^i_{jk}\right)\dot{x}^j\dot{x}^k\\ =-\frac{\partial f_\mu}{\partial x^0}\ddot{x}^0-2\frac{\partial^2 f_\mu}{\partial x^j\partial x^0}\dot{x}^j\dot{x}^0-\frac{\partial^2 f_\mu}{\partial x^0\partial x^0}\dot{x}^0\dot{x}^0.\end{aligned}$$

With these relations it will be possible to determine k of the multipliers of constraints since, because of (3.8), the square of the gradient

$$g^{ij}\frac{\partial f_\mu}{\partial x^i}\frac{\partial f_\mu}{\partial x^j}$$

of each constraint $f_\mu = 0$ is not zero.

c) If holonomic rheonomic constraints are given in parametric form (3.15) and velocities of particles of the system are determined by vectors (3.17) from D'Alembert principle, and from equation (6.2) respectively, then $n+1 = 3N-k+1$ differential equations on $(n+1)$-dimensional manifolds may be isolated. The scalar product of each equation (6.2) by corresponding vectors $\partial\mathbf{r}_\nu/\partial q^i$ and summation by index ν

$$(6.12)\qquad \sum_{\nu=1}^{N}(\mathbf{F}_\nu+\mathbf{R}_\nu+\mathbf{I}_\nu)\cdot\frac{\partial\mathbf{r}_\nu}{\partial q^i}=0\quad(i=0,1,\dots,n)$$

should give scalar differential equation of motion in generalized coordinates $q^0, q^1, \dots, q^n$. The first addends are covector coordinates of generalized force

$$(6.13)\qquad Q_i=\sum_{\nu=1}^{N}\mathbf{F}_\nu\cdot\frac{\partial\mathbf{r}_\nu}{\partial q^i}.$$

Covector of reaction generalized forces is

$$(6.14)\qquad R_i=\sum_{\nu=1}^{N}\mathbf{R}_\nu\cdot\frac{\partial\mathbf{r}_\nu}{\partial q^i}.$$

However, if the decomposing of reaction forces is adopted as in the preceding section under b) and they are reduced to reactions of ideal constraints (6.4), n coordinates R_α will be zero due to the orthogonality of vectors $\mathrm{grad}_{\mathbf{r}_\nu} f$ and $\partial\mathbf{r}_\nu/\partial q^\alpha$

$$(6.15)\qquad R_\alpha=\sum_{\nu=1}^{N}\sum_{\mu=1}^{k}\lambda_\mu\,\mathrm{grad}_\nu f_\mu\cdot\frac{\partial\mathbf{r}_\nu}{\partial q^\alpha}=0.$$

Different from zero can be the coordinate

$$R_0 = \sum_{\nu=1}^{N} \mathbf{R}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^0} \tag{6.16}$$

because in general $\mathrm{grad}_\nu f_\mu \cdot (\partial \mathbf{r}_\nu / \partial q^0) \neq 0$.

The sum of inertia forces (6.1) of ν-th points in a system and scalar product by base vectors $\partial \mathbf{r}_\nu / \partial q^i$, with (3.45) in mind, is the acceleration covector coordinate with changed sign, i.e.

$$J_i = \sum_{\nu=1}^{N} \mathbf{J}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^i} = -\sum_{\nu=1}^{N} m_\nu \mathbf{a}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^i} = -a_i = -a_{ij} a^j, \tag{6.17}$$

where a^j is determined by expression (3.44). By substituting (6.17), (6.16), (6.15) and (6.14) into relations (6.12) $n+1$ differential equations for system motion are obtained in the form

$$Q_i + R_i - a_i = 0, \qquad R_1 = \cdots = R_n = 0 \tag{6.18}$$

or

$$a_{\alpha j} \frac{D\dot{q}^j}{dt} = Q_\alpha \qquad (\alpha = 1, 2, \ldots, n) \tag{6.19}$$

$$a_{0j} \frac{D\dot{q}^j}{dt} = Q_0 + R_0. \tag{6.20}$$

If generalized forces Q_α $(\alpha = 1, \ldots, n)$ are known in the system of equations (6.19) then n coordinates $q^\alpha = q^\alpha(t)$ can be determined because the required function is $q^0 = q^0(t)$, and then the function R_0 can be determined from equation (6.20).

An example of these differential equations of motion (6.19) and (6.20) are differential equations of point motion on $2+1$ dimensional surface (2.44), (2.45) and (2.49).

7. Principle of possible displacement

The notion of "possible displacement" here means an infinitely small displacement of a particle which could occur if the given constraints allow that. Thus the idea of "possible displacement" denoted δ by Lagrange is in concordance with Langrange's "possible velocities" $\mathbf{v}$ of points "in the course of an infinitely small time interval δt in which possible displacement might take place"

$$\delta \mathbf{r}_\nu = \mathbf{v}_\nu^* \, \delta t. \tag{7.1}$$

Belonging to a set of displacements of a particle is the actual displacement which would take place within time dt, so in the case of rheonomic constraints which themselves are changing with time, the number of possible displacements exceeds the number of those displacements of a point under the action of scleronomic constraints. But, independent from the above the principle of possible displacement equally applies to the system of forces acting on the points whose displacements are limited both by scleronomic and rheonomic constraints. The essence of this principle of possible displacement refers to eventual or possible work of forces which could be realized on possible displacements and asserts: in order to retain the mechanical system of points in equilibrium it is necessary and sufficient to have the total work of all forces upon possible displacements non-positive.

More frequently this principle is used for the systems with retaining constraints for which it reads: a mechanical system of points is in equilibrium if the total work of all forces upon possible displacements is zero. According to this principle a necessary and sufficient number of conditions required to determine the equilibrium of a rheonomic system of points can be derived. To this end it is further assumed that there are N dynamical points M_ν. On each ν-th point acts the resultant force $\mathbf{F}_\nu$ which can be considered as a vector sum of resultant $\mathcal{F}_\nu$ of active forces and resultant $\mathcal{R}_\nu$ of reaction forces. The displacement of points is limited by means of k rheonomic holonomic constraints given in

a) vector relations (3.2) or forces only,

b) in scalar form (3.6), or

c) in parametric form (3.16),

which physically is always be the same but different one from another in the mathematical approach

a) each μ-th constraint $f_\mu = 0$ can always be abstracted on the ν-th point and substituted by the constraint reaction forces $\mathbf{R}_{\nu\mu}$ so that the resultant reaction force acts on the ν-th point

$$\mathbf{R}_\nu = \sum_{\mu=1}^{k} \mathbf{R}_{\nu\mu}.$$

In this sense the point M_ν is free and there are possible displacements $\delta\mathbf{r}_\nu$ of its position $\mathbf{r}_\nu$. Possible work δA of all forces of the ν-th point upon elementary displacements is $\delta A_\nu = \mathbf{F}_\nu \cdot \delta\mathbf{r}_\nu$ and the total possible work of all forces upon possible displacements according to the quoted principle is

$$\delta A = \sum_{\nu=1}^{N} \mathbf{F}_\nu \cdot \delta\mathbf{r}_\nu = \sum_{\nu=1}^{N} (\mathcal{F}_\nu + \mathbf{R}_\nu) \cdot \delta\mathbf{r}_\nu = 0 \tag{7.2}$$

If (7.1) is taken into consideration, the principle of possible velocities is contained in the following relation

$$\sum_{\nu=1}^{N} (\mathcal{F}_\nu + \mathbf{R}_\nu) \cdot \mathbf{v}_\nu^* \delta t = 0 \tag{7.3}$$

From the set of possible velocities $\mathbf{v}_\nu^* = \delta\mathbf{r}_\nu/\delta t$ the real velocities $\mathbf{v}_\nu = d\mathbf{r}_\nu/dt$ can be removed particularly in rheonomic systems requiring that the later ones all equal zero, so that possible displacements remain arbitrary. Hence from (7.2) or (7.3) the conditions of rest of points in the system are

$$\mathcal{F}_\nu + \mathbf{R}_\nu = 0, \quad \frac{d\mathbf{r}_\nu}{dt} = 0 \tag{7.4}$$

which is identical to the conditions (5.5).

b) If the analytical form of smooth holonomic rheonomic constraints (6.9) is taken into account, then the set of "all forces" means active forces F_i, and their total work on possible displacements is

$$\delta A = \sum_{i=1}^{3N} F_i\, \delta x^i = 0, \tag{7.5}$$

provided that the possible displacements satisfy the constraints (6.4), i.e.

$$\delta f_\mu = \frac{\partial f_\mu}{\partial x^0}\,\delta x^0 + \frac{\partial f_\mu}{\partial x^1}\,\delta x^1 + \cdots + \frac{\partial f_\mu}{\partial x^{3N}}\,\delta x^{3N} = 0. \tag{7.6}$$

The rheonomic coordinate x^0 has the same treatment as other coordinates as regards possible displacements. Since the function of time $x^0 = x^0(t)$ is known, we can write

$$\delta x^0 = \frac{\partial x^0}{\partial t}\,\delta t. \tag{7.7}$$

If Lagrange's indefinite multipliers of constraints are introduced, the principle of possible displacement for the constraint form (6.4) can be determined by the relation

$$\delta A + \sum_{\mu=1}^{k} \lambda_\mu\, \delta f_\mu = 0. \tag{7.8}$$

The consequences of this relation can easily be seen if (7.5) and (7.6) are substituted into (7.8) i.e.

$$\sum_{i=1}^{3N}\left(\mathcal{F}_i + \sum_{\mu=1}^{k}\lambda_\mu \frac{\partial f_\mu}{\partial x^i}\right)\delta x^i + \sum_{\mu=1}^{k}\lambda_\mu \frac{\partial f_\mu}{\partial x^0}\,\delta x^0 = 0. \tag{7.9}$$

By using the Lagrange's method of indefinite multipliers of constraints $3N+1$ needed conditions of rheonomic systems will be obtained, namely

$$\mathcal{F}_i + \sum_{\mu=1}^{k}\lambda_\mu \frac{\partial f_\mu}{\partial x^i} = 0, \qquad (i = 1, \ldots, 3N) \tag{7.10}$$

(7.11) $$\sum_{\mu=1}^{k} \lambda_\mu \frac{\partial f_\mu}{\partial x^0} = 0,$$

which together with k of the constraints

(7.12) $$f_\mu(x^0, x^1, \dots, x^{3N}) = 0$$

determine the conditions of equilibrium for the rheonomic system of points.

It suffices for condition (7.11) to be satisfied in equilibrium if partial derivatives of constraints $f_\mu = 0$ on a rheonomic coordinate or in time equal zero

(7.13) $$\frac{\partial f_\mu}{\partial x^0} = 0 \implies \frac{\partial f_\mu}{\partial t} = 0.$$

Thus these relations and equations (7.10) represent sufficient conditions for equilibrium of a rheonomic system. In the case of scleronomic constraints, therefore, equations (7.10) represent necessary and sufficient conditions of system equilibrium. Conditions (7.13) correspond to the requirement that constraints should satisfy velocities $\mathbf{v}_\nu = 0$, because in this case, as seen in (5.6),

$$\frac{\partial f_\mu}{\partial x^0} = \frac{\partial f_\mu}{\partial t}\frac{\partial t}{\partial x^0} = 0 \implies \frac{\partial f}{\partial t} = 0.$$

This is only so for the equilibrium of the ν-th point $M_\nu \in f = 0$ for which $\lambda_\mu = 0$.

Example. A double pendulum weighing $\mathbf{G}_1$ and $\mathbf{G}_2$ of variable lengths $l_1(t)$ and $l_2(t)$ can oscillate in a vertical plane. Determine the position and conditions of system equilibrium.

If the form of constraint (6.8) is chosen for this example then

$$f_1 = y_1^2 + y_2^2 - l_1^2(t) = 0,$$
$$f_2 = (y_3 - y_1)^2 + (y_4 - y_2)^2 - l_2(t) = 0.$$

Equations (7.10) for coordinates $x^i = y^i$ are

$$G_1 + 2\lambda_1 y_1 - 2\lambda_2(y_3 - y_1) = 0$$
$$G_2 + 0 + 2\lambda_2(y_3 - y_1) = 0$$
$$2\lambda_1 y_1 - 2\lambda_2(y_1 - y_2) = 0$$
$$2\lambda_2(y_4 - y_2) = 0.$$

The condition (7.1) is

$$\lambda_1 \frac{\partial l_1}{\partial t} + \lambda_2 \frac{\partial l_2}{\partial t} = \lambda_1 \dot{l}_1 + \lambda_1 \dot{l}_2 = 0$$

or (7.13)

$$\dot{l}_1 = 0 \quad \text{and} \quad \dot{l}_2 = 0.$$

Then the anticipated results come: the pendulum is in equilibrium in the position $y_1 = l_1$, $y_2 = l_1 + l_2$ on the condition that l_1 and l_2 are not changing with time and that $\lambda_1 = (G_1 + G_2)/(2l_1)$, $\lambda_2 = -G_2/(2l_2)$.

If polar cylindrical system of coordinates is chosen $x^1 = \rho_1$, $x^2 = \varphi_1$, $x^3 = \rho_2$, $x^4 = \varphi_2$, the constraints (6.9) are

$$f_1 = \rho_1 - l_1(t) = 0, \quad f_2 = \rho_2 - l_2(t) = 0.$$

Equations (7.10) in this case are

$$\begin{aligned} &G_1 \cos\varphi_1 + G_2 \cos\varphi_2 + \lambda_1 = 0 \longrightarrow 4.\ \lambda = -(G_1 + G_2) \\ (7.14)\qquad &-G_1\rho_1 \sin\varphi_1 - G_2\rho_1 \sin\varphi_1 = 0 \longrightarrow 3.\ \varphi_1 = 0, \pi \\ &G_2 \cos\varphi_2 + \lambda_2 = 0 \longrightarrow 2.\ \lambda_2 = -G_2 \\ (7.15)\qquad &G_2\rho_2 \sin\varphi_2 = 0 \longrightarrow 1.\ \varphi_2 = 0, \pi \end{aligned}$$

and relation (7.11)

$$\lambda_1 \frac{\partial l_1}{\partial t} + \lambda_2 \frac{\partial l_2}{\partial t} = 0 \tag{7.16}$$

or

$$\dot{l}_1 = 0 \quad \text{and} \quad \dot{l}_2 = 0.$$

The same conclusion follows as in the preceding coordinate system, but in this coordinate system it is easily noticed that the reactions of rheonomic constraints are directed along the variables $\rho_i = l_i$.

c) On an expanded $n+1$ dimensional manifold i.e. where the constraints are holonomic and are given in parametric form of (3.15)

$$\mathbf{r}_\nu = \mathbf{r}_\nu(q^0, q^1, \dots, q^n) \tag{7.17}$$

where $q^1, \dots, q^n$ are Lagrange's generalized independent coordinates and $q^0 = q^0(t)$ is the known rheonomic coordinate, possible displacements (7.1) will be

$$\delta\mathbf{r}_\nu = \frac{\partial \mathbf{r}_\nu}{\partial q^0}\delta q^0 + \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha}\delta q^\alpha, \quad (\alpha = 1, \dots, n), \tag{7.18}$$

since the possible point velocities are

$$\mathbf{v}_\nu^* = \frac{\partial \mathbf{r}_\nu}{\partial q^0}\dot{q}_*^0 + \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha}\dot{q}_*^\alpha. \tag{7.19}$$

Possible "generalized velocities" here are, as seen in (7.1) $\dot{q}^i_* = \delta q^i/\delta t$. Real velocities also belong to them. By substituting (7.18) into the mathematical expression for the principle (7.2) we obtain

$$\sum_{\nu=1}^{N}\left(\mathcal{F}_\nu\cdot\frac{\partial \mathbf{r}_\nu}{\partial q^0}+\mathbf{R}_\nu\cdot\frac{\partial \mathbf{r}_\nu}{\partial q^0}\right)\delta q^0+\sum_{\nu=1}^{N}\left(\mathcal{F}_\nu\cdot\frac{\partial \mathbf{r}_\nu}{\partial q^\alpha}+\mathbf{R}_\nu\cdot\frac{\partial \mathbf{r}_\nu}{\partial q^\alpha}\right)\delta q^\alpha=0. \tag{7.20}$$

If it is taken in consideration that covector of generalized forces (6.13) is

$$Q_\alpha=\sum_{\nu=1}^{N}\mathcal{F}_\nu\cdot\frac{\partial \mathbf{r}_\nu}{\partial q^\alpha}\qquad(\alpha=1,2,\dots,n) \tag{7.21}$$

$$\text{and}\quad Q_0^*=\sum_{\nu=1}^{N}\mathcal{F}_\nu\cdot\frac{\partial \mathbf{r}_\nu}{\partial q^0} \tag{7.22}$$

and that generalized forces of constraint reactions (6.14) same as in (6.15) and (6.16), $R_\alpha = 0$ and

$$R_0=\sum_{\nu=1}^{N}\mathbf{R}_\nu\cdot\frac{\partial \mathbf{r}_\nu}{\partial q^0}=\sum_{\nu=1}^{N}\left(\sum_{\mu=1}^{k}\lambda_\mu\,\mathrm{grad}_\nu f_\mu\right)\cdot\frac{\partial \mathbf{r}_\nu}{\partial q^0} \tag{7.23}$$

then relation (7.20) is reduced to

$$(Q_0^*+R_0)\,\delta q^0+Q_\alpha\,\delta q^\alpha=0,\qquad(\alpha=1,\dots,n). \tag{7.24}$$

As all possible displacements δq^α are mutually independent there follow $n+1$ conditions of equilibrium of a system of points on a $(n+1)$-dimensional manifold namely

$$Q_\alpha=0\qquad(\alpha=1,\dots,n) \tag{7.25}$$

$$Q_0^*+R_0=0. \tag{7.26}$$

For a scleronomic system of points with retaining constraints the necessary and sufficient conditions of equilibrium are (7.25). However, they are not sufficient for a rheonomic system. Corresponding to the rheonomic coordinate q^0 of the $(n+1)$-dimensional manifold is equation (7.26), whose structure should be analyzed in detail. But before that, it is necessary to note that conditions (7.25) determine the values of generalized coordinates q^α of configurational manifold M_n which correspond to an equilibrium. It means that they are also at rest with respect to this manifold, under conditions $\dot{q}^1 = \dot{q}^2 = \cdots = \dot{q}^n = 0$. However, such equilibrium on manifold M_n does not include the manifold itself. This invokes the need to distinguish the notions "rest of particles of a system" for which $\mathbf{v} = 0$ from "rest of a system of points on a manifold". Before any further analysis of equation (7.26) it is useful to do the above problem of a double pendulum.

Let $q^0 = t$, $q^1 = \varphi^1$, $q^2 = \varphi^2$. The parametric form of constraints (7.17) in this case is

$$\mathbf{r}_1 = l_1(t)(\cos\varphi_1\mathbf{e}_1 + \sin\varphi\mathbf{e}_2)$$
$$\mathbf{r}_2 = \mathbf{r}_1 + l_2(t)(\cos\varphi_2\mathbf{e}_1 + \sin\varphi_2\mathbf{e}_2),$$

so corresponding to the conditions of equilibrium (7.25) two of which exist ($\alpha = 1, 2$) are the equations (7.14) and (7.15) namely

$$Q_1 = -(G_1 + G_2)l_1\sin\varphi_1 = 0$$
$$Q_2 = G_2 l_2 \sin\varphi_2 = 0,$$

and hence the positions of equilibrium are $\varphi_1 = \varphi_2 = 0, \pi$.

With the values $\varphi_1 =$ const. and $\varphi_2 =$ const. the configuration is constant and at rest as it follows that $\dot\varphi_1 = 0$ and $\dot\varphi_2 = 0$, too. However, it is evident that the points do not remain at rest with respect to the expanded space since the lengths $l_1(t)$ and $l_2(t)$ are changing. The conditions of rest of points (7.4) contain the conditions

$$\frac{d\mathbf{r}_\nu}{dt} = \frac{\partial\mathbf{r}_\nu}{\partial q^\alpha}\dot q^\alpha + \frac{\partial\mathbf{r}_\nu}{\partial q^0}\dot q^0, \tag{7.27}$$

equations (7.10) and also conditions (7.13). In addition to the conditions (7.25), as it can be seen, condition (7.26) is needed as it corresponds to equilibrium of the $(n+1)$-dimensional configuration of the system. Namely the same n-dimensional configurational manifold M_n to which one variable coordinate over time $\dot q^0 \neq 0$ is added. Corresponding to this coordinate q^0 is the equation (7.26) as a condition under which equilibrium can be realized for the system of points on a configurational rheonomic manifold,. Therefore, if generalized force Q_0 is nullified by force R_0 the system of forces will remain in equilibrium in which it had been before the action of forces Q_0 and R_0 during the whole time $t \geq t_0$.

Concretely for the example of a double pendulum the power of constraint change is

$$R_0 = -Q_0^* = (G_1 + G_2)\dot l_1 + G_2\dot l_2. \tag{7.28}$$

This may be interpreted as if the action of power $(G_1 + G_2)\dot l_1$ on the first pendulum and power $G_2\dot l_2$ on the second pendulum than the system would be balanced. Otherwise for $\dot l_1 = 0$ and $\dot l_2 = 0$ the force Q_0 would be zero. In general the requirement for velocities of particles to be zero also produce a condition that the $n+1$ coordinates of generalized force are zero. Indeed it follows from the conditions needed for the rest of all points in system (7.27) that

$$\frac{\partial\mathbf{r}_\nu}{\partial q^0} = -\frac{\partial\mathbf{r}_\nu}{\partial q^\alpha}\frac{\dot q^\alpha}{\dot q^0}.$$

By substituting into (7.22) we get that during the rest of a system of points

$$Q_0\dot{q}^0 = -Q_\alpha \dot{q}^\alpha.$$

In equilibrium, during rest of a body, the effect of work of all generalized forces, as it can be seen in relation (7.25) is zero, and hence it follows that either the velocity of a rheonomic coordinate is zero or $n+1$ coordinates of generalized force are zero i.e.

$$Q_0^* = 0, \qquad R_0 = 0, \tag{7.29}$$

which is in agreement with relation (7.26). This shows that the power of rheonomic constraints in equilibrium of the system is zero which means that constraints in equilibrium are not changing with time. For such equilibrium example (7.28) goes together with the conditions $\dot{l}_1 = 0$, $\dot{l}_2 = 0$ which coincide with general conditions (7.13).

Example. The conditions of rest of a simple pendulum of weight G, whose point of suspension moves vertically following the law $h_0 \sin \omega t$, $\omega =$ const. and the length of pendulum l is a function of time, $l = l(t)$.

Parametric equations of the constraint are

$$\begin{aligned} y^1 &= h_0 \sin \omega t + l(t) \cos \varphi \\ y^2 &= l(t) \sin \varphi. \end{aligned} \tag{7.30}$$

The conditions of pendulum resting in equilibrium according to (7.25) and (7.26) are

$$Q = -Gl \sin \varphi = 0 \implies \varphi = 0, \pi \tag{7.31}$$

$$Q_0 = -Gh_0(\omega \cos \omega t + \dot{l} \cos \varphi) = 0 \tag{7.32}$$

Hence it follows that the rest of an object M in equilibrium $\varphi = 0$ is realized if the pendulum length is changing

$$\dot{l}(t) = -h_0 \cos \omega t. \tag{7.33}$$

Therefore to have the load at rest under the action of the given rheonomic constraint the pendulum length should be changing at velocity (7.33) or according to the law $l(t) = -h_0 \sin \omega t + l_0$.

For equilibrium of a nonholonomic rheonomic system the influences of constraints of form (4.7) are considered. The conditions for possible displacement of constraints (4.7) can be written in the form

$$b_{\mu i}\, \delta q^i = 0 \tag{7.34}$$

namely

$$b_{\mu 0}\, dq^0 + b_{\mu\alpha}\, dq^\alpha = 0, \qquad (\mu = 1, \dots, l) \tag{7.35}$$

If multiplied by indefinite multipliers λ_μ and summed together with (7.24) we get

$$\left(Q_0^* + R_0 + \sum_{\mu=1}^{k} \lambda_\mu b_{\mu 0}\right) \delta q^0 + \left(Q_\alpha + \sum_{\mu=1}^{k} \lambda_\mu b_{\mu\alpha}\right) \delta q^\alpha = 0. \tag{7.36}$$

Following Langrange's method of indefinite multipliers the $n+1$ conditions for the determination of equilibrium of a rheonomic system of points is obtained as follows

$$Q_\alpha + \sum_{\mu=1}^{k} \lambda_\mu b_{\mu\alpha} = 0 \tag{7.37}$$

$$Q_0^* + R_0 + \sum_{\mu=1}^{k} \lambda_\mu b_{\mu 0} = 0. \tag{7.38}$$

For condition (7.27) equation (7.38) is reduced, because of (7.31), to

$$\sum_{\mu=1}^{k} \lambda_\mu b_{\mu 0} = 0. \tag{7.39}$$

Therefore it comes out that the conditions in relation (7.37) and $b_{\mu 0} = 0$ are sufficient for equilibrium of a rheonomic system.

8. Invariance of principle of lost forces

D'Alembert's principle (6.2) is expressed by means of the effect of forces upon possible displacements in Lagrange's form (7.2) and is known under the name of a general "equation" in dynamics or D'Alembert's principle. The very fact that D'Alembert's principle can be formulated in terms of sum of force vectors, makes it invariant with respect to coordinate transformations because the vector itself makes a vectorial invariance. Similarly, the scalar product of vectors in expressions (7.2) and (7.3) make scalar invariances. Because of that, D'Alembert's principle (6.2) written in Lagrange's form (7.2) i.e.

$$\sum_{\nu=1}^{N} (\mathcal{F}_\nu + \mathbf{R}_\nu + \mathbf{J}_\nu) \cdot \delta \mathbf{r}_\nu = 0 \tag{8.1}$$

represents a scalar invariance. "Invariance" here means that property of a mathematical expression or relation which retains its form during different transformations and keeps the physical properties of the original form. Therefore, an invariant relation (8.1) should produce at least all results obtained from D'Alambert's principle (6.2). However, the first look upon relation (8.1) reveals that relations (6.2) and (8.1) are not equivalent because D'Alembert's principle in Lagrange's form (8.1) incompasses the work of only those components of forces which lie in the same plane as vectors of possible displacements $\delta\mathbf{r}_\nu$. The scalar product of vectors of forces (6.2) by vectors of possible displacements is limiting and means to project relations (6.2) into the space made by possible displacements $\delta\mathbf{r}_\nu$.

Through the scalar product of vectors of forces (6.2) by vectors of possible displacements, those components of forces which are orthogonal on the space TM^{n+1} are lost. If all forces from (6.2) are decomposed into two components $\mathbf{F}_T$ and $\mathbf{I}_T$ which belong to vectorial space TM^{n+1}, and $\mathbf{F}_N$ and $\mathbf{J}_N$ which are normal to that space, we can write

$$\mathbf{F}_\nu = \mathbf{F}_{\nu T} + \mathbf{P}_{\nu N}, \tag{8.2}$$

where $\mathbf{F}_{\nu T} = \mathcal{F}_{\nu T} + \mathbf{R}_{\nu T}$, $\mathbf{P}_{\nu N} = \mathcal{F}_{\nu N} + \mathbf{R}_{\nu N}$ and

$$\mathbf{J}_\nu = \mathbf{J}_{\nu T} + \mathbf{J}_{\nu N} \tag{8.3}$$

With regard to the fact that $\delta\mathbf{r}_\nu \perp \mathbf{F}_{\nu N}$ and $\delta\mathbf{r}_\nu \perp \mathbf{J}_{\nu N}$ it follows that

$$(\mathbf{F}_{\nu N} + \mathbf{J}_{\nu N}) \cdot \delta\mathbf{r}_\nu = 0$$

for each ν and therefore their sum is zero, too, same as the scalar product of orthogonal vectors

$$\sum_{\nu=1}^{N} (\mathbf{F}_{\nu N} + \mathbf{J}_{\nu N}) \cdot \delta\mathbf{r}_\nu = 0. \tag{8.4}$$

This basically is D'Alembert's principle in Lagrange's form which reads: *The work of all lost forces upon possible displacements of retaining constraints is zero.*

Relation (8.1) follows from relation (8.4) and the principle is more often written in form (8.1) than in form (8.4). Taking into account (8.2) and (8.3) D'Alembert's principle (6.2) leads to equations

$$\mathbf{F}_{\nu N} + \mathbf{J}_{\nu N} = -(\mathbf{F}_{\nu T} + \mathbf{J}_{\nu T}).$$

Substitution into (8.4) gives that

$$\sum_{\nu=1}^{N} (\mathbf{F}_{\nu T} + \mathbf{J}_{\nu T}) \cdot \delta\mathbf{r}_\nu = 0 \implies \sum_{\nu=1}^{N} (\mathbf{F}_\nu + \mathbf{J}_\nu) \cdot \delta\mathbf{r}_\nu = 0 \tag{8.5}$$

and this relation follows from (8.1) if (8.4) is taken into account. Because of that, D'Alembert's principle which speaks of balancing of lost forces is most often reduced to the form (8.1). Hence as a consequence comes the other formulation of the principle: *the work of all forces upon possible displacements, which make retaining constraints possible, equals zero.*

Therefore, it is worth repeating, that the relation (8.1) contains only those attributes of motion which belong to configurational, that is, to the expanded configurational space of the a system. That is why it is meaningful to discuss invariance of the principle within the domain of configurational manifolds namely:

a) "Removal" of constraints by substituting the reaction forces $\mathbf{R}_\nu$, the displacements $\delta\mathbf{r}_\nu$ may be considered independent so that differential equations of motion (6.2) and (6.3) respectively will follow from (8.1).

b) In case constraint reactions are given in the form (6.4) equations (6.5) will follow. For the scalar form of holonomic rheonomic constraints (6.8) relation (8.1) will have the form

$$\sum_{i=1}^{3N}\left(Y_i + \sum_{\mu=1}^{k}\lambda_\mu \frac{\partial f_\mu}{\partial y^i} - m_i \ddot{y}_i\right)\delta y^i = 0. \tag{8.6}$$

Hence it is possible for the $3N$ independent variations δy^i to obtain the equation of system motion (6.7). When the constraints are given in curvilinear coordinates by relations (6.9), the main relation of this principle (8.1) has the form

$$\sum_{i=1}^{3N}\left(\mathcal{F}_i + \sum_{\mu=1}^{k}\lambda_\mu \frac{\partial f}{\partial x^i} - g_{ij}\frac{D\dot{x}^j}{dt}\right)\delta x^j = 0 \tag{8.7}$$

which is equivalent to the preceding relation (8.6), since in the rectilinear Cartesian coordinate system the acceleration coordinates are $D\dot{y}^i/dt = dy^i/dt$ and $g_{ij} = m_i\delta_{ij}$.

Both expressions (8.7) and (8.6) show that all forces in space E^{3N} are conserved and that they are balanced by means of constraint reactions. In this space, therefore, forces are not lost and it may be said that there are no lost forces.

c) When rheonomic constraints are required in parametric form (7.17) and displacements are possible in relations (7.18) D'Alembert-Lagrange's principle (8.1) for the system with rheonomic constraints (7.17) will be

$$\sum_{\nu=1}^{N}(\boldsymbol{\mathcal{F}}_\nu + \mathbf{R}_\nu + \mathbf{J}_\nu)\cdot\frac{\partial \mathbf{r}_\nu}{\partial q^0}\,dq^0 + \sum_{\nu=1}^{N}(\boldsymbol{\mathcal{F}}_\nu + \mathbf{R}_\nu + \mathbf{J}_\nu)\cdot\frac{\partial \mathbf{r}_\nu}{\partial q^\alpha}\,dq^\alpha = 0 \tag{8.8}$$

or shorter

$$\sum_{\nu=1}^{N}(\boldsymbol{\mathcal{F}}_\nu + \mathbf{R}_\nu + \mathbf{J}_\nu)\cdot\frac{\partial \mathbf{r}_\nu}{\partial q^i}\delta q^i = 0, \quad (i = 0, 1, \ldots, n). \tag{8.9}$$

Taking into account formulas (6.17), (7.21), (7.22) and (7.23) the general equation (8.8) which reflects D'Alembert's principle in Lagrange's form is reduced to the scalar invariance in the form

$$(Q_0^* + J_0 + R_0)\,dq^0 + (J_\alpha + Q_\alpha)\delta q^\alpha = 0 \quad . \tag{8.10}$$

or

$$(Q_i + R_i)\delta q^i = 0 \qquad (i = 0, 1, \ldots, n) \tag{8.11}$$

where

$$Q_0 = Q_0^* + R_0, \tag{8.12}$$

and J_i are the covariant coordinates of inertia force (6.17). Since possible displacements δq^α are independent from relation (8.11) then differential equations of motion (6.19) and (6.20) follow.

However when passing from D'Alembert's principle (6.2) to its expression (8.1) the coordinates of forces that figure in relation (8.4) are lost in the final calculation and do not cause acceleration of points in space TM^{n+1}. Invariant relations (8.6), (8.7) and (8.11) sufficiently clearly show the worthiness of D'Alembert's principle in Lagrange's form for the rheonomic system. Relation (8.10) shows that for a scleronomic system the first addend vanishes because $\delta q^0 = 0$ so that this relation as well gets the form of relation (8.11) with only an addend less.

As the set of possible displacements δq^i also includes real displacements dq^i relation (8.11) for actual displacement will be

$$(Q_i + J_i)dq^i = 0 \tag{8.13}$$

or

$$a_{ij}\frac{D\dot{q}^j}{dt}\,dq^i = Q_i\,dq^i . \tag{8.14}$$

The left hand side of this expression can still be simplified

$$a_{ij}\dot{q}^i\,D\dot{q}^j = D\left(\tfrac{1}{2}a_{ij}\dot{q}^i\dot{q}^j\right) = d\left(\tfrac{1}{2}a_{ij}\dot{q}^i\dot{q}^j\right), \tag{8.15}$$

because the absolute differential of the invariance equals its ordinary differential.

The expression in parenthesis in the preceding relation represents kinetic energy of a rheonomic system (3.31). Thus relation (8.14) represents the law of change of energy of a mechanical system which can be written in the form

$$dT = Q_i\,dq^i \tag{8.16}$$

or

$$dT = (Q_0^*)\, dq^0 + Q_\alpha\, dq^\alpha \tag{8.17}$$

For the system of forces

$$Q_\alpha = -\frac{\partial \Pi}{\partial q^\alpha}, \qquad Q_0^* = -\frac{\partial \Pi}{\partial q^0} \tag{8.18}$$

with potential

$$\Pi = \Pi(q^0, q^1, \dots, q^n) \tag{8.19}$$

relation (8.17) is reduced to

$$dT = -\frac{\partial \Pi}{\partial q^i}\, dq^i + R_0\, dq^0.$$

If potential Π satisfies the conditions $\dfrac{\partial^2 \Pi}{\partial q^j \partial q^i} = \dfrac{\partial^2 \Pi}{\partial q^i \partial q^j}$ the differential expressions on both sides of the equation

$$d\Pi = \frac{\partial \Pi}{\partial q^i}\, dq^i$$

represent total differentials and the law of energy may be written in the integral form

$$T + \Pi = \int R_0\, dq^0 + C. \tag{8.20}$$

In this way the worthiness of D'Alembert's principle in Lagrange's form for a rheonomic system also confirms the law of energy of rheonomic system motion with several degrees of freedom.

9. Principle of least constraint

From among all possible motions this actual motion least deviates from the motion which would be free from any constraint.

According to this principle which is known as Gauss principle of least constraint, the constraints which limit motion compel the particles in a system to deviate from motion which would occur at moment t if the particles were released from constraints. The measure (quadratic) of this deviation according to Gauss is

$$Z = \frac{1}{2} \sum_{\nu=1}^{N} m_\nu (\mathbf{a}_\nu - \mathbf{F}_\nu / m_\nu)^2 \tag{9.1}$$

and is called *constraint*. (Zwang). Other notations are ordinary ones: m_ν — mass and $\mathbf{a}_\nu$ — acceleration of the ν-th material particle of the system, $\mathbf{F}_\nu$ — resultant of all forces which act upon the ν-th point. That is why this Gauss principle is often called "principle of least constraint" and is formulated as: *In a set of all possible movements Gauss constraint Z has a minimum in actual motion.* Relations in which the first variation of the constraint is zero corresponds to this statement i.e.

$$\delta Z = 0 \tag{9.2}$$

As $\mathbf{F}_\nu/m_\nu$ in formula (9.1) also represents acceleration $\mathbf{a}_\nu^*$, the constraint may vary with accelerations $\delta\mathbf{a}_\nu$. Because of that with relation (9.2) one should have in mind that $\delta\mathbf{r}_\nu = 0$ and $\delta\mathbf{v}_\nu = 0$ for each particle in a system. The constraint dimension is the same as the dimension of acceleration function (3.72)

$$[\dim Z] = \mathrm{ML^2T^{-4}}. \tag{9.3}$$

Since vectors of acceleration $\mathbf{a}_\nu$ and force $\mathbf{F}_\nu$ go out of tangential spaces on configurational manifolds, the invariant analytical expression for the formula of constraint (9.1) should be looked for in the expanded configurational space E^{3N}. To this end it can be assumed that the constraints are given in form of (3.53). For this reason, acceleration vectors $\mathbf{a}_\nu$ can be decomposed as in (3.66) or (3.67). In the same way vectors $\mathbf{F}_\nu/m_\nu$ may be represented with $m+k$ components. First the resultant force $\mathbf{F}_\nu$ is decomposed into active forces $\boldsymbol{\mathcal{F}}_\nu$ and constraint reactions $\boldsymbol{\mathcal{R}}_\nu$ i.e.

$$\mathbf{F}_\nu/m_\nu = (1/m_\nu)(\boldsymbol{\mathcal{F}}_\nu + \boldsymbol{\mathcal{R}}_\nu) \tag{9.4}$$

then analogous to expression (3.67) the following can be written

$$\boldsymbol{\mathcal{F}}_\nu/m_\nu = \mathcal{F}_\kappa \ni_\nu^\kappa + Q^\alpha \mathbf{g}_{\nu\alpha} \tag{9.5}$$

$$\boldsymbol{\mathcal{R}}_\nu/m_\nu = \mathcal{R}_\kappa \ni_\nu^\kappa \tag{9.6}$$

since constraint reactions $\mathbf{R}_\nu$ do not have components in tangential space TM^n. By substituting (3.67), (9.5) and (9.6) into (9.1) we get

$$Z = \frac{1}{2}\sum_{\nu=1}^{N} m_\nu \left[(a^\alpha - Q^\alpha)\mathbf{g}_{\nu\alpha} + (a_\kappa - \mathcal{F}_{\nu\kappa} - \mathcal{R}_{\nu\kappa})\ni_\nu^\kappa\right]^2. \tag{9.7}$$

Since $\mathbf{g}_{\nu\alpha} \cdot \ni_\nu^\kappa = \delta_\alpha^\kappa = 0$ $(\alpha \neq \kappa)$ it follows further that

$$Z = \frac{1}{2}\sum_{\nu=1}^{N} m_\nu \left[(a^\alpha - Q^\alpha)\mathbf{g}_{\nu\alpha}\right]^2 + \frac{1}{2}\sum_{\nu=1}^{N} m_\nu \left[(a_\kappa - \mathcal{F}_\kappa - \mathcal{R}_\kappa)\ni_\nu^\kappa\right]^2. \tag{9.8}$$

Bearing in mind (3.94) and (3.95) the constraints can be written as two quadratic forms

$$Z = \tfrac{1}{2}a_{\alpha\beta}(a^\alpha - Q^\alpha)(a^\beta - Q^\beta) + \tfrac{1}{2}g^{\mu\kappa}(a_\mu - \mathcal{F}_\mu - \mathcal{R}_\mu)(a_\kappa - \mathcal{F}_\kappa - \mathcal{R}_\kappa) \tag{9.9}$$

where

$$Z_1 = \tfrac{1}{2}a_{\alpha\beta}(a^\alpha - Q^\alpha)(a^\beta - Q^\beta) \tag{9.10}$$

is the constraint on configurational manifold M_n, and

$$Z_2 = \tfrac{1}{2}g^{\mu\kappa}(a_\mu - \mathcal{F}_\mu - \mathcal{R}_\mu)(a_\kappa - \mathcal{F}_\kappa - \mathcal{R}_\kappa) \tag{9.11}$$

is the constraint in the space E^{3N-n}.

According to principle (9.2) for which $\delta q^\alpha = 0$, $\delta \dot{q}^\alpha = 0$ we get

$$\frac{\partial Z}{\partial a^\alpha}\,\delta a^\alpha + \frac{\partial Z}{\partial a_\mu}\,\delta a_\mu = 0. \tag{9.12}$$

Due to independence of variations δa^α and δa_μ, the $n+k$ equations follow

$$\frac{\partial Z}{\partial a^\alpha} = 0, \qquad \alpha = 1, \dots, n \tag{9.13}$$

$$\frac{\partial Z}{\partial a_\mu} = 0, \qquad \mu = n+1, \dots, 3N. \tag{9.14}$$

These are differential equations of motion of a holonomic system. Indeed, partial derivatives of constraint (9.9) on coordinates of acceleration vectors reduce the equations (9.13) to n covariant differential equations of motion

$$g_{\alpha i}a^i = Q_\alpha \tag{9.15}$$

It follows from (9.14) another k equations

$$g^{\mu\kappa}(a_\kappa - \mathcal{F}_\kappa - \mathcal{R}_\kappa) = 0 \tag{9.16}$$

with the help of which it is possible to determine the reactions of smooth holonomic retaining constraints

$$\mathcal{R}^\mu = -\mathcal{F}^\mu + a^\mu, \tag{9.17}$$

as motion $q^\alpha = q^\alpha(t)$ can be determined from n equations (9.15).

Example. A double pendulum of constant masses m_1 and m_2 and of variable lengths $l_1(t)$ and $l_2(t)$ in a vertical plane $z = 0$ (example after (7.13)).

Position vectors of the particles could be written in the form

$$\mathbf{r}_1 = l_1(\mathbf{e}_1 \cos\varphi_1 + \mathbf{e}_2 \sin\varphi_1)$$
$$\mathbf{r}_2 = \mathbf{r}_1 + l_2(\mathbf{e}_1 \cos\varphi_2 + \mathbf{e}_2 \sin\varphi_1).$$

Let coordinates be: $x^1 = q^1 = \varphi_1$, $x^2 = q^2 = \varphi_2$, $x^3 = l_1$, $x^4 = l_2$. Inertia tensor (3.62) in this case is

$$g_{ij} = \begin{Bmatrix} m_1 l_1^2 + m_2 l_2^2 & m_2 l_1 l_2 \cos(\varphi_2-\varphi_1) & 0 & m_2 l_1 \sin(\varphi_2-\varphi_1) \\ m_2 l_1 l_2 \cos(\varphi_2-\varphi_1) & m_2 l_2^2 & m_2 l_2 \sin(\varphi_2-\varphi_1) & 0 \\ 0 & m_2 l_2 \sin(\varphi_2-\varphi_1) & m_1+m_2 & m_2 \cos(\varphi_2-\varphi_1) \\ m_2 l_1 \sin(\varphi_2-\varphi_1) & 0 & m_2 \cos(\varphi_2-\varphi_1) & m_2 \end{Bmatrix}$$

Generalized forces are:

$$Q_1 = -gl_1(m_1+m_2)\sin\varphi_1, \qquad Q_2 = -gl_2 m_2 \sin\varphi_2,$$
$$\mathcal{F}_3 = (m_1+m_2)gl_1\cos\varphi_1, \qquad \mathcal{F}_4 = m_2 g l_2 \cos\varphi_2.$$

Therefore the double constraint (9.9) is

$$2Z = (a_1 - Q_1)(a^1 - Q^1) + (a_2 - Q_2)(a^2 - Q^2)$$
$$+ (a^3 - \mathcal{F}^3 - \mathcal{R}^3)(a_3 - F_3 - \mathcal{R}_3) + (a^4 - \mathcal{F}^4 - \mathcal{R}^4)(a_4 - \mathcal{F}_4 - \mathcal{R}_4).$$

Differential equations of motion (9.13) or (9.15) are

$$\frac{\partial Z}{\partial a^1} = a_1 - Q_1 = 0 \quad \text{and} \quad \frac{\partial Z}{\partial a^2} = a_2 - Q_2 = 0$$

or in a more developed form

$$(l_1^2 m_1 + l_2^2 m_2)\frac{D\dot\varphi_1}{dt} + m_2 l_1 l_2 \cos(\varphi_2 - \varphi_1)\frac{D\dot\varphi_2}{dt} + m_2 l_2 \sin(\varphi_2 - \varphi_1)\frac{D\dot l_2}{dt}$$
$$= gl_1(m_1+m_2)\sin\varphi_1,$$
$$m_2 l_1 l_2 \cos(\varphi_2 - \varphi_1)\frac{D\dot\varphi_1}{dt} + m_2 l_2^2 \frac{D\dot\varphi_2}{dt} + m_2 l_2 \sin(\varphi_2 - \varphi_1)\frac{D\dot l_1}{dt}$$
$$= gl_2 m_2 \sin\varphi_2.$$

Equations (9.14) or (9.17) may be reduced to covariant form

$$\mathcal{R}_\mu = a_\mu - \mathcal{F}_\mu \qquad (\mu = 3, 4) \tag{9.18}$$

or in developed form

$$\mathcal{R}_3 = m_2 l_2 \sin(\varphi_2 - \varphi_1)\frac{D\dot\varphi_2}{dt} + (m_1 + m_2)\frac{D\dot l_1}{dt}$$

$$+ m_2 \cos(\varphi_2 - \varphi_1)\frac{D\dot{l}_2}{dt} - gl_1(m_1 + m_2)\cos\varphi_1,$$

$$\mathcal{R}_4 = m_2\left[l_1 \sin(\varphi_2 - \varphi_1)\frac{D\dot{\varphi}_1}{dt} + \cos(\varphi_2 - \varphi_1)\frac{D\dot{l}_1}{dt} + \frac{D\dot{l}_2}{dt} - gl_2\cos\varphi_2\right].$$

In general the equation (9.18) produces constraint reactions

$$\mathcal{R}_\sigma = g_{\sigma\alpha}\frac{D\dot{q}^\alpha}{dt} + g_{\sigma\mu}\frac{D\dot{x}^\mu}{dt} - \mathcal{F}_\sigma, \quad (\mu, \sigma = 1, \ldots, k;\ \alpha = 1, \ldots, n). \tag{9.19}$$

If the constraints are invariable, $x^\sigma = c^\sigma = \text{const.}$, constraint reactions will be

$$\mathcal{R}_\sigma = g_{\sigma\alpha}\frac{D\dot{q}^\alpha}{dt} + g_{\sigma\mu}\Gamma^\mu_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta - \mathcal{F}_\sigma. \tag{9.20}$$

If all coordinates $x^\sigma \in E^{3N-n}$ are reduced to functions of rheonomic coordinate q^0, i.e. to the constraint $x^\sigma = x^\sigma(q^0)$ for which velocities are reduced to $\dot{x}^\sigma = \frac{\partial x^\sigma}{\partial q^0}\dot{q}^0$ and acceleration to $\frac{D\dot{x}^\sigma}{dt} = \frac{\partial x^\sigma}{\partial q^0}\frac{D\dot{q}^0}{dt}$ the relations (9.19) become

$$\mathcal{R}_\sigma = g_{\sigma\alpha}\frac{D\dot{q}^\alpha}{dt} + g_{\sigma\mu}\frac{\partial x^\mu}{\partial q^0}\frac{D\dot{q}^0}{dt} - \mathcal{F}_\sigma.$$

By composing this with $\partial x^\sigma/\partial q^0$ it follows that

$$\mathcal{R}_0 = g_{0\alpha}\frac{D\dot{q}^\alpha}{dt} + g_{00}\frac{D\dot{q}^0}{dt} - Q_0^* = g_{i0}\frac{D\dot{q}^i}{dt} - Q_0^*.$$

This equation together with n equations (9.15) creates a system of $n+1$ differential equation of motion of a rheonomic system which are equivalent to the system of equations (6.18). In this case the constraint (9.9) is reduced to the form (9.13) and the vectors have one coordinate more i.e.

$$Z = \tfrac{1}{2}a_{ij}(a^i - Q^i)(a^j - Q^j) \qquad (i, j = 0, 1, \ldots, n).$$

The principle of least constraint is applicable to nonholonomic systems as well. If it is assumed that the system of particles is acted upon by holonomic constraints and l nonholonomic retaining constraints of the form

$$\varphi_\xi(\mathbf{r}_1, \ldots, \mathbf{r}_N, \mathbf{v}_1, \ldots, \mathbf{v}_N) = 0 \qquad (\xi = 1, \ldots, l < n) \tag{9.21}$$

then it is necessary that they satisfy the conditions of acceleration

$$\operatorname{grad}_{\mathbf{v}_\nu}\varphi_\xi \cdot \mathbf{a}_\nu + \Theta(\mathbf{v}, \mathbf{r}) = 0.$$

The conditions of variations according to Gauss $\delta\mathbf{r} = 0$, $\delta\mathbf{v} = 0$ will be

$$\text{grad}_{\mathbf{v}_\nu} \varphi_\xi \cdot \delta\mathbf{a}_\nu = 0$$

or, with regard to (3.67),

$$\text{grad}_{\mathbf{v}_\nu} \varphi_\xi \cdot \ni^{\chi}_{\nu} \, \delta a_\chi + \text{grad}_{\mathbf{v}_\nu} \varphi_\xi \cdot \mathbf{g}_{\nu\alpha} \, \delta a^\alpha = 0.$$

If these relations are summed by index ν and notations

$$b^{\chi}_{\xi} = \sum_{\nu=1}^{N} \text{grad}_{\mathbf{v}_\nu} \varphi_\xi \cdot \ni^{\chi}_{\nu}; \quad b_{\xi\alpha} = \sum_{\nu=1}^{N} \text{grad}_{\mathbf{v}_\nu} \varphi_\xi \cdot \mathbf{g}_{\nu\alpha}$$

are introduced, it follows

(9.22) $$b^{\mu}_{\xi} \, \delta a_\mu + b_{\xi\alpha} \, \delta a^\alpha = 0, \quad (\xi = 1, \dots, l < n; \ \alpha = 1, \dots, n; \ \mu = 1, \dots, k).$$

With regard to the fact that the basic relations of principle (9.12) contain the same variations of acceleration δa_μ and δa^α, their dependence is obvious. By the method of Langrange's multipliers of constraints λ_ξ the relations (9.12) and (9.22) are reduced to

$$\left(\frac{\partial Z}{\partial a^\alpha} - \sum_{\xi=1}^{l} \lambda_\xi b_{\xi\alpha}\right)\delta a^\alpha + \left(\frac{\partial Z}{\partial a_\mu} - \sum_{\xi=1}^{l} \lambda_\xi b^{\mu}_{\xi}\right)\delta a_\mu = 0.$$

Hence it follows that $n + k$ differential equations of motion of a nonholonomic system are of the form

(9.23) $$\frac{\partial Z}{\partial a^\alpha} = \sum_{\xi=1}^{l} \lambda_\xi b_{\xi\alpha}$$

(9.24) $$\frac{\partial Z}{\partial a_\mu} = \sum_{\xi=1}^{l} \lambda_\xi b^{\mu}_{\xi}$$

Then l nonholonomic constraints (9.21) should be joined to them, namely $\varphi_\xi(q^1, \dots, q^k; x^1, \dots, x^k; \dot{q}^1, \dots, \dot{q}^k; \dot{x}^1, \dots, \dot{x}^k) = 0$ so that the multipliers of constraints λ_ξ might be determined. However, by determining dependent variations of acceleration in (9.22), by means of independent variations and by their elimination; in this way from (6.12) it is possible to avoid unknown multipliers of constraints λ_ξ. In relations (9.21) it is possible to separate dependent variations of acceleration δa^η from the independent ones δa^γ $(\gamma = l + 1, \dots, n)$ and it follows that

$$b_{\xi\eta} \, \delta a^\eta + b_{\xi\gamma} \, \delta a^\gamma + b^{\mu}_{\xi} \, \delta a_\mu = 0,$$

where $|b_{\xi\eta}|_l^l \neq 0$. With this condition it is always possible to determine dependent variations δa^η by means of others and

$$\delta a^\eta = c_\gamma^\eta\, \delta a^\gamma + b^{\mu\eta}\, \delta a_\mu \tag{9.25}$$

where

$$c_\gamma^\eta = -b_{\xi\gamma} b^{\xi\eta}, \quad b^{\mu\eta} = -b_\xi^\mu b^{\xi\eta}, \quad b^{\xi\eta} = B_{\xi\eta}/|b_{\xi\eta}|$$

when $B_{\xi\eta}$ is a cofactor of element $b_{\xi\eta}$ of determinant $|b_{\xi\eta}|$. If dependent variations are separated from independent ones in relation (9.12) there will be

$$\frac{\partial Z}{\partial a^\eta}\delta a^\eta + \frac{\partial Z}{\partial a^\gamma}\delta a^\gamma + \frac{\partial Z}{\partial a_\mu}\delta a_\mu = 0$$

or with regard to (9.25)

$$\left(\frac{\partial Z}{\partial a^\gamma} + c_\gamma^\xi \frac{\partial Z}{\partial a^\xi}\right)\delta a^\gamma + \left(\frac{\partial Z}{\partial a_\mu} + b^{\xi\mu}\frac{\partial Z}{\partial a^\xi}\right)\delta a_\mu = 0.$$

Hence follow two systems of differential equations of motion of nonholonomic systems released from constraint multipliers namely

$$\frac{\partial Z}{\partial a^\gamma} + c_\gamma^\xi \frac{\partial Z}{\partial a^\xi} = 0 \quad (\gamma = l+1,\dots,n;\ \xi = 1,\dots,l) \tag{9.26}$$

$$\frac{\partial Z}{\partial a_\mu} + b^{\xi\mu}\frac{\partial Z}{\partial a^\xi} = 0 \quad (\xi = 1,\dots,l;\ \mu = 1,\dots,k), \tag{9.27}$$

which together with constraint equations

$$\varphi_\xi(q^1,\dots,q^n; x^1,\dots,x^k; \dot q^1,\dots,\dot q^n; \dot x^1,\dots,\dot x^k) = 0.$$

solve the problem of motion.

Example. Motion of a heavy ball of mass m and radius r over an immobile circular cylinder of radius $R =$ const. Nonholonomic constraints in a cylindric coordinate system $x^1 = \rho$, $q^2 = \chi$, $q^3 = \zeta$ and Euler angles $q^4 = \varphi$, $q^5 = \psi$, $q^6 = \theta$ are given by the equations [6, p. 98]

$$(R - r)\dot\chi + r\dot\psi + r\dot\varphi\cos\theta = 0,$$
$$\dot\zeta - r\dot\theta\sin(\psi - \chi) - r\dot\varphi\sin\theta\cos(\psi - \chi) = 0.$$

To solve the problem it is necessary to determine the generalized forces

$$\mathcal{F}_\rho = 0, \quad Q_\chi = Q_\varphi = Q_\psi = Q_\theta = 0, \quad Q_3 = -mg$$

and inertia tensor

$$g_{\chi\kappa} = \begin{Bmatrix} m & 0 & 0 & 0 & 0 & 0 \\ 0 & m\rho^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 & 0 \\ 0 & 0 & 0 & I & I\cos\theta & 0 \\ 0 & 0 & 0 & I\cos\theta & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{Bmatrix}$$

where I is the main and central moment of the ball inertia.

The function of constraint

$$Z = \tfrac{1}{2}\mathfrak{g}^{11}(a_1 - \mathcal{F}_1 - \mathcal{R}_1)^2 + \tfrac{1}{2}g_{\alpha\beta}(a^\alpha - Q^\alpha)(a^\beta - Q^\beta).$$

is therefore

$$2Z = (a_1 - \mathcal{R}_1)^2 + a_2a^2 + (a_3 + mg)(a^3 + g) + a_4a^4 + a_5a^5 + a_6a^6.$$

Differential equation of motion (9.23) are now simply reduced to five equations:

$$\begin{aligned} a_\chi &= \lambda_1(R - r) \\ a_5 &= mg = \lambda_2 \\ a_\varphi &= \lambda_1 r\cos\theta - \lambda_2 r\sin\theta\cos(\psi - \chi) \\ a_\psi &= \lambda_1 r \\ a_\theta &= -\lambda_2 r\sin(\psi - \chi) \end{aligned}$$

or three equations equivalent to them (9.26)

$$\begin{aligned} a_\chi &= a_\psi(R - r)/r \\ a_\varphi &= a_\psi\cos\theta - (a_\zeta + mg)r\cos\theta\cos(\psi - \chi) \\ a_\theta &= -(a_\zeta + mg)r\sin(\psi - \chi), \end{aligned}$$

where

$$\begin{aligned} a_\chi &= m(R - r)^2\ddot{\chi}, \quad a_z = m\ddot{\zeta}, \\ a_\psi &= I\frac{d}{dt}(\dot{\psi} + \dot{\varphi}\cos\theta) \\ a_\varphi &= I\frac{d}{dt}(\dot{\varphi} + \dot{\psi}\cos\theta), \qquad a_\theta = I(\ddot{\theta} + \dot{\varphi}\dot{\psi}\sin\theta). \end{aligned}$$

Equation (9.24) which coresponds to coordinate $x^1 = \text{const.}$ is

$$\frac{\partial Z}{\partial a^1} = \mathfrak{g}^{11}(a_1 - R_1) = \lambda_1 b_1^1 + \lambda_2 b_2^1,$$

hence it follows

$$\mathcal{R}_1 = a_1 = a_\rho = -mR\dot{\chi}^2$$

since $b_1^1 = b_2^1 = 0$, $a_\rho = a_1 = \Gamma_{\alpha\beta,1}\dot{q}^\alpha\dot{q}^\beta = \Gamma_{22,1}\dot{q}^2\dot{q}^2 = -m\rho\dot{\chi}^2$.

Since the function of constraint has a physical dimensions of acceleration function (3.77), which is indicated by relations (3.72) and (9.3), it is easy to establish an analytical relation between them. Constraint (9.8) may be written in a more developed form

$$Z = \tfrac{1}{2}(a_{\alpha\beta}a^\alpha a^\beta + \mathfrak{g}^{ij}a_i a_j) + a_{\alpha\beta}a^\alpha Q^\beta + \mathfrak{g}^{ij}a_i X_j + \Phi,$$

where $\Phi = a_{\alpha\beta}Q^\alpha Q^\beta + \mathfrak{g}^{ij}X_i X_j$, $X_j = \mathcal{F}_j + \mathcal{R}_j$.

If acceleration function (3.77) and constraint function (9.28) are compared, then it is obvious that the relation

(9.29) $$Z = S - a^\alpha Q_\alpha + a_i X^i + \Phi(q, x, \dot{q}, \dot{x}).$$

can be written.

Ensuing from (9.13) and (9.14) are the differential equations for the motion of a holonomic system in the form

(9.30) $$\frac{\partial S}{\partial a^\alpha} = Q_\alpha$$

and

(9.31) $$\frac{\partial S}{\partial a_\mu} = \mathcal{F}^\mu + \mathcal{R}^\mu.$$

Similarly for the nonholonomic.system, equations (9.26) and (9.27) are reduced to

(9.32) $$\frac{\partial S}{\partial a^\gamma} = Q_\gamma - c_\gamma^\mu\left(\frac{\partial S}{\partial a^\mu} - Q_\mu\right),$$

and

(9.33) $$\frac{\partial S}{\partial a_i} = \mathcal{F}^i + \mathcal{R}^i - b^{i\mu}\left(\frac{\partial S}{\partial a^\mu} - Q_\mu\right).$$

Since we have that

$$\frac{\partial S}{\partial \ddot{q}^\beta} = \frac{\partial S}{\partial a^\alpha}\frac{\partial a^\alpha}{\partial \ddot{q}^\beta} = \delta_\beta^\alpha\frac{\partial S}{\partial a^\alpha} = \frac{\partial S}{\partial a^\beta}$$

relations (9.30) may be reduced to equations

$$\frac{\partial S}{\partial \ddot{q}^\alpha} = Q_\alpha$$

and relations (9.32) to equations

$$\frac{\partial S}{\partial \ddot{q}^{\alpha}} = Q_{\alpha} - c_{\alpha}^{\mu}\left(\frac{\partial S}{\partial q^{\mu}} - Q_{\mu}\right)$$

which are known as Appell's or Gibbs' equations of motion.

10. Principle of least action

"Action" is the central concept in integral variational principles of mechanics, to which the principle of least action also belongs. Yet, some authors do not use these terms still. Because of this and because it is characteristic of rheonomic systems, it is necessary to clarify this important concept of analytical mechanics. If one starts with the introduction of the concept "action"* it is useful to quote the following sentences: "When a body moves from one position to another, some kind of action is required. And this action is dependent on both the velocity of the body and on the path through which the body passes, but not on either one separately." "The quantity of motion is directly proportional to the velocity and the distance covered." "The least quantity of action gives, at the same time, the shortest path and shortest time." "The quantity of action** is the product of the body's mass, its velocity and the total distance covered by the body. When the body moves from one position to another the action is directly proportional to the mass, velocity and total distance covered."

The quantity of action of the forces*** that are present on point M can be determined in the following way: "Each force V should be multiplied by the differential of the line $MC = v$, oriented in the direction that force is applied, and an integral of the product $V\,dv$ should be formed and then the sum of all these integrals

$$V\,dv + V'\,dv' + V''\,dv'' + \cdots$$

gives the quantity of action of all forces on point M.

From Euler a more general statement can be found about the functional $\int \Phi\, dt$: "As the expression appears in what Maupertuis calls the action of the body during

* As far as I know, the concept of action was introduced in physics by Pierre Louis Moreau de Maupertuis in his work "Accord de différentes lois de la Nature qui avaient jusqu'ici paru incompatibles" about which he reported to the Academy of Sciences in Paris April 15th, 1744 and later published in "Histoire de l'Académie des Sciences de Paris" 1744.

** Maupertuis, P., Les lois de mouvement et du repos deduites d'un Principe Metaphysique; "Histoire de l'Académie Royale des Sciences et Belles Lettres," 1745, Berlin.

*** Leonhard Euler, "Réflexions sur quelques lois générales de la nature qui s'observent dans les effects des forces quelconques," "Histoire de l'Académie Royale des Sciences et Belles Lettres," année 1748, Berlin.

an infinitely short time, so it can be rightfully said that Φ denotes instantaneous actions, if time is not considered, in which case Φ corresponds to what is called 'the live force'"('the live force' = double the kinetic energy — Leibnitz's formulation). D'Alembert wrote in his "Cosmology" "Mr Wolf ("Memoire de St. Petersburg") got an idea to multiply 'the raw force' and time and called that product action..." Later on, that concept evolved into the integral $\int 2E_k\, dt$ where E_k is the kinetic energy of the system. This integral will be marked by the letter J. i.e.

$$J \overset{\text{def}}{=} \int 2E_k\, dt. \tag{10.1}$$

It is easiest to see from this that the physical dimension of action is

$$[\dim J] = \mathrm{ML^2T^{-1}} \tag{10.2}$$

which is equal to the dimension of the moment of momentum. The quantity of action between two moments close in time t_1 and t_2 will be also designated by the letter

$$J \overset{\text{def}}{=} \int_{t_1}^{t_2} 2E_k\, dt. \tag{10.3}$$

In general, this integral cannot be calculated because the kinetic energy of the system, as seen from (3.31) or (3.34) is a function of a more compound structure, so the quantity of action represents a function in the form

$$J = \int_1^2 f(q, \dot{q})\, dt. \qquad (f = 2E_k = 2T). \tag{10.4}$$

If one keeps in mind that doubled kinetic energy of the system can be written, because of (3.26), in the form

$$2E_k = p_i \dot{q}^i \tag{10.5}$$

and so the quantity of action can be written in the following way

$$J = \int_1^2 p_i\, dq^i = \int_1^2 a_{ij} \dot{q}^i \dot{q}^j\, dt, \quad (i, j = 0, 1, \dots, n) \tag{10.6}$$

where now it is obvious that the bounds 1 and 2 relate to the configuration of system M^{n+1}. In the case of constant kinetic energy $E_k = \text{const.}$ the quantity of action is proportional to the interval of time $t_2 - t_1$. For equal intervals of time, in that case, action is constant, and for $t_2 > t_1$ is not negative, because kinetic energy is greater than zero in each instant of time. It can be seen from relations (8.6) and (8.20) that kinetic energy of systems will always be constant if no forces are applied to the system of points. Then the kinetic energy of the system is at the same time the total mechanical system energy E, so it makes sense to consider function (10.6) separately in the absence of the potential of other forces. Then

$$J = \int_1^2 a_{\alpha\beta}\dot{q}^\beta \, dq^\alpha \tag{10.7}$$

the constraints are absent, and the system is free, or, if we speak of smooth holonomic scleronomic retaining constraints, then the functional of action on a smooth multidimensional manifold is in question. The principle of least action, from the historical point of view, just referred to in function (10.7) and it was established that the action (10.7) has stationary value during real motion. It means that the first variation of function (10.7) equals zero, i.e.

$$\delta J = \delta \int_{q_1}^{q_2} a_{\alpha\beta}\dot{q}^\beta \, dq^\alpha = 0; \tag{10.8}$$

$$\delta q_1^\alpha = 0, \quad \delta q_2^\alpha = 0 \quad \text{and} \quad \delta dq^\alpha = d\delta q^\alpha \tag{10.9}$$

Relation (10.8) independent of the others, will not be changed if multiplied by any positive natural number; it is of no importance whether

$$\delta \int_{t_1}^{t_2} 2E_k \, dt = \delta \int_{t_1}^{t_2} 2T \, dt = \delta \int a_{\alpha\beta}\dot{q}^\beta \, dq^\alpha$$

or

$$\delta \int_{t_1}^{t_2} T \, dt = \delta \int_1^2 \tfrac{1}{2} a_{\alpha\beta}\dot{q}^\beta \, dq^\alpha \qquad (\alpha, \beta = 1, 2, \dots, n).$$

That is why the action (10.3) can be modified by 3/2 and adopt for action the function

$$J = \frac{1}{2}\int_1^2 a_{ij}\dot{q}^i \, dq^j = \frac{1}{2}\int_1^2 p_j \, dq^j = \int_1^2 T \, dt \tag{10.10}$$

where, as it is written, T is kinetic energy E_k. Regarding that here the main attention is paid to systems with rheonomic constraint then indices $i, j, k = 0, 1, 2, \dots .n$. Even without variational calculus it can be understood that function (10.10) can have a minimum but not a maximum during real motion since $J \geq 0$ and is equal to zero only when $E_k = 0$, which means it is in a state of rest for $\dot{q}^i = 0$. Really, the differential equations of motion (6.19) and (6.20) in the absence of forces are reduced to

$$a_{\alpha\beta}\frac{D\dot{q}^\beta}{dt} = 0 \implies \frac{Dp_\alpha}{dt} = 0. \tag{10.11}$$

The first variation of functional (10.10) for conditions (10.9) is

$$\delta J = \frac{1}{2}\int_1^2 \delta p_\alpha \, dq^\alpha + p_\alpha \, \delta dq^\alpha,$$

or

$$\delta J = \frac{1}{2}\int_1^2 \left(\frac{\partial a_{\alpha\beta}}{\partial q^\gamma}\dot{q}^\beta\, dq^\alpha \delta q^\gamma - 2a_{\alpha\beta}\dot{q}^\beta\, d\delta q^\alpha\right)$$

$$= \int_1^2 \left(\frac{\partial a_{\alpha\beta}}{\partial q^\gamma}\dot{q}^\beta\, dq^\alpha \delta q^\nu - \frac{\partial a_{\alpha\beta}}{\partial q^\gamma}\dot{q}^\beta\, dq^\gamma \delta q^\alpha - a_{\alpha\beta}\, dq^\beta \delta q^\alpha\right).$$

Since the second sum can be transformed as

$$\frac{\partial a_{\alpha\beta}}{\partial q^\gamma}\dot{q}^\beta\, dq^\gamma \delta q^\alpha = \frac{1}{2}\left(\frac{\partial a_{\gamma\beta}}{\partial q^\alpha} + \frac{\partial a_{\gamma\alpha}}{\partial q^\beta}\right)\dot{q}^\beta \dot{q}^\gamma\, \delta q^\alpha dt$$

it follows that

(10.12) $$\delta J = -\int_1^2 a_{\alpha\beta}\frac{D\dot{q}^\beta}{dt}\,\delta q^\alpha dt = -\int_1^2 \frac{Dp_\alpha}{dt}\,\delta q^\alpha dt$$

because

$$\frac{Dp_\alpha}{dt} = a_{\alpha\beta}\frac{D\dot{q}^\beta}{dt}$$

considering that the inertial tensor is covariantly constant for constant masses of particles, $Da_{\alpha\beta} = 0$. But, as the differential equations (10.11) are valid for the observed motion of the system, it is obvious that the first variation of the function (10.10) equals zero, i.e.

(10.13) $$\delta J = 0.$$

The conclusion that the extremum of the function is minimum is proved by Legendre's rule, because

$$\frac{\partial^2 E_k}{\partial q^\alpha \partial q^\beta} = a_{\alpha\beta}$$

is a covariant-constant positively definite tensor.

It is not easy to accept the concept of "action" without connecting them with the concept of forces which cause a change of state of the motion or rest of the body. For a free body other principles establish uniform motion along the shortest line, but for restricted bodies the only constraints are sources of forces. That is why the question is imposed, what is the importance of this principle with the action of rheonomic constraints that both cause and restrict motion by a force from a change of constraints. Before giving a general answer, it will be instructive to examine the example of action of a rheonomic constraint, let us assume (1.17) i.e.

$$f(y^1, y^2, y^3, y^0) = 0, \qquad y^0 = \tau(t).$$

For this example, in accordance with (10.12) and (10.13) we have

(10.14) $$\int_1^2 m\ddot{y}_k\, \delta y^k dt = 0, \qquad (k = 1, 2, 3)$$

but the variations must satisfy the constraints, i.e.

$$\delta f = \frac{\partial f}{\partial y^k}\delta y^k + \frac{\partial f}{\partial y^0}\delta y^0 = 0. \tag{10.15}$$

Relation (10.15) can be written without changing the essence, in the form

$$\int_1^2 \left(\lambda\frac{\partial f}{\partial y^k}\delta y^k + \lambda\frac{\partial f}{\partial y^0}\delta y^0\right) dt = 0,$$

where λ is the indefinite multiplier of constraint. Further,

$$\int_1^2 \left[\left(-m\ddot{y}_k + \lambda\frac{\partial f}{\partial y^k}\right)\delta y^k + \lambda\frac{\partial f}{\partial y^0}\delta y^0\right] dt = 0. \tag{10.16}$$

Next, the necessary conditions for the minimum of functional (10.14) are

$$m\ddot{y}_k = \lambda\frac{\partial f}{\partial y^k} \qquad \text{and} \tag{10.17}$$

$$0 = \lambda\frac{\partial f}{\partial y^0} \implies \frac{\partial f}{\partial y^0} = 0, \quad \text{if } \lambda \neq 0. \tag{10.18}$$

However, if one introduces the curvilinear coordinate system it is possible to write the same rheonomic constraint in the form (1.31), i.e.

$$f = x^3 - \tau^0(t) = 0 \implies \frac{\partial f}{\partial x^3}\delta x^3 = \frac{\partial f}{\partial \tau^0}\delta\tau^0 = 0$$

relation (10.16) is reduced to

$$\int_1^2 \left(-g_{kl}\frac{D\dot{x}^l}{dt} + \lambda\frac{\partial f}{\partial x^k}\right)_{|x^3=\tau(t)} \delta x^k\, dt = 0.$$

From there result three equations

$$g_{1l}\frac{D\dot{x}^l}{dt} = 0, \quad g_{2l}\frac{D\dot{x}^l}{dt} = 0 \qquad (l = 1, 2, 3) \tag{10.19}$$

$$g_{3l}\frac{D\dot{x}^l}{dt} = \lambda \tag{10.20}$$

which are equivalent to equations (10.17)) and (10.18).

For generalized coordinates q^i it is best to choose the just mentioned curvilinear coordinates x^1 and x^2 and $q^0 = \tau(t)$. Thus, the given example shows that the indefinite function $\lambda(\partial f/\partial x^0) = R_0$ corresponds to the rheonomic constraint. Further, while looking for extremum of function (10.6) the "rheonomic coordinate"

$q^0 = \tau(t)$ should be considered a constraint, to which corresponds some "reaction of rheonomic constraint" R_0. Then the first variation of action (10.6) equals

$$\delta J = \int_1^2 (\delta T + \mathcal{R}_0\, \delta q^0)\, dt \tag{10.21}$$

or

$$\int_1^2 \left(-\frac{Dp_i}{dt}\, \delta q^i + \mathcal{R}_0\, \delta q^0 \right) dt = 0, \quad (i = 0, 1, \dots, n). \tag{10.22}$$

From here the system of differential equations of motion follows

$$\frac{Dp_\alpha}{dt} = 0 \quad \text{and} \quad \frac{Dp_0}{dt} = R_0. \tag{10.23}$$

Principle of stationary action. The principle of least action can be generalized to the motion of a system in the field of potential forces (8.18) by the assertion: Action $J = \int_1^2 T\, dt$ reaches a minimum when the function

$$\int_1^2 \Pi\, dt, \tag{10.24}$$

which can be called the action of potential forces, reaches a stationary value. For the minimum action of a rheonomic system relation (10.21) is needed, and for a stationary value of function (10.24) the following function is needed

$$\int_1^2 \frac{\partial \Pi}{\partial q^i}\, \delta q^i\, dt = 0. \tag{10.25}$$

Relations (10.21) and (10.25) according to statement of the principle should be equated, and so

$$\int_1^2 \left(R_0\, \delta q^0 - \frac{Dp_i}{dt} \delta q^i \right) dt = \int_1^2 \frac{\partial \Pi}{\partial q^i}\, \delta q^i\, dt.$$

This relation is further reduced to

$$\int_1^2 \left[\mathcal{R}_0\, \delta q^0 + \delta(T - \Pi) \right] dt = 0$$

or more concisely

$$\int_1^2 (\delta L + \mathcal{R}_0\, \delta q^0)\, dt = 0 \tag{10.26}$$

where $L = T - \Pi$ is kinetic potential. Therefore we have arrived at the principle of stationary action which maintains: *the action*

$$J = \int_{t_1}^{t_2} L\,dt \tag{10.27}$$

under the conditions (10.9) *on a real path is stationary*. This also proves the validity of this principle for rheonomic systems. Moreover, it is possible to reduce (10.26) to the form

$$\int_{t_1}^{t_2} \delta\mathcal{L}\,dt = 0 \tag{10.28}$$

where it is obvious only the kinetic potential $\mathcal{L}$ is modified. Introducing the function

$$P = -\int \mathcal{R}_0\,dq^0, \qquad \mathcal{R}_0 = \mathcal{R}(q^0) \tag{10.29}$$

under the name rheonomic potential, the potential of forces can be extended to the sum of the natural Π and rheonomic potential (10.29), i.e.

$$V = \Pi + \mathcal{P} = V(q^0, q^1, \dots, q^n). \tag{10.30}$$

Therefore, the action for the rheonomic system should be

$$J = \int_{t_1}^{t_2} \mathcal{L}\,dt, \tag{10.31}$$

which, concerning the form, corresponds to the classic function (10.27) but under the condition the function

$$\mathcal{L} = T - V = T - \Pi - P = L - P. \tag{10.32}$$

is introduced for the Lagrangian.

By substituting $\mathcal{L}$ into (10.28) we get, after varying, (10.26), which is the condition for stationary action, according to Hamilton. Concerning the operation of varying there is no distinction between (10.26) and (10.28). Variation of kinetic potential is

$$\begin{aligned} \delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\dot{q}^i}\delta\dot{q}^i + \frac{\partial\mathcal{L}}{\partial q^i}\delta q^i \\ &= \frac{\partial\mathcal{L}}{\partial\dot{q}^\alpha}\delta\dot{q}^\alpha + \frac{\partial\mathcal{L}}{\partial q^\alpha}\delta q^\alpha + \frac{\partial\mathcal{L}}{\partial\dot{q}^0}\delta\dot{q}^0 + \frac{\partial\mathcal{L}}{\partial q^0}\delta q^0. \end{aligned}$$

That is why, under conditions (10.9), given previously, i.e. under conditions $\delta q^i(t_1) = 0$, $\delta q^i(t_2) = 0$, relation (10.28) is reduced to

$$\int_{t_1}^{t_2} \left(\frac{\partial\mathcal{L}}{\partial\dot{q}^i}\,d\delta q^i + \frac{\partial\mathcal{L}}{\partial q^i}\,\delta q^i\,dt \right) = 0,$$

that is, after partial integration of the first sum

$$\int_{t_1}^{t_2}\left(\frac{\partial\mathcal{L}}{\partial q^i}-\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}^i}\right)\delta q^i\,dt=0.$$

Next, $n+1$ differential equations of motion in Lagrange's form follow

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}^i}-\frac{\partial\mathcal{L}}{\partial q^i}=0,\qquad(i=0,1,\dots,n) \tag{10.33}$$

where the function $\mathcal{L}$ is determined by formula (10.23). Based on (10.29) and (10.32) from (10.33) it follows

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{q}^\alpha}-\frac{\partial L}{\partial q^\alpha}=0,\quad(\alpha=1,\dots,n) \tag{10.34}$$

and

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{q}^0}-\frac{\partial L}{\partial q^0}=\mathcal{R}_0. \tag{10.35}$$

These differential equations of motion are equivalent to the covariant equations (6.19) and (6.20) for the potential of forces of the system

$$-\frac{\partial\Pi}{\partial q^\alpha}=-\frac{\partial V}{\partial q^\alpha}=Q_\alpha$$

and

$$-\frac{\partial V}{\partial q^0}=-\frac{\partial\Pi}{\partial q^0}-\frac{\partial\mathcal{P}}{\partial q^0}=Q_0+\mathcal{R}_0.$$

The system of differential equations (10.34) and (10.35) contain $n+1$ unknowns: n coordinates q depending on time and the generalized "reaction of rheonomic constraints" R_0. From the differential equations (10.34) for known force potentials, it is possible to determine, independently of equation (10.35), n generalized coordinates depending on time and the initial conditions. By substituting them into the left-hand side of equation (10.35) we get $R_0=R_0(t)$, also as a function of time or a function of rheonomic coordinate q^0, where both are in accordance with each other. In case $q^0=t$ the magnitude R_0 has the dimensions of power

$$[\dim R_0]=\mathrm{ML^2T^{-3}}$$

which can be seen from (10.35). In that case all the derivatives in equation (10.35) should be calculated with q^0 and $\dot{q}^0$ and after differentiation substitute $q^0=t$, i.e.

$$\mathcal{R}_0=\left(\frac{d}{dt}\frac{\partial T}{\partial\dot{q}^0}-\frac{\partial L}{\partial q^0}\right)_{q^0=t}. \tag{10.36}$$

Law of energy. The law of energy (8.20) can be obtained by integration of differential equations (10.33). Really, if we compose equations (10.33) with the vector $dq^i = \dot{q}^i\, dt$ we get

$$d\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i}\dot{q}^i\right) - \frac{\partial \mathcal{L}}{\partial q^i}dq^i - \frac{\partial \mathcal{L}}{\partial \dot{q}^i}d\dot{q}^i$$

that is, under the condition that

$$d\mathcal{L} = \frac{\partial \mathcal{L}}{\partial q^i}dq^i - \frac{\partial \mathcal{L}}{\partial \dot{q}^i}d\dot{q}^i$$

is the total differential of kinetic potential L of the mechanical system, we get

$$d\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i}\dot{q}^i - \mathcal{L}\right) = 0$$

so then

$$\frac{\partial \mathcal{L}}{\partial \dot{q}^i}\dot{q}^i - \mathcal{L} = c = \text{const.} \tag{10.37}$$

This integral is equivalent to the law of energy (8.20) under the condition that one knows either the function R_0 or the rheonomic potential $P(q^0)$ which depends on the rheonomic coordinate q^0. This becomes clearer if we take into consideration the potential (10.29) because integral (8.20) in that case can be written

$$T + \Pi + P = C, \tag{10.38}$$

and that is also the contents of integral (10.37) under the quoted conditions. That is easy to prove. Regarding (10.32) and (3.31) it follows that $\partial\mathcal{L}/\partial\dot{q}^i = a_{ij}\dot{q}^j$, so we obtain $(\partial\mathcal{L}/\partial\dot{q}^i)\dot{q}^i = a_{ij}\dot{q}^i\dot{q}^j = 2T$. By returning the obtained values for the partial derivatives in (10.37) and taking into consideration definition (10.32) integral (10.38) is obtained.

Similar to integral (10.37) from the n differential equations of motion (10.30) as is known, Jacobi's integral

$$\frac{\partial L}{\partial \dot{q}^\alpha}\dot{q}^\alpha - L = C_1 \tag{10.39}$$

is obtained under the condition that L is not explicitly dependent on time. It is instructive to check that because of the subsequent proofs. If equations (10.30) are composed with vector $dq^\alpha = \dot{q}^\alpha\, dt$ it follows

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\alpha}\right)\dot{q}^\alpha\, dt - \frac{\partial L}{\partial q^\alpha}\, dq^\alpha = 0,$$

which gives

$$d\left(\frac{\partial L}{\partial \dot{q}^\alpha}\dot{q}^\alpha\right) - \left(\frac{\partial L}{\partial \dot{q}^\alpha}d\dot{q}^\alpha + \frac{\partial L}{\partial q^\alpha}dq^\alpha\right) = 0.$$

If the partial derivative $-(\partial L/\partial t)dt$ is added to both sides, considering that the Lagrangian L in general is dependent on n coordinates of position q^α, velocity $\dot{q}^\alpha$ and time t, then the previous expression is reduced to

$$d\left(\frac{\partial L}{\partial \dot{q}^\alpha}\dot{q}^\alpha - L\right) = -\frac{\partial L}{\partial t}dt$$

or

$$\frac{\partial L}{\partial \dot{q}^\alpha}\dot{q}^\alpha - L = -\int \frac{\partial L}{\partial t}\,dt. \tag{10.40}$$

Under the condition that $\partial L/\partial t = 0$ when the rheonomic coordinate is not introduced and equation (10.35) is not used, we get integral (10.39). However, it should be noted that laws (10.40) and (8.20) are quite different as well as integrals (10.39) and (10.37). From (10.37) follows the law of conservation of mechanical energy (10.38). Integral (10.39) is reduced to

$$T_2 - T_0 + \Pi = C_1. \tag{10.41}$$

In this integral total kinetic energy is not contained

$$T = \tfrac{1}{2}a_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta + a_{0\alpha}\dot{q}^\alpha + \tfrac{1}{2}a_{00} = T_2 + T_1 + T_0, \tag{10.42}$$

where

$$T_2 = \tfrac{1}{2}a_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta,$$
$$T_1 = a_{0\alpha}\dot{q}^\alpha,$$
$$T_0 = \tfrac{1}{2}a_{00}.$$

This originates from $(\partial L/\partial \dot{q}^\alpha)\dot{q}^\alpha \neq 2T = (\partial \mathcal{L}/\partial \dot{q}^i)\dot{q}^i$, and that means the partial derivatives of kinetic energy $\partial T/\partial \dot{q}^0$ and $\partial T/\partial q^0$ are not contained in equations (10.34). The essential and formal distinction between equations (10.33) and (10.34) as well as between their relative integrals (10.38) and (10.41) can be clearly seen from the schematic survey (see Schema 1).

Jacobi's integral (10.41) is better known as the "generalized integral of energy". This term is abandoned here because: 1. it does not contain the total mechanical energy of the system, 2. integral (10.38) is more general, and 3. other integrals of the form of (10.41) can exist, so as such it is a partial or cocyclic integral of energy. Integral (10.41) comes from integrating a reduced number of differential equations of motion. We get it from the system of differential equations of motion (10.33) in

$$\begin{cases} \dfrac{d}{dt}(m_\nu \mathbf{v}_\nu) = \mathbf{F}_\nu + \displaystyle\sum_{\mu=1}^{k} \lambda_\mu \operatorname{grad}_\nu f_\mu, \\ f_\mu(\mathbf{r}_1, \dots, \mathbf{r}_\nu, t) = 0 \end{cases}$$

$$\left\{ \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^i} - \frac{\partial \mathcal{L}}{\partial q^i} = 0 \right. \iff \begin{cases} \dfrac{d}{dt}\dfrac{\partial T}{\partial \dot{q}^0} - \dfrac{\partial T}{\partial q^0} = -\dfrac{\partial V}{\partial q^0} \\ \dfrac{d}{dt}\dfrac{\partial T}{\partial \dot{q}^\alpha} - \dfrac{\partial T}{\partial q^\alpha} = -\dfrac{\partial \Pi}{\partial q^\alpha} \end{cases}$$

$$T_2 - T_0 + \Pi = \int \left(\frac{\partial \Pi}{\partial t} - \frac{\partial T}{\partial t} \right) dt,$$

"energy integral" $T_2 - T_0 + \Pi = c = \text{const.}$

$$T = \underbrace{\tfrac{1}{2} a_{\alpha\beta} \dot{q}^\alpha \dot{q}^\beta}_{T_2} + \underbrace{a_{\alpha 0} \dot{q}^\alpha \dot{q}^0}_{T_1} + \underbrace{\tfrac{1}{2} a_{00} \dot{q}^0 \dot{q}^0}_{T_0} = \tfrac{1}{2} a_{ij} \dot{q}^i \dot{q}^j$$

the law of energy

$$T + \Pi = \int R_0(q^0)\, dq^0 \implies T + \Pi = \int \mathcal{R}_0\, dt + C$$

$$T_2 + T_1 + T_0 + \Pi + P = h = \text{const.}$$

P — potential of rheonomic constraints

$\mathcal{R}_0(t)$ — power of change of the rheonomic constraints

the case that $\mathcal{L}$ is not dependent on time or on the rheonomic coordinate that is a linear function of time. In that case $R_0 = 0$, so from equation (10.35) follows the first cyclic integral $\partial L/\partial \dot{q}^0 = \text{const.}$, and from the other n differential equations (10.34) integral (10.41). It is not difficult to prove. If we assume, for better clarity, that the system is scleronomic and that the Lagrangian L is not dependent on time nor on any cyclic coordinate which is a linear function of time, for example q^1, then Lagrangian L can be decomposed as follows

(10.43)
$$\begin{aligned} L = L(q^2, \dots, q^n; \dot{q}^1, \dot{q}^2, \dots, \dot{q}^n) &= \tfrac{1}{2} a_{11} \dot{q}^1 \dot{q}^1 + a_{1\mu} \dot{q}^1 \dot{q}^\mu \\ &+ \tfrac{1}{2} a_{\mu\nu} \dot{q}^\mu \dot{q}^\nu - \Pi(q^2, q^3, \dots, q^n) = T_1 + T_{12} + T_2 - \Pi \end{aligned}$$

where

(10.44)
$$T_1 = \tfrac{1}{2} a_{11} \dot{q}^1 \dot{q}^1, \quad T_{12} = a_{1\mu} \dot{q}^1 \dot{q}^\mu, \quad T_2 = \tfrac{1}{2} a_{\mu\nu} \dot{q}^\mu \dot{q}^\nu.$$

In that case the system of n independent differential equations of motion is separated into one cyclic integral

$$\frac{\partial L}{\partial \dot{q}^1} = a_{11} \dot{q}^1 + a_{1\mu} \dot{q}^\mu = p_1 = C_1 = \text{const.}$$

and $n-1$ differential equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\mu} - \frac{\partial L}{\partial q^\mu} = 0. \tag{10.45}$$

By multiplying each of these equations by the corresponding differential $dq^\mu = \dot{q}^\mu\, dt$ and summing by index μ we get, under the quoted conditions,

$$d\left(\frac{\partial L}{\partial \dot{q}^\mu}\dot{q}^\mu - L\right) = 0.$$

Thus, from $n-1$ differential equations of motion, it follows that

$$\frac{\partial L}{\partial \dot{q}^\mu}\dot{q}^\mu = T_2 - T_1 + \Pi(q^2, \dots, q^n) = C_1 = \text{const.} \tag{10.46}$$

The related integral

$$T_2 - T_k + \Pi(q^1, \dots, q^{k-1}, q^{k+1}, \dots, q^n) = C_k = \text{const.} \tag{10.47}$$

will correspond to any cyclic coordinate q^k which is a linear function of time.

Coupled differential equations. For the set of generalized coordinates $q^1, \dots, q^n$ and generalized momenta $p_1, \dots, p_n$ the broad term "canonical variables" is used and related to them in well developed analytical mechanics of systems. A clearer and more adequate term for these coordinates would be conjugate coordinates. Namely, in the third chapter ((3.17)–(3.30)) generalized velocities and generalized momenta were introduced and it was concluded that among them there exists a connection of the form $p_i = a_{ij}\dot{q}^j$

$$\frac{dq^j}{dt} = a^{ji}p_i. \tag{10.48}$$

This differential conjugation of generalized coordinates and generalized momenta originates from accepted definitions of the velocity vector $\mathbf{v} = d\mathbf{r}/dt$ and the momentum vector $\mathbf{p} \stackrel{\text{def}}{=} m\mathbf{v}$. Therefore, the relations (10.48) a priori exist in kinetics. They can be determined one by the other only if the other independent relations, based on the laws of motion or on the principles of mechanics, are not joined to those relations. If we take, also defined, the concept of kinetic energy (3.34), it is seen that the relations (10.48) can be enlarged as

$$\frac{dq^j}{dt} = a^{ji}p_i = \frac{\partial T}{\partial p_i}, \tag{10.49}$$

but new quality is not obtained. This is not realized even if we add an arbitrary function of coordinates and time to the kinetic energy. Let the potential energy be $V = V(t, q^0, q^1, \dots, q^n)$. Then it is again

$$\frac{dq^j}{dt} = a^{ji} p_i = \frac{\partial T}{\partial p_i} = \frac{\partial E}{\partial p_i} \tag{10.50}$$

where E, also a defined concept, is total mechanical energy of a system with potential

$$E = T + \Pi + P(q^0) = \tfrac{1}{2} a^{ij} p_i p_j + \Pi + P(q^0). \tag{10.51}$$

The law of motion of potential system is reflected by the differential equations of motion (10.33) in which kinetic potential $\mathcal{L}$ is determined, as can be seen from (10.32), by expression

$$\mathcal{L} = \tfrac{1}{2} a_{ij} \dot{q}^i \dot{q}^j - V, \qquad V = \Pi + P. \tag{10.52}$$

Corresponding partial derivatives in agreement with relation (10.50) are

$$\frac{\partial \mathcal{L}}{\partial \dot{q}^i} = a_{ij} \dot{q}^j = p_i \tag{10.53}$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial q^i} &= \frac{1}{2} \frac{\partial a_{jk}}{\partial q^i} \dot{q}^j \dot{q}^k - \frac{\partial V}{\partial q^i} = \frac{1}{2} \frac{\partial a_{jk}}{\partial q^i} a^{jl} p_l a^{km} p_m - \frac{\partial V}{\partial q^i} \\ &= -\frac{1}{2} \frac{\partial a^{lm}}{\partial q^i} p_l p_m - \frac{\partial V}{\partial q^i} = -\frac{\partial T}{\partial q^i} - \frac{\partial V}{\partial q^i} = -\frac{\partial E}{\partial q^i}. \end{aligned} \tag{10.54}$$

Substitution (10.35) and (10.54) into (10.33) gives the differential equations of system motion in the form

$$\frac{dp_i}{dt} = -\frac{\partial E}{\partial q^i}. \tag{10.55}$$

While deriving these equations, coupling was used between the coordinates and momenta (10.50), so together they make a system of $2n+2$ independent conjugated (canonical) differential equations of motion of first order

$$\frac{dq^i}{dt} = \frac{\partial E}{\partial p_i}, \tag{10.56}$$

$$\frac{dp_i}{dt} = -\frac{\partial E}{\partial q^i}. \tag{10.57}$$

where E, the energy of the contemplated system, is given in the form of (10.51).

If we introduce the Hamiltonian by the modified definition

$$H = \tfrac{1}{2} a^{ij} p_i p_j + \Pi \tag{10.58}$$

the system of differential equations (10.56) and (10.57) can be written in a more developed form

(10.56a) $$\frac{dq^\alpha}{dt} = \frac{\partial H}{\partial p_\alpha}, \qquad \frac{dq^0}{dt} = -\frac{\partial H}{\partial p_0}$$

(10.57b) $$\frac{dp_\alpha}{dt} = -\frac{\partial H}{\partial q^\alpha} \qquad \frac{dp_\alpha}{dt} = -\frac{\partial H}{\partial q^0} + R_0$$

With the help of the differential of generalized coordinates dq^i, generalized momenta p_i, and total mechanical energy E, the function of action

$$J = \int_{t_1}^{t_2} \mathcal{L}\, dt = \int_{t_1}^{t_2} (T - V)\, dt = \int_{t_1}^{t_2} [2T - (T + V)]\, dt$$

can be easily transformed into the form

(10.59) $$J = \int_{t_1}^{t_2} (p_i\, dq^i - E\, dt).$$

Therefore, relation (10.28) also shows that the first variation of function

(10.60) $$\delta \int_{t_1}^{t_2} (p_i\, dq^i - E\, dt) = 0$$

is equal to zero for real motion, i.e. motion described by differential equations (10.56) and (10.57). Really, since energy E is a function of coordinates q^i and momenta p_i, it follows that

$$\delta J = \int_1^2 \left[\delta p_i\, dq^i + p_i \delta dq^i - \left(\frac{\partial E}{\partial q^i} \delta q^i + \frac{\partial E}{\partial p_i} \delta p_i \right) dt \right].$$

By partial integration, under the conditions $\delta q_1^i = \delta q_2^i = 0$, we get

$$\int_1^2 p_i\, \delta dq^i = \int_1^2 p_i\, d\delta q^i = -\delta q^i\, dp_i.$$

Substitution into the previous integral gives

(10.61) $$\delta J = \int_1^2 \left[\left(dq^i - \frac{\partial E}{\partial p_i} dt \right) \delta p_i - \left(dp_i + \frac{\partial E}{\partial q^i} dt \right) \delta q^i \right].$$

For the motion, whose equations are (10.56) and (10.57) it follows that $\delta J = 0$. If deviations $\delta q^i = \xi^i$ and $\delta p_i = \eta_i$ exist between trajectories, it is possible to write variation (10.61) in the form

(10.62) $$\delta J = \int_1^2 (\eta_i\, dq^i - \xi^i\, dp_i - \mathcal{H})\, dt, \quad (i = 0, 1, \dots, n)$$

where

$$\mathcal{H} = \frac{\partial E}{\partial q^i}\xi^i + \frac{\partial E}{\partial p_i}\eta_i = \mathcal{H}(q,p,\xi,\eta) \tag{10.63}$$

is the variation of mechanical energy E. The second variation of action, therefore, is

$$\delta^2 J = \int_1^2 \Big[\delta\eta_i dq^i - d\eta_i\delta q^i - \delta\xi^i dp_i + d\xi^i\delta p_i - \left(\frac{\partial\mathcal{H}}{\partial q^i}\delta q^i + \frac{\partial\mathcal{H}}{\partial p_i}\delta p_i + \frac{\partial\mathcal{H}}{\partial\xi^i}\delta\xi^i + \frac{\partial\mathcal{H}}{\partial\eta}\delta\eta\right) dt\Big],$$

or when it is put into the order of similar variations

$$\delta^2 J = \int_1^2 \Big\{\left(dq^i - \frac{d\mathcal{H}}{dt}dt\right)\delta\eta_i + \left(d\xi^i - \frac{\partial\mathcal{H}}{\partial p_i}dt\right)\delta p_i - \left(dp_i + \frac{\partial\mathcal{H}}{\partial\xi^i}dt\right)\delta\xi^i - \left(d\eta_i + \frac{d\mathcal{H}}{dq^i}dt\right)\delta q^i\Big\}.$$

Under the condition that the perturbed motion slightly deviates from the real one, than the second variation of action is equal to zero. The expressions in parenthesis next to $\delta\eta_i$ and $\delta\xi^i$ equal zero because of the equations of motion (10.56) and (10.57), and the expressions in parenthesis next to δq^i and δp_i become invalid because of the differential equations of perturbation. Inversely from the requirement that function (10.62) has extremum we get $4n+4$ differential equations of motion and differential equations of perturbation and in coupled form

$$\dot{q}^i = \frac{\partial\mathcal{H}}{\partial\eta_i}, \qquad \dot{p}_i = -\frac{\partial\mathcal{H}}{\partial\xi^i}$$
$$\dot{\xi}^i = \frac{\partial\mathcal{H}}{\partial p_i}, \qquad \dot{\eta}_i = -\frac{\partial\mathcal{H}}{\partial q^i}$$

where $\mathcal{H}$ is the coupled function (10.63).

EPILOGUE

Initial detailed explanation of the introduction of rheonomic coordinates had as an aim to make an easier and clearer understanding of the meaning of the function $q^0 = \tau(t)$. As it was shown, this coordinate is introduced by means of the rheonomic constraints, so as it is, it represents constraint $f = q^0 - \tau(t)$, to which, in equations of motion, correspond the force R_0. In all variational relations this magnitude is chosen in such a way to make all variations of coordinates independent. In the absence of rheonomic constraints, the "rheonomic coordinate" also disappears and all phrases, formulas, equations, and principles retain invariant form and physical meaning. Time t in this monograph retains the status of independent basic concept of Newton's mechanics. That is why it is possible in this approach that dependence of dynamic and kinetic magnitudes on both time and rheonomic coordinate appear, the dependence of generalized forces $Q = Q(t, q^0, q^1, \ldots, q^n, \dot{q}^0, \ldots, \dot{q}^n)$ for example. This comes from the nature of the forces and from calculating generalized forces by means of parametric equations of rheonomic constraints. The rheonomic coordinate also appears as an argument of various functions through transformation of coordinates and transformation of constraints in parametric form. Even if we take time as the rheonomic coordinate, that corodinate should be distinguished from the time which figures in equations independently from rheonomic constraints. When choosing rheonomic constraint q^0 one must consider its dimension, because the dimension of function R_0 is the reflection of the dimension of the rheonomic coordinate.

If time is chosen for the rheonomic coordinate q^0, $q^0 = t$, where $[\dim q^0] = T$ then function R_0 has dimension of power, $[\dim R_0] = \mathrm{MLT}^{-3}$. When the rheonomic coordinate is chosen with the dimension of length, then the function R_0 has the dimension of force, $[\dim q^0] = \mathrm{L}$, and $[\dim R_0] = \mathrm{MLT}^{-2}$. In the case that an angle is chosen for the rheonomic coordinate, function R_0 has the dimension of moment of force, $\mathrm{ML}^2\mathrm{T}^{-2}$.

Therefore, the rheonomic coordinate is chosen according to the condition of rheonomic constraints and the wish to describe an attribute of the change of constraints. It can also be understood if all rheonomic holonomic constraints were substituted by the one constraint $q^0 - \tau(t) = 0$, to which corresponds the generalized force R_0 of constraint change.

REFERENCES

[1] T. P. Angelitch, *Tensorkalkül nebst Anwendungen*, in: *Mathematische Hilfsmittel des Ingenieurs*, Springer-Verlag, Berlin-Heidelberg-New York 1968, pp. 167–230.

[2] V. Čović, *On the first integrals of rheonomous holonomic mechanical systems*, Z. Angew. Math. Mech. **67** (1987), 5.

[3] Б. А. Дубровин, С. П. Новиков и А. Т. Фоменко, *Современная геометрия*, Наука, Москва, 1979.

[4] N. Jovanović, *On the first integrals of the mechanical rheonomous systems with the non-holonomic constraints*, XVII Yugoslav Congress of Theoretical and Applied Mechanics, A – Opšta mehanika, Zadar, 1986, 93–98.

[5] M. Kažić, *Mechanical energy of rheonomical systems; energy integral*, XVII Yugoslav Congress of Theoretical and Applied Mechanics, A – Opšta mehanika, Zadar, 1986, 27–32.

[6] Ю. И. Неймарк и Н. А. Фуфаев, *Динамика неголономных систем*, Наука, Москва, 1967.

[7] В. В. Румянцев, *О различных формах теоремы о кинетической энергии*, Теор. Примен. Механика, **11**, (1985), 147–154.

[8] В. В. Румянцев, *К динамике Лагранжевых реономных систем со связями*, Прикл. Мат. Мех. **48** 1984, 540—550.

[9] B. Vujanović, *A study of conservation laws of dynamical systems by means of the differential variational principles of Jourdain and Gauss*, Acta Mech. **65** (1986), 63–80.

[10] B. Vujanović and S. E. Jones, *Variational Methods in Nonconservative Phenomena*, Academic Press, Boston–Toronto, 1989.

[11] V. A. Vujičić, *Kovarijantno diferenciranje skalara preko vektora*, Mat. Vesnik **3 (16) (31)** (1979), 357–360.

[12] В. А. Вуйичич, *Об интеграле энергии систем отнесенных к нестационарным связями*, Теор. Примен. Мех. **6**, (1980), 133—143.

[13] V. A. Vujičić, *The energy integral of rheonomous systems*, Acad. Serbe Sci. Arts Glas. **47** (1981), 37–51.

[14] V. A. Vujičić, *Kovarijantna dinamika*, Matematički institut, Beograd, 1981.

[15] V. A. Vujičić, *The optimal control of the motion of a holonomic system*, Acad. Serbe Sci. Arts Glas **47** (1981), 17–27.

[16] V. A. Vujičić, *The potential and power of rheonomous constraints*, Bull. Appl. Math. Budapest **61** (1986), 365–381.

[17] В. А. Вуйичич, *Одно следствие инвариантности принципа Гаусса*, Прикл. Мат. Мех. **51** (1987), 735–740.

[18] В. А. Вуйичич, *К принципу возможных перемещений для реономных систем*, Прикл. Мех. **24** (2) (1988), 125–127.

[19] V. A. Vujičić, *On Hamilton's principle for the rheonomous system*, Bull. Acad. Serbe Sci. Arts Cl. Sci. Math. Natur. (N.S.) **16** (1988), 37–50.

[20] V. A. Vujičić, *The modification of analytical dynamics of rheonomic systems*, Tensor (N.S.) **46** (1987), 418–431.