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Dissertation title: Semi-Fredholm operators on Hilbert C^*-modules

Abstract: In the first part of the thesis, we establish the semi-Fredholm theory on Hilbert C^*-modules as a continuation of the Fredholm theory on Hilbert C^*-modules which was introduced by Mishchenko and Fomenko. Starting from their definition of C^*-Fredholm operator, we give definition of semi-C^*-Fredholm operator and prove that these operators correspond to one-sided invertible elements in the Calkin algebra. Also, we give definition of semi-C^*-Weyl operators and semi-C^*-B-Fredholm operators and obtain in this connection several results generalizing the counterparts from the classical semi-Fredholm theory on Hilbert spaces. Finally, we consider closed range operators on Hilbert C^*-modules and give necessary and sufficient conditions for a composition of two closed range C^*-operators to have closed image. The second part of the thesis is devoted to the generalized spectral theory of operators on Hilbert C^*-modules. We introduce generalized spectra in C^*-algebras of C^*-operators and give description of such spectra of shift operators, unitary, self-adjoint and normal operators on the standard Hilbert C^*-module. Then we proceed further by studying generalized Fredholm spectra (in C^*-algebras) of operators on Hilbert C^*-modules induced by various subclasses of semi-C^*-Fredholm operators. In this setting we obtain generalizations of some of the results from the classical spectral semi-Fredholm theory such as the results by Zemanek regarding the relationship between the spectra of an operator and the spectra of its compressions. Also, we study 2×2 upper triangular operator matrices acting on the direct sum of two standard Hilbert C^*-modules and describe the relationship between semi-C^*-Fredholmness of these matrices and of their diagonal entries.

Keywords: Hilbert C^*-module, semi-C^*-Fredholm operator, semi-C^*-Weyl operator, semi-C^*-B-Fredholm operator, essential spectrum, Weyl spectrum, perturbation of spectra, compression

Research area: Mathematics

Research sub-area: Analysis, operator theory and operator algebra

Наслов дисертације: Полу-Фредхолмови оператори на Хилбертовим C^*-модулима

Резиме: У првом делу тезе успостављамо полу-Фредхолмову теорију на Хилбертовим C^*-модулима као наставак Фредхолмове теорије на Хилбертовим C^*-модулима коју су увели Мишченко и Фоменко. Полазећи од њихове дефиниције C^*-Фредхолмових оператора, дајемо дефиницију полу-C^*-Фредхолмовог оператора и доказујемо да ти оператори одговарају једнострано инвертибилним елементима у Калкиновој алгебри. Такође, дајемо дефиницију полу-C^*-Вајлових оператора и полу-C^*-$Б$-Фредхолмових оператора и добијамо с тим у вези више резултата који генерализују пандане из класичне полу-Фредхолмове теорије на Хилбертовим просторима. На крају, разматрамо операторе са затвореном сликом на Хилбертовим C^*-модулима и дајемо потребне и довољне услове да композиција два C^*-оператора са затвореном сликом има затворену слику. Други део тезе посвећен је генерализованој спектралној теорији оператора на Хилбертовим C^*-модулима. За C^*-операторе дефинишемо генерализоване спектре у C^*-алгебри и дајемо опис таквих спектара у конкретном случају оператора помака, унитарних, самоадјонгованих и нормалних оператора на стандардном Хилбертовом C^*-модулу. Затим настављамо даље проучавајући генерализоване Фредхолмове спектре (у C^*-алгебрама) оператора на Хилбертовим C^*-модулима индукованим различитим подкласама полу-C^*-Фредхолмових оператора. У овом контексту добијамо уопштење неких резултата из класичне спектралне полу-Фредхолмове теорије, као што су Земанекови резултати у вези релација између спектара оператора и спектара њихових компресија. Такође, проучавамо 2×2 горње тријангуларне операторске матрице које делују на директној суми два стандардна Хилбертова C^*-модула и описујемо однос између полу-C^*-Фредхолмности ових матрица и њихових дијагоналних елемената.

Кључне речи: Хилбертов C^*-модул, полу-C^*-Фредхолмов оператор, полу-C^*-Вајлов оператор, полу-C^*-$Б$-Фредхолмов оператор, есенцијални спектар, Вајлов спектар, пертурбације спектра, компресије.

Научна област: Математика

Ужа научна област: Анализа, теорија оператора и алгебре оператора

Contents

Chapter 1

Introduction

The Fredholm and semi-Fredholm theory on Hilbert and Banach spaces started by studying the integral equations introduced in the pioneering work by Fredholm in 1903 in [12]. After that the abstract theory of Fredholm and semi-Fredholm operators on Banach spaces was further developed in numerous papers and books such as [2], [3] and [56]. Some recent results in the classical semi-Fredholm theory can be found in [55]. Now, Fredholm theory on Hilbert C^*-modules as a generalization of Fredholm theory on Hilbert spaces was started by Mishchenko and Fomenko in [40]. They have introduced the notion of a Fredholm operator on the standard module and proved the generalization of the Atkinson theorem. Our aim is to study more general operators than the Fredholm ones, namely a generalization of semi-Fredholm operators. In this thesis we give the definition of those and establish several properties as an analogue or a generalized version of the properties of the classical semi-Fredholm operators on Hilbert and Banach spaces.
Recall that if H is a Hilbert space, then F is a semi-Fredholm operator on H, denoted by $F \in \Phi_{\pm}(H)$ if $F \in B(H)$ and ImF is closed, that is, if there exists a decomposition

$$H = (\ker F)^{\perp} \oplus \ker F \xrightarrow{F} ImF \oplus (ImF)^{\perp} = H$$

with respect to which F has the matrix $\begin{bmatrix} F_1 & 0 \\ 0 & 0 \end{bmatrix}$, where F_1 is an isomorphism, and either

$$\dim \ker F < \infty \text{ or } \dim (ImF)^{\perp} < \infty.$$

If $\dim \ker F < \infty$, then F is called an upper semi-Fredholm operator on H, denoted by $F \in \Phi_+(H)$, whereas if $\dim (ImF)^{\perp} < \infty$, then F is called a lower semi-Fredholm operator on H, denoted by $F \in \Phi_-(H)$. If F is both an upper and a lower semi-Fredholm operator on H, then F is said to be a Fredholm operator on H, denoted by $F \in \Phi(H)$. In the case when $F \in \Phi(H)$, the index of F is defined as index $F = \dim \ker F - \dim (ImF)^{\perp}$.

Now, Hilbert C^*-modules are a natural generalization of Hilbert spaces when the field of scalars is replaced by an arbitrary C^*-algebra. Some recent results in the theory of Hilbert C^*-modules can be found in [11], [16], [34], [41]. In [40] Mishchenko and Fomenko consider a standard Hilbert C^*-module over a unital C^*-algebra $\mathcal{A}$, denoted by $H_{\mathcal{A}}$, and they define an $\mathcal{A}$-Fredholm operator F on $H_{\mathcal{A}}$ as a generalization of a Fredholm operator on Hilbert space H in the following way (see [40, Definition]): A (bounded $\mathcal{A}-$linear) operator $F : H_{\mathcal{A}} \to H_{\mathcal{A}}$ is called $\mathcal{A}$-Fredholm if
1) it is adjointable;
2) there exists a decomposition of the domain $H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1$, and the range, $H_{\mathcal{A}} = M_2 \tilde{\oplus} N_2$, where M_1, M_2, N_1, N_2 are closed $\mathcal{A}$-modules and N_1, N_2 have a finite number of generators in algebraic sense, such that F has the matrix form

$$\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}$$

with respect to these decompositions and $F_1 : M_1 \to M_2$ is an isomorphism.
It is then proved in [40] that some of the main results from the classical Fredholm theory on Hilbert spaces also hold when one considers this generalization of Fredholm operator on $H_{\mathcal{A}}$.
The idea in this thesis is to go further in this direction, to give, in a similar way, a definition of semi-Fredholm operators on Hilbert C^*-modules over unital C^*-algebras, to investigate and prove generalized version in this setting of significantly many results from the classical semi-Fredholm theory on Hilbert and Banach spaces.

Let us mention a few words on the motivation for studying semi-Fredholm operators on Hilbert C^*-modules.
People have over long time been interested in solving equations of the form $Ax = y$ for $x \in X$, $y \in Y$ when X and Y are Banach spaces and $A \in B(X,Y)$. The simplest case is when A is invertible and the fomula for A^{-1} is known. In this case the solution is unique and is given by $x = A^{-1}y$. Unfortunately, A is in general not invertible in such equations. Therefore, people have studied more general situations in which A can happen to be non-invertible, but still regular, i.e. ImA is closed in Y and $\ker A$ and ImA are complementable in the respective Banach spaces or, in other words, A admits generalized inverse. In these situations we can still solve the equations of the form $Ax = y$, although not uniquely. More precisely, if A is regular, then we have decompositions $X = (\ker A)^\circ \oplus \ker A$ and $Y = ImA \oplus (ImA)^\circ$, where $(\ker A)^\circ$ and $(ImA)^\circ$ denote the complements $\ker A$, and ImA, respectively. Let $\sqcap$ denote the projection onto ImA along $(ImA)^\circ$ (that is $\sqcap(u + v) = u$ for all $u \in ImA, v \in (ImA)^\circ$). The equation $Ax = y$ has a solution if and only if $\sqcap y = y$ and in this case the solutions are given by $x = A'y + z$, where $z \in \ker A$ and A' is generalized inverse of A. So, in the situation when A is regular, it is still possible to handle the equations of the form $Ax = y, x \in X, y \in Y$.
Now, a natural generalization of linear operators on Hilbert spaces are $\mathcal{A}$-linear operators on Hilbert modules over a C^*-algebra $\mathcal{A}$. One of the reasons for studying $\mathcal{A}$-linear operators is that sometimes they may give a better description of non-linear phenomena in the real life than ordinary linear operators. Indeed, it is well known that the motivation for linear analysis comes partly from studying local linear approximations of non-linear phenomena. In the case of Hilbert spaces, the equation $Ax = y$ induces a (possibly infinite) system of equations in $\mathbb{C}$ when x and y are represented as coordinate vectors with respect to an orthonormal basis for the respective Hilbert space and A is given by a matrix with respect to this basis. For more details we refer to [51]. However, if F is an $\mathcal{A}$-linear, bounded operator on the standard module $H_{\mathcal{A}}$ over a C^*-algebra $\mathcal{A}$, then the equation $Fx = y, x, y \in H_{\mathcal{A}}$ induces an infinite system of equations in $\mathcal{A}$. Since $\mathcal{A}$ is an arbitrary unital C^*-algebra, thus it could be an algebra of functions or operators, such system of equations may sometimes give a better description of non-linear phenomena in the real life than the system of equations with constant coefficients. Therefore, we may sometimes obtain more information by studying $\mathcal{A}$-linear operators than by just studying classical linear operators. On the other hand it turns out that $\mathcal{A}$-linear operators, especially adjointable ones, still keep many of the "nice" properties of the classical bounded, linear operators on Hilbert spaces. All this together gives one of the reasons for studying $\mathcal{A}$-linear, bounded operators on $H_{\mathcal{A}}$ (where $\mathcal{A}$ is a unital C^*-algebra).

We may hence consider regular, $\mathcal{A}$-linear, bounded operators on $H_{\mathcal{A}}$ for solving the equations of the form $Fx = y$, where $x, y \in H_{\mathcal{A}}$. It turns out that if F is adjointable and ImF is closed, then F is automatically regular, since ImF and $\ker F$ are orthogonally complementable in this case. Thus, in this case F has the matrix $\begin{bmatrix} F_1 & 0 \\ 0 & 0 \end{bmatrix}$ with respect to the decomposition

$$H_{\mathcal{A}} = (\ker F)^\perp \oplus \ker F \xrightarrow{F} ImF \oplus (ImF)^\perp = H_{\mathcal{A}},$$

where F_1 is an isomorphism. If $(ImF)^{\perp}$ is finitely generated, then it is easily checked (even without computing the explicit formula for the orthogonal projection onto ImF) whether the equation $Fx = y$ has a solution. Indeed, this equation has a solution if and only if y is orthogonal to all generators of $(ImF)^{\perp}$, which are finitely many in this case. On the other hand, if $\ker F$ is finitely generated and we have an explicit formula for F_1^{-1}, then we can also give an explicit expression for solutions. Namely, the solutions in this case are given by $x = F_1^{-1}y + \sum_{k=1}^{n} z_k \cdot \alpha_k$, where the set $\{z_1, \ldots, z_n\}$ generates $\ker F$ and $\alpha_1, \ldots, \alpha_n$ are arbitrary elements of $\mathcal{A}$. Therefore, we are in particular interested in those regular, $\mathcal{A}$-linear, bounded operators on $H_{\mathcal{A}}$ for which either complement of the kernel or complement of the image is finitely generated. This leads us to study more general class of operators than regular ones, namely the class of those $\mathcal{A}$-linear, bounded operators F for which there exists a decomposition $H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$ with respect to which F has the matrix $\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}$, where F_1 is an isomorphism and either N_1 or N_2 is finitely generated. We denote this class by semi $\mathcal{A}$-Fredholm operators. The inspiration for considering such operators comes from the definition of $\mathcal{A}$-Fredholm operators given by Mishchenko and Fomenko .

In fact we are in particular interested in operators that arise from natural cases, e.g. (pseudo) differential operators acting on manifolds. The classical theory works nice for compact manifolds, but not for general ones. Even operators on Euclidean spaces are hard to study, e.g. Laplacian is not Fredholm. However, they can become Fredholm when we look at them as operators on a torus with coefficients in the group C^*-algebra of the integers (as the torus is the quotient of the Euclidean space modulo the action of integers). Kernels and cokernels of many operators are infinite-dimensional as Banach spaces, but become finitely generated viewed as Hilbert modules. This is the most important reason for studying semi-$\mathcal{A}$-Fredholm operators.

Let us give an overview of the main results in each of the chapters in the thesis.

In the second chapter we recall the results from the general theory of Hilbert C^*-modules and from $\mathcal{A}$-Fredholm theory on $H_{\mathcal{A}}$ that are needed in the rest of the thesis.

In the third chapter we define adjointable upper and lower semi-$\mathcal{A}$-Fredholm operators and prove that they correspond to one-sided invertible elements modulo compact operators. We establish several properties of these operators as an analogue of the properties of classical semi-Fredholm operators given in [56], such as openess of the set of proper semi-$\mathcal{A}$-Fredholm operators and Schechter characterization. Moreover, we consider various new classes of operators on Hilbert C^*-modules as generalizations of the class of semi-Weyl operators on Hilbert spaces. We prove that these new classes are open, invariant under compact perturbations and several other results generalizing in this setting the results from [56, Section 1.9]. Such operators will be called semi-$\mathcal{A}$-Weyl operators.

Next, in addition to adjointable semi-$\mathcal{A}$-Fredholm operators, we consider also non-adjointable semi-$\mathcal{A}$-Fredholm operators in the third chapter as a continuation of Mishchenko's work on non-adjointable $\mathcal{A}$-Fredholm operators in [17].

One of the challenges with working with non-adjointable operators is that one does not necessarily have complementability of the kernel and the image of the closed range operators as one has for adjointable operators where the kernel and the image of a closed range operator are even orthogonally complementable (we recall that not all closed submodules of a Hilbert C^*- modules are complementable, which is one of the big differences between Hilbert spaces and Hilbert C^*-modules in general). Moreover, in the case of adjointable operators on Hilbert C^*-modules, one can sometimes easily obtain a symmetric version of certain results simply by taking the adjoint, while it is not possible to do that with non-adjointable operators. Because of all these facts, the theory of non-adjointable operators sometimes differs from the theory of adjointable operators and is more challenging. Therefore, it is interesting to investigate non-adjointable operators in addition to adjointable operators and this is the reason why in this thesis sometimes we treat separately the case of non-adjointable operators. Moreover, non-adjointable operators occur more often in applications than adjointable ones, so this is also one of the reasons why we are especially interested in non-adjointable operators.

Finally, at the end of the third chapter we introduce examples of semi-$\mathcal{A}$-Fredholm operators.

The generalized versions in the setting of Hilbert C^*-modules of the results from the classical semi-Fredholm theory on Banach and Hilbert spaces, which are presented here in this thesis, usually demand different proofs from the classical ones. However, the techniques used in these proofs are to a certain extent inspired by the techniques used in the proofs of some of the results in [40]. In the last section of the third chapter we also show how these techniques can be applied to the special class of operators on infinite-dimensional Hilbert spaces, so called generalized Fredholm operators on Hilbert spaces, which are the operators with image that contains a closed, infinite-dimensional subspace.

Several special properties of $\mathcal{A}$-Fredholm operators in the case of W^*-algebra were described in [38, Section 3.6]. The idea in the fourth chapter of the thesis is to go further in this direction and establish more special properties of $\mathcal{A}$-Fredholm operators defined in [40] and of semi$-\mathcal{A}$-Fredholm operator in the case when $\mathcal{A}$ is a W^*-algebra, the properties that are closer related to the properties of the classical semi-Fredholm operators on Hilbert spaces than in the general case, when $\mathcal{A}$ is an arbitrary C^*-algebra. Using the assumption that $\mathcal{A}$ is a W^*-algebra (and not an arbitrary C^*-algebra) we obtain various results such as a generalization of Schechter-Lebow characterization of semi-Fredholm operators and a generalization of the "punctured neighbourhood" theorem, as well as some other results that generalize their classical counterparts. We consider both adjointable and non-adjointable semi-Fredholm operators over W^*-algebras. At the end of this chapter we consider the special case of self-dual Hilbert W^*-modules and prove that the set of semi-$\mathcal{A}$-Fredholm operators and the set of semi-$\mathcal{A}$-Weyl opertors on self-dual Hilbert $\mathcal{A}$-modules form semigroups under the multiplication.

Various generalizations of classical semi-Fredholm operators such as generalized Weyl operators defined by Đorđević in [8] and semi-B-Fredholm operators defined by Berkani in [4] and [5] have been considered earlier. In the fifth chapter we construct in a similar way generalizations of semi-$\mathcal{A}$-Fredholm operators and investigate some of their properties. Those operators will be called generalized $\mathcal{A}$-Weyl operators and semi-$\mathcal{A}$-B-Fredholm operators. We prove that these classes of operators are under certain conditions closed under the multiplication and invariant under the finitely generated perturbations. Again, we consider both adjointable and non-adjointable operators. Moreover, we apply also the techniques from our proofs in this chapter to extend the results from [8] to the case of regular operators on Banach spaces and give thus partly an answer to the open question from [8] regarding whether the results from [8] could be extended from the case of operators on Hilbert spaces to the case of operators on Banach spaces. At the end of this chapter we give an example of a semi-$\mathcal{A}$-B-Fredholm operator.
The main technique in the proofs in this chapter is application of exact sequences which allows us not only to obtain new results for operators on Hilbert C^*-modules, but also to provide generalizations and extensions of the classical results for operators on Banach spaces.

It turns out that closed range operators are very important in semi-Fredholm theory on Hilbert C^*-modules. In the sixth chapter we present equivalent conditions for a composition of two closed range adjointable operators to have closed image. We also give a simplification of the results by Sharifi in [49] and we give a sufficient condition in terms of Dixmier angle for a composition of two non-adjointable closed range $\mathcal{A}$-Fredholm operators to have closed image. One of the main differences between classical Fredholm operators on Hilbert spaces and

$\mathcal{A}$-Fredholm operators in general is that $\mathcal{A}$-Fredholm operators may happen to have non-closed image, whereas classical Fredholm operators always have closed image. In the sixth chapter we give examples of $\mathcal{A}$-Fredholm operators with non-closed image. We also give an example of an $\mathcal{A}$-Fredholm operator F satisfying that ImF is closed and ImF^2 is not closed.

Next, given an $\mathcal{A}$-linear, bounded, adjointable operator F on $H_{\mathcal{A}}$, we consider the operators of the form $F-\alpha I$ as α varies over $\mathcal{A}$, and this gives rise to a different kind of spectra of F in $\mathcal{A}$ as a generalization of ordinary spectra of F in $\mathbb{C}$. The aim of the seventh chapter is to provide basic results regarding generalized spectra in $\mathcal{A}$ of operators on Hilbert $\mathcal{A}$-modules and hence make first step into a new spectral theory of operators on Hilbert C^*-modules in the setting of generalized spectra in C^*-algebras. It turns out that some of the results in this context are valid only in the case of commutative C^*-algebras, so we provide counterexamples in the case when $\mathcal{A} = B(H)$. At the end of the seventh chapter we also show by an example how these results can be applied on operators on the Hilbert space $L^2((0,1))$ by considering the spectra in $C([0,1])$ or in $L^\infty((0,1))$.

In the eighth chapter of the thesis we study perturbations of the generalized spectra in $\mathcal{A}$. However, the main topic of the eighth chapter are upper triangular operator 2×2 matrices acting on two copies of $H_{\mathcal{A}}$. We describe the relationship between semi-$\mathcal{A}$-Fredholmness of such matrices and their diagonal entries. Also, we consider the perturbations of the spectra in $\mathcal{A}$ of such matrices, generalizing thus the results from [7].

In the ninth chapter we define several special subclasses of semi-$\mathcal{A}$-Weyl operators and we provide examples of such operators at the end of the thesis. As already observed in the third chapter, this shows that the class of classical semi-Weyl operators on Hilbert spaces has several different generalizations in the setting of operators on Hilbert C^*-modules. We consider then generalized spectra in $\mathcal{A}$ of operators on $H_{\mathcal{A}}$ induced by these special subclasses of semi-$\mathcal{A}$-Weyl operators, and give a description of such spectra in $\mathcal{A}$ in terms of the intersection of the $\mathcal{A}$-valued spectra of the compressions of operators. Thus, we generalize in this setting the well known results by Zemanek in [54]. Moreover, we show by an example how our proofs can be applied to operators on infinite-dimensional Hilbert spaces in order to extend Zemanek's results.

Semi-$\mathcal{A}$-Fredholm operators have been considered in [1] and [15]. In [1] semi-$\mathcal{A}$-Fredholm operators are defined to be those that are one-sided invertible modulo compact operators. However, in this thesis, inspired by the definition of $\mathcal{A}-$Fredholm operator on $H_{\mathcal{A}}$ given by Mishchenko and Fomenko, we define semi-$\mathcal{A}$- Fredholm operators in terms of decompositions, as explained above. It turns out that these operators are exactly those that are one-sided invertible modulo compact operators when we consider the standard module $H_{\mathcal{A}}$, so in this case our definition coincide with the definition given in [1]. However, this does not need to hold in the case of arbitrary Hilbert C^*-modules. In the last chapter we give an overview of the results from the thesis that are valid in the case of arbitrary Hilbert C^*-modules (and not just the standard module).

At the end we would like to recall that a unital C^*-algebra is a Hilbert module over itself and left multipliers on this algebra are examples of bounded operators that are linear with respect to this C^*-algebra. Thus, our results should be of interest also in this particular case.

Chapter 2

Preliminaries

Throughout this thesis we always assume that $\mathcal{A}$ is a unital C^*-algebra. The material in this chapter is mainly taken from [38].

For a right module $\mathcal{M}$ over a unital C^*-algebra $\mathcal{A}$, we shall denote an action of an element $a \in \mathcal{A}$ on $\mathcal{M}$ by $x \cdot a$ where $x \in \mathcal{M}$. As a generalization of the classical inner product on Hilbert spaces, an $\mathcal{A}$-valued inner product on an $\mathcal{A}$-module $\mathcal{M}$ is constructed as follows.

Definition 2.0.1. [38, Definition 1.2.1.] A pre-Hilbert $\mathcal{A}$-module is a (right) $\mathcal{A}$-module $\mathcal{M}$ equipped with a sesquilinear form $\langle \cdot, \cdot \rangle : \mathcal{M} \times \mathcal{M} \to \mathcal{A}$ with the following properties:
$(i)\langle x, x \rangle \geq 0$ for any $x \in \mathcal{M}$;
$(ii)\langle x, x \rangle = 0$ implies that $x = 0$;
$(iii)\langle y, x \rangle = \langle x, y \rangle^*$ for any $x, y \in \mathcal{M}$;
$(iv)\langle x, ya \rangle = \langle x, y \rangle a$ for any $x, y \in \mathcal{M}$ and any $a \in \mathcal{A}$.

The map $\langle \cdot, \cdot \rangle$ is called an $\mathcal{A}$-valued inner product.

Below are some examples.

Example 2.0.2. [38, Example 1.2.2] Let $J \subset \mathcal{A}$ be a right ideal. Then J can be equipped with the structure of a pre-Hilbert $\mathcal{A}$-module with the inner product of elements $x, y \in J$ defined by $\langle x, y \rangle := x^* y$.

Example 2.0.3. [38, Example 1.2.3] Let $\{J_i\}$ be a countable set of right ideals of a unital C^*-algebra $\mathcal{A}$ and let $\mathcal{M}$ be the linear space of all sequences $(x_i), x_i \in J_i$ satisfying the condition $\sum_i \parallel x_i \parallel^2 < \infty$. Then $\mathcal{M}$ becomes a right $\mathcal{A}$-module if the action of $\mathcal{A}$ is defined by $(x_i) \cdot a := (x_i a)$ for $(x_i) \in \mathcal{M}, a \in \mathcal{A}$, and becomes a pre-Hilbert $\mathcal{A}$-module if the inner product of elements $(x_i), (y_i) \in \mathcal{M}$ is defined by $\langle (x_i), (y_i) \rangle := \sum_i x_i^* y_i$.

In the similar way as in the case of Hilbert spaces, the $\mathcal{A}$-valued inner product on $\mathcal{M}$ induces a norm on $\mathcal{M}$ given by $\parallel x \parallel_{\mathcal{M}} = \parallel \langle x, x \rangle \parallel^{\frac{1}{2}}$ for all $x \in \mathcal{M}$.

Proposition 2.0.4. *[43] [38, Proposition 1.2.4] The function $\parallel \cdot \parallel_{\mathcal{M}}$ is a norm on $\mathcal{M}$ and satisfies the following properties:*
(i) $\parallel x \cdot a \parallel_{\mathcal{M}} \leq \parallel x \parallel_{\mathcal{M}} \cdot \parallel a \parallel$ for any $x \in \mathcal{M}, a \in \mathcal{A}$;
(ii)$\langle x, y \rangle \langle y, x \rangle \leq \parallel y \parallel_{\mathcal{M}}^2 \langle x, x \rangle$ for any $x, y \in \mathcal{M}$;
(iii) $\parallel \langle x, y \rangle \parallel \leq \parallel x \parallel_{\mathcal{M}} \parallel y \parallel_{\mathcal{M}}$ for any $x, y \in \mathcal{M}$.

Note that the properties (ii) and (iii) generalize Cauchy–Bunyakovsky–Schwarz inequality for inner product on Hilbert spaces.

Definition 2.0.5. [38, Definition 1.3.2] A pre-Hilbert $\mathcal{A}$-module $\mathcal{M}$ is called a Hilbert C^*-module if it is complete with respect to the norm $\parallel \cdot \parallel_{\mathcal{M}}$.

Below are some examples.

Example 2.0.6. [38, Example 1.3.3] If $J \subset \mathcal{A}$ is a closed right ideal, then the pre-Hilbert module J is complete with respect to the norm $\| \cdot \|_J = \| \cdot \|$. In particulaar, the unital C^*-algebra $\mathcal{A}$ itself is a free Hilbert $\mathcal{A}$-module with one generator.

Example 2.0.7. [38, Example 1.3.4] If $\{\mathcal{M}_i\}$ is a finite set of Hilbert $\mathcal{A}$-modules, then one can define the direct sum $\oplus \mathcal{M}_i$. The inner product on $\oplus \mathcal{M}_i$ is given by the formula $\langle x, y \rangle := \sum_i \langle x_i, y_i \rangle$ where $x = (x_i), y = (y_i) \in \oplus \mathcal{M}_i$. We denote the direct sum of n copies of a Hilbert module $\mathcal{M}$ by $\mathcal{M}^n$ or $L_n(\mathcal{M})$.

In the case when $\mathcal{M} = \mathcal{A}$, we will simply denote $L_n(\mathcal{A})$ by L_n in the rest of the thesis.

Example 2.0.8. [38, Example 1.3.5] If $\{\mathcal{M}_i\}, i \in \mathbb{N}$, is a countable set of Hilbert $\mathcal{A}$-modules, then one can define their direct sum $\oplus \mathcal{M}_i$ to be the set of all sequences $x = (x_i) : x_i \in \mathcal{M}_i$, such that the series $\sum_i \langle x_i, y_i \rangle$ is norm-convergent in the C^*-algebra $\mathcal{A}$. Then we define the inner product by

$$\langle x, y \rangle := \sum_i \langle x_i, y_i \rangle \text{ for } x, y \in \oplus \mathcal{M}_i.$$

With respect to this inner product $\oplus \mathcal{M}_i$ is a Hilbert $\mathcal{A}$-module. If each $\mathcal{M}_i = \mathcal{A}$, then we will denote $\oplus \mathcal{M}_i$ by $H_{\mathcal{A}}$. **This module is called the standard module over $\mathcal{A}$. So, in other words $\mathbf{H}_{\mathcal{A}} = \mathbf{l}^2(\mathcal{A})$.** If $\mathcal{A}$ is unital, then $H_{\mathcal{A}} = l^2(\mathcal{A})$ has natural orthonormal basis $\{e_j\}_{j \in \mathbb{N}}$.

There are also some other interesting examples of C^*-modules. We can for example consider $L^2(\Omega, \mu, B(H))$, for more details we refer to [26] and [27].

Let $\mathcal{N} \subset \mathcal{M}$ be a closed submodule of a Hilbert C^*-module $\mathcal{M}$. In the same way as for Hilbert spaces, we define the orthogonal complement $\mathcal{N}^{\perp}$ by the formula

$$\mathcal{N}^{\perp} = \{y \in \mathcal{M} : \langle x, y \rangle = 0 \text{ for all } x \in \mathcal{N}\}.$$

By Proposition 2.0.4 part (iii) it follows that $\mathcal{N}^{\perp}$ is a closed submodule of the Hilbert C^*-module $\mathcal{M}$. However, the important difference from Hilbert spaces is that the equality $\mathcal{M} = \mathcal{N} \oplus \mathcal{N}^{\perp}$ does not always hold, as the following example shows.

Example 2.0.9. [38, Example 1.3.7] Let $\mathcal{A} = C[0, 1]$ be the C^*-algebra of all continuous function on the segment $[0, 1]$. Consider, in the Hilbert $\mathcal{A}$-module $\mathcal{M} = \mathcal{A}$, the submodule $\mathcal{N} = C_0(0, 1)$ of functions that vanish at the end points of the segment. Then, obviously, $\mathcal{N}^{\perp} = 0$.

By the symbol $\tilde{\oplus}$ we denote the direct sum of modules as given in [38].

Thus, if M is a Hilbert C^*-module and M_1, M_2 are two closed submodules of M, we write $M = M_1 \tilde{\oplus} M_2$ if $M_1 \cap M_2 = \{0\}$ and $M_1 + M_2 = M$. If, in addition M_1 and M_2 are mutually orthogonal, then we write $M = M_1 \oplus M_2$.

If M and N are two Hilbert C^*-modules over a unital C^*-algebra $\mathcal{A}$, then a map $T : M \to N$ is called an $\mathcal{A}$-linear operator if $T(x \cdot \alpha) = T(x) \cdot \alpha$ for all $x \in M$ and $\alpha \in \mathcal{A}$. In particular this means that T is linear because

$$T(\lambda x) = T(x \cdot \lambda 1) = Tx \cdot \lambda 1 = \lambda Tx$$

for all $\lambda \in \mathbb{C}$. The set of all bounded, $\mathcal{A}$-linear operators from M into N will be denoted by $B(M, N)$.

To simplify notation, for $F \in B(M, N)$ we will throughout this thesis simply write $ImF^{\perp}$ and $\ker F^{\perp}$ instead of $(ImF)^{\perp}$ and $(\ker F)^{\perp}$, respectively.

Lemma 2.0.10. *Let M be a Hilbert C^*- module and suppose that $M = M_1\tilde{\oplus}M_2$ for some Hilbert submodules M_1 and M_2. Then $M \cong M_1 \oplus M_2$, where we consider the direct sum of M_1 and M_2 in the sense of Example 2.0.7.*

Proof. We define in a natural way the map $\iota : M \to M_1 \oplus M_2$ given by $\iota(x) = (\sqcap x, (I - \sqcap)(x))$ where $\sqcap$ denotes the projection of M onto M_1 along M_2. This map is well defined and bijective since $M = M_1\tilde{\oplus}M_2$ by assumption. Moreover, it is $\mathcal{A}-$linear. It remains to show that ι is bounded. However, by the definition of direct sum of Hilbert modules given in Example 2.0.7, for all $x \in M$, we have

$$\begin{gathered} \| \iota(x) \|^2 = \| \langle \sqcap x, \sqcap x\rangle + \langle (I-\sqcap)x, (I-\sqcap)x\rangle \| \\ \leq \| \langle \sqcap x, \sqcap xy\rangle \| + \| \langle (I-\sqcap)x, (I-\sqcap)x\rangle \| \\ = \| \sqcap x \|^2 + \| (I-\sqcap)x \|^2 \leq (\| \sqcap \|^2 + \| I-\sqcap \|^2) \| x \|^2 . \end{gathered}$$

□

An operator $T \in B(M,N)$ is said to be adjointable if there exists an $\mathcal{A}$-linear operator $T^* : N \to M$ such that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \text{ for all } x \in M, y \in N.$$

It turns out that in this case T^* is also bounded. The set of all adjointable, bounded, $\mathcal{A}$-linear operators from M into N will be denoted by $B^a(M,N)$. It can be shown that $B^a(M)$ is a C^*-algebra, for more details see [38, Section 2.2].

The next example shows that there exist non-adjointable operators on Hilbert C^*-modules.

Example 2.0.11. [38, Example 2.1.2] Let $\mathcal{A}$ be a unital C^*-algebra. As above, the standard basis of the Hilbert module $H_\mathcal{A}$ consists of the elements $e_i = (0,\cdots,0,1,0,\cdots)$, where 1 is the i-th entry. To each operator $T \in B(H_\mathcal{A})$ one can associate an infinite matrix with respect to this basis,

$$[t_{i,j}], t_{i,j} = \langle e_i, Te_j\rangle.$$

Then the adjoint operator, if it exists, has the matrix $[t^*_{i,j}]$.

Let $\mathcal{A} = C([0,1])$ and let the functions $\varphi_i \in \mathcal{A}, i = 1,2,\cdots$, be defined by the formula

$$\varphi_i = \begin{cases} 0 & \text{on } \left[0, \dfrac{1}{i+1}\right] \text{ and } \left[\dfrac{1}{i}, 1\right], \\ 1 & \text{at the point } x_i = \frac{1}{2}\left(\dfrac{1}{i} + \dfrac{1}{i+1}\right), \\ \text{is linear} & \text{on } \left[\dfrac{1}{i+1}, x_i\right] \text{ and } \left[x_i, \dfrac{1}{i}\right]. \end{cases}$$

Let T be the operator which has the matrix

$$\begin{pmatrix} \varphi_1 & \varphi_2 & \varphi_3 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

(actually it is an operator from the module $H_\mathcal{A}$ to $\mathcal{A}$, thus an $\mathcal{A}$-functional). It is easy to verify that T is bounded. However, the operator T^* is not well defined since it should have the matrix

$$\begin{pmatrix} \varphi_1^* & 0 & 0 & \cdots \\ \varphi_2^* & 0 & 0 & \cdots \\ \varphi_3^* & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

and the image of the basis element e_1 should be an element of $H_\mathcal{A}$ having the first column as its coordinates and it has to be an element of $H_\mathcal{A}$, which is impossible since the series $\sum \varphi_i \varphi_i^*$ is not norm-convergent in the C^*-algebra $\mathcal{A}$.

Definition 2.0.12. [38, Definition 1.4.1] Hilbert C^*-module $\mathcal{M}$ is called *finitely generated* if there exists a finite set $\{x_i\} \subset \mathcal{M}$ such that $\mathcal{M}$ equals the linear span (over $\mathbb{C}$ and $\mathcal{A}$) of this set. A Hilbert C^*-module $\mathcal{M}$ is called *countably generated* if there exists a countable set $\{x_i\} \subset \mathcal{M}$ such that $\mathcal{M}$ equals the norm-closure of the linear span (over $\mathbb{C}$ and $\mathcal{A}$) of this set.

Theorem 2.0.13. *(Kasparov stabilization theorem) [29] [38, Theorem 1.4.2] Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{M}$ a countably generated Hilbert $\mathcal{A}$-module. Then $\mathcal{M} \oplus H_\mathcal{A} \cong H_\mathcal{A}$.*

Definition 2.0.14. [38, Definition 1.4.4] A Hilbert $\mathcal{A}$-module $\mathcal{M}$ is called a *finitely generated projective* $\mathcal{A}$-module if there exists a Hilbert $\mathcal{A}$-module $\mathcal{N}$ such that $\mathcal{M} \oplus \mathcal{N} \cong L_n(\mathcal{A})$ for some n.

As explained in [38, Section 1.4] an element x of a Hilbert C^*-module N is called nonsingular if $\langle x, x \rangle$ is invertible in the respective C^*-algebra.

Theorem 2.0.15. *(Dupré - Fillmore, [9] [38, Theorem 1.4.5]) Let $\mathcal{A}$ be a unital C^*-algebra and let $\mathcal{M}$ be a finite-dimensional projective $\mathcal{A}$-submodule in the standard Hilbert $\mathcal{A}$-module $H_\mathcal{A}$. Then*
(i) The nonsingular elements of the module $\mathcal{M}^\perp$ are dense in $\mathcal{M}^\perp$;
(ii) $H_\mathcal{A} = \mathcal{M} \oplus \mathcal{M}^\perp$;
(iii) $\mathcal{M}^\perp \cong H_\mathcal{A}$.

Theorem 2.0.16. *[9], [38, Theorem 1.4.6] Let $\mathcal{A}$ be a unital C^*-algebra and let $\mathcal{M}$ be a finitely generated projective Hilbert submodule in an arbitrary Hilbert $\mathcal{A}$-module $\mathcal{N}$. Then $\mathcal{N} = \mathcal{M} \oplus \mathcal{M}^\perp$.*

Let $\mathcal{M}$ be a Hilbert C^*-module over a unital C^*-algebra $\mathcal{A}$. We set $\mathcal{K}^*(\mathcal{M})$ to be the closure in the norm topology of the linear span of the operators $\theta_{x,y}$, where $x, y \in \mathcal{M}$ and $\theta_{x,y}(z) = x < y, z >$ for all $z \in \mathcal{M}$. In [38, Section 2.2] the operators $\theta_{x,y}$ are called elementary operators. The set $\mathcal{K}^*(\mathcal{M})$ is a closed, two sided self-adjoint ideal in the C^*-algebra $B^a(\mathcal{M})$, see [38, Section 2.2].

Proposition 2.0.17. *[38, Proposition 2.2.1] Let $H_\mathcal{A}$ be the standard Hilbert module over a unital C^*-algebra $\mathcal{A}$ and let $L_n(\mathcal{A}) \subset H_\mathcal{A}$ be the free submodule generated by the first n elements of the standard basis. An operator $K \in B^a(H_\mathcal{A})$ is compact if and only if the norms of restrictions of K onto the orthogonal complements $L_n(\mathcal{A})^\perp$ of the submodules $L_n(\mathcal{A})$ vanish as $n \to \infty$.*

Definition 2.0.18. [38, Definition 2.3.1] A closed submodule $\mathcal{N}$ in a Hilbert C^*-module $\mathcal{M}$ is called (topologically) complementable if there exists a closed submodule $\mathcal{L}$ in $\mathcal{M}$ such that $\mathcal{N} + \mathcal{L} = \mathcal{M}, \mathcal{N} \cap \mathcal{L} = 0$.

The following example shows that there exist topologically complementable submodules that are not orthogonally complementable, which again illustrates the difference from Hilbert spaces.

Example 2.0.19. [38, Example 2.3.2] Let $J \subset \mathcal{A}$ be a closed ideal such that the equality $Ja = 0, a \in \mathcal{A}$ implies that $a = 0$. Put $\mathcal{M} := \mathcal{A} \oplus J$,

$$\mathcal{N} := \{(b, b) : b \in J\}.$$

Then

$$\mathcal{N}^\perp := \{(c, -c) : c \in J\}.$$

Therefore $\mathcal{N} \oplus \mathcal{N}^{\perp} = J \oplus J \neq \mathcal{M}$. However, the submodule

$$\mathcal{L} = \{(a,0) : a \in \mathcal{A}\} \subset \mathcal{M}$$

is a topological complement to $\mathcal{N}$ in $\mathcal{M}$.

The next theorem is going to be one of the main tools in our proofs. Moreover, this theorem has several useful corollaries given below.

Theorem 2.0.20. *[37] [38, Theorem 2.3.3] Let $\mathcal{M}, \mathcal{N}$ be Hilbert A-modules and $T \in B^a(\mathcal{M},\mathcal{N})$ an operator with closed image. Then*
(i) $\ker T$ is an orthogonally complementable submodule in $\mathcal{M}$
(ii) ImT is an orthogonally complementable submodule in $\mathcal{N}$.

From the proof of Theorem 2.0.20 it follows that ImT is closed if and only if ImT^* is closed. This fact will be used in several proofs later.

Remark 2.0.21. If M and N are two Hilbert C^*-modules and $F \in B^a(M,N)$ with the property that F is invertible, then F^{-1} is also adjointable. In order to see this, we use that, by the observation above, ImF^* is closed since ImF is closed. Moreover, $\ker F^* = ImF^{\perp} = \{0\}$ and $ImF^{*\perp} = \ker F = \{0\}$, hence, by Theorem 2.0.20, $ImF^* = M$ since ImF^* is closed. Thus, F^* is invertible by the Banach open mapping theorem. For any $x \in M$ and $y \in N$ we have then

$$\langle F^{-1}y, x\rangle = \langle F^{-1}y, F^*(F^*)^{-1}x\rangle = \langle FF^{-1}y, (F^*)^{-1}x\rangle = \langle y, (F^*)^{-1}x\rangle,$$

so F^{-1} is adjointable and $(F^{-1})^* = (F^*)^{-1}$.

Corollary 2.0.22. *[38, Corollary 2.3.4] If $P \in B^a(\mathcal{M})$ is an idempotent, then its image ImP is an orthogonally complementable submodule in $\mathcal{M}$.*

Corollary 2.0.23. *[38, Corollary 2.3.5] Let $\mathcal{M}, \mathcal{N}$ be Hilbert $\mathcal{A}$-modules and let $F : \mathcal{M} \to \mathcal{N}$ be a topologically injective (i.e. $\| Fx \| \geq \delta \| x \|$ for some $\delta > 0$ and for all $x \in \mathcal{M}$) adjointable $\mathcal{A}$-homomorphism. Then $F(\mathcal{M}) \oplus F(\mathcal{M})^{\perp} = \mathcal{N}$.*

Corollary 2.0.24. *[38, Corollary 2.3.6] Let $\mathcal{M}$ be a Hilbert $\mathcal{A}$–module and let J be a selfadjoint topologically injective $\mathcal{A}$-homomorphism. Then J is an isomorphism.*

Lemma 2.0.25. *[37] [38, Lemma 2.3.7] Let $\mathcal{M}$ be a finitely generated Hilbert submodule in a Hilbert module $\mathcal{N}$ over a unital C^*-algebra. Then $\mathcal{M}$ is an orthogonal direct summand in $\mathcal{N}$.*

Corollary 2.0.26. *Let $\mathcal{A}$ be a unital C^*-algebra. Suppose that M_1 and N_1 are closed submodules of $H_{\mathcal{A}}$ such that $H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1$. If N_1 is finitely generated, then $M_1 \cong H_{\mathcal{A}}$.*

Proof. If N_1 is finitely generated, then by Lemma 2.0.25 we have $H_{\mathcal{A}} = N_1 \oplus N_1^{\perp}$. Now, by the Dupre-Filmore Theorem 2.0.15. we get $N_1^{\perp} \cong H_{\mathcal{A}}$. Hence

$$M_1 \cong N_1^{\perp} \cong H_{\mathcal{A}}.$$

□

As explained in the beginning of [38, Section 2.5], for a Hilbert $\mathcal{A}$-module $\mathcal{M}$ we denote by $\mathcal{M}'$ the set of all bounded $\mathcal{A}$-linear maps from $\mathcal{M}$ to $\mathcal{A}$. The formula

$$(f \cdot a)(x) = a^* f(x),$$

where $a \in \mathcal{A}$, introduces the structure of right $\mathcal{A}$-module on $\mathcal{M}'$. This module is complete with respect to the norm $\| f \| = \sup\{\| f(x) \| : \| x \| \leq 1\}$. Such modules are called *dual (Banach)* modules. The elements of the module $\mathcal{M}'$ are called *functionals* on the Hilbert module $\mathcal{M}$. Note that there is an obvious isometric inclusion $\mathcal{M} \subset \mathcal{M}'$, which is defined by the formula $x \mapsto \langle x, \cdot \rangle = \widehat{x}$.

Definition 2.0.27. [38, Definition 2.5.1] A Hilbert module $\mathcal{M}$ is called self-dual if $\mathcal{M} = \mathcal{M}'$.

Proposition 2.0.28. *[43] [38, Proposition 2.5.2] Let $\mathcal{M}$ be a self-dual Hilbert $\mathcal{A}$-module, $\mathcal{N}$ an arbitrary Hilbert $\mathcal{A}$-module and $T \in B(\mathcal{M}, \mathcal{N})$. Then there exists an operator $T^* : \mathcal{N} \to \mathcal{M}$ such that the equality $\langle x, T^*y\rangle = \langle Tx, y\rangle$ holds for all $x \in \mathcal{M}, y \in \mathcal{N}$.*

Corollary 2.0.29. *[38, Corollary 2.5.3] Let $\mathcal{M}$ be a self-dual Hilbert $\mathcal{A}$-module. Then $B^a(\mathcal{M}) = B(\mathcal{M})$.*

Lemma 2.0.30. *Let M be a self-dual Hilbert C^*-module and suppose that $M = M_1 \tilde{\oplus} M_2$ for some Hilbert submodules M_1 and M_2. Then, both M_1 and M_2 are self-dual.*

Proof. We have that M_1 and M_2 are the kernels of the bounded, $\mathcal{A}$-linear projections. By combining Corollary 2.0.29 and Theorem 2.0.20 it follows that M_1 and M_2 are orthogonally complementable in M. Let P denote the orthogonal projection onto M_1. If $\varphi \in M_1'$, then $\varphi \circ P \in M'$. Hence there exists an $y \in M$ such that $(\varphi \circ P)(x) = \langle y, x\rangle$ for all $x \in M$. In particular, $\varphi(z) = (\varphi \circ P)(z) = \langle y, z\rangle$ for all $z \in M_1$ since $z = Pz$ in this case. Thus, for all $z \in M_1$ we have $\varphi(z) = \langle y, z\rangle = \langle Py, z\rangle$.

□

Lemma 2.0.31. *Let M and N be two Hilbert modules over a C^*-algebra $\mathcal{A}$. Suppose that M is self-dual and $M \cong N$. Then N is self-dual as well.*

Proof. Let $U : M \to N$ be an isomorphism. If $\varphi \in N'$, then $\varphi \circ U \in M'$. Hence there exists an $x_0 \in M$ such that $\varphi(U(x)) = \langle x_0, x\rangle$ for all $x \in M$. This gives $\varphi(y) = \varphi(UU^{-1}y) = \langle x_0, U^{-1}y\rangle$ for all $y \in N$. Since M is self-dual by assumption, U is adjointable by Proposition 2.0.28. Moreover, by Remark 2.0.21 we have that U^{-1} is then adjointable and $(U^{-1})^* = (U^*)^{-1}$. Hence we get $\varphi(y) = \langle x_0, U^{-1}y\rangle = \langle (U^*)^{-1}x_0, y\rangle$ for all $y \in N$. Since $\varphi \in N'$ was chosen arbitrary, it follows that N is self-dual. □

Proposition 2.0.32. *[38, Proposition 2.5.4] Let $\mathcal{M}$ be a self-dual Hilbert $\mathcal{A}$-module and let $\mathcal{M} \subset \mathcal{N}$. Then $\mathcal{N} = \mathcal{M} \oplus \mathcal{M}^\perp$.*

Lemma 2.0.33. *Let M be a Hilbert C^*-module and suppose that $M = M_1 \tilde{\oplus} M_2$ where M_1 and M_2 are self-dual. Then M is self-dual as well.*

Proof. By Proposition 2.0.32 we have $M = M_1 \oplus M_1^\perp$. Clearly, $M_1^\perp \cong M_2$, hence, by Lemma 2.0.31, $M_1^\perp$ is also self-dual. Let P denote the orthogonal projection onto M_1. If $\varphi \in M'$, then $\varphi = \varphi \circ P + \varphi \circ (I - P)$. Since $\varphi_{|_{M_1}} \in M_1'$, there exists an $x_1 \in M_1$ such that $\varphi(x) = \langle x_1, x\rangle$ for all $x \in M_1$. Hence $\varphi(P(y)) = \langle x_1, Py\rangle$ for all $y \in M$. Similarly, there exists an $x_1^\perp \in M_1^\perp$ such that $\varphi((I - P)(y)) = \langle x_1^\perp, (I - P)y\rangle$ for all $y \in M$. Therefore, for all $y \in M$ we get

$$\varphi(y) = \varphi(P(y)) + \varphi((I - P)y)$$

$$= \langle x_1, Py\rangle + \langle x_1^\perp, (I - P)y\rangle = \langle x_1, y\rangle + \langle x_1^\perp, y\rangle = \langle x_1 + x_1^\perp, y\rangle.$$

□

The next theorem is originally given in [40].

Theorem 2.0.34. *[38, Theorem 2.7.5] Let $H_{\mathcal{A}} \cong \mathcal{M} \tilde{\oplus} \mathcal{N}$ where $\mathcal{M}$ and $\mathcal{N}$ are closed $\mathcal{A}$-modules and $\mathcal{N}$ has a finite number of generators $a_1, \cdots, a_s$. Then $\mathcal{N}$ is a projective $\mathcal{A}$-module of finite type.*

This theorem has several consequences and is also going to be one of the main tools in our proofs.

Corollary 2.0.35. *Let $P \in B^a(H_{\mathcal{A}})$ and suppose that P is projection onto a finitely generated closed submodule. Then $P \in \mathcal{K}^*(H_{\mathcal{A}})$.*

Proof. Let $M = ImP$, then M is finitely generated. By Lemma 2.0.25 we have $H_{\mathcal{A}} = M \oplus M^{\perp}$. Then, by Theorem 2.0.34 there exists an $n \in \mathbb{N}$ such that $p_{n_{|_M}}$ is an isomorphism onto $p_n(M)$, where p_n stands for the orthogonal projection onto L_n.
Now, since M is orthogonaly complementable in $H_{\mathcal{A}}$, we have $p_{n_{|_M}} \in B^a(M, p_n(M))$. Hence $(p_{n_{|_M}})^{-1} \in B^a(p_n(M), M)$ by Remark 2.0.21. Moreover, since M is orthogonally complementable, we have $J_M \in B^a(M, H_{\mathcal{A}})$ where J_M stands for the inclusion of M. Next, since M is finitely generated, by Remark 2.0.68 $p_n(M)$ is finitely generated, hence it is orthogonally complementable by Lemma 2.0.25. Let Q denote the orthogonal projection onto $p_n(M)$. Then we obtain

$$P = J_M (p_{n_{|_M}})^{-1} Q p_{n_{|_M}} P = J_M (p_{n_{|_M}})^{-1} Q p_n P.$$

By Proposition 2.0.17 it follows that $p_n \in \mathcal{K}^*(H_{\mathcal{A}})$, hence $P \in \mathcal{K}^*(H_{\mathcal{A}})$ since $\mathcal{K}^*(H_{\mathcal{A}})$ is a two-sided ideal in $B^a(H_{\mathcal{A}})$. $\square$

Next, we recall the definition of the K-group of a C^*-algebra $\mathcal{A}$.

Definition 2.0.36. [30] [38, Definition 2.7.1] Let M be an abelian monoid. Consider the Cartesian product $M \times M$ and its quotient monoid with respect to the equivalence relation

$$(m, n) \sim (m', n') \Leftrightarrow \exists p, q : (m, n) + (p, p) = (m', n') + (q, q).$$

This quotient monoid is a group, which is denoted by $S(M)$ and is called the symmetrization of M. Consider now the additive category $\mathcal{P}(\mathcal{A})$ of projective modules over a unital C^*-algebra $\mathcal{A}$ and denoted by $[\mathcal{M}]$ the isomorphism class of an object $\mathcal{M}$ from $\mathcal{P}(\mathcal{A})$. The set $\phi(\mathcal{P}(\mathcal{A}))$ of these classes has the structure of an Abelian monoid with respect to the operation $[\mathcal{M}] + [\mathcal{N}] = [\mathcal{M} \oplus \mathcal{N}]$. In this case the group $S(\phi(\mathcal{P}(\mathcal{A})))$ is denoted by $K(\mathcal{A})$ or $K_0(\mathcal{A})$ and is called the K-group of $\mathcal{A}$ or the Grothendieck group of the category $\mathcal{P}(\mathcal{A})$.

As regards the K-group $K_0(\mathcal{A})$, it is worth mentioning that it is not true in general that $[M] = [N]$ implies that $M \cong N$ for two finitely generated Hilbert modules M, N over $\mathcal{A}$. If $K_0(\mathcal{A})$ satisfies the property that $[N] = [M]$ implies that $N \cong M$ for any two finitely generated, Hilbert modules M, N over $\mathcal{A}$, then $K_0(\mathcal{A})$ is said to satisfy "the cancellation property", see [53, Section 6.2].

Finally we are ready to recall the definition of a Fredholm operator on a Hilbert C^*-module originally given by Mishchenko and Fomenko in [40].

Definition 2.0.37. [38, Definition 2.7.4] A (bounded $\mathcal{A}$-linear) operator $F : H_{\mathcal{A}} \to H_{\mathcal{A}}$ is called (adjointable) $\mathcal{A}$-Fredholm if
(i) it is adjointable;
(ii) there exists a decomposition of the domain, $H_{\mathcal{A}} = \mathcal{M}_1 \tilde{\oplus} \mathcal{N}_1$, and the range $H_{\mathcal{A}} = \mathcal{M}_2 \tilde{\oplus} \mathcal{N}_2$ (where $\mathcal{M}_1, \mathcal{M}_2, \mathcal{N}_1, \mathcal{N}_2$ are closed $\mathcal{A}$-modules and $\mathcal{N}_1, \mathcal{N}_2$ have a finite number of generators), such that F has the matrix form $F = \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}$ with respect to these decompositions and $F_1 = F_{|_{\mathcal{M}_1}} : \mathcal{M}_1 \to \mathcal{M}_2$ is an isomorphism.

Theorem 2.0.38. *[52] , [38, Theorem 2.7.6] In the decomposition mentioned in the Definition 2.0.37 one always can assume that the modules $\mathcal{M}_1$ and $\mathcal{M}_2$ are orthogonally complementable. More precisely, there exist decompositions for F,*

$$\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix} : H_{\mathcal{A}} = V_0 \tilde{\oplus} W_0 \to V_1 \tilde{\oplus} W_1 = H_{\mathcal{A}}$$

such that $V_0^{\perp} \oplus V_0 = H_{\mathcal{A}}$ or such that projections $V_0 \tilde{\oplus} W_0 \to V_0$ and $V_1 \tilde{\oplus} W_1 \to V_1$ are adjointable.

Proof. Although the proof of this theorem is already given in [38], we will provide here a slightly different proof. Let $H_{\mathcal{A}} = M_0 \tilde{\oplus} N_0 \xrightarrow{F} M_1 \tilde{\oplus} N_1 = H_{\mathcal{A}}$ be a Fredholm decomposition for F. Observe first that, since N_0 is orthogonally complementable by Lemma 2.0.25, then

$$H_{\mathcal{A}} = M_0 \tilde{\oplus} N_0 = N_0 \oplus N_0^{\perp},$$

so $\sqcap_{M_0 |_{N_0^{\perp}}}$ is an isomorphism from $N_0^{\perp}$ onto M_0, where $\sqcap_{M_0 |_{N_0^{\perp}}}$ stands for the projection onto M_0 along N_0 restricted to $N_0^{\perp}$. Observe next that, since $F(M_0) = M_1$ and $F(N_0) \subseteq N_1$, we have $\sqcap_{M_1} F_{|_{N_0^{\perp}}} = F \sqcap_{M_0 |_{N_0^{\perp}}}$, where $\sqcap_{M_1}$ stands for the projection onto M_1 along N_1. Since $F_{|_{M_0}}$ is an isomorphism, it follows that $\sqcap_{M_1} F_{|_{N_0^{\perp}}} = F \sqcap_{M_0 |_{N_0^{\perp}}}$ is an isomorphism as a composition of isomorphisms. Hence, with respect to the decomposition

$$H_{\mathcal{A}} = N_0^{\perp} \oplus N_0 \xrightarrow{F} M_1 \tilde{\oplus} N_1 = H_{\mathcal{A}},$$

F has the matrix

$$\begin{bmatrix} \tilde{F}_0 & 0 \\ \tilde{F}_3 & F_4 \end{bmatrix},$$

where $\tilde{F}_0 = \sqcap_{M_1} F_{|_{N_0^{\perp}}}$ is an isomorphism. Using the technique of diagonalization as in the proof of [38, Lemma 2.7.10], we deduce that there exists an isomorphism V such that

$$H_{\mathcal{A}} = N_0^{\perp} \oplus N_0 \xrightarrow{F} V(M_1) \tilde{\oplus} V(N_1) = H_{\mathcal{A}}$$

is a Fredholm decomposition of F. Moreover, by the construction of V we have $V(N_1) = N_1$. Hence

$$H_{\mathcal{A}} = F(N_0^{\perp}) \tilde{\oplus} N_1.$$

□

Definition 2.0.39. [38, Definition 2.7.8] Let the conditions of Definition 2.0.37 hold. We define the index of F by

$$\text{index } F = [\mathcal{N}_1] - [\mathcal{N}_2] \in K_0(\mathcal{A}).$$

Theorem 2.0.40. *[38, Theorem 2.7.9] The index is well defined.*

Proof. Although the proof of this theorem is already given in [38], we will provide here a slightly different proof. As in the proof of [38, Theorem 2.7.5] we can find an $n \in \mathbb{N}$ such that

$$L_n = P \tilde{\oplus} p_n(N_1) = P' \tilde{\oplus} p_n(N_1'), p_n(N_1) \cong N_1,$$

$$p_n(N_1') \cong N_1', P = M_1 \cap L_n, P' = M_1' \cap L_n,$$

where

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}},$$

$$H_{\mathcal{A}} = M_1' \tilde{\oplus} N_1' \xrightarrow{F} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}},$$

are two Fredholm decompositions of F and P, P' are finitely generated. We obtain new Fredholm decompositions for F,

$$H_{\mathcal{A}} = L_n^{\perp} \tilde{\oplus} (P \tilde{\oplus} N_1) \xrightarrow{F} F(L_n^{\perp}) \tilde{\oplus} (F(P) \tilde{\oplus} V^{-1}(N_2)) = H_{\mathcal{A}},$$

$$H_{\mathcal{A}} = L_n^{\perp}\tilde{\oplus}(P'\tilde{\oplus}N_1') \xrightarrow{F} F(L_n^{\perp})\tilde{\oplus}(F(P')\tilde{\oplus}V'^{-1}(N_2')) = H_{\mathcal{A}},$$

where V, V' are isomorphisms. Moreover,

$$L_n^{\perp} \cong F(L_n^{\perp}), P \cong F(P),\ F(P') \cong P'.$$

This works as in the proof of [38, Lemma 2.7.11]. Since

$$H_{\mathcal{A}} = F(L_n^{\perp})\tilde{\oplus}F(P)\tilde{\oplus}V^{-1}(N_2) = F(L_n^{\perp})\tilde{\oplus}F(P')\tilde{\oplus}V'^{-1}(N_2'),$$

we deduce that

$$(F(P)\tilde{\oplus}V^{-1}(N_2)) \cong (F(P')\tilde{\oplus}V'^{-1}(N_2')),$$

hence

$$[F(P)] + [N_2] = [F(P')] + [N_2'].$$

Moreover,

$$[P] + [N_1] = [P'] + [N_1'] = [L_n]$$

and

$$[F(P)] = [P], [F(P')] = [P'].$$

Therefore, $[N_1] - [N_2] = [N_1'] - [N_2']$. □

In order to generalize the sign of the index when the index takes values in the K-group, we are going to introduce the following definition and notation.

Definition 2.0.41. [21, Definition 2] For two closed submodules N_1, N_2 of a Hilbert C^*-module M we write $N_1 \preceq N_2$ when N_1 is isomorphic to a closed submodule of N_2.

The idea for this concept is originally taken from [7] where this concept was introduced in connection with Banach spaces. More precisely, our Definition 2.0.41 is inspired by [7, Definition 4.2].

Next we recall some important properties of $\mathcal{A}$-Fredholm operators.

Lemma 2.0.42. *[38, Lemma 2.7.10] Let an operator $F : H_{\mathcal{A}} \to H_{\mathcal{A}}$ be adjointable $\mathcal{A}$-Fredholm. Then there exists a number $\epsilon > 0$ such that any adjointable operator D satisfying the condition $\parallel F - D \parallel < \epsilon$ is an $\mathcal{A}$-Fredholm operator and*

$$\text{index } D = \text{index } F.$$

Lemma 2.0.43. *[38, Lemma 2.7.11] Let F and D be $\mathcal{A}$-Fredholm operators,*

$$F : H_{\mathcal{A}} \to H_{\mathcal{A}},\ D : H_{\mathcal{A}} \to H_{\mathcal{A}}.$$

Then $DF : H_{\mathcal{A}} \to H_{\mathcal{A}}$ is an $\mathcal{A}$-Fredholm operator and

$$\text{index } DF = \text{index } D + \text{index } F.$$

Lemma 2.0.44. *[38, Lemma 2.7.12] Let $K : H_{\mathcal{A}} \to H_{\mathcal{A}}$ belong to $K \in \mathcal{K}^*(H_{\mathcal{A}})$. Then $I + K$ is an $\mathcal{A}$-Fredholm operator and* index $(I + K) = 0$.

Lemma 2.0.45. *[38, Lemma 2.7.13] Consider an $\mathcal{A}$-Fredholm operator $F : H_{\mathcal{A}} \to H_{\mathcal{A}}$ and let $K \in \mathcal{K}^*(H_{\mathcal{A}})$. Then the operator $F + K$ is $\mathcal{A}$-Fredholm and* index $(F + K) =$ index F.

These results regarding $\mathcal{A}$-Fredholm operators are originally given in [40].

Now we are going to recall some special properties of Hilbert W^*-modules.

Theorem 2.0.46. *[43], [38, Theorem 3.2.1] Let $\mathcal{M}$ be a Hilbert $\mathcal{A}$-module where $\mathcal{A}$ is a W^*-algebra. An $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle$ admits an extension to the Banach module $\mathcal{M}'$, making it a self-dual Hilbert $\mathcal{A}$-module. In particular, the extended inner product satisfies the equality $\langle f, \widehat{x} \rangle = f(x)$ for all $x \in \mathcal{M}, f \in \mathcal{M}'$.*

The next results are originally given in [10].

Lemma 2.0.47. *[38, Lemma 3.6.1] Let $\mathcal{M}$ be a self-dual Hilbert C^*-module over a W^*-algebra $\mathcal{A}$. For each closed submodule $\mathcal{N} \subseteq \mathcal{M}$ the biorthogonal set $\mathcal{N}^{\perp\perp} \subseteq \mathcal{M}$ is a Hilbert $\mathcal{A}$-submodule and is a direct summand of $\mathcal{M}$, as well as its orthogonal complement $\mathcal{N}^{\perp}$.*

Lemma 2.0.48. *[38, Lemma 3.6.2] Let ϕ be a bounded $\mathcal{A}$-module morphism of a self-dual module $\mathcal{M}$ over a W^*-algebra $\mathcal{A}$. Then the kernel $\ker(\phi)$ of the map ϕ is a direct summand in $\mathcal{M}$ and satisfies the equality $\ker \phi = \ker(\phi)^{\perp\perp}$.*

Example 2.0.49. [38, Example 3.6.3] Note that the kernel of a bounded $\mathcal{A}$-linear operator on a Hilbert $\mathcal{A}$-module over an arbitrary C^*-algebra $\mathcal{A}$ need not be a direct summand. For example, consider the C^*-algebra $\mathcal{A} = C([0,1])$ of all continuous functions on the segment $[0,1]$ as a Hilbert $\mathcal{A}$-module over itself equipped with the standard inner product $\langle a, b \rangle_{\mathcal{A}} = a^* b$. Define the mapping φ_g by the formula $\varphi_g(f) = g \cdot f$ for the fixed function

$$g(x) = \begin{cases} -2x + 1 & \text{if } x \leq \frac{1}{2} \\ 0 & \text{if } x \geq \frac{1}{2} \end{cases}$$

and for every $f \in \mathcal{A}$. Then $\ker \phi_g$ equals the Hilbert $\mathcal{A}$-submodule and the (left) ideal

$$\{f \in \mathcal{A} : f(x) = 0 \text{ for } x \in [0, \frac{1}{2}]\}$$

is not a direct summand of $\mathcal{A}$, but coincides, nevertheless, with its bi-orthogonal complement in $\mathcal{A}$.

Corollary 2.0.50. *[38, Corollary 3.6.4] Let $\phi : \mathcal{M} \to \mathcal{N}$ be a bounded $\mathcal{A}$-linear mapping of self-dual modules over a W^*-algebra $\mathcal{A}$. Then the kernel of ϕ is a direct summand of $\mathcal{M}$ and has the property $\ker \phi = \ker(\phi)^{\perp\perp}$.*

Remark 2.0.51. The assumption in Corollary 2.0.50 that N is self-dual may be omitted. Indeed, we recall that there is an isometric inclusion from N into N'. Let J denote this isometry and consider the map $J \circ \varphi : M \to N'$. By Theorem 2.0.46 N' is a self-dual Hilbert W^*-module, hence, by Corollary 2.0.50, $\ker J \circ \varphi$ is a direct summmand in M. However, $\ker J \circ \varphi = \ker \varphi$ since J is an isometry. Therefore, throughout the thesis whenever we apply Corollary 2.0.50 we will not assume that N is self-dual.

The next lemma is a modified version of [38, Corollary 3.6.7].

Lemma 2.0.52. *Let $\mathcal{M}$ and $\mathcal{N}$ be self-dual Hilbert $\mathcal{A}$-modules (where $\mathcal{A}$ is a W^*-algebra). If there exists an injective module mapping α from $\mathcal{M}$ into $\mathcal{N}$, then there exists a Hilbert $\mathcal{A}$-module isomorphism between $\mathcal{M}$ and a direct summand of $\mathcal{N}$.*

Proposition 2.0.53. *[10], [38, Proposition 3.6.8], Let $\mathcal{M}$ and $\mathcal{N}$ be countably generated Hilbert $\mathcal{A}$-modules over a W^*-algebra $\mathcal{A}$ and let $F : \mathcal{M} \to \mathcal{N}$ be an $\mathcal{A}$-Fredholm operator. Then $\ker F$ and $(ImF)^{\perp}$ are projective finitely generated $\mathcal{A}$-submodules and $\text{index } F = [\ker F] - [(ImF)^{\perp}]$ in $K_0(\mathcal{A})$.*

This proposition shows that Fredholm operators over a W^*-algebra behave more similarly to the classical Fredholm operators on Hilbert spaces than in the general C^*-algebra case.

Now we are going to recall the results on non-adjointable compact and Fredholm operators on Hilbert C^*-modules. We start with the following definition.

Definition 2.0.54. [17, Definition 1] An $\mathcal{A}$-operator $K : H_{\mathcal{A}} \to H_{\mathcal{A}}$ is called a finitely generated $\mathcal{A}$-operator if it can be represented as a composition of bounded $\mathcal{A}$-operators f_1 and f_2:

$$K : H_{\mathcal{A}} \xrightarrow{f_1} M \xrightarrow{f_2} H_{\mathcal{A}},$$

where M is a finitely generated Hilbert C^*-module. The set $\mathcal{FG}(\mathcal{A}) \subset B(H_{\mathcal{A}})$ of all finitely generated $\mathcal{A}$-operators forms a two sided ideal. By definition, an $\mathcal{A}$-operator K is called compact if it belongs to the closure

$$\mathcal{K}(H_{\mathcal{A}}) = \overline{\mathcal{FG}(\mathcal{A})} \subset B(H_{\mathcal{A}}),$$

which also forms two sided ideal.

As observed in [17], in general, the set $\mathcal{FG}(\mathcal{A}) \subset B(H_{\mathcal{A}})$ is not a closed subset. For example, in classical case, when $\mathcal{A} = \mathbb{C}$ the set $\mathcal{FG}(\mathcal{A})$ consists of all finite rank operators, while not all compact operators are finite rank operators if the space is infinite-dimensional.

Lemma 2.0.55. *[17, Lemma 1] The ideal $\mathcal{K}(H_{\mathcal{A}})$ is a proper ideal.*

Theorem 2.0.56. *[17, Theorem 2] A bounded $\mathcal{A}$-operator $K : H_{\mathcal{A}} \to H_{\mathcal{A}}$ is a compact $\mathcal{A}$-operator iff for any $\epsilon > 0$ there exists a number N such that for any $m > N$ we have $\| q_m K \| \leq \epsilon$, where q_m denotes the orthogonal projection onto $L_m^{\perp}$.*

Corollary 2.0.57. *[17, Corollary 1]Let $K : H_{\mathcal{A}} \to H_{\mathcal{A}}$ be a compact $\mathcal{A}$-operator. Then for any $\epsilon > 0$ there exists a number N such that for any $m > N$ we have $\| q_m K q_m \| \leq \epsilon$.*

Definition 2.0.58. [17, Definition 2] A bounded $\mathcal{A}$-operator $H_{\mathcal{A}} \longrightarrow H_{\mathcal{A}}$ is called a Fredholm $\mathcal{A}$-operator if there exists a bounded $\mathcal{A}$-operator G such that

$$\mathbf{id} - FG \in \mathcal{K}(H_{\mathcal{A}}), \mathbf{id} - GF \in \mathcal{K}(H_{\mathcal{A}}).$$

Definition 2.0.59. [17, Definition 3] We say that a bounded $\mathcal{A}$-operator $F : H_{\mathcal{A}} \longrightarrow H_{\mathcal{A}}$ admits an inner (Noether) decomposition if there is a decomposition of the preimage and the image $H_{\mathcal{A}} = M_1 \oplus N_1, H_{\mathcal{A}} = M_2 \oplus N_2$, respectively, where C^*-modules N_1 and N_2 are finitely generated Hilbert C^*-modules and if F has the following matrix form

$$F = \begin{bmatrix} F_1 & F_2 \\ 0 & F_4 \end{bmatrix} : M_1 \oplus N_1 \longrightarrow M_2 \oplus N_2,$$

where $F_1 : M_1 \longrightarrow M_2$ is an isomorphism.

Definition 2.0.60. [17, Definition 4] We put by definition index $F = [N_2] - [N_1] \in K_0(\mathcal{A})$.

Definition 2.0.61. [17, Definition 5] We say that a bounded $\mathcal{A}$-operator $F : H_{\mathcal{A}} \longrightarrow H_{\mathcal{A}}$ admits an external (Noether) decomposition if there exist two finitely generated C^*-modules X_1 and X_2 and two bounded $\mathcal{A}$-operators E_2, E_3 such that the matrix operator
$F_0 = \begin{bmatrix} F & E_2 \\ E_3 & 0 \end{bmatrix} : H_{\mathcal{A}} \oplus X_1 \longrightarrow H_{\mathcal{A}} \oplus X_2$, is an invertible operator.

Theorem 2.0.62. *[17, Theorem 3] A bounded $\mathcal{A}$-operator $F : H_{\mathcal{A}} \to H_{\mathcal{A}}$ admits an external (Noether) decomposition iff it admits an inner (Noether) decomposition.*

Corollary 2.0.63. *[17, Corollary 2] The index constructed by inner or external decomposition does not depend on the method of decomposition.*

Theorem 2.0.64. *[17, Theorem 4] Let $K : H_{\mathcal{A}} \to H_{\mathcal{A}}$ be a compact operator in the sense of definition 2.0.54. Then the operator $id + K$ admits an inner (Noether) decomposition.*

Theorem 2.0.65. *[17, Theorem 5] Any Fredholm operator in the sense of Definition 2.0.58 admits both the inner and external (Noether) decomposition.*

At the end of this chapter we introduce the following auxiliary technical lemma which will be useful later in the proofs.

Lemma 2.0.66. *Let M be a Hilbert C^*-module and M_1, M_2 be closed submodules of M such that $M_1 \subseteq M_2$ and $M = M_1 \tilde{\oplus} M_1'$ for some Hilbert submodule M_1'. Then $M_2 = M_1 \tilde{\oplus} (M_1' \cap M_2)$.*

Proof. Since $M = M_1 \tilde{\oplus} M_1'$ by assumption and $M_2 \subseteq M$, any $z \in M_2$ can be written as $z = x + y$ for some $x \in M_1$ and $y \in M_1'$. Now, since $M_1 \subseteq M_2$ by assumption, we have $y = z - x \in M_2$. Thus, $y \in M_1' \cap M_2$. □

Remark 2.0.67. Lemma 2.0.66 is a slightly modifed version of [19, Lemma 2.6].

Remark 2.0.68. Note that a direct summand in a finitely generated Hilbert module is also finitely generated. Indeed, if M is a finitely generated Hilbert C^*-module and $M = M_1 \tilde{\oplus} N_1$, let $\sqcap$ denote the projection onto M_1 along N_1. If $\{x_1, \dots, x_n\}$ is a generating set of M, then, clearly, $\{\sqcap x_1, \dots, \sqcap x_n\}$ is a generating set of M_1 as $\sqcap x = x$ for all $x \in M_1$, so M_1 is finitely generated. In general, it F is any $\mathcal{A}$-linear operator on M, it follows that $\{F x_1, \dots, F x_n\}$ is a generating set for $F(M)$. We are going to use these properties frequently in the proofs throughout the thesis.

Remark 2.0.69. [21, Remark 8] If M is a countably generated Hilbert C^*-module, then by the Kasparov stabilization Theorem 2.0.13, $M \oplus H_{\mathcal{A}} \cong H_{\mathcal{A}}$. Given an operator $F \in B^a(M)$, we may consider the induced operator $F' \in B^a(M \oplus H_{\mathcal{A}})$ given by the operator matrix $\begin{bmatrix} F & 0 \\ 0 & I \end{bmatrix}$. It is then clear that if $M = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = M$ is a decomposition with respect to which F has the matrix $\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}$, where F_1 is an isomorphism, then F' has the matrix $\begin{bmatrix} F_1' & 0 \\ 0 & F_4' \end{bmatrix}$ with respect to the decomposition

$$M \oplus H_{\mathcal{A}} = (M_1 \oplus H_{\mathcal{A}}) \tilde{\oplus} (N_1 \oplus \{0\}) \xrightarrow{F'} (M_2 \oplus H_{\mathcal{A}}) \tilde{\oplus} (N_2 \oplus \{0\}) = M \oplus H_{\mathcal{A}},$$

where F_1' is an isomorphism. It follows then that any $\mathcal{A}$-Fredholm decomposition for F gives rise in a natural way to an $\mathcal{A}$-Fredholm decomposition of F'. Moreover, F' can be viewed as an operator in $B^a(H_{\mathcal{A}})$, as $M \oplus H_{\mathcal{A}} \cong H_{\mathcal{A}}$. It follows easily that index F is well defined, since index F' is so, and in this case index F = index F'. Thus, Theorem 2.0.40 holds for F. Similarly, Lemma 2.0.43 also holds for F.

Chapter 3

Semi-C^*-Fredholm operators

3.1 Adjointable semi-C^*-Fredholm operators

In this section we define adjointable semi-$\mathcal{A}$-Fredholm operators on the standard module $H_{\mathcal{A}}$ and prove some of the main properties and results concerning these operators. Most of the results in this section are generalizations of the results in [56, Section 1.2] and [56, Section 1.3] in the setting of operators on $H_{\mathcal{A}}$.

Definition 3.1.1. [18, Definition 2.1] Let $F \in B^a(H_{\mathcal{A}})$. We say that F is an upper semi-$\mathcal{A}$-Fredholm operator if there exists a decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

with respect to which F has the matrix

$$\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix},$$

where F_1 is an isomorphism, M_1, M_2, N_1, N_2 are closed submodules of $H_{\mathcal{A}}$ and N_1 is finitely generated. Similarly, we say that F is a lower semi-$\mathcal{A}$-Fredholm operator if all the above conditions hold except that in this case we assume that N_2 (and not N_1) is finitely generated.

Set

$$\mathcal{M}\Phi_+(H_{\mathcal{A}}) = \{F \in B^a(H_{\mathcal{A}}) \mid F \text{ is upper semi-}\mathcal{A}\text{-Fredholm }\},$$

$$\mathcal{M}\Phi_-(H_{\mathcal{A}}) = \{F \in B^a(H_{\mathcal{A}}) \mid F \text{ is lower semi-}\mathcal{A}\text{-Fredholm }\},$$

$$\mathcal{M}\Phi(H_{\mathcal{A}}) = \{F \in B^a(H_{\mathcal{A}}) \mid F \text{ is } \mathcal{A}\text{-Fredholm operator on } H_{\mathcal{A}}\}.$$

Then, obviously, $\mathcal{M}\Phi(H_{\mathcal{A}}) \subseteq \mathcal{M}\Phi_+(H_{\mathcal{A}}) \cap \mathcal{M}\Phi_-(H_{\mathcal{A}})$. We are going to show later in this section that actually "=" holds.

Next we set $\mathcal{M}\Phi_\pm(H_{\mathcal{A}}) = \mathcal{M}\Phi_+(H_{\mathcal{A}}) \cup \mathcal{M}\Phi_-(H_{\mathcal{A}})$. Notice that if M, N are two arbitrary Hilbert modules C^*-modules, the definition above could be generalized to the classes $\mathcal{M}\Phi_+(M,N)$ and $\mathcal{M}\Phi_-(M,N)$.

Theorem 3.1.2. *[18, Theorem 2.2] Let $F \in B^a(H_{\mathcal{A}})$. The following statements are equivalent.*
1) $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$.
2) There exists $D \in B^a(H_{\mathcal{A}})$ such that $DF = I + K$ for some $K \in \mathcal{K}^*(H_{\mathcal{A}})$.

Proof. 2)⇒ 1) If 2) holds, then $DF \in \mathcal{M}\Phi(H_{\mathcal{A}})$ by Lemma 2.0.44. Let

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{DF} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

be a decomposition with respect to which DF has the matrix

$$\begin{bmatrix} (DF)_1 & 0 \\ 0 & (DF)_4 \end{bmatrix},$$

where $(DF)_1$ is an isomorphism and N_1, N_2 are finitely generated. We wish to show that $F(M_1)$ is closed and we will do it by showing that $F_{|_{M_1}}$ is bounded below. Suppose that this is not the case. Then there exists a sequence $\{x_n\} \subseteq M_1$ such that $||x_n|| = 1$ for all n and $Fx_n \to 0$ as $n \to \infty$. Since D is bounded, we must have that $DFx_n \to 0$ as $n \to +\infty$. However, this would mean that DF is not bounded below on M_1 as $||x_n|| = 1$ for all n. This is a contradiction since $DF_{|_{M_1}}$ is an isomorphism. Hence we must have that F is bounded below on M_1, which means that $F(M_1)$ is closed.
Now, by Theorem 2.0.38, we may assume that M_1 is orthogonally complementable in $H_{\mathcal{A}}$. Hence $F_{|_{M_1}}$ is adjointable, so, by Theorem 2.0.20, $ImF_{|_{M_1}}$ is orthogonally complementable in $H_{\mathcal{A}}$. Hence $H_{\mathcal{A}} = F(M_1) \oplus F(M_1)^{\perp}$. With respect to the decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} F(M_1) \oplus F(M_1)^{\perp} = H_{\mathcal{A}},$$

F has the matrix $\begin{bmatrix} F_1 & F_2 \\ 0 & F_4 \end{bmatrix}$, where F_1 is an isomorphism. If we let

$$U = \begin{bmatrix} 1 & -F_1{}^{-1}F_2 \\ 0 & 1 \end{bmatrix}$$

with respect to the decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{U} M_1 \tilde{\oplus} N_1 = H_{\mathcal{A}},$$

then U is an isomorphism and with respect to the decomposition

$$H_{\mathcal{A}} = U(M_1) \tilde{\oplus} U(N_1) \xrightarrow{F} F(M_1) \oplus F(M_1)^{\perp} = H_{\mathcal{A}}$$

F has the matrix $\begin{bmatrix} F_1 & 0 \\ 0 & \tilde{F}_4 \end{bmatrix}$. Since N_1 is finitely generated, $U(N_1)$ is finitely generated also, hence $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$.
1) ⇒ 2)
Let

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

be a decomposition with respect to which F has the matrix $\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}$, where F_1 is an isomorphism and N_1 is finitely generated. Since N_1 is finitely generated, it is orthogonally complementable in $H_{\mathcal{A}}$ by Lemma 2.0.25. Then, by the proof of Theorem 2.0.38 , we can deduce that $F_{|_{N_1^{\perp}}}$ is an isomorphism onto $F(N_1{}^{\perp})$. Now, $F(N_1{}^{\perp}) = ImFP_{N_1{}^{\perp}}$, where $P_{N_1{}^{\perp}}$ denotes the orthogonal projection onto $N_1{}^{\perp}$. Since $FP_{N_1{}^{\perp}} \in B^a(H_{\mathcal{A}})$ and $F(N_1{}^{\perp})$ is closed as $F_{|_{N_1{}^{\perp}}}$ is an isomorphism, by Theorem 2.0.20 it follows that $F(N_1{}^{\perp})$ is orthogonally complementable. With respect to the decomposition

$$H_{\mathcal{A}} = N_1{}^{\perp} \oplus N_1 \xrightarrow{F} F(N_1{}^{\perp}) \oplus F(N_1{}^{\perp})^{\perp} = H_{\mathcal{A}}$$

F has the matrix $\begin{bmatrix} \tilde{F}_1 & \tilde{F}_2 \\ 0 & \tilde{F}_4 \end{bmatrix}$, where $\tilde{F}_1$ is an isomorphism. Clearly, $\tilde{F}_1, \tilde{F}_2$ and $\tilde{F}_4$ are then adjointable.
Let D be the operator which has the matrix $\begin{bmatrix} \tilde{F}_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$ with respect to the decomposition

$$H_{\mathcal{A}} = F({N_1}^{\perp}) \oplus F({N_1}^{\perp})^{\perp} \xrightarrow{D} {N_1}^{\perp} \oplus N_1 = H_{\mathcal{A}}.$$

Then $D \in B^a(H_{\mathcal{A}})$ and $DF = \begin{bmatrix} 1 & \tilde{F}_1^{-1}\tilde{F}_2 \\ 0 & 0 \end{bmatrix}$ with respect to the decomposition

$$H_{\mathcal{A}} = {N_1}^{\perp} \oplus N_1 \xrightarrow{DF} {N_1}^{\perp} \oplus N_1 = H_{\mathcal{A}}.$$

Let $K = \begin{bmatrix} 0 & \tilde{F}_1^{-1}\tilde{F}_2 \\ 0 & -1 \end{bmatrix}$ with respect to the same decomposition. Since N_1 is finitely generated, by Corollary 2.0.35 we have $P_{N_1} \in \mathcal{K}^*(H_{\mathcal{A}})$, where P_{N_1} denotes the orthogonal projection onto N_1. Now, since $KP_{N_1} = K$ and $K \in \mathcal{K}^*(H_{\mathcal{A}})$ is a two-sided ideal in $B^a(H_{\mathcal{A}})$, we have $K \in \mathcal{K}^*(H_{\mathcal{A}})$. Moreover, $DF = I + K$. □

Lemma 3.1.3. *Let M be a Hilbert C^*-module and $F \in B(M)$. Suppose that*

$$M = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = M$$

is a decomposition with respect to which F has the matrix $\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}$, where F_1 is an isomorphism. Then $N_1 = F^{-1}(N_2)$.

Proof. Obviously, $N_1 \subseteq F^{-1}(N_2)$. Assume now that $x \in F^{-1}(N_2)$. Then $x = m_1 + n_1$ for some $m_1 \in M_1$ and $n_1 \in N_1$. We get $Fx = Fm_1 + Fn_1 \in N_2$. Since $Fm_1 \in M_2$ and $Fn_1 \in N_2$, we must have $Fm_1 = 0$. As $F_{|_{M_1}}$ is an isomorphism, we deduce that $m_1 = 0$. Hence $x = n_1 \in N_1$. □

Notice that Lemma 3.1.3 also holds if we consider arbitrary Banach spaces and not just Hilbert C^*-modules.

Theorem 3.1.4. *[18, Theorem 2.3] Let $D \in B^a(H_{\mathcal{A}})$. Then the following statements are equivalent.*
1) $D \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$.
2) There exist $F \in B^a(H_{\mathcal{A}}), K \in \mathcal{K}^*(H_{\mathcal{A}})$ such that $DF = I + K$.

Proof. 2) $\Rightarrow$ 1)
Let

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{I+K} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}.$$

be an $\mathcal{M}\Phi$-decomposition for $I + K$. As in the proof of Theorem 3.1.2, we deduce that $F(M_1)$ is closed and orthogonally complementable in $H_{\mathcal{A}}$.
With respect to the decomposition

$$H_{\mathcal{A}} = F(M_1) \oplus F(M_1)^{\perp} \xrightarrow{D} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}},$$

D has the matrix $\begin{bmatrix} D_1 & D_2 \\ 0 & D_4 \end{bmatrix}$, where D_1 is an isomorphism. As in the proof of Theorem 3.1.2, part 2) $\Rightarrow$ 1), we deduce then that D has the matrix $\begin{bmatrix} D_1 & 0 \\ 0 & \tilde{D}_4 \end{bmatrix}$ with respect to the decomposition

$$H_{\mathcal{A}} = \tilde{U}(F(M_1)) \tilde{\oplus} \tilde{U}(F(M_1)^{\perp}) \xrightarrow{D} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}},$$

where $\tilde{U}$ is an isomorphism. Since N_2 is finitely generated, it follows that
$D \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$.
1) $\Rightarrow$ 2)
Let

$$H_{\mathcal{A}} = M_1'\tilde{\oplus}N_1' \xrightarrow{D'} M_2'\tilde{\oplus}N_2' = H_{\mathcal{A}}$$

be an $\mathcal{M}\Phi_-$-decomposition for D (so that N_2' is finitely generated). Since N_2' is finitely generated, it is orthogonally complementable by Lemma 2.0.25. Now, since

$$H_{\mathcal{A}} = M_2'\tilde{\oplus}N_2' = N_2'^{\perp}\tilde{\oplus}N_2',$$

we have that $P_{N_2'^{\perp}}{}_{|_{M_2'}}$ is an isomorphism from M_2' onto $N_2'^{\perp}$, where $P_{N_2'^{\perp}}$ denotes the orthogonal projection onto $N_2'^{\perp}$. Since D has the matrix $\left[\begin{array}{cc} D_1 & 0 \\ 0 & D_4 \end{array}\right]$ with respect to the decomposition

$$H_{\mathcal{A}} = M_1'\tilde{\oplus}N_1' \xrightarrow{D} M_2'\tilde{\oplus}N_2' = H_{\mathcal{A}},$$

where D_1 is an isomorphism, by Lemma 3.1.3 it follows that $D^{-1}(N_2') = N_1'$. Therefore,

$$\ker P_{N_2'^{\perp}}D = D^{-1}(N_2') = N_1'$$

and moreover, $ImP_{N_2'^{\perp}}D = P_{N_2'^{\perp}}(M_2') = N_2'^{\perp}$ which is closed. By Theorem 2.0.20, $\ker P_{N_2'^{\perp}}D = N_1'$ is orthogonally complementable, so $H_{\mathcal{A}} = N_1'^{\perp} \oplus N_1'$. Hence $\sqcap_{M_1'}{}_{|_{N_1'^{\perp}}}$ is an isomorphism from $N_1'^{\perp}$ onto M_1', where $\sqcap_{M_1'}$ denotes the projection onto M_1' along N_1'. Therefore, $P_{N_2'^{\perp}}D\sqcap_{M_1'}{}_{|_{N_1'^{\perp}}}$ is an isomorphism from $N_1'^{\perp}$ onto $N_2'^{\perp}$ as a composition of isomorphisms. However, since $\ker P_{N_2'^{\perp}}D = N_1'$ and $H_{\mathcal{A}} = M_1'\tilde{\oplus}N_1'$, it follows that

$$P_{N_2'^{\perp}}D = P_{N_2'^{\perp}}D \sqcap_{M_1'} .$$

Hence

$$P_{N_2'^{\perp}}D_{|_{N_1'^{\perp}}} = P_{N_2'^{\perp}}D \sqcap_{M_1'}{}_{|_{N_1'^{\perp}}} .$$

Therefore, $P_{N_2'^{\perp}}D_{|_{N_1'^{\perp}}}$ is an isomorphism from $N_1'^{\perp}$ onto $N_2'^{\perp}$, so with respect to the decomposition

$$H_{\mathcal{A}} = N_1'^{\perp} \oplus N_1' \xrightarrow{D} N_2'^{\perp} \oplus N_2' = H_{\mathcal{A}},$$

D has the matrix $\left[\begin{array}{cc} \tilde{D}_1 & 0 \\ \tilde{D}_3 & \tilde{D}_4 \end{array}\right]$, where $\tilde{D}_1$ is an isomorphism.
Let $F = \left[\begin{array}{cc} (\tilde{D}_1)^{-1} & 0 \\ 0 & 0 \end{array}\right]$ with respect to the decomposition

$$H_{\mathcal{A}} = N_2'^{\perp} \oplus N_2' \xrightarrow{F} N_1'^{\perp} \oplus N_1' = H_{\mathcal{A}}.$$

Then $F \in B^a(H_{\mathcal{A}})$ and $DF = \left[\begin{array}{cc} 1 & 0 \\ \tilde{D}_3\tilde{D}_1^{-1} & 0 \end{array}\right]$ with respect to the decomposition

$$H_{\mathcal{A}} = N_2'^{\perp} \oplus N_2' \xrightarrow{DF} N_2'^{\perp} \oplus N_2' = H_{\mathcal{A}}.$$

Since N_2' is finitely generated, it follows that if we let the operator K be given by the operator matrix $\left[\begin{array}{cc} 0 & 0 \\ \tilde{D}_3(\tilde{D}_1)^{-1} & -1 \end{array}\right]$ with respect to the decomposition above, then $K \in \mathcal{K}^*(H_{\mathcal{A}})$. This is because $P_{N_2'}K = K$, where $P_{N_2'}$ is the orthogonal projection onto N_2' and $P_{N_2'} \in \mathcal{K}^*(H_{\mathcal{A}})$ by Corollary 2.0.35. Moreover, $DF = I + K$. □

Lemma 3.1.5. *[22, Lemma 2.2] Let M, N, W be Hilbert C^*-modules over a unital C^*-algebra $\mathcal{A}$. If $F \in B^a(M,N), D \in B^a(N,W)$ and $DF \in \mathcal{M}\Phi(M,W)$, then there exists a chain of decompositions*

$$M = M_2^{\perp} \oplus M_2 \xrightarrow{F} F(M_2^{\perp}) \oplus R \xrightarrow{D} W_1 \tilde{\oplus} W_2 = W$$

with respect to which F, D have the matrices $\begin{bmatrix} F_1 & F_2 \\ 0 & F_4 \end{bmatrix}$, $\begin{bmatrix} D_1 & D_2 \\ 0 & D_4 \end{bmatrix}$, respectively, where F_1, D_1 are isomorphisms, $F(M_2^{\perp}) \oplus R = N$ and in addition

$$M = M_2^{\perp} \oplus M_2 \xrightarrow{DF} W_1 \tilde{\oplus} W_2 = W$$

is an $\mathcal{M}\Phi$-decomposition for DF.

Proof. By the proof of Theorem 2.0.38 applied to the operator

$$DF \in \mathcal{M}\Phi(M,W),$$

there exists an $\mathcal{M}\Phi$-decomposition

$$M = M_2^{\perp} \oplus M_2 \xrightarrow{DF} W_1 \tilde{\oplus} W_2 = W$$

for DF. This is because the proof of Theorem 2.0.38 also holds when we consider arbitrary Hilbert C^*-modules M and W over a unital C^*-algebra $\mathcal{A}$ and not only the standard module $H_{\mathcal{A}}$. Then we can proceed as in the proof of Theorem 3.1.2, part 2) implies 1). □

Lemma 3.1.6. *[22, Lemma 2.3] Let M be a Hilbert C^*-module over a unital C^*-algebra $\mathcal{A}$. If $D \in \mathcal{M}\Phi_-(M)$, then there exists an $\mathcal{M}\Phi_-$-decomposition for D*

$$M = N_1'^{\perp} \oplus N_1' \xrightarrow{D} M_2 \tilde{\oplus} N_2' = M.$$

Similarly, if $F \in \mathcal{M}\Phi_+(M)$, then there exists an $\mathcal{M}\Phi_+$-decomposition for F

$$M = M_1 \tilde{\oplus} N_1 \xrightarrow{F} N_2^{\perp} \oplus N_2 = M.$$

Proof. From the proof of Theorem 3.1.2 part 1) implies 2) it follows that if $F \in \mathcal{M}\Phi_+(M)$, then there exists a decomposition

$$M = N_1^{\perp} \oplus N_1 \xrightarrow{F} F(N_1^{\perp}) \oplus F(N_1^{\perp})^{\perp} = M$$

with respect to which F has the matrix $\begin{bmatrix} F_1 & F_2 \\ 0 & F_4 \end{bmatrix}$, where F_1 is an isomorphism and N_1 is finitely generated. Hence

$$M = N_1^{\perp} \tilde{\oplus} U(N_1) \xrightarrow{F} F(N_1^{\perp}) \oplus F(N_1^{\perp})^{\perp} = M$$

is an $\mathcal{M}\Phi_+$-decomposition for F, where U is an isomorphism of M.

Similarly, if $D \in \mathcal{M}\Phi_-(M)$, then, from the proof of Theorem 3.1.4 part 1) implies 2), we get that there exists a decomposition

$$M = N_1'^{\perp} \oplus N_1' \xrightarrow{D} N_2'^{\perp} \oplus N_2' = M$$

with respect to which D has the matrix $\begin{bmatrix} D_1 & 0 \\ D_3 & D_4 \end{bmatrix}$, where D_1 is an isomorphism and N_2' is finitely generated. It follows that

$$M = N_1'^{\perp} \oplus N_1' \xrightarrow{D} V(N_2') \tilde{\oplus} N_2' = M$$

is an $\mathcal{M}\Phi$-decomposition for D where V is an isomorphism of M.

□

Lemma 3.1.7. *Let $F \in B^a(M)$, where M is a Hilbert C^*-module over a unital C^*-algebra $\mathcal{A}$. Suppose that $F \in \mathcal{M}\Phi_+(M)$ and let $M = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = M$ be an $\mathcal{M}\Phi_+$-decomposition for F. Then $M = N_1^{\perp} \oplus N_1 \xrightarrow{F} F(N_1^{\perp}) \tilde{\oplus} N_2 = M$ is also an $\mathcal{M}\Phi_+$-decomposition for F.*

Proof. This can be shown by exactly the same arguments as in the proof of Theorem 2.0.38. $\square$

The key lemma for proving the next results is the following lemma.

Lemma 3.1.8. *Let M be a Hilbert C^*-module and $F \in B(M)$. Suppose that there are decompositions*

$$M = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = M,$$

$$M = M_1' \tilde{\oplus} N_1' \xrightarrow{F} M_2' \tilde{\oplus} N_2' = M,$$

with respect to which F has matrices $\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}$ and $\begin{bmatrix} F_1' & 0 \\ 0 & F_4' \end{bmatrix}$, respectively, where F_1, F_1' are isomorphisms and N_1, N_2' are finitely generated. Then N_2 and N_1' are finitely generated as well.

Proof. We show first that N_2 is finitely generated. Let $\sqcap$ denote the projection onto N_2 along M_2 and consider the direct sum of modules $N_1 \oplus N_2'$ in the sense of Example 2.0.7. We claim that the map $\iota : N_1 \oplus N_2' \to N_2$ given by $\iota(x, y') = Fx + \sqcap y'$ is an epimorphism. To see this, let $y \in N_2$. Then $y = y_1' + y_2'$ for some $y_1' \in M_2'$ and $y_2' \in N_2'$. Since $F_{|_{M_1'}}$ is an isomorphism onto M_2', there exists an $m_1' \in M_1'$ such that $Fm_1' = y_1'$. We can write m_1' as $m_1' = m_1 + n_1$ for some $m_1 \in M_1$ and $n_1 \in N_1$. Then we obtain $y = Fm_1 + Fn_1 + y_2'$. Hence we get $y = \sqcap y = \sqcap Fm_1 + \sqcap Fn_1 + \sqcap y_2' = Fn_1 + \sqcap y_2'$. Since $y \in N_2$ was chosen arbitrary, it follows that ι is an epimorphism. However, $N_1 \oplus N_2'$ is finitely generated since both N_1 and N_2' are so by assumption, hence, by Remark 2.0.68, we must have that N_2 is finitely generated as well.

Next we show that N_1' is finitely generated. Let $\sqcap_{M_2}, \sqcap_{M_2'}, \sqcap_{N_1'}$ and $\sqcap_{N_2'}$ denote the projections onto M_2 along N_2, onto M_2' along N_2', onto N_1' along M_1' and onto N_2' along M_2', respectively. We claim that the map $\iota' : N_2' \oplus N_1 \longrightarrow N_1'$ given by

$$\iota'(n_2', n_1) = \sqcap_{N_1'} F_1^{-1} \sqcap_{M_2} (n_2' - \sqcap_{M_2'} Fn_1) + \sqcap_{N_1'} n_1$$

is an epimorphism. In order to show this, let $y = N_1'$. Then $y = m_1 + n_1$ for some $m_1 \in M_1$ and $n_1 \in N_1$. Set $m_2 = Fm_1$, then $m_1 = F_1^{-1} m_2$. We get $Fy = m_2 + Fn_1$. Now, since $\sqcap_{N_1'} y = y$ and $F \sqcap_{N_1'} = \sqcap_{N_2'} F$, we get

$$Fy = F \sqcap_{N_1'} y = \sqcap_{N_2'} Fy = \sqcap_{N_2'} m_2 + \sqcap_{N_2'} Fn_1.$$

Hence $m_2 + Fn_1 = \sqcap_{N_2'}(m_2 + Fn_1)$ which gives $\sqcap_{M_2'}(m_2 + Fn_1) = 0$, so $\sqcap_{M_2'} m_2 = - \sqcap_{M_2'} Fn_1$. Therefore, we get

$$m_2 = \sqcap_{N_2'} m_2 + \sqcap_{M_2'} m_2 = \sqcap_{N_2'} m_2 - \sqcap_{M_2'} Fn_1.$$

So we derive that

$$y = m_1 + n_1 = F_1^{-1} m_2 + n_1 = F_1^{-1}(\sqcap_{N_2'} m_2 - \sqcap_{M_2'} Fn_1) + n_1$$

$$= F_1^{-1} \sqcap_{M_2} (\sqcap_{N_2'} m_2 - \sqcap_{M_2'} Fn_1) + n_1 = F_1^{-1} \sqcap_{M_2} (n_2' - \sqcap_{M_2'} Fn_1) + n_1,$$

where we put $n_2' = \sqcap_{N_2'} m_2$. Recalling that $\sqcap_{N_1'} y = y$, we obtain that y can be written as

$$y = \sqcap_{N_1'} F_1^{-1} \sqcap_{M_2} (n_2' - \sqcap_{M_2'} Fn_1) + \sqcap_{N_1'} n_1,$$

where $n_2' \in N_2'$ and $n_1 \in N_1$. Since $y \in N_1'$ was chosen arbitrary, it follows that ι' is an epimorphism from $N_2' \oplus N_1$ onto N_1', hence, by Remark 2.0.68, N_1' is finitely generated. $\square$

Remark 3.1.9. From the proof of Lemma 3.1.8 it follows that there exist epimorphisms from $N_1 \oplus N_2'$ onto N_2 and onto N_1' also in the case when N_1 and N_2' are not finitely generated. Moreover, this holds in the case of arbitrary Banach spaces and not just Hilbert C^*-modules.

Corollary 3.1.10. *For any Hilbert C^*-module M, we have*

$$\mathcal{M}\Phi(M) = \mathcal{M}\Phi_+(M) \cap \mathcal{M}\Phi_-(M).$$

Proof. It suffices to show " $\supseteq$ ". However, if $F \in \mathcal{M}\Phi_+(M) \cap \mathcal{M}\Phi_-(M)$ and

$$M = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = M,$$

$$M = M_1' \tilde{\oplus} N_1' \xrightarrow{F} M_2' \tilde{\oplus} N_2' = M$$

are an $\mathcal{M}\Phi_+$-decomposition and an $\mathcal{M}\Phi_-$-decomposition for F, respectively, then from Lemma 3.1.8 it follows that both these decompositions are $\mathcal{M}\Phi$-decompositions for F. □

Recall from preliminaries that $B^a(H_\mathcal{A})$ is a C^*-algebra and $\mathcal{K}^*(H_\mathcal{A})$ is a closed two sided ideal in $B^a(H_\mathcal{A})$. Hence $B^a(H_\mathcal{A})/\mathcal{K}^*(H_\mathcal{A})$ is also a C^*-algebra, equipped with the quotient norm. We will call this algebra the *Calkin* algebra.

Remark 3.1.11. From Theorem 3.1.2, Theorem 3.1.4 and Corollary 3.1.10 it follows that $\mathcal{A}$-Fredholm operators on $H_\mathcal{A}$ are exactly those that are invertible in the Calkin algebra $B^a(H_\mathcal{A})/\mathcal{K}^*(H_\mathcal{A})$, which is well known from before and given in [38, Theorem 2.7.14]. However, notice that Lemma 3.1.8 and Corollary 3.1.10 hold for arbitrary Hilbert C^*-modules and not just the standard module.

Corollary 3.1.12. *Let M be a Hilbert C^*-module and $F \in \mathcal{M}\Phi(M)$. Then any $\mathcal{M}\Phi_+$-decomposition or $\mathcal{M}\Phi_-$-decomposition for F is an $\mathcal{M}\Phi$-decomposition for F.*

Proof. Let

$$M = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = M$$

be an $\mathcal{M}\Phi_+$-decomposition for F. Since $F \in \mathcal{M}\Phi(M)$ by assumption, there exists an $\mathcal{M}\Phi$-decomposition for F

$$M = M_1' \tilde{\oplus} N_1' \xrightarrow{F} M_2' \tilde{\oplus} N_2' = M.$$

In particular, N_1 and N_2' are finitely generated. We may hence apply Lemma 3.1.8 on these two decompositions for F and deduce that N_2 is finitely generated. The proof of the second statement is similar. □

The next lemma is a generalization of [56, Lemma 2.10.1], originally given in [44].

Lemma 3.1.13. *[20, Lemma 1], [21, Lemma 13] Let $F \in B(H_\mathcal{A})$ and suppose that $P \in B(H_\mathcal{A})$ is an adjointable projection such that $\ker P$ is finitely generated. Then $F \in \mathcal{M}\Phi(H_\mathcal{A})$ if and only if $PF_{|_{ImP}} \in \mathcal{M}\Phi(ImP)$ and in this case*

$$\text{index } PF_{|_{ImP}} = \text{index } F.$$

Proof. Observe that, since $\ker P$ is finitely generated and $H_\mathcal{A} = ImP \tilde{\oplus} \ker P$, it follows by Corollary 2.0.26 that $ImP \cong H_\mathcal{A}$. Hence the index of $PF_{|_{ImP}}$ is well defined.

Suppose first that $F \in \mathcal{M}\Phi(H_\mathcal{A})$. Since $\ker P$ is finitely generated, we have $P \in \mathcal{M}\Phi(H_\mathcal{A})$ also. Hence $PFP \in \mathcal{M}\Phi(H_\mathcal{A})$ by Lemma 2.0.43.
Let

$$H_\mathcal{A} = M \tilde{\oplus} N \xrightarrow{PFP} M' \tilde{\oplus} N' = H_\mathcal{A}$$

be a decomposition with respect to which PFP has the matrix

$$\left[\begin{array}{cc} (PFP)_1 & 0 \\ 0 & (PFP)_4 \end{array}\right],$$

where $(PFP)_1$ is an isomorphism and N, N' are finitely generated. By the proof of Theorem 3.1.2 part 2) $\Rightarrow$ 1) we know that $P(M)$ is closed. Moreover, by Theorem 2.0.38 we may assume that M is orthogonally complementable. Hence $P_{|_M}$ could be viewed as an adjointable operator from M into ImP with closed image. By Theorem 2.0.20 $P(M)$ is then orthogonallly complementable in ImP, that is $P(M) \oplus \tilde{N} = ImP$ for some closed submodule $\tilde{N}$. With respect to the decomposition

$$H_{\mathcal{A}} = M \tilde{\oplus} N \xrightarrow{P} P(M) \tilde{\oplus} (\tilde{N} \tilde{\oplus} \ker P) = H_{\mathcal{A}},$$

P has the matrix $\left[\begin{array}{cc} P_1 & P_2 \\ 0 & P_4 \end{array}\right]$, where P_1 is an isomorphism. Hence P_1 has the matrix $\left[\begin{array}{cc} P_1 & 0 \\ 0 & \tilde{P}_4 \end{array}\right]$ with respect to the decomposiotion

$$H_{\mathcal{A}} = U(M) \tilde{\oplus} U(N) \xrightarrow{P} P(M) \tilde{\oplus} (\tilde{N} \tilde{\oplus} \ker P) = H_{\mathcal{A}},$$

where U is an isomorphism. Since $P \in \mathcal{M}\Phi(H_{\mathcal{A}})$ and $U(N)$ is finitely generated, by Corollary 3.1.12 it follows that $\tilde{N} \tilde{\oplus} \ker P$ is finitely generated. Hence $\tilde{N}$ is finitely generated by Remark 2.0.68.

Next, $PF_{|_{P(M)}}$ is an isomorhism from $P(M)$ onto M' . Since $P(M)$ is closed, $P(M)$ is then also orthogonally complementable in $H_{\mathcal{A}}$ by Theorem 2.0.20 (because $P_{|_M} \in B^a(M, H_{\mathcal{A}})$, as M is orthogonally complementable in $H_{\mathcal{A}}$ and P is adjointable). It follows again that $PF_{|_{P(M)}}$ can be viewed as an adjointable operator from $P(M)$ into ImP, so M' is orthogonally complementable in ImP by Theorem 2.0.20 (since $M' = Im(PF_{|_{P(M)}})$). Thus, $M' \oplus \tilde{N}' = ImP$ for some closed submodule $\tilde{N}'$. Now,

$$H_{\mathcal{A}} = M' \tilde{\oplus} N' = M' \tilde{\oplus} \tilde{N}' \tilde{\oplus} \ker P,$$

so it follows that $(\tilde{N}' \tilde{\oplus} \ker P) \cong N'$. Since N' is finitely generated, we get that $\tilde{N}'$ is finitely generated also. With respect to the decomposition

$$ImP = P(M) \oplus \tilde{N} \xrightarrow{PF} M' \oplus \tilde{N}' = ImP,$$

$PF_{|_{ImP}}$ has the matrix $\left[\begin{array}{cc} (PF)_1 & (PF)_2 \\ 0 & (PF)_4 \end{array}\right]$, where $(PF)_1$ is an isomorphism. Then $PF_{|_{ImP}}$ has the matrix $\left[\begin{array}{cc} (PF)_1 & 0 \\ 0 & (\tilde{PF})_4 \end{array}\right]$ with respect to the decomposition

$$ImP = \tilde{U}(P(M)) \tilde{\oplus} \tilde{U}(\tilde{N}) \xrightarrow{PF} M' \oplus \tilde{N}' = ImP,$$

where $\tilde{U}$ is an isomorphism of ImP onto ImP. Since $\tilde{N}, \tilde{N}'$ and thus also $\tilde{U}(\tilde{N})$ are finitely generated, it follows that $PF_{|_{ImP}} \in \mathcal{M}\Phi(ImP)$.

Conversely, suppose that $PF_{|_{ImP}} \in \mathcal{M}\Phi(ImP)$. Let

$$ImP = M \tilde{\oplus} N \xrightarrow{PF} M' \tilde{\oplus} N' = ImP$$

be a decomposition with respect to which $PF_{|_{ImP}}$ has the matrix

$$\left[\begin{array}{cc} (PF)_1 & 0 \\ 0 & (PF)_4 \end{array}\right],$$

where N, N' are finitely generated and $(PF)_1$, is an isomorphism. It follows that with respect to the decomposition

$$H_{\mathcal{A}} = M\tilde{\oplus}(N\tilde{\oplus}\ker P) \xrightarrow{F} M'\tilde{\oplus}(N'\tilde{\oplus}\ker P) = H_{\mathcal{A}},$$

F has the matrix $\begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix}$, where F_1 is an isomorphism, as $F_1 = (PF)_1$. Indeed, we have $F_1 = \sqcap_{M'} F_{|_M}$, where $\sqcap_{M'}$ denotes the projection onto M' along $N'\tilde{\oplus}\ker P$. However, since PF maps M isomorphically onto M' and $ImP = M'\tilde{\oplus}N'$, it follows that $PF_{|_M} = \sqcap_{M'} F_{|_M}$. Therefore, $F_1 = \sqcap_{M'} F_{|_M} = PF_{|_M}$ is an isomorphism from M onto M'. Using the technique of diagonalization from the proof of Lemma 2.0.42 and the fact that $N\tilde{\oplus}\ker P$ and $N'\tilde{\oplus}\ker P$ are finitely generated, we deduce that $F \in \mathcal{M}\Phi(H_{\mathcal{A}})$.

It remains to show that index $PF_{|_{ImP}} =$ index F. Now, since $P \in \mathcal{M}\Phi(H_{\mathcal{A}})$, from Lemma 2.0.43 we get index $PFP =$ index $P+$index $F+$index $P =$ index F, as index $P = 0$. We recall that by the above arguments there exists a decomposition

$$ImP = P(M) \oplus \tilde{N} \xrightarrow{PF} M' \oplus \tilde{N}' = ImP$$

with respect to which PF has the matrix

$$\begin{bmatrix} (PF)_1 & (PF)_2 \\ 0 & (PF)_4 \end{bmatrix},$$

where $(PF)_1$ is an isomorphism and $\tilde{N}, \tilde{N}'$ are finitely generated Hilbert submodules. In addition, it also follows that P has the matrix $\begin{bmatrix} P_1 & P_2 \\ 0 & P_4 \end{bmatrix}$ with respect to the decomposition

$$H_{\mathcal{A}} = M\tilde{\oplus}N \xrightarrow{P} P(M)\tilde{\oplus}(\tilde{N}\tilde{\oplus}\ker P) = H_{\mathcal{A}},$$

where P_1 is an isomorphism and N is a finitely generated Hilbert submodule. Moreover,

$$H_{\mathcal{A}} = M\tilde{\oplus}N \xrightarrow{PFP} M'\tilde{\oplus}N' = H_{\mathcal{A}}$$

is an $\mathcal{M}\Phi$-decomposition for PFP and $N' \cong \tilde{N}'\tilde{\oplus}\ker P$.

Since index $PFP =$ index F, it follows that $[N] - [N'] =$ index F in $K_0(\mathcal{A})$. Next, by diagonalizing the matrix $\begin{bmatrix} P_1 & P_2 \\ 0 & P_4 \end{bmatrix}$ as in the proof of Lemma 2.0.42, it is easily seen that

$$[N] - [\tilde{N}] - [\ker P] = [N] - [\tilde{N}\tilde{\oplus}\ker P] = \text{index } P = 0.$$

Similarly, by diagonalizing the matrix $\begin{bmatrix} (PF)_1 & (PF)_2 \\ 0 & (PF)_4 \end{bmatrix}$, we obtain that

$$\text{index } (PF_{|_{ImP}}) = [\tilde{N}] - [\tilde{N}'].$$

Finally, $[\tilde{N}'] + [\ker P] = [N']$ since $\tilde{N}'\tilde{\oplus}\ker P \cong N'$. Combining all this facts together, we obtain that

$$\begin{aligned} \text{index } (PF_{|_{ImP}}) &= [\tilde{N}] - [\tilde{N}'] = [\tilde{N}] + [\ker P] - [\tilde{N}'] - [\ker P] \\ &= [\tilde{N}\tilde{\oplus}\ker P] - [\tilde{N}'\tilde{\oplus}\ker P] = [N] - [N'] = \text{index } F. \end{aligned}$$

□

From Theorem 3.1.2 and Theorem 3.1.4 we get nice algebraic descriptions of the classes $\mathcal{M}\Phi_+(H_\mathcal{A})$ and $\mathcal{M}\Phi_-(H_\mathcal{A})$ in terms of the left and the right invertible elements in the Calkin algebra $B^a(H_\mathcal{A})/\mathcal{K}^*(H_\mathcal{A})$, respectively. This directly leads to several useful corollaries, as given below.

Corollary 3.1.14. *[18, Corollary 2.5] $\mathcal{M}\Phi_+(H_\mathcal{A})$ and $\mathcal{M}\Phi_-(H_\mathcal{A})$ are semigroups under the multiplication.*

Proof. The statement follows directly from Theorem 3.1.2 and Theorem 3.1.4, as $\mathcal{M}\Phi_+(H_\mathcal{A})$ consists of all elements that are left invertible in the Calkin algebra whereas $\mathcal{M}\Phi_+(H_\mathcal{A})$ consists of all elements that are right invertible in the Calkin algebra. $\square$

Corollary 3.1.15. *[18, Corollary 2.6] Let $F, D \in B^a(H_\mathcal{A})$. If $DF \in \mathcal{M}\Phi_+(H_\mathcal{A})$, then $F \in \mathcal{M}\Phi_+(H_\mathcal{A})$. If $DF \in \mathcal{M}\Phi_-(H_\mathcal{A})$, then $D \in \mathcal{M}\Phi_-(H_\mathcal{A})$.*

Proof. Suppose that $DF \in \mathcal{M}\Phi_+(H_\mathcal{A})$. By Theorem 3.1.2 there exists some $C \in B^a(H_\mathcal{A})$ and $K \in \mathcal{K}^*(H_\mathcal{A})$ such that $CDF = I + K$. Again, by Theorem 3.1.2 it follows that $F \in \mathcal{M}\Phi_+(H_\mathcal{A})$. The proof of the second statement of Corollary 3.1.15 is similar. $\square$

Corollary 3.1.16. *[18, Corollary 2.7] Let $F, D \in B^a(H_\mathcal{A})$. If $DF \in \mathcal{M}\Phi_+(H_\mathcal{A})$ and $F \in \mathcal{M}\Phi(H_\mathcal{A})$, then $D \in \mathcal{M}\Phi_+(H_\mathcal{A})$. If $DF \in \mathcal{M}\Phi_-(H_\mathcal{A})$ and $D \in \mathcal{M}\Phi(H_\mathcal{A})$, then $F \in \mathcal{M}\Phi_-(H_\mathcal{A})$.*

Proof. Suppose that $DF \in \mathcal{M}\Phi_+(H_\mathcal{A})$ and $F \in \mathcal{M}\Phi(H_\mathcal{A})$. By Theorem 3.1.2 there exist some $C \in B^a(H_\mathcal{A}), K \in \mathcal{K}^*(H_\mathcal{A})$ such that $CDF = I + K$, as $DF \in \mathcal{M}\Phi_+(H_\mathcal{A})$ by assumption. Moreover, since $F \in \mathcal{M}\Phi(H_\mathcal{A})$, by Theorem 3.1.4 there exist some $F' \in B^a(H_\mathcal{A}), K' \in \mathcal{K}^*(H_\mathcal{A})$ such that $FF' = I + K'$. Hence

$$CDFF' = (CDF)F' = (I + K)F' = F' + KF',$$

$$CDFF' = CD(FF') = CD(I + K') = CD + CDK'.$$

Therefore, $FF' + FKF' = FCD + FCDK'$. So we get that

$$FCD = FF' + FKF' - FCDK' = I + K' + FKF' - FCDK'.$$

Since $K' + FKF' - FCDK' \in \mathcal{K}^*(H_\mathcal{A})$, by Theorem 3.1.2 it follows that $D \in \mathcal{M}\Phi_+(H_\mathcal{A})$. The proof of the second statement of Corollary 3.1.16 is similar. $\square$

Corollary 3.1.17. *[18, Corollary 2.8] Let $F, D \in B^a(H_\mathcal{A})$. If $D \in \mathcal{M}\Phi_+(H_\mathcal{A})$ and $DF \in \mathcal{M}\Phi(H_\mathcal{A})$, then $D \in \mathcal{M}\Phi(H_\mathcal{A})$. If $F \in \mathcal{M}\Phi_-(H_\mathcal{A})$ and $DF \in \mathcal{M}\Phi(H_\mathcal{A})$, then $F \in \mathcal{M}\Phi(H_\mathcal{A})$.*

Proof. Let $D \in \mathcal{M}\Phi_+(H_\mathcal{A})$ and $DF \in \mathcal{M}\Phi(H_\mathcal{A})$. Since $DF \in \mathcal{M}\Phi(H_\mathcal{A})$, by Theorem 3.1.4 there exist some $C \in B^a(H_\mathcal{A}), K \in \mathcal{K}^*(H_\mathcal{A})$, such that $DFC = I + K$. By Theorem 3.1.4, we have then that $D \in \mathcal{M}\Phi_-(H_\mathcal{A})$. So $D \in \mathcal{M}\Phi_+(H_\mathcal{A}) \cap \mathcal{M}\Phi_-(H_\mathcal{A})$. However, by Corollary 3.1.10, $\mathcal{M}\Phi(H_\mathcal{A}) = \mathcal{M}\Phi_+(H_\mathcal{A}) \cap \mathcal{M}\Phi_-(H_\mathcal{A})$, so $D \in \mathcal{M}\Phi(H_\mathcal{A})$.
The proof of the second statement of Corollary 3.1.17 is similar. $\square$

Corollary 3.1.18. *[18, Corollary 2.9] If $D \in \mathcal{M}\Phi(H_\mathcal{A})$ and $DF \in \mathcal{M}\Phi(H_\mathcal{A})$, then $F \in \mathcal{M}\Phi(H_\mathcal{A})$. If $F \in \mathcal{M}\Phi(H_\mathcal{A})$ and $DF \in \mathcal{M}\Phi(H_\mathcal{A})$, then $D \in \mathcal{M}\Phi(H_\mathcal{A})$.*

Proof. Suppose that $D \in \mathcal{M}\Phi(H_\mathcal{A})$ and $DF \in \mathcal{M}\Phi(H_\mathcal{A})$. Since $DF \in \mathcal{M}\Phi(H_\mathcal{A})$, by Theorem 3.1.4 there exist some $C \in B^a(H_\mathcal{A}), K \in \mathcal{K}^*(H_\mathcal{A})$ such that $DFC = I + K$. Moreover, since $D \in \mathcal{M}\Phi(H_\mathcal{A})$, by the Theorem 3.1.2 there exist some $D' \in B^a(H_\mathcal{A}), K' \in \mathcal{K}^*(H_\mathcal{A})$ such that $D'D = I + K'$. Hence

$$D'DFC = D'(DFC) = D'(I + K) = D' + D'K,$$

$$D'DFC = (D'D)FC = (I + K')FC = FC + K'FC.$$

Thus $D'+D'K = FC+K'FC$. Hence $D'D+D'KD = FCD+K'FCD$. However, $D'D = I+K'$, so we obtain $I+K'+D'KD = FCD+K'FCD$. So, $FCD = I+K'+D'KD-K'FCD$. Since $(K'+D'KD-K'FCD) \in \mathcal{K}^*(H_{\mathcal{A}})$, by Theorem 3.1.4 we have that $F \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$. Now, since $DF \in \mathcal{M}\Phi(H_{\mathcal{A}}) \subseteq \mathcal{M}\Phi_+(H_{\mathcal{A}})$, by Corollary 3.1.15 it follows that $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$ also. Hence $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}}) \cap \mathcal{M}\Phi_-(H_{\mathcal{A}}) = \mathcal{M}\Phi(H_{\mathcal{A}})$ by Corollary 3.1.10.
The proof of the second statement of Corollary 3.1.18 is similar. □

Corollary 3.1.19. *[18, Corollary 2.10] It holds that*

$$\mathcal{M}\Phi_+(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}}) \text{ and } \mathcal{M}\Phi_-(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$$

are two sided ideals in $\mathcal{M}\Phi_+(H_{\mathcal{A}})$ and $\mathcal{M}\Phi_-(H_{\mathcal{A}})$, respectively. In particular, they are semigroups under the multiplication.

Proof. Let $F, D \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$ and suppose first that $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$. Since $\mathcal{M}\Phi_+(H_{\mathcal{A}})$ is closed under the multiplication by Corollary 3.1.14, it follows that $DF \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$. Now, if $DF \in \mathcal{M}\Phi(H_{\mathcal{A}})$, by Corollary 3.1.17 we have $D \in \mathcal{M}\Phi(H_{\mathcal{A}})$. Then, by Corollary 3.1.18, it would follow that $F \in \mathcal{M}\Phi(H_{\mathcal{A}})$, which is a contradiction. Thus we must have that $DF \in \mathcal{M}\Phi_+(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$.

Suppose next that $D \in \mathcal{M}\Phi_+(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$. Again, if $DF \in \mathcal{M}\Phi(H_{\mathcal{A}})$, then, since $D \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$, by Corollary 3.1.17 we would have that $D \in \mathcal{M}\Phi(H_{\mathcal{A}})$, which is impossible. So, also in this case, we must have that $DF \in \mathcal{M}\Phi_+(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$.

Similarly, one can prove the statement for $\mathcal{M}\Phi_-(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$. □

In the corollaries above we give pure algebraic proofs by using that semi-$\mathcal{A}-$Fredholm operators on $H_{\mathcal{A}}$ correspond to one-sided invertible elements in the Calkin algebra $B^a(H_{\mathcal{A}})/\mathcal{K}^*(H_{\mathcal{A}})$. It is also possible to give direct proofs of these corollaries by only using the definition of semi-$\mathcal{A}$-Fredholm operators and Lemma 3.1.8. We provide these proofs in Section 3.5. The advantage of such approach is that it can also be applied to the case of arbitrary Hilbert C^*-modules and not just $H_{\mathcal{A}}$.

Corollary 3.1.20. *[18, Corollary 2.11] Let $F \in B^a(M,N)$ where M and N are Hilbert C^*-modules over a unital C^*-algebra. Then $F \in \mathcal{M}\Phi_+(M,N)$ if and only if $F^* \in \mathcal{M}\Phi_-(N,M)$. Moreover, if $F \in \mathcal{M}\Phi(H_{\mathcal{A}})$, then $F^* \in \mathcal{M}\Phi(H_{\mathcal{A}})$ and* index $F = -$index F^*.

Proof. From the proof of Theorem 3.1.2 part 1) ⇒ 2) it follows that if $F \in \mathcal{M}\Phi_+(M,N)$, then for F and consequently for F^* there exist decompositions

$$M = M_1 \oplus M_1^{\perp} \xrightarrow{F} M_2 \oplus M_2^{\perp} = N,$$

$$N = M_2 \oplus M_2^{\perp} \xrightarrow{F^*} M_1 \oplus M_1^{\perp} = M,$$

with respect to which F and F^* have matrices

$$\begin{bmatrix} F_1 & F_2 \\ 0 & F_4 \end{bmatrix}, \begin{bmatrix} F_1^* & 0 \\ F_2^* & F_4^* \end{bmatrix},$$

respectively, where F_1, F_1^* are isomorphisms and $M_1^{\perp}$ is finitely generated. Using the technique of diagonalization as in the proof of Lemma 2.0.42, we deduce that $F^* \in \mathcal{M}\Phi_-(N,M)$ since $M_1^{\perp}$ is finitely generated. The proof is analogue when $F \in \mathcal{M}\Phi_-(N,M)$, only in this case $M_2^{\perp}$ is finitely generated and we apply the proof of Theorem 3.1.4 part 1) ⇒ 2) instead of the proof of Theorem 3.1.2 part 1) ⇒ 2). If in addition F is in $\mathcal{M}\Phi(H_{\mathcal{A}})$, then both $M_1^{\perp}$ and $M_2^{\perp}$ are finitely generated. Using again the technique of diagonalization, one deduces easily that $F^* \in \mathcal{M}\Phi(H_{\mathcal{A}})$ in this case and index $F = [M_1^{\perp}] - [M_2^{\perp}]$, index $F^* = [M_2^{\perp}] - [M_1^{\perp}]$, so index $F = -$index F^*. □

Closed range semi-$\mathcal{A}$-Fredholm operators can be described in a more similar way as classical semi-Fredholm operators on Hilbert spaces than arbitrary semi-$\mathcal{A}$-Fredholm operators, as the following lemma shows.

Lemma 3.1.21. *[21, Lemma 12] Let $F \in B^a(M)$ where M is a Hilbert C^*-module and suppose that ImF is closed. Then the following statements hold:*
a) $F \in \mathcal{M}\Phi_+(M)$, if and only if $\ker F$ is finitely generated;
b) $F \in \mathcal{M}\Phi_-(M)$, if and only if $ImF^\perp$ is finitely generated.

Proof. a) Let

$$M = M_1 \tilde{\oplus} M_2 \xrightarrow{F} M_1' \tilde{\oplus} M_2' = M$$

be an $\mathcal{M}\Phi_+$-decomposition for F. By Lemma 3.1.3 we have that $\ker F \subseteq F^{-1}(M_2') = M_2$. Now, by Theorem 2.0.20, $\ker F$ is orthogonally complementable in M. Hence, $\ker F$ is orthogonally complementable in M_2, since $\ker F \subseteq M_2$. This follows from Lemma 2.0.66. Since M_2 is finitely generated, it follows that $\ker F$ is finitely generated, being a direct summand in M_2.

Conversely, if $\ker F$ is finitely generated, then

$$H_{\mathcal{A}} = \ker F^\perp \oplus \ker F \xrightarrow{F} ImF \oplus ImF^\perp = H_{\mathcal{A}}$$

is an $\mathcal{M}\Phi_+$-decomposition for F. Here we use that ImF is closed, which by Theorem 2.0.20 gives

$$H_{\mathcal{A}} = ImF \oplus ImF^\perp = \ker F^\perp \oplus \ker F.$$

b) This can be shown by passing to the adjoints and using a). Use that ImF^* is closed if and only if ImF is closed by the proof of Theorem 2.0.20 part ii). Moreover, $F \in \mathcal{M}\Phi_-(M)$ if and only if $F^* \in \mathcal{M}\Phi_+(M)$ by Corollary 3.1.20 and $ImF^\perp = \ker F^*$. □

Lemma 3.1.22. *[18, Lemma 2.12] Let M be a closed submodule of $H_{\mathcal{A}}$ such that $H_{\mathcal{A}} = M \tilde{\oplus} N$ for some finitely generated Hilbert submodule N. Let $F \in B^a(H_{\mathcal{A}})$, J_M be the inclusion map from M into $H_{\mathcal{A}}$ and suppose that $FJ_M \in \mathcal{M}\Phi_+(M, H_{\mathcal{A}})$. Then $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$.*

Proof. Consider a decomposition

$$M = M_1 \tilde{\oplus} M_2 \xrightarrow{FJ_M} \tilde{M}_1 \tilde{\oplus} \tilde{M}_2 = H_{\mathcal{A}}$$

with respect to which

$$FJ_M = \begin{bmatrix} (FJ_M)_1 & 0 \\ 0 & (FJ_M)_4 \end{bmatrix},$$

where $(FJ_M)_1$ is an isomorphism and M_2 is finitely generated. Then F has the matrix

$$\begin{bmatrix} F_1 & F_2 \\ 0 & F_4 \end{bmatrix}$$

with respect to the decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} (M_2 \tilde{\oplus} N) \xrightarrow{F} \tilde{M}_1 \tilde{\oplus} \tilde{M}_2 = H_{\mathcal{A}},$$

where F_1 is an isomorphism. Using the technique of diagonalization as in the proof of Lemma 2.0.42 and the fact that $M_2 \tilde{\oplus} N$ is finitely generated since both M_2 and N are so, we deduce that $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$. □

Lemma 3.1.23. *[18, Corollary 2.18] Let $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$ and let*

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}},$$

$$H_{\mathcal{A}} = \tilde{M}_1 \tilde{\oplus} \tilde{N}_1 \xrightarrow{F} \tilde{M}_2 \tilde{\oplus} \tilde{N}_2 = H_{\mathcal{A}},$$

be two $\mathcal{M}\Phi_+$-decompositions for F. Then there exist some finitely generated Hilbert submodules P and $\tilde{P}$ such that $(N_2 \oplus P) \cong (\tilde{N}_2 \oplus \tilde{P})$.

Proof. Since N_1 and ${N_1}'$ are finitely generated, by Theorem 2.0.34 there exists an $n \in \mathbb{N}$ such that

$$L_n = P \tilde{\oplus} p_n(N_1), P = M_1 \cap L_n, p_n(N_1) \cong N_1 \text{ and}$$

$$L_n = P' \tilde{\oplus} p_n(N_1') \ , \ P' = M_1' \cap L_n \ , \ p_n(N_1') \cong N_1',$$

where p_n denotes the orthogonal projection onto L_n.
Then

$$H_{\mathcal{A}} = L_n^{\perp} \tilde{\oplus} P \tilde{\oplus} N_1 = L_n^{\perp} \tilde{\oplus} P' \tilde{\oplus} N_1'.$$

Consequently, $\sqcap_{M_1}|_{(L_n^{\perp} \tilde{\oplus} P)}$ and $\sqcap_{{M_1}'}|_{(L_n^{\perp} \tilde{\oplus} P')}$ are isomorphisms from $L_n^{\perp} \tilde{\oplus} P$ onto M_1 and from $L_n^{\perp} \tilde{\oplus} P'$ onto M_1', respectively, where $\sqcap_{M_1}|_{(L_n^{\perp} \tilde{\oplus} P)}$ and $\sqcap_{{M_1}'}|_{(L_n^{\perp} \tilde{\oplus} P')}$ denote the restrictions of projections onto M_1 and M_1' along N_1 and N_1' restricted to $L_n^{\perp} \tilde{\oplus} P$ and $L_n^{\perp} \tilde{\oplus} P'$, respectively. Since

$$F(M_1) = M_2, \ F(N_1) \subseteq N_2 \text{ and } H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1,$$

it follows that

$$\sqcap_{M_2} F|_{(L_n^{\perp} \tilde{\oplus} P)} = F \sqcap_{M_1}|_{(L_n^{\perp} \tilde{\oplus} P)},$$

where $\sqcap_{M_2}$ denotes the projection onto M_2 along N_2. Hence $\sqcap_{M_2} F|_{(L_n^{\perp} \tilde{\oplus} P)}$ is an isomorphism as a composition of isomorphisms. Similarly, $\sqcap_{M_2'} F|_{(L_n^{\perp} \tilde{\oplus} P')}$ is an isomorphism, where $\sqcap_{M_2'}$ denotes the projection onto M_2' along N_2'. We get then that F has the matrices

$$\begin{bmatrix} \tilde{F}_1 & 0 \\ \tilde{F}_3 & F_4 \end{bmatrix}, \begin{bmatrix} \tilde{F}_1' & 0 \\ \tilde{F}_3' & F_4' \end{bmatrix}$$

with respect to the decompositions

$$H_{\mathcal{A}} = (L_n^{\perp} \tilde{\oplus} P) \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}},$$

$$H_{\mathcal{A}} = (L_n^{\perp} \tilde{\oplus} P') \tilde{\oplus} N_1' \xrightarrow{F} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}},$$

respectively, where $\tilde{F}_1 = \sqcap_{M_2} F|_{(L_n^{\perp} \tilde{\oplus} P)}$ and $\tilde{F}_1' = \sqcap_{M_2'} F|_{(L_n^{\perp} \tilde{\oplus} P')}$ are isomorphisms. As in the proof of Lemma 2.0.43, we let

$$V = \begin{bmatrix} 1 & 0 \\ -\tilde{F}_3 \tilde{F}_1^{-1} & 1 \end{bmatrix}, V' = \begin{bmatrix} 1 & 0 \\ -\tilde{F}_3' \tilde{F}_1'^{-1} & 1 \end{bmatrix}$$

with respect to the decompositions

$$H_{\mathcal{A}} = M_2 \tilde{\oplus} N_2 \xrightarrow{V} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}},$$

$$H_{\mathcal{A}} = M_2' \tilde{\oplus} N_2' \xrightarrow{V} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}}, \text{ respectively.}$$

Then F has the matrices

$$\begin{bmatrix} \tilde{\tilde{F}}_1 & 0 \\ 0 & \tilde{\tilde{F}}_4 \end{bmatrix}, \begin{bmatrix} \tilde{\tilde{F}}_1' & 0 \\ 0 & \tilde{\tilde{F}}_4' \end{bmatrix}$$

with respect to the decompositions

$$H_{\mathcal{A}} = (L_n^{\perp} \tilde{\oplus} P) \tilde{\oplus} N_1 \xrightarrow{F} V^{-1}(M_2) \tilde{\oplus} V^{-1}(N_2) = H_{\mathcal{A}},$$

$$H_{\mathcal{A}} = (L_n^{\perp} \tilde{\oplus} P') \tilde{\oplus} N_1' \xrightarrow{F} V'^{-1}(M_2') \tilde{\oplus} V'^{-1}(N_2') = H_{\mathcal{A}},$$

respectively, where $\tilde{\tilde{F}}_1, \tilde{\tilde{F}}_1'$ are isomorphisms. Since

$$H_{\mathcal{A}} = F(L_n^{\perp}) \tilde{\oplus} (F(P) \tilde{\oplus} V^{-1}(N_2)) = F(L_n^{\perp}) \tilde{\oplus} (F(P') \tilde{\oplus} V'^{-1}(N_2')) = H_{\mathcal{A}},$$

clearly, we have

$$(F(P) \tilde{\oplus} V^{-1}(N_2)) \cong (F(P') \tilde{\oplus} V'^{-1}(N_2')).$$

Hence $P \oplus N_2 \cong F(P) \tilde{\oplus} V^{-1}(N_2) \cong F(P') \tilde{\oplus} V'^{-1}(N_2') \cong P' \oplus N_2'$. □

Remark 3.1.24. The proof of Corollary 3.1.23 is similar to the proof of [18, Lemma 2.16].

3.2 Generalized Schechter characterization

In this section, we describe the set of upper semi-$\mathcal{A}$-Fredholm operators in terms of some equivalent conditions, generalizing in this settings Schechter characterization of the classical upper semi-Fredholm operators given in [47] and [56, Section 1.4].

Lemma 3.2.1. *[18, Lemma 3.1] Let $F \in B^a(M,N)$ where M and N are Hilbert modules over a unital C^*-algebra $\mathcal{A}$. Then $F \in \mathcal{M}\Phi_+(M,N)$ if and only if there exists a closed, orthogonally complementable submodule $M' \subseteq M$ such that $F_{|_{M'}}$ is bounded below and $M'^{\perp}$ is finitely generated.*

Proof. If such M' exists, then $F(M')$ is closed in N. Moreover, since M' is orthogonally complementable, $F_{|_{M'}}$ is adjointable. By Theorem 2.0.20, $F(M')$ is orthogonally complementable in N. Then, with respect to the decomposition

$$M = M' \oplus M'^{\perp} \xrightarrow{F} F(M') \oplus F(M')^{\perp} = N,$$

F has the matrix

$$\begin{bmatrix} F_1 & F_2 \\ 0 & F_4 \end{bmatrix},$$

where F_1 is an isomorphism.Using the technique of diagonalization as in the proof of Lemma 2.0.42 and the fact that $M'^{\perp}$ is finitely generated, we deduce that $F \in \mathcal{M}\Phi_+(M,N)$. On the other hand, if $F \in \mathcal{M}\Phi_+(M,N)$, then by the similar arguments as in the proof of Theorem 2.0.38 we may assume that there exists a decomposition

$$M = M' \oplus M'^{\perp} \xrightarrow{F} N' \tilde{\oplus} N'' = N,$$

with respect to which F has the matrix

$$\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix},$$

where F_1 is an isomorphism and $M'^{\perp}$ is finitely generated. □

Lemma 3.2.2. *[18, Lemma 3.2] Let $F \in B^a(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi_+(H_{\mathcal{A}})$. Then there exists a sequence $\{x_k\} \subseteq H_{\mathcal{A}}$ and an increasing sequence $\{n_k\} \subseteq \mathbb{N}$ such that*

$$x_k \in L_{n_k} \cap L_{n_{k-1}}^{\perp}, \quad \| x_k \|= 1$$

and

$$\| Fx_k \| \leq 2^{1-2k} \textit{ for all } k \ \in \mathbb{N}.$$

Proof. If $F = 0$, then the lemma follows trivially.

Suppose that $F \in B^a(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi_+(H_{\mathcal{A}})$ and $F \neq 0$. If $F \in B^a(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi_+(H_{\mathcal{A}})$, then F is not bounded below by Lemma 3.2.1, hence we may in fact find an $\tilde{\tilde{x}}_1 \in H_{\mathcal{A}}$ such that $\| \tilde{\tilde{x}}_1 \|= 1 + \frac{1}{4\|F\|}$ and $\| F\tilde{\tilde{x}}_1 \| \leq \frac{1}{4}$. As $\| P_{L_n^{\perp}} \tilde{\tilde{x}}_1 \| \longrightarrow 0$ when $n \to \infty$, there exists an $n_1 \in \mathbb{N}$ such that $\| P_{L_{n_1}^{\perp}} \tilde{\tilde{x}}_1 \| \leq \frac{1}{4\|F\|}$, hence, for $\tilde{x}_1 := P_{L_{n_1}} \tilde{\tilde{x}}_1$, we have

$$\| F\tilde{x}_1 \|=\| FP_{L_{n_1}} \tilde{\tilde{x}}_1 \| \leq \| F\tilde{\tilde{x}}_1 \| + \| FP_{L_{n_1}^{\perp}} \tilde{\tilde{x}}_1 \| \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Now,

$$\| \tilde{x}_1 \| \geq \| \tilde{\tilde{x}}_1 \| - \| P_{L_{n_1}^{\perp}} \tilde{\tilde{x}}_1 \| \geq 1 + \frac{1}{4 \| F \|} - \frac{1}{4 \| F \|} = 1.$$

Set $x_1 = \frac{1}{\|\tilde{x}_1\|}\tilde{x}_1$. Then $\| x_1 \|= 1$ and $\| Fx_1 \|= \frac{1}{\|\tilde{x}_1\|} \| F\tilde{x}_1 \| \leq \| F\tilde{x}_1 \| \leq \frac{1}{2}$.

Suppose next that there exists $x_1, \dots, x_k \in H_{\mathcal{A}}$, $n_1 \leq n_2 \leq \cdots \leq n_k$ that satisfy the hypotthesis of the lemma. Since F is not bounded below on $L_{n_k}^{\perp}$ by Lemma 3.2.1, we can actually find some $\tilde{\tilde{x}}_{k+1} \in L_{n_k}^{\perp}$ such that

$$\| \tilde{\tilde{x}}_{k+1} \|= 1 + \frac{1}{\| F \|} 2^{-(k+1)} \text{ and } \| F\tilde{\tilde{x}}_{k+1} \| \leq 2^{-2(k+1)}.$$

We choose an $n_{k+1} \in \mathbb{N}$ such that $n_{k+1} > n_k$ and

$$\| P_{L_{n_{k+1}}^{\perp}} \tilde{\tilde{x}}_{k+1} \| \leq \frac{1}{\| F \|} 2^{-2(k+1)}.$$

Then, if we set $\tilde{x}_{k+1} := P_{L_{n_{k+1}}} \tilde{\tilde{x}}_{k+1}$, by the same a arguments as above we deduce that

$$\| \tilde{x}_{k+1} \| \geq 1 \text{ and } \| F\tilde{x}_{k+1} \| \leq 2^{1-2(k+1)},$$

hence, for $x_{k+1} := \frac{1}{\|\tilde{x}_{k+1}\|}\tilde{x}_{k+1}$, we get $\| x_{k+1} \|= 1$ and in addition

$$\| Fx_{k+1} \| \leq \| F\tilde{x}_{k+1} \| \leq 2^{1-2(k+1)}.$$

Moreover, $x_{k+1} \in L_{n_{k+1}} \cap L_{n_k}^{\perp}$. By induction, the lemma follows. □

The next lemma is a generalization of [25, Chapter XI, Theorem 2.3(d)].

Lemma 3.2.3. *Let $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$. Then there is no sequence of unit vectors $\{x_n\}$ in $H_{\mathcal{A}}$ such that $\langle e_k, x_n \rangle \to 0$ as $n \to \infty$ for all $k \in \mathbb{N}$ and $\lim_{n\to\infty} \| Fx_n \|= 0$.*

Proof. Let $D \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$ and $K \in \mathcal{K}^*(H_{\mathcal{A}})$ be such that $DF = I + K$. If $K = 0$, then $DF = I$, which in particularly means that F is bounded below. Since $\| x_n \|= 1$ for all $n \in \mathbb{N}$, it follows that $Fx_n \not\to 0$ as $n \to \infty$.

Suppose next that $K \neq 0$. Then

$$|1- \| DFx_n \| \, | = | \, \| x_n \| - \| DFx_n \| \, | \leq \| (I - DF)x_n \|=\| Kx_n \| .$$

Here we have applied the same arguments as in the proof of [25, Chapter XI, Theorem 2.3] part $(a) \Rightarrow (d)$. Given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $\| K_{|_{L_n^\perp}} \| < \frac{\epsilon}{2}$ for all $n \geq N$, since $K \in \mathcal{K}^*(H_{\mathcal{A}})$. This follows from Proposition 2.0.17. If $\langle e_k, x_n \rangle \overset{n\to\infty}{\longrightarrow} 0$ for all $k \in \{1, 2, \cdots, N\}$, then we may choose an $M \in \mathbb{N}$ such that $\| \langle e_k, x_n \rangle \| < \dfrac{\epsilon}{2 \| K \| N}$ for all $n \geq M$ and for all $k \in \{1, \ldots, N\}$. Let P_N denote the orthogonal projection onto $L_{N+1}^\perp$. Then, for all $n \geq M$, we have

$$\| Kx_n \| \leq \| KP_N x_n \| + \sum_{k=1}^{N} \| Ke_k \cdot \langle e_k, x_n \rangle \| \leq \frac{\epsilon}{2} + \sum_{k=1}^{N} \| K \| \quad \| \langle e_k, x_n \rangle \| < \epsilon.$$

Thus, $\| Kx_n \| \to 0$, so from the above calculations it follows that $\| DFx_n \| \to 1$ as $n \to \infty$. Therefore we can not have that $\| Fx_n \| \to 0$ as $n \to \infty$. ☐

Corollary 3.2.4. *If $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$, then $Fe_n \not\to 0$ as $n \to \infty$.*

Corollary 3.2.5. *Let $F \in B^a(H_{\mathcal{A}})$. Then $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$ if and only if there is no sequence of unit vectors $\{x_k\}_{k\in\mathbb{N}}$ in $H_{\mathcal{A}}$ satisfying the conditions of Lemma 3.2.2.*

Proof. The implication in one direction follows from Lemma 3.2.2. Let us prove the implication in the other direction. To this end, suppose that $F \in B^a(H_{\mathcal{A}})$ and that there exists a sequence of unit vectors $\{x_n\}_{n\in\mathbb{N}} \subseteq H_{\mathcal{A}}$ satisfying the conditions of Lemma 3.2.2. By these conditions, it follows then that $\lim_{n\to\infty} \langle e_k, x_n \rangle = 0$ for all $k \in \mathbb{N}$ and moreover, $\lim_{n\to\infty} \| Fx_n \| = 0$. Hence, by Lemma 3.2.3, we deduce that $F \in B^a(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi_+(H_{\mathcal{A}})$, which shows the implication in the other direction. ☐

Example 3.2.6. If we consider $\mathcal{A}$ as a Hilbert module over itself, then, in general, we can find closed submodules of $\mathcal{A}$ that are not finitely generated. As an example, if $\mathcal{A} = C([0,1])$, then $C_0([0,1])$ is a Hilbert submodule of $\mathcal{A}$ that is not finitely generated. Similarly, if $\mathcal{A} = B(H)$ where H is a Hilbert space, then the closed ideal of compact operators on H is an example of a Hilbert submodule that is not finitely generated. Let P denote the orthogonal projection onto $L_1^\perp$. Then $P \in \mathcal{M}\Phi(H_{\mathcal{A}})$ and $\ker P = L_1$. It follows that $\ker P$ contains a Hilbert submodule that is not finitely generated in the case when $\mathcal{A} = C([0,1])$ or when $\mathcal{A} = B(H)$. Compared to [25, Chapter XI, Theorem 2.3], this illustrates that $\mathcal{A}$-Fredholm operators may behave differently from the classical Fredholm operators on Hilbert spaces.

In chapter 6 we shall give some examples of $\mathcal{A}$-Fredholm operators with non-closed image, which once again illustrates the difference between classical Fredholm operators and $\mathcal{A}$-Fredholm operators in general.

3.3 Openness of the set of semi-C^*-Fredholm operators

In this section we prove that the set of proper semi $\mathcal{A}$-Fredholm operators is open in the norm topology, as an analogue of the result in [46]. Also, we derive some consequences. The results in this section generalize the results from [56, Section 1.6].

Recall that $\mathcal{M}\Phi(H_{\mathcal{A}})$ is open in the norm topology by Lemma 2.0.42.

Theorem 3.3.1. *[18, Theorem 4.1] The sets $\mathcal{M}\Phi_+(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$, $\mathcal{M}\Phi_-(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$ are open in $B^a(H_{\mathcal{A}})$, where $B^a(H_{\mathcal{A}})$ is equipped with the norm topology.*

Proof. Let $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$. Then there exists a decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

with respect to which F has the matrix

$$\left[\begin{array}{cc} F_1 & 0 \\ 0 & F_4 \end{array}\right],$$

where F_1 is an isomorphism, N_1 is closed, finitely generated, and N_2 is closed, but not finitely generated. If $D \in B^a(H_{\mathcal{A}})$ such that $\| D \| < \epsilon$, then for ϵ small enough we may (by the same arguments as in the proof of Lemma 2.0.42) find isomorphisms U_1, U_2 such that $F + D$ has the matrix

$$\left[\begin{array}{cc} (F+D)_1 & 0 \\ 0 & (F+D)_4 \end{array}\right]$$

with respect to the decomposition

$$H_{\mathcal{A}} = U_1(M_1)\tilde{\oplus}U_1(N_1) \xrightarrow{F+D} U_2^{-1}(M_2)\tilde{\oplus}U_2^{-1}(N_2) = H_{\mathcal{A}},$$

where $(F+D)_1$ is an isomorphism. Since U_2 is an isomorphism and N_2 is not finitely generated, it follows that $U_2^{-1}(N_2)$ is not finitely generated. Now, as $F + D$ has the matrix

$$\left[\begin{array}{cc} (F+D)_1 & 0 \\ 0 & (F+D)_4 \end{array}\right]$$

with respect to the above decomposition, where$(F + D)_1$ is an isomorphism, $U_1(N_1)$ is finitely generated whereas $U_2^{-1}(N_2)$ is not finitely generated, it follows by Corollary 3.1.12 that

$$(F + D) \in \mathcal{M}\Phi_+(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$$

(because, by Corollary 3.1.12, if $F + D$ was $\mathcal{A}$-Fredholm, then $U_2^{-1}(N_2)$ would be finitely generated, which is a contradiction). The first part of the theorem follows, whereas the second part can be proved in the analogue way or can be deduced directly from the first part by passing to the adjoints and using Corollary 3.1.20. □

Remark 3.3.2. We recall from Theorem 3.1.2 and Theorem 3.1.4 that the sets

$$\mathcal{M}\Phi_+(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}}) \text{ and } \mathcal{M}\Phi_-(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$$

can be identified with the set of left invertible, but not invertible elements and with the set of right invertible, but not invertible elements in the Calkin algebra $B^a(H_{\mathcal{A}})\ /\ \mathcal{K}^*(H_{\mathcal{A}})$, respectively.

More precisely,

$$\mathcal{M}\Phi_+(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}}) = \pi^{-1}(G_l(B^a(H_{\mathcal{A}})\ /\ \mathcal{K}^*(H_{\mathcal{A}})) \setminus G(B^a(H_{\mathcal{A}})\ /\ \mathcal{K}^*(H_{\mathcal{A}})))$$

and

$$\mathcal{M}\Phi_-(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}}) = \pi^{-1}(G_r(B^a(H_{\mathcal{A}})\ /\ \mathcal{K}^*(H_{\mathcal{A}})) \setminus G(B^a(H_{\mathcal{A}})\ /\ \mathcal{K}^*(H_{\mathcal{A}}))),$$

where $\pi : B^a(H_{\mathcal{A}}) \to B^a(H_{\mathcal{A}})\ /\ \mathcal{K}^*(H_{\mathcal{A}})$ denotes the quotient map and G_l, G_r and G denotes the sets of left invertible , right invertible and invertible elements, respectively. Recalling that $G_l \setminus G$ and $G_r \setminus G$ are open in Banach algebras, and that π is continuous, we can deduce Theorem 3.3.1 also by these arguments. However, our proof of Theorem 3.3.1 can be applied to arbitrary Hilbert C^*-modules and not just $H_{\mathcal{A}}$, so Theorem 3.3.1 holds also in the case of arbitrary Hilbert C^*-modules.

Corollary 3.3.3. *[18, Corollary 4.2] If $F \in B^a(H_{\mathcal{A}})$ belongs to the boundary of $\mathcal{M}\Phi(H_{\mathcal{A}})$ in $B^a(H_{\mathcal{A}})$, then $F \notin \mathcal{M}\Phi_{\pm}(H_{\mathcal{A}})$.*

Proof. The statement follows by the same arguments as in the proof of [56, Corollary 1.6.10] since

$$\mathcal{M}\Phi_{\pm}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}}) = (\mathcal{M}\Phi_{+}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})) \cup (\mathcal{M}\Phi_{-}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}}))$$

is open in $B^a(H_{\mathcal{A}})$ by Theorem 3.3.1. □

Remark 3.3.4. By exactly the same arguments as in the proof of Lemma 2.0.42 it can be shown that $\mathcal{M}\Phi_{+}(M)$ and $\mathcal{M}\Phi_{-}(M)$ are open (where M is a Hilbert C^*-module).
Next recall that $\mathcal{M}\Phi(M) = \mathcal{M}\Phi_{+}(M) \cap \mathcal{M}\Phi_{-}(M)$ by Corollary 3.1.10. It follows that

$$\mathcal{M}\Phi_{\pm}(M) \setminus \mathcal{M}\Phi_{-}(M) = \mathcal{M}\Phi_{+}(M) \setminus \mathcal{M}\Phi(M),$$

$$\mathcal{M}\Phi_{\pm}(M) \setminus \mathcal{M}\Phi_{+}(M) = \mathcal{M}\Phi_{-}(M) \setminus \mathcal{M}\Phi(M),$$

which are both open by Theorem 3.3.1, hence we can in a similar way as in the proof of Corollary 3.3.3 deduce that

$$\partial\mathcal{M}\Phi_{+}(M) \cap \mathcal{M}\Phi_{\pm}(M) = \emptyset \text{ and } \partial\mathcal{M}\Phi_{-}(M) \cap \mathcal{M}\Phi_{\pm}(M) = \emptyset.$$

Corollary 3.3.5. *[18, Corollary 4.3] Let $f : [0,1] \to B^a(H_{\mathcal{A}})$ be continuous and assume that $f([0,1]) \subseteq \mathcal{M}\Phi_{\pm}(H_{\mathcal{A}})$. Then the following statements hold.*
1) If $f(0) \in \mathcal{M}\Phi_{+}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$, then $f(1) \in \mathcal{M}\Phi_{+}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$.
2) If $f(0) \in \mathcal{M}\Phi_{-}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$, then $f(1) \in \mathcal{M}\Phi_{-}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$.
3) If $f(0) \in \mathcal{M}\Phi(H_{\mathcal{A}})$, then $f(1) \in \mathcal{M}\Phi(H_{\mathcal{A}})$ and $\operatorname{index} f(0) = \operatorname{index} f(1)$.

Proof. We have that $\mathcal{M}\Phi_{\pm}(H_{\mathcal{A}})$ is a disjoint union of $\mathcal{M}\Phi_{+}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$, $\mathcal{M}\Phi(H_{\mathcal{A}})$ and $\mathcal{M}\Phi_{-}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$. The first two sets are open by Theorem 3.3.1, whereas $\mathcal{M}\Phi(H_{\mathcal{A}})$ is open by Lemma 2.0.42. By assumption in the corollary, we have that $f([0,1]) \subseteq \mathcal{M}\Phi_{\pm}(H_{\mathcal{A}})$. Since f is continuous by assumption, $f([0,1])$ must be connected in $B^a(H_{\mathcal{A}})$, hence $f([0,1])$ must be completely contained in one of these three sets $\mathcal{M}\Phi_{+}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}}), \mathcal{M}\Phi_{-}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$ or $\mathcal{M}\Phi(H_{\mathcal{A}})$(otherwise we would get a separation of $f([0,1])$ which is impossible). Thus $1), 2)$ and the first part of 3) follows.
For the second part of 3), use the additional fact that the index is locally constant on $\mathcal{M}\Phi(H_{\mathcal{A}})$ by Lemma 2.0.42 . Again, since $f([0,1])$ is connected and $f(0) \in \mathcal{M}\Phi(H_{\mathcal{A}})$ by assumption, it follows that $f([0,1]) \subseteq \mathcal{M}\Phi(H_{\mathcal{A}})$ and $\operatorname{index} f(t) = \operatorname{index} f(0)$ for all $t \in [0,1]$. □

Recall Definition 2.0.41 from Preliminaries. The next lemma is a generalization of [56, Theorem 1.6.8].

Lemma 3.3.6. *[19, Lemma 3.22] Let $F \in \mathcal{M}\Phi(M)$ be such that ImF is closed, where M is a Hilbert C^*-module. Then there exists an $\epsilon > 0$ such that for every $D \in B^a(M)$ with $\parallel D \parallel < \epsilon$, we have*

$$\ker(F+D) \preceq \ker F \text{ and } Im(F+D)^{\perp} \preceq ImF^{\perp}.$$

Proof. Since $F \in \mathcal{M}\Phi(M)$ has closed image, by Theorem 2.0.20 F has the matrix $\begin{bmatrix} F_1 & 0 \\ 0 & 0 \end{bmatrix}$ with respect to the decomposition

$$M = \ker F^{\perp} \tilde{\oplus} \ker F \xrightarrow{F} ImF \tilde{\oplus} ImF^{\perp} = M,$$

where F_1 is an isomorphism. By the proof of Lemma 2.0.42, there exists an $\epsilon > 0$ such that if $\parallel F - \tilde{D} \parallel < \epsilon$ for some $\tilde{D} \in B^a(M)$, then $\tilde{D}$ has the matrix

$$\begin{bmatrix} \tilde{D}_1 & 0 \\ 0 & \tilde{D}_4 \end{bmatrix}$$

with respect to the decomposition

$$M = U_1(\ker F^{\perp})\tilde{\oplus}U_1(\ker F) \xrightarrow{\tilde{D}} U_2^{-1}(ImF)\tilde{\oplus}U_2^{-1}(ImF^{\perp}) = M,$$

where U_1, U_2 and $\tilde{D}_1$ are isomorphisms. Then, by Lemma 3.1.3 it follows that

$$\ker \tilde{D} \subseteq U_1(\ker F) \cong \ker F.$$

Set $D = \tilde{D} - F$, then $\tilde{D} = F + D$. Hence $\ker(F + D) \preceq \ker F$.
Next, by the proof of Theorem 2.0.20, ImF^* is closed if ImF is closed. Hence, by the same arguments as above, we can choose $\epsilon > 0$ sufficiently small such that if $\| D^* \| < \epsilon$, then it holds that $\ker(F^* + D^*) \preccurlyeq \ker F^*$. However, we have

$$\ker(F^* + D^*) = Im(F + D)^{\perp} \ , \ \ker F^* = ImF^{\perp} \text{ and } \ \| D \| = \| D^* \| .$$

Therefore, it suffices to choose a sufficiently small $\epsilon > 0$ such that if $\| D \| = \| D^* \| < \epsilon$, then

$$\ker(F + D) \preccurlyeq \ker F \text{ and } \ker(F^* + D^*) \preccurlyeq \ker F^*.$$

□

3.4 Adjointable semi-C^*-Weyl operators

In this section we construct certain classes of operators on $H_{\mathcal{A}}$ as a generalization of upper and lower semi-Weyl operators on Hilbert spaces. Then we investigate and prove several properties concerning these new classes of operators. The results in this section generalize the results from [56, Section 1.9].

Definition 3.4.1. [18, Definition 5.1] Let $F \in \mathcal{M}\Phi(H_{\mathcal{A}})$. We say that $F \in \tilde{\mathcal{M}}\Phi_+^-(H_{\mathcal{A}})$ if there exists a decomposition

$$H_{\mathcal{A}} = M_1\tilde{\oplus}N_1 \xrightarrow{F} M_2\tilde{\oplus}N_2 = H_{\mathcal{A}}$$

with respect to which F has the matrix

$$\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix},$$

where F_1 is an isomorphism, N_1, N_2 are closed, finitely generated and $N_1 \preceq N_2$, that is N_1 is isomorphic to a closed submodule of N_2. We define similarly the class $\tilde{\mathcal{M}}\Phi_-^+(H_{\mathcal{A}})$, the only difference in this case is that $N_2 \preceq N_1$. Then we set

$$\mathcal{M}\Phi_+^-(H_{\mathcal{A}}) = (\tilde{\mathcal{M}}\Phi_+^-(H_{\mathcal{A}})) \cup (\mathcal{M}\Phi_+(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}}))$$

and

$$\mathcal{M}\Phi_-^+(H_{\mathcal{A}}) = (\tilde{\mathcal{M}}\Phi_-^+(H_{\mathcal{A}})) \cup (\mathcal{M}\Phi_-(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})).$$

Further, we define $\mathcal{M}\Phi_0(H_{\mathcal{A}})$ to be the set of all $F \in \mathcal{M}\Phi(H_{\mathcal{A}})$ for which there exists an $\mathcal{M}\Phi$-decomposition

$$H_{\mathcal{A}} = M_1\tilde{\oplus}N_1 \xrightarrow{F} M_2\tilde{\oplus}N_2 = H_{\mathcal{A}},$$

where $N_1 \cong N_2$.

Remark 3.4.2. Notice that Definition 3.4.1 can be extended to the case when $F \in B^a(M,N)$ and M, N are two arbitrary Hilbert C^*-modules.

Lemma 3.4.3. *[18, Lemma 5.2] Suppose that $K_0(\mathcal{A})$ satisfies the cancellation property. If $F \in \tilde{\mathcal{M}}\Phi_+^-(H_{\mathcal{A}})$, then for any decomposition*

$$H_{\mathcal{A}} = M_1' \tilde{\oplus} N_1' \xrightarrow{F} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}}$$

with respect to which F has the matrix

$$\begin{bmatrix} F_1' & 0 \\ 0 & F_4' \end{bmatrix},$$

where F_1' is an isomorphism and N_1', N_2' are finitely generated, we have $N_1' \preceq N_2'$. Similarly, $N_2' \preceq N_1'$ if $F \in \tilde{\mathcal{M}}\Phi_-^+(H_{\mathcal{A}})$.

Proof. Given $F \in \tilde{\mathcal{M}}\Phi_+^-(H_{\mathcal{A}})$, choose a decomposition for F

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

as described in Definition 3.4.1. Then $N_1 \cong N_{2,1} \preceq N_2$ for some closed submodule $N_{2,1}$ of N_2. Since N_1 is finitely generated, so is $N_{2,1}$; therefore, $N_{2,1}$ is orthogonally complementable in N_2 by Lemma 2.0.25. So $N_2 = N_{2,1} \oplus N_{2,2}$ for some closed submodule $N_{2,2}$ of N_2.
Hence

$$\text{index } F = [N_1] - [N_2] = [N_{2,1}] - [N_{2,1}] - [N_{2,2}] = -[N_{2,2}].$$

If $H_{\mathcal{A}} = M_1' \tilde{\oplus} N_1' \xrightarrow{F} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}}$ is any other $\mathcal{M}\Phi$-decomposition for F, then we must have

$$\text{index } F = [N_1'] - [N_2'] = -[N_{2,2}].$$

Taking the inverses on the both sides of the equality in $K_0(\mathcal{A})$, we get

$$[N_2'] - [N_1'] = [N_{2,2}],$$

so

$$[N_2'] = [N_1'] + [N_{2,2}].$$

Since

$$[N_1'] + [N_{2,2}] = [N_1' \oplus N_{2,2}] = [N_2'],$$

it follows that

$$(N_1' \oplus N_{2,2}) \cong N_2'$$

as $K_0(\mathcal{A})$ satisfies the cancellation property.
Let $\tilde{\iota} : N_1' \oplus N_{2,2} \longrightarrow N_2'$ be an isomorphism, then, since $N_1' \oplus \{0\}$ is a closed submodule of the module $N_1' \oplus N_{2,2}$, it follows that $\tilde{\iota}(N_1' \oplus \{0\})$ is a closed submodule of N_2'. Thus we get $(N_1' \oplus \{0\}) \preceq N_2'$. However, $N_1' \oplus \{0\} \cong N_1'$, so $N_1' \preceq N_2'$. One treats analogously the case when $F \in \tilde{\mathcal{M}}\Phi_-^+(H_{\mathcal{A}})$. □

Proposition 3.4.4. *Let $K \in \mathcal{K}^*(H_{\mathcal{A}})$ and $T \in B^a(H_{\mathcal{A}})$. Suppose that T is invertible and that $K_0(\mathcal{A})$ satisfies the cancellation property. Then the equation $(T+K)x = y$ has a solution for every $y \in H_{\mathcal{A}}$ if and only if $T+K$ is bounded below. In this case the solution of the above equation is unique.*

Proof. Since T is invertible, by Lemma 2.0.45 it follows that index $(T+K)=0$. Now, if the equation $(T+K)x=y$ has a solution for each $y\in H_{\mathcal{A}}$, this simply means that $T+K$ is surjective. Then, by Theorem 2.0.20, $\ker(T+K)$ is orthogonally complementable in $H_{\mathcal{A}}$. Therefore, by Lemma 3.1.21 we have that

$$H_{\mathcal{A}}=\ker(T+K)^{\perp}\oplus\ker(T+K)\xrightarrow{T+K}H_{\mathcal{A}}\oplus\{0\}=H_{\mathcal{A}}$$

is also an $\mathcal{M}\Phi$-decomposition for $T+K$ and, thus, index $(T+K)=[\ker(T+K)]$. However, index $(T+K)=0$. Since $K_0(\mathcal{A})$ satisfies the cancellation property by assumption, it follows that $\ker(T+K)=\{0\}$, so $T+K$ is invertible, thus bounded below.
Conversely, if $T+K$ bounded below, then, by Theorem 2.0.20, $Im(T+K)$ is orthogonally complementable in $H_{\mathcal{A}}$. Thus, again by Lemma 3.1.21 we have that

$$H_{\mathcal{A}}\oplus\{0\}\xrightarrow{T+K}Im(T+K)\oplus Im(T+K)^{\perp}=H_{\mathcal{A}}$$

is an $\mathcal{M}\Phi$-decomposition for $T+K$. By the same argument as above, since index $(T+K)=0$ and $K_0(\mathcal{A})$ satisfies the cancellation property, it follows that $Im(T+K)^{\perp}=\{0\}$. ☐

Example 3.4.5. Let $\mathcal{A}=B(H)$, where H is an infinite-dimensional, separable Hilbert space. If H_1 is any infinite-dimensional subspace of H, then there exists an isometric isomorphism U of H onto H_1. Set $\tilde{U}$ to be the operator on $\mathcal{A}$ given by $\tilde{U}(F)=JUF$ for all $F\in\mathcal{A}$ where J is the inclusion of H_1 into H. Then $\tilde{U}\in B^a(\mathcal{A})$ and moreover, $\tilde{U}$ is an isometry. Put T to be the operator with the matrix $\begin{bmatrix}1&0\\0&\tilde{U}\end{bmatrix}$ with respect to the decomposition

$$H_{\mathcal{A}}=L_1^{\perp}\oplus L_1\xrightarrow{T}L_1^{\perp}\oplus L_1=H_{\mathcal{A}}.$$

Then $T\in B^a(H_{\mathcal{A}})$ and T is bounded below. Moreover, $ImT^{\perp}=Span_{\mathcal{A}}\{(P,0,0,0,\dots)\}$, where P is the orthogonal projection of H onto $H_1^{\perp}$. However, $T=I+K$ where $K=\begin{bmatrix}0&0\\0&\tilde{U}-1\end{bmatrix}$ with respect to the decomposition $L_1^{\perp}\oplus L_1\to L_1^{\perp}\oplus L_1$, hence $K\in\mathcal{K}^*(H_{\mathcal{A}})$. This shows that the assumption that $K_0(\mathcal{A})$ satisfies the cancellation property in Proposition 3.4.4 is really needed.

For $a\in\mathcal{A}$ we may let αI be the operator on $H_{\mathcal{A}}$ given by

$$\alpha I(x_1,x_2,\dots)=(\alpha x_1,\alpha x_2,\dots).$$

It is straightforward to check that αI is an $\mathcal{A}$-linear operator on $H_{\mathcal{A}}$. Moreover, αI is bounded and $\parallel\alpha I\parallel=\parallel\alpha\parallel$. Finally, αI is adjointable and its adjoint is given by $(\alpha I)^*=\alpha^* I$.
We give then the following generalization of the well known Fredholm alternative stated in [28, Chapter VII, Corollary 7.10].

Corollary 3.4.6. *Let $K\in\mathcal{K}^*(H_{\mathcal{A}})$ and $\alpha\in G(\mathcal{A})$. Suppose that $K_0(\mathcal{A})$ satisfies the cancellation property. Then the equation $(K-\alpha I)\ x=y$ has a solution for every $y\in H_{\mathcal{A}}$ if and only if $K-\alpha I$ is bounded below. In this case the solution of the above equation is unique.*

Next we present the following lemma.

Lemma 3.4.7. *[18, Lemma 5.3] It holds that $\tilde{\mathcal{M}\Phi}_{+}^{-}(H_{\mathcal{A}})$ and $\tilde{\mathcal{M}\Phi}_{-}^{+}(H_{\mathcal{A}})$ are semigroups under the multiplication.*

Proof. Let $F,D\in\tilde{\mathcal{M}\Phi}_{+}^{-}(H_{\mathcal{A}})$. Then there exist decompositions

$$H_{\mathcal{A}}=M_1\tilde{\oplus}N_1\xrightarrow{F}M_2\tilde{\oplus}N_2=H_{\mathcal{A}},$$

$$H_{\mathcal{A}} = M_1'\tilde{\oplus}N_1' \xrightarrow{D} M_2'\tilde{\oplus}N_2' = H_{\mathcal{A}}$$

with respect to which F, D have matrices $\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}$, $\begin{bmatrix} D_1 & 0 \\ 0 & D_4 \end{bmatrix}$, respectively, where F_1, D_1 are isomorphisms, N_1, N_2, N_1', N_2' are finitely generated and moreover $N_1 \preceq N_2, N_1' \preceq N_2'$. By the proof of Lemma 2.0.43, with respect to the decomposition

$$H_{\mathcal{A}} = \overline{\overline{M_1}}\tilde{\oplus}\overline{\overline{N_1}} \xrightarrow{DF} \overline{\overline{M_2'}}\tilde{\oplus}\overline{\overline{N_2'}} = H_{\mathcal{A}},$$

DF has the matrix $\begin{bmatrix} (DF)_1 & 0 \\ 0 & (DF)_4 \end{bmatrix}$, where $(DF)_1$ is an isomorphism,

$$\overline{\overline{N_1}} = U(F_1^{-1}(P)\tilde{\oplus}N_1), \overline{\overline{N_2'}} = D(P')\tilde{\oplus}N_2', (P\tilde{\oplus}N_2) \cong (P\tilde{\oplus}N_1') \cong L_n$$

for some n, $D_{|_P}, F_{|_{P'}}$ and U are isomorphisms. Since N_1 is isomorphic to a closed submodule of N_2 and $F_1^{-1}(P) \cong P$, it follows that $F_1^{-1}(P) \oplus N_1$ is isomorphic to a closed submodule of $P \oplus N_2$. However, since there are natural isomorphisms between $((F_1^{-1}(P)\tilde{\oplus}N_1))$ and $((F_1^{-1}(P) \oplus N_1))$, between $(P\tilde{\oplus}N_2)$ and $(P \oplus N_2)$, it follows that $F_1^{-1}(P)\tilde{\oplus}N_1$ is isomorphic to a closed submodule of $(P\tilde{\oplus}N_2)$. As U is an isomorphism, it follows that $\overline{\overline{N_1}} = U(F_1^{-1}(P)\tilde{\oplus}N_1)$ is isomorphic to a closed submodule of $P\tilde{\oplus}N_2$. Now, $P\tilde{\oplus}N_2$ is isomorphic to $P'\tilde{\oplus}N_1'$, so $\overline{\overline{N_1}}$ is isomorphic to a closed submodule of $P'\tilde{\oplus}N_1'$. Next, using that $P' \cong D(P')$ and that N_1' is isomorphic to a closed submodule of N_2', by the same arguments as above (considering direct sums of modules), we can deduce that $(P'\tilde{\oplus}N_1')$ is isomorphic to a closed submodule of $(D(P')\tilde{\oplus}N_2') = \overline{\overline{N_2'}}$, so $\overline{\overline{N_1}} \preceq (P'\tilde{\oplus}N_1') \preceq \overline{\overline{N_2'}}$. Thus, $DF \in \tilde{\mathcal{M}\Phi}_{-}^{+}(H_{\mathcal{A}})$. Similarly one can show that $\tilde{\mathcal{M}\Phi}_{-}^{+}(H_{\mathcal{A}})$ is a semigroup. □

Lemma 3.4.8. *[18, Lemma 5.4] It holds that $\mathcal{M}\Phi_{+}^{-}(H_{\mathcal{A}})$ and $\mathcal{M}\Phi_{-}^{+}(H_{\mathcal{A}})$ are semigroups under the multiplication.*

Proof. Let $F, D \in \mathcal{M}\Phi_{+}^{-}(H_{\mathcal{A}})$. We consider four possible cases.

1) If $F, D \in \tilde{\mathcal{M}\Phi}_{+}^{-}(H_{\mathcal{A}})$, by Lemma 3.4.7 it follows that $DF \in \tilde{\mathcal{M}\Phi}_{+}^{-}(H_{\mathcal{A}})$.

2) If $D, F \in \mathcal{M}\Phi_{+}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$, then $DF \in \mathcal{M}\Phi_{+}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$ by Corollary 3.1.19.

3) If $F \in \mathcal{M}\Phi_{+}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$ and $D \in \tilde{\mathcal{M}\Phi}_{+}^{-}(H_{\mathcal{A}})$, then in particular $D \in \mathcal{M}\Phi(H_{\mathcal{A}})$ as $\tilde{\mathcal{M}\Phi}_{+}^{-}(H_{\mathcal{A}}) \subseteq \mathcal{M}\Phi(H_{\mathcal{A}})$ by definition. By Corollary 3.1.18, it follows that DF can <u>not</u> be in $\mathcal{M}\Phi(H_{\mathcal{A}})$ as $F \notin \mathcal{M}\Phi(H_{\mathcal{A}})$. Now, by definition, $\mathcal{M}\Phi_{+}^{-}(H_{\mathcal{A}}) \subset \mathcal{M}\Phi_{+}(H_{\mathcal{A}})$, so then $F, D \in \mathcal{M}\Phi_{+}(H_{\mathcal{A}})$. Hence, by Corollary 3.1.14 we have that $DF \in \mathcal{M}\Phi_{+}(H_{\mathcal{A}})$ which gives that $DF \in \mathcal{M}\Phi_{+}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$.

4) If $D \in \mathcal{M}\Phi_{+}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$, it is clear that DF can not be an element of $\mathcal{M}\Phi(H_{\mathcal{A}})$. Indeed, if $DF \in \mathcal{M}\Phi(H_{\mathcal{A}})$, then by Corollary 3.1.15 we would get that $D \in \mathcal{M}\Phi_{-}(H_{\mathcal{A}})$ since $\mathcal{M}\Phi(H_{\mathcal{A}})) \subseteq \mathcal{M}\Phi_{-}(H_{\mathcal{A}})$. Hence $D \in \mathcal{M}\Phi_{-}(H_{\mathcal{A}}) \cap \mathcal{M}\Phi_{+}(H_{\mathcal{A}}))$ which is a contradiction as $\mathcal{M}\Phi_{-}(H_{\mathcal{A}}) \cap \mathcal{M}\Phi_{+}(H_{\mathcal{A}}) = \mathcal{M}\Phi(H_{\mathcal{A}})$ by Corollary 3.1.10. Again, since $DF \in \mathcal{M}\Phi_{+}(H_{\mathcal{A}})$ by Corollary 3.1.14. it follows that $DF \in \mathcal{M}\Phi_{+}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$.

Collecting all these arguments together, we deduce that $\mathcal{M}\Phi_{+}^{-}(H_{\mathcal{A}})$ is a semigroup under the multiplication. Similarly one can show that $\mathcal{M}\Phi_{-}^{+}(H_{\mathcal{A}})$ is a semigroup under the multiplication. □

Lemma 3.4.9. *[18, Lemma 5.5] It holds that $\tilde{\mathcal{M}\Phi}_{+}^{-}(H_{\mathcal{A}})$ and $\tilde{\mathcal{M}\Phi}_{-}^{+}(H_{\mathcal{A}})$ are open.*

Proof. Given $F \in \tilde{\mathcal{M}\Phi}_{+}^{-}(H_{\mathcal{A}})$, let

$$H_{\mathcal{A}} = M_1\tilde{\oplus}N_1 \xrightarrow{F} M_2\tilde{\oplus}N_2 = H_{\mathcal{A}}$$

be a decomposition with respect to which

$$F = \begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix},$$

where F_1 is an isomorphism, N_1, N_2 are finitely generated and $N_1 \preceq N_2$. By the proof of Lemma 2.0.42, there exists an $\epsilon > 0$ such that if $\parallel F - D \parallel < \epsilon$, then there exists a decomposition

$$H_{\mathcal{A}} = M_1' \tilde{\oplus} N_1' \xrightarrow{D} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}}$$

with respect to which

$$D = \begin{bmatrix} D_1 & 0 \\ 0 & D_4 \end{bmatrix},$$

where D_1 is an isomorphism, and moreover,

$$M_1 \cong M_1', N_1 \cong N_1', M_2 \cong M_2' \text{ and } N_2 \cong N_2'.$$

Let

$$U_1 : N_1' \to N_1, U_2 : N_2 \to N_2'$$

be these isomorphisms. Since $N_1 \preceq N_2$, there exists an isomorphism $\tilde{\iota}$ from N_1 onto some closed submodule $\tilde{\iota}(N_1) \subseteq N_2$. Then $U_2 \tilde{\iota} U_1$ is an isomorphism from N_1' onto $(U_2 \tilde{\iota} U_1)(N_1)$ which is a closed submodule of N_2'. Thus, $N_1' \preceq N_2'$ (and also N_1', N_2' are finitely generated as N_1, N_2 are so). Therefore, $D \in \tilde{\mathcal{M}\Phi}_+^-(H_{\mathcal{A}})$.

Similarly, we can show that $\tilde{\mathcal{M}\Phi}_-^+(H_{\mathcal{A}})$ is open. □

Corollary 3.4.10. *The sets $\mathcal{M}\Phi_+^-(H_{\mathcal{A}})$ and $\mathcal{M}\Phi_-^+(H_{\mathcal{A}})$ are open in the norm topology of $B^a(H_{\mathcal{A}})$.*

Proof. Combine Theorem 3.3.1 and Lemma 3.4.9. □

Definition 3.4.11. [18, Definition 5.6] Let $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$. We say that $F \in \mathcal{M}\Phi_+^{-'}(H_{\mathcal{A}})$ if there exists a decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

with respect to which

$$F = \begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix},$$

where F_1 is an isomorphism, N_1 is closed, finitely generated and $N_1 \preceq N_2$. Similarly, we define the class $\mathcal{M}\Phi_-^{+'}(H_{\mathcal{A}})$, only in this case $F \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$, N_2 is finitely generated and $N_2 \preceq N_1$. Such operators will be called semi-$\mathcal{A}$-Weyl operators throughout the thesis.

Proposition 3.4.12. *[18, Proposition 5.7] We have*

$$\tilde{\mathcal{M}\Phi}_+^-(H_{\mathcal{A}}) = \mathcal{M}\Phi_+^{-'}(H_{\mathcal{A}}) \cap \mathcal{M}\Phi(H_{\mathcal{A}}) \text{ and } \tilde{\mathcal{M}\Phi}_-^+(H_{\mathcal{A}}) = \mathcal{M}\Phi_-^{+'}(H_{\mathcal{A}}) \cap \mathcal{M}\Phi(H_{\mathcal{A}}).$$

Proof. By the definition of the class $\tilde{\mathcal{M}\Phi}_+^-(H_{\mathcal{A}})$, the inclusion " $\subseteq$ " is obvious. Let us show the other inclusion. To this end, choose some $D \in \mathcal{M}\Phi_+^{-'}(H_{\mathcal{A}}) \cap \mathcal{M}\Phi(H_{\mathcal{A}})$. Since $D \in \mathcal{M}\Phi_+^{-'}(H_{\mathcal{A}})$, there exists a decomposition

$$H_{\mathcal{A}} = M_1' \tilde{\oplus} N_1' \xrightarrow{D} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}}$$

with respect to which D has the matrix $\begin{bmatrix} D_1 & 0 \\ 0 & D_4 \end{bmatrix}$, where D_1 is an isomorphism, N_1' is finitely generated and $N_1' \preceq N_2'$. On the other hand, since $D \in \mathcal{M}\Phi(H_{\mathcal{A}})$, by Corollary 3.1.12, N_2' must be then finitely generated. Hence $D \in \tilde{\mathcal{M}\Phi}_+^{-'}(H_\mathcal{A})$. Similarly, using Corollary 3.1.12, one can show that

$$\tilde{\mathcal{M}\Phi}_-^+(H_\mathcal{A}) = \mathcal{M}\Phi_-^{+'}(H_\mathcal{A}) \cap \mathcal{M}\Phi(H_\mathcal{A}).$$

□

Remark 3.4.13. [18, Remark 5.8] Notice that by Proposition 3.4.12 we get

$$\mathcal{M}\Phi_+^{-'}(H_\mathcal{A}) = (\mathcal{M}\Phi_+^{-'}(H_\mathcal{A}) \cap \mathcal{M}\Phi(H_\mathcal{A})) \cup (\mathcal{M}\Phi_+^{-'}(H_\mathcal{A}) \setminus \mathcal{M}\Phi(H_\mathcal{A}))$$

$$= \tilde{\mathcal{M}\Phi}_+^-(H_\mathcal{A}) \cup (\mathcal{M}\Phi_+^{-'}(H_\mathcal{A}) \setminus \mathcal{M}\Phi(H_\mathcal{A}))$$

$$\subseteq \tilde{\mathcal{M}\Phi}_+^-(H_\mathcal{A}) \cup (\mathcal{M}\Phi_+(H_\mathcal{A}) \setminus \mathcal{M}\Phi(H_\mathcal{A})) = \mathcal{M}\Phi_+^-(H_\mathcal{A}).$$

Similarly, we obtain that $\mathcal{M}\Phi_-^{+'}(H_\mathcal{A}) \subseteq \mathcal{M}\Phi_-^+(H_\mathcal{A})$.

On Hilbert spaces "=" holds due to that given any finite dimensional subspace N_1 and infinite-dimensional subspace N_2, then N_1 is isomorphic to a closed subspace of N_2. Observe also that Proposition 3.4.12 holds in the case of arbitrary Hilbert C^*-modules and not just $H_\mathcal{A}$.

Lemma 3.4.14. *[18, Lemma 5.9] The sets $\mathcal{M}\Phi_-^{+'}(H_\mathcal{A})$ and $\mathcal{M}\Phi_+^{-'}(H_\mathcal{A})$ are open. Moreover, if $F \in \mathcal{M}\Phi_+^{-'}(H_\mathcal{A})$ and $K \in \mathcal{K}^*(H_\mathcal{A})$, then*

$$(F+K) \in \mathcal{M}\Phi_+^{-'}(H_\mathcal{A}).$$

If $F \in \mathcal{M}\Phi_-^{+'}(H_\mathcal{A})$ and $K \in \mathcal{K}^*(H_\mathcal{A})$, then

$$(F+K) \in \mathcal{M}\Phi_-^{+'}(H_\mathcal{A}).$$

Proof. Suppose $F \in \mathcal{M}\Phi_+^{-'}(H_\mathcal{A})$ and choose a decomposition

$$H_\mathcal{A} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_\mathcal{A}$$

such that $N_1 \preceq N_2$ as described in the Definition 3.4.11. Then, again by the proof of Lemma 2.0.42, there exists an $\epsilon > 0$ such that if $\| F - D \| < \epsilon$, then there exists a decomposition

$$H_\mathcal{A} = M_1' \tilde{\oplus} N_1' \xrightarrow{D} M_2' \tilde{\oplus} N_2' = H_\mathcal{A},$$

with respect to which D has the matrix

$$\begin{bmatrix} D_1 & 0 \\ 0 & D_4 \end{bmatrix},$$

where D_1 is an isomorphism and $N_1' \cong N_1, N_2' \cong N_2$. Therefore, by the same arguments as in the proof of Lemma 3.4.9, we have $N_1' \preceq N_2'$ since $N_1 \preceq N_2$. Thus, D is in $\mathcal{M}\Phi_+^-(H_\mathcal{A})'$ also, so $\mathcal{M}\Phi_+^-(H_\mathcal{A})'$ is open.

Next, let $K \in \mathcal{K}^*(H_\mathcal{A})$. By the proof of Lemma 2.0.45 we may without loss of generality assume that there exists an $n \in \mathbb{N}$ such that $F+K$ has the matrix

$$\begin{bmatrix} (F+K)_1 & 0 \\ 0 & (F+K)_4 \end{bmatrix}$$

with respect to the decomposition

$$H_\mathcal{A} = U_1'(L_n^\perp) \tilde{\oplus} U_1'(P \tilde{\oplus} N_1) \xrightarrow{F+K} U_2'^{-1} F(L_n^\perp) \tilde{\oplus} U_2'^{-1}(F(P) \tilde{\oplus} N_2) = H_\mathcal{A},$$

where $(F+K)_1, U_1', U_2'$ are isomorphisms, $L_n = N_1 \tilde{\oplus} P$, $P = M_1 \cap L_n, P \cong F(P)$ for some closed, finitely generated submodule P (here F, N_1, N_2 are as given above). Indeed, if

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

is an $\mathcal{M}\Phi_+^{-\prime}$-decomposition for F, by Theorem 2.0.34 there exists an $n \in \mathbb{N}$ such that we have $H_{\mathcal{A}} = L_n^{\perp} \tilde{\oplus} P \tilde{\oplus} N_1$ for some finitely generated Hilbert submodule P. Since it holds that $\sqcap_{M_2} F_{|_{L_n^{\perp} \tilde{\oplus} P}} = F \sqcap_{M_1 |_{L_n^{\perp} \tilde{\oplus} P}}$ where $\sqcap_{M_1}$ and $\sqcap_{M_2}$ stand for the projections onto M_1 along N_1 and onto M_2 along N_2, respectively, it follows easily that F has the matrix $\begin{bmatrix} F_1 & 0 \\ F_3 & F_4 \end{bmatrix}$ with respect to the decomposition

$$H_{\mathcal{A}} = (L_n^{\perp} \tilde{\oplus} P) \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

where F_1 is an isomorphism, so there exists an isomorphism V such that

$$H_{\mathcal{A}} = (L_n^{\perp} \tilde{\oplus} P) \tilde{\oplus} N_1 \xrightarrow{F} V(M_2) \tilde{\oplus} V(N_2) = H_{\mathcal{A}}$$

is an $\mathcal{M}\Phi_+$-decomposition for F. Then we have $N_1 \preceq N_2 \cong V(N_2)$, so this is actually an $\mathcal{M}\Phi_+^{-\prime}$-decomposition for F. Hence we can proceed in the same way as in the proof of Lemma 2.0.45 to obtain the decomposition given above for the operator $F+K$. Now, since N_1 is isomorphic to a closed submodule of N_2, then clearly $P \tilde{\oplus} N_1$ is isomorphic to a closed submodule of $F(P) \tilde{\oplus} N_2$ as $P \cong F(P)$. Therefore, $(P \tilde{\oplus} N_1) \preceq (F(P) \tilde{\oplus} N_2)$. Since U_1', U_2' are isomorphisms, then $U_1'(P \tilde{\oplus} N_1) \preceq U_2'^{-1}(F(P) \tilde{\oplus} N_2)$, so $(F+K) \in \mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}})$. Similarly one proves the statements for $\mathcal{M}\Phi_-^{+\prime}(H_{\mathcal{A}})$. □

Remark 3.4.15. Lemma 3.4.8 follows also from Proposition 3.4.12 and the first statement in Lemma 3.4.14.

All the results about the classes $\tilde{\mathcal{M}}\Phi_+^-(H_{\mathcal{A}}), \tilde{\mathcal{M}}\Phi_-^+(H_{\mathcal{A}}), \mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}}), \mathcal{M}\Phi_-^{+\prime}(H_{\mathcal{A}})$ such as Lemma 3.4.7, Lemma 3.4.8, Lemma 3.4.9 and Lemma 3.4.14 are also valid for the class $\mathcal{M}\Phi_0(H_{\mathcal{A}})$ and can be proved in a similar way.

Lemma 3.4.16. *The sets* $\mathcal{M}\Phi_+(H_{\mathcal{A}}) \backslash \mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}})$, $\mathcal{M}\Phi_-(H_{\mathcal{A}}) \backslash \mathcal{M}\Phi_-^{+\prime}(H_{\mathcal{A}})$, $\mathcal{M}\Phi(H_{\mathcal{A}}) \backslash \mathcal{M}\Phi_0(H_{\mathcal{A}})$ *are open.*

Proof. Let $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}})$ and

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

be an $\mathcal{M}\Phi_+$-decomposition for F. By the proof of Lemma 2.0.42 there exists an $\epsilon > 0$ such that if $\parallel F - D \parallel < \epsilon$, then D has an $\mathcal{M}\Phi_+-$decomposition

$$H_{\mathcal{A}} = M_1' \tilde{\oplus} N_1{}' \xrightarrow{D} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}},$$

where $M_1 \cong M_1', N_1 \cong N_1', M_2 \cong M_2'$ and $N_2 \cong N_2'$. Suppose that $D \in \mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}})$. Then there exists an $\mathcal{M}\Phi_+^{-\prime}-$decomposition for D,

$$H_{\mathcal{A}} = M_1'' \tilde{\oplus} N_1'' \xrightarrow{D} M_2'' \tilde{\oplus} N_2'' = H_{\mathcal{A}},$$

which means in particular that $N_1{}''$ is finitely generated and $N_1{}'' \preceq N_2{}''$. By the proof of Lemma 2.0.43 there exists an $n \in \mathbb{N}$ and finitely generated Hilbert submodules P', P'' such that

$$H_{\mathcal{A}} = L_n^{\perp} \tilde{\oplus} (P' \tilde{\oplus} N_1') \xrightarrow{D} D(L_n^{\perp}) \tilde{\oplus} (D(P') \tilde{\oplus} V'(N_2')) = H_{\mathcal{A}}$$

and
$$H_{\mathcal{A}} = L_n^{\perp}\tilde{\oplus}(P''\tilde{\oplus}N_1'') \xrightarrow{D} D(L_n^{\perp})\tilde{\oplus}(D(P'')\tilde{\oplus}V''(N_2'')) = H_{\mathcal{A}}$$
are two $\mathcal{M}\Phi_+-$decompositions for D, where V and V'' are isomorphisms. It follows that
$$P'\tilde{\oplus}N_1' \cong P''\tilde{\oplus}N_1'' \text{ and } D(P')\tilde{\oplus}V'(N_2') \cong D(P'')\tilde{\oplus}V''(N_2'').$$
Moreover, $M_1' \cong L_n^{\perp}\tilde{\oplus}P', M_1'' \cong L_n^{\perp}\tilde{\oplus}P'', M_2' \cong D(L_n^{\perp})\tilde{\oplus}D(P'), M_2'' \cong D(L_n^{\perp})\tilde{\oplus}D(P''), D(P') \cong P'$ and $D(P'') \cong P''$. Since $N_1'' \preceq N_2''$, we get that
$$P''\tilde{\oplus}N_1'' \preceq D(P'')\tilde{\oplus}V''(N_2'').$$

Hence we obtain that
$$P'\tilde{\oplus}N_1' \cong P''\tilde{\oplus}N_1'' \preceq D(P'')\tilde{\oplus}V''(N_2'') \cong D(P')\tilde{\oplus}V'(N_2').$$
Now, we have $M_1 \cong M_1' \cong L_n^{\perp} \oplus P'$ and $M_2 \cong M_2' \cong D(L_n^{\perp})\tilde{\oplus}D(P') \cong L_n^{\perp} \oplus P'$. Therefore, there exist isomorphisms U_1 and U_2 such that
$$M_1 = U_1(L_n^{\perp})\tilde{\oplus}U_1(P'),\ M_2 = U_2(L_n^{\perp})\tilde{\oplus}U_2(P').$$
With respect to the decomposition
$$H_{\mathcal{A}} = U_1(L_n^{\perp})\tilde{\oplus}(U_1(P')\tilde{\oplus}N_1) \xrightarrow{F} F(U_1(L_n^{\perp}))\tilde{\oplus}(F(U_1(P')))\tilde{\oplus}N_2) = H_{\mathcal{A}},$$
the operator F has the matrix $\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}$, where F_1 is an isomorphism and $F(U_1(P')) \cong P'$.

Hence, $(F(U_1(P')\tilde{\oplus}N_2)) \cong D(P')\tilde{\oplus}V'(N_2')$ since
$$F(U_1(P')) \cong P' \cong D(P') \text{ and } N_2 \cong N_2' \cong V'(N_2').$$
Moreover, $U_1(P')\tilde{\oplus}N_1 \cong P'\tilde{\oplus}N_1'$ since $N_1 \cong N_1'$ and U_1 is an isomorphism. Since we have from above that $P'\tilde{\oplus}N_1' \preceq D(P')\tilde{\oplus}V'(N_2')$, we deduce that $U_1(P')\tilde{\oplus}N_1 \preceq F(U_1(P'))\tilde{\oplus}N_2$. So
$$H_{\mathcal{A}} = U_1(L_n^{\perp})\tilde{\oplus}(U_1(P')\tilde{\oplus}N_1) \xrightarrow{F} F(U_1(L_n^{\perp}))\tilde{\oplus}(F(U_1(P'))\tilde{\oplus}N_2) = H_{\mathcal{A}}$$
is an $\mathcal{M}\Phi_+^{-\prime}$-decomposition for F. We get a contradiction since we assumed that $F \notin \mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}})$. Thus, we must have that $D \notin \mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}})$, which means that $\mathcal{M}\Phi_+(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}})$ is open. The proofs of the other statements are similar. □

Corollary 3.4.17. *Let $f : [0,1] \to B^a(H_{\mathcal{A}})$ be a continuous map such that $f([0,1]) \subseteq \mathcal{M}\Phi_{\pm}(H_{\mathcal{A}})$. Then*
1) If $f(0) \in \mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}})$, then $f(1) \in \mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}})$.
2) If $f(0) \in \mathcal{M}\Phi_+(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}})$, then $f(1) \in \mathcal{M}\Phi_+(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}})$.
3) If $f(0) \in \mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}})$, then $f(1) \in \mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}})$.
4) If $f(0) \in \mathcal{M}\Phi_-^{+\prime}(H_{\mathcal{A}})$, then $f(1) \in \mathcal{M}\Phi_-^{+\prime}(H_{\mathcal{A}})$.
5) If $f(0) \in \mathcal{M}\Phi_-(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi_-^{+\prime}(H_{\mathcal{A}})$, then $f(1) \in \mathcal{M}\Phi_-(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi_-^{+\prime}(H_{\mathcal{A}})$.
6) If $f(0) \in \mathcal{M}\Phi_0(H_{\mathcal{A}})$, then $f(1) \in \mathcal{M}\Phi_0(H_{\mathcal{A}})$.
7) If $f(0) \in \mathcal{M}\Phi(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi_0(H_{\mathcal{A}})$, then $f(1) \in \mathcal{M}\Phi(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi_0(H_{\mathcal{A}})$.

Proof. By applying Lemma 3.4.16 we can proceed in the same way as in the proof of Corollary 3.3.5. □

Theorem 3.4.18. *[18, Theorem 5.10] Let $F \in B^a(H_{\mathcal{A}})$. The following statements are equivalent:*
1) $F \in \mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}})$,
2) There exist $D \in B^a(H_{\mathcal{A}}), K \in \mathcal{K}^*(H_{\mathcal{A}})$ such that D is bounded below and $F = D + K$.

Proof. 1) $\rightarrow$ 2)
Let $F \in \mathcal{M}\Phi_+^{-\prime}(H_\mathcal{A})$ and let

$$H_\mathcal{A} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_\mathcal{A}$$

be a decomposition as given in the Definition 3.4.11, so that N_1 is finitely generated, $N_1 \preceq N_2$, and $F_{|_{M_1}}$ is an isomorphism onto M_2. Since N_1 is finitely generated, by the proof of Theorem 2.0.38 we may assume that $M_1 = N_1^\perp$. Let ι be the isomorphism from N_1 onto a closed submodule $\iota(N_1) \subseteq N_2$. Set $D = F + (\iota - F)P_{N_1}$, where P_{N_1} is the orthogonal projection onto N_1. Note that ιP_{N_1} is adjointable. Indeed, since $\iota : N_1 \rightarrow \iota(N_1) \subseteq N_2$ and N_1 is self-dual being finitely generated, then by Proposition 2.0.28, ι is adjointable. Moreover, since $\iota(N_1)$ is finitely generated being isomorphic to N_1, it follows that $\iota(N_1)$ is an orthogonal direct summand in $H_\mathcal{A}$ by Lemma 2.0.25. Hence the inclusion $J_{\iota(N_1)} : \iota(N_1) \rightarrow H_\mathcal{A}$ is adjointable. Also, P_{N_1} is adjointable, so $\iota P_{N_1} = J_{\iota(N_1)} \iota P_{N_1} \in B^a(H_\mathcal{A})$. Then $(\iota - F)P_{N_1}$ is in $K \in \mathcal{K}^*(H_\mathcal{A})$ and, in addition, $D = F + (\iota - F)P_{N_1} = FP_{M_1} + \iota P_{N_1}$. Since $F_{|_{M_1}}$ is an isomorphism from M_1 onto M_2, ι is an isomorphism from N_1 onto $\iota(N_1) \subseteq N_2$ and $H_\mathcal{A} = M_2 \tilde{\oplus} N_2$, it follows that D is bounded below as an isomorphism of $H_\mathcal{A}$ onto $M_2 \tilde{\oplus} \iota(N_1)$, which is a closed submodule of $H_\mathcal{A}$. Moreover, $F = D + (F - \iota)P_{N_1}$ and $(F - \iota)P_{N_1}$ is compact.
2) $\Rightarrow$ 1)
If $D \in B^a(H_\mathcal{A})$ is bounded below, then it follows from Theorem 2.0.20 that $D \in \mathcal{M}\Phi_+^{-\prime}(H_\mathcal{A})$. Since $K \in \mathcal{K}^*(H_\mathcal{A})$, by Lemma 3.4.14 we get that $(D + K) \in \mathcal{M}\Phi_+^{-\prime}(H_\mathcal{A})$. $\square$

Proposition 3.4.19. *[18, Proposition 5.11] We have the following:*
1)$F \in \mathcal{M}\Phi_+^{-\prime}(H_\mathcal{A}) \Leftrightarrow F^* \in \mathcal{M}\Phi_-^{+\prime}(H_\mathcal{A})$,
2) $F \in \tilde{\mathcal{M}}\Phi_+^{-}(H_\mathcal{A}) \Leftrightarrow F^* \in \tilde{\mathcal{M}}\Phi_-^{+}(H_\mathcal{A})$,
3) $F \in \mathcal{M}\Phi_+^{-}(H_\mathcal{A}) \Leftrightarrow F^* \in \mathcal{M}\Phi_-^{+}(H_\mathcal{A})$.

Proof. 1) Let $F \in \mathcal{M}\Phi_+^{-\prime}(H_\mathcal{A})$ and choose a decomposition

$$H_\mathcal{A} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_\mathcal{A}$$

with respect to which F has the matrix

$$\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix},$$

where F_1 is an isomorphism, $N_1 \preceq N_2$ and N_1 is finitely generated. Again, by the proof of Theorem 2.0.38, we may assume that $M_1 = N_1^\perp$. With respect to the decomposition

$$H_\mathcal{A} = {N_1}^\perp \oplus N_1 \xrightarrow{F} F(N_1^\perp) \oplus F(N_1^\perp)^\perp = H_\mathcal{A},$$

F has the matrix

$$\begin{bmatrix} \tilde{F}_1 & \tilde{F}_2 \\ 0 & \tilde{F}_4 \end{bmatrix},$$

where $\tilde{F}_1$ is an isomorphism and $\tilde{F}_1, \tilde{F}_2, \tilde{F}_4$ are adjointable, so

$$F^* = \begin{bmatrix} \tilde{F}_1^* & 0 \\ \tilde{F}_2^* & \tilde{F}_4^* \end{bmatrix}$$

with respect to the decomposition

$$H_\mathcal{A} = F(N_1^\perp) \oplus F(N_1^\perp)^\perp \xrightarrow{F^*} {N_1}^\perp \oplus N_1 = H_\mathcal{A}.$$

This follows from the proof of Theorem 3.1.2 part 1) implies 2). Since

$$M_2 = F(M_1) = F(N_1^{\perp}) \text{ and } H_{\mathcal{A}} = F(N_1^{\perp}) \oplus F(N_1^{\perp})^{\perp} = M_2 \tilde{\oplus} N_2,$$

we clearly have that $F(N_1^{\perp})^{\perp} \cong N_2$. Therefore, $N_1 \preceq F(N_1^{\perp})^{\perp}$. Moreover, since $\tilde{F}_1^*$ is an isomorphism, F^* has the matrix

$$\begin{bmatrix} \tilde{\tilde{F}}_1^* & 0 \\ 0 & \tilde{F}_4^* \end{bmatrix}$$

with respect to the decomposition

$$H_{\mathcal{A}} = F(N_1^{\perp}) \oplus F(N_1^{\perp})^{\perp} \xrightarrow{F^*} V^{-1}({N_1}^{\perp}) \tilde{\oplus} V^{-1}(N_1) = H_{\mathcal{A}},$$

where V is an isomorphism and also, $\tilde{\tilde{F}}_1^*$ is an isomorphism. Now, since V is an isomorphism and there exists an isomorphism $\iota : N_1 \to \iota(N_1) \subseteq F(N_1^{\perp})^{\perp}$ (as $N_1 \preceq F(N_1^{\perp})^{\perp}$), we get that $\iota V : V^{-1}(N_1) \to \iota(N_1) \subseteq F(N_1^{\perp})^{\perp}$ is an isomorphism, so $V^{-1}(N_1) \preceq F(N_1^{\perp})^{\perp}$. Moreover, $V^{-1}(N_1)$ finitely generated as N_1 is so. Therefore, $F^* \in \mathcal{M}\Phi_-^{+'}(H_{\mathcal{A}})$.

Conversely, if $F \in \mathcal{M}\Phi_-^{+'}(H_{\mathcal{A}})$, let

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

be an $\mathcal{M}\Phi_-^{+'}$-decomposition for F, then $N_2 \preceq N_1$ and N_2 is finitely generated. By the proof of Theorem 3.1.4 part 1) $\Rightarrow$ 2) F has the matrix

$$\begin{bmatrix} \tilde{F}_1 & 0 \\ \tilde{F}_3 & \tilde{F}_4 \end{bmatrix}$$

with respect to the decomposition

$$H_{\mathcal{A}} = {N_1}^{\perp} \oplus N_1 \xrightarrow{F} {N_2}^{\perp} \oplus N_2 = H_{\mathcal{A}},$$

where $\tilde{F}_1, \tilde{F}_3, \tilde{F}_4$ are adjointable and $\tilde{F}_1$ is an isomorphism. Then F^* has the matrix

$$\begin{bmatrix} \tilde{F}_1^* & \tilde{F}_2^* \\ 0 & \tilde{F}_4^* \end{bmatrix}$$

with respect to the decomposition

$$H_{\mathcal{A}} = {N_2}^{\perp} \oplus N_2 \xrightarrow{F^*} {N_1}^{\perp} \oplus N_1 = H_{\mathcal{A}},$$

and $\tilde{F}_1^*$ is an isomorphism. Hence

$$F^* = \begin{bmatrix} \tilde{F}_1^* & 0 \\ 0 & \tilde{\tilde{F}}_4^* \end{bmatrix}$$

with respect to the decomposition

$$H_{\mathcal{A}} = U({N_2}^{\perp}) \tilde{\oplus} U(N_2) \xrightarrow{F^*} {N_1}^{\perp} \oplus N_1 = H_{\mathcal{A}},$$

where U is an isomorphism.
If $\iota : N_2 \Rightarrow \iota(N_2) \subseteq N_1$ is an isomorphism, then $\iota U^{-1} : U(N_2) \to \iota(N_2) \subseteq N_1$ is also an isomorphism, so $U(N_2) \preceq N_1$. Thus, $F^* \in \mathcal{M}\Phi_+^{-'}(H_{\mathcal{A}})$.
2) Use 1) together with the fact that

$$F \in \mathcal{M}\Phi(H_{\mathcal{A}}) \Leftrightarrow F^* \in \mathcal{M}\Phi(H_{\mathcal{A}})$$

by Corollary 3.1.20 and the fact that

$$\tilde{\mathcal{M}\Phi}_-^+(H_{\mathcal{A}}) = {\mathcal{M}\Phi_-^+}'(H_{\mathcal{A}}) \cap \mathcal{M}\Phi(H_{\mathcal{A}}),$$

$$\tilde{\mathcal{M}\Phi}_+^-(H_{\mathcal{A}}) = {\mathcal{M}\Phi_+^-}'(H_{\mathcal{A}}) \cap \mathcal{M}\Phi(H_{\mathcal{A}})$$

by Proposition 3.4.12.
3) Use 2) together with the fact that

$$F \in \mathcal{M}\Phi_+(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}}) \Leftrightarrow F^* \in \mathcal{M}\Phi_-(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$$

by Corollary 3.1.20 and the fact that

$$\mathcal{M}\Phi_+^-(H_{\mathcal{A}}) = \tilde{\mathcal{M}\Phi}_+^-(H_{\mathcal{A}}) \cup (\mathcal{M}\Phi_+(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})),$$

$$\mathcal{M}\Phi_-^+(H_{\mathcal{A}}) = \tilde{\mathcal{M}\Phi}_-^+(H_{\mathcal{A}}) \cup (\mathcal{M}\Phi_-(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}}))$$

by Definition 3.4.1. □

Definition 3.4.20. [18, Definition 5.12] We set

$$M^a(H_{\mathcal{A}}) = \{F \in B^a(H_{\mathcal{A}}) \mid F \text{ is bounded below } \},$$

$$Q^a(H_{\mathcal{A}}) = \{D \in B^a(H_{\mathcal{A}}) \mid D \text{ is surjective } \}.$$

Lemma 3.4.21. *[18, Lemma 5.13] Let $B^a(H_{\mathcal{A}})$. Then $F \in M^a(H_{\mathcal{A}})$ if and only if $F^* \in Q^a(H_{\mathcal{A}})$.*

Proof. Let $F \in M^a(H_{\mathcal{A}})$. By the proof of Theorem 2.0.20, as ImF is closed in this case, we have that ImF^* is also closed. Moreover, by the proof of Theorem 2.0.20, since ImF^* is closed, we also have $H_{\mathcal{A}} = \ker F \oplus ImF^*$. Since $\ker F = \{0\}$, it follows that $H_{\mathcal{A}} = ImF^*$.
Conversely, if $F^* \in Q^a(H_{\mathcal{A}})$, then $\ker F = ImF^{*\perp} = \{0\}$, so F is injective. Moreover, since $ImF^* = H_{\mathcal{A}}$, which is closed, then ImF is closed also, (again by the proof of Theorem 2.0.20). By the Banach open mapping theorem, it follows that F is an isomorphism from $H_{\mathcal{A}}$ onto its image. Thus, F is bounded below. □

Corollary 3.4.22. *[18, Corollary 5.14] Let $D \in B^a(H_{\mathcal{A}})$. The following statements are equivalent:*
1) $D \in {\mathcal{M}\Phi_-^+}'(H_{\mathcal{A}})$,
2) There exist $Q \in Q^a(H_{\mathcal{A}}), K \in \mathcal{K}^*(H_{\mathcal{A}})$ such that $D = Q + K$.

Proof. Follows from Theorem 3.4.18, Proposition 3.4.19 part 1) and Lemma 3.4.21 by passing to the adjoints. □

Corollary 3.4.23. *${\mathcal{M}\Phi_+^-}'(H_{\mathcal{A}})$ and ${\mathcal{M}\Phi_-^+}'(H_{\mathcal{A}})$ are semigroups under the multiplication.*

Proof. By using the fact that a composition of two operators that are bounded below is an operator that is bounded below and a composition of two surjective operators is a surjective operator, together with the fact that $\mathcal{K}^*(H_{\mathcal{A}})$ is a two sided ideal in $B^a(H_{\mathcal{A}})$, we deduce the statement from Theorem 3.4.18 and Corollary 3.4.22. □

Remark 3.4.24. By using Corollary 3.4.23 together with Proposition 3.4.12 and with the fact that $\mathcal{M}\Phi(H_{\mathcal{A}})$ is also a semigrop under the multiplication, we can directly deduce Lemma 3.4.7 without proving it separately.

Recalling that the sets $M^a(H_\mathcal{A})$ and $Q^a(H_\mathcal{A})$ are open in the norm topology, it follows from Theorem 3.3.1 that the sets

$$M^a(H_\mathcal{A}) \setminus \mathcal{M}\Phi_-(H_\mathcal{A}) = M^a(H_\mathcal{A}) \cap (\mathcal{M}\Phi_+(H_\mathcal{A}) \setminus \mathcal{M}\Phi_-(H_\mathcal{A})),$$

$$Q^a(H_\mathcal{A}) \setminus \mathcal{M}\Phi_-(H_\mathcal{A}) = Q^a(H_\mathcal{A}) \cap (\mathcal{M}\Phi_-(H_\mathcal{A}) \setminus \mathcal{M}\Phi_+(H_\mathcal{A}))$$

are also open in the norm topology, which is an analogue of [56, Lemma 1.6.6] in the setting of operators on Hilbert C^*-modules. Moreover, this holds for arbitrary Hilbert C^*-modules and not just $H_\mathcal{A}$.

The next theorem can be proved in a similar way as Theorem 3.4.18.

Theorem 3.4.25. *Let $B^a(H_\mathcal{A})$. Then the following statements are equivalent:*
1) $F \in \mathcal{M}\Phi_0(H_\mathcal{A})$,
2) There exist an invertible $D \in B^a(H_\mathcal{A})$ and $K \in \mathcal{K}^*(H_\mathcal{A})$ such that $F = D + K$.

Proposition 3.4.26. *Let $F \in \mathcal{M}\Phi_+^{-'}(H_\mathcal{A}) \cap \mathcal{M}\Phi_-^{+'}(H_\mathcal{A})$. Then there exists an $\mathcal{M}\Phi$-decomposition*

$$H_\mathcal{A} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_\mathcal{A}$$

for F with the property that $N_1 \preceq N_2$ and $N_2 \preceq N_1$.

Proof. Let

$$H_\mathcal{A} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_\mathcal{A},$$

$$H_\mathcal{A} = M_1' \tilde{\oplus} N_1' \xrightarrow{F} M_2' \tilde{\oplus} N_2' = H_\mathcal{A}$$

be an $\mathcal{M}\Phi_+^{-'}$ and an $\mathcal{M}\Phi_-^{+'}$-decomposition for F, respectively. By Corollary 3.1.12 it follows that both these decompositions are actually $\mathcal{M}\Phi$-decompositions for F. Hence, both N_1 and N_1' are finitely generated. Therefore, by Theorem 2.0.34 there exists an $n \in \mathbb{N}$ such that $H_\mathcal{A} = L_n^\perp \tilde{\oplus} P \tilde{\oplus} N_1 = L_n^\perp \tilde{\oplus} P' \tilde{\oplus} N_1'$. By the proof of Lemma 2.0.43 given in [38], there exists then isomorphisms V and V' such that

$$H_\mathcal{A} = L_n^\perp \tilde{\oplus} (P \tilde{\oplus} N_1) \xrightarrow{F} F(L_n^\perp) \tilde{\oplus} (F(P) \tilde{\oplus} V(N_2) = H_\mathcal{A},$$

$$H_\mathcal{A} = L_n^\perp \tilde{\oplus} (P' \tilde{\oplus} N_1') \xrightarrow{F} F(L_n^\perp) \tilde{\oplus} (F(P') \tilde{\oplus} V'(N_2') = H_\mathcal{A}$$

are two $\mathcal{M}\Phi$-decompositions for F and moreover, $P \cong F(P)$, $P' \cong F(P')$. Since $N_1 \preceq N_2$, we get that $(P \tilde{\oplus} N_1) \preceq (F(P) \tilde{\oplus} V(N_2))$. Similarly, we have $(F(P') \tilde{\oplus} V'(N_2')) \preceq (P' \tilde{\oplus} N_1')$ since $N_2' \preceq N_1'$. Finally,

$$P \tilde{\oplus} N_1 \cong P' \tilde{\oplus} N_1', \;\; F(P) \tilde{\oplus} V(N_2) \cong F(P') \tilde{\oplus} V'(N_2').$$

Hence, $(F(P) \tilde{\oplus} V(N_2)) \preceq (P \tilde{\oplus} N_1)$. □

3.5 Non-adjointable semi-C^*-Fredholm operators

We define now general, (not necessarily adjointable) semi-$\mathcal{A}$-Fredholm operators in exactly the same way as adjointable semi-$\mathcal{A}$-Fredholm operators, only without assuming adjointablity.

Definition 3.5.1. Let $F \in B(H_{\mathcal{A}})$, where $B(H_{\mathcal{A}})$ is the set of all bounded, (not necessarily adjointable) $\mathcal{A}$-linear operators on $H_{\mathcal{A}}$. We say that F is an upper semi-$\mathcal{A}$-Fredholm operator if there exists a decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

with respect to which F has the matrix

$$\left[\begin{array}{cc} F_1 & 0 \\ 0 & F_4 \end{array}\right],$$

where F_1 is an isomorphism, M_1, M_2, N_1, N_2 are closed submodules of $H_{\mathcal{A}}$ and N_1 is finitely generated. Similarly, we say that F is a lower semi-$\mathcal{A}$-Fredholm operator if all the above conditions hold except that in this case we assume that N_2 (and not N_1) is finitely generated.

Set

$$\widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}}) = \{F \in B(H_{\mathcal{A}}) \mid F \text{ is upper semi-}\mathcal{A}\text{-Fredholm }\},$$

$$\widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}}) = \{F \in B(H_{\mathcal{A}}) \mid F \text{ is lower semi-}\mathcal{A}\text{-Fredholm }\},$$

$$\widehat{\mathcal{M}\Phi}(H_{\mathcal{A}}) = \{F \in B(H_{\mathcal{A}}) \mid F \text{ is } \mathcal{A}\text{-Fredholm operator on } H_{\mathcal{A}}\}.$$

Then, by definition we have

$$\mathcal{M}\Phi_+(H_{\mathcal{A}}) = \widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}}) \cap B^a(H_{\mathcal{A}}),$$

$$\mathcal{M}\Phi_-(H_{\mathcal{A}}) = \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}}) \cap B^a(H_{\mathcal{A}})$$

and

$$\mathcal{M}\Phi(H_{\mathcal{A}}) = \widehat{\mathcal{M}\Phi}(H_{\mathcal{A}}) \cap B^a(H_{\mathcal{A}}).$$

Remark 3.5.2. Recall Definition 2.0.59. If

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}},$$

is an inner (Noether) decomposition for the operator F in $B(H_{\mathcal{A}})$, it follows from the proof of Lemma 2.0.42 that F has the matrix $\left[\begin{array}{cc} F_1 & 0 \\ 0 & \tilde{F}_4 \end{array}\right]$ with respect to the decomposition

$$H_{\mathcal{A}} = U(M_1) \tilde{\oplus} U(N_1) \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}},$$

where F_1 and U are isomorphisms. Obviously, such operators are invertible in $B(H_{\mathcal{A}})/\mathcal{K}(H_{\mathcal{A}})$.

Set

$$\widehat{\mathcal{M}\Phi}_-(H_{\mathcal{A}}) = \{G \in B(H_{\mathcal{A}}) \mid \text{ there exist closed submodules } M, N, M' \text{ of } H_{\mathcal{A}}$$

$$\text{such that } H_{\mathcal{A}} = M \tilde{\oplus} N, N \text{ is finitely generated and } G_{|_{M'}} \text{ is an isomorphism onto } M\}.$$

We have the following lemma.

Lemma 3.5.3. *It holds that* $\widehat{\mathcal{M}\Phi}_-(H_{\mathcal{A}}) = \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}})$.

Proof. Obviously, we have $\widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}}) \subseteq \widehat{\mathcal{M}\Phi}_-(H_{\mathcal{A}})$, so it suffices to prove the opposite inclusion. Let $G \in \widehat{\mathcal{M}\Phi}_-(H_{\mathcal{A}})$ and choose Hilbert submodules M, N and M' such that $H_{\mathcal{A}} = M\tilde{\oplus}N$, N is finitely generated and $G_{|_{M'}}$ is an isomorphism onto M. We wish to show that

$$H_{\mathcal{A}} = M'\tilde{\oplus}G^{-1}(N).$$

To this end, choose an $x \in H_{\mathcal{A}}$. Since $H_{\mathcal{A}} = M\tilde{\oplus}N$, there exist some $m \in M$ and $n \in N$ such that $Gx = m + n$. Now, since $G_{|_{M'}}$ is an isomorphism onto M, there exists an $m' \in M'$ such that $Gm' = m$. So, we have $Gx = Gm' + n$. On the other hand, $Gx = Gm' + G(x - m')$, hence $n = G(x - m')$. It follows that $x - m' \in G^{-1}(N)$ and $x = m' + (x - m')$, which gives $H_{\mathcal{A}} = M' + G^{-1}(N)$. Finally, $M' \cap G^{-1}(N) = \{0\}$ because $G(M') = M$, $M \cap N = \{0\}$ and $G_{|_{M'}}$ is an isomorphism, thus injective.
Therefore, G has the matrix $\begin{bmatrix} G_1 & 0 \\ 0 & G_4 \end{bmatrix}$ with respect to the decomposition

$$H_{\mathcal{A}} = M'\tilde{\oplus}G^{-1}(N) \xrightarrow{G} M\tilde{\oplus}N = H_{\mathcal{A}},$$

where G_1 is an isomorphism. □

Clearly, any operator $F \in \widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}})$ is also left invertible in $B(H_{\mathcal{A}})/\mathcal{K}(H_{\mathcal{A}})$, whereas any operator $G \in \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}})$ is right invertible in $B(H_{\mathcal{A}})/\mathcal{K}(H_{\mathcal{A}})$. The converse also holds.

Proposition 3.5.4. *[19, Proposition 2.3] We have the following.*
1) If F is left invertible in $B(H_{\mathcal{A}})/\mathcal{K}(H_{\mathcal{A}})$, then $F \in \widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}})$.
2) If F is right invertible in $B(H_{\mathcal{A}})/\mathcal{K}(H_{\mathcal{A}})$, then $F \in \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}})$.

Proof. Suppose that $GF = I + K$ for some $G, F \in B(H_{\mathcal{A}})$ and $K \in \mathcal{K}(H_{\mathcal{A}})$. Let

$$H_{\mathcal{A}} = M_1\tilde{\oplus}N_1 \xrightarrow{GF} M_2\tilde{\oplus}N_2 = H_{\mathcal{A}}$$

be an $\widehat{\mathcal{M}\Phi}$-decomposition for GF. Since $GF_{|M_1}$ is an isomorphism onto M_2, it is readily verified that $F_{|M_1}$ is an isomorphism onto $F(M_1)$ and $G_{|F(M_1)}$ is an isomorphism onto M_2. From the proof of Lemma 3.5.3 it follows that $H_{\mathcal{A}} = F(M_1)\tilde{\oplus}G^{-1}(N_2)$. Since $M_1\tilde{\oplus}N_1 \xrightarrow{GF} M_2\tilde{\oplus}N_2$ is an $\widehat{\mathcal{M}\Phi}$-decomposition for GF, we must have that F has the matrix $\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}$ with respect to the decomposition $H_{\mathcal{A}} = M_1\tilde{\oplus}N_1 \xrightarrow{F} F(M_1)\tilde{\oplus}G^{-1}(N_2) = H_{\mathcal{A}}$ and G has the matrix $\begin{bmatrix} G_1 & 0 \\ 0 & G_4 \end{bmatrix}$ with respect to the decomposition

$$H_{\mathcal{A}} = F(M_1)\tilde{\oplus}G^{-1}(N_2) \xrightarrow{G} M_2\tilde{\oplus}N_2 = H_{\mathcal{A}},$$

where F_1 and G_1 are isomorphisms. □

Corollary 3.5.5. *[19, Corollary 2.4] The sets $\widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}})$ and $\widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}})$ are closed under the multiplication.*

The next lemma can be proved in the similar way as Proposition 3.5.4.

Lemma 3.5.6. *Let M be a Hilbert C^*-module and $F, G \in B(M)$. Suppose that there exists a decomposition*

$$M = M_1\tilde{\oplus}N_1 \xrightarrow{GF} M_2\tilde{\oplus}N_2 = M$$

with respect to which GF has the matrix $\left[\begin{array}{cc} (GF)_1 & 0 \\ 0 & (GF)_4 \end{array}\right]$, *where $(GF)_1$ is an isomorphism. Then we have $M = F(M_1)\tilde{\oplus}G^{-1}(N_2)$ and moreover, with respect to the decompositions*

$$M = M_1\tilde{\oplus}N_1 \xrightarrow{F} F(M_1)\tilde{\oplus}G^{-1}(N_2) = M,$$

$$M = F(M_1)\tilde{\oplus}G^{-1}(N_2) \xrightarrow{G} M_2\tilde{\oplus}N_2 = M,$$

the operators F and G have the matrices $\left[\begin{array}{cc} F_1 & 0 \\ 0 & F_4 \end{array}\right]$ *and* $\left[\begin{array}{cc} G_1 & 0 \\ 0 & G_4 \end{array}\right]$, *respectively, where F_1 and G_1 are isomorphisms.*

We recall now that Lemma 3.1.8, Corollary 3.1.10 and Corollary 3.1.12 are also valid in the case of non-adjointable operators. Moreover, Lemma 3.5.6 is valid in the case of general bounded linear operators on arbitrary Banach spaces.

Corollary 3.5.7. *The analogue of Corollary 3.1.15 holds in the case of non-adjointable operators on arbitrary Hilbert C^*-modules.*

Proof. Suppose that M is a Hilbert C^*-module and $DF \in \widehat{\mathcal{M}\Phi}_l(M)$. If

$$M = M_1\tilde{\oplus}N_1 \xrightarrow{DF} M_2\tilde{\oplus}N_2 = M$$

is an $\widehat{\mathcal{M}\Phi}_l$-decomposition for DF, then, by Lemma 3.5.6 , F has the matrix $\left[\begin{array}{cc} F_1 & 0 \\ 0 & F_4 \end{array}\right]$ with respect to the decomposition

$$M = M_1\tilde{\oplus}N_1 \xrightarrow{F} F(M_1)\tilde{\oplus}D^{-1}(N_2) = M,$$

whereas D has the matrix $\left[\begin{array}{cc} D_1 & 0 \\ 0 & D_4 \end{array}\right]$ with respect to the decomposition

$$M = F(M_1)\tilde{\oplus}D^{-1}(N_2) \xrightarrow{D} M_2\tilde{\oplus}N_2 = M,$$

where F_1 and D_1 are isomorphisms. Since N_1 is finitely generated, the first statement follows. The proof of the second statement is similar. ☐

Corollary 3.5.8. *The analogue of Corollary 3.1.16 holds in the case of non-adjointable operators on arbitrary Hilbert C^*-modules.*

Proof. Let M be a Hilbert C^*-module and $DF \in \widehat{\mathcal{M}\Phi}_l(M)$. Suppose that $F \in \widehat{\mathcal{M}\Phi}(M)$ and let

$$M = M_1\tilde{\oplus}N_1 \xrightarrow{DF} M_2\tilde{\oplus}N_2 = M$$

be an $\widehat{\mathcal{M}\Phi}_l$-decomposition for DF. By Lemma 3.5.6 we have that

$$M = M_1\tilde{\oplus}N_1 \xrightarrow{F} F(M_2)\tilde{\oplus}D^{-1}(N_2) = M$$

is an $\widehat{\mathcal{M}\Phi}_l$-decomposition for F and D has the matrix $\left[\begin{array}{cc} D_1 & 0 \\ 0 & D_4 \end{array}\right]$ with respect to the decomposition

$$M = F(M_1)\tilde{\oplus}D^{-1}(N_1) \xrightarrow{D} M_2\tilde{\oplus}N_2 = M,$$

where D_1 is an isomorphism. Now, since

$$M = M_1\tilde{\oplus}N_1 \xrightarrow{F} F(M_1)\tilde{\oplus}D^{-1}(N_2) = M$$

is an $\widehat{\mathcal{M}\Phi}_l$-decomposition for F, from Corollary 3.1.12 it follows that $D^{-1}(N_2)$ must be finitely generated since $F \in \widehat{\mathcal{M}\Phi}(M)$. Hence,

$$M = F(M_1)\tilde{\oplus}D^{-1}(N_2) \xrightarrow{D} M_2\tilde{\oplus}N_2 = M$$

is an $\widehat{\mathcal{M}\Phi}_l$-decomposition for D, so $D \in \widehat{\mathcal{M}\Phi}_l(M)$. By applying Corollary 3.1.12 on the operator D instead of F and using the similar arguments, we obtain the second statement in the corollary. □

Corollary 3.5.9. *The analogue of Corollary 3.1.17 holds in the setting of non-adjointable operators on arbitrary Hilbert C^*-modules.*

Proof. Let M be a Hilbert C^*-module and $D, F \in B(M)$. Suppose that $D \in \widehat{\mathcal{M}\Phi}_l(M)$ and $DF \in \widehat{\mathcal{M}\Phi}(M)$. If

$$M = M_1\tilde{\oplus}N_1 \xrightarrow{DF} M_2\tilde{\oplus}N_2 = M$$

is an $\widehat{\mathcal{M}\Phi}$-decomposition for DF, then, by Lemma 3.5.6, we have that

$$M = F(M_1)\tilde{\oplus}D^{-1}(N_2) \xrightarrow{D} M_2\tilde{\oplus}N_2 = M$$

is an $\widehat{\mathcal{M}\Phi}_r$-decomposition for D. Hence, by Corollary 3.1.10 we get that

$$D \in \widehat{\mathcal{M}\Phi}_r(M) \cap \widehat{\mathcal{M}\Phi}_l(M) = \widehat{\mathcal{M}\Phi}(M).$$

In the similar way we can deduce the second statement of Corollary 3.1.17. □

Corollary 3.5.10. *The analogue of Corollary 3.1.18 holds in the settings of non-adjointable operators on arbitrary Hilbert C^*-modules.*

Proof. Let M be a Hilbert C^*-module. Suppose that $D \in \widehat{\mathcal{M}\Phi}(M)$ and $DF \in \widehat{\mathcal{M}\Phi}(M)$. If

$$M = M_1\tilde{\oplus}N_1 \xrightarrow{DF} M_2\tilde{\oplus}N_2 = M$$

is an $\widehat{\mathcal{M}\Phi}$-decomposition for DF, then, by Lemma 3.5.6,

$$M = F(M_1)\tilde{\oplus}D^{-1}(N_2) \xrightarrow{D} M_2\tilde{\oplus}N_2 = M$$

is an $\widehat{\mathcal{M}\Phi}_r$-decomposition for D. Since $D \in \widehat{\mathcal{M}\Phi}(M)$, by Corollary 3.1.12 we have that $D^{-1}(N_2)$ is finitely generated. It follows by Lemma 3.5.6 that

$$M = M_1\tilde{\oplus}N_1 \xrightarrow{F} F(M_1)\tilde{\oplus}D^{-1}(N_2) = M$$

is an $\widehat{\mathcal{M}\Phi}$-decomposition for F, so $F \in \widehat{\mathcal{M}\Phi}(M)$.

The case when $F \in \widehat{\mathcal{M}\Phi}(M)$ and $DF \in \widehat{\mathcal{M}\Phi}(M)$ can be treated similarly. □

Many of the results on adjointable semi-$\mathcal{A}$-Fredholm operators that are presented so far can in a similar be proved for non-adjointable semi-$\mathcal{A}$-Fredholm operators. However, for non-adjointable operators we do not have Theorem 2.0.20 at disposition. Therefore, we now need to give different proofs or to slightly modify the statements in the results where we apply Theorem 2.0.20 in order to hold in the case of non-adjointable semi-$\mathcal{A}$-Fredholm operators.

In the next results we always assume that M is a Hilbert C^*-module over a unital C^*-algebra.

The next proposition is a modified version of [19, Proposiotion 3.1].

Proposition 3.5.11. *Let $F \in B(M)$. If $F \in \widehat{\mathcal{M}\Phi}_l(M)$ and ImF is closed, then $\ker F$ and ImF are complementable in M.*
In this case F has the matrix $\begin{bmatrix} F_1 & 0 \\ 0 & 0 \end{bmatrix}$ with respect to the decomposition

$$M = \ker F^\circ \tilde{\oplus} \ker F \xrightarrow{F} ImF \tilde{\oplus} ImF^\circ = M,$$

where F_1 is an isomorphism and $\ker F^\circ$, ImF° denote the complements of $\ker F$ and ImF, respectively. If $F \in \widehat{\mathcal{M}\Phi}(M)$ and ImF is closed, then the index of F is well-defined.

Proof. Let
$$M = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = M$$
be an $\widehat{\mathcal{M}\Phi}_l$-decomposition for F. If ImF is closed, then it is easy to see that $F(N_1)$ must be closed. This is because $F(N_1) = ImF \cap N_2$. Indeed, if $y \in ImF \cap N_2$, then $y = Fx$ for some $x \in F^{-1}(N_2)$. However, by Lemma 3.1.3 we have $F^{-1}(N_2) = N_1$, hence $x \in N_1$, which gives $y \in F(N_1)$, so $ImF \cap N_2 \subseteq F(N_1)$. The opposite inclusion is obvious. Since N_1 is self-dual, by Proposition 2.0.28 we have that $F_{|_{N_1}}$ is adjointable. Thus we are in the position to apply Theorem 2.0.20 to deduce that

$$N_1 = \ker F \oplus \tilde{N}_1 \text{ and } N_2 = F(\tilde{N}_1) \oplus \tilde{N}_2$$

for some closed submodules $\tilde{N}_1$ and $\tilde{N}_2$. Then we get

$$M = M_2 \tilde{\oplus} F(\tilde{N}_1) \tilde{\oplus} \tilde{N}_2 = ImF \tilde{\oplus} \tilde{N}_2,$$

so ImF is complementable in M. Moreover, $M = M_1 \tilde{\oplus} \tilde{N}_1 \tilde{\oplus} \ker F$, so $\ker F$ is also complementable in M. It follows by the Banach open mapping theorem that F has the matrix $\begin{bmatrix} F_1 & 0 \\ 0 & 0 \end{bmatrix}$ with respect to the decomposition

$$M = \ker F^\circ \tilde{\oplus} \ker F \xrightarrow{F} ImF \tilde{\oplus} ImF^\circ = M,$$

where F_1 is an isomorphism. Since N_1 is finitely generated, then $\ker F$ is finitely generated as a direct summand in N_1 and similarly, if N_2 is finitely generated, then $ImF^\circ = \tilde{M}_2$ is finitely generated.

Finally, $\tilde{N}_1 \cong F(\tilde{N}_1)$, so, in the case when

$$M = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = M$$

is an $\widehat{\mathcal{M}\Phi}$-decomposition for F, we have $[N_1] - [N_2] = [\ker F] - [ImF^\circ]$. Although ImF can have several different complemented submodules in M, it is clear that they are all mutually isomorphic to each other, hence the index of F is well-defined in this case. □

Corollary 3.5.12. *If $F \in \widehat{\mathcal{M}\Phi}_l(M) \setminus \widehat{\mathcal{M}\Phi}(M)$ and ImF is closed, then the complement of ImF is not finitely generated.*

Proposition 3.5.13. *[19, Proposition 3.2] If $F \in \widehat{\mathcal{M}\Phi}_r(M)$ and ImF is closed and complementable in M, then the decomposition from Proposition 3.5.11 exists for the operator F. In this case, instead of $\ker F$, we have that ImF° is finitely generated where ImF° is the complement of ImF.*

Proof. Suppose that $F \in \widehat{\mathcal{M}\Phi}_r(M)$. Let

$$M = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = M$$

be an $\widehat{\mathcal{M}\Phi}_r$-decomposition for F. Then N_2 is finitely generated. Since ImF is closed by assumption, it follows that $F(N_1)$ is closed by the same arguments as in the proof of Proposition 3.5.11. As ImF is complementable by assumption, we obtain that $F(N_1)$ is complementable in N_2.
More precisely, we have

$$M = ImF \tilde{\oplus} ImF^\circ = M_2 \tilde{\oplus} F(N_1) \tilde{\oplus} ImF^\circ,$$

where ImF° stands for the complement of ImF. Hence, $F(N_1)$ is complementable in M, so, by Lemma 2.0.66, $F(N_1)$ is complementable in N_2 since $F(N_1) \subseteq N_2$. Therefore, $F(N_1)$ is finitely generated projective, being a direct summand in a finitely generated, projective module N_2. Since the operator $F_{|_{N_1}} : N_1 \to F(N_1)$ is an epimorphism, there exists a decomposition $N_1 = N_1' \tilde{\oplus} \ker F$, where $N_1' \cong F(N_1)$. □

Corollary 3.5.14. *If $F \in \widehat{\mathcal{M}\Phi}_r(M) \setminus \widehat{\mathcal{M}\Phi}(M)$ and in addition ImF is closed and complementable, then $\ker F$ is not finitely generated.*

Corollary 3.5.15. *Let $F \in B(M)$ and suppose that F is regular, that is ImF is closed and $\ker F, ImF$ are complementable. Then the following statements hold.*
a) $F \in \widehat{\mathcal{M}\Phi}_l(M)$ if and only if $\ker F$ is finitely generated.
b) $F \in \widehat{\mathcal{M}\Phi}_r(M)$ if and only if the complement of ImF is finitely generated.

Lemma 3.5.16. *Let $F \in B(H_{\mathcal{A}})$ and suppose that F is a regular operator. If $F \in \widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}})$ or if $F \in \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}})$, then $ImF \cong H_{\mathcal{A}}$.*

Proof. Consider the decomposition

$$H_{\mathcal{A}} = \ker F^\circ \tilde{\oplus} \ker F \xrightarrow{F} ImF \tilde{\oplus} ImF^\circ = H_{\mathcal{A}}.$$

If $F \in \widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}})$, then by Corollary 3.5.15. $\ker F$ is finitely generated. By Lemma 2.0.25 we have $H_{\mathcal{A}} = \ker F^\perp \oplus \ker F$ and then, by the Dupre-Filmore Theorem 2.0.15, we get $\ker F^\perp \cong H_{\mathcal{A}}$. Hence we deduce that

$$ImF \cong \ker F^\circ \cong \ker F^\perp \cong H_{\mathcal{A}}.$$

If $F \in \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}})$, then by Corollary 3.5.15 ImF° is finitely generated. By the same arguments as above we obtain $ImF \cong (ImF^\circ)^\perp \cong H_{\mathcal{A}}$. □

Inspired by Definition 2.0.61 we give now the following definition which is a slightly modified version of [21, Definition 13].

Definition 3.5.17. Let $F \in B(H_{\mathcal{A}})$. We say that F admits an upper external (Noether) decomposition if there exist closed C^*-modules X_1, X_2 and bounded $\mathcal{A}-$linear operators E_2, E_3 such that the matrix operator

$$F_0 = \begin{bmatrix} F & E_2 \\ E_3 & 0 \end{bmatrix} : H_{\mathcal{A}} \oplus X_1 \to H_{\mathcal{A}} \oplus X_2$$

is an invertible operator and X_2 is finitely generated. Similarly we say that F admits a lower external (Noether) decomposition if all the above conditions hold, only in this case we assume that X_1 (and not X_2) is finitely generated.

The next proposition is a slightly modified version of [21, Proposition 5].

Proposition 3.5.18. *Let $F \in B(H_{\mathcal{A}})$. Then $F \in \widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}})$ if and only if F admits an upper external (Noether) decomposition and similarly, $F \in \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}})$ if and only if F admits a lower external (Noether) decomposition.*

Proof. Suppose that F admits an upper external (Noether) decomposition. If $G_0 = F_0^{-1}$ and G_0 has the matrix $\begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix}$ with respect to the decomposition $H_{\mathcal{A}} \oplus X_2 \xrightarrow{G_0} H_{\mathcal{A}} \oplus X_1$, then by the same arguments as in the proof of Theorem 2.0.62 we deduce that $id_{X_2} = E_3 G_2$ and $id_{X_1} = G_3 E_2$. By Lemma 3.5.6 we get

$$H_{\mathcal{A}} = ImG_2 \tilde{\oplus} \ker E_3 = ImE_2 \tilde{\oplus} \ker G_3.$$

Hence we may let

$$M_1 = \ker E_3, N_1 = ImG_2, M_2 = \ker G_3, N_2 = ImE_2$$

and proceed as in the proof of Theorem 2.0.62.

The proof for the case when F admits lower external (Noether) decomposition is similar.

The proof for the implication in the other direction is exactly the same as the proof of Theorem 2.0.62. □

Notice that Definition 3.5.17 and Proposition 3.5.18 can be generalized from the standard module case to arbitrary Hilbert C^*-modules.

The next three lemmas present a generalization of [56, Theorem 1.2.7] in the setting of operators on Hilbert C^*-modules.

Lemma 3.5.19. *[18, Lemma 2.13] Suppose that $D, F \in B(H_{\mathcal{A}})$ $DF \in \widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}})$ and ImF is closed. Then $DJ_{ImF} \in \widehat{\mathcal{M}\Phi}_l(ImF, H_{\mathcal{A}})$.*

Proof. Let

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{DF} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

be a decomposition with respect to which DF has the matrix

$$\begin{bmatrix} (DF)_1 & 0 \\ 0 & (DF)_4 \end{bmatrix},$$

where $(DF)_1$ is an isomorphism and N_1 is finitely generated. By Lemma 3.5.6 we have that $H_{\mathcal{A}} = F(M_1) \tilde{\oplus} D^{-1}(N_2)$. Hence, by Lemma 2.0.66, we get that

$$ImF = F(M_1) \tilde{\oplus} (D^{-1}(N_2) \cap ImF).$$

With respect to the decomposition

$$ImF = F(M_1) \tilde{\oplus} (D^{-1}(N_2) \cap ImF) \xrightarrow{DJ_{ImF}} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}},$$

DJ_{ImF} has the matrix

$$\begin{bmatrix} (DJ_{ImF})_1 & 0 \\ 0 & (DJ_{ImF})_4 \end{bmatrix},$$

where $(DJ_{ImF})_1$ is an isomorphism. Now, since DF has the matrix

$$\left[\begin{array}{cc} (DF)_1 & 0 \\ 0 & (DF)_4 \end{array}\right]$$

with respect to the decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{DF} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}},$$

by Lemma 3.5.6 it follows that F has the matrix $\left[\begin{array}{cc} F_1 & 0 \\ 0 & F_4 \end{array}\right]$ with respect to the decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} F(M_1) \tilde{\oplus} D^{-1}(N_2) = H_{\mathcal{A}},$$

where F_1 is an isomorphism. By the same arguments as in the proof of Proposition 3.5.11 we get that $D^{-1}(N_2) \cap ImF = F(N_1)$ which is finitely generated by Remark 2.0.68. We are done. $\square$

Lemma 3.5.20. *[21, Lemma 7] Let V be a finitely generated Hilbert submodule of $H_{\mathcal{A}}$ and $F \in B(H_{\mathcal{A}})$. Suppose that $P_{V^\perp}F \in \widehat{\mathcal{M}\Phi}(H_{\mathcal{A}}, V^\perp)$, where $P_{V^\perp}$ denotes the orthogonal projection onto $V^\perp$ along V. Then $F \in \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}})$.*

Proof. Since V is finitely generated, by Lemma 2.0.25 it follows that V is an orthogonal direct summand in $H_{\mathcal{A}}$, so $H_{\mathcal{A}} = V \oplus V^\perp$. Consider the decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{P_{V^\perp}F} M_2 \tilde{\oplus} N_2 = V^\perp$$

with respect to which $P_{V^\perp}F$ has the matrix

$$\left[\begin{array}{cc} (P_{V^\perp}F)_1 & 0 \\ 0 & P_{V^\perp}F)_4 \end{array}\right],$$

where N_1, N_2 are finitely generated Hilbert submodules and $(P_{V^\perp}F)_1$ is an isomorphism. Since $(P_{V^\perp}F)_1 = P_{M_2}^{V^\perp} P_{V^\perp}F_{|_{M_1}}$, where $P_{M_2}^{V^\perp}$ stands for the projection of $V^\perp$ onto M_2 along N_2, it follows that $P_{M_2}^{V^\perp} P_{V^\perp}F_{|_{M_1}}$ is an isomorphism of M_1 onto M_2. However, $H_{\mathcal{A}} = M_2 \tilde{\oplus} N_2 \tilde{\oplus} V$, so $P_{M_2}^{V^\perp} P_{V^\perp} = P_{M_2}$, where P_{M_2} stands for the projection of $H_{\mathcal{A}}$ onto M_2 along $N_2 \oplus V$. Hence, F has the matrix

$$\left[\begin{array}{cc} F_1 & F_2 \\ F_3 & F_4 \end{array}\right]$$

with respect to the decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} (N_2 \tilde{\oplus} V) = H_{\mathcal{A}},$$

where $F_1 = P_{M_2}F_{|_{M_1}}$ is an isomorphism. Then, with respect to the decomposition

$$H_{\mathcal{A}} = U_1(M_1) \tilde{\oplus} U_1(N_1) \xrightarrow{F} U_2^{-1}(M_2) \tilde{\oplus} U_2^{-1}(N_2 \tilde{\oplus} V) = H_{\mathcal{A}},$$

F has the matrix

$$\left[\begin{array}{cc} \tilde{F}_1 & 0 \\ 0 & \tilde{F}_4 \end{array}\right],$$

where U_1, U_2 and $\tilde{F}_1$ are isomorphisms. Now, $N_2 \tilde{\oplus} V$ is finitely generated, hence, $U_2^{-1}(N_2 \tilde{\oplus} V)$ is finitely generated . $\square$

Lemma 3.5.21. *[21, Lemma 8] Let $G, F \in B(H_{\mathcal{A}})$ and suppose that ImG is closed. Assume in addition that $\ker G$ and ImG are complementable in $H_{\mathcal{A}}$. If $GF \in \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}})$, then*

$$\sqcap F \in \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}}, N),$$

where $\ker G \tilde{\oplus} N = H_{\mathcal{A}}$ and $\sqcap$ denotes the projection onto N along $\ker G$.

Proof. Let $H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{GF} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$ be an $\widehat{\mathcal{M}\Phi}_r$-decomposition for GF. From Lemma 3.5.6 it follows that F and G have the matrices $\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}$ and $\begin{bmatrix} G_1 & 0 \\ 0 & G_4 \end{bmatrix}$ with respect to the decompositions

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} F(M_1) \tilde{\oplus} G^{-1}(N_2) = H_{\mathcal{A}},$$

$$H_{\mathcal{A}} = F(M_1) \tilde{\oplus} G^{-1}(N_2) \xrightarrow{G} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}},$$

respectively, where F_1 and G_1 are isomorphisms.
Since $\ker G \tilde{\oplus} N = H_{\mathcal{A}}$ and $\ker G \subseteq G^{-1}(N_2)$, by Lemma 2.0.66 we get that

$$G^{-1}(N_2) = \ker G \tilde{\oplus} (G^{-1}(N_2) \cap N).$$

As ImG is closed and $H_{\mathcal{A}} = F(M_1) \tilde{\oplus} (G^{-1}(N_2) \cap N) \tilde{\oplus} \ker G$, we get that $G_{|_{(G^{-1}(N_2) \cap N) \tilde{\oplus} F(M_1))}}$ is an isomorphism onto ImG by the Banach open mapping theorem. Thus,

$$ImG = M_2 \tilde{\oplus} G(G^{-1}(N_2) \cap N).$$

Since ImG is complementable in $H_{\mathcal{A}}$, we have that $G(G^{-1}(N_2) \cap N)$ is complementable in $H_{\mathcal{A}}$. As $G(G^{-1}(N_2) \cap N) \subseteq N_2$, it follows that $G(G^{-1}(N_2) \cap N)$ is complementable in N_2 by Lemma 2.0.66. However, N_2 is finitely generated, hence, $G(G^{-1}(N_2) \cap N)$ must be finitely generated as a direct summand in N_2. Therefore, $G^{-1}(N_2) \cap N$ is finitely generated, being isomorphic to $G(G^{-1}(N_2) \cap N)$.

With respect to the decomposition $M_1 \tilde{\oplus} N_1 \xrightarrow{F} F(M_1) \tilde{\oplus} G^{-1}(N_2)$, F has the matrix $\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}$, where F_1 is an isomorphism. Moreover, since

$$H_{\mathcal{A}} = F(M_1) \tilde{\oplus} (G^{-1}(N_2) \cap N) \tilde{\oplus} \ker G,$$

it follows that $\sqcap_{|_{(F(M_1) \tilde{\oplus} (G^{-1}(N_2) \cap N))}}$ is an isomorphism onto N (recall that $\sqcap$ is the projection onto N along $\ker G$). Therefore, we get that

$$N = \sqcap(F(M_1)) \tilde{\oplus} (G^{-1}(N_2) \cap N).$$

It is then easy to see that $\sqcap F$ has the matrix $\begin{bmatrix} (\sqcap F)_1 & 0 \\ 0 & (\sqcap F)_4 \end{bmatrix}$ with respect to the decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{\sqcap F} \sqcap(F(M_1)) \tilde{\oplus} (G^{-1}(N_2) \cap N) = N,$$

where $(\sqcap F)_1$ is an isomorphism. Now, $G^{-1}(N_2) \cap N$ is finitely generated. □

Lemma 3.5.22. *Let $F \in \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}})$ and suppose that*

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}},$$

$$H_{\mathcal{A}} = M_1' \tilde{\oplus} N_1' \xrightarrow{F} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}},$$

are two $\widehat{\mathcal{M}\Phi}_r$-decompositions for F. Then there exist some finitely generated, closed submodules P and P' such that $P \oplus N_1 \cong P' \oplus N_1'$.

Proof. Let $H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$ be an $\widehat{\mathcal{M}\Phi}_r$-decomposition for F. As in the proof of Lemma 2.0.43 we may without loss of generality assume that

$$M_2 = L_n^{\perp} \oplus P, L_n = P' \tilde{\oplus} p_n(N_2'),\ P' = M_2' \cap L_n,\ p_n(N_2') \cong N_2'$$

for some $n \in \mathbb{N}$ and some finitely generated Hilbert submodules P, P', where p_n denotes the orthogonal projection onto L_n. Indeed, from Theorem 2.0.34 it follows that there exists an $n \in \mathbb{N}$ such that $H_{\mathcal{A}} = L_n^{\perp} \tilde{\oplus} P \tilde{\oplus} N_2$ for some finitely generated Hilbert submodule P. If we let $\sqcap$ denote the projection onto $L_n^{\perp} \tilde{\oplus} P$ along N_2 and the operator V be given by the operator matrix $\begin{bmatrix} \sqcap & 0 \\ 0 & 1 \end{bmatrix}$ with respect to the decomposition

$$H_{\mathcal{A}} = M_2 \tilde{\oplus} N_2 \xrightarrow{V} L_n^{\perp} \tilde{\oplus} P \tilde{\oplus} N_2 = H_{\mathcal{A}},$$

then V is an isomorphism. Hence $VF \in \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}})$ and

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{VF} (L_n^{\perp} \tilde{\oplus} P) \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

is an $\widehat{\mathcal{M}\Phi}_r$-decomposition for VF. Moreover,

$$H_{\mathcal{A}} = M_1' \tilde{\oplus} N_1' \xrightarrow{VF} V(M_2') \tilde{\oplus} V(N_2') = H_{\mathcal{A}}$$

is also an $\widehat{\mathcal{M}\Phi}_r$-decomposition for VF, since V is an isomorphism. Now, since $V(N_2')$ is finitely generated, by Theorem 2.0.34 there exists an $m \geq n$ such that $L_m = P' \tilde{\oplus} p_m(V(N_2'))$ where $P' = V(M_2') \cap L_m$ and $p_m(V(N_2')) \cong V(N_2')$. Then $L_n^{\perp} \tilde{\oplus} P = L_m^{\perp} \tilde{\oplus} \widetilde{P}$, where $\widetilde{P} = P \oplus (L_m \cap L_n^{\perp})$. By considering the operator VF instead F, we see that we may in fact without loss of generality assume that $M_2 = L_n^{\perp} \oplus P$, $L_n = P' \tilde{\oplus} p_n(N_2')$ for some $n \in \mathbb{N}$, where $P' = M_2' \cap L_n$ and $p_n(N_2') \cong N_2'$.

Therefore, we obtain that

$$H_{\mathcal{A}} = L_n^{\perp} \tilde{\oplus} P \tilde{\oplus} N_2 = L_n^{\perp} \tilde{\oplus} P' \tilde{\oplus} N_2'.$$

We put $F_1 = F_{|_{M_1}}, F_1' = F_{|_{M_1'}}$ and claim that $H_{\mathcal{A}} = F_1^{-1}(L_n^{\perp}) \tilde{\oplus} F_1'^{-1}(P') \tilde{\oplus} N_1'$. Indeed, let $x \in H_{\mathcal{A}}$. Then $Fx = y + y' + z_2'$ for some $y \in L_n^{\perp}, y' \in P'$ and $z_2' \in N_2'$. Since F_1 is an isomorphism of $F_1^{-1}(L_n^{\perp})$ onto $L_n^{\perp}$ and F_1' is an isomorphism of $F_1'^{-1}(P')$ onto P', there exist some $u \in F_1^{-1}(L_n^{\perp})$ and $v \in F_1'^{-1}(P')$ such that $y = F_1 u$ and $y' = F_1' v$. Hence, $Fx = F_1 u + F_1' v + z_2'$. It follows that

$$z_2' = Fx - F_1 u - F_1' v = F(x - u - v) \in ImF \cap N_2'.$$

Hence, $(x - u - v) \in F^{-1}(N_2')$. Since F has the matrix $\begin{bmatrix} F_1' & 0 \\ 0 & F_4' \end{bmatrix}$ with respect to the decomposition

$$H_{\mathcal{A}} = M_1' \tilde{\oplus} N_1' \xrightarrow{F} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}},$$

where F_1' is an isomorphism, by Lemma 3.1.3 it follows that $F^{-1}(N_2') = N_1'$. Now,

$$x = (x - u - v) + u + v,\ u \in F_1^{-1}(L_n^{\perp}),\ v \in F_1'^{-1}(P'), (x - u - v) \in F^{-1}(N_2') = N_1'.$$

Since $x \in H_{\mathcal{A}}$ was arbitrary, it follows that $H_{\mathcal{A}} = F_1^{-1}(L_n^{\perp}) + F_1'^{-1}(P') + N_1'$. Moreover, since the submodules $F_1^{-1}(L_n^{\perp}), F_1'^{-1}(P'), N_1'$ obviously mutually intersects trivially (here we also use that $F_{|_{L_n^{\perp}}}$ and $F_{|_{P'}}$ are isomorphisms, thus injective), it follows that

$$H_{\mathcal{A}} = F_1^{-1}(L_n^{\perp}) \tilde{\oplus} F_1'^{-1}(P') \tilde{\oplus} N_1'.$$

Hence, as we also have $H_{\mathcal{A}} = F_1^{-1}(L_n^{\perp}) \tilde{\oplus} F_1^{-1}(P) \tilde{\oplus} N_1$, it follows that

$$F_1^{-1}(P) \tilde{\oplus} N_1 \cong F_1'^{-1}(P') \tilde{\oplus} N_1'.$$

□

Remark 3.5.23. The proof of Lemma 3.5.22 is exactly the same as the proof of [21, Lemma 9].

Remark 3.5.24. Lemma 3.1.13 holds also for non-adjointable operators. Indeed, if $P \in B(H_{\mathcal{A}})$ and P a projection with finitely generated kernel, then $P \in \widehat{\mathcal{M}\Phi}(H_{\mathcal{A}})$. If in addition we have $F \in \widehat{\mathcal{M}\Phi}(H_{\mathcal{A}})$, then by Corollary 3.5.5 we get $PFP \in \widehat{\mathcal{M}\Phi}(H_{\mathcal{A}})$. Let

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{PFP} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

be an $\widehat{\mathcal{M}\Phi}$-decomposition for PFP. By Lemma 3.5.6 we have that both $P(M_1)$ and $PF(M_1)$ are closed and complementable in $H_{\mathcal{A}}$. Indeed,

$$H_{\mathcal{A}} = P(M_1) \tilde{\oplus} (PF)^{-1}(N_2) = PF(M_1) \tilde{\oplus} P^{-1}(N_2).$$

By Lemma 2.0.66 it follows then that $P(M_1)$ and $PF(M_1)$ are complementable in ImP. Hence, by applying these facts instead of Theorem 2.0.20 we can proceed in the same way as in the proof of Lemma 3.1.13.

3.6 Non-adjointable semi-C^*-Weyl operators

Recall now Definition 3.4.11 of the classes $\mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}}), \mathcal{M}\Phi_+^{+\prime}(H_{\mathcal{A}})$. We are going to use the same notation here, only without assuming the adjointability of operators.

Lemma 3.6.1. *[21, Lemma 10] Let $F \in B(H_{\mathcal{A}})$. Then F admits an upper external (Noether) decomposition with the property that $X_2 \preceq X_1$ if and only if $F \in \mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}})$. Similarly, F admits a lower external (Noether) decomposition with the property that $X_1 \preceq X_2$ if and only if $F \in \mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}})$.*

Proof. The statements can be shown in a similar way as in the proof of Proposition 3.5.18. □

Lemma 3.6.2. *[21, Lemma 11] Let $F \in \mathcal{M}\Phi_-^{+\prime}(H_{\mathcal{A}})$. Then $F + K \in \mathcal{M}\Phi_-^{+\prime}(H_{\mathcal{A}})$ for all $K \in \mathcal{K}(H_{\mathcal{A}})$.*

Proof. Let $H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$ be an $\mathcal{M}\Phi_-^{+\prime}$ -decomposition for F. Then N_2 is finitely generated and $N_2 \preceq N_1$. We may assume that

$$N_2 \subseteq L_n, \;\; L_n = N_2 \tilde{\oplus} P \text{ and } M_2 = L_n^{\perp} \oplus P$$

for some $n \in \mathbb{N}$ and some finitely generated Hilbert submodule P. Indeed, by the proof of Theorem 2.0.34, $L_n = (M_2 \cap L_n) \tilde{\oplus} p_n(N_2)$ where $p_n(N_2) \cong N_2$. Hence we get

$$H_{\mathcal{A}} = L_n^{\perp} \tilde{\oplus} P \tilde{\oplus} p_n(N_2) = L_n^{\perp} \tilde{\oplus} P \tilde{\oplus} N_2$$

where $P = M_2 \cap L_n$. Let $\sqcap$ denote the projection onto $L_n^{\perp} \tilde{\oplus} P$ along N_2 and $V = \begin{bmatrix} \sqcap & 0 \\ 0 & p_n \end{bmatrix}$ with respect to the decomposition

$$H_{\mathcal{A}} = M_2 \tilde{\oplus} N_2 \longrightarrow (L_n^{\perp} \oplus P) \tilde{\oplus} p_n(N_2) = H_{\mathcal{A}}.$$

Then V is an isomorphism, hence,

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{VF} (L_n^{\perp} \oplus P) \tilde{\oplus} p_n(N_2) = H_{\mathcal{A}}$$

is an $\mathcal{M}\Phi_-^{+'}-$ decomposition for VF. If we can show that $VF + K \in \mathcal{M}\Phi_-^{+\prime}(H_{\mathcal{A}})$ for all $K \in \mathcal{K}(H_{\mathcal{A}})$, it would follow that $F + V^{-1}K \in \mathcal{M}\Phi_-^{+\prime}(H_{\mathcal{A}})$ for all $K \in \mathcal{K}(H_{\mathcal{A}})$ since V is an isomorphism. However, since $\mathcal{K}(H_{\mathcal{A}})$ is two sided ideal in $B(H_{\mathcal{A}})$, we have $V^{-1}\mathcal{K}(H_{\mathcal{A}}) = \mathcal{K}(H_{\mathcal{A}})$, hence it suffices to consider the operator VF instead of F.

Moreover, we may choose an n big enough such that $\| q_n K \| < \| F_1^{-1} \|^{-1}$. This is possible by Theorem 2.0.56. Then we may proceed as in the proof of Lemma 2.0.45 and use that $N_2 \preceq N_1$ in order to deduce the lemma. □

Lemma 3.6.3. *Let $F \in \mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}})$ and $K \in \mathcal{K}(H_{\mathcal{A}})$. Then $F + K \in \mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}})$.*

Proof. Let $F \in \mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}}), K \in \mathcal{K}(H_{\mathcal{A}})$ and

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

be an $\mathcal{M}\Phi_+^{-\prime}$-decomposition for F. Set $F_1 = F_{|_{M_1}}$ and consider the operator G given by the operator matrix $\begin{bmatrix} F_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$ with respect to the decomposition

$$H_{\mathcal{A}} = M_2 \tilde{\oplus} N_2 \longrightarrow M_1 \tilde{\oplus} N_1 = H_{\mathcal{A}}.$$

Then GF has the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ with respect to the decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{GF} M_1 \tilde{\oplus} N_1 = H_{\mathcal{A}}.$$

Now, as in the proof of Lemma 2.0.45, we may without loss of generality assume that there exists some $m \in \mathbb{N}$ such that for all $k \geq m$ we have $M_1 = L_k^{\perp} \tilde{\oplus} P$ and $L_k = P \tilde{\oplus} N_1$, since N_1 is finitely generated. Indeed, by the proof Theorem 2.0.34 there exists some $m \in \mathbb{N}$ such that for all $k \geq m$, we have $L_k = P \tilde{\oplus} p_k(N_1)$, where $P = M_1 \cap L_k$ and $p_k(N_1) \cong N_1$ (here p_k denotes the orthogonal projection onto L_k). Therefore, we have $H_{\mathcal{A}} = L_k^{\perp} \tilde{\oplus} P \tilde{\oplus} p_k(N_1) = L_k^{\perp} \tilde{\oplus} P \tilde{\oplus} N_1$. This holds for all $k \geq m$. Let Q denote the projection onto M_1 along N_1 and W be the operator that has the matrix $\begin{bmatrix} Q & 0 \\ 0 & {p_{k|_{N_1}}}^{-1} \end{bmatrix}$ with respect to the decomposition

$$H_{\mathcal{A}} = (L_k^{\perp} \tilde{\oplus} P) \tilde{\oplus} p_k(N_1) \xrightarrow{W} M_1 \tilde{\oplus} N_1 = H_{\mathcal{A}}.$$

Then W is an isomorphism. The operator FW has the matrix $\begin{bmatrix} (FW)_1 & 0 \\ 0 & (FW)_4 \end{bmatrix}$ with respect to the decomposition

$$H_{\mathcal{A}} = (L_k^{\perp} \tilde{\oplus} P) \tilde{\oplus} p_k(N_1) \xrightarrow{FW} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}},$$

where $(FW)_1$ is an isomorphism. Thus, we may consider the operator FW instead of the operator F. If we can show that $FW + K \in \mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}})$ for all $K \in \mathcal{K}(H_{\mathcal{A}})$, it would follow that for all $K \in \mathcal{K}(H_{\mathcal{A}})$ we have $(FW + K)W^{-1} = F + KW^{-1} \in \mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}})$. Now, since $\mathcal{K}(H_{\mathcal{A}})$ is two sided ideal in $B(H_{\mathcal{A}})$, we have $\mathcal{K}(H_{\mathcal{A}}) = \mathcal{K}(H_{\mathcal{A}})W^{-1}$, so we may in fact without loss of generality assume that F has $\mathcal{M}\Phi_+^{-\prime}$-decomposition

$$H_{\mathcal{A}} = (L_m^{\perp} \tilde{\oplus} P) \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

for some $m \in \mathbb{N}$ and some finitely generated Hilbert submodule P satisfying $L_m = P \tilde{\oplus} N_1$, i.e. we may assume that $M_1 = L_m^{\perp} \tilde{\oplus} P$ and $L_m = P \tilde{\oplus} N_1$.
Let now $K \in \mathcal{K}(H_{\mathcal{A}})$. Again, since $\mathcal{K}(H_{\mathcal{A}})$ is a two sided ideal in $B(H_{\mathcal{A}})$, we have $GK \in \mathcal{K}(H_{\mathcal{A}})$. By Theorem 2.0.56 there exists some $k \geq m$ such that $\| q_k GK \| < 1$. Then we observe that $M_1 = L_m^{\perp} \oplus P = L_k^{\perp} \oplus \tilde{P}$, where $\tilde{P} = P \oplus (L_m^{\perp} \setminus L_k^{\perp})$. It follows that GF has the matrix $\begin{bmatrix} 1 & 0 \\ 0 & \sqcap \end{bmatrix}$ with respect to the decomposition $L_k^{\perp} \oplus L_k \xrightarrow{GF} L_k^{\perp} \oplus L_k$, where $\sqcap$ denotes the projection onto $\tilde{P}$ along N_1. Then, with respect to the decomposition

$$H_{\mathcal{A}} = L_k^{\perp} \tilde{\oplus} L_k \xrightarrow{GF+GK} L_k^{\perp} \tilde{\oplus} L_k = H_{\mathcal{A}},$$

the operator $GF+GK$ has the matrix $\begin{bmatrix} (GF+GK)_1 & (GF+GK)_2 \\ (GF+GK)_3 & (GF+GK)_4 \end{bmatrix}$, where $(GF+GK)_1$ is an isomorphism, since $\| q_k GK_{|_{L_k^\perp}} \| \leq \| q_k GK \| < 1$. Hence $GF+GK$ has the matrix

$$\begin{bmatrix} \overline{(GF+GK)_1} & 0 \\ 0 & \overline{(GF+GK)_4} \end{bmatrix}$$

with respect to the decomposition

$$H_{\mathcal{A}} = L_k^\perp \tilde{\oplus} U(L_k) \xrightarrow{GF+GK} V^{-1}(L_k^\perp) \tilde{\oplus} L_k = H_{\mathcal{A}},$$

where $\overline{(GF+GK)_1}, U, V$ are isomorphisms. From this (using that $GF+GK = G(F+K)$) and by Lemma 3.5.6 we obtain that G has the matrix $\begin{bmatrix} G_1 & 0 \\ 0 & G_4 \end{bmatrix}$ with respect to the decomposition

$$H_{\mathcal{A}} = (F+K)L_k^\perp \tilde{\oplus} N \xrightarrow{G} V^{-1}(L_k^\perp) \tilde{\oplus} L_k = H_{\mathcal{A}},$$

where $N = G^{-1}(L_k)$ and G_1 is an isomorphism. Also, we obtain that $F+K$ has the matrix

$$\begin{bmatrix} (F+K)_1 & 0 \\ 0 & (F+K)_4 \end{bmatrix}$$

with respect to the decomposition

$$H_{\mathcal{A}} = L_k^\perp \tilde{\oplus} U(L_k) \xrightarrow{F+K} (F+K)L_k^\perp \tilde{\oplus} N = H_{\mathcal{A}},$$

where $(F+K)_1$ is an isomorphism.
However, since G has the matrix $\begin{bmatrix} F_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$ with respect to the decomposition

$$H_{\mathcal{A}} = M_2 \tilde{\oplus} N_2 \xrightarrow{G} M_1 \tilde{\oplus} N_1 = H_{\mathcal{A}},$$

it follows that G has the matrix $\begin{bmatrix} \tilde{\tilde{G}}_1 & 0 \\ 0 & \tilde{\tilde{G}}_4 \end{bmatrix}$ with respect to the decomposition

$$H_{\mathcal{A}} = F(L_k^\perp) \tilde{\oplus} (F(\tilde{P}) \tilde{\oplus} N_2) \xrightarrow{G} L_k^\perp \tilde{\oplus} L_k = H_{\mathcal{A}},$$

where $\tilde{\tilde{G}}_1 = F_1^{-1}|_{F(L_k^\perp)}$ is an isomoprhism (observe that $M_2 = F(L_k^\perp) \tilde{\oplus} F(\tilde{P})$ since $M_1 = L_k^\perp \oplus \tilde{P}$). From Lemma 3.1.3 it follows that $F(\tilde{P}) \tilde{\oplus} N_2 = N = G^{-1}(L_k)$. Since $N_1 \preceq N_2$ and $F_{|_{\tilde{P}}}$ is an isomorphism, we get that

$$L_k = \tilde{P} \tilde{\oplus} N_1 \preceq F(\tilde{P}) \tilde{\oplus} N_2 = N.$$

Moreover, $L_k \cong U(L_k)$ and, as we have seen above, $F+K$ has the matrix $\begin{bmatrix} (F+K)_1 & 0 \\ 0 & (F+K)_4 \end{bmatrix}$ with respect to the decomposition

$$H_{\mathcal{A}} = L_k^\perp \tilde{\oplus} U(L_k) \xrightarrow{F+K} (F+K)L_k^\perp \tilde{\oplus} N = H_{\mathcal{A}},$$

where $(F+K)_1$ is an isomorphism. $\square$

Recall Definition 3.4.1. Let us again use the same notation here, but without assuming the adjointability of operators. It can be proved similarly as in the proof of Lemma 3.6.2 that the classes $\tilde{\mathcal{M}\Phi}_-^+(H_\mathcal{A})$, $\tilde{\mathcal{M}\Phi}_+^-(H_\mathcal{A})$ are invariant under compact perturbations or, more precisely, under perturbations by operators belonging to the class $\mathcal{K}(H_\mathcal{A})$. Since $\widehat{\mathcal{M}\Phi}_l(H_\mathcal{A})\backslash\widehat{\mathcal{M}\Phi}(H_\mathcal{A})$ and $\widehat{\mathcal{M}\Phi}_r(H_\mathcal{A})\backslash\widehat{\mathcal{M}\Phi}(H_\mathcal{A})$ correspond to the set of all left invertible, but not invertible elements and the set of all right invertible, but not invertible elements in the Calkin algebra $B(H_\mathcal{A})/\mathcal{K}(H_\mathcal{A})$, respectively, it follows that these sets are also invariant under compact perturbations. Thus, also in the setting of non-adjointable operators, the classes $\mathcal{M}\Phi_+^-(H_\mathcal{A})$ and $\mathcal{M}\Phi_-^+(H_\mathcal{A})$ are invariant under compact perturbations, being the union of the sets which are invariant under compact perturbations.

Definition 3.6.4. We set

$$M(H_\mathcal{A}) = \{D \in B(H_\mathcal{A}) \mid F \text{ is bounded below and } ImF \text{ is complementable in } H_\mathcal{A}\},$$

$$Q(H_\mathcal{A}) = \{G \in B(H_\mathcal{A}) \mid G \text{ is surjective and } \ker G \text{ is complementable in } H_\mathcal{A}\}.$$

Then we have the following propositions which give a description of the sets $\mathcal{M}\Phi_+^{-'}(H_\mathcal{A})$ and $\mathcal{M}\Phi_-^{+'}(H_\mathcal{A})$ in terms of compact perturbations.

Proposition 3.6.5. *Let $F \in B(H_\mathcal{A})$. Then the following statements are equivalent:*
(1) $F \in \mathcal{M}\Phi_+^{-'}(H_\mathcal{A})$,
(2) There exist $D \in M(H_\mathcal{A})$ and $K \in \mathcal{K}(H_\mathcal{A})$ such that $F = D + K$.

Proof. From Lemma 3.6.3 it follows that (2) $\Longrightarrow$ (1), since $M(H_\mathcal{A}) \subseteq \mathcal{M}\Phi_+^{-'}(H_\mathcal{A})$. Let us show the implication (1) $\Longrightarrow$ (2). If $H_\mathcal{A} = M_1\tilde{\oplus}N_1 \xrightarrow{F} M_2\tilde{\oplus}N_2 = H_\mathcal{A}$ is an $\mathcal{M}\Phi_+^{-'}$-decomposition for F, then there exists an isomorphism ι of N_1 onto a closed submodule of N_2. Since N_1 is finitely generated, we have that $\iota(N_1)$ is finitely generated as well. By Lemma 2.0.25 there exists a closed submodule N of N_2 such that $\iota(N_1)\tilde{\oplus}N = N_2$. Let $\sqcap$ denote the projection onto N_1 along M_1 and D be the operator having the matrix $\begin{bmatrix} F_{|_{M_1}} & 0 \\ 0 & \iota \end{bmatrix}$ with respect to the decomposition

$$H_\mathcal{A} = M_1\tilde{\oplus}N_1 \xrightarrow{D} M_2\tilde{\oplus}N_2 = H_\mathcal{A}.$$

Then D is bounded below and $ImD\tilde{\oplus}N = M_2\tilde{\oplus}\iota(N_1)\tilde{\oplus}N = H_\mathcal{A}$, so $D \in M(H_\mathcal{A})$. Moreover, $D + (F-\iota)\sqcap = F$ and $(F-\iota)\sqcap \in \mathcal{K}(H_\mathcal{A})$ since $\sqcap \in \mathcal{K}(H_\mathcal{A})$. □

Proposition 3.6.6. *Let $F \in B(H_\mathcal{A})$. Then the following statements are equivalent:*
(1) $F \in \mathcal{M}\Phi_-^{+'}(H_\mathcal{A})$,
(2) There exist $G \in Q(H_\mathcal{A})$ and $K \in \mathcal{K}(H_\mathcal{A})$ such that $F = G + K$.

Proof. From Lemma 3.6.2 we have that (2) implies (1) since $Q(H_\mathcal{A}) \subseteq \mathcal{M}\Phi_-^{+'}(H_\mathcal{A})$, so it suffices to prove the opposite implication .
Let $H_\mathcal{A} = M_1\tilde{\oplus}N_1 \xrightarrow{F} M_2\tilde{\oplus}N_2 = H_\mathcal{A}$ be an $\mathcal{M}\Phi_-^{+'}$-decomposition for F. Then there exist Hilbert submodules N' and N'' such that $N_1 = N'\tilde{\oplus}N''$ and $N' \cong N_2$. Indeed, since $N_2 \preccurlyeq N_1$, there exists a closed submodule N' of N_1 such that $N_2 \cong N'$. As N_2 is finitely generated, it follows that N' is finitely generated also. Hence, by Lemma 2.0.24, we have that $N_1 = N' \oplus N''$ for some closed submodule N''.

Set ι to be isomorphism of N' onto N_2 and P be the projection onto N' along N''. Let G be the operator with the matrix $\left[\begin{array}{cc} F_{|_{M_1}} & 0 \\ 0 & \iota P \end{array}\right]$ with respect to the decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{G} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}.$$

Then G is obviously surjective and $\ker G = N''$, which complementable in $H_{\mathcal{A}}$. Moreover, $F = G + (F - \iota P)\sqcap$, where $\sqcap$ stands for the projection onto N_1 along M_1. Put $\tilde{\sqcap}$ to be the projection onto N_2 along M_2. Then $\tilde{\sqcap} \in \mathcal{K}(H_{\mathcal{A}})$ since N_2 is finitely generated. We have $(F - \iota P)\sqcap = \tilde{\sqcap}(F - \iota P)\sqcap \in \mathcal{K}(H_{\mathcal{A}})$. $\square$

Let $\mathcal{M}\Phi_0(H_{\mathcal{A}})$ have the same meaning as in Definition 3.4.1, only without assuming the adjointability of operators. Similarly as in the propositions above, we can prove that the following statements are equivalent:
(1) $F \in \mathcal{M}\Phi_0(H_{\mathcal{A}})$,
(2) There exist $T \in B(H_{\mathcal{A}})$ and $K \in \mathcal{K}(H_{\mathcal{A}})$ such that T is invertible and $F = T + K$.
In addition we have the analogue of Proposition 3.4.26 in the setting of non-adjointable operators.

Next, we introduce the following auxiliary result.

Lemma 3.6.7. *The sets $M(H_{\mathcal{A}})$ and $Q(H_{\mathcal{A}})$ are semigroups under the multiplication.*

Proof. Let $D, D' \in M(H_{\mathcal{A}})$. Since D and D' are both bounded below, it follows that $D'D$ is also bounded below. Now, since D' is bounded below and ImD, ImD' are both complementable in $H_{\mathcal{A}}$, we get

$$H_{\mathcal{A}} = ImD' \tilde{\oplus} ImD'^{\circ} = (D'(ImD \tilde{\oplus} ImD^{\circ})) \tilde{\oplus} ImD'^{\circ}$$
$$= D'(ImD) \tilde{\oplus} D'(ImD^{\circ}) \tilde{\oplus} ImD'^{\circ} = ImD'D \tilde{\oplus} (D'(ImD^{\circ})) \tilde{\oplus} ImD^{\circ}.$$

Thus, $ImD'D$ is complementable in $H_{\mathcal{A}}$, so $D'D \in M(H_{\mathcal{A}})$.
Next, let $G, G' \in Q(H_{\mathcal{A}})$. Obviously, $G'G$ is surjective. So, since

$$H_{\mathcal{A}} = \ker G'^{\circ} \tilde{\oplus} \ker G' = \ker G^{\circ} \tilde{\oplus} \ker G$$

and $G_{|_{\ker G^{\circ}}}$ is an isomorphism onto $H_{\mathcal{A}}$, it follows that $\ker G^{\circ} = R \tilde{\oplus} R^{\circ}$ for some Hilbert submodules R and R° where $G_{|_R}$ and $G_{|_{R^{\circ}}}$ are isomorphisms onto $\ker G'$ and $\ker G'^{\circ}$, respectively. Therefore, $\ker G'G = \ker G \tilde{\oplus} R$, so $\ker G'G$ is complementable in $H_{\mathcal{A}}$ since

$$H_{\mathcal{A}} = \ker G \tilde{\oplus} \ker G^{\circ} = \ker G \tilde{\oplus} R \tilde{\oplus} R^{\circ}.$$

Thus, $G'G \in Q(H_{\mathcal{A}})$. $\square$

Lemma 3.6.8. *The sets $M(H_{\mathcal{A}})$ and $Q(H_{\mathcal{A}})$ are open in the norm topology.*

Proof. Let $\tilde{M}(H_{\mathcal{A}})$ and $\tilde{Q}(H_{\mathcal{A}})$ denote the sets of bounded below operators and surjective operators on $H_{\mathcal{A}}$, respectively. Then these sets are open. Since

$$M(H_{\mathcal{A}}) = \tilde{M}(H_{\mathcal{A}}) \cap \widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}}) \text{ and } Q(H_{\mathcal{A}}) = \tilde{Q}(H_{\mathcal{A}}) \cap \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}}),$$

which holds by Proposition 3.5.11 and Proposition 3.5.13, respectively, it follows that $M(H_{\mathcal{A}})$ and $Q(H_{\mathcal{A}})$ are open. Moreover, the lemma holds also in the case of arbitrary Hilbert C^*-modules and not just $H_{\mathcal{A}}$. $\square$

Corollary 3.6.9. *$\mathcal{M}\Phi_+^{-'}(H_{\mathcal{A}})$ and $\mathcal{M}\Phi_-^{+'}(H_{\mathcal{A}})$ are semigroups under the multiplication.*

Corollary 3.6.10. *The sets* $M(H_{\mathcal{A}}) \setminus \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}})$ *and* $Q(H_{\mathcal{A}}) \setminus \widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}})$ *are open.*

Lemma 3.6.11. *Let M be a Hilbert C^*-module and $F \in \mathcal{M}\Phi_+^{-'}(M)$. If*

$$M = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = M$$

is an $\mathcal{M}\Phi_+^{-'}$-decomposition for F and $D \in B(M)$ is such that $\sqcap(D+F)_{|_{M_1}} \in \mathcal{M}\Phi_+^{-'}(M_1, M_2)$ where $\sqcap$ stands for the projection onto M_2 along N_2, then $D + F \in \mathcal{M}\Phi_+^{-'}(M)$. Similar statements hold for the classes $\mathcal{M}\Phi_-^{+'}, \mathcal{M}\Phi_+$, $\mathcal{M}\Phi_-$, $\mathcal{M}\Phi$, $\mathcal{M}\Phi_0$, $\tilde{\mathcal{M}\Phi}_+^-$, and $\tilde{\mathcal{M}\Phi}_-^+$.

Proof. Let

$$M_1 = \tilde{M}_1 \tilde{\oplus} \tilde{N}_1 \xrightarrow{F} \tilde{M}_2 \tilde{\oplus} \tilde{N}_2 = M_2$$

be an $\mathcal{M}\Phi_+^{-'}$-decomposition for $\sqcap(D+F)_{|_{M_1}}$. Then $\tilde{N}_1$ is finitely generated, $\tilde{N}_1 \preceq \tilde{N}_2$ and $\sqcap(D+F)_{|_{\tilde{M}_1}}$ is an isomorphism onto $\tilde{M}_2$. If we let $\tilde{\sqcap}$ denote the projection onto $\tilde{M}_2$ along $\tilde{N}_2 \tilde{\oplus} N_2$, then $\tilde{\sqcap}(D+F)_{|_{\tilde{M}_1}} = \sqcap(D+F)_{|_{\tilde{M}_1}}$. Hence $D+F$ has the matrix $\left[\begin{array}{cc} (D+F)_1 & (D+F)_2 \\ (D+F)_3 & (D+F)_4 \end{array}\right]$ with respect to the decomposition

$$M = \tilde{M}_1 \tilde{\oplus} (\tilde{N}_1 \tilde{\oplus} N_1) \xrightarrow{D+F} \tilde{M}_2 \tilde{\oplus} (\tilde{N}_2 \tilde{\oplus} N_2) = M,$$

where $(D+F)_1$ is an isomorphism. Moreover, since $N_1 \preceq N_2$, $\tilde{N}_1 \preceq \tilde{N}_2$ and N_1, N_2 are finitely generated, it follows that $N_1 \tilde{\oplus} \tilde{N}_1$ is finitely generated and $N_1 \tilde{\oplus} \tilde{N}_1 \preceq N_2 \tilde{\oplus} \tilde{N}_2$. Then we can proceed in the same way as in the proof of Lemma 2.0.42 to deduce that there exist isomorphisms U and V such that

$$M = \tilde{M}_1 \tilde{\oplus} U(\tilde{N}_1 \tilde{\oplus} N_1) \xrightarrow{D+F} V(\tilde{M}_2) \tilde{\oplus} (\tilde{N}_2 \tilde{\oplus} N_2) = M$$

is an $\mathcal{M}\Phi_+^{-'}$-decomposition for $D+F$.

The proofs for the other cases are similar.

□

3.7 Examples of semi-C^*-Fredholm operators

At the end of this chapter we introduce some examples of semi-$\mathcal{A}$-Fredholm operators.

Example 3.7.1. Let $F \in B^a(H_{\mathcal{A}})$ satisfying that $F(e_k) = e_{2k}$ for all $k \in \mathbb{N}$.
Then $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$.

Example 3.7.2. Let $D \in B^a(H_{\mathcal{A}})$ satisfying that $D(e_{2k-1}) = 0, D(e_{2k}) = e_k$ for all $k \in \mathbb{N}$.
Then $D \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$.

Example 3.7.3. In general, let $\iota : \mathbb{N} \to \iota(\mathbb{N})$ be a bijection such that $\iota(\mathbb{N}) \subseteq \mathbb{N}$ and $\mathbb{N} \setminus \iota(\mathbb{N})$ is infinite. Moreover, we may define ι in a such way that $\iota(1) < \iota(2) < \iota(3) < \dots$. Then, if we define an $\mathcal{A}$-linear bounded operator F as $F(e_k) = e_{\iota(k)}$ for all k, we get that $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$. Also, if we define an $\mathcal{A}$-linear operator D as

$$D(e_k) = \begin{cases} e_{\iota^{-1}(k)}, & \text{for } k \in \iota(\mathbb{N}), \\ 0, & \text{else}, \end{cases}$$

then $D \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$.

Those examples are also valid in the case when $\mathcal{A} = \mathbb{C}$, that is when $H_{\mathcal{A}} = H$ is a Hilbert space. We will now introduce examples where we use the structure of $\mathcal{A}$ itself in the case when $\mathcal{A} \neq \mathbb{C}$.

Example 3.7.4. Let $\mathcal{A} = L^\infty([0,1],\mu)$, where μ is the Lebesgue measure. Set

$$F(f_1, f_2, f_3, \dots) = (\mathcal{X}_{[0,\frac{1}{2}]} f_1, \mathcal{X}_{[\frac{1}{2},1]} f_1, \mathcal{X}_{[0,\frac{1}{2}]} f_2, \mathcal{X}_{[\frac{1}{2},1]} f_2, \dots) \ .$$

Then F is a bounded $\mathcal{A}-$ linear operator, $\ker F = \{0\}$,

$$ImF = Span_{\mathcal{A}}\{\mathcal{X}_{[0,\frac{1}{2}]} e_1, \mathcal{X}_{[\frac{1}{2},1]} e_2, \mathcal{X}_{[0,\frac{1}{2}]} e_3, \mathcal{X}_{[\frac{1}{2},1]} e_4, \dots\},$$

and, clearly, $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$. Actually, F is an isometry onto its image.

Example 3.7.5. Let again $\mathcal{A} = (L^\infty([0,1]),\mu)$. Set

$$D(g_1, g_2, g_3, \dots) = (\mathcal{X}_{[0,\frac{1}{2}]} g_1 + \mathcal{X}_{[\frac{1}{2},1]} g_2, \mathcal{X}_{[0,\frac{1}{2}]} g_3 + \mathcal{X}_{[\frac{1}{2},1]} g_4, \dots) \ .$$

Then $\ker D = ImF^\perp$, D is an $\mathcal{A}$-linear, bounded operator and $ImD = H_{\mathcal{A}}$. Thus, $D \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$. Indeed, $D = F^*$, where F is the operator from Example 3.7.4.

Example 3.7.6. Let $\mathcal{A} = B(H)$, where H is a Hilbert space and let P be an orthogonal projection on H. Set

$$F(T_1, T_2, \dots) = (PT_1, (I-P)T_1, PT_2, (I-P)T_2, \dots),$$

$$D(S_1, S_2, \dots) = (PS_1 + (I-P)S_2, PS_3 + (I-P)S_4, \dots).$$

Then, by the similar arguments as in Example 3.7.4 and Example 3.7.5, we have $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$ and $D \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$. Moreover, $D = F^*$.

Example 3.7.7. In general, supose that $\{p_j^i\}_{j,i\in\mathbb{N}}$ is a family of projections in $\mathcal{A}$ such that $p_{j_1}^i p_{j_2}^i = 0$ for all i, whenever $j_1 \neq j_2$, and $\sum\limits_{j=1}^{k} p_j^i = 1$ for all i and some $k \in \mathbb{N}$.
Set

$$F'(\alpha_1, \dots, \alpha_n, \dots) = (p_1^1\alpha_1, p_2^1\alpha_1, \dots p_k^1\alpha_1, p_2^1\alpha_2, p_2^2\alpha_2, \dots p_k^2\alpha_2, \dots),$$

$$D'(\beta_1, \dots, \beta_n, \dots) = (\sum_{i=1}^{k} p_i^1 \beta_i, \sum_{i=1}^{k} p_i^2 \beta_{i+k}, \dots).$$

Then $F' \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$ and $D' \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$.

Later, in Section 7.1, when we introduce the generalized spectra in $\mathcal{A}$ of operators on $H_{\mathcal{A}}$, we calculate in Example 7.1.27 the generalized spectra of semi-$\mathcal{A}$-Fredholm operators from Example 3.7.4 and Example 3.7.5. Notice that all these examples of $\mathcal{M}\Phi_+$ operators so far are actually examples of operators that are bounded below, whereas all our examples of $\mathcal{M}\Phi_-$ operators so far are examples of surjective operators. Since

$$M^a(H_{\mathcal{A}}) \subseteq \mathcal{M}\Phi_+^{-'}(H_{\mathcal{A}}) \text{ and } Q^a(H_{\mathcal{A}}) \subseteq \mathcal{M}\Phi_-^{+'}(H_{\mathcal{A}}),$$

it follows that these operators are actually examples of semi-$\mathcal{A}$-Weyl operators. In Section 9.2 we shall give some more examples of semi-$\mathcal{A}$-Weyl operators.

Recalling now that a composition of two $\mathcal{M}\Phi_+$ operators on $H_{\mathcal{A}}$ is again an $\mathcal{M}\Phi_+$ operator on $H_{\mathcal{A}}$ and that the same is true for $\mathcal{M}\Phi_-$ operators, we may take suitable compositions of operators from these examples in order to construct more $\mathcal{M}\Phi_\pm$ operators.
Even more $\mathcal{M}\Phi_\pm$ operators can be obtained by composing these operators with isomorphisms of $H_{\mathcal{A}}$. We will present here also some isomorphisms of $H_{\mathcal{A}}$.

Example 3.7.8. Let $j : \mathbb{N} \to \mathbb{N}$ be a bijection. Then the operator U given by $U(e_k) = e_{j(k)}$ for all k is an isomorphism of $H_{\mathcal{A}}$. This is a classical well known example of an isomorphism.

Example 3.7.9. Let $(\alpha_1, \dots, \alpha_n, \dots) \in \mathcal{A}^{\mathbb{N}}$ be a sequence of invertible elements in $\mathcal{A}$ such that $\| \alpha_k \|, \| \alpha_k^{-1} \| \leq M$ for all $k \in \mathbb{N}$ and some $M > 0$. If the operator V is given by

$$V(x_1, \cdots, x_n, \cdots) = (\alpha_1 x_1 \cdots, \alpha_n x_n, \cdots) \text{ for all } (x_1, \cdots, x_n, \cdots) \in H_{\mathcal{A}},$$

then V is an isomorphism of $H_{\mathcal{A}}$.

We will now apply some of the techniques and the ideas from the proofs of the results in $\mathcal{A}$-Fredholm theory and semi-$\mathcal{A}$-Fredholm theory in order to extend the results from the classical semi-Fredholm theory on infinite-dimensional Hilbert spaces to a new, greater class of operators on infinite-dimensional Hilbert spaces.

Definition 3.7.10. Let H be a separable infinite-dimensional Hilbert space. We set $g\mathcal{M}\Phi(H)$ to be the class of all operators $F \in B(H)$ for which there exists a decomposition

$$H = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H$$

with respect to which F has the matrix $\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}$, where F_1 is an isomorphism and M_1, M_2 are infinite-dimensional, closed subspaces of H.

Remark 3.7.11. Note that we only require that M_1 and M_2 are infinite-dimensional, closed subspaces, but we do not require that N_1 or N_2 to be finite dimensional. Thus, this class of operators on H is strictly greater than the class of semi-Fredholm operators on H and includes the class of semi-Fredholm operators. In the rest of this section we always assume that H is a separable, infinite-dimensional Hilbert space. We have the following characterization of $g\mathcal{M}\Phi$-operators.

Lemma 3.7.12. *Let $F \in B(H)$. Then $F \in g\mathcal{M}\Phi(H)$ if and only if ImF contains an infinite-dimensional closed subspace.*

Proof. Suppose that there exists an infinite-dimensional closed subspace $M \subseteq ImF$. We set $\tilde{F} = F_{|_{F^{-1}(M)}}$. Then $\tilde{F} \in B(F^{-1}(M), M)$ and $\tilde{F}$ is surjective. Let M_1 denote the orthogonal complement of $\ker \tilde{F}$ in $F^{-1}(M)$. It follows that $\tilde{F}_{|_{M_1}}$ is an isomorphism onto M. With respect to the decomposition

$$H = M_1 \oplus M_1^{\perp} \xrightarrow{F} M \oplus M^{\perp} = H,$$

F has the matrix $\begin{bmatrix} F_1 & F_2 \\ 0 & F_4 \end{bmatrix}$, where F_1 is an isomorphism. Using the technique of diagonalization as in the proof of Lemma 2.0.42 we easily obtain that $F \in g\mathcal{M}\Phi(H)$, because M and thus M_1 are infinite-dimensional.
The "only if" part follows from the definition of the class $g\mathcal{M}\Phi(H)$. □

Lemma 3.7.13. *Let $F, D \in g\mathcal{M}\Phi(H)$, Then $DF \in g\mathcal{M}\Phi(H)$ if and only if there exist two $g\mathcal{M}\Phi$-decompositions*

$$H = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H,$$

$$H = M_1' \tilde{\oplus} N_1' \xrightarrow{D} M_2' \tilde{\oplus} N_2' = H$$

for F and D, respectively, such that $M_2 \cap M_1'$ is an infinite-dimensional subspace.

Proof. Suppose first that such decompositions exist and set $F_1 := F_{|_{M_1}}$. Then $DF_{|_{F_1^{-1}(M_2 \cap M_1')}}$ is an isomorphism onto $D(M_2 \cap M_1')$ which is a closed infinite-dimensional subspace of $ImDF$. From Lemma 3.7.12 the implication in one direction follows.
Assume now that $DF \in g\mathcal{M}\Phi(H)$ and let

$$H = M_1 \tilde{\oplus} N_1 \xrightarrow{DF} M_2 \tilde{\oplus} N_2 = H$$

be a $g\mathcal{M}\Phi$-decomposition for DF. From Lemma 3.5.6 it follows that with respect to the decompositions

$$H = M_1 \tilde{\oplus} N_1 \xrightarrow{F} F(M_1) \tilde{\oplus} D^{-1}(N_2) = H,$$

$$H = F(M_1) \tilde{\oplus} D^{-1}(N_2) \xrightarrow{D} M_2 \tilde{\oplus} N_2 = H,$$

F and D have the matrices $\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}$, and $\begin{bmatrix} D_1 & 0 \\ 0 & D_4 \end{bmatrix}$, respectively, where F_1 and D_1 are isomorphisms. Since M_1 is infinite-dimensional, it follows that $F(M_1)$ is infinite-dimensional also. This proves the implication in the opposite direction.

□

Note that for proving the implication in the opposite direction, we haven't used the assumption that $F, D \in g\mathcal{M}\Phi(H)$. Therefore, we obtain the following corollary.

Corollary 3.7.14. *Let $F, D \in B(H)$ and suppose that $DF \in g\mathcal{M}\Phi(H)$. Then $F, D \in g\mathcal{M}\Phi(H)$.*

Let now $\{e_j\}_{j \in \mathbb{N}}$ be an orthonormal basis for H. For each n, we set $L_n = Span\{e_1, \dots, e_n\}$.

Lemma 3.7.15. *If $F \in g\mathcal{M}\Phi(H)$, then $F + K \in g\mathcal{M}\Phi(H)$ for every compact operator K.*

Proof. Let $H = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H$ be a $g\mathcal{M}\Phi$-decomposition for F and choose an $n \in \mathbb{N}$ such that $\|K_{|_{L_n^\perp}}\| < \|F_1^{-1}\|^{-1}$ where $F_1 := F_{|_{M_1}}$. We have $M_1 = (M_1 \cap L_n^\perp) \oplus P$ for some finite dimensional subspace P. Indeed, if we denote by p_n the orthogonal projection onto L_n, then, since ${p_n}_{|_P}$ is injective , it follows that P is finite dimensional. Now, since M_1 is infinite-dimensional and P is finite dimensional, we must have that $M_1 \cap L_n^\perp$ is infinite-dimensional. As in the proof of Lemma 2.0.45 we can proceed further and deduce that $F + K$ has the matrix $\begin{bmatrix} (F+K)_1 & 0 \\ 0 & (F+K)_4 \end{bmatrix}$ with respect to the decomposition

$$H = \overline{M_1} \tilde{\oplus} \overline{N_1} \xrightarrow{F+K} \overline{M_2} \tilde{\oplus} \overline{N_2} = H,$$

where $\overline{M_1} \cong M_1 \cap L_n^\perp$ and $(F+K)_1$ is an isomorphism. □

In exactly the same way as in the proof of Lemma 2.0.42 we can show that the set $g\mathcal{M}\Phi(H)$ is open in the norm topology. Moreover, in the same way as in the proof of the Corollary 3.1.20, passing to the orthogonal decompositions, we can show that $F^* \in g\mathcal{M}\Phi(H)$ if and only if $F \in g\mathcal{M}\Phi(H)$.
Next, the following results can be proved in exactly the same way as the corresponding results for semi-$\mathcal{A}$-Fredholm operators.

Lemma 3.7.16. *Let M be a closed infinite-dimensional subspace of H and J_M denote the inclusion map. If $FJ_M \in g\mathcal{M}\Phi(M, H)$, then $F \in g\mathcal{M}\Phi(H)$.*

Lemma 3.7.17. *Suppose that $D, F \in B(H)$, ImF is closed and $DF \in g\mathcal{M}\Phi(H)$. Then $DJ_{ImF} \in g\mathcal{M}\Phi(ImF, H)$.*

Corollary 3.7.18. *Let V be a closed subspace of H such that dim $V^{\perp} = \infty$. Suppose that $F \in B(H)$ and $P_{V^{\perp}} F \in g\mathcal{M}\Phi(H, V^{\perp})$. Then $F \in g\mathcal{M}\Phi(H)$.*

Corollary 3.7.19. *Let $D, F \in B(H)$ and suppose that ImD^* is closed. If $DF \in g\mathcal{M}\Phi(H)$, then $P_{\ker D^{\perp}} F \in g\mathcal{M}\Phi(H, ImD^*)$.*

Lemma 3.7.20. *Let $F \in B(H)$. Then $F \in g\mathcal{M}\Phi(H)$ if and only if there exists a closed, infinite-dimensional subspace M of H such that $F_{|_M}$ is bounded below.*

Lemma 3.7.21. *The analogue of Lemma 3.6.11 holds in the setting of $g\mathcal{M}\Phi$ operators.*

Remark 3.7.22. The operators belonging to the class $g\mathcal{M}\Phi(H)$ can still be useful for solving the equation of the form $Fx = y$ because, if

$$H = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H$$

is a $g\mathcal{M}\Phi$-decomposition for F, then we can at least solve the equation when $y \in M_2$.

Chapter 4

Semi-Fredholm operators over W^*-algebras

Throughout this chapter we will assume that $\mathcal{A}$ is a W^*- algebra. We will show that in this case semi-$\mathcal{A}$-Fredholm operators have several properties more similar to the properties of classical semi Fredholm operators than in the general C^*-algebra case. More precisely, we give a generalization in this setting of Schechter Lebow characterization of semi-Fredholm operators, punctured neighbourhood theorem etc.. Main tools in proving these results are the results from preliminaries regarding Hilbert W^*-modules. Therefore, we assume in this chapter that $\mathcal{A}$ is a W^*-algebra.

We start first with the following auxiliary lemma.

Lemma 4.0.1. *Let M be a Hilbert W^*-module and $F \in \widehat{\mathcal{M}\Phi}_l(M)$. Then* $\ker F$ *is finitely generated.*

Proof. Consider an $\widehat{\mathcal{M}\Phi}_l$-decomposition $M = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = M$ for F. Then N_1 is finitely generated and $\ker F = \ker F_{|_{N_1}}$. Since N_1 is self-dual, from Corollary 2.0.50 it follows that $\ker F_{|_{N_1}}$ is an orthogonal direct summand in N_1. Hence, $\ker F = \ker F_{|_{N_1}}$ is finitely generated. □

Then we obtain the following generalization of Schechter-Lebow characterization given in [56, Theorem 1.4.4] and [56, Theorem 1.4.5].

Corollary 4.0.2. *Let $\widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}})$. Then* $\ker(F-K)$ *is finitely generated for all $K \in \mathcal{K}(H_{\mathcal{A}})$.*

Proof. The statement follows from Lemma 4.0.1 since $F - K \in \widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}})$ for all $K \in \mathcal{K}(H_{\mathcal{A}})$ by Proposition 3.5.4. □

Proposition 4.0.3. *[19, Proposition 3.10] Let $G \in \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}})$. Then for every $K \in \mathcal{K}(H_{\mathcal{A}})$ there exists an inner product equivalent to the initial one such that the orthogonal complement of $\overline{Im(G+K)}$ with respect to this new inner product is finitely generated.*

Proof. Let $H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{G} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$ be an $\widehat{\mathcal{M}\Phi}_r$-decomposition for G. Then N_2 is finitely generated, so, by the proof of Theorem 2.0.34, there exists an $n \in \mathbb{N}$ such that $L_n = (M_2 \cap L_n) \tilde{\oplus} \overline{N}$, $\overline{N} \cong N_2$ and $H_{\mathcal{A}} = M_2 \tilde{\oplus} \overline{N}$. Moreover, $G_{|_{M_1}}$ is an isomorphism onto M_2. To simplify notation, we let $M = M_2$, $M' = M_1$ and $N = \overline{N}$. Since N is then a finitely generated Hilbert submodule of L_n (being a direct summand in L_n), by Lemma 2.0.25 we deduce that $L_n = P \oplus N$ for some closed submodule P. Hence, $H_{\mathcal{A}} = M \tilde{\oplus} N = L_n^{\perp} \oplus P \oplus N$.
Denote by $\sqcap$ the orthogonal projection onto $L_n^{\perp} \oplus P$ along N. It follows that $\sqcap_{|_M}$ is an isomorphism onto $L_n^{\perp} \oplus P$. Hence $\sqcap G_{|M'}$ is an isomorphism of M' onto $L_n^{\perp} \oplus P$. If $K \in \mathcal{K}(H_{\mathcal{A}})$,

then, by Theorem 2.0.56, there exists an $m \geq n$ such that $\|q_m K\| < \|(\sqcap G_{|M'})^{-1}\|^{-1}$. Let $M'' = (\sqcap G_{|M'})^{-1}(L_m^{\perp})$. Then $\sqcap G_{|M''} = q_m \sqcap G_{|M''} = q_m G_{|M''}$ since $q_m \sqcap = q_m$. Moreover, $q_m(G-K)_{|M''}$ is an isomorphism onto $L_m^{\perp}$. Now, $M' = M'' \tilde{\oplus} N''$, where $N'' = (\sqcap G_{|M'})^{-1}(P \oplus (L_m \cap L_n^{\perp}))$. With respect to the decomposition

$$M' = M'' \tilde{\oplus} N'' \xrightarrow{G-K} L_m^{\perp} \oplus L_m = H_{\mathcal{A}},$$

$G-K$ has the matrix $\begin{bmatrix} (G-K)_1 & (G-K)_2 \\ (G-K)_3 & (G-K)_4 \end{bmatrix}$, where $(G-K)_1 = q_m(G-K)_{|_{M''}}$ is an isomorphism. Hence, by the same arguments as in the proof of Lemma 2.0.42, there exists an isomorphism $U : M' \longrightarrow M'$ and an isomorphism $V : H_{\mathcal{A}} \longrightarrow H_{\mathcal{A}}$ such that $G-K$ has the matrix $\begin{bmatrix} \overbrace{(G-K)_1} & 0 \\ 0 & \overbrace{(G-K)_4} \end{bmatrix}$ with respect to the decomposition

$$M' = U(M'') \tilde{\oplus} U(N'') \xrightarrow{G-K} V(L_m^{\perp}) \tilde{\oplus} V(L_m) = H_{\mathcal{A}},$$

where $\overbrace{(G-K)_1}$ is an isomorphism. Moreover, V satisfies the equality $V(L_m) = L_m$ by the construction of V from the proof of Lemma 2.0.42. Since

$$V(L_m^{\perp}) \subseteq Im(G-K) \subseteq \overline{Im(G-K)} \text{ and } H_{\mathcal{A}} = V(L_m^{\perp}) \tilde{\oplus} L_m,$$

we obtain that $\overline{Im(G-K)} = V(L_m^{\perp}) \tilde{\oplus} (L_m \cap \overline{Im(G-K)})$. This follows from Lemma 2.0.66. On $H_{\mathcal{A}}$ we may replace the inner product by an equivalent one, in such a way that $V(L_m^{\perp})$ and L_m form an orthogonal direct sum with respect to this new inner product.

Let us consider from now on this new inner product. We will therefore in the rest of the proof denote $L_m^{\perp}$ by L_m^* in order to avoid possible confusion regarding orthogonlity with respect to the old and the new inner product.

Since L_m is finitely generated and $L_m \cap \overline{Im(G-K)}$ is a closed submodule of L_m, we have from Lemma 2.0.47 that

$$L_m = (L_m \cap \overline{Im(G-K)})^{\perp\perp} \oplus (L_m \cap \overline{Im(G-K)})^{\perp}.$$

Then it follows that $(L_m \cap \overline{Im(G-K)})^{\perp}$ is finitely generated. Since

$$\overline{Im(G-K)} = V(L_m^*) \oplus (L_m \cap \overline{Im(G-K)}),$$

we see that $\overline{Im(G-K)}^{\perp} \subseteq L_m$, since $\overline{Im(G-K)}^{\perp}$ is orthogonal to $V(L_m^*)$ with respect to this new inner product and $(V(L_m^*))^{\perp} = L_m$. Therefore,

$$\overline{Im(G-K)}^{\perp} = L_m \cap \overline{Im(G-K)}^{\perp},$$

so $\overline{Im(G-K)}^{\perp}$ is finitely generated. □

Note that from the proof of Proposition 4.0.3 it follows that if $\overline{Im(F-K)}$ is complementable in $H_{\mathcal{A}}$ (for $F \in \widehat{\mathcal{M}\Phi_r}, (H_{\mathcal{A}}), K \in \mathcal{K}(H_{\mathcal{A}})$), then the complement must be finitely generated. Indeed, in the proof of Proposition 4.0.3 we have obtained that

$$\overline{Im(G-K)} = V(L_m^{\perp}) \tilde{\oplus} (L_m \cap \overline{Im(G-K)}) \text{ and } H_{\mathcal{A}} = V(L_m^{\perp}) \tilde{\oplus} V(L_m).$$

Hence, if $\overline{Im(G-K)}^{\circ}$ is complement of $\overline{Im(G-K)}$, we get

$$H_{\mathcal{A}} = V(L_m^{\perp}) \tilde{\oplus} (L_m \cap \overline{Im(G-K)}) \tilde{\oplus} \overline{Im(G-K)}^{\circ},$$

which gives that

$$(L_m \cap \overline{Im(G-K)}) \tilde{\oplus} \overline{Im(G-K)}^{\circ} \cong V(L_m) \cong L_m.$$

It follows that $\overline{Im(G-K)}^{\circ}$ is finitely generated being isomorphic to a direct summand in L_m.

Definition 4.0.4. [19, Definition 3.24] Let M be a Hilbert W^*- module. For $F \in B^a(M)$, we say that F satisfies the condition (*) if the following holds:
1) ImF^n is closed for all n,
2) $F(\bigcap_{n=1}^{\infty} Im(F^n)) = \bigcap_{n=1}^{\infty} Im(F^n)$.

If we have a decreasing sequence of complementable submodules $N_k's$, then their intersection in general (for C^*-algebras) is not complementable, but it is complementable for W^*-algebras. This is true due to the possibility to define a w^*-(or weak) direct sum of submodules, as opposed to the standard l_2 construction. Let $N_{k-1} = N_k \oplus L_k$. Then we can define $w^* - \oplus_k L_k$ as the set of sequences (x_k), $x_k \in L_k$, such that the sum $\sum_{k=1}^{\infty} \langle x_k, x_k \rangle$ is convegent in $\mathcal{A}$ with respect to the *-strong topology, as opposed to the norm topology. Then it is easy to see that $N_0 = \bigcap_{k=1}^{\infty} N_k \oplus (w^* - \oplus_k L_k)$.
Note that if H is an ordinary Hilbert space, then (*) is always satisfied for any $F \in \Phi(H)$ by [56, Theorem 1.1.9]. There are also other examples of Hilbert W^*-modules for which the condition (*) is automatically satisfied for a W^*-linear, bounded operator F as long as F has closed image.

Example 4.0.5. [19, Example 3.25] Let $\mathcal{A}$ be a commutative von Neumann algebra with a cyclic vector, that is $\mathcal{A} \cong L^{\infty}(X, \mu)$, where X is a compact topological space and u is a Borel probability measure. Consider $\mathcal{A}$ as a Hilbert module over itself. If F is an $\mathcal{A}$-linear operator on $\mathcal{A}$, it is easily seen that $Im(F^k) = Span_{\mathcal{A}}\{(F(1))^k\}$ for all k. Let $S = (F(1)^{-1}(\{0\}))^c$. Then one can show that $ImF = ImF^k = Span_{\mathcal{A}}\{\chi_S\}$ for all k if we assume that $F(1)$ is bounded away from 0 on S and hence invertible on S. However, if F has closed image, then this is the case. Indeed,

$$\ker F = \{f \in \mathcal{A} \mid f_{|_S} = 0 \ \mu \ -a.e. \ \text{ on } \ S\} = Span_{\mathcal{A}}\{\chi_{S^c}\}, \text{ so } \ker F^{\perp} = Span_{\mathcal{A}}\{\chi_S\}.$$

Now, if F has closed image, then F is bounded below on $\ker F^{\perp}$, hence we have

$$\| F(f) \|_{\infty} = \| fF(1) \|_{\infty} \geq C \| f \|_{\infty}$$

for all f vanishing μ-almost everywhere on S^c and for some constant $C > 0$. However, if

$$\mu(F(1)^{-1}((B(0, \frac{1}{n})) \cap S)) > 0 \ \forall n,$$

then, letting

$$f_n = \chi_{F(1)^{-1}((B(0,\frac{1}{n}))\cap S)},$$

we get $\| f_n \|_{\infty} = 1$ for all n and

$$F(f_n) = f_n F(1) = \chi_{F(1)^{-1}((B(0,\frac{1}{n}))\cap S)} F(1),$$

so $\| F(f_n) \|_{\infty} \leq \dfrac{1}{n}$ for all n. Moreover, $f_n \chi_{S^c} = 0$ for all n. It follows that F is not bounded below on $(\ker F)^{\perp}$ in this case, which is a contradiction. Thus we must have that $F(1)$ is bounded away from 0 on S if F has closed image. Hence, in this case we get

$$Im(F) = Im(F^k) = Span_{\mathcal{A}}\{\chi_S\} = (\ker F)^{\perp} \forall k,$$

so $ImF = F(Im^{\infty}(F)) = Im^{\infty}(F)$, where $Im^{\infty}(F)$ denotes $\bigcap_{k=1}^{\infty} Im(F^k)$.

For each $n \in \mathbb{N}$, let F_n be the operator on $L_n(\mathcal{A})$ given by

$$F_n(x_1, \ldots, x_n) = (Fx_1, \ldots, Fx_n),$$

where F is the operator on $\mathcal{A}$ from above. If U is an isomorphism of $L_n(\mathcal{A})^\perp$ onto $L_n(\mathcal{A})^\perp$, then the operator $\tilde{F_n}$ on $H_\mathcal{A}$ given by the operator matrix $\begin{bmatrix} U & 0 \\ 0 & F_n \end{bmatrix}$ with respect to the decomposition

$$H_\mathcal{A} = L_n(\mathcal{A})^\perp \oplus L_n(\mathcal{A}) \xrightarrow{\tilde{F_n}} L_n(\mathcal{A})^\perp \oplus L_n(\mathcal{A}) = H_\mathcal{A},$$

satisfies the condition $(*)$.
Next, if $F \in M^a(H_\mathcal{A})$ or if $F \in Q^a(H_\mathcal{A})$, then obviously such operator F satisfies the condition (*). We recall once again that $\mathcal{M}\Phi_+$ and $\mathcal{M}\Phi_-$ operators from our examples are actually examples of the operators belonging to the class $M^a(H_\mathcal{A})$ and to the class $Q^a(H_\mathcal{A})$, respectively.

Proposition 4.0.6. *Let M be a Hilbert W^*-module and $F \in \mathcal{M}\Phi(M)$. Then index of F is well defined.*

Proof. Let $M = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = M$ be an $\mathcal{M}\Phi$-decompostion for F. Then there exists an isomorphism U such that

$$M = N_1^\perp \tilde{\oplus} U(N_1) \xrightarrow{F} F(N_1^\perp) \oplus F(N_1^\perp)^\perp = M$$

is also an $\mathcal{M}\Phi$-decompostion for F and $N_2 \cong F(N_1^\perp)^\perp$. Indeed, by the proof of Theorem 2.0.38 it follows that

$$M = N_1^\perp \oplus N_1 \xrightarrow{F} F(N_1^\perp) \tilde{\oplus} N_2 = M$$

is also an $\mathcal{M}\Phi$-decomposition for F. Moreover, as explained in the proof of Theorem 3.1.2 part 1) $\Rightarrow$ 2), we have that $F(N_1^\perp)$ is orthogonally complementable in M. Obviously, it holds that $(F(N_1^\perp))^\perp \cong N_2$. With respect to the decomposition

$$M = N_1^\perp \oplus N_1 \xrightarrow{F} F(N_1^\perp) \oplus F(N_1^\perp)^\perp = M,$$

F has the matrix $\begin{bmatrix} F_1 & F_2 \\ 0 & F_4 \end{bmatrix}$, where F_1 is an isomorphism, hence there exists an isomorphism U such that

$$M = N_1^\perp \tilde{\oplus} U(N_1) \xrightarrow{F} F(N_1^\perp) \oplus F(N_1^\perp)^\perp = M$$

is an $\mathcal{M}\Phi$- decomposition for F. Since $U(N_1)$ and $F(N_1^\perp)^\perp$ are self-dual being finitely generated, as in the proof of Proposition 2.0.53 we can apply Corollary 2.0.50 and Lemma 2.0.52 in order to deduce that there exists a Hilbert submodule $\tilde{N}_1$ such that

$$U(N_1) = \tilde{N}_1 \oplus \ker F \text{ and } F(N_1^\perp)^\perp = \overline{F(\tilde{N}_1)} \oplus ImF^\perp,$$

where $\tilde{N}_1 \cong \overline{F(\tilde{N}_1)}$ and $\overline{F(\tilde{N}_1)}$ denotes the closure of $F(\tilde{N}_1)$ in τ_1-topology (for more details about this topology, see [38, Section 3.5]). It follows that

$$[N_1] - [N_2] = [U(N_1)] - [F(N_1^\perp)^\perp] = [\ker F] - [ImF^\perp].$$

Since this holds for <u>any</u> $\mathcal{M}\Phi$-decomposition of F, the statement follows. $\square$

Remark 4.0.7. Proposition 4.0.6 shows that Proposition 2.0.53 is valid also in the case of arbitrary Hilbert W^*-modules and not just countably generated ones.

Recall that for a W^*-algebra $\mathcal{A}$, $G(\mathcal{A})$ denotes the set of all invertible elements in $\mathcal{A}$ and $Z(\mathcal{A}) = \{\beta \in \mathcal{A} \mid \beta\alpha = \alpha\beta \text{ for all } \alpha \in \mathcal{A}\}$. For a Hilbert $\mathcal{A}$-module M and $\alpha \in Z(\mathcal{A})$ we let αI denote the operator on M given by $\alpha I(x) = x \cdot \alpha$ for all $x \in M$. We notice that this definition differs from the definition of the operator αI on $H_{\mathcal{A}}$ given in Section 3.4, however, this definition is applicable in the case of arbitrary Hilbert-$\mathcal{A}$-modules. The limitation of this definition (compared to the definition of αI from Section 3.4) is that it requires that $\alpha \in Z(\mathcal{A})$, however, in the case when $\mathcal{A}$ is commutative, then this definition coincides with the definition of αI from Section 3.4.

We have then the following generalization of the punctured neighbourhood theorem stated in [56, Theorem 1.7.7].

Theorem 4.0.8. *[19, Theorem 3.26] Let $F \in \mathcal{M}\Phi(\tilde{M})$ where $\tilde{M}$ is a Hilbert $\mathcal{A}$-module over a W^*-algebra $\mathcal{A}$ and suppose that F satisfies the condition (*). Then there exists an $\epsilon > 0$ such that if $\alpha \in Z(\mathcal{A}) \cap G(\mathcal{A})$ and $\parallel \alpha \parallel < \epsilon$, then*

$$[\ker(F - \alpha I)] + [N_1] = [\ker F] \text{ and } [Im(F - \alpha I)^{\perp}] + [N_1] = [Im(F)^{\perp}]$$

for some fixed, finitely generated closed submodule N_1.

Proof. Since $F \in \mathcal{M}\Phi(\tilde{M})$ has closed image, then by Lemma 3.3.6 and Lemma 2.0.42, there exists an $\epsilon_1 > 0$ such that if $\parallel \alpha \parallel < \epsilon_1, \alpha \in Z(\mathcal{A}) \cap G(\mathcal{A})$, then

$$\ker(F - \alpha I) \preceq \ker F, Im(F - \alpha I)^{\perp} \preceq Im F^{\perp}$$

and index $(F - \alpha I) =$ index F. Now, since $\alpha \in G(\mathcal{A}) \cap Z(\mathcal{A})$, we have that

$$\ker(F - \alpha I) \subseteq Im^{\infty}(F) := \bigcap_{n=1}^{\infty} Im(F^n).$$

This works as in by the proof of [56, Theorem 1.7.7]. As $Im^{\infty}(F)$ is orthogonally complementable in $\tilde{M}$, there exists orthogonal projection $P_{Im^{\infty}(F)^{\perp}}$ onto $Im^{\infty}(F)^{\perp}$ along $Im^{\infty}(F)$, hence

$$(\ker F \cap Im^{\infty}(F)) = \ker P_{Im^{\infty}(F)^{\perp}|_{\ker F}}.$$

Since $\ker F$ is self dual being finitely generated, then, by Corollary 2.0.50, $\ker F \cap Im^{\infty}(F)$ is an orthogonal direct summand in $\ker F$, so

$$\ker F = (\ker F \cap Im^{\infty}(F)) \oplus N_1$$

for some closed submodule N_1. Set $M = \bigcap_{n=1}^{\infty} Im(F^n)$ and $F_0 = F_{|_M}$. Then $\ker F_0 = \ker F \cap M$ is finitely generated as a direct summand in $\ker F$ (which is finitely generated itself by Lemma 3.1.21). Since $\ker F \cap M$ is finitely generated, by Lemma 2.0.25 $\ker F \cap M$ is orthogonally complementable in M, so $M = (\ker F \cap M) \oplus M'$ for some closed submodule M'. On M', the mapping F_0 is an isomorphism from M' onto M, since $F(M) = M$ by assumption as F satisfies (*) condition. Therefore, $F_0 \in \mathcal{M}\Phi(M)$. By Lemma 3.3.6 and Lemma 2.0.42 there exists an $\epsilon_2 > 0$ such that if $\parallel \alpha \parallel < \epsilon_2$, $\alpha \in G(\mathcal{A}) \cap Z(\mathcal{A})$, then

$$\ker(F_0 - \alpha I_{|M}) \preceq \ker F_0, Im(F_0 - \alpha I_{|M})^{\perp} \preceq Im F_0^{\perp}$$

in M and

$$\text{index } (F_0 - \alpha I) = \text{index } F_0 = [\ker F_0],$$

because F_0 is surjective. Since $Im F_0^{\perp} = \{0\}$ (in M), we have

$$Im(F_0 - \alpha I)^{\perp} {=} 0 \text{ for all } \parallel \alpha \parallel < \epsilon_2, \ , \alpha \in G(\mathcal{A}) \cap Z(\mathcal{A}),$$

as $Im(F_0 - \alpha I_{|_M})^{\perp} \preceq ImF_0^{\perp}$ for all $\parallel \alpha \parallel < \epsilon_2, \alpha \in G(\mathcal{A}) \cap Z(\mathcal{A})$.
Recall that $\ker(F - \alpha I) \subseteq Im^{\infty}(F) = M$. Therefore,

$$[\ker(F - \alpha I)] = [\ker(F_0 - \alpha I_{|_M})] = \text{index } (F_0 - \alpha I_{|_M}) = \text{index } F_0 = [\ker F_0].$$

This holds whenever $\parallel \alpha \parallel < \epsilon_2, \ , \alpha \in G(\mathcal{A}) \cap Z(\mathcal{A})$.
Now, $\ker F_0 = \ker F \cap M$ and $\ker F = (\ker F \cap M) \oplus N_1$. Therefore, if $\alpha \in G(\mathcal{A}) \cap Z(\mathcal{A})$ and $\parallel \alpha \parallel < \epsilon_2$, then

$$[\ker F] = [\ker F \cap M] + [N_1] = [\ker F_0] + [N_1] = [\ker(F - \alpha I)] + [N_1]$$

whenever $\parallel \alpha \parallel < \epsilon_2$, $\alpha \in G(\mathcal{A}) \cap Z(\mathcal{A})$. If, in addition $\parallel \alpha \parallel < \epsilon_1$, then as we have seen at the beginning of this proof, by the choice of ϵ_1, we have that

$$\text{index } (F - \alpha I) = \text{index } F.$$

So, if $\parallel \alpha \parallel < min\{\epsilon_1, \epsilon_2\}$ for $\alpha \in G(\mathcal{A}) \cap Z(\mathcal{A})$, then

$$\text{index } (F - \alpha I) = \text{index } F \text{ and } [\ker F] = [\ker(F - \alpha I)] + [N_1].$$

It follows by Proposition 2.0.53 and Proposition 4.0.6 that

$$[ImF^{\perp}] = [Im(F - \alpha I)^{\perp}] + [N_1].$$

□

Remark 4.0.9. [19, Remark 3.27] If $\mathcal{A}$ is a factor, then Theorem 4.0.8 is of interest in the case of finite factors, since $K_0(\mathcal{A})$ is trivial otherwise.

Self-dual W^*-modules have several special and nice properties, as described in [38, Chapter 3] and in preliminaries. We recall that there are also examples of self-dual Hilbert W^*-modules that are not finitely generated. On such self-dual Hilbert W^*-modules semi-$\mathcal{A}$-Fredholm theory might still be of interest. At the end of this section we give some results regarding semi-$\mathcal{A}$-Fredholm operators on self-dual Hilbert modules over a W^*-algebra $\mathcal{A}$.

Lemma 4.0.10. *Let M be a self-dual Hilbert module over a W^*-algebra $\mathcal{A}$. Then the classes $\mathcal{M}\Phi_+(M)$, $\mathcal{M}\Phi_-(M)$, $\mathcal{M}\Phi_0(M), \mathcal{M}\Phi_+^{-'}(M)$ and $\mathcal{M}\Phi_-^{-'}(M)$ are semigroups under the multiplication.*

Proof. Suppose that $D, F \in \mathcal{M}\Phi_+^{-'}(M)$ and let

$$M = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = M,$$

$$M = M_1' \tilde{\oplus} N_1' \xrightarrow{D} M_2' \tilde{\oplus} N_2' = M$$

be two $\mathcal{M}\Phi_+^{-'}$ -decompositions for F and D, respectively. Then $M_2 \cap M_1' = \ker \sqcap_{|_{M_1'}}$, where $\sqcap$ stands for the projection onto N_2 along M_2. By Lemma 2.0.30 M_1' is self-dual. Hence $\sqcap_{|_{M_1'}}$ is a bounded $\mathcal{A}$-linear mapping between self-dual Hilbert $\mathcal{A}$-modules. From Corollary 2.0.50 it follows that $\ker \sqcap_{|_{M_1'}} \oplus M_1'' = M_1'$ (where M_1'' is the orthogonal complement of $\ker \sqcap_{|_{M_1'}}$.) With respect to the decomposition

$$M = (M_2 \cap M_1') \tilde{\oplus} (M_1'' \tilde{\oplus} N_1') \xrightarrow{D} D(M_2 \cap M_1') \tilde{\oplus} (D(M_1'') \tilde{\oplus} N_2') = M,$$

the operator D has the matrix $\begin{bmatrix} D_1 & 0 \\ 0 & D_4 \end{bmatrix}$, where D_1 is an isomorphism and moreover, we have $M_1'' \cong D(M_1'')$. It follows that $M_1'' \tilde{\oplus} N_1' \preceq D(M_1'') \tilde{\oplus} N_2'$ since $N_1' \preceq N_2'$.

Next, since $M_2 \cap M_1'$ is complementable in M and $M_2 \cap M_1' \subseteq M_2$, by Lemma 2.0.66 we have that $(M_2 \cap M_1')\tilde{\oplus}\tilde{M}_2 = M_2$ for some Hilbert submodule $\tilde{M}_2$. Set $F_1 = F_{|M_1}$, then $F_{|M_1}$ is an isomorphism from M_1 onto M_2. So we get

$$M = F_1^{-1}(M_2 \cap M_1')\tilde{\oplus}F_1^{-1}(\tilde{M}_2)\tilde{\oplus}N_1,$$

where $F_1^{-1}(M_2 \cap M_1') \cong M_2 \cap M_1'$ and $F_1^{-1}(\tilde{M}_2) \cong \tilde{M}_2$. Hence $F_1^{-1}(\tilde{M}_2)\tilde{\oplus}N_1 \preceq \tilde{M}_2\tilde{\oplus}N_2$ because $N_1 \preceq N_2$. Since

$$M = (M_2 \cap M_1')\tilde{\oplus}(\tilde{M}_2\tilde{\oplus}N_2) = (M_2 \cap M_1')\tilde{\oplus}(M_1''\tilde{\oplus}N_1'),$$

it follows that $\tilde{M}_2\tilde{\oplus}N_2 \cong M_1''\tilde{\oplus}N_1'$. Therefore, we get $F_1^{-1}(\tilde{M}_2)\tilde{\oplus}N_1 \preceq M_1''\tilde{\oplus}N_1'$. With respect to the decomposition

$$M = F_1^{-1}(M_2 \cap M_1')\tilde{\oplus}(F_1^{-1}(\tilde{M}_2)\tilde{\oplus}N_1) \xrightarrow{F} (M_2 \cap M_1')\tilde{\oplus}(M_1''\tilde{\oplus}N_1') = M,$$

F has the matrix $\begin{bmatrix} F_1 & F_2 \\ 0 & F_4 \end{bmatrix}$, where F_1 is an isomorphism. Hence, by the proof of Lemma 2.0.42, there exists an isomorphism U such that F has the matrix $\begin{bmatrix} F_1 & 0 \\ 0 & \tilde{F}_4 \end{bmatrix}$ with respect to the decomposition

$$M = F_1^{-1}(M_2 \cap M_1')\tilde{\oplus}U(F_1^{-1}(\tilde{M}_2)\tilde{\oplus}N_1) \xrightarrow{F} (M_2 \cap M_1')\tilde{\oplus}(M_1''\tilde{\oplus}N_1') = M.$$

Then, with respect to the decomposition

$$M = F_1^{-1}(M_2 \cap M_1')\tilde{\oplus}U(F_1^{-1}(\tilde{M}_2)\tilde{\oplus}N_1) \xrightarrow{DF} D(M_2 \cap M_1')\tilde{\oplus}(D(M_1'')\tilde{\oplus}N_2') = M,$$

the operator DF has the matrix $\begin{bmatrix} (DF)_1 & 0 \\ 0 & (DF)_4 \end{bmatrix}$, where $(DF)_1$ is an isomorphism. Moreover,

$$F_1^{-1}(\tilde{M}_2)\tilde{\oplus}N_1 \preceq M_1''\tilde{\oplus}N_1' \preceq D(M_1'')\tilde{\oplus}N_2'.$$

It remains to show that $F_1^{-1}(\tilde{M}_2)$ is finitely generated. To this end, observe that we have that $M_2 \cap M_1' = \ker \sqcap'_{|M_2}$, where $\sqcap'$ stands for the projection onto N_1' along M_1'. Hence $\sqcap'_{|M_2}$ is injective on $\tilde{M}_2$. Now, by Lemma 2.0.30, $\tilde{M}_2$ is self-dual, so, by Lemma 2.0.52, we deduce that $\tilde{M}_2$ is isomorphic to a direct summand in N_1'. Since N_1' is finitely generated, it follows that $\tilde{M}_2$ is finitely generated as well. Hence $F_1^{-1}(\tilde{M}_2)$ is finitely generated, since $F_1^{-1}(\tilde{M}_2) \cong \tilde{M}_2$. We have then obtained an $\mathcal{M}\Phi_+^{-'}$-decomposition for DF, so $DF \in \mathcal{M}\Phi_+^{-'}(M)$.

The proofs for the other cases are similar. □

Corollary 4.0.11. *Let M be a self-dual Hilbert W^*-module. Then $\tilde{\mathcal{M}}\Phi_+^-(M)$ and $\tilde{\mathcal{M}}\Phi_-^+(M)$ are semigroups under the multiplication.*

Proof. We use Lemma 4.0.10 together with Proposition 3.4.12 (which is valid for arbitrary Hilbert C^*-modules by Remark 3.4.13) and obtain the result. □

Corollary 4.0.12. *Let M be a self-dual Hilbert W^*-module. Then analogue of Corollary 3.1.19 holds in this case.*

Proof. By applying Lemma 4.0.10, Corollary 3.5.9 and Corollary 3.5.10 instead of Corollary 3.1.14, Corollary 3.1.17 and Corollary 3.1.18, respectively, we deduce the desired result. □

Corollary 4.0.13. *Let M be a self-dual Hilbert W^*-module. Then $\mathcal{M}\Phi_+^-(M)$ and $\mathcal{M}\Phi_-^+(M)$ are semigroups under the multiplication.*

Proof. By applying Lemma 4.0.10 instead of Corollary 3.1.14, Corollary 3.5.7 instead of Corollary 3.1.15, Corollary 3.5.10 instead of Corollary 3.1.18, Corollary 4.0.12 instead of Corollary 3.1.19 and Corollary 4.0.11 instead of Lemma 3.4.7, we can argue in exactly the same way as in the proof of Lemma 3.4.8. □

Remark 4.0.14. Notice that Lemma 3.1.13 also holds in the case of arbitrary self-dual Hilbert W^*-modules. Indeed, by applying Lemma 4.0.10 instead of Lemma 2.0.43 and recalling that by Proposition 4.0.6 the index is well defined on arbitrary Hilbert W^*-modules, we can argue in this case in exactly the same way as in the proof of Lemma 3.1.13.

Lemma 4.0.15. *Let M be a self-dual Hilbert W^*-module.*
Then $\mathcal{M}\Phi_+(M) \setminus \mathcal{M}\Phi_+^{-\prime}(M)$, $\mathcal{M}\Phi_-(M) \setminus \mathcal{M}\Phi_-^{+\prime}(M)$ and $\mathcal{M}\Phi(M) \setminus \mathcal{M}\Phi_0(M)$ are open.

Proof. Let $F \in \mathcal{M}\Phi_+(M) \setminus \mathcal{M}\Phi_+^{-\prime}(M)$ and

$$M = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = M$$

be an $\mathcal{M}\Phi_+$-decomposition for F. As in the proof of Lemma 3.4.14, for a sufficiently small $\epsilon > 0$ we can find some $D \in \mathcal{M}\Phi_+(M)$ such that

$$M = M_1' \tilde{\oplus} N_1' \xrightarrow{D} M_2' \tilde{\oplus} N_2' = M$$

is an $\mathcal{M}\Phi_+$-decomposition for D and in addition $M_1 \cong M_1'$, $N_1 \cong N_1'$, $M_2 \cong M_2'$ and $N_2 \cong N_2'$. If $D \in \mathcal{M}\Phi_+^{-\prime}(M)$, we can find an $\mathcal{M}\Phi_+^{-\prime}$-decomposition for D,

$$M = M_1'' \tilde{\oplus} N_1'' \xrightarrow{D} M_2'' \tilde{\oplus} N_2'' = M.$$

Then, by the same arguments as in the proof of Lemma 4.0.10, we deduce that there exists finitely generated Hilbert submodules P' and P'' such that

$$M_1' = (M_1' \cap M_1'') \oplus P' \text{ and } M_1'' = (M_1' \cap M_1'') \oplus P''.$$

Hence we can proceed in the same way as in the proof of Lemma 3.4.14 to conclude that there exists an isomorphism U_1 such that

$$M = U_1(M_1' \cap M_1'') \tilde{\oplus} (U_1(P') \tilde{\oplus} N_1) \xrightarrow{F} F(U_1(M_1' \cap M_1'')) \tilde{\oplus} (F(U_1(P')) \tilde{\oplus} N_2) = M$$

is an $\mathcal{M}\Phi_+^{-\prime}$-decomposition for F. □

Chapter 5

Generalizations of semi-C^*-Fredholm operators

Various generalizations of classical semi-Fredholm operators such as generalized Weyl operators defined by Đorđević in [8] and semi-B-Fredholm operators defined by Berkani in [4] and [5] have been considered earlier. In this chapter we are going to construct in a similar way generalizations of semi-$\mathcal{A}$-Fredholm operators and investigate some of their properties. Moreover, we shall apply some of these results to the classical case of regular operators on Banach spaces.

5.1 Generalized semi-C^*-Weyl operators

We start with the following definition.

Definition 5.1.1. [21, Definition 11] Let $F \in B^a(H_{\mathcal{A}})$.
1) We say that $F \in \mathcal{M}\Phi^{gc}(H_{\mathcal{A}})$ if ImF is closed and in addition $\ker F$ and $ImF^{\perp}$ are self-dual.
2) We say that $F \in \mathcal{M}\Phi_0^{gc}(H_{\mathcal{A}})$ if ImF is closed and $kerF \cong ImF^{\perp}$ (here we do <u>not</u> require the self-duality of $\ker F$, $ImF^{\perp}$).

The operators belonging to $\mathcal{M}\Phi^{gc}(H_{\mathcal{A}})$ will be called the generalized $\mathcal{A}$-Fredholm operators, whereas the operators belonging to $\mathcal{M}\Phi_0^{gc}(H_{\mathcal{A}})$ will be called the generalized $\mathcal{A}$-Weyl operators.

Remark 5.1.2. [21, Remark 10] Recall that if H is a Hilbert space, then an operator $F \in B(H)$ is called a Weyl operator if F is a Fredholm operator with index 0. Now, as we have mentioned in preliminaries, for $\mathcal{A}$-Fredholm operators we wish to generalize the sign of the index by considering monomorphism between the submodules. For the operators in $B^a(H_{\mathcal{A}})$ with closed image we may obtain a generalization of the index by considering the monomorphisms between their kernel and the orthogonal complement of their image. Thus, a natural generalization in this setting of Weyl operators on Hilbert spaces would be the operators in $B^a(H_{\mathcal{A}})$ with closed image such that their kernel is isomorphic to the orthogonal complement of their image, in other words the operators belonging to $\mathcal{M}\Phi_0^{gc}(H_{\mathcal{A}})$.

We have the following proposition.

Proposition 5.1.3. *[21, Proposition 3] Let $F, D \in \mathcal{M}\Phi_0^{gc}(H_{\mathcal{A}})$ and suppose that $ImDF$ is closed. Then $DF \in \mathcal{M}\Phi_0^{gc}(H_{\mathcal{A}})$.*

Proof. Since $ImDF$ is closed, by Theorem 2.0.20 there exists a closed submodule X such that $ImD = ImDF \oplus X$ because DF can be viewed as an adjointable operator from $H_{\mathcal{A}}$ into ImD.

Next, considering the map $D_{|_{ImF}}$ and again using the assumption that $ImDF$ is closed, we have that $\ker D_{|_{ImF}}$, which is equal to $\ker D \cap ImF$, is orthogonally complementable in ImF by Theorem 2.0.20. Indeed, ImF is orthogonally complementable in $H_{\mathcal{A}}$ by Theorem

2.0.20, hence $D_{|_{ImF}}$ is an adjointable operator from ImF onto $ImDF$. Therefore, we can apply Theorem 2.0.20 on the operator $D_{|_{ImF}}$. It follows that $ImF = W \oplus (\ker D \cap ImF)$ for some closed submodule W. Now, since $(\ker D \cap ImF) \oplus W \oplus ImF^{\perp} = H_{\mathcal{A}}$ and $(\ker D \cap ImF) \subseteq \ker D$, by Lemma 2.0.66 it follows that

$$\ker D = (\ker D \cap ImF) \oplus (\ker D \cap (W \oplus ImF^{\perp})).$$

Set $M = \ker D \cap (W \oplus ImF^{\perp})$, then $\ker D = (\ker D \cap ImF) \oplus M$.

Clearly, $D_{|\ker D^{\perp}}$ is an isomorphism from $\ker D^{\perp}$ onto ImD. Let $S = (D_{|_{\ker D^{\perp}}})^{-1}$ and $P_{\ker D^{\perp}}$ denote the orthogonal projection onto $\ker D^{\perp}$. Then $P_{\ker D^{\perp}} = SDP_{\ker D^{\perp}}$ and $P_{\ker D^{\perp}{}_{|_W}}$ is an isomorphism from W onto $S(ImDF)$. Indeed, since $D_{|_W}$ is injective and $D(W) = ImDF$ is closed, by the Banach open mapping theorem $D_{|_W}$ is an isomorphism onto $ImDF$. This actually means that $DP_{\ker D^{\perp}{}_{|_W}}$ is an isomorphism onto $ImDF$, as $D_{|_W} = DP_{\ker D^{\perp}{}_{|_W}}$. It follows that

$$P_{\ker D^{\perp}}(W) = SDP_{\ker D^{\perp}}(W) = S(ImDF).$$

Since $DP_{\ker D^{\perp}{}_{|_W}}$ is an isomorphism onto $ImDF$, it follows that $P_{\ker D^{\perp}{}_{|_W}}$ is an isomorphism onto $S(ImDF)$. Recall that

$$\ker D^{\perp} = S(ImD) = S(ImDF)\tilde{\oplus}S(X).$$

Therefore, we get that $H_{\mathcal{A}} = W\tilde{\oplus}S(X)\tilde{\oplus}\ker D$. Indeed, let $x \in H_{\mathcal{A}}$, then, since

$$H_{\mathcal{A}} = S(ImDF)\tilde{\oplus}S(X)\tilde{\oplus}\ker D,$$

we have $x = y + z + u$ for some $y \in S(ImDF)$, $z \in S(X)$ and $u \in \ker D$. As $P_{\ker D^{\perp}{}_{|_W}}$ is an isomorphism onto $S(ImDF)$, there exists some $w \in W$ such that $P_{\ker D^{\perp}}(w) = y$. This means that $w = y + u'$ for some $u' \in \ker D$. Hence we get

$$x = y + z + u = y + u' + z + u - u' \in W + S(X) + \ker D,$$

because $u - u' \in \ker D$. Thus, $H_{\mathcal{A}} = W + S(X) + \ker D$. Since $P_{\ker D^{\perp}{}_{|_W}}$ is injective, we have $W \cap \ker D = \{0\}$. If $v \in W \cap S(X)$ for some $v \in H_{\mathcal{A}}$, then, as $S(X) \subseteq \ker D^{\perp}$, we get

$$v = P_{\ker D^{\perp}}(v) \in S(ImDF) \cap S(X) = \{0\}.$$

Hence $W \cap S(X) = 0$, so we obtain $H_{\mathcal{A}} = W\tilde{\oplus}S(X)\tilde{\oplus}\ker D$.

Thus, we have

$$H_{\mathcal{A}} = W\tilde{\oplus}S(X)\tilde{\oplus}(\ker D \cap ImF)\tilde{\oplus}M = W \oplus (\ker D \cap ImF) \oplus ImF^{\perp}.$$

This gives $S(X)\tilde{\oplus}M \cong ImF^{\perp}$.

Next, by Theorem 2.0.20 applied on the operator DF, we obtain that $\ker DF$ is orthogonally complementable in $H_{\mathcal{A}}$. Hence, $F_{|_{\ker DF}}$ is adjointable. Moreover, $Im(F_{|_{\ker DF}}) = ImF \cap \ker D$, which is closed. Now, $\ker F = \ker\ (F_{|_{\ker DF}})$, as $\ker F \subseteq \ker DF$. It follows by Theorem 2.0.20 that $\ker F$ is orthogonally complementable in $\ker DF$, so $\ker DF = \ker F \oplus \tilde{W}$ for some closed submodule $\tilde{W}$. On $\tilde{W}$, F is an isomorphism onto $\ker D \cap ImF$, so $\tilde{W} \cong (\ker D \cap ImF)$.

Therefore, we get

$$\ker DF \cong (\ker F \oplus (\ker D \cap ImF)) \cong ImF^{\perp} \oplus (\ker D \cap ImF)$$

$$\cong S(X) \oplus M \oplus (\ker D \cap ImF) \cong S(X) \oplus \ker D \cong X \oplus ImD^{\perp} \cong ImDF^{\perp}$$

(here $\oplus$ denotes now the direct sum in the sense of Example 2.0.6). □

Remark 5.1.4. In the proof of Proposition 5.1.3 we have obtained the relation

$$H_{\mathcal{A}} = W\tilde{\oplus}S(X)\tilde{\oplus}(\ker D \cap ImF)\tilde{\oplus}M.$$

Since $\ker D = (\ker D \cap ImF) \oplus M$ and $ImF = W \oplus (\ker D \cap ImF)$, we deduce from the above relation that

$$H_{\mathcal{A}} = S(X)\tilde{\oplus}(\ker D + ImF).$$

Thus, $\ker D + ImF$ is closed and complementable in this case.

Proposition 5.1.3 is a generalization of [8, Theorem 1]. Indeed, our proof is also valid in the case when $F \in \mathcal{M}\Phi_0^{gc}(M,N), D \in \mathcal{M}_0^{gc}(N,R)$, where M,N,R are arbitrary Hilbert C^*-modules over a unital C^*-algebra $\mathcal{A}$.

Next, by our proof we easily obtain a generalization of Harte's ghost theorem in [14].

Corollary 5.1.5. *[21, Corollary 1] Let $F, D \in B^a(H_{\mathcal{A}})$ and suppose that ImF, ImD, $ImDF$ are closed. Then*

$$\ker F \oplus \ker D \oplus ImDF^{\perp} \cong ImD^{\perp} \oplus ImF^{\perp} \oplus \ker DF.$$

Proof. We keep the notation from the previous proof. In that proof we have shown the relation

$$ImF^{\perp} \cong S(X) \oplus M.$$

Moreover,

$$D = (\ker D \cap ImF) \oplus M, \;\; ImDF^{\perp} = ImD^{\perp} \oplus X,$$
$$\ker DF \cong \ker F \oplus (\ker D \cap ImF).$$

This gives

$$\ker F \oplus \ker D \oplus ImDF^{\perp} \cong \ker F \oplus \ker D \oplus ImD^{\perp} \oplus X$$
$$\cong \ker F \oplus (\ker D \cap ImF) \oplus M \oplus ImD^{\perp} \oplus X \cong \ker DF \oplus M \oplus S(X) \oplus ImD^{\perp}$$
$$\cong \ker DF \oplus ImF^{\perp} \oplus ImD^{\perp}.$$

□

Inspired by the definition of the exact sequences in Banach spaces, we give now the following definition.

Definition 5.1.6. [21, Definition 12] Let $M_1, \dots, M_n$ be Hilbert submodules of $H_{\mathcal{A}}$. We say that the sequence $0 \to M_1 \to M_2 \to \cdots \to M_n \to 0$ is exact if for each $k \in \{2, \dots, n-1\}$ there exist closed submodules M_k' and M_k'' such that the following holds:
1) $M_k = M_k'\tilde{\oplus}M_k''$ for all $k \in \{2, \dots, n-1\}$;
2) $M_2' \cong M_1$ and $M_{n-1}'' \cong M_n$;
3) $M_k'' \cong M_{k+1}'$ for all $k \in \{2, \dots, n-2\}$.

Then we have the following lemma.

Lemma 5.1.7. *[21, Lemma 2] Let $F, D \in B^a(H_{\mathcal{A}})$ and suppose that ImF, ImD, $ImDF$ are closed. Then the sequence*

$$0 \to \ker F \to \ker DF \to \ker D \to ImF^{\perp} \to ImDF^{\perp} \to ImD^{\perp} \to 0$$

is exact.

Proof. From the proof of Proposition 5.1.3 and using the same notation, we obtain the following:

$$\ker DF = \ker F \oplus \tilde{W}, \text{ where } \tilde{W} \cong \ker D \cap ImF,$$

$$\ker D = (\ker D \cap ImF) \oplus M, \; ImF^{\perp} \cong X \oplus M,$$

$$ImDF^{\perp} = X \oplus ImD^{\perp}.$$

□

Using the fact that a direct summand in a self-dual module is again a self-dual submodule, the fact that a direct sum of two self-dual modules is a self-dual module itself and the fact that the self-duality is preserved under isomorphisms, which follows from Lemma 2.0.30, Lemma 2.0.31 and Lemma 2.0.33, we easily obtain the next result as a corollary of Lemma 5.1.7.

Corollary 5.1.8. *[21, Lemma 3] Let $F, D \in \mathcal{M}\Phi^{gc}(H_{\mathcal{A}})$ and suppose that $ImDF$ is closed. Then $DF \in \mathcal{M}\Phi^{gc}(H_{\mathcal{A}})$.*

Lemma 5.1.9. *[21, Lemma 4] Let $F \in B^a(H_{\mathcal{A}})$. Then $F \in \mathcal{M}\Phi^{gc}(H_{\mathcal{A}})$ if and only if $F^* \in \mathcal{M}\Phi^{gc}(H_{\mathcal{A}})$.*

Proof. By the proof of Theorem 2.0.20 part ii), ImF^* is closed if ImF is closed. Next, we use that $\ker F = ImF^{*\perp}$ and $\ker F^* = ImF^{\perp}$. □

Proposition 5.1.10. *[21, Proposition 4] Let $F, D \in B^a(H_{\mathcal{A}})$, suppose that ImF, ImD are closed and $DF \in \mathcal{M}\Phi^{gc}(H_{\mathcal{A}})$. Then the following statements hold:*
a) $D \in \mathcal{M}\Phi^{gc}(H_{\mathcal{A}}) \Leftrightarrow F \in \mathcal{M}\Phi^{gc}(H_{\mathcal{A}})$;
b) if $\ker D$ is self-dual, then $F, D \in \mathcal{M}\Phi^{gc}(H_{\mathcal{A}})$;
c) if $ImF^{\perp}$ is self-dual, then $F, D \in \mathcal{M}\Phi^{gc}(H_{\mathcal{A}})$.

Proof. Part b) follows from Lemma 5.1.7, Lemma 2.0.30, Lemma 2.0.31 and Lemma 2.0.33. By passing to the adjoints and using Lemma 5.1.9 one may obtain c). To deduce a), use b) and c). □

Lemma 5.1.11. *[21, Lemma 5] Let $F \in B^a(H_{\mathcal{A}})$ and suppose that ImF is closed. Moreover, assume that there exist operators $D, D' \in B^a(H_{\mathcal{A}})$ with closed images such that $D'F, FD \in \mathcal{M}\Phi^{gc}(H_{\mathcal{A}})$. Then $F \in \mathcal{M}\Phi^{gc}(H_{\mathcal{A}})$.*

Proof. By Lemma 5.1.7, since FD is in $\mathcal{M}\Phi^{gc}(H_{\mathcal{A}})$ and ImF, ImD are closed, it follows that $ImF^{\perp}$ is self-dual. Now, by passing to the adjoints and using Lemma 5.1.9, we obtain that $F^*(D')^* \in \mathcal{M}\Phi^{gc}(H_{\mathcal{A}})$, as $D'F \in \mathcal{M}\Phi^{gc}(H_{\mathcal{A}})$. Moreover, by the proof of Theorem 2.0.20 part ii), $ImF^*, (ImD')^*$ are closed, as ImF, ImD' are closed by assumption. Hence, using the previous arguments, we deduce that $ImF^{*\perp} = \ker F$ is self-dual. □

Remark 5.1.12. Our results on generalized $\mathcal{A}$-Fredholm operators are motivated and inspired by Yang's results and work in [31] on generalized Fredholm operators on Banach spaces.

We are now going to apply the proofs of Proposition 5.1.3 and Lemma 5.1.7 to the case of operators on Banach spaces and extend to this case Theorem 1 of [8] as well as some other results from the classical semi-Fredholm theory.

Definition 5.1.13. Let X, Y be Banach spaces and $T \in B(X, Y)$. Then T is called a regular operator if $T(X)$ is closed in Y and in addition $T^{-1}(0)$ and $T(X)$ are complementable in X and Y, respectively.

Remark 5.1.14. It is not hard to see that T is a regular operator if and only if T admits a generalized inverse, that is if and only if there exists some $T' \in B(Y, X)$ such that $TT'T = T$. In this case we have that TT' and $T'T$ are the projections onto $T(X)$ and complement of $T^{-1}(0)$, respectively, and moreover, $T'TT' = T'$. Thus, Definition 5.1.13 corresponds to the definition of regular operators on Banach spaces given in [14].

Definition 5.1.15. [8] Let X, Y be Banach spaces and $T \in B(X, Y)$. Then we say that T is generalized Weyl, if $T(X)$ is closed in Y, and $T^{-1}(0)$ and $Y/T(X)$ are mutually isomorphic Banach spaces.

We give then the following proposition as an extension of [8, Theorem 1] to the case of regular operators on Banach spaces.

Proposition 5.1.16. *Let X, Y, Z be Banach spaces and let $T \in B(X, Y), S \in B(Y, Z)$. Suppose that T, S, ST are regular, that is $T(X), S(Y), ST(X)$ are closed and T, S, ST admit generalized inverse. If T and S are generalized Weyl operators, then ST is a generalized Weyl operator.*

Proof. Since T, S, ST are regular by assumption, their kernels and ranges are complementable in the respective Banach spaces X, Y, Z. Moreover, observe that $S_{|_{T(X)}}$ is regular. Indeed, if U denotes the generalized inverse of ST, then for any x in X, we have $STUST(x) = ST(x)$, so it is easily seen that TU is generalized inverse of $S_{|_{T(X)}}$. Hence $(S_{|_{T(X)}})^{-1}(0)$ is complementable in $T(X)$. However, we have $(S_{|_{T(X)}})^{-1}(0) = S^{-1}(0) \cap T(X)$. Since $T(X)$ is complementable in Y, because T is regular, it follows that $S^{-1}(0) \cap T(X)$ is complementable in Y. By Lemma 2.0.66 we have that $S^{-1}(0) \cap T(X)$ is then complementable in $S^{-1}(0)$. Moreover, $ST(X)$ is complementable in $S(Y)$ by Lemma 2.0.66, since $ST(X)$ is complementable in Z. Finally, since $T^{-1}(0)$ is complementable in X, because T is regular, and $T^{-1}(0) \subseteq ST^{-1}(0)$, it follows again from Lemma 2.0.66 that $T^{-1}(0)$ is complementable in $ST^{-1}(0)$. Then we are in the position to apply exactly the same proof as in Proposition 5.1.3. □

Remark 5.1.17. In general, if X, Y, Z are Banach spaces and $F \in B(X, Y), G \in B(Y, Z)$, $GF \in B(X, Z)$ are regular operators, then we have that the sequence

$$0 \to \ker F \to \ker GF \to \ker G \to ImF^\circ \to ImGF^\circ \to ImG^\circ \to 0$$

is exact, where ImF°, ImG° and $ImGF^\circ$ denote the complements of ImF, ImG and $ImGF$ in the respective Banach spaces. This can be deduced from the proof of Proposition 5.1.3 and Proposition 5.1.16 or from [31, Proposition 2.1] and [31, Theorem 2.7]. If G, F, GF are regular operators, then all the subspaces in the above sequence are complementable in the respective Banach spaces. From the exactness of the above sequence we may deduce as direct corollaries various results such as [8, Theorem 1] and index theorem, Harte's ghost theorem in [14] etc. Recalling from Proposition 3.5.11 that the index of closed range operators $\mathcal{A}$-Fredholm operators on arbitrary Hilbert $\mathcal{A}$-modules is well-defined, from the exact sequence in Lemma 5.1.7 we obtain that Lemma 2.0.43 remains valid for closed range $\mathcal{A}$-Fredholm operators on arbitrary Hilbert $\mathcal{A}$-modules.

Recall Definition 2.0.41 from preliminaries. The next proposition is another generalization of the well-known index theorem [56, Theorem 1.2.4].

Proposition 5.1.18. *[19, Proposition 3.19] Let $F, G \in \widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}})$ with closed images and suppose that $ImGF$ is closed. Then ImF, ImG and $ImGF$ are complementable in $H_{\mathcal{A}}$. Moreover, if $ImF^\circ, ImG^\circ, ImGF^\circ$ denote the complements of $ImF, ImG, ImGF$, respectively, then*

$$ImGF^\circ \preceq ImF^\circ \oplus ImG^\circ,$$

$$\ker GF \preceq \ker G \oplus \ker F.$$

If $F, G \in \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}})$ and $ImF, ImG, ImGF$ are closed, then the above statement holds under the additional assumption that $ImF, ImG, ImGF$ are complementable in $H_{\mathcal{A}}$.

Proof. Since $F \in \widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}})$, from Proposition 3.5.11 it follows that ImF is complementable in $H_{\mathcal{A}}$ because ImF is closed by assumption. Similarly, since $ImG, ImGF$ are closed, we have that $ImG^\circ, ImGF^\circ$ exist, because $G, GF \in \widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}})$. Here we use the fact that $GF \in \widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}})$ by Corollary 3.5.5 since $F, G \in \widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}})$. Moreover, if $G, F \in \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}})$, we have $GF \in \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}})$ by Corollary 3.5.5. In the first case, when $G, F, GF \in \widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}})$, we have by Proposition 3.5.11 that F, G and GF are regular operators, whereas in the second case, when $G, F, GF \in \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}})$, we have by Proposition 3.5.13 that F, G and GF are regular operators. By Remark 5.1.17 we can apply the exact sequence from Lemma 5.1.7 provided that we replace the orthogonal complements by the respective complemented submodules. □

Lemma 5.1.19. *[19, Lemma 3.20] Let M be a Hilbert C^*-module and $F, D \in B^a(M)$. Suppose that ImF, ImD and $ImDF$ are closed. Then*

$$ImDF^\perp \preceq ImF^\perp \oplus ImD^\perp,$$

$$\ker DF \preceq \ker D \oplus \ker F.$$

Proof. If $F, D \in B^a(M)$ and ImF, ImD are closed, by Theorem 2.0.20 F and D are then regular operators. Hence we can apply the exact sequence from Lemma 5.1.7 . □

Lemma 5.1.20. *Let M be an arbitrary Hilbert W^*-module and $G, F \in \widehat{\mathcal{M}\Phi}(M)$. Suppose that ImG, ImF and $ImGF$ are closed. If $ImGF$ is complementable in M, then $GF \in \widehat{\mathcal{M}\Phi}(M)$ and the relations from Proposition 5.1.18 hold in this case as well.*

Proof. Since ImG and ImF are closed by assumption, from Proposition 3.5.11 it follows that

$$M = \ker F^\circ \tilde{\oplus} \ker F \xrightarrow{F} ImF \tilde{\oplus} ImF^\circ = M,$$

$$M = \ker G^\circ \tilde{\oplus} \ker G \xrightarrow{G} ImG \tilde{\oplus} ImG^\circ = M$$

are two $\widehat{\mathcal{M}\Phi}$-decomposition for F and G, respectively. In particular, $\ker F$ and $\ker G$ are finitely generated. Let $\sqcap$ stand for the projection onto ImF° along ImF. Since $\ker G \cap ImF = \ker \sqcap_{|_{\ker G}}$ and $\ker G$ is self-dual, from Corollary 2.0.50 it follows that $(\ker G \cap ImF) \oplus M' = \ker G$ for some Hilbert submodule M'. Hence $\ker G \cap ImF$ is finitely generated as a direct summand in $\ker G$. By Lemma 2.0.25 we obtain $ImF = (\ker G \cap ImF) \oplus M$ for a Hilbert submodule M. Hence

$$\ker F^\circ = (F_{|_{\ker F^\circ}})^{-1}(M) \tilde{\oplus} (F_{|_{\ker F^\circ}})^{-1}(\ker G \cap ImF).$$

We have that

$$\ker GF = \ker F \tilde{\oplus} (F_{|_{\ker F^\circ}})^{-1}(\ker G \cap ImF),$$

so this implies that $\ker GF$ is complementable in M. Since $ImGF$ is closed and complementable in M by assumption, we get that GF is a regular operator. From the exactness of the sequence given in Remark 5.1.17 we deduce then the desired results. □

Corollary 5.1.21. *Let X, Y, Z be Banach spaces and $F \in B(X, Y), G \in B(Y, Z)$ be regular operators. Suppose that GF is also a regular operator. Then*

$$ImGF^\circ \preceq ImG^\circ \oplus ImF^\circ \text{ and } \ker GF \preceq \ker G \oplus \ker F,$$

where ImF°, ImG° and $ImGF^\circ$ denote the complements of ImF, ImG and $ImGF$ in the respective Banach spaces.

Definition 5.1.22. Let X, Y be Banach spaces and $T \in B(X, Y)$ be a regular operator. Then T is said to be a generalized upper semi-Weyl operator if $\ker T \preceq Y/Im(T)$. Similarly, T is said to be a generalized lower semi-Weyl operator if $Y/Im(T) \preceq \ker T$.

Lemma 5.1.23. *Let $T \in B(X,Y)$, $S \in B(Y,Z)$ and suppose that S, T, ST are regular. If S and T are upper (or lower) generalized semi-Weyl operators, then ST is an upper (or respectively lower) generalized semi-Weyl operator.*

Proof. This follows from the exactness of the sequence given in Remark 5.1.17. □

Lemma 5.1.24. *Let $F \in B(M)$ where M is a Hilbert C^*-module and suppose that F and F^2 are regular. If $F \in \widehat{\mathcal{M}\Phi}_l(M)$, then $F_{|_{ImF}} \in \widehat{\mathcal{M}\Phi}_l(ImF)$. Similarly, if $F \in \widehat{\mathcal{M}\Phi}_r(M)$, then $F_{|_{ImF}} \in \widehat{\mathcal{M}\Phi}_r(ImF)$. Finally, if $F \in \widehat{\mathcal{M}\Phi}(M)$, then $F_{|_{ImF}} \in \widehat{\mathcal{M}\Phi}(ImF)$ and in this case* $\text{index } F_{|_{ImF}} = \text{index } F$.

Proof. We can apply Lemma 5.1.7 and Corollary 3.5.15 to deduce the lemma. Indeed, by Remark 5.1.17 we have that Lemma 5.1.7 can be generalized to regular operators. For the third statement in the lemma we recall also that the index of regular $\mathcal{A}$-Fredholm operators is well-defined on arbitrary Hilbert C^*-modules by Proposition 3.5.11. □

Corollary 5.1.25. *Let $F \in B^a(M)$ and suppose that ImF and ImF^2 are closed.*
1) If $F \in \mathcal{M}\Phi_+(M)$, then $F_{|_{ImF}} \in \mathcal{M}\Phi_+(ImF)$.
2) If $F \in \mathcal{M}\Phi_-(M)$, then $F_{|_{ImF}} \in \mathcal{M}\Phi_-(ImF)$.
3) If $F \in \mathcal{M}\Phi(M)$, then $F_{|_{ImF}} \in \mathcal{M}\Phi(ImF)$ and index $F =$ index $F_{|_{ImF}}$.

Proof. We just need to observe that F and F^2 are regular operators by Theorem 2.0.20. Moreover, since ImF is orthogonally complementable in M by Theorem 2.0.20, it follows that $F_{|_{ImF}} \in B^a(ImF)$. Then we can proceed further as in the proof of Lemma 5.1.24. □

Definition 5.1.26. For two Hilbert C^*-modules M and M' we set $\tilde{\mathcal{M}}\Phi_0^{gc}(M,M')$ to be the class of all closed range operators $F \in B^a(M,M')$ for which there exist finitely generated Hilbert submodules $N, \tilde{N}$ with the property that

$$N \oplus \ker F \cong \tilde{N} \oplus ImF^{\perp}.$$

Then we obtain the following generalization of [8, Theorem 2].

Lemma 5.1.27. *Let $T \in \tilde{\mathcal{M}}\Phi_0^{gc}(H_{\mathcal{A}})$ and $F \in B^a(H_{\mathcal{A}})$ such that ImF is closed, finitely generated. Suppose that $Im(T+F), T(\ker F), P(\ker T), P(\ker(T+F))$ are closed, where P denotes the orthogonal projection onto $\ker F^{\perp}$. Then*

$$T + F \in \tilde{\mathcal{M}}\Phi_0^{gc}(H_{\mathcal{A}}).$$

Proof. Since ImT and $Im(T+F)$ are closed by assumption, by Theorem 2.0.20 we have $H_{\mathcal{A}} = ImT \oplus ImT^{\perp}$ and $H_{\mathcal{A}} = Im(T+F) \oplus Im(T+F)^{\perp}$. Similarly, since ImF is closed by assumption, from Theorem 2.0.20 we get that $H_{\mathcal{A}} = \ker F^{\perp} \oplus \ker F$. Hence $T_{|_{\ker F}}$ is an adjointable operator from $\ker F$ into ImT (and $(T+F)_{|_{\ker F}} = T_{|_{\ker F}}$ is an adjointable operator from $\ker F$ into $Im(T+F)$). Now, since $T(\ker F)$ is closed by assumption, again by applying Theorem 2.0.20 on the operator $T_{|_{\ker F}}$, we deduce that

$$ImT = T(\ker F) \oplus N \text{ and } \; Im(T+F) = T(\ker F) \oplus N'$$

for some Hilbert submodules N, N'. Hence

$$ImT^{\perp} \oplus N = Im(T+F)^{\perp} \oplus N' = T(\ker F)^{\perp}.$$

Thus, $T(\ker F)$ is orthogonally complementable in $H_{\mathcal{A}}$. Let Q denote the orthogonal projection onto $T(\ker F)^{\perp}$. It turns out that N and N' are finitely generated. Indeed, we have

$$ImT = T(\ker F) + T(\ker F^{\perp}) \text{ and } Im(T+F) = T(\ker F) + (T+F)(\ker F^{\perp}).$$

As $F_{|_{\ker F^{\perp}}}$ is an isomorphism onto ImF by the Banach open mapping theorem and ImF is finitely generated by assumption, it follows that $\ker F^{\perp}$ is finitely generated. Hence $QT(\ker F^{\perp})$ and $Q(T+F)(\ker F^{\perp})$ are finitely generated. However, we have

$$N = Q(ImT) = QT(\ker F^{\perp}) \text{ and } N' = Q(Im(T+F)) = Q(T+F)(\ker F^{\perp}).$$

Furthermore, since $P(\ker T)$ is closed by assumption and $P_{|_{\ker T}}$ is adjointable (as $\ker T$ is orthogonally complementable by Theorem 2.0.20), then $\ker P_{|_{(\ker T)}} = \ker F \cap \ker T$ is orthogonally complementable in $\ker T$, so

$$\ker T = (\ker F \cap \ker T) \oplus M$$

for some closed submodule M. We have that $P_{|_M}$ is an isomorphism onto $P(\ker T)$. Since $P_{|_{\ker T}}$ is adjointable and $P(\ker T)$ is closed, by Theorem 2.0.20 $P(\ker T)$ is orthogonally complementable in $\ker F^{\perp}$. As $\ker F^{\perp}$ is finitely generated, it follows that $P(\ker T)$ is finitely generated. Thus, M must be finitely generated because $P_{|_M}$ is an isomorphism onto $P(\ker T)$.
By similar arguments as above, using that $P(\ker(T+F))$ is closed by assumption, we obtain that

$$\ker(T+F) = (\ker(T+F) \cap \ker F) \oplus M',$$

where M' is a finitely generated Hilbert submodule. Now, if $x \in \ker(T+F)$ and $Fx = 0$, then obviously $Tx = 0$ as well, hence $\ker(T+F) \cap \ker F \subseteq \ker T \cap \ker F$. On the other hand, $\ker T \cap \ker F \subseteq \ker(T+F) \cap \ker F$, so we get

$$\ker T \cap \ker F = \ker(T+F) \cap \ker F.$$

Thus,

$$\ker(T+F) = (\ker T \cap \ker F) \oplus M'.$$

Finally, since $T \in \tilde{\mathcal{M}\Phi}_0^{gc}(H_{\mathcal{A}})$, there exist finitely generated Hilbert submodules R and R' such that $R \oplus \ker T \cong R' \oplus ImT^{\perp}$. Combining all this together, we deduce that

$$\ker(T+F) \oplus M \oplus N \oplus R \cong (\ker T \cap \ker F) \oplus M' \oplus M \oplus N \oplus R$$

$$\cong \ker T \oplus M' \oplus N \oplus R \cong ImT^{\perp} \oplus M' \oplus N \oplus R' \cong Im(T+F)^{\perp} \oplus M' \oplus N' \oplus R'.$$

$\square$

Corollary 5.1.28. *Let $T \in \mathcal{M}\Phi_0^{gc}(H_{\mathcal{A}})$ and suppose that $\ker T \cong ImT^{\perp} \cong H_{\mathcal{A}}$. If $F \in B^a(H_{\mathcal{A}})$ satisfies the assumptions of Lemma 5.1.27, then*

$$\ker(T+F) \cong Im(T+F)^{\perp} \cong H_{\mathcal{A}}.$$

In particular, $T+F \in \mathcal{M}\Phi_0^{gc}(H_{\mathcal{A}})$.

Proof. Notice that, since $T \in \mathcal{M}\Phi_0^{gc}(H_{\mathcal{A}})$ by hypothesis, we already have that $\ker T \cong ImT^{\perp}$, so the additonal assumption is that $\ker T$ and $ImT^{\perp}$ are isomorphic to $H_{\mathcal{A}}$. By the proof of Lemma 5.1.27 (and using the same notation), since $\mathcal{M}\Phi_0^{gc}(H_{\mathcal{A}}) \subseteq \tilde{\mathcal{M}\Phi}_0^{gc}(H_{\mathcal{A}})$, we have

$$\ker(T+F) \oplus M \oplus N \oplus R \cong \ker T \oplus M' \oplus N \oplus R$$

$$\cong ImT^{\perp} \oplus M' \oplus N \oplus R' \cong Im(T+F)^{\perp} \oplus M' \oplus N' \oplus R'.$$

Since M, N, R, M', N', R' are finitely generated Hilbert submodules and $\ker T \cong ImT^{\perp} \cong H_{\mathcal{A}}$ by assumption, by the Kasparov stabilization Theorem 2.0.13 we have

$$H_{\mathcal{A}} \cong \ker T \oplus M' \oplus N \oplus R \cong ImT^{\perp} \oplus M' \oplus N \oplus R'.$$

Hence
$$H_{\mathcal{A}} \cong \ker(T+F) \oplus M \oplus N \oplus R \cong Im(T+F)^{\perp} \oplus M' \oplus N' \oplus R'.$$
By the Dupre-Filmore Theorem 2.0.15, it follows easily that
$$\ker(T+F) \cong Im(T+F)^{\perp} \cong H_{\mathcal{A}}.$$

□

Lemma 5.1.29. *Let $T \in \mathcal{M}\Phi(H_{\mathcal{A}})$ and suppose that ImT is closed. Then $T \in \tilde{\mathcal{M}}\Phi_0^{gc}(H_{\mathcal{A}})$.*

Proof. By Lemma 3.1.21, since ImT is closed and $T \in \mathcal{M}\Phi(H_{\mathcal{A}})$, we have that $\ker T$ and $ImT^{\perp}$ are then finitely generated. By Theorem 2.0.34 we can find an $n \in \mathbb{N}$ such that
$$L_n = P\tilde{\oplus}p_n(\ker T) = P'\tilde{\oplus}p_n(ImT^{\perp}) \text{ and } p_n(\ker T) \cong \ker T, p_n(ImT^{\perp}) \cong ImT^{\perp},$$
where P and P' are finitely generated Hilbert submodules and p_n denotes the orthogonal projection onto L_n. It follows that $P \oplus \ker T \cong P' \oplus ImT^{\perp}$. □

Lemma 5.1.30. *Let $\tilde{M}$ be a Hilbert C^*-module and $F, D \in \tilde{\mathcal{M}}\Phi_0^{gc}(\tilde{M})$. If $ImDF$ is closed, then $DF \in \tilde{\mathcal{M}}\Phi_0^{gc}(\tilde{M})$.*

Proof. Since $F, D \in \tilde{\mathcal{M}}\Phi_0^{gc}(\tilde{M})$ by assumption, there exist finitely generated Hilbert submodules $N, \tilde{N}, N'$ and $\tilde{N}'$ such that
$$N \oplus \ker F \cong \tilde{N} \oplus ImF^{\perp} \text{ and } N' \oplus \ker D \cong \tilde{N}' \oplus ImD^{\perp}.$$
By applying the arguments from the proof of Proposition 5.1.3 and using the same notation, we obtain the following chain of isomorphisms:
$$\ker DF \oplus N \oplus N' \cong \ker F \oplus (\ker D \cap \ker F) \oplus N \oplus N'$$
$$\cong ImF^{\perp} \oplus (\ker D \cap ImF) \oplus \tilde{N} \oplus N' \cong S(X) \oplus M \oplus (\ker D \cap F)\tilde{\oplus}\tilde{N} \oplus N'$$
$$\cong S(X) \oplus \ker D \oplus \tilde{N} \oplus N' \cong X \oplus ImD^{\perp} \oplus \tilde{N} \oplus \tilde{N}' \cong ImDF^{\perp} \oplus \tilde{N} \oplus \tilde{N}'.$$

□

We can apply the arguments from the proof of Lemma 5.1.27 to obtain an extension of [8, Theorem 2] to the case of regular operators on Banach spaces.
First we give the following definition.

Definition 5.1.31. Let X, Y be Banach spaces. We set $\Phi_0^{gc}(X,Y)$ to be the set of all regular operators $F \in B(X,Y)$ satisfying that there exist finite dimensional Banach spaces Z_1 and Z_2 with the property that $\ker F \oplus Z_1 \cong ImF^{\circ} \oplus Z_2$, where ImF° stands for the complement of ImF in Y.

Then we present the following extension of [8, Theorem 2] to the case of regular operators on Banach spaces.

Lemma 5.1.32. *Let X, Y be Banach spaces and $T \in \Phi_0^{gc}(X,Y)$. Suppose that F is a finite rank operator from X into Y. Then $T + F \in \Phi_0^{gc}(X,Y)$.*

Proof. Since F is finite rank operator, it is regular, i.e. ImF is closed, $\ker F$ and ImF are complementable in X and Y, respectively. Let $\ker F^\circ$ denote complement of $\ker F$ in X. As ImT is closed by assumption and $ImT = T(\ker F) + T(\ker F^\circ)$, it follows that $T(\ker F)$ has finite co-dimension in ImT, so, by the Kato Theorem [56, Corollary 1.1.7], we have that $T(\ker F)$ is closed (as $T(\ker F) = ImT_{|_{\ker F}}$ and $\ker F^\circ$ is finite dimensional). Hence, again using that $T(\ker F)$ has finite co-dimension, by part b) in [45, Lemma 4.21] we obtain that $ImT = T(\ker F)\tilde{\oplus}N$, where N is a finite dimensional subspace. Now, since $T(\ker F)$ is closed and

$$Im(T + F) = T(\ker F) + (T + F)(\ker F^\circ),$$

by [56, Lemma 1.1.2] we get that $Im(T+F)$ is closed as $(T+F)(\ker F^\circ)$ is finite dimensional. By the similar arguments as above, we deduce then that $Im(T + F) = T(\ker F)\tilde{\oplus}N'$ for some finite dimensional subspace N'. Since

$$Y = ImT\tilde{\oplus}ImT^\circ = T(\ker F)\tilde{\oplus}N\tilde{\oplus}ImT^\circ,$$

where ImT° stands for the complement of ImT in Y, we see that $T(\ker F)$ is complementable in Y. Let $T(\ker F)^\circ$ denote complement of $T(\ker F)$ in Y and Q be the projection onto $T(\ker F)^\circ$ along $T(\ker F)$. Then $Q_{|_{N'}}$ is injective. As N' is finite dimensional, so is $Q(N')$, hence $Q(N')$ is closed and $T(\ker F)^\circ = Q(N')\tilde{\oplus}V$ for some closed subspace V. This follows by part a) in [45, Lemma 4.21] . Since $Q_{|_{N'}}$ is then an isomorphism onto $Q(N')$, by the same arguments as in the proof of Proposition 5.1.3 we deduce that

$$Y = T(\ker F)\tilde{\oplus}N'\tilde{\oplus}V = Im(T + F)\tilde{\oplus}V,$$

so $Im(T + F)$ is complementable.

Next, let P denote the projection onto $\ker F^\circ$ along $\ker F$. Then $P_{|_{\ker T}}$ and $P_{|_{\ker(T+F)}}$ are finite rank operators, hence regular. It follows that their kernels are complementable, hence by the same arguments as in the proof of Lemma 5.1.27 we deduce that

$$\ker T = (\ker T \cap \ker F)\tilde{\oplus}M \text{ and } \ker(T + F) = (\ker T \cap \ker F)\tilde{\oplus}M'$$

for some finite dimensional subspaces M and M'. Since $\ker T$ is complementable in X as T is regular, then $\ker T \cap \ker F$ is complementable in X, so by the similar arguments as above we can deduce that $\ker(T + F)$ is complementable in X. Hence $T + F$ is a regular operator. Moreover, proceeding in the same way as in the proof of Lemma 5.1.27 by considering chain of isomorphisms, we conclude that $T + F \in \Phi_0^{gc}(X, Y)$. □

Remark 5.1.33. If H is a Hilbert space, it follow that if $F \in \Phi_0^{gc}(H)$ and $\ker F$ or $ImF^\perp$ are infinite-dimensional, then $\ker F \cong ImF^\perp$. Hence it is not hard to see that Lemma 5.1.32 is indeed an extension of [8, Theorem 2].

Remark 5.1.34. As explained in the proof of Proposition 5.1.16 and Remark 5.1.17, the proof of Proposition 5.1.3 applies in the case of regular operators on Banach spaces. By combining this fact with the proof of Lemma 5.1.30 we can deduce that if $T \in \Phi_0^{gc}(X, Y), S \in \Phi_0^{gc}(Y, Z)$ and ST is regular, then $ST \in \Phi_0^{gc}(X, Z)$ (where X, Y and Z are Banach spaces).

5.2 Semi-C^*-B-Fredholm operators

In this section we are going to construct a generalization of B-Fredholm and semi-B-Fredholm operators on Hilbert and Banach spaces defined in [5] and [4] in the setting of semi-$\mathcal{A}$-Fredholm operators. We give the following definition.

Definition 5.2.1. [21, Definition 16] Let $F \in B^a(H_{\mathcal{A}})$. Then F is said to be an upper semi-$\mathcal{A}$-B-Fredholm operator if there exists some $n \in \mathbb{N}$ such that ImF^m is closed for all $m \geq n$ and $F_{|_{ImF^n}}$ is an upper semi-$\mathcal{A}$-Fredholm operator. Similarly, F is said to be a lower semi-$\mathcal{A}$-B-Fredholm operator if the above conditions hold except that in this case we assume that $F_{|_{ImF^n}}$ is a lower semi-$\mathcal{A}$-Fredhlom operator and not an upper semi-$\mathcal{A}$-Fredholm operator.

Remark 5.2.2. [21, Remark 7] If $\mathcal{A} = \mathbb{C}$, that is if $H_{\mathcal{A}} = H$ is a Hilbert space, then the assumption that $F_{|_{ImF^n}}$ is an $\mathcal{A}$-Fredholm operator would automatically imply that ImF^m is closed for all $m \geq n$. Indeed, since $\mathcal{A} = \mathbb{C}$, it is not difficult to see that the property of being $\mathcal{A}$-Fredholm in the sense of Definition 2.0.37 is the same as the property of being Fredholm in the ordinary sense. Since $F_{|_{ImF^n}}$ is then a Fredholm operator, it follows that $F^k_{|_{ImF^n}}$ is Fredholm for all k, as the composition of Fredholm operators is again a Fredholm operator . Then $F^k(ImF^n)$ is closed for all k, as the image of a Fredholm operator is closed. However, $F^k(ImF^n) = ImF^{n+k}$.

Proposition 5.2.3. *[21, Proposition 7] If F is an upper semi-$\mathcal{A}$-B-Fredholm operator (respectively, a lower semi-$\mathcal{A}$-B-Fredholm operator), $n \in \mathbb{N}$ is such that ImF^m is closed for all $m \geq n$ and $F_{|_{ImF^n}}$ is an upper semi-$\mathcal{A}$-Fredholm operator (respectively, a lower semi-$\mathcal{A}$-Fredholm operator), then $F_{|_{ImF^m}}$ is an upper semi-$\mathcal{A}$-Fredholm operator (respectively, a lower semi-$\mathcal{A}$-Fredholm operator) for all $m \geq n$. Moreover, if F is an $\mathcal{A}$-B-Fredholm operator, $n \in \mathbb{N}$ is such that ImF^m is closed for all $m \geq n$ and $F_{|_{ImF^n}}$ is an $\mathcal{A}$-Fredholm operator, then $F_{|_{ImF^m}}$ is an $\mathcal{A}$ Fredholm operator and* index $F_{|_{ImF^m}} =$ index $F_{|_{ImF^n}}$ *for all $m \geq n$.*

Proof. By applying Corollary 5.1.25 on the operator $F_{|_{ImF^n}}$ and proceeding inductively, we deduce the proposition. □

For an $\mathcal{A}$-B-Fredholm operator F we set index $F =$ index $F_{|_{ImF^n}}$, where n is such that ImF^m is closed for all $m \geq n$ and such that $F_{|_{ImF^n}}$ is an $\mathcal{A}$-Fredholm operator.

Inspired by [5, Theorem 3.2] we state the following proposition.

Proposition 5.2.4. *Let M be a Hilbert-module and $F, D \in B^a(M)$ satisfying that $FD = DF$. Suppose that there exists an $n \in \mathbb{N}$ such that $Im(DF)^m$ is closed for all $m \geq n$ and in addition for each $m \geq n$ we have that $ImF^{m+1}D^m$ and $ImD^{m+1}F^m$ are closed. If F and D are upper (lower) semi-$\mathcal{A}$-B-Fredholm, then DF is upper (lower) semi-$\mathcal{A}$-B-Fredholm. If F and D are $\mathcal{A}$-B-Fredholm, then DF is $\mathcal{A}$-B-Fredholm and* index $DF =$ index $D +$ index F.

Proof. If F and D are upper semi-$\mathcal{A}$-B-Fredholm, then by Proposition 5.2.3 we can choose an $n \in \mathbb{N}$ sufficiently large such that n satisfies the assumption in the proposition and in addition satisfies that ImD^m, ImF^m are closed and $F_{|_{ImF^m}}, D_{|_{ImF^m}}$ are upper semi-$\mathcal{A}$-Fredholm for all $m \geq n$. As $ImF^{n+1}D^n = ImF(DF)^n$, $ImD^{n+1}F^n = ImD(DF)^n$, $Im(DF)^n$ and $Im(DF)^{n+1}$ are all closed by assumption, we have that $F_{|_{Im(DF)^n}}, D_{|_{Im(DF)^n}}$ and $DF_{|_{Im(DF)^n}}$ are regular operators. This follows from Theorem 2.0.20. Hence we can apply the exact sequence from Lemma 5.1.7. Since $F_{|_{ImF^n}}$ and $D_{|_{ImD^n}}$ are upper semi-$\mathcal{A}$-Fredholm, we have that

$$\ker F_{|_{ImF^n}} = \ker F \cap ImF^n \text{ and } \ker D_{|_{ImD^n}} = \ker D \cap ImD^n$$

are both finitely generated by Lemma 3.1.21. As $F_{|_{Im(DF)^n}}$ and $D_{|_{Im(DF)^n}}$ are regular operators, it follows that

$$\ker F_{|_{Im(DF)^n}} = \ker F \cap Im(DF)^n \text{ and } \ker D_{|_{Im(DF)^n}} = \ker D \cap Im(DF)^n$$

are both orthogonally complementable in $Im(DF)^n$. However, $Im(DF)^n$ is orthogonally complementable in M by Theorem 2.0.20, so $\ker F \cap Im(DF)^n$ and $\ker D \cap Im(DF)^n$ are orthogonally complementable in M. Since

$$Im(DF)^n = ImD^nF^n = ImF^nD^n \subseteq ImF^n \cap ImD^n,$$

we get that

$$\ker D \cap Im(DF)^n \subseteq \ker D \cap ImD^n \text{ and } \ker F \cap Im(DF)^n \subseteq \ker F \cap ImF^n.$$

By Lemma 2.0.66 we obtain that $\ker F \cap Im(DF)^n$ and $\ker D \cap Im(DF)^n$ are orthogonally complementable in $\ker F \cap ImF^n$ and $\ker D \cap ImD^n$, respectively. As $\ker F \cap ImF^n$ and $\ker D \cap ImD^n$ are finitely generated, it follows that $\ker F \cap Im(DF)^n$ and $\ker D \cap Im(DF)^n$ are both finitely generated. By applying the exact sequence from Lemma 5.1.7 on the operators $F_{|_{Im(DF)^n}}, D_{|_{Im(DF)^n}}$ and $DF_{|_{Im(DF)^n}}$ we deduce that $\ker DF_{|_{Im(DF)^n}}$ is finitely generated. Hence, $DF_{|_{Im(DF)^n}}$ is upper semi-$\mathcal{A}$-Fredholm by Lemma 3.1.21. Proceeding inductively we obtain that $DF_{|_{Im(DF)^m}}$ is upper semi-$\mathcal{A}$-Fredholm for all $m \geq n$.

Suppose next that $F_{|_{ImF^n}}$ and $D_{|_{ImD^n}}$ are lower semi-$\mathcal{A}$-Fredholm. Then, by Lemma 3.1.21,

$$ImF^n = ImF^{n+1} \oplus N \text{ and } ImD^n = ImD^{n+1} \oplus N'$$

for some finitely generated Hilbert submodules N and N'. It follows that

$$ImD^nF^n = ImD^nF^{n+1} + D^n(N) \text{ and } ImF^nD^n = ImF^nD^{n+1} + F^n(N').$$

Since $ImF^{n+1}D^n = ImF(DF)^n$ and $ImD^{n+1}F^n = ImD(DF)^n$ are both closed by assumption, by Theorem 2.0.20 we have that $ImF^{n+1}D^n$ and $ImD^{n+1}F^n$ are orthogonally complementable in $ImF^nD^n = ImD^nF^n = Im(DF)^n$, so

$$Im(DF)^n = ImF(DF)^n \oplus \tilde{N} \text{ and } Im(DF)^n = ImD(DF)^n \oplus \tilde{N}'$$

for some Hilbert submodules $\tilde{N}$ and $\tilde{N}'$. Let P and P' stand for the orthogonal projections onto $\tilde{N}$ and $\tilde{N}'$, respectively. As $ImF^{n+1}D^n = ImD^nF^{n+1}$ and $ImD^{n+1}F^n = ImF^nD^{n+1}$, it follows that $\tilde{N} = PD^n(N)$ and $\tilde{N}' = P'F^n(N')$, hence $\tilde{N}$ and $\tilde{N}'$ are finitely generated since N and N' are so. Thus, the orthogonal complement of $ImF(DF)^n$ and the orthogonal complement of $ImD(DF)^n$ in $Im(DF)^n$ are both finitely generated. By applying again the exact sequence from Lemma 5.1.7 on the operators $F_{|_{Im(DF)^n}}, D_{|_{Im(DF)^n}}$ and $DF_{|_{Im(DF)^n}}$, we obtain by Lemma 3.1.21 that $DF_{|_{Im(DF)^n}}$ is lower semi-$\mathcal{A}$-Fredholm. Proceeding inductively we obtain that $DF_{|_{Im(DF)^m}}$ is lower semi-$\mathcal{A}$-Fredholm for all $m \geq n$.

The proof in the case when F and D are $\mathcal{A}$-B-Fredholm is similar, or more precisely, a combination of the previous proofs for the cases when D and F were upper or lower semi-$\mathcal{A}$-B-Fredholm. Moreover, by applying the exact sequence from Lemma 5.1.7 in this case, we can also deduce that

$$\text{index } DF = \text{index } D + \text{index } F.$$

□

Remark 5.2.5. If F and D are operators on a Hilbert space and both F and D are B-Fredholm, then from [5, Theorem 3.2] we know that DF is B-Fredholm if D and F mutually commute. Hence, there exists an $n \in \mathbb{N}$ such that $Im(DF)^m$ is closed for all $m \geq n$. Now, if we choose $n \in \mathbb{N}$ such that in addition $F_{|_{ImF^n}}$ and $D_{|_{ImD^n}}$ are Fredholm, then by the arguments from the proof of Proposition 5.2.4 we get that the co-dimension of $ImF(DF)^n$ and the co-dimension of $ImD(DF)^n$ in $Im(DF)^n$ are finite. Since $Im(DF)^n$ is closed, by the Kato Theorem [56, Corollary 1.1.7] we must have that $ImF(DF)^n = ImF^{n+1}D^n$ and $ImD(DF)^n = ImD^{n+1}F^n$ are both closed. Proceeding inductively we obtain that $ImF^{m+1}D^m$ and $ImD^{m+1}F^m$ are closed for all $m \geq n$. Thus Proposition 5.2.4 can in a certain way be considered as a generalization of [5, Theorem 3.2] to the case of operators on Hilbert C^*-modules.

Theorem 5.2.6. *[21, Theorem 8] Let T be an $\mathcal{A}$-B-Fredholm operator on $H_{\mathcal{A}}$ and suppose that $m \in \mathbb{N}$ is such that $T_{|_{ImT^m}}$ is an $\mathcal{A}$-Fredholm operator and ImT^n is closed for all $n \geq m$. Let F be in the linear span of elementary operators and suppose that $Im(T+F)^n$ is closed for all $n \geq m$. Finally, assume that $ImT^m \cong H_{\mathcal{A}}$, $Im(\tilde{F}), T^m(\ker \tilde{F})$ are closed and that $Im\tilde{F}$ is finitely generated, where $\tilde{F} = (T+F)^m - T^m$. Then $T+F$ is an $\mathcal{A}$-B-Fredholm operator and* index $(T+F) =$ index T.

Proof. Since $\tilde{F} \in B^a(H_{\mathcal{A}})$ and $Im\tilde{F}$ is closed by assumption, by Theorem 2.0.20 we have that $\ker \tilde{F}$ is orthogonally complementable in $H_{\mathcal{A}}$. Hence, $T^m_{|_{\ker \tilde{F}}}$ is adjointable. Since $T^m(\ker \tilde{F})$ is closed by assumption, again by Theorem 2.0.20 we have that $T^m(\ker \tilde{F})$ is orthogonally complementable in $H_{\mathcal{A}}$.

Observe that, since $\widetilde{F} = (T+F)^m - T^m$ by definition, it follows that $(T+F)^m_{|_{\ker \tilde{F}}} = T^m_{|_{\ker \tilde{F}}}$, so $T^m(\ker \widetilde{F}) \subseteq Im(T+F)^m$. Hence, since $T^m(\ker \tilde{F}) \subseteq ImT^m \cap Im(T+F)^m$, by Lemma 2.0.66 it follows that

$$ImT^m = T^m(\ker \tilde{F}) \oplus N \text{ and } \ Im(T+F)^m = T^m(\ker \tilde{F}) \oplus N'$$

for some closed submodules N, N'.

Now, since $Im\tilde{F}$ is finitely generated, it follows that $\ker \tilde{F}^{\perp}$ is also finitely generated, as $\tilde{F}_{|_{\ker \tilde{F}^{\perp}}}$ is an isomorphism onto $Im\tilde{F}$. Moreover,

$$ImT^m = T^m(\ker \tilde{F}) + T^m(\ker \tilde{F}^{\perp}),$$

$$Im(T+F)^m = T^m(\ker \tilde{F}) + (T+F)^m(\ker \tilde{F}^{\perp}).$$

Let Q denote the orthogonal projection onto $T^m(\ker \tilde{F})^{\perp}$. It is then clear that

$$N = Q(ImT^m) = Q(T^m(\ker \tilde{F}^{\perp})) \text{ and } N' = Q(Im(T+F)^m) = Q((T+F)^m(\ker \tilde{F}^{\perp})).$$

Since $\ker \tilde{F}^{\perp}$ is finitely generated, it follows that N, N' are also finitely generated.

As $T_{|_{ImT^m}}$ is an $\mathcal{A}$-Fredholm operator, by Lemma 3.1.13 it follows that $\sqcap T_{|_{T^m(\ker \tilde{F})}}$ is an $\mathcal{A}$-Fredholm operator, where $\sqcap$ denotes the orthogonal projection onto $T^m(\ker \tilde{F})$ along N. Here we use that $ImT^m \cong H_{\mathcal{A}}$ by assumption, so we are indeed in the position to apply Lemma 3.1.13.

Let $P = I - Q$. Since $T^m(\ker \tilde{F})^{\perp} = N \oplus ImT^{m\perp}$, we have that $PT_{|_{T^m(\ker \tilde{F})}}$ is an $\mathcal{A}$-Fredholm operator on $T^m(\ker \tilde{F})$, as $PT_{|_{T^m(\ker \tilde{F})}} = \sqcap T_{|_{T^m(\ker \tilde{F})}}$ (because $TT^m(\ker \tilde{F}) \subseteq ImT^{m+1} \subseteq ImT^m$). By Lemma 3.1.13, since $ImT^m \cong H_{\mathcal{A}}$ by assumption, it follows that

$$\text{index } T = \text{index } T_{|_{ImT^m}} = \text{index } \sqcap T_{|_{T^m(\ker \tilde{F})}} = \text{index } PT_{|_{T^m(\ker \tilde{F})}}.$$

Now, since $ImT^m \cong H_{\mathcal{A}}$, $ImT^m = T^m(\ker \tilde{F}) \oplus N$ and N is finitely generated, by the Dupre-Filmore Theorem 2.0.15 it follows that $T^m(\ker \tilde{F}) \cong H_{\mathcal{A}}$. Since $PF_{|_{T^m(\ker \tilde{F})}} \in \mathcal{K}^*(T^m(\ker \tilde{F}))$, it follows from Lemma 2.0.45 that $P(T+F)_{|_{T^m(\ker \tilde{F})}}$ is an $\mathcal{A}$-Fredholm operator on $T^m(\ker \tilde{F})$ and

$$\text{index } PT_{|_{T^m(\ker \tilde{F})}} = \text{index } P(T+F)_{|_{T^m(\ker \tilde{F})}}.$$

Moreover,

$$Im(T+F)^m = T^m(\ker \tilde{F}) \oplus N',$$

where N' is finitely generated Hilbert submodule. Hence, $P(T+F)_{|_{T^m(\ker \tilde{F})}} = \tilde{\sqcap}(T+F)_{|_{T^m(\ker \tilde{F})}}$, where $\tilde{\sqcap}$ denotes the orthogonal projection onto $T^m(\ker \tilde{F})$ along N', as

$$(T+F)(T^m(\ker \tilde{F})) = (T+F)^{m+1}(\ker \tilde{F}) \subseteq Im(T+F)^{m+1} \subseteq Im(T+F)^m.$$

In addition, since N' is finitely generated and $T^m(\ker \tilde{F}) \cong H_{\mathcal{A}}$, by the Kasparov stabilization Theorem 2.0.13 it follows that $Im(T+F)^m \cong H_{\mathcal{A}}$.
Since $\tilde{\sqcap}(T+F)_{|_{T^m(\ker \tilde{F})}}$ is an $\mathcal{A}$-Fredholm operator on $T^m(\ker \tilde{F})$, $Im(T+F)^m \cong H_{\mathcal{A}}$ and N' is finitely generated, by Lemma 3.1.13 it follows that $(T+F)_{|_{Im(T+F)^m}}$ is an $\mathcal{A}$-Fredholm operator and

$$\text{index } (T+F) = \text{index } (T+F)_{|_{Im(T+F)^m}} = \text{index } \tilde{\sqcap}(T+F)_{|_{T^m(\ker \tilde{F})}}.$$

□

Remark 5.2.7. [21, Remark 9] When $A = \mathbb{C}$, that is when $H_{\mathcal{A}} = H$ is a Hilbert space, then Theorem 5.2.6 reduces to [4, Proposition 3.3]. Indeed, since F and hence $\tilde{F}$ are finite rank operators, then $Im\tilde{F}$ and $\ker \tilde{F}^{\perp}$ are finite dimensional in this case. Hence, we have that $T^m(\ker \tilde{F}^{\perp})$ and $(T+F)^m(\ker \tilde{F}^{\perp})$ are finite dimensional, so all these subspaces are closed, being finite dimensional. Moreover,

$$Im(T+F)^m = Im(T^m + \tilde{F})$$

$$= T^m(\ker \tilde{F}) + (T+\tilde{F})(\ker \tilde{F}^{\perp}) = T^m(\ker \tilde{F}) + (T+F)^m(\ker \tilde{F}^{\perp}).$$

Since ImT^m is closed, $ImT^m = T^m(\ker \tilde{F}) + T^m(\ker \tilde{F}^{\perp})$ and $T^m(\ker \tilde{F}^{\perp})$ is finite dimensional, it follows that $T^m(\ker \tilde{F})$ is closed. This follows from the Kato theorem [56, Corollary 1.1.7] applied on the operator $T^m_{|_{\ker \tilde{F}}} : \ker \tilde{F} \to ImT^m$.
Now, since

$$Im(T+F)^m = T^m(\ker \tilde{F}) + (T+F)^m(\ker \tilde{F}^{\perp}),$$

$T^m(\ker \tilde{F})$ is closed and $(T+F)^m(\ker \tilde{F}^{\perp})$ is finite dimensional, we obtain that $Im(T+F)^m$ is closed by [56, Lemma 1.1.2]. By the same arguments it follows that $Im(T+F)^n$ is closed for all $n \geq m$, whenever $Im(T^n)$ is closed for all $n \geq m$ (and this is going to be the case when $T_{|_{ImT^m}}$ is Fredholm) . Finally, if $Im(T^m)$ is closed and infinite-dimensional, then $Im(T^m) \cong H$.

Now we are going to consider non-adjointable semi-$\mathcal{A}$-B-Fredholm operators.

Proposition 5.2.8. *Let $F \in B(H_{\mathcal{A}})$. If $n \in \mathbb{N}$ is such that ImF^n closed, $ImF^n \cong H_{\mathcal{A}}$, $F_{|_{ImF^n}}$ is upper semi-$\mathcal{A}$-Fredholm and ImF^m is closed for all $m \geqslant n$, then $F_{|_{ImF^m}}$ is upper semi-$\mathcal{A}$- Fredholm and $ImF^m \cong H_{\mathcal{A}}$ for all $m \geqslant n$. If $n \in \mathbb{N}$ is such that ImF^n is closed, $ImF^n \cong H_{\mathcal{A}}, ImF^m$ is closed and complementable in ImF^n for all $m \geqslant n$ and $F_{|_{ImF^n}}$ is lower semi-$\mathcal{A}$-Fredholm, then $F_{|_{ImF^m}}$ is lower semi-$\mathcal{A}$-Fredholm and $ImF^m \cong H_{\mathcal{A}}$ for all $m \geqslant n$.*

Finally, if $n \in \mathbb{N}$ is such that ImF^m is closed for all $m \geq n$, $ImF^n \cong H_{\mathcal{A}}$ and $F_{|_{ImF^n}}$ is in $\widehat{\mathcal{M}\Phi}(ImF^n)$, then $ImF^m \cong H_{\mathcal{A}}$, $F_{|_{ImF^m}} \in \widehat{\mathcal{M}\Phi}(ImF^m)$ and index $F_{|_{ImF^m}} =$ index $F_{|_{ImF^n}}$ for all $m \geq n$.

Proof. Recall that if $F \in \widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}})$ and ImF is closed, or if $F \in \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}})$ and ImF is closed and complementable, then by Proposition 3.5.11 and Proposition 3.5.13 F is a regular operator.

Next, if $F_{|_{ImF^n}} \in \widehat{\mathcal{M}\Phi}(ImF^n)$, then, since $ImF^n \cong H_{\mathcal{A}}$ by assumption, it follows from Corollary 3.5.5 that $F^2_{|_{ImF^n}} \in \widehat{\mathcal{M}\Phi}(ImF^n)$. The proof is similar in the case when we have $F_{|_{ImF^n}} \in \widehat{\mathcal{M}\Phi}_r(ImF^n)$ or when $F_{|_{ImF^n}} \in \widehat{\mathcal{M}\Phi}_l(ImF^n)$. Combining all this together we deduce that $F_{|_{ImF^n}}$ and $F^2_{|_{ImF^n}}$ are regular operators on ImF^n. We can then apply Lemma 5.1.24 to deduce that $F_{|_{ImF^{n+1}}} \in \widehat{\mathcal{M}\Phi}(ImF^{n+1})$ when $F_{|_{ImF^n}} \in \widehat{\mathcal{M}\Phi}(ImF^n)$ and in this case index $F_{|_{ImF^{n+1}}} =$ index $F_{|_{ImF^n}}$. The proof for the case when $F_{|_{ImF^n}} \in \widehat{\mathcal{M}\Phi}_l(ImF^n)$ or when $F_{|_{ImF^n}} \in \widehat{\mathcal{M}\Phi}_r(ImF^n)$ is similar. Also, we wish to argue that $ImF^{n+1} \cong H_{\mathcal{A}}$, however, this follows from Lemma 3.5.16. Then we can proceed inductively to deduce the proposition. □

Remark 5.2.9. By applying Corollary 3.5.15 instead of Lemma 3.1.21, one can show that the proof of Proposition 5.2.4 remains valid also for non-adjointable operators provided that we assume in addition that $Im(DF)^n$ is complementable and that $F_{|_{Im(DF)^m}}, D_{|_{Im(DF)^m}}, DF_{|_{Im(DF)^m}}$ are regular operators for all $m \geq n$.

Lemma 5.2.10. *Let $F \in \widehat{\mathcal{M}\Phi}(H_{\mathcal{A}}), K \in \mathcal{K}(H_{\mathcal{A}})$. Then* index $F =$ index $(F+K)$.

Proof. Let
$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$
be an $\widehat{\mathcal{M}\Phi}$-decomposition for F. We may without loss of generality assume that there exists an $n \in \mathbb{N}$ such that $F+K$ has the matrix $\begin{bmatrix} (F+K)_1 & (F+K)_2 \\ (F+K)_3 & (F+K)_4 \end{bmatrix}$ with respect to the decomposition
$$H_{\mathcal{A}} = F_1^{-1}(L_n^{\perp}) \tilde{\oplus} (F_1^{-1}(P) \tilde{\oplus} N_1) \xrightarrow{F+K} L_n^{\perp} \oplus L_n = H_{\mathcal{A}},$$
where $(F+K)_1$ is an isomorphism. Here P is finitely generated Hilbert $\mathcal{A}$-module such that $L_n = N_2 \tilde{\oplus} P, M_2 = L_n^{\perp} \oplus P$. By diagonalizing the operator matrix $\begin{bmatrix} (F+K)_1 & (F+K)_2 \\ (F+K)_3 & (F+K)_4 \end{bmatrix}$ as in the proof of Lemma 2.0.42, we easily obtain that index $(F+K) =$ index F. □

Lemma 5.2.11. *Let $T \in B(H_{\mathcal{A}})$ and $F \in \mathcal{K}(H_{\mathcal{A}})$. Suppose that there exists an $m \in \mathbb{N}$ satisfying the conditions of Theorem 5.2.6 and assume in addition that $T^m(\ker \tilde{F})$ is complementable in $H_{\mathcal{A}}$. Then the analogue of Theorem 5.2.6 holds in this case.*

Proof. By assumption $Im\tilde{F}$ is closed since m satisfies the condition of Theorem 5.2.6, so $Im\tilde{F}$ is a finitely generated, projective Hilbert $\mathcal{A}$-module. It follows that $\ker \tilde{F}$ is complementable in $H_{\mathcal{A}}$ since $\tilde{F} : H_{\mathcal{A}} \to Im\tilde{F}$ is an epimorphism. If we let $\ker \tilde{F}^{\circ}$ denote the complement of $\ker \tilde{F}$, it follows that $\ker \tilde{F}^{\circ}$ is finitely generated. Using that $T^m(\ker \tilde{F})$ is complementable in $H_{\mathcal{A}}$ by assumption, we may proceed in the same way as in the proof of Theorem 5.2.6. The projections $P, \sqcap, \tilde{\sqcap}$ and Q are no longer orthogonal projections, but rather skew projections. By applying Corollary 2.0.26 instead of the Dupre-Filmore Theorem 2.0.15 we can show that $T^m(\ker \tilde{F}) \cong H_{\mathcal{A}}$. Let $U : H_{\mathcal{A}} \to T^m(\ker \tilde{F})$ be an isomorphism and $J : T^m(\ker \tilde{F}) \to: H_{\mathcal{A}}$ be the inclusion. Then $U^{-1}PFU = U^{-1}PFJU \in \mathcal{K}(H_{\mathcal{A}})$ since $F \in \mathcal{K}(H_{\mathcal{A}})$ and $\mathcal{K}(H_{\mathcal{A}})$ is a two-sided ideal in $B(H_{\mathcal{A}})$. By applying Lemma 5.2.10 instead of Lemma 2.0.45 and recalling that Lemma 3.1.13 also holds for non-adjointable operators by Remark 3.5.24, we obtain that
$$\text{index } PT_{|_{T^m(\ker \tilde{F})}} = \text{index } U^{-1}PTU$$
$$= \text{index } U^{-1}P(T+F)U = \text{index } P(T+F)_{|_{T^m(\ker \tilde{F})}}.$$
Then we can proceed in exactly the same way as in the proof of Theorem 5.2.6. □

Corollary 5.2.12. *Let M be an arbitrary Hilbert W^*-module and $F \in B(M)$. If $n \in \mathbb{N}$ is such that ImF^m is closed and complementable for every $m \geq n$ and such that $F_{|_{ImF^n}}$ is an $\mathcal{A}$-Fredholm operator, then $F_{|_{ImF^m}}$ is an $\mathcal{A}$-Fredholm operator and* index $F_{|_{ImF^m}} =$ index $F_{|_{ImF^n}}$ *for all $m \geq n$.*

Proof. Since $ImF^{n+2} = ImF^2_{|_{ImF^n}}$ is complementable in ImF^n, which follows from the assumption in the corollary and Lemma 2.0.66, we deduce from Lemma 5.1.20 that $F^2_{|_{ImF^n}}$ belongs to $\widehat{\mathcal{M}\Phi}(ImF^n)$, as $F_{|_{ImF^n}} \in \widehat{\mathcal{M}\Phi}(ImF^n)$ by assumption. Then we can proceed in the same way as in the proof of Proposition 5.2.8. □

We are now going to give some examples of semi-$\mathcal{A}$-B-Fredholm operators. Before that we wish to introduce some examples of nilpotent operators on Hilbert submodules of $H_{\mathcal{A}}$ There are various ways of constructing such operators. Of course, the zero operator is certainly a nilpotent operator, however, we wish to give here also some non-trivial examples of nilpotent operators on $H_{\mathcal{A}}$.

Example 5.2.13. Let $\mathcal{A} = B(H)$, choose a nilpotent operator $\tilde{C} \in B(H)$ and let

$$C'(A_1, A_2, A_3, ...) = (\tilde{C}A_1, \tilde{C}A_2, \tilde{C}A_3, ...) \text{ for all } (A_1, A_2, A_3, ...) \in H_{\mathcal{A}}.$$

Then, if $\tilde{C}^j = 0$ for some $j \in \mathbb{N}$, it follows that $C'^j = 0$ also. Hence, if $N \cong H_{\mathcal{A}}$ and $V : N \to H_{\mathcal{A}}$ is an isomorphism, then $V^{-1}C'V$ is a nilpotent operator on N.

Example 5.2.14. Consider now a more general situation where $\mathcal{A}$ is an arbitrary unital C^*-algebra and N is a closed submodule of $H_{\mathcal{A}}$ not necessarily isomorphic to $H_{\mathcal{A}}$. If we may write N as $N = N_1 \oplus N_2$ where $N_1 \cong L_n(\mathcal{A})$ for some n, then we may let $C = C_1 \oplus C_2$, where C_1 is a nilpotent operator on $N_1 \cong L_n(\mathcal{A})$ and $C_2 = 0$. Such operators can easily be constructed, as there are a plenty of nilpotent operators on $L_n(\mathcal{A})$. For example, if
$$F(e_k) = \begin{cases} 0, & k = 1 \\ e_{k-1}, & k \in \{2, 3, ..., n\}, \end{cases}$$
then F is an example of a nilpotent operator on $L_n(\mathcal{A})$. In general, if F is given by $n \times n$ matrix with coefficients in $\mathcal{A}$ and 0 on the main diagonal, with respect to the standard orthonormal basis $\{e_j\}_{1 \leq j \leq n}$, then F is nilpotent.

Then we are ready to construct some semi-$\mathcal{A}$-B-Fredholm operators.

Example 5.2.15. Let $H_{\mathcal{A}} = M \oplus N$ be a decomposition where $M \cong H_{\mathcal{A}}$ and let U denote the isomorphism from M onto $H_{\mathcal{A}}$. Choose an operator $T \in \mathcal{M}\Phi_{\pm}(H_{\mathcal{A}})$ such that ImT^k is closed for all k. Again, such operators have been constructed in our previous examples. Hence, if $T \in \mathcal{M}\Phi_{\pm}(H_{\mathcal{A}})$ such that ImT^k is closed for all k, then $U^{-1}TU \in \mathcal{M}\Phi_{\pm}(M)$ and $Im(U^{-1}TU)^k$ is closed for all k. Observe also that, since $U^{-1}TU \in \mathcal{M}\Phi_{\pm}(M), Im(U^{-1}TU)^k$ is closed for all k and $M \cong H_{\mathcal{A}}$, it follows by applying inductively Corollary 5.1.25 that $U^{-1}TU_{|_{Im(U^{-1}TU)^k}}$ is in $\mathcal{M}\Phi_{\pm}(Im(U^{-1}TU)^k)$ for all k. Next, choose $C \in B^a(N)$ such that C is nilpotent. Let F be the operator having the matrix
$$\begin{bmatrix} U^{-1}TU & 0 \\ 0 & C \end{bmatrix}$$
with respect to the decomposition $H_{\mathcal{A}} = M \oplus N$. Then F is a semi-$\mathcal{A}$-B-Fredholm operator.

Chapter 6

Closed range operators over C^*-algebras

In several results in previous chapters we have assumed that the image of an operator is closed. This shows that closed range operators are important in semi-Fredholm theory on Hilbert C^*-modules. Therefore, we will devote this chapter to studying closed range operators and their properties.
We start with the following lemma.

Lemma 6.0.1. *[19, Lemma 3.13] Let $\tilde{M}$ be a Hilbert C^*-module, $F, D \in B^a(\tilde{M})$ and suppose that ImF, ImD are closed. If $ImF + \ker D$ is closed, then $ImF + \ker D$ is orthogonally complementable.*

Proof. Suppose that $ImF + \ker D$ is closed. Since $ImF \oplus ImF^\perp = \tilde{M}$ by Theorem 2.0.20, we have that $ImF + \ker D = ImF \oplus M''$, where

$$M'' = (ImF + \ker D) \cap ImF^\perp, \text{ as } ImF \subseteq ImF + \ker D.$$

This follows from Lemma 2.0.66. Let P denote the orthogonal projection onto $ImF^\perp$. Then $M'' = P(ImF + \ker D) = P(ImF) + P(\ker D) = P(\ker D)$. Thus, $Im(P_{|_{\ker D}}) = M''$. Now, since ImD is closed, again by Theorem 2.0.20 $\ker D$ is orthogonally complementable in $\tilde{M}$. Hence $P_{|_{\ker D}}$ is an adjointable operator from $\ker D$ into $ImF^\perp$ and its image is closed. Applying once again Theorem 2.0.20 on the operator $P_{|_{\ker D}}$, we obtain that $ImP_{|_{\ker D}}$ is orthogonally complementable in $ImF^\perp$, hence $ImF^\perp = M'' \oplus N''$. Therefore,

$$\tilde{M} = ImF \oplus M'' \oplus N'' = (ImF + \ker D) \oplus N''.$$

□

Corollary 6.0.2. *[19, Corollary 3.14] Let $\tilde{M}$ be a Hilbert C^*-module, $F, D \in B^a(\tilde{M})$ and suppose that ImF, ImD are closed. Then $ImDF$ is closed if and only if $ImF + \ker D$ is closed and orthogonally complementable.*

Proof. By [42, Corollary 1], $ImDF$ is closed if and only if $ImF + \ker D$ is closed. Now we use Lemma 6.0.1. □

Remark 6.0.3. [19, Remark 3.15] The statement of Corollary 6.0.2 was already proved in [49], however, we have given here another, shorter proof.

We recall the definition of the Dixmier angle between two closed submodules of a Hilbert C^*-module, given in [49].

Definition 6.0.4. [49], [19, Definition 3.16] Given two closed submodules M, N of a Hilbert C^*-module $\widetilde{M}$, we set

$$c_0(M, N) = \sup\{\| \langle x, y\rangle \| \ \mid x \in M, y \in N, \| x \|, \| y \| \leq 1\}.$$

We say that the Dixmier angle between M and N is positive if $c_0(M, N) < 1$.

Lemma 6.0.5. *[19, Lemma 3.17] Let M, N be two closed, submodules of a Hilbert C^*-module $\widetilde{M}$ over a C^*-algebra $\mathcal{A}$. Assume that M orthogonally complementable and suppose that $M \cap N = \{0\}$. Then $M+N$ is closed if the Dixmier angle between M and N is positive.*

Proof. Suppose that the Dixmier angle between M and N is positive. If $c_0(M,N)=0$, then $M \perp N$. It follows that $M+N$ is closed in this case.

Now let $c_0(M,N)=\delta$ where $\delta \in (0,1)$. We wish first to show that in this case there exists some constatnt $C>0$ such that whenever $x \in M, y \in N$ satisfy $\| x+y \| \leq 1$, then $\| x \| \leq C$. To this end, observe first that, since M is orthogonally complementable in $\widetilde{M}$, there exist some $y' \in M, y'' \in M^{\perp}$ such that $y = y' + y''$ for $y \in N$. Then we have

$$\sup\{\| \langle y, z \rangle \| \mid z \in M, \| z \| = 1\} = \| y' \| \leq \| y \| \, \delta.$$

Indeed,

$$\sup\{\| \langle y, z \rangle \| \mid z \in M, \| z \| = 1\} = \sup\{\| \langle y', z \rangle \| \mid z \in M, \| z \| = 1\},$$

hence, by Proposition 2.0.4 part (iii), it follows that

$$\sup\{\| \langle y, z \rangle \| \mid z \in M, \| z \| = 1\} \leq \| y' \| .$$

On the other hand, if $y'=0$, then $\langle y, z \rangle = 0$ for all $z \in M$. If $y' \neq 0$, then $\frac{y'}{\|y'\|} \in M$ and

$$\| y' \| = \| \langle y, \frac{y'}{\| y' \|} \rangle \| \leq \sup\{\| \langle y, z \rangle \| \mid z \in M, \| z \| = 1\}.$$

It follows that

$$\| y'' \| = \| y - y' \| \geq \| y \| - \| y' \| \geq (1-\delta) \| y \| = \frac{1-\delta}{\delta} \delta \| y \| \geq \frac{1-\delta}{\delta} \| y' \| .$$

Now observe that for $x \in M$ we have that

$$\langle x+y, x+y \rangle = \langle x+y', x+y' \rangle + \langle y'', y'' \rangle.$$

By taking the supremum over all states on $\mathcal{A}$, we obtain that

$$\| x+y \| \geq \max\{\| x+y' \|, \| y'' \|\}.$$

Thus, if $\| x+y \| \leq 1$, then $\| x+y' \|, \| y'' \| \leq 1$. However, if $\| y'' \| \leq 1$, then by the above calculation, we get that $\| y' \| \leq \dfrac{\delta}{1-\delta}$. If in addition $\| x+y' \| \leq 1$, then

$$1 \geq \| x \| - \| y' \| \geq \| x \| - \frac{\delta}{1-\delta}.$$

Hence we get $\| x \| \leq 1 + \dfrac{\delta}{1-\delta}$, so we may set $C = 1 + \dfrac{\delta}{1-\delta} = \dfrac{1}{1-\delta}$.
Assume now that $\{x_n + y_n\}_n$ is a Cauchy sequence in $M+N$ (here $x_n \in M, y_n \in N$ for all n). By the above arguments we have that $\{x_n\}_n$ must be then a Cauchy sequence in M. Indeed, if $\{x_n + y_n\}$ is a Cauchy sequence, then given $\epsilon > 0$, there exists some $N_0 \in \mathbb{N}$ such that $\| (x_n - x_m) + (y_n - y_m) \| < \frac{\epsilon}{C}$ for all $n, m \geq N_0$. By the above arguments it follows then that $\| x_n - x_m \| < \epsilon$ for all $n, m \geq N_0$. Since M is closed, $x_n \to x$ for some $x \in M$. However, then $\{y_n\}_n$ must be also convergent as the difference of two convergent sequences, so $y_n \to y$ for some $y \in N$ since N is closed. Hence $x_n + y_n \to x + y \in M+N$ as $n \to \infty$. Thus, $M+N$ is closed. ∎

Corollary 6.0.6. *[19, Corollary 3.18] Let $\widetilde{M}$ be a Hilbert C^*-module, $F, D \in B^a(\widetilde{M})$ and suppose that ImF, ImD are closed. Assume that $\ker D \cap ImF$ is orthogonally complementable. Set $M = ImF \cap (\ker D \cap ImF)^{\perp}$, $M' = \ker D \cap (\ker D \cap ImF)^{\perp}$. Then $ImDF$ is closed if the Dixmier angle between M' and ImF is positive, (or if the Dixmier angle between M and $\ker D$ is positive).*

Proof. Since $(\ker D \cap ImF)$ is orthogonally complementable by assumption, by Lemma 2.0.66 we have that

$$ImF = (\ker D \cap ImF) \oplus M,$$

$$\ker D = (\ker D \cap ImF) \oplus M'.$$

Then it follows that $ImF + \ker D = ImF + M' = \ker D + M$. Moreover, M and M' are orthogonally complementable being orthogonal direct summands of ImF and $\ker D$, respectively, which are orthogonally complementable by Theorem 2.0.20. Finally, $M \cap \ker D = M' \cap ImF = \{0\}$. Then we apply Lemma 6.0.5 and [42, Corollary 1] . □

Remark 6.0.7. It is easy to see that the requirement that $\ker D \cap ImF$ is orthogonally complementable is satisfied if the condition in [49] that $\overline{\ker F^* + ImD^*}$ is orthogonally complementable holds. Indeed, if $(\overline{\ker F^* + ImD^*}) \oplus N = \widetilde{M}$ for some closed submodule N, then in particular $N \subseteq \ker F^{*\perp}$ and $N \subseteq ImD^{*\perp}$. By the proof of Theorem 2.0.20 $\ker F^{*\perp} = ImF$ and $ImD^{*\perp} = \ker D$ since ImF and ImD are closed by assumption. Hence $N \subseteq \ker D \cap ImF$. On the other hand, since $(\ker D \cap ImF) \subseteq \overline{\ker F^* + ImD^*}^{\perp}$ by the linearity and the continuity of the inner product, it follows that $\ker D \cap ImF \subseteq N$. Thus,

$$\ker D \cap ImF = N = \overline{\ker F^* + ImD^*}^{\perp}.$$

Hence Lemma 6.0.5 and Corollary 6.0.6 are indeed a simplification of the result in [49].

Lemma 6.0.8. *Let M and N be two closed submodules of a Hilbert C^*-module $\widetilde{M}$ over a C^*-algebra $\mathcal{A}$. Suppose that M is orthogonally complementable in $\widetilde{M}$ and that $M \cap N = \{0\}$. Then $M + N$ is closed if and only if $P_{|_N}$ is bounded below, where P denotes the orthogonal projection onto $M^{\perp}$.*

Proof. Suppose first that $P_{|_N}$ is bounded below and let $\delta = m(P_{|_N})$. Then $\delta > 0$. As in the proof of Lemma 6.0.5 we wish to argue that in this case, there exists a constant $C > 0$ such that if $x \in M$ and $y \in N$ satisfy $\| x + y \| \leq 1$, then $\| x \| \leq C$. Now, since M is orthogonally complementable, given $y \in N$, we may write y as $y = y' + y''$, where $y' \in M, y'' \in M^{\perp}$. Observe that $\langle y, y \rangle = \langle y', y' \rangle + \langle y'', y'' \rangle$. By taking the supremum over all states on $\mathcal{A}$ we obtain that $\| y \| \geq \max\{\| y' \|, \| y'' \|\}$. Hence $\| y'' \| = \| P_{|_N}(y) \| \geq \delta \| y \| \geq \delta \| y' \|$. Then, by the same arguments as in the proof of Lemma 6.0.5, we obtain that if $\| x + y \| \leq 1$ and $x \in M$, then $\| x \| \leq 1 + \dfrac{1}{\delta} = \dfrac{\delta + 1}{\delta}$. It follows that $M + N$ is closed.

Conversely, if $M + N$ is closed, then, by Lemma 2.0.66, $M + N = M \oplus M'$, where $M' = M^{\perp} \cap (M + N)$. Hence $P(M + N) = M'$, which is closed. However, $P(M + N) = P(N)$. Moreover, since $M \cap N = \{0\}$, we have that $P_{|_N}$ is injective. By the Banach open mapping theorem it follows that $P_{|_N}$ is an isomorphism onto M', hence $P_{|_N}$ is bounded below. □

Finally we are ready to give the conditions that are both necessary and sufficient for a composition of two closed range operators to have closed image.

Corollary 6.0.9. *Let $\widetilde{M}$ be a Hilbert C^*-module, $F, D \in B^a(\widetilde{M})$ and suppose that ImF, ImD are closed. Then $ImDF$ is closed if and only if $\ker D \cap ImF$ is orthogonally complementable and $P_{|_{ImF \cap (\ker D \cap ImF)^{\perp}}}$ is bounded below, (or, equivalently, $Q_{|_{\ker D \cap (\ker D \cap ImF)^{\perp}}}$ is bounded below), where P and Q denote the orthogonal projections onto $\ker D^{\perp}$ and $ImF^{\perp}$, respectively.*

Proof. If $\ker D \cap ImF$ is orthogonally complementable, then from Lemma 2.0.66 it follows that

$$\ker D = (\ker D \cap ImF) \oplus (\ker D \cap (\ker D \cap ImF)^{\perp})$$

and

$$ImF = (\ker D \cap ImF) \oplus (ImF \cap (\ker D \cap ImF)^{\perp}).$$

Hence

$$\ker D + ImF = \ker D + (ImF \cap (\ker D \cap ImF)^{\perp}) = ImF + (\ker D \cap (\ker D \cap ImF)^{\perp}).$$

If in addition $P_{ImF \cap (\ker D \cap ImF)^{\perp}}$ or $Q_{|_{\ker D \cap (\ker D \cap ImF)^{\perp}}}$ is bounded below, from Lemma 6.0.8 (as both $\ker D$ and ImF are orthogonally complementable by Theorem 2.0.20) we deduce that $\ker D + ImF$ is closed. Then, from [42, Corollary 1] it follows that $ImDF$ is closed.

Conversely, if $ImDF$ is closed, then $D_{|_{ImF}}$ is an adjointable operator with closed image. Indeed, since ImF is closed, by Theorem 2.0.20 ImF is orthogonally complementable, hence $D_{|_{ImF}}$ is adjointable. From Theorem 2.0.20 it follows that $\ker D_{|_{ImF}}$ is orthogonally complementable in ImF. However, $\ker D_{|_{ImF}} = \ker D \cap ImF$. Since ImF is orthogonally complementable in $\widetilde{M}$ and $\ker D \cap ImF \subseteq ImF$, we get that $\ker D \cap ImF$ is orthogonally complementable in $\widetilde{M}$. Moreover, $\ker D + ImF$ is closed by [42, Corollary 1] since $ImDF$ is closed. By the previous arguments we have that

$$\ker D = (\ker D \cap ImF) \oplus (\ker D \cap (\ker D \cap ImF)^{\perp}),$$

$$ImF = (\ker D \cap ImF) \oplus (ImF \cap (\ker D \cap ImF)^{\perp}),$$

so we are then in the position to apply Lemma 6.0.8 which gives us the implication in the opposite direction. □

Remark 6.0.10. If H is a Hilbert space and M, N are closed subspaces of H such that $M \cap N = \{0\}$, it is not hard to see that if P denotes the orthogonal projection onto $M^{\perp}$, then $P_{|_N}$ is bounded below if and only if the Dixmier angle between M and N is positive. Thus, Corollary 6.0.9 is a proper generalization of Bouldin's result in [6]. Indeed, since H is a Hilbert space, for each $y \in N$ we have that $||y||^2 = ||P_{|_N} y||^2 + ||(I - P_{|_N})y||^2$. So, $||(I-P_{|_N})y|| = \sqrt{||y||^2 - ||P_{|_N} y||^2}$ for every $y \in N$, in particular $||(I - P_{|_N})y|| = \sqrt{1 - ||P_{|_N} y||^2}$ for every $y \in N$ with $||y|| = 1$. Next, for each $y \in N$, we have $\sup \{|\langle x, y\rangle| \mid x \in M \text{ and } ||x|| \leq 1\} = ||(I - P_{|_N})y||$. This is because $|\langle x, y\rangle| = |\langle x, (I - P_{|_N})y\rangle| \leq ||(I - P_{|_N})y||$ when $x \in M$ with $\| x \| \leq 1$, and, on the other hand, $|\langle y', y\rangle| = ||(I - P_{|_N})(y)||$, where

$$y' = \begin{cases} \frac{(I-P_{|_N})y}{||(I-P_{|_N})y||} & \text{if } (I - P_{|_N})y \neq 0, \\ \\ 0 & \text{if } (I - P_{|_N})y = 0. \end{cases}$$

Thus, $||y'|| \leq 1$ and $y' \in M$. Therefore,

$$\sup \{|\langle x, y\rangle| \mid x \in M, ||y'|| \leq 1\} = ||(I - P_{|_N})y||$$

for every $y \in N$. Combining all this together, we deduce that

$$c_0(M, N) = \sup \{\sqrt{1 - \| P_{|_N} y \|^2} \mid y \in N, \| y \| = 1\},$$

hence $c_0(M, N) < 1$ if and only if $P_{|_N}$ is bounced below.

In the case when we deal with non-adjointable operators, it is more challenging to describe necessary and sufficient conditions for a composition of two closed range operators to have closed image since we do not have in this case Theorem 2.0.20 at disposition. We provide in the next lemma (which is an extended version of [19, Lemma 3.21]) such conditions for a composition of two $\widehat{\mathcal{M}\Phi}_l$ closed range operators.

Lemma 6.0.11. *Let $F, G \in \widehat{\mathcal{M}\Phi}_l(H_\mathcal{A})$ and suppose that ImG and ImF are closed. Then $ImGF$ is closed if and only if $ImF + \ker G$ is closed and complementable.*
If $F, G \in \widehat{\mathcal{M}\Phi}_l(H_\mathcal{A})$, ImF, ImG are closed and $\ker G \cap ImF$ is complementable and if in addition the Dixmier angle between $\ker G$ and $ImF \cap (\ker G \cap ImF)^\circ$ is positive, (or, if the Dixmier angle between ImF and $\ker G \cap (\ker G \cap ImF)^\circ$ is positive), where $(\ker G \cap ImF)^\circ$ denotes the complement of $\ker G \cap ImF$, then $ImGF$ is closed.

Proof. If $ImF + \ker G$ is closed, from [42, Corollary 1] we have that $ImGF$ is closed. Conversely, if $ImGF$ is closed, then it follows from Corollary 3.5.5 and Proposition 3.5.11 that F, G, GF are regular operators. Since the proof of Proposition 5.1.3 extends to regular operators (as explained in the proof of Proposition 5.1.16), we deduce from that proof that $\ker G + ImF$ is closed and complementable, as noticed in Remark 5.1.4.

Now, if $G \in \widehat{\mathcal{M}\Phi}_l(H_\mathcal{A})$, then $\ker G$ is finitely generated by Proposition 3.5.11, hence it is orthogonally complementable by Lemma 2.0.25. If $\ker G \cap ImF$ is complementable, then in the similar way as in the proof of Corollary 6.0.9 we obtain that

$$ImF + \ker G = \ker G + (ImF \cap (\ker G \cap ImF)^\circ).$$

Hence we may apply Lemma 6.0.5. Further, again since $\ker G \cap ImF$ is complementable, it follows by similar arguments as in the proof of Corollary 6.0.9 that

$$ImF + \ker G = ImF + (\ker G \cap (\ker G \cap ImF)^\circ).$$

Now, from Lemma 2.0.66 we have $\ker G = (\ker G \cap ImF) \tilde{\oplus} (\ker G \cap (\ker G \cap ImF)^\circ)$. We deduce that $\ker G \cap (\ker G \cap ImF)^\circ$ is finitely generated since it is a direct summand in $\ker G$, which is finitely generated itself. Hence it is orthogonally complementable by Lemma 2.0.25. Since $ImF + \ker G = ImF + (\ker G \cap (\ker G \cap ImF)^\circ)$, we are again in the position to apply Lemma 6.0.5.

□

Corollary 6.0.12. *Let $\mathcal{A}$ be a W^*-algebra and M be a Hilbert module over $\mathcal{A}$. Suppose that $G, F \in \widehat{\mathcal{M}\Phi}(M)$ and that ImG, ImF are closed. Then $ImGF$ is closed and complementable in M if and only if $ImF + \ker G$ is closed and complementable in M. Moreover, if G, F are closed range $\mathcal{A}$-Fredholm operators on M, then $\ker G \cap ImF$ is complementable in M. If in addition the Dixmier angle between $\ker G$ and $ImF \cap (\ker G \cap ImF)^\circ$ is positive (or if the Dixmier angle between ImF and $\ker G \cap (\ker G \cap ImF)^\circ$ is positive), where $(\ker G \cap ImF)^\circ$ denotes the complement of $\ker G \cap ImF$ in M, then $ImGF$ is closed.*

Proof. We recall again that $\ker G \cap ImF$ is finitely generated. Indeed, since $G, F \in \widehat{\mathcal{M}\Phi}(M)$, from Proposition 3.5.11 we have that ImF is complementable in M and $\ker G$ is finitely generated. Since $\ker G \cap ImF = \ker \sqcap_{|_{\ker G}}$, where $\sqcap$ stands for the projection onto ImF° along ImF, from Corollary 2.0.50 it follows that $\ker G = (\ker G \cap ImF) \oplus M'$ for some Hilbert submodule M'. Hence $\ker G \cap ImF$ is finitely generated as a direct summand in $\ker G$, so from Lemma 2.0.25 it follows that $ImF = (\ker G \cap ImF) \oplus M''$ for some Hilbert submodule M''. Moreover, from Lemma 2.0.25 it also follows that $\ker G \cap ImF$ is orthogonally complementable in M.

If $ImF + \ker G$ is closed and complementable, from the above equations we get that

$$ImF + \ker G = M'' \tilde{\oplus} M' \tilde{\oplus} (\ker G \cap ImF),$$

which gives
$$M = M''\tilde{\oplus}M'\tilde{\oplus}(\ker G \cap ImF)\tilde{\oplus}N,$$
where N stands for the complement of $\ker G + ImF$ in M. So, $M = \ker G\tilde{\oplus}M''\tilde{\oplus}N$, therefore, $ImG = G(M'')\tilde{\oplus}G(N)$ (as $G_{|_{(M''\tilde{\oplus}N)}}$ is an isomorphism onto ImG because ImG is closed). Hence
$$M = G(M'')\tilde{\oplus}G(N)\tilde{\oplus}ImG^{\circ} = ImGF\tilde{\oplus}G(N)\tilde{\oplus}ImG^{\circ},$$
since $ImGF = G(M'')$.

Conversely, if $ImGF$ is closed and complementable, then from Lemma 5.1.20 it follows that $GF \in \widehat{\mathcal{M}\Phi}(M)$ and GF is a regular operator. Hence we can proceed in the same way as in the proof of Lemma 6.0.11 to deduce that $\ker G + ImF$ is closed and complementable.

Next, since $\ker G \cap ImF$ is complementable in M when F and G are closed range $\mathcal{A}$-Fredholm operators on M, which follows from the above arguments, we can proceed in exactly the same way as in the proof of Lemma 6.0.11 in order to prove the second statement in the lemma. $\square$

Now we give some examples of $\mathcal{A}$-Fredholm operators with non-closed image.

Example 6.0.13. Let $\mathcal{A} = L^{\infty}((0,1),\mu)$ and consider the operator $F : \mathcal{A} \to \mathcal{A}$ given by $F(f) = f \cdot id$ (where $id(x) = x$ for all $x \in (0,1)$). Then F is an $\mathcal{A}$-linear, bounded operator on $\mathcal{A}$ and, since $\mathcal{A}$ is finitely generated considered as Hilbert $\mathcal{A}$-module over itself, it follows that F is $\mathcal{A}$-Fredholm. However, ImF is not closed. Indeed, $\| F(\mathcal{X}_{(0,\frac{1}{n})}) \|_{\infty} = \frac{1}{n}$ for all n whereas $\| (\mathcal{X}_{(0,\frac{1}{n})}) \|_{\infty} = 1$ for all n, so F is not bounded below.
Consider now the operator $\tilde{F} \in B^a(H_{\mathcal{A}})$ given by $\tilde{F} = Q + JFP$, where Q denotes the orthogonal projection onto $L_1^{\perp}$, $P = I - Q$ and $J(\alpha) = (\alpha, 0, 0, 0, \dots)$ for all $\alpha \in \mathcal{A}$. Then it is easy to see that $\tilde{F} \in \mathcal{M}\Phi(H_{\mathcal{A}})$ and $Im\tilde{F}$ is not closed.

Example 6.0.14. Let $\mathcal{A} = B(H)$ where H is a Hilbert space. Choose an $S \in B(H)$ such that ImS is not closed. Then S is not bounded below, so there exists a sequence of unit vectors $\{x_n\}_{n\in\mathbb{N}}$ in H such that $\| Sx_n \| \to 0$ as $n \to \infty$. Choose an $x \in H$ such that $\| x \| = 1$ and define the operators $B_n \in B(H)$ to be given as $B_n x = x_n$ and $B_{n|Span\{x\}^{\perp}} = 0$ for all n. Then we have that $\| B_n \| = \| B_n x \| = \| x_n \| = 1$ for all n. However, since $SB_{n|Span\{x\}^{\perp}} = 0$ for all n and $\| x \| = 1$, it follows that $\| SB_n \| = \| SB_n x \| = \| Sx_n \|$ for all n. Thus, $\| SB_n \| \to 0$ as $n \to \infty$. If we consider the operator $F : \mathcal{A} \to \mathcal{A}$ given by $F(T) = ST$ for all $T \in B(H)$, then F is an $\mathcal{A}$-linear, bounded operator on $\mathcal{A}$ (when $\mathcal{A}$ is viewed as a Hilbert $\mathcal{A}$-module over itself), but ImF is not closed. This also follows from [28, Theorem 7]. Using the operator F, it is easy to construct an operator $\tilde{F} \in \mathcal{M}\Phi(H_{\mathcal{A}})$ in the same way as in the previous example such that $Im\tilde{F}$ is not closed.

Notice that if $S \in B(H)$ is such that ImS is closed , but ImS^2 is not closed, then $Im\tilde{F}$ will be closed , whereas $Im\tilde{F}^2$ will <u>not</u> be closed. Now we will give another example of an $\mathcal{A}$-Fredholm operator F with the property that ImF is closed, but ImF^2 is not closed.

Example 6.0.15. Let H be an infinite-dimensional Hilbert space, M and N be closed, infinite-dimensional subspaces of H such that $M + N$ is not closed. Denote by p and q the orthogonal projections onto M and N, respectively. If we let $\mathcal{A} = B(H)$, then $\tilde{M} = Span_{\mathcal{A}}\{(p, 0, 0, 0, \dots)\}$ and $\tilde{N} = Span_{\mathcal{A}}\{(q, 0, 0, 0, \dots)\}$ are finitely generated Hilbert submodules of $H_{\mathcal{A}}$. Moreover, $\tilde{M} + \tilde{N}$ is not closed. Indeed, since $M + N$ is not closed, there exists a sequence $\{x_n + y_n\}$ in H such that $x_n \in M, y_n \in N$ for all n and $x_n + y_n \to z$ for some $z \in H \setminus (M + N)$. Choose an $x \in H$ such that $\| x \| = 1$ and let, for each n, T_n and S_n be the operators in $B(H)$ defined by $T_n x = x_n$, $S_n x = y_n$ and $T_{n|Span\{x\}^{\perp}} = S_{n|Span\{x\}^{\perp}} = 0$. Since $x_n \in M$ and $y_n \in N$ for all n, it follows that $T_n \in p\mathcal{A}$ and $S_n \in q\mathcal{A}$ for all n. Moreover, $\| S_n + T_n - S_m - T_m \| = \| (S_n + T_n - S_m - T_m)x \|$ for all m, n. Since $(S_n + T_n)x = x_n + y_n$ for all n, it follows that $\{S_n + T_n\}_n$ is a Cauchy sequence in

$B(H)$, hence $S_n+T_n \to T$ for some $T \in B(H)$. Then $x_n+y_n = S_nx+T_nx \to Tx = z$ as $n \to \infty$. Now, $S_n + T_n \in p\mathcal{A} + q\mathcal{A}$ for all n. If also $T \in p\mathcal{A} + q\mathcal{A}$, then $Tx \in M + N$. However, then $z \in M+N$, which is a contradiction. Thus, $T \notin p\mathcal{A}+q\mathcal{A}$, so $p\mathcal{A}+q\mathcal{A}$ is not closed in $\mathcal{A}$. It follows easily that $\tilde{M}+\tilde{N}$ is not closed. Also, $(L_1^\perp \oplus \tilde{M})+\tilde{N}$ is not closed. Since $\tilde{N}$ is finitely generated, by the Dupre-Filmore Theorem 2.0.15 we have that $\tilde{N}^\perp \cong H_\mathcal{A}$. Moreover, $L_1^\perp \oplus \tilde{M} \cong H_\mathcal{A}$, hence $L_1^\perp \oplus \tilde{M} \cong \tilde{N}^\perp$. Let $U : \tilde{N}^\perp \to L_1^\perp \oplus \tilde{M}$ be an isomorphism, set $F = JUP$, where P is the orthogonal projection onto $\tilde{N}^\perp$ and J is the inclusion from $L_1^\perp \oplus \tilde{M}$ into $H_\mathcal{A}$. Then $\ker F = \tilde{N}$ and $ImF = L_1^\perp \oplus \tilde{M}$, so F is $\mathcal{A}$-Fredholm. Now, since $ImF + \ker F$ is not closed, it follows from [42, Corollary 1] that ImF^2 is not closed.

These examples show that semi-$\mathcal{A}-$Fredholm operators may behave differently from classical semi-Fredholm operators on Hilbert spaces. Indeed, classical semi-Fredholm operators always have closed image and are therefore regular operators on Hilbert spaces.

For $F \in B^a(H_\mathcal{A})$ let L_F and R_F denote the left and the right multiplier by F, respectively, i.e. $L_F(D) = FD$ and $R_F(D) = DF$ for all $D \in B^a(H_\mathcal{A})$. By exactly the same arguments as in the proof of [28, Theorem 7] we can prove the following.

Proposition 6.0.16. *Let $F \in B^a(H_\mathcal{A})$. Then the following statements are equivalent.*
1) ImF is closed in $H_\mathcal{A}$.
2) ImL_F is closed in $B^a(H_\mathcal{A})$.
3) ImR_F is closed in $B^a(H_\mathcal{A})$.

Proof. Assume that ImF is closed in $H_\mathcal{A}$. Then by Theorem 2.0.20, ImF is orthogonally complementable. In the same way as in the proof of [28, Theorem 7] part (1) $\Rightarrow$ (3), we may define the map λ from ImF into $H_\mathcal{A}$ and extend it to a map $\lambda' : H_\mathcal{A} \to H_\mathcal{A}$ by letting $\lambda'(y_1 + y_2) = \lambda(y_1)$ for $y_1 \in ImF$ and $y_2 \in ImF^\perp$. By the same arguments as in the proof of [28, Theorem 7] one can show that λ is well defined and $\mathcal{A}$-linear in this case here.

Moreover, since ImF is closed, $\ker F$ is orthogonally complementable in $H_\mathcal{A}$ by Theorem 2.0.20. Hence $F_{|_{\ker F^\perp}}$ is an isomorphism onto ImF. Therefore, there exists a positive constant C such that $\| Fx \| \geq C \| x \|$ for all $x \in \ker F^\perp$. Hence, given $y \in H_\mathcal{A}$, we have

$$C \| P_{\ker F^\perp} y \| \leq \| FP_{\ker F^\perp} y \| = \| Fy \|,$$

where $P_{\ker F^\perp}$ stands for the orthogonal projection onto $\ker F^\perp$. Then, using this fact we are in the position to apply the same arguments as in the proof of [28, Theorem 7].

In order to prove the implication (3) $\Rightarrow$ (2) we can proceed in exactly the same way as in the proof of [28, Theorem 7]. We just need to observe that, since $B^a(H_\mathcal{A})$ is a C^*-algebra, then for any closed subset S of $B^a(H_\mathcal{A})$ we have that S^* is also a closed subset of $B^a(H_\mathcal{A})$.

In order to show (2) $\Rightarrow$ (1), as in the proof of [28, Theorem 7], we choose a sequence $\{x_n\} \subseteq ImF$ such that $x_n \to y_0$ where $y_0 \notin ImF$. For each n we set F_n to be the operator given by $F_n(x) = x_n \cdot \langle e_1, x \rangle$ and we set $D(x) = y_0 \cdot \langle e_1, x \rangle$. Then, $D(e_1) = y_0 \cdot 1_\mathcal{A} = y_0$, so $y_0 \in ImD$. Moreover, by [28, Theorem 7] it follows that $FF_n \to D$ in $B^a(H_\mathcal{A})$. Then we proceed as in the proof of [28, Theorem 7]. □

Recall Definition 3.4.20 of the class $M^a(H_\mathcal{A})$. We have the following lemma as an analogue of [56, Lemma 1.6.5] in the setting of operators on Hilbert C^*-modules.

Lemma 6.0.17. *Let $F \in M^a(H_\mathcal{A})$. If there exists a sequence $\{F_n\} \subseteq \mathcal{M}\Phi(H_\mathcal{A})$ of constant index such that $F_n \to F$ in the operator norm, then $F \subset \mathcal{M}\Phi(H_\mathcal{A})$ and* index $F =$ index F_n *for all n.*

Proof. Since $M^a(H_{\mathcal{A}})$ is open in $B^a(H_{\mathcal{A}})$ in the norm topology, we may without loss of generality assume that $\{F_n\} \subseteq M^a(H_{\mathcal{A}})$, as $F \in M^a(H_{\mathcal{A}})$ and $F_n \to F$. By Theorem 2.0.20, ImF_n is orthogonally complementable in $H_{\mathcal{A}}$ for all n. Since $F_n \in \mathcal{M}\Phi(H_{\mathcal{A}})$ for all n, by Lemma 3.1.21 we must then have that $ImF_n^{\perp}$ is finitely generated for all n. Thus, for each n there exists an orthogonal projection P_n such that $\ker P_n = ImF_n$ and $ImP_n = ImF_n^{\perp}$, which is finitely generated. It follows that $\parallel P_n \parallel = 1$ and $P_n \in \mathcal{K}^*(H_{\mathcal{A}})$ for all n by Corollary 2.0.35. Then we can proceed in exactly the same way as in the proof of [56, Lemma 1.6.5]. □

Proposition 6.0.18. *Let $F \in B(H_{\mathcal{A}})$ be bounded below and suppose that there exists a sequence $\{F_n\} \subseteq \widehat{\mathcal{M}\Phi}(H_{\mathcal{A}})$ of constant index and such that $F_n \to F$. Suppose also that for each n there exists an $\widehat{\mathcal{M}\Phi}$-decomposition for F_n*

$$H_{\mathcal{A}} = M_1^{(n)} \tilde{\oplus} N_1^{(n)} \xrightarrow{F_n} M_2^{(n)} \tilde{\oplus} N_2^{(n)} = H_{\mathcal{A}}$$

such that the sequence of projections $\{\sqcap_n\}$ is uniformly bounded, where $\sqcap_n$ denotes the projection onto $N_2^{(n)}$ along $M_2^{(n)}$ for each n. Then $F \in \widehat{\mathcal{M}\Phi}(H_{\mathcal{A}})$ and index F_n = index F *for all n.*

Proof. As in the proof of Lemma 6.0.17, we may without loss of generality assume that F_n is bounded below for all n. It follows that $F_n(N_1^{(n)})$, which is a submodule of $N_2^{(n)}$, is closed. Hence ${F_n}_{|_{N_1^{(n)}}}$ is a closed range operator from $N_1^{(n)}$ into $N_2^{(n)}$. Since $N_1^{(n)}$ and $N_2^{(n)}$ are finitely generated, they are self-dual Hilbert $\mathcal{A}$-modules, hence, by Proposition 2.0.28 and Theorem 2.0.20, $F_n(N_1^{(n)})$ is orthogonally complementable in $N_2^{(n)}$. Then $H_{\mathcal{A}} = M_2^{(n)} \tilde{\oplus} (F_n(N_1^{(n)}) \oplus \tilde{N}^{(n)})$, where $\tilde{N}^{(n)}$ stands for the orthogonal complement of $F_n(N_1^{(n)})$ in $N_2^{(n)}$. Let Q_n be the projection onto $\tilde{N^{(n)}}$ along $M_2^{(n)} \tilde{\oplus} F_n(N_1^{(n)})$. Then, $Q_n = P_n \sqcap_n$, where P_n stands for the orthogonal projection of $N_2^{(n)}$ onto $\tilde{N}^{(n)}$. Since $\{\sqcap_n\}$ is uniformly bounded and $\parallel P_n \parallel = 1$ for all n, it follows that $\{Q_n\}$ is uniformly bounded. Hence we may proceed in the same way as in the proof of [56, Lemma 1.6.5]. □

Lemma 6.0.19. *Let X, Y be Banach spaces and $F \in M(X,Y)$. Suppose that there exists a sequence $\{F_n\}$ of regular operators in $B(X,Y)$ such that $F_n \to F$. Moreover, assume that there exists a sequence of projections $\{\sqcap_n\}$ in $B(Y)$ which is uniformly bounded in the norm and such that $Im(I - \sqcap_n) = ImF_n$ for all n. Then F is a regular operator, i.e. ImF is complementable in Y.*

Proof. We may proceed in exactly the same way as in the proof of [56, Lemma 1.6.5] in order to deduce that $DF = I_X$ for an operator $D \in B(Y,X)$. This is because the sequence $\{\sqcap_n\}$ is uniformly bounded in the norm by assumption, so the arguments from the proof of [56, Lemma 1.6.5] applies. Further, the operator FD is then a projection onto $ImF \subseteq Y$ because $FDFD = FI_XD = FD$ and $ImFD = F(D(Y)) = F(X) = ImF$ since $D(Y) = X$ because $DF = I_X$. Therefore, ImF is complementable in Y. □

Remark 6.0.20. Lemma 6.0.17 is valid in the case of arbitrary Hilbert C^*-modules and not just $H_{\mathcal{A}}$. Indeed, we recall from Proposition 3.5.11 that the index of closed range $\mathcal{A}$-Fredholm operator is well-defined on arbitrary Hilbert $\mathcal{A}$-modules. Let us consider now an arbitrary Hilbert C^*-module N, suppose that $F \in M^a(N)$ and that $\{F_n\} \subseteq \mathcal{M}\Phi(N)$ satisfies the assumption of Lemma 6.0.17. Then, as explained in the proof of Lemma 6.0.17, for each n we can consider the orthogonal projection onto $ImF_n^{\perp}$ and proceed in the same way as in the proof of [56, Lemma 1.6.5]. Hence we obtain that G_nF is invertible for large enough n, where G_n is generalized inverse of F_n that satisfies $\ker G_n = ImF_n^{\perp}$. If we set $G := (G_nF)^{-1}G_n$, then G is surjective since G_n is so. Moreover, $\ker G = \ker G_n = ImF_n^{\perp}$ and $GF = I$. However, $ImF_n^{\perp} = -$index F_n since F_n is bounded below, which follows from Proposition 3.5.11. In particular, $ImF_n^{\perp}$ is finitely generated for all $n \in \mathbb{N}$ because $F_n \in \mathcal{M}\Phi(N)$ for all $n \in \mathbb{N}$. Since $GF = I$, from Lemma 3.5.6 it follows that $ImF \tilde{\oplus} \ker G = N$, hence $F \in \mathcal{M}\Phi(N)$ and index $F = -[\ker G]$.

Chapter 7

Generalized spectra of operators over C^*-algebras

We recall the definition of the operator αI on $H_{\mathcal{A}}$ from Section 3.4.
Our starting question is the following: If $\mathcal{A}$ is a C^*-algebra, then for $\alpha \in \mathcal{A}$ could we consider the generalized spectra in $\mathcal{A}$ of operators in $B^a(H_{\mathcal{A}})$ by setting for every $F \in B^a(H_{\mathcal{A}})$

$$\sigma^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid F - \alpha I \text{ is not invertible in } B^a(H_{\mathcal{A}})\} \ ?$$

The main topic from now on and in the rest of the thesis will be to obtain generalization of some results from spectral theory of operators on Hilbert spaces in the setting of generalized spectra in C^*-algebras of operators on Hilbert C^*-modules.
We introduce first the following notion:
$\sigma^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid F - \alpha I \text{ is not invertible in } B^a(H_{\mathcal{A}})\}$;
$\sigma_p^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid \ker(F - \alpha I) \neq \{0\}\}$;
$\sigma_{rl}^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid F - \alpha I \text{ is bounded below, but not surjective on } H_{\mathcal{A}}\}$;
$\sigma_{cl}^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid Im(F - \alpha I) \text{ is } \underline{\text{not}} \text{ closed }\}$.
It is understood that $F \in B^a(H_{\mathcal{A}})$.

Recall that not all closed submodules of $H_{\mathcal{A}}$ are orthogonally complementable in $H_{\mathcal{A}}$, which differs from the situation of Hilbert spaces. It may happen that $\overline{Im(F-\alpha I)} \oplus Im(F-\alpha I)^{\perp} \subsetneqq H_{\mathcal{A}}$. However, if $Im(F - \alpha I)$ is closed, then $Im(F^* - \alpha^* I)$ is closed and we also have

$$H_{\mathcal{A}} = Im(F - \alpha I) \oplus \ker(F^* - \alpha^* I) = \ker(F - \alpha I) \oplus Im(F^* - \alpha^* I)$$

whenever $F \in B^a(H_{\mathcal{A}})$, which follows from the proof of Theorem 2.0.20.

Therefore, it is more convinient in this setting to work with $\sigma_{rl}^{\mathcal{A}}(F)$ and $\sigma_{cl}^{\mathcal{A}}(F)$ for $F \in B^a(H_{\mathcal{A}})$ instead of the residual and the continuous spectrum.

Note that we obviously have

$$\sigma^{\mathcal{A}}(F) = \sigma_p^{\mathcal{A}}(F) \cup \sigma_{rl}^{\mathcal{A}}(F) \cup \sigma_{cl}^{\mathcal{A}}(F) \text{ and } \sigma^{\mathcal{A}}(F^*) = (\sigma^{\mathcal{A}}(F))^*.$$

The challenges which arise are the following:

1) $\mathcal{A}$ may be non commutative;
2) If $\mathcal{A}$ is a non trivial C^*-algebra, then there exists certainly nonzero non-invertible elements by the Gelfand-Mazur Theorem [25, Chapter VII, Theorem 8.1]. Moreover, even if $\alpha \in \mathcal{A} \cap G(\mathcal{A})$, we do not have in general that $\| \alpha^{-1} \| = \dfrac{1}{\| \alpha \|}$. Therefore, $\sigma^{\mathcal{A}}(F)$ may be unbounded. (However, $\sigma^{\mathcal{A}}(F)$ is always closed in $\mathcal{A}$).

7.1 Generalized spectra of shift operators, unitary, self-adjoint and normal operators

In this section we shall give description of the generalized spectra of shift operators, unitary, self-adjoint and normal operators on $H_{\mathcal{A}}$ and investigate some further properties of these spectra. Most of the results in this section are generalizations of the results from [50, Chapter 4] .
We start with the following proposition.

Proposition 7.1.1. *Let $\mathcal{A}$ be a unital C^*-algebra, $\{e_k\}_{k\in\mathbb{N}}$ denote the standard orthonormal basis of $H_{\mathcal{A}}$ and S be the operator defined by $Se_k = e_{k+1}, k \in \mathbb{N}$, that is S is a unilateral shift and $S^*e_{k+1} = e_k$ for all $k \in \mathbb{N}$. If $\mathcal{A} = L^\infty((0,1),\mu)$ where μ is the Lebesgue measure, or if $\mathcal{A} = C([0,1])$, then*

$$\sigma^{\mathcal{A}}(S) = \{\alpha \in \mathcal{A} \mid \inf|\alpha| \leq 1\},$$

where in the case when $\mathcal{A} = L^\infty((0,1),\mu)$, we set

$$\inf|\alpha| = \inf\{C > 0 \mid \mu(|\alpha|^{-1}([0,C])) > 0\} = \sup\{K > 0 \mid |\alpha| > K \text{ a.e. on } [0,1]\}.$$

Moreover, $\sigma_p^{\mathcal{A}}(S) = \varnothing$ in both cases.

Proof. We have two cases.
Case 1: In this case we consider $\mathcal{A} = C([0,1])$. Let $\alpha \in \mathcal{A}$ and suppose that $\inf|\alpha| < 1$. Since $|\alpha|$ is continuous, we may find an open interval $(t_1,t_2) \subseteq (0,1)$ such that $|\alpha(t)| < 1-\epsilon$ for all $t \in (t_1,t_2)$, where $0 < \epsilon < 1 - \inf|\alpha|$. We may find some $g \in \mathcal{A}$ such that $\operatorname{supp} g \subseteq (t_1,t_2)$ and $0 \leq g \leq 1$. Consider

$$x_\alpha = (g, \overline{\alpha}g, \overline{\alpha}^2 g, \cdots).$$

Then, obviously, $x_\alpha \in H_{\mathcal{A}}$ and $\langle(\alpha I - S)e_k, x_\alpha\rangle = \overline{\alpha}^k g - \overline{\alpha}^k g = 0$. Hence $x_\alpha \in Im(\alpha I - S)^\perp$ and $x_\alpha \neq 0$, which gives that $\alpha \in \sigma^{\mathcal{A}}(S)$. Therefore, $\{\alpha \in \mathcal{A} \mid \inf|\alpha| < 1\} \subseteq \sigma^{\mathcal{A}}(S)$.
Since $\sigma^{\mathcal{A}}(S)$ is closed in the norm topolgy in $\mathcal{A}$, it follows that

$$\{\alpha \in \mathcal{A} \mid \inf|\alpha| \leq 1\} \subseteq \sigma^{\mathcal{A}}(S).$$

On the other hand, if $\alpha \in \mathcal{A}$ and $\inf|\alpha| > 1$, then α is invertible and $\sup|\alpha^{-1}| = \parallel \alpha^{-1} \parallel < 1$. It follows that $\parallel \alpha^{-1}S \parallel \leq \parallel \alpha^{-1} \parallel \parallel S \parallel < 1$, so $\alpha I - S = \alpha(I - \alpha^{-1}S)$ is invertible in $B^a(H_{\mathcal{A}})$.

Next, suppose that $(\alpha I - S)(x) = 0$ for some $\alpha \in \mathcal{A}$ and $x \in H_{\mathcal{A}}$. This gives the following system of equations coordinatewise: $\alpha x_1 = 0, \alpha x_2 - x_1 = 0, \alpha x_3 - x_2 = 0, \cdots$. Since $\alpha x_1 = 0$, we deduce that $x_{1|\operatorname{supp}\alpha} = 0$. However, since $\alpha x_2 - x_1 = 0$, it follows that $x_{1|(\operatorname{supp}\alpha)^c} = 0$ also. Hence $x_1 = 0$. However, then $\alpha x_2 = 0$ and $\alpha x_3 - x_2 = 0$. Using the same argument we obtain that $x_2 = 0$. Proceeding inductively, we obtain that $x_k = 0$ for all k, so $x = 0$. Since $\alpha \in \mathcal{A}$ was arbitrary chosen, we conclude that $\sigma_p^{\mathcal{A}}(S) = \varnothing$.

Case 2: In this case we consider $\mathcal{A} = L^\infty((0,1),\mu)$. Let $\alpha \in \mathcal{A}$ and assume that $\inf|\alpha| < 1$. This means that $\mu(|\alpha|^{-1}([0,1-\epsilon])) > 0$, where $0 < \epsilon < 1 - \inf|\alpha|$. Set $M_\epsilon = |\alpha|^{-1}([0,1-\epsilon])$, then $\chi_{M_\epsilon} \neq 0$. Letting χ_{M_ϵ} play the role of the function g in the previous proof, (which is possible since $x_\alpha = (\chi_{M_\epsilon}, \overline{\alpha}\chi_{M_\epsilon}, \overline{\alpha}^2\chi_{M_\epsilon}, \dots) \in H_{\mathcal{A}}$ because $|\alpha| \leq 1-\epsilon$ on M_ϵ), we deduce by the same arguments that

$$\sigma^{\mathcal{A}}(S) = \{\alpha \in \mathcal{A} \mid \inf|\alpha| \leq 1\}.$$

Next, assume that $(\alpha I - S)(x) = 0$ for some $\alpha \in \mathcal{A}$ and $x \in H_{\mathcal{A}}$. As in the previous proof we get the system of equations $\alpha x_1 = 0, \alpha x_2 - x_1 = 0, \alpha x_3 - x_2 = 0, \cdots$. The first equation gives that $x_1 = 0$ a.e. on $|\alpha|^{-1}(0,\infty)$, whereas the second equation gives $x_1 = 0$ a.e. on $\alpha^{-1}(\{0\})$. Hence $x_1 = 0$. Proceeding inductively as in the previous proof, we get $x = 0$, hence $\sigma_p^{\mathcal{A}}(S)$ is empty also in this case. $\square$

Lemma 7.1.2. *Let $\mathcal{A} = B(H)$, $T \in B(H)$ and suppose that T is invertible. Then the equation $(T \cdot I - S)x = y$ has a solution in $H_{\mathcal{A}}$ for all $e_k, k \in \mathbb{N}$, if and only if the sequence $(T^{-1}, T^{-2}, \cdots, T^{-k}, \cdots)$ belongs to $H_{\mathcal{A}}$.*

Proof. For $k = 1$, if $(T \cdot I - S)x = e_1$, then we must have $TB_1 = I$, where $x = (B_1, B_2, \cdots)$. Hence $B_1 = T^{-1}$. Next, $TB_2 - B_1 = 0$, so $TB_2 = B_1 = T^{-1}$ which gives $B_2 = T^{-2}$. Proceeding inductively, we obtain that $B_k = T^{-k}$ for all k. So the equation $(T \cdot I - S)x = e_1$ has a solution in $H_{\mathcal{A}}$ if and only if the sequence $(T^{-1}, T^{-2}, \cdots)$ belongs to $H_{\mathcal{A}}$.

Now, if $(T^{-1}, T^{-2}, \cdots) \in H_{\mathcal{A}}$, then the sequence $x^{(k)}$ in $H_{\mathcal{A}}$ given by
$$x_n^{(k)} = \begin{cases} 0 & \text{if } n \in \{1, \cdots, k-1\} \\ T^{-(n-k+1)} & \text{for } n \in \{k, k+1, \cdots\} \end{cases}$$
is the solution of the equation $(T \cdot I - S)x = e_k$ for each $k \in \mathbb{N}$. □

Set $\tilde{\sigma}_{cl}^{\mathcal{A}}(S) = \{\alpha \in \sigma_{cl}^{\mathcal{A}}(S) \mid \overline{Im(\alpha I - S)} = H_{\mathcal{A}}\}$. We have the following corollary.

Corollary 7.1.3. *Let $\mathcal{A}$ be a commutative unital C^*-algebra. Then*
$$\sigma^{\mathcal{A}}(S) = (\mathcal{A} \setminus G(\mathcal{A})) \cup \{\alpha \in G(\mathcal{A}) | (\alpha^{-1}, \alpha^{-2}, \cdots, \alpha^{-k}, \cdots) \notin H_{\mathcal{A}}\} \cup \tilde{\sigma}_{cl}^{\mathcal{A}}(S).$$

Proof. Since $\mathcal{A}$ is commutative, then the set of right invertible elemnts coincides with $G(\mathcal{A})$. Hence we can apply the arguments from the proof of Lemma 7.1.2. □

Corollary 7.1.4. *Let $\mathcal{A}$ be a unital C^*-algebra. If $1_{\mathcal{A}}$ denotes the unit in $\mathcal{A}$, then $1_{\mathcal{A}} \in \sigma^{\mathcal{A}}(S)$.*

Proof. We obviously have that the sequence $(1_{\mathcal{A}}, 1_{\mathcal{A}}, 1_{\mathcal{A}}, \cdots) = (1_{\mathcal{A}}^{-1}, 1_{\mathcal{A}}^{-2}, 1_{\mathcal{A}}^{-3}, \cdots)$ is not an element of $H_{\mathcal{A}}$. Then we apply the arguments from the proof of Lemma 7.1.2. □

Example 7.1.5. We may also consider a weighted shift S_w on $H_{\mathcal{A}}$ given by $S_w(x)_{j+1} = w_j x_j$, where $w = (w_1, w_2, \cdots)$ is a bounded sequence in $\mathcal{A}$. In this case, if α has a common right annihilator as w_j for some $j \in \mathbb{N}$, then the sequence having this right annihilator in its j-th coordinate and 0 elsewhere belongs to the kernel of $\alpha I - S_w$. Hence $\alpha \in \sigma^{\mathcal{A}}(S_w)$ in this case.

Example 7.1.6. Let $\mathcal{A} = L^\infty((0,1), \mu)$. Set
$$\tilde{S}(f_1, f_2, \cdots) = (f_1 \chi_{(0,\frac{1}{2})}, f_2 \chi_{(0,\frac{1}{2})} + f_1 \chi_{(\frac{1}{2},1)}, f_3 \chi_{(0,\frac{1}{2})} + f_2 \chi_{(\frac{1}{2},1)}, \cdots).$$
Then $\tilde{S}$ has the matrix $\begin{bmatrix} 1 & 0 \\ 0 & S \end{bmatrix}$ with respect to the decomposition $(H_{\mathcal{A}} \cdot \chi_{(0,\frac{1}{2})}) \oplus (H_{\mathcal{A}} \cdot \chi_{(\frac{1}{2},1)})$. It follows that
$$\sigma^{\mathcal{A}}(\tilde{S}) = \{\alpha \in \mathcal{A} \mid \inf\{C > 0 \mid \mu(|\alpha|^{-1}([0, C]) \cap (\frac{1}{2}, 1))\} \leq 1\}$$
$$\cup \{\alpha \in \mathcal{A} \mid (\alpha - 1) \cdot \chi_{(0,\frac{1}{2})} \text{ is not invertible in } L^\infty((0, \frac{1}{2}), \mu)\}.$$

Proposition 7.1.7. *Let $\alpha \in \mathcal{A}$. We have*
1. If $\alpha I - F$ is bounded below and $F \in B^a(H_{\mathcal{A}})$, then $\alpha \in \sigma_{rl}^{\mathcal{A}}(F)$ if and only if $\alpha^* \in \sigma_p^{\mathcal{A}}(F^*)$;
2. If $F, D \in B^a(H_{\mathcal{A}})$ and $D = U^*FU$ for some unitary operator U, then
$$\sigma^{\mathcal{A}}(F) = \sigma^{\mathcal{A}}(D), \sigma_p^{\mathcal{A}}(F) = \sigma_p^{\mathcal{A}}(D), \sigma_{cl}^{\mathcal{A}}(F) = \sigma_{cl}^{\mathcal{A}}(D) \text{ and } \sigma_{rl}^{\mathcal{A}}(F) = \sigma_{rl}^{\mathcal{A}}(D).$$

Proof. 1) Suppose first that $F - \alpha I$ is bounded below and $\alpha \in \sigma_{rl}^{\mathcal{A}}(F)$. Then $Im(F - \alpha I)$ is closed. Hence, by Theorem 2.0.20 we have that $H_{\mathcal{A}} = Im(F - \alpha I) \oplus Im(F - \alpha I)^\perp$ which gives that $Im(F - \alpha I)^\perp \neq \{0\}$ as $Im(F - \alpha I) \neq H_{\mathcal{A}}$. Since $Im(F - \alpha I)^\perp = \ker(F^* - \alpha^* I)$, it follows that $\alpha^* \in \sigma_p^{\mathcal{A}}(F^*)$.

Conversely, suppose that $\alpha^* \in \sigma_p^{\mathcal{A}}(F^*)$ and that $F - \alpha I$ is bounded below. Then, again, $Im(F - \alpha I)$ is closed and moreover, $Im(F - \alpha I)^\perp = \ker(F^* - \alpha^* I) \neq \{0\}$. It follows that $\alpha \in \sigma_{rl}^{\mathcal{A}}(F)$.

It is straightforward to prove the statement 2. □

Now we are going to describe the generalized spectrum of a unitary operator on $H_{\mathcal{A}}$.

Proposition 7.1.8. *Let $U \in B^a(H_{\mathcal{A}})$ be unitary. Then*

$$\sigma^{\mathcal{A}}(U) \subseteq \{\alpha \in \mathcal{A} \mid \ \parallel \alpha \parallel \geq 1\},$$

$$\sigma^{\mathcal{A}}(U) \cap G(\mathcal{A}) \subseteq \{\alpha \in G(\mathcal{A}) \mid \ \parallel \alpha^{-1} \parallel, \parallel \alpha \parallel \geq 1\}.$$

Proof. We have $\alpha I - U = ((\alpha I)U^* - I)U$ and $\parallel U^* \parallel = \parallel U \parallel = 1$. □

Consider again the orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$ for $H_{\mathcal{A}}$. We may enumerate this basis by indexes in $\mathbb{Z}$. Then we get orthonormal basis $\{e_j\}_{j\in\mathbb{Z}}$ for $H_{\mathcal{A}}$ and we can consider a bilateral shift operator V with respect to this basis, i.e. $Ve_k = e_{k+1}$ all $k \in \mathbb{Z}$, which gives $V^*e_k = e_{k-1}$ for all $k \in \mathbb{Z}$.

Proposition 7.1.9. *Let V be the bilateral shift operator on $H_{\mathcal{A}}$. Then the following holds:*
1) If $\mathcal{A} = C([0,1])$, then $\sigma^{\mathcal{A}}(V) = \{f \in \mathcal{A} \mid |f|([0,1]) \cap \{1\} \neq \varnothing\}$,
2) If $\mathcal{A} = L^\infty((0,1),\mu)$, then

$$\sigma^{\mathcal{A}}(V) = \{f \in \mathcal{A} \mid \mu(|f|^{-1}((1-\epsilon, 1+\epsilon))) > 0 \ \forall \epsilon > 0\}.$$

In both cases $\sigma_p^{\mathcal{A}}(V) = \varnothing$.

Proof. Case 1:

In this case we consider $\mathcal{A} = C([0,1])$. Suppose that $\alpha \in \mathcal{A}$ and $|\alpha(\tilde{t})| = 1$ for some $\tilde{t} \in [0,1]$. Choose a function $y \in \mathcal{A}$ such that $y(\tilde{t}) = 1$. If $\alpha I - V$ is surjective, then there exists an $x \in H_{\mathcal{A}}$ such that $(\alpha I - V)x = e_1 \cdot y$. Now, $x(\tilde{t}) \in l_2$ since $x \in H_{\mathcal{A}}$. If we let $\tilde{V}$ denote the ordinary bilateral shift on l_2, we get that $\alpha(\tilde{t})x(\tilde{t}) - \tilde{V}(x(\tilde{t})) = (1,0,0,\cdots)$, since $y(\tilde{t}) = 1$. However, this is not possible since $|\alpha(\tilde{t})| = 1$ (for more details, see [50, Chapter 4, Proposition 19]). We conclude that $\alpha I - V$ can not be surjective, so $\alpha \in \sigma^{\mathcal{A}}(V)$.

On the other hand, if $\alpha \in \mathcal{A}$ and $|\alpha|([0,1]) \cap \{1\} = \varnothing$, then either $|\alpha(t)| \geq C > 1$ or $|\alpha(t)| \leq K < 1$ for all $t \in [0,1]$ and some constants C or K (here we use that $|\alpha|$ is continuous). If $|\alpha(t)| \geq C > 1$ for all $t \in [0,1]$, then α is invertible in $\mathcal{A}$ and $\parallel \alpha^{-1} \parallel \leq \dfrac{1}{C} < 1$. Since $\parallel V \parallel = 1$, it follows that $\alpha \notin \sigma^{\mathcal{A}}(V)$. If $|\alpha(t)| \leq K < 1$ for all $t \in [0,1]$, then $\parallel \alpha \parallel \leq K < 1$, so, by Proposition 7.1.8 it follows then that $\alpha \notin \sigma^{\mathcal{A}}(V)$. Hence

$$\sigma^{\mathcal{A}}(V) = \{\alpha \in \mathcal{A} \mid |\alpha|([0,1]) \cap \{1\} \neq \varnothing\}.$$

Next, if $(\alpha I - V)x = 0$ for some $x \in H_{\mathcal{A}}$, then we must have $\alpha(t)x(t) - \tilde{V}x(t) = 0$ for all $t \in [0,1]$. This means that $x(t) = 0$ for all $t \in [0,1]$ since $\sigma_p(\tilde{V}) = \emptyset$ by [50, Chapter 4, Proposition 19] .

Case 2:

Let now $\mathcal{A} = L^\infty((0,1),\mu)$ and $\alpha \in \mathcal{A}$ be such that $\mu(|\alpha|^{-1}((1-\epsilon,1+\epsilon))) > 0$ for all $\epsilon > 0$. If $(\alpha I - V)x = e_0$ for some $x \in H_{\mathcal{A}}$, then we must have $\alpha x_k - x_{k-1} = 0$ for all $k \neq 0$ and $\alpha x_0 - x_{-1} = 1_{\mathcal{A}}$. For small $\epsilon > 0$ set $M_\epsilon = |\alpha|^{-1}((1-\epsilon,1+\epsilon))$, $M_\epsilon^- = |\alpha|^{-1}((1-\epsilon,1))$ and $M_\epsilon^+ = |\alpha|^{-1}((1,1+\epsilon))$, so $M_\epsilon = M_\epsilon^- \cup M_\epsilon^+$ and $\chi_{M_\epsilon} \neq 0$. From the first equation above we get $x_k = \alpha^{-(k+1)}x_{-1}$ for all $k \leq -1$. Moreover, $x_k = \alpha^{-k}x_0$ for all $k \geq 0$ a.e. on any subset of $(0,1)$ on which $|\alpha|$ is bounded below, thus in particular on M_ϵ. Hence $x_k\chi_{M_\epsilon} = x_0\alpha^{-k}\chi_{M_\epsilon}$ for all $k \geq 0$ where for all k we let $\alpha^{-k}\chi_{M_\epsilon}$ denote the function given by

$$\alpha^{-k}\chi_{M_\epsilon}(t) = \begin{cases} \alpha^{-k}(t) & \text{for } t \in M_\epsilon, \\ 0 & \text{else.} \end{cases}$$

Since $x \in H_{\mathcal{A}}$, it follows that $x_k \chi_{M_\epsilon^+} = 0$ for all $k \leq -1$ and $x_k \chi_{M_\epsilon^-} = 0$ for all $k \geq 0$. Setting this into the second equation above, we get $\alpha x_0 \chi_{M_\epsilon^+} - x_{-1}\chi_{M_\epsilon^-} = \chi_{M_\epsilon}$, which gives $x_0 \chi_{M_\epsilon} = \alpha^{-1}\chi_{M_\epsilon^+}$ and $x_{-1}\chi_{M_\epsilon} = -\chi_{M_\epsilon^-}$. Hence $x_k \chi_{M_\epsilon} = \alpha^{-(k+1)}\chi_{M_\epsilon^+}$ for all $k \geq 0$ and $x_k\chi_{M_\epsilon} = -\alpha^{-(k+1)}\chi_{M_\epsilon^-}$ for all $k \leq -1$. This gives $|x_k| \geq (1+\epsilon)^{-(k+1)}\chi_{M_\epsilon^+}$ for all $k \geq 0$ and $|x_k| \geq (1-\epsilon)^{-(k+1)}\chi_{M_\epsilon^-}$ for all $k \leq -1$. Since this holds for all $\epsilon > 0$ and moreover, we have that either $\chi_{M_\epsilon^-}$ or $\chi_{M_\epsilon^+}$is non-zero (because χ_{M_ϵ} is non-zero for all $\epsilon > 0$), we get that the infinite sum $\sum_{k\in\mathbb{Z}} x_k^* x_k$ diverge in $\mathcal{A}$, otherwise $\| \sum_{k\in\mathbb{Z}} x_k^* x_k \| \geq \min \left\{ \sum_{k=0}^{\infty} \frac{1}{(1+\epsilon)^{k+1}}, \sum_{k=0}^{\infty}(1-\epsilon)^k \right\}$ for all $\epsilon > 0$, a contradiction. Hence x can not be an element of $H_{\mathcal{A}}$. We conclude that $e_0 \notin Im(\alpha I - V)$, so $\alpha \in \sigma^{\mathcal{A}}(V)$ in this case.

On the other hand, if $\mu(|\alpha|^{-1}((1-\epsilon, 1+\epsilon))) = 0$ for $\alpha \in \mathcal{A}$ and some $\epsilon > 0$, then we have $(0,1) = N_\epsilon^- \cup N_\epsilon^+$, where $N_\epsilon^- = |\alpha|^{-1}((0, 1-\epsilon))$ and $N_\epsilon^+ = |\alpha|^{-1}((1+\epsilon, +\infty))$. Since the decomposition $H_{\mathcal{A}} = H_{\mathcal{A}} \cdot \chi_{N_\epsilon^+} \oplus H_{\mathcal{A}} \cdot \chi_{N_\epsilon^-}$ clearly reduces the operator $\alpha I - V$ and the restrictions of $\alpha I - V$ on both these submodules are invertible, (as the restriction of V to both these submodules acts as a unitary operator on these submodules), it follows that $\alpha I - V$ is invertible, so $\alpha \notin \sigma^{\mathcal{A}}(V)$. □

Example 7.1.10. Let $\{\alpha_1, \alpha_2, \cdots\}$ be a sequence in a unital C^*-algebra $\mathcal{A}$ such that each α_k is a unitary element of $\mathcal{A}$. Then the operator V defined by

$$V(x_1, x_2, \cdots) = (\alpha_1 x_1, \alpha_2 x_2, \cdots)$$

is a unitary operator on $H_{\mathcal{A}}$. If $\mathcal{A} = C([0,1])$ or if $\mathcal{A} = L^\infty((0,1), \mu)$ and J_1, J_2 are two closed subintervals of $(0,1)$ such that $J_1 \cap J_2 = \varnothing$, then we may easily find a function $\beta \in \mathcal{A}$ such that $\beta = \alpha_1$ on J_1 and $|\beta(t)| > 1$ for all $t \in J_2$. Hence $\| \beta \| > 1$, but we also have $\beta \in \sigma^{\mathcal{A}}(V)$ since $\ker(\beta I - V) \neq \{0\}$. Similarly, if $\mathcal{A} = B(H)$ where H is an infinite-dimensional Hilbert space, then we may easily find two closed suspaces H_1 and H_2 such that $H_1 \perp H_2$ and $T \in B(H)$ satisfying $T_{|_{H_1}} = \alpha_{1|_{H_1}}$ and $\| T_{|_{H_2}} \| > 1$. Hence, again $T \in \sigma^{\mathcal{A}}(V)$ and $\| T \| > 1$. So, if V is a unitary operator on $H_{\mathcal{A}}$, we do <u>not</u> have in general that

$$\sigma^{\mathcal{A}}(V) \subseteq \{\alpha \in \mathcal{A} \mid \ \| \alpha \| = 1\}.$$

Next we are going to describe and investigate some properties of generalized spectra of self-adjoint operators on $H_{\mathcal{A}}$.

Lemma 7.1.11. *Let $\mathcal{A}$ be a commutative C^*-algebra. If F is a self-adjoint operator on $H_{\mathcal{A}}$, then $\sigma_p^{\mathcal{A}}(F)$ is a self-adjoint subset of $\mathcal{A}$, that is $\alpha \in \sigma_p^{\mathcal{A}}(F)$ if and only if $\alpha^* \in \sigma_p^{\mathcal{A}}(F)$.*

Proof. Since $F - \alpha I$ and $F - \alpha^* I = F^* - \alpha^* I$ mutually commute because $\mathcal{A}$ is commutative, we can deduce that $\| (F - \alpha I)x \| = \| (F - \alpha^* I)x \|$ for all $x \in H_{\mathcal{A}}$. □

Example 7.1.12. Let $\mathcal{A} = C([0,1])$ or $\mathcal{A} = L^\infty((0,1), \mu)$. If G is the operator on $H_{\mathcal{A}}$ given by $G(f_1, f_2, \cdots) = (g_1 f_1, g_2 f_2, \cdots)$, where $\{g_1, g_2, \cdots\}$ is a bounded sequence of real valued functions in $\mathcal{A}$, then G is a self-adjoint operator. Suppose that there are two mutually disjoint, closed subintervals J_1 and J_2 of $(0,1)$ such that $g_{1|_{J_1}} \neq 0$ and $g_{1|_{J_2}} = 0$. Set $\tilde{g} = i g_1$. Then, if we choose a function f in $\mathcal{A}$ such that supp $f \subseteq J_2$, we get that $(\tilde{g} I - G)(f, 0, 0, \cdots) = 0$. However, $\tilde{g} \neq \overline{\tilde{g}}$, so we do <u>not</u> have that $\sigma_p^{\mathcal{A}}(G)$ is included in the set of self-adjoint elements of $\mathcal{A}$.

Example 7.1.13. Let $\mathcal{A} = B(H)$ where H is a separable infinite-dimensional Hilbert space and let $\{e_j\}_{j\in\mathbb{N}}$ be an orthonormal basis for H. If P denotes the orthogonal projection onto $Span\{e_1\}$, then the operator $P \cdot I$ is a self-adjoint operator on $H_{\mathcal{A}}$. Now, if S is the unilateral shift operator on H with respect to the orthonormal basis $\{e_j\}$, then $S - P$ is injective whereas

$S^* - P$ is not injective because $(S^* - P)(e_1 + e_2) = 0$. It follows that $(S - P) \cdot I$ is an injective operator on $H_{\mathcal{A}}$, whereas $(S^* - P) \cdot I = ((S - P) \cdot I)^*$ is **not** an injective operator on $H_{\mathcal{A}}$, since $(S^* - P) \cdot I(Q, 0, 0, 0, \cdots) = 0$, where Q is the orthogonal projection onto $Span\{e_1 + e_2\}$.

Hence, if $\mathcal{A} = B(H)$, we do not have in general that $\sigma_p^{\mathcal{A}}(F)$ is a self-adjoint subset of $\mathcal{A}$ when $F = F^*$. It follows that the assumption that $\mathcal{A}$ is commutative is indeed necessary in Lemma 7.1.11.

Lemma 7.1.14. *Let $\mathcal{A}$ be a commutative C^*-algebra. If F is a self-adjoint operator on $H_{\mathcal{A}}$ and $\alpha \in \mathcal{A} \setminus \sigma_p^{\mathcal{A}}(F)$, then $Im(F - \alpha I)^{\perp} = \{0\}$. Hence, if $\alpha \in \mathcal{A}$ and $F - \alpha I$ is bounded below, then $\alpha \in \mathcal{A} \setminus \sigma^{\mathcal{A}}(F)$.*

Proof. Suppose that $\alpha \in \mathcal{A} \setminus \sigma_p^{\mathcal{A}}(F)$. If $y \in Im(F - \alpha I)^{\perp}$, then $y \in \ker(F^* - \alpha^* I)$. By the proof of Lemma 7.1.11 we obtain that $(F - \alpha I)y = 0$. Since $\alpha \notin \sigma_p^{\mathcal{A}}(F)$ by the choice of α, we get that $y = 0$. Thus, $Im(F - \alpha I)^{\perp} = \{0\}$, when $\alpha \in \mathcal{A} \setminus \sigma_p^{\mathcal{A}}(F)$.

Suppose next that $\alpha \in \mathcal{A}$ is such that $F - \alpha I$ is bounded below. Then $\alpha \in \mathcal{A} \setminus \sigma_p^{\mathcal{A}}(F)$, so from the previous arguments we deduce that $Im(F - \alpha I)^{\perp} = \{0\}$. Moreover, since $Im(F - \alpha I)$ is then closed and $F - \alpha I \in B^a(H_{\mathcal{A}})$, from Theorem 2.0.20 it follows that $Im(F - \alpha I)$ is orthogonally complementable in $H_{\mathcal{A}}$. However, since $Im(F - \alpha I)^{\perp} = \{0\}$, we must have that $Im(F - \alpha I) = H_{\mathcal{A}}$. Hence $F - \alpha I$ is invertible in $B^a(H_{\mathcal{A}})$, so α is in $\mathcal{A} \setminus \sigma^{\mathcal{A}}(F)$. □

Corollary 7.1.15. *Let $\mathcal{A}$ be a unital commutative C^*-algebra and F be a self-adjoint operator on $H_{\mathcal{A}}$. If $\alpha \in \mathcal{A}$ and $\alpha - \alpha^* \in G(\mathcal{A})$, then $F - \alpha I$ is invertible. In this case,*

$$\parallel (F - \alpha I)^{-1} \parallel \leq 2 \parallel (\alpha - \alpha^*)^{-1} \parallel .$$

Proof. If $\alpha \in \mathcal{A}$, then, since $\mathcal{A}$ is commutative, we get

$$\langle x, Fx - \alpha Ix \rangle - \langle Fx - \alpha Ix, x \rangle = \alpha^* \langle x, x \rangle - \langle x, x \rangle \alpha = (\alpha^* - \alpha) \langle x, x \rangle.$$

From the triangle inequality and the Cauchy-Schwartz inequality for the inner product we obtain $\|(\alpha - \alpha^*)\langle x, x \rangle\| \leq 2\|x\|\|Fx - \alpha Ix\|$. Since $(\alpha - \alpha^*)$ is invertible by assumption, we get from this inequality

$$\begin{aligned} \parallel x \parallel^2 = \parallel \langle x, x \rangle \parallel &\leq \parallel (\alpha - \alpha^*)^{-1} \parallel \parallel (\alpha - \alpha^*)\langle x, x \rangle \parallel \\ &\leq 2 \cdot \parallel x \parallel \parallel (F - \alpha I)x \parallel \parallel (\alpha - \alpha^*)^{-1} \parallel, \end{aligned}$$

which gives

$$\frac{\parallel x \parallel}{2 \cdot \parallel (\alpha - \alpha^*)^{-1} \parallel} \leq \parallel (F - \alpha I)x \parallel$$

for all $x \in H_{\mathcal{A}}$. From Lemma 7.1.14 it follows that $F - \alpha I$ is invertible. □

Remark 7.1.16. Let $\mathcal{A} = C([0, 1])$ or $\mathcal{A} = L^{\infty}((0, 1), \mu)$. As we have seen in Example 7.1.12, the operator $\tilde{g}I - G$ is not invertible, whereas $\tilde{g} - \overline{\tilde{g}} = 2ig_1 \neq 0$. Therefore, it is not sufficient only to assume that $\alpha - \alpha^* \neq 0$, so the requirement that $a - a^*$ is invertible is indeed necessary in Corollary 7.1.15.

Example 7.1.17. Let $\mathcal{A} = M_2(\mathbb{C})$ and $T_1, T_2 \in \mathcal{A}$ be given by $T_1 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$, $T_2 = \begin{bmatrix} 0 & i \\ i & i \end{bmatrix}$. Then T_1 is self-adjoint and $T_2 - T_2^* = 2i \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, so $T_2 - T_2^*$ is invertible. Now, $T_1 - T_2 = \begin{bmatrix} 2 & 1 - i \\ 1 - i & -i \end{bmatrix}$, so $det(T_1 - T_2) = 0$, which gives that $T_1 - T_2$ is not invertible. Hence the operator $F := T_1 \cdot I$ is a self-adjoint operator on $H_{\mathcal{A}}$, but $F - T_2 \cdot I = (T_1 - T_2) \cdot I$ is not invertible. This shows that the assumption that $\mathcal{A}$ is commutative in Corollary 7.1.15 is indeed necessary.

For a self-adjoint operator F on $H_{\mathcal{A}}$, set

$$M(F) = \sup\{\| \langle Fx, x\rangle \| \mid \| x \|= 1\} \text{ and } m(F) = \inf\{\| \langle Fx, x\rangle \| \mid \| x \|= 1\}.$$

We have the following corollary.

Corollary 7.1.18. *If $\mathcal{A} = C([0, 1])$ and F is a self-adjoint operator on $H_{\mathcal{A}}$, then*

$$\sigma^{\mathcal{A}}(F) \subseteq \{f \in \mathcal{A} \mid \ |f|([0, 1]) \cap [m, M] \neq \varnothing\}.$$

If $\mathcal{A} = L^\infty((0, 1), \mu)$ and F is a self-adjoint operator on $H_{\mathcal{A}}$, then

$$\sigma^{\mathcal{A}}(F) \subseteq \{f \in \mathcal{A} \mid \mu(|f|^{-1}([m - \epsilon, M + \epsilon])) > 0 \textit{ for all } \epsilon > 0\}.$$

Proof. Let $\mathcal{A} = L^\infty((0, 1), \mu)$, F be a self-adjoint operator on $H_{\mathcal{A}}$ and $\alpha \in \mathcal{A}$ be such that there exists an $\epsilon = \epsilon(\alpha)$ with the property that $\mu(|\alpha|^{-1}([m - \epsilon, M + \epsilon])) = 0$. Then $(0, 1) = M_1 \cup M_2$, where M_1 and M_2 are Lebesgue measurable, mutually disjoint subsets of $(0, 1)$ satisfying $|\alpha|\chi_{M_1} \geq (M + \epsilon)\chi_{M_1}$ and $|\alpha|\chi_{M_2} \leq (m - \epsilon)\chi_{M_2}$ a.e. Hence, for all $x \in H_{\mathcal{A}}$, we have

$$\langle (F - \alpha I)x, x\rangle = \langle (F - \alpha I)x, x\rangle \cdot \chi_{M_1} + \langle (F - \alpha I)x, x\rangle \cdot \chi_{M_2}.$$

Now, we have

$$\begin{aligned}
&\| \langle (F - \alpha I)x, x\rangle \| \geq \| \langle (F - \alpha I)x, x\rangle\chi_{M_1} \| \\
&\geq \| \overline{\alpha}\langle x, x\rangle\chi_{M_1} \| - \| \langle Fx, x\rangle\chi_{M_1} \| = \| \overline{\alpha}\chi_{M_1}\langle x, x\rangle\chi_{M_1} \| - \| \chi_{M_1}\langle Fx, x\rangle\chi_{M_1} \| \\
&= \| \overline{\alpha}\chi_{M_1}\langle x, x\rangle\chi_{M_1} \| - \| \langle Fx \cdot \chi_1, x \cdot \chi_{M_1}\rangle \| \\
&\geq (M + \epsilon) \| \langle x, x\rangle\chi_{M_1} \| - \| \langle F(x \cdot \chi_{M_1}), x \cdot \chi_{M_1}\rangle \| \\
&\geq (M + \epsilon) \| \langle x, x\rangle\chi_{M_1} \| - M \| \langle x \cdot \chi_{M_1}, x \cdot \chi_{M_1}\rangle \| \\
&= (M + \epsilon) \| \langle x, x\rangle\chi_{M_1} \| - M \| \chi_{M_1}\langle x, x\rangle\chi_{M_1} \| = \epsilon \| \langle x, x\rangle\chi_{M_1} \|
\end{aligned}$$

(where we have used that

$$\| \langle Fy, y\rangle \| = \| y \|^2 \| \langle F(\frac{y}{\| y \|}), \frac{y}{\| y \|}\rangle \| \leq \| \langle y, y\rangle \| M).$$

Similarly we obtain

$$\begin{aligned}
&\| \langle (F - \alpha I)x, x\rangle \| \geq \| \langle (F - \alpha I)x, x\rangle\chi_{M_2} \| \geq \| \langle Fx, x\rangle\chi_{M_2} \| - \| \overline{\alpha}\langle x, x\rangle\chi_{M_2} \| \\
&= \| \langle F(x \cdot \chi_{M_2}), x \cdot \chi_{M_2}\rangle \| - \| \overline{\alpha}\langle x, x\rangle\chi_{M_2} \| \\
&\geq m \| \langle x \cdot \chi_{M_2}, x \cdot \chi_{M_2}\rangle \| - (m - \epsilon) \| \langle x, x\rangle\chi_{M_2} \| = \epsilon \| \langle x, x\rangle\chi_{M_2} \| .
\end{aligned}$$

Hence $\| \langle (F - \alpha I)x, x\rangle \| \geq \epsilon \max\{\| \langle x, x\rangle\chi_{M_2} \|, \| \langle x, x\rangle\chi_{M_1} \|\} = \epsilon \| \langle x, x\rangle \|$.

Thus, $\| (F - \alpha I)x \| \| x \| \geq \| \langle (F - \alpha I)x, x\rangle \| \geq \epsilon \| x \|^2$ for all $x \in H_{\mathcal{A}}$. It follows that $F - \alpha I$ is bounded below, hence, from Lemma 7.1.14 we deduce that $F - \alpha I$ is invertible in $B^a(H_{\mathcal{A}})$.

The proof in the case when $\mathcal{A} = C([0, 1])$ is similar, but more simple, because if $\alpha \in \mathcal{A}$ and $|\alpha|([0, 1]) \cap [m, M] = \varnothing$, then by the continuity of $|\alpha|$ we must either have that $|\alpha| < m$ or $|\alpha| > M$ that on the whole interval $[0, 1]$. Moreover, there exists then an $\epsilon > 0$ such that $|\alpha| \leq m - \epsilon$ or $|\alpha| \geq M + \epsilon$ on the whole $[0, 1]$. Hence we may proceed in the same way as in the above proof. ☐

Finally, we are going to study the properties of generalized spectra of normal operators on $H_{\mathcal{A}}$.

Lemma 7.1.19. *Let $\mathcal{A}$ be a commutative unital C^*-algebra and F be a normal operator on $H_{\mathcal{A}}$, that is $FF^* = F^*F$. If $\alpha_1, \alpha_2 \in \sigma_p^{\mathcal{A}}(F)$ and $\alpha_1 - \alpha_2$ is not a zero divisor in $\mathcal{A}$, then* $\ker(F - \alpha_1 I) \perp \ker(F - \alpha_2 I)$.

Proof. Since F commutes with F^* and $\mathcal{A}$ is a commutative unital C^*-algebra, then $F - \alpha_2 I$ and $F^* - \alpha_2^* I$ mutually commute. Hence $\ker(F - \alpha_2 I) = \ker(F^* - \alpha_2^* I)$. For $x_1 \in \ker(F - \alpha_1 I)$ and $x_2 \in \ker(F - \alpha_2 I) = \ker(F^* - \alpha_2^* I)$, we get

$$\langle x_2, x_1 \rangle (\alpha_1 - \alpha_2) = \langle x_2, x_1 \rangle \alpha_1 - \alpha_2 \langle x_2, x_1 \rangle = \langle x_2, F x_1 \rangle - \langle F^* x_2, x_1 \rangle = 0$$

(where we have used that $\mathcal{A}$ is commutative, so $\langle x_2, x_1 \rangle\ \alpha_2 = \alpha_2 \langle x_2, x_1 \rangle$). Since $(\alpha_1 - \alpha_2)$ is not a zero divisor by assumption, it follows that $\langle x_2, x_1 \rangle = 0$. $\square$

Example 7.1.20. Let $\mathcal{A} = C([0,1])$ or $\mathcal{A} = L^\infty((0,1), \mu)$ and consider the self-adjoint operator G from Example 7.1.12. For any function f in $\mathcal{A}$ with the support contained in J_2, we have $(f, 0, 0, \cdots) \in \ker G \cap \ker(\tilde{g} I - G)$. However, $\tilde{g} = i g_1 \neq 0$ and $f \neq 0$, but $\tilde{g}$ is <u>not</u> invertible in $\mathcal{A}$, so it is not sufficient only to assume that $\alpha_1 - \alpha_2 \neq 0$ and the assumption that $\alpha_1 - \alpha_2$ is not a zero divisor in $\mathcal{A}$ is indeed necessary.

Example 7.1.21. Let $\mathcal{A} = B(H)$ and $T \in \mathcal{A}$ be a normal and invertible operator. If H_1 and H_2 are two closed subspaces of H such that $H = H_1 \tilde{\oplus} H_2$ and $H_1 \neq H_2^\perp$ (that is H_1 and H_2 are not mutually orthogonal), then $T\sqcap$ and $T(1-\sqcap)$ are elements of $\sigma_p^{\mathcal{A}}(T \cdot I)$, where $\sqcap$ stands for the skew projection onto H_1 along H_2. Moreover, the operator $T \cdot I$ is normal operator on $H_{\mathcal{A}}$ and $T\sqcap - T(1-\sqcap)$ is invertible in $\mathcal{A}$ because $T\sqcap - T(1-\sqcap)$ has the matrix $\begin{bmatrix} T & 0 \\ 0 & -T \end{bmatrix}$ with respect to the decomposition $H = H_1 \tilde{\oplus} H_2 \to T(H_1) \tilde{\oplus} T(H_2) = H$. However, if P_1 and P_2 denote the orthogonal projections onto H_1 and H_2, respectively, then, for all j,

$$e_j \cdot P_1 \in \ker(T\sqcap \cdot I - T \cdot I) \text{ and } e_j \cdot P_2 \in \ker(T(1-\sqcap) \cdot I - T \cdot I),$$

since $\sqcap P_1 = P_1$ and $(1-\sqcap)P_2 = P_2$. Moreover, $P_1 P_2 \neq 0$. So the assumption that $\mathcal{A}$ is commutative is indeed necessary in Lemma 7.1.19.

Lemma 7.1.22. *Let $\mathcal{A}$ be a commutative C^*-algebra and F be a normal operator on $H_{\mathcal{A}}$. Then $\sigma_{rl}^{\mathcal{A}}(F) = \varnothing$, hence $\sigma^{\mathcal{A}}(F) = \sigma_p^{\mathcal{A}}(F) \cup \sigma_{cl}^{\mathcal{A}}(F)$.*

Proof. Suppose that $\alpha \in \sigma_{rl}^{\mathcal{A}}(F)$. Then $F - \alpha I$ is bounded below. Again, since $F - \alpha I$ and $F^* - \alpha^* I$ mutually commute, we get that $\ker(F - \alpha I) = \ker(F^* - \alpha^* I) = \{0\}$. Next, since $Im(F - \alpha I)$ is closed, by Theorem 2.0.20 we have that

$$H_{\mathcal{A}} = \ker(F^* - \alpha^* I) \oplus Im(F - \alpha I) = Im(F - \alpha I).$$

So $F - \alpha I$ is surjective, thus invertible, which gives that $\sigma_{rl}^{\mathcal{A}}(F) = \varnothing$. $\square$

Example 7.1.23. Let $\mathcal{A} = B(H)$ and S, P be as in Example 7.1.13. Then $P \cdot I$ is a normal operator on $H_{\mathcal{A}}$ being self-adjoint and $(S - P) \cdot I$ is bounded below on $H_{\mathcal{A}}$. Indeed, we have that $\| (S-P)x \| \geq \| x \|$ for all $x \in H$, hence $m(S-P) \geq 1$. Therefore, since

$$T^*(S-P)^*(S-P)T \geq (m(S-P))^2 T^* T$$

for all $T \in B(H)$, it is not hard to see that $(S-P) \cdot I$ is bounded below on $H_{\mathcal{A}}$. However, $Im((S-P) \cdot I)^\perp = \ker((S^* - P) \cdot I)$ and $\ker((S^* - P) \cdot I) \neq \{0\}$ as we have seen in Example 7.1.13. Hence $P \cdot I$ is a normal operator on $H_{\mathcal{A}}$ and $S \in \sigma_{rl}^{\mathcal{A}}(P \cdot I)$, which shows that the assumption that $\mathcal{A}$ is commutative is indeed necessary in Lemma 7.1.22. Moreover, this also shows that the assumption that $\mathcal{A}$ is commutative is indeed necessary in Lemma 7.1.14 as well, because $S \in \mathcal{A} \setminus \sigma_p^{\mathcal{A}}(P \cdot I)$, however, $Im((S-P) \cdot I)^\perp \neq \{0\}$.

The next lemma is a generalization of [25, Chapter XI, Proposition 1.1] . For $F \in B^a(H_\mathcal{A})$, set

$$\sigma_a^\mathcal{A}(F) = \{\alpha \in \mathcal{A} \mid F - \alpha I \text{ is not bounded below } \},$$

$$\sigma_l^\mathcal{A}(F) = \{\alpha \in \mathcal{A} \mid F - \alpha I \text{ is not left invertible in } B^a(H_\mathcal{A})\},$$

$$\sigma_r^\mathcal{A}(F) = \{\alpha \in \mathcal{A} \mid F - \alpha I \text{ is not right invertible in } B^a(H_\mathcal{A})\}.$$

Lemma 7.1.24. *Let $F \in B^a(H_\mathcal{A})$. Then the following statements are equivalent.*
a) $\alpha \in \mathcal{A} \setminus \sigma_a^\mathcal{A}(F)$.
b) $\alpha \in \mathcal{A} \setminus \sigma_l^\mathcal{A}(F)$.
c) $\alpha^* \in \mathcal{A} \setminus \sigma_r^\mathcal{A}(F^*)$.
d) $Im(\alpha^* I - F^*) = H_\mathcal{A}$.

Proof. This proof is similar to the proof of [25, Chapter XI, Proposition 1.1]. Indeed, if $F - \alpha I$ is bounded below, then $Im(F - \alpha I)$ is orthogonally complementable in $H_\mathcal{A}$ by Theorem 2.0.20. The operator $F - \alpha I$ is invertible viewed as an operator from $H_\mathcal{A}$ onto $Im(F - \alpha I)$. This follows by the Banach open mapping theorem. Hence $(F - \alpha I)^{-1} \in B^a(Im(F - \alpha I), H_\mathcal{A})$ by Remark 2.0.21. Let P denote the orthogonal projection onto $Im(F - \alpha I)$, then $(F - \alpha I)^{-1}P$ is a left inverse of $F - \alpha I$ in $B^a(H_\mathcal{A})$. Next, $F - \alpha I$ has left inverse if and only if $F^* - \alpha^* I$ has right inverse in $B^a(H_\mathcal{A})$, so $(b) \Rightarrow (c)$. Part $(c) \Rightarrow (d)$ is obvious. Finally, if $Im(\alpha^* I - F^*) = H_\mathcal{A}$, then $\ker(F - \alpha I) = Im(F^* - \alpha^* I)^\perp = \{0\}$. Moreover, from the proof of Theorem 2.0.20 we have that $Im(F - \alpha I)$ is closed since $Im(F^* - \alpha^* I)$ is closed. Therefore, $F - \alpha I$ is bounded below. □

The next two propositions can be proved in exactly the same way as for operators on Hilbert spaces, see [50, Chapter 4, Proposition 20] and [50, Chapter 4, Proposition 21].

Proposition 7.1.25. *For $F \in B^a(H_\mathcal{A})$, we have that $\sigma_a^\mathcal{A}(F)$ is a closed subset of $\mathcal{A}$ in the norm topology and $\sigma^\mathcal{A}(F) = \sigma_a^\mathcal{A}(F) \cup \sigma_{rl}^\mathcal{A}(F)$.*

Proof. The statement follows since $M^a(H_\mathcal{A})$ is open in $B^a(H_\mathcal{A})$ in the norm topology. Next, if $F - \alpha_0 I$ is bounded below, it is easy to see that either $\alpha_0 \in \sigma_{rl}^\mathcal{A}(F)$ or $F - \alpha_0 I$ is invertible. □

Proposition 7.1.26. *Let $\mathcal{A}$ be a commutative C^*-algebra. If $F \in B^a(H_\mathcal{A})$, then $\partial\sigma^\mathcal{A}(F) \subseteq \sigma_a^\mathcal{A}(F)$. Moreover, if M is a closed submodule of $H_\mathcal{A}$ invariant with respect to F and $F_0 = F_{|_M}$, then we have $\partial\sigma^\mathcal{A}(F_0) \subseteq \sigma_a^\mathcal{A}(F)$ and $\sigma^\mathcal{A}(F_0) \cap \rho^\mathcal{A}(F) = \sigma_{rl}^\mathcal{A}(F_0)$, where $\rho^\mathcal{A}(F) = \mathcal{A} \setminus \sigma^\mathcal{A}(F)$.*

Proof. Let $\alpha_0 \in \partial\sigma^\mathcal{A}(F)$. Then there exists a sequence $\{\alpha_n\} \subseteq \mathcal{A} \setminus \sigma^\mathcal{A}(F)$ such that $\alpha_n \to \alpha_0$ in $\mathcal{A}$, hence $F - \alpha_n I \longrightarrow F - \alpha_0 I$ in the norm. From a well known result for operators on Banach spaces stated in [50, Chapter 4, Proposition 12], there exists a subsequence α_{n_k} such that $||(F - \alpha_{n_k} I)^{-1}|| \longrightarrow \infty$ as $k \longrightarrow \infty$ since $F - \alpha_0 I$ is not invertible. Hence, there exists a sequence of unit vectors $\{x_k\} \subseteq H_\mathcal{A}$ such that $||(F - \alpha_{n_k} I)^{-1} x_k|| \longrightarrow \infty$ as $k \longmapsto \infty$. For each k, set $y_k = (F - \alpha_{n_k} I)^{-1} x_k$ and $v_k = \dfrac{y_k}{\parallel y_k \parallel}$. Then we have that

$$\parallel (F - \alpha_0 I)v_k \parallel \leq \parallel (\alpha_0 - \alpha_{n_k}) I v_k \parallel + \parallel (F - \alpha_{nk} I) v_k \parallel \leq \parallel \alpha_0 - \alpha_{n_k} \parallel + \frac{1}{\parallel y_k \parallel},$$

which gives that $\parallel (F - \alpha_0 I)v_k \parallel \longrightarrow 0$, so $\alpha_0 \in \sigma_a^\mathcal{A}(F)$. This shows the first statement in the proposition. However, then we have that

$$\partial\sigma^\mathcal{A}(F_0) \subseteq \sigma_a^\mathcal{A}(F_0) \subseteq \sigma_a^\mathcal{A}(F) \subseteq \sigma^\mathcal{A}(F).$$

□

Example 7.1.27. We may also consider the operators on $H_{\mathcal{A}}$ defined by

$$W(e_k) = e_{2k} \text{ and } W'(e_k) = e_{2k-1} \text{ for all } k \in \mathbb{N}.$$

Also for these operators we have $\sigma^{\mathcal{A}}(W) = \sigma^{\mathcal{A}}(W') = \{\alpha \in \mathcal{A} \mid \inf |\alpha| \leq 1\}$ in the case when $\mathcal{A} = C([0,1])$ or when $\mathcal{A} = L^{\infty}((0,1),\mu)$. Suppose now that $\mathcal{A} = L^{\infty}((0,1),\mu)$ and consider the operator F on $H_{\mathcal{A}}$ given by

$$F(f_1, f_2, f_3, \cdots) = (\chi_{(0,\frac{1}{2})} f_1, \chi_{(\frac{1}{2},1)} f_1, \chi_{(0,\frac{1}{2})} f_2, \chi_{(\frac{1}{2},1)} f_2, \cdots).$$

It follows that F has the matrix $\begin{bmatrix} W' & 0 \\ 0 & W \end{bmatrix}$ with respect to the decomposition

$$H_{\mathcal{A}} = (H_{\mathcal{A}} \cdot \chi_{(0,\frac{1}{2})}) \oplus (H_{\mathcal{A}} \cdot \chi_{(\frac{1}{2},1)}) \xrightarrow{F} (H_{\mathcal{A}} \cdot \chi_{(0,\frac{1}{2})}) \oplus (H_{\mathcal{A}} \cdot \chi_{(\frac{1}{2},1)}) = H_{\mathcal{A}}.$$

Therefore, $\sigma^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid \inf |\alpha| \leq 1\}$. Next we have that

$$\sigma_p^{\mathcal{A}}(W) = \varnothing, \sigma_p^{\mathcal{A}}(W') = \{\alpha \in \mathcal{A} \mid \alpha = 1 \text{ on some closed subinterval } J \subseteq [0,1]\}$$

in the case when $\mathcal{A} = C([0,1])$ and $\sigma_p^{\mathcal{A}}(W') = \{\alpha \in \mathcal{A} \mid \mu(\{t \in (0,1) \mid \alpha(t) = 1\}) > 0\}$ in the case when $\mathcal{A} = L^{\infty}((0,1),\mu)$. Hence, we get that

$$\sigma_p^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid \mu(\{t \in (0,\frac{1}{2}) \mid \alpha(t) = 1\}) > 0\}.$$

Consider next the operators

$$Z(e_j) = \begin{cases} e_k & \text{when } j = 2k \\ 0 & \text{else} \end{cases}, \ k \in \mathbb{N}$$

$$Z'(e_j) = \begin{cases} e_k & \text{when } j = 2k-1 \\ 0 & \text{else} \end{cases}, \ k \in \mathbb{N}$$

Then $\sigma^{\mathcal{A}}(Z) = \sigma^{\mathcal{A}}(Z') = \{\alpha \in \mathcal{A} \mid \inf |\alpha| \leq 1\}$. This follows since $Z = W^*$ and $Z' = W'^*$. Moreover, we have

$$\sigma_p^{\mathcal{A}}(Z) = \{\alpha \in \mathcal{A} \mid \inf |\alpha| < 1\}$$

both in the case when $\mathcal{A} = C([0,1])$ and when $\mathcal{A} = L^{\infty}((0,1),\mu)$.

In the case when $\mathcal{A} = L^{\infty}((0,1),\mu)$ we have that

$$\sigma_p^{\mathcal{A}}(Z') = \{\alpha \in \mathcal{A} \mid \inf |\alpha| < 1 \text{ or } \mu(\{t \in (0,1) \mid \alpha(t) = 1\}) > 0\}$$

and in the case when $\mathcal{A} = C([0,1])$, we have that

$$\sigma_p^{\mathcal{A}}(Z') = \{\alpha \in \mathcal{A} \mid \inf |\alpha| < 1 \text{ or } \alpha = 1 \text{ on some closed subinterval } J \subseteq [0,1]\}.$$

Let the operator D on $H_{\mathcal{A}}$ be given by

$$D(g_1, g_2, g_3, \dots) = (g_1 \chi_{(0,\frac{1}{2})} + g_2 \chi_{(\frac{1}{2},1)}, g_3 \chi_{(0,\frac{1}{2})} + g_4 \chi_{(\frac{1}{2},1)}, \cdots)$$

when $\mathcal{A} = L^{\infty}((0,1),\mu)$. Then $D = F^*$ and D has the matrix $\begin{bmatrix} Z' & 0 \\ 0 & Z \end{bmatrix}$ with respect to the decomposition $H_{\mathcal{A}} \cdot \chi_{(0,\frac{1}{2})} \oplus H_{\mathcal{A}} \cdot \chi_{(\frac{1}{2},1)}$. It follows that

$$\sigma^{\mathcal{A}}(D) = \{\alpha \in \mathcal{A} \mid \inf |\alpha| \leq 1\},$$

$$\sigma_p^{\mathcal{A}}(D) = \{\alpha \in \mathcal{A} \mid \inf |\alpha| < 1 \text{ or } \mu(\{t \in (0,\frac{1}{2}) \mid \alpha(t) = 1\}) > 0\}.$$

Note that the operators F and D here are actually the operators from Example 3.7.4 and Example 3.7.5, respectively.

7.2 Generalized Fredholm spectra of operators over C^*-algebras

Various subclasses of semi-$\mathcal{A}$-Fredholm operators induce various corresponding generalized spectra in $\mathcal{A}$ of operators in $B^a(H_\mathcal{A})$. We shall investigate several properties of such spectra and the relationship between them. Most of the results in this section are generalizations in this setting of the results from [56, Section 2.2] and [56, Section 2.3] .
We start with the following definition.

Definition 7.2.1. We set $ms_\Phi(F) = \inf\{\| \alpha \| \ \mid \alpha \in \mathcal{A}, F - \alpha I \notin \mathcal{M}\Phi(H_\mathcal{A})\}$,

$$ms(F) = \inf\{\| \alpha \| \ \mid \alpha \in \mathcal{A}, F - \alpha I \notin \mathcal{M}\Phi_\pm(H_\mathcal{A})\},$$

$$ms_+(F) = \inf\{\| \alpha \| \ \mid \alpha \in \mathcal{A}, F - \alpha I \notin \mathcal{M}\Phi_+(H_\mathcal{A})\},$$

$$ms_-(F) = \inf\{\| \alpha \| \ \mid \alpha \in \mathcal{A}, F - \alpha I \notin \mathcal{M}\Phi_-(H_\mathcal{A})\}.$$

It follows that $ms_\Phi(F) = \max\{\epsilon \geq 0 \mid \ \| \alpha \| < \epsilon \Rightarrow F - \alpha I \in \mathcal{M}\Phi(H_\mathcal{A})\}$,

$$ms_+(F) = \max\{\epsilon \geq 0 \mid \ \| \alpha \| < \epsilon \Rightarrow F - \alpha I \in \mathcal{M}\Phi_+(H_\mathcal{A})\},$$

$$ms_-(F) = \max\{\epsilon \geq 0 \mid \ \| \alpha \| < \epsilon \Rightarrow F - \alpha I \in \mathcal{M}\Phi_-(H_\mathcal{A})\},$$

$$ms(F) = \max\{\epsilon \geq 0 \mid \ \| \alpha \| < \epsilon \Rightarrow F - \alpha I \in \mathcal{M}\Phi_\pm(H_\mathcal{A})\},$$

From Lemma 2.0.42 and Theorem 3.3.1 it follows that

$$ms_\Phi(F) > 0 \Leftrightarrow F \in \mathcal{M}\Phi(H_\mathcal{A}),$$

$$ms_+(F) > 0 \Leftrightarrow F \in \mathcal{M}\Phi_+(H_\mathcal{A}), ms_-(F) > 0 \Leftrightarrow F \in \mathcal{M}\Phi_-(H_\mathcal{A}),$$

$$ms(F) > 0 \Leftrightarrow F \in \mathcal{M}\Phi_\pm(H_\mathcal{A}).$$

From Corollary 3.1.20 it follows that

$$ms_+(F) = ms_-(F^*), ms_\Phi(F) = ms_\Phi(F^*),$$

$$ms(F) = ms(F^*).$$

Lemma 7.2.2. *Let $F \in B^a(H_\mathcal{A})$. If $ms_+(F) > 0$ and $ms_-(F) > 0$, then $ms_+(F) = ms_-(F)$.*

Proof. Since $ms_+(F)$ and $ms_-(F)$ are strictly positive by assumption, then, by Corollary 3.1.10, $F \in \mathcal{M}\Phi_+(H_\mathcal{A}) \cap \mathcal{M}\Phi_-(H_\mathcal{A}) = \mathcal{M}\Phi(H_\mathcal{A})$. If $ms_+(F) > ms_-(F)$, then, obviously, there exists an $\alpha \in \mathcal{A}$ such that $\| \alpha \| \in (ms_-(F), ms_+(F))$, and $(F - \alpha I) \in \mathcal{M}\Phi_+(H_\mathcal{A}) \setminus \mathcal{M}\Phi_-(H_\mathcal{A})$. However, if we consider the map $f : [0,1] \to B^a(H_\mathcal{A})$ given by $f(t) = F - t\alpha I$, then f is continuous. Since $\| \alpha \| < ms_+(F)$, it follows that $f([0,1]) \subseteq \mathcal{M}\Phi_+(H_\mathcal{A}) \subseteq \mathcal{M}\Phi_\pm(H_\mathcal{A})$. By Corollary 3.3.5 we deduce that $f(1) \in \mathcal{M}\Phi(H_\mathcal{A})$ since $f(0) \in \mathcal{M}\Phi(H_\mathcal{A})$. However, we have that $f(1) = F - \alpha I \notin \mathcal{M}\Phi_-(H_\mathcal{A})$. Since $\mathcal{M}\Phi(H_\mathcal{A}) \subseteq \mathcal{M}\Phi_-(H_\mathcal{A})$, we get a contradiction. Thus, $ms_+(F) = ms_-(F)$ in this case. Similarly, if $ms_-(F) \geq ms_+(F)$, we can show that actually $ms_-(F) = ms_+(F)$. □

Lemma 7.2.3. *Let $F \in B^a(H_\mathcal{A})$. Then*
1) $ms_\Phi(F) = \min\{ms_+(F), ms_-(F)\}$,
2) $ms(F) = \max\{ms_+(F), ms_-(F)\}$.

Proof. First we prove 1). If $0 = \min\{ms_+(F), ms_-(F)\}$, then either $ms_+(F) = 0$ or $ms_-(F) = 0$. Suppose that $ms_+(F) = 0$. Then, by the above arguments, since $\mathcal{M}\Phi_+(H_{\mathcal{A}})$ is open, we must have that $F \notin \mathcal{M}\Phi_+(H_{\mathcal{A}})$. Hence $F \notin \mathcal{M}\Phi(H_{\mathcal{A}})$, so $ms_\Phi(F) = 0$. Similarly, if $ms_-(F) = 0$, it follows that $ms_\Phi(F) = 0$, since $\mathcal{M}\Phi_-(H_{\mathcal{A}})$ is open and $\mathcal{M}\Phi(H_{\mathcal{A}}) \subseteq \mathcal{M}\Phi_-(H_{\mathcal{A}})$.
Suppose now that

$$0 < \min\{ms_+(F), ms_-(F)\} = ms_+(F).$$

By Lemma 7.2.2 we have $ms_+(F) = ms_-(F)$. Applying Corollary 3.1.10 we easily deduce that $ms_\Phi(F) = ms_+(F) = ms_-(F)$.

Next we prove 2). If $\max\{ms_+(F), ms_-(F)\} = 0$, then $F \notin \mathcal{M}\Phi_\pm(H_{\mathcal{A}})$, hence $ms(F) = 0$, as in the proof of [56, (2.3.8.2.)].
Suppose that $0 < \max\{ms_+(F), ms_-(F)\} = ms_+(F)$. Obviously, $ms(F) \geq ms_+(F)$. If we have $ms(F) > ms_+(F)$, then for any $r \in (ms_+(F), ms(F))$, the set

$$C_r := \{F - \alpha I \mid \alpha \in \mathcal{A} \,|\, \parallel \alpha \parallel \leq r\}$$

would intersect both $\mathcal{M}\Phi_+(H_{\mathcal{A}})$ and $\mathcal{M}\Phi_-(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi_+(H_{\mathcal{A}})$, which are both open by Theorem 3.3.1 and Remark 3.3.4. Hence the sets $\mathcal{M}\Phi_+(H_{\mathcal{A}}) \cap C_r$ and $(\mathcal{M}\Phi_-(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi_+(H_{\mathcal{A}})) \cap C_r$ would form a separation of C_r, since $C_r \subseteq \mathcal{M}\Phi_\pm(H_{\mathcal{A}})$. Indeed, since $r > \max\{ms_+(F), ms_-(F)\}$, we can not have that $C_r \subseteq \mathcal{M}\Phi_+(H_{\mathcal{A}})$ or $C_r \subseteq \mathcal{M}\Phi_-(H_{\mathcal{A}})$. On the other hand, since $r < ms(F)$, we must have that $C_r \subseteq \mathcal{M}\Phi_\pm(H_{\mathcal{A}})$. Therefore, it follows that $C_r \cap \mathcal{M}\Phi_+(H_{\mathcal{A}}) \neq \varnothing$ and $C_r \cap (\mathcal{M}\Phi_-(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi_+(H_{\mathcal{A}})) \neq \varnothing$. This is a contradiction since C_r is connected. Hence we must have $ms(F) = ms_+(F)$.
The case when $\max\{ms_+(F), ms_-(F)\} = ms_-(F)$ can be treated analogously. $\square$

Definition 7.2.4. Let $F \in B^a(H_{\mathcal{A}})$. We set

$$\sigma_{ew}^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid (F - \alpha I) \notin \mathcal{M}\Phi_0(H_{\mathcal{A}})\},$$

$$\sigma_{e\,uf}^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid (F - \alpha I) \notin \mathcal{M}\Phi_+(H_{\mathcal{A}})\},$$

$$\sigma_{e\,lf}^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid (F - \alpha I) \notin \mathcal{M}\Phi_-(H_{\mathcal{A}})\},$$

$$\sigma_{ek}^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid (F - \alpha I) \notin \mathcal{M}\Phi_\pm(H_{\mathcal{A}})\},$$

$$\sigma_{ef}^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid (F - \alpha I) \notin \mathcal{M}\Phi(H_{\mathcal{A}})\}.$$

Lemma 7.2.5. *Let $F \in B(H_{\mathcal{A}})$ and suppose that $K_0(\mathcal{A})$ satisfies the cancellation property. Then $\sigma^{\mathcal{A}}(F) = \sigma_{ew}^{\mathcal{A}}(F) \cup \sigma_p^{\mathcal{A}}(F) \cup \sigma_{cl}^{\mathcal{A}}(F)$.*

Proof. It suffices to show " $\subseteq$ ". Suppose that $\alpha \in \sigma^{\mathcal{A}}(F) \setminus (\sigma_{cl}^{\mathcal{A}}(F) \cup \sigma_{ew}^{\mathcal{A}}(F))$. Then $Im(F - \alpha I)$ is closed and $(F - \alpha I) \in \mathcal{M}\Phi_0(H_{\mathcal{A}})$. By Proposition 3.5.11 the operator $F - \alpha I$ has the matrix $\begin{bmatrix} (F - \alpha I)_1 & 0 \\ 0 & 0 \end{bmatrix}$ with respect to the decomposition

$$H_{\mathcal{A}} = \ker(F - \alpha I)^{\circ} \tilde{\oplus} \ker(F - \alpha I) \xrightarrow{F - \alpha I} Im(F - \alpha I) \tilde{\oplus} Im(F - \alpha I)^{\circ} = H_{\mathcal{A}},$$

where $(F - \alpha I)_1$ is an isomorphism by the Banach open mapping theorem. Since we have $(F - \alpha I) \in \mathcal{M}\Phi_0(H_{\mathcal{A}})$, then it holds that

$$0 = \text{index } (F - \alpha I) = [\ker(F - \alpha I)] - [Im(F - \alpha I)^{\circ}],$$

so $[\ker(F-\alpha I)] = [Im(F-\alpha I)^\circ]$. If $[\ker(F-\alpha I)] = 0$, then $\ker(F-\alpha I) = \{0\}$, since $K_0(\mathcal{A})$ satisfies the cancellation property by assumption. By the same reason we would have $Im(F-\alpha I)^\circ = \{0\}$, so $F-\alpha I$ is then invertible, which is a contradiction, since $\alpha \in \sigma^{\mathcal{A}}(F)$. Thus, we must have $\ker(F-\alpha I) \neq \{0\}$, so $\alpha \in \sigma_p^{\mathcal{A}}(F)$. □

Example 7.2.6. Let $\mathcal{A} = B(H)$ where H is a separable, infinite-dimensional Hilbert space and consider the operator T from Example 3.4.5. Obviously, $T \in \mathcal{M}\Phi_0(H_{\mathcal{A}})$ and moreover, T is bounded below, but T is not surjective, thus not invertible. Hence

$$0 \in (\sigma_{rl}^{\mathcal{A}}(T) \setminus \sigma_{ew}^{\mathcal{A}}(T)) \subseteq (\sigma^{\mathcal{A}}(T) \setminus (\sigma_{ew}^{\mathcal{A}}(T) \cup \sigma_p^{\mathcal{A}}(T) \cup \sigma_{cl}^{\mathcal{A}}(T))).$$

This shows that the assumption that $K_0(\mathcal{A})$ satisfies the cancellation property is indeed necessary in Lemma 7.2.5.

Recall Definition 5.1.1 of the class $\mathcal{M}\Phi_0^{gc}(H_{\mathcal{A}})$. We have the following lemma.

Lemma 7.2.7. *For $F \in B^a(H_{\mathcal{A}})$ set $\sigma_{ewgc}^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid (F-\alpha I) \notin \mathcal{M}\Phi_0^{gc}(H_{\mathcal{A}})\}$. Then $\sigma^{\mathcal{A}}(F) = \sigma_{ewgc}^{\mathcal{A}}(F) \cup \sigma_p^{\mathcal{A}}(F)$.*

Proof. Again it suffices to show " $\subseteq$ ". Suppose that $\alpha \in \sigma^{\mathcal{A}}(F) \setminus \sigma_{ewgc}^{\mathcal{A}}(F)$. Then $Im(F-\alpha I)$ is closed and $\ker(F-\alpha I) \cong Im(F-\alpha I)^\perp$. Moreover, $H_{\mathcal{A}} = Im(F-\alpha I) \oplus Im(F-\alpha I)^\perp$ by Theorem 2.0.20. Since $\alpha \in \sigma^{\mathcal{A}}(F)$, it follows that $\ker(F-\alpha I) \neq \{0\}$. □

For $F \in B^a(H_{\mathcal{A}})$ we set

$$\mathcal{M}\Phi_+(F) = \{\alpha \in \mathcal{A} \mid F-\alpha I \in \mathcal{M}\Phi_+(H_{\mathcal{A}})\},$$

$$\mathcal{M}\Phi_-(F) = \{\alpha \in \mathcal{A} \mid F-\alpha I \in \mathcal{M}\Phi_-(H_{\mathcal{A}})\},$$

$$\mathcal{M}\Phi(F) = \{\alpha \in \mathcal{A} \mid F-\alpha I \in \mathcal{M}\Phi(H_{\mathcal{A}})\},$$

$$\mathcal{M}\Phi_\pm(F) = \{\alpha \in \mathcal{A} \mid F-\alpha I \in \mathcal{M}\Phi_\pm(H_{\mathcal{A}})\} \text{ and}$$

$$\mathcal{M}\Phi_0(F) = \{\alpha \in \mathcal{A} \mid F-\alpha I \in \mathcal{M}\Phi_0(H_{\mathcal{A}})\}.$$

The next two results are generalizations of [25, Chapter XI, Proposition 4.9] .

Proposition 7.2.8. *If $F \in B^a(H_{\mathcal{A}})$, then the components of $\mathcal{A} \setminus (\sigma_{e\,uf}^{\mathcal{A}}(F) \cap \sigma_{e\,lf}^{\mathcal{A}}(F))$ are either completely contained in one of the sets*

$$\mathcal{M}\Phi_+(F) \setminus \mathcal{M}\Phi(F),\ \mathcal{M}\Phi_-(F) \setminus \mathcal{M}\Phi(F)$$

or they are completely contained in $\mathcal{M}\Phi(F)$ and in this case index $(F-\alpha I)$ *is constant on them.*

Proof. Let C be a component of $\mathcal{A} \setminus (\sigma_{e\,uf}^{\mathcal{A}}(F) \cap \sigma_{e\,lf}^{\mathcal{A}}(F))$. Then either $C \cap \mathcal{M}\Phi_+(F) \neq \varnothing$ or $C \cap \mathcal{M}\Phi_-(F) \neq \varnothing$. Hence we must have that either $C \subseteq \mathcal{M}\Phi_-(F)$ or $C \subseteq \mathcal{M}\Phi_+(F)$ because otherwise the sets $C \cap \mathcal{M}\Phi_-(F)$ and $C \cap (\mathcal{M}\Phi_+(F) \setminus \mathcal{M}\Phi_-(F))$ would form a separation of C, which is a contradiction. Indeed, it follows straightforward from Theorem 3.3.1 and Remark 3.3.4 that the sets $\mathcal{M}\Phi_-(F)$ and $\mathcal{M}\Phi_+(F) \setminus \mathcal{M}\Phi_-(F)$ are open in the norm topology of $\mathcal{A}$.

Assume that $C \subseteq \mathcal{M}\Phi_+(F)$. If $C \cap \mathcal{M}\Phi(F) \neq \varnothing$, then $C \subseteq \mathcal{M}\Phi(F)$ because otherwise the sets $\mathcal{M}\Phi(F)$ and $\mathcal{M}\Phi_+(F) \setminus \mathcal{M}\Phi(F)$ would form a separation of C, since it follows straightforward from Lemma 2.0.42 and Theorem 3.3.1 that $\mathcal{M}\Phi(F)$ and $\mathcal{M}\Phi_+(F) \setminus \mathcal{M}\Phi(F)$ are open. So, either $C \subseteq \mathcal{M}\Phi_+(F) \setminus \mathcal{M}\Phi(F)$ or $C \subseteq \mathcal{M}\Phi(F)$. Now, if $C \subseteq \mathcal{M}\Phi(F)$, then index $(F-\alpha I)$ must be constant on C, since index is locally constant by Lemma 2.0.42.

The case when $C \subseteq \mathcal{M}\Phi_-(F)$ can be treated similarly. □

Lemma 7.2.9. *Let $F \in B^a(H_{\mathcal{A}})$. If $\alpha \in \partial\sigma^{\mathcal{A}}(F) \setminus (\sigma^{\mathcal{A}}_{e\,uf}(F) \cap \sigma^{\mathcal{A}}_{e\,lf}(F))$, then $\alpha \in \mathcal{M}\Phi_0(F)$.*

Proof. Let $\alpha \in \partial\sigma^{\mathcal{A}}(F) \setminus (\sigma^{\mathcal{A}}_{e\,uf}(F) \cap \sigma^{\mathcal{A}}_{e\,lf}(F))$. Then $\alpha \in \mathcal{M}\Phi_{\pm}(F)$. Since $\alpha \in \partial\sigma^{\mathcal{A}}(F)$, each open neighbourhood of α in $\mathcal{A}$ intersects $\mathcal{M}\Phi_0(F)$ non-empty. Since $\mathcal{M}\Phi_+(F) \setminus \mathcal{M}\Phi(F)$ and $\mathcal{M}\Phi_-(F) \setminus \mathcal{M}\Phi(F)$ are open, it follows that α must be an element of $\mathcal{M}\Phi(F)$. Now, since $\alpha \in \partial\sigma^{\mathcal{A}}(F)$ and $\mathcal{M}\Phi(F) \setminus \mathcal{M}\Phi_0(F)$ is open (this follows from Lemma 3.4.16), we must have that $\alpha \in \mathcal{M}\Phi_0(F)$. □

Theorem 7.2.10. *[20, Theorem 4] Let $F \in B^a(H_{\mathcal{A}})$. Then the following inclusions hold:*

$$\partial\sigma^{\mathcal{A}}_{ew}(F) \subseteq \partial\sigma^{\mathcal{A}}_{ef}(F) \subseteq \begin{array}{c}\partial\sigma^{\mathcal{A}}_{e\,lf}(F)\\ \partial\sigma^{\mathcal{A}}_{e\,uf}(F)\end{array} \subseteq \partial\sigma^{\mathcal{A}}_{ek}(F).$$

Proof. We will show this by proving the following inclusions:

$$\partial\sigma^{\mathcal{A}}_{ew}(F) \subseteq \sigma^{\mathcal{A}}_{ef}(F),$$

$$\partial\sigma^{\mathcal{A}}_{ef}(F) \subseteq (\sigma^{\mathcal{A}}_{e\,uf}(F) \cap \sigma^{\mathcal{A}}_{e\,lf}(F)) = \sigma^{\mathcal{A}}_{ek}(F),$$

$$\partial\sigma^{\mathcal{A}}_{e\,uf}(F) \subseteq \sigma^{\mathcal{A}}_{ek}(F) \text{ and } \partial\sigma^{\mathcal{A}}_{e\,lf}(F) \subseteq \sigma^{\mathcal{A}}_{ek}(F).$$

Since, obviously,

$$\sigma^{\mathcal{A}}_{ek}(F) \subseteq \begin{array}{c}\sigma^{\mathcal{A}}_{e\,uf}(F)\\ \sigma^{\mathcal{A}}_{e\,lf}(F)\end{array} \subseteq \sigma^{\mathcal{A}}_{ef}(F) \subseteq \sigma^{\mathcal{A}}_{ew}(F),$$

if we prove the inclusions above, the theorem would follow. Here we use the property that if $S, S' \subseteq \mathcal{A}$, $S \subseteq S'$ and $\partial S' \subseteq S$, then $\partial S' \subseteq \partial S$.

The first inclusion follows by the same arguments as in the classical case (the proof of [56, Theorem 2.2.2.3]) since $\sigma^{\mathcal{A}}_{ew}(F) \setminus \sigma^{\mathcal{A}}_{ef}(F)$ is open in $\mathcal{A}$ by Lemma 3.4.16.

Next, if $\alpha \in \partial\sigma^{\mathcal{A}}_{ef}(F)$, then, obviously, $F - \alpha I$ is in $\partial\mathcal{M}\Phi(H_{\mathcal{A}})$. By applying Corollary 3.3.3, we deduce that $(F - \alpha I) \notin \mathcal{M}\Phi_+(H_{\mathcal{A}}) \cup \mathcal{M}\Phi_-(H_{\mathcal{A}})$. This works as in the proof of [56, 2.2.2.4] and [56, 2.2.2.5]. Hence,

$$\partial\sigma^{\mathcal{A}}_{ef}(F) \subseteq (\sigma^{\mathcal{A}}_{e\,uf}(F) \cap \sigma^{\mathcal{A}}_{e\,lf}(F)).$$

Suppose now that $\alpha \in \partial\sigma_{e\,uf}(F)$. If $\alpha \notin \sigma^{\mathcal{A}}_{e\,lf}(F)$, then $(F - \alpha I) \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$. Since $\alpha \in \partial\sigma_{e\,uf}(F)$, it follows that $(F - \alpha I) \in \partial\mathcal{M}\Phi_+(H_{\mathcal{A}})$. Hence $(F - \alpha I) \notin \mathcal{M}\Phi_+(H_{\mathcal{A}})$, because $\mathcal{M}\Phi_+(H_{\mathcal{A}})$ is open in the norm topology. Thus, if $\alpha \in \partial\sigma^{\mathcal{A}}_{e\,uf}(F)\sigma^{\mathcal{A}}_{e\,lf}(F)$, then $\alpha \in \mathcal{M}\Phi_-(F)$ and $\alpha \notin \mathcal{M}\Phi_+(F)$. Now, since α belongs to the boundary of $\sigma^{\mathcal{A}}_{e\,uf}(F)$, it follows that any open ball around α in $\mathcal{A}$ intersects $\mathcal{M}\Phi_+(F)$ non–empty. On the other hand, $\mathcal{M}\Phi_-(F) \setminus \mathcal{M}\Phi_+(F)$ is open in $\mathcal{A}$, which follows from Theorem 3.3.1, and $\alpha \in \mathcal{M}\Phi_-(F) \setminus \mathcal{M}\Phi_+(F)$, a contradiction. Thus, we must have $\partial\sigma_{e\,uf}(F) \subseteq \sigma_{e\,lf}(F)$. Hence $\partial\sigma^{\mathcal{A}}_{e\,uf}(F) \subseteq \sigma^{\mathcal{A}}_{e\,uf}(F) \cap \sigma^{\mathcal{A}}_{e\,lf}(F) = \sigma^{\mathcal{A}}_{ek}(F)$. Similarly, we can show that $\partial\sigma^{\mathcal{A}}_{e\,lf}(F) \subseteq \sigma^{\mathcal{A}}_{ek}(F)$. □

Now we consider the following spectra for $F \in B^a(H_{\mathcal{A}})$:

$$\sigma^{\mathcal{A}}_{e\tilde{a}}(F) = \{\alpha \in \mathcal{A} \mid (F - \alpha I) \notin \tilde{\mathcal{M}}\Phi^-_+(H_{\mathcal{A}})\},$$

$$\sigma^{\mathcal{A}}_{ea}(F) = \{\alpha \in \mathcal{A} \mid (F - \alpha I) \notin \mathcal{M}\Phi^-_+(H_{\mathcal{A}})\}.$$

Clearly, $\sigma^{\mathcal{A}}_{e\,uf}(F) \subseteq \sigma^{\mathcal{A}}_{ea}(F) \subseteq \sigma^{\mathcal{A}}_{e\tilde{a}}(F) \subseteq \sigma^{\mathcal{A}}_{ew}(F)$. We have the following theorem.

Theorem 7.2.11. *[20, Theorem 5] Let $F \in B^a(H_{\mathcal{A}})$. Then*

$$\partial\sigma^{\mathcal{A}}_{ew}(F) \subseteq \partial\sigma^{\mathcal{A}}_{e\tilde{a}}(F) \subseteq \partial\sigma^{\mathcal{A}}_{ea}(F) \subseteq \partial\sigma^{\mathcal{A}}_{e\,uf}(F).$$

Proof. Again it suffices to show

$$\partial\sigma^{\mathcal{A}}_{ew}(F) \subseteq \sigma^{\mathcal{A}}_{e\tilde{a}}(F), \partial\sigma^{\mathcal{A}}_{e\tilde{a}}(F) \subseteq \sigma^{\mathcal{A}}_{ea}(F) \text{ and } \partial\sigma^{\mathcal{A}}_{ea}(F) \subseteq \sigma^{\mathcal{A}}_{e\,uf}(F).$$

The first inclusion follows as in the proof of [56, Theorem 2.7.5], since $\partial\sigma^{\mathcal{A}}_{ew}(F) \subseteq \partial\sigma^{\mathcal{A}}_{ek}(F)$ by Theorem 7.2.10 and since $\partial\sigma^{\mathcal{A}}_{ek}(F) \subseteq \sigma^{\mathcal{A}}_{ek}(F) \subseteq \sigma^{\mathcal{A}}_{e\tilde{a}}(F)$.

To deduce the second inclusion, assume first that $\alpha \in \partial\sigma^{\mathcal{A}}_{e\tilde{a}}(F) \setminus \sigma^{\mathcal{A}}_{ea}(F)$. Then we have that $(F-\alpha I) \in \mathcal{M}\Phi^{-}_{+}(H_{\mathcal{A}})$ and $(F-\alpha I) \notin \tilde{\mathcal{M}}\Phi^{-}_{+}(H_{\mathcal{A}})$ since $\tilde{\mathcal{M}}\Phi^{-}_{+}(H_{\mathcal{A}})$ is open by Lemma 3.4.9. It follows by Definition 3.4.1 that $F-\alpha I$ is in $\mathcal{M}\Phi_{+}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$. However, since $\mathcal{M}\Phi_{+}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$ is open by Theorem 3.3.1 and $\tilde{\mathcal{M}}\Phi^{-}_{+}(H_{\mathcal{A}}) \subseteq \mathcal{M}\Phi(H_{\mathcal{A}})$ by definition, we must have $(F-\alpha I) \notin \partial\tilde{\mathcal{M}}\Phi^{-}_{+}(H_{\mathcal{A}})$. This contradicts the choice of $\alpha \in \partial\sigma^{\mathcal{A}}_{e\tilde{a}}(F)$. Hence $\partial\sigma^{\mathcal{A}}_{e\tilde{a}}(F) \subseteq \sigma^{\mathcal{A}}_{ea}(F)$.

For the last inclusion, assume that $\tilde{\alpha} \in \partial\sigma^{\mathcal{A}}_{ea}(F)$ and that $\tilde{\alpha} \notin \sigma^{\mathcal{A}}_{e\,uf}(F)$. Then it follows that $(F-\tilde{\alpha}I) \in \mathcal{M}\Phi_{+}(H_{\mathcal{A}})$ and $(F-\tilde{\alpha}I) \notin \mathcal{M}\Phi^{-}_{+}(H_{\mathcal{A}})$. This means by the definitions of $\mathcal{M}\Phi^{-}_{+}(H_{\mathcal{A}})$, $\mathcal{M}\Phi_{+}(H_{\mathcal{A}})$ and by Proposition 3.4.12 that $(F-\tilde{\alpha}I) \in \mathcal{M}\Phi(H_{\mathcal{A}})$ and that

$$(F-\tilde{\alpha}I) \notin \tilde{\mathcal{M}}\Phi^{-}_{+}(H_{\mathcal{A}}) = \mathcal{M}\Phi^{-'}_{+}(H_{\mathcal{A}}) \cap \mathcal{M}\Phi(H_{\mathcal{A}}).$$

Thus, $(F-\tilde{\alpha}I) \in \mathcal{M}\Phi(H_{\mathcal{A}}) \setminus \tilde{\mathcal{M}}\Phi^{-}_{+}(H_{\mathcal{A}}) = \mathcal{M}\Phi(H_{\mathcal{A}}) \cap (\mathcal{M}\Phi_{+}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi^{-'}_{+}(H_{\mathcal{A}}))$. By Lemma 2.0.42 and Lemma 3.4.16 the set $\mathcal{M}\Phi(H_{\mathcal{A}}) \cap (\mathcal{M}\Phi_{+}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi^{-'}_{+}(H_{\mathcal{A}}))$ is open in the norm topology. Hence, there exists an $\epsilon > 0$ such that $(F-\tilde{\alpha}'I) \in \mathcal{M}\Phi(H_{\mathcal{A}}) \setminus \tilde{\mathcal{M}}\Phi^{-}_{+}(H_{\mathcal{A}})$ for all $\tilde{\alpha}' \in \mathcal{A}$ satisfying that $\| \tilde{\alpha} - \tilde{\alpha}' \| < \epsilon$. It follows that $\tilde{\alpha} \notin \partial\sigma^{\mathcal{A}}_{ea}(F)$, which is a contradiction. Thus, we must have that $\tilde{\alpha} \in \sigma^{\mathcal{A}}_{e\,uf}(F)$, so $\partial\sigma^{\mathcal{A}}_{ea}(F) \subseteq \sigma^{\mathcal{A}}_{e\,uf}(F)$.

□

As mentioned in [20], in a similar way as in Theorem 7.2.11, one can show that

$$\partial\sigma^{\mathcal{A}}_{ew}(F) \subseteq \partial\sigma^{\mathcal{A}}_{e\tilde{b}}(F) \subseteq \partial\sigma^{\mathcal{A}}_{eb}(F) \subseteq \partial\sigma^{\mathcal{A}}_{e\,lf}(F),$$

where

$$\sigma^{\mathcal{A}}_{e\tilde{b}}(F) = \{\alpha \in \mathcal{A} \mid (F-\alpha I) \notin \mathcal{M}\tilde{\Phi}^{+}_{-}(H_{\mathcal{A}})\}$$

and

$$\sigma^{\mathcal{A}}_{eb}(F) = \{\alpha \in \mathcal{A} \mid (F-\alpha I) \notin \mathcal{M}\Phi^{+}_{-}(H_{\mathcal{A}})\}.$$

By applying the arguments from the proof of Theorem 7.2.11 we obtain the following.

Corollary 7.2.12. *The sets $\mathcal{M}\Phi(H_{\mathcal{A}}) \setminus \tilde{\mathcal{M}}\Phi^{-}_{+}(H_{\mathcal{A}})$ and $\mathcal{M}\Phi(H_{\mathcal{A}}) \setminus \tilde{\mathcal{M}}\Phi^{+}_{-}(H_{\mathcal{A}})$ are open in the norm topology.*

Next we introduce the following spectra for $F \in B^a(H_{\mathcal{A}})$:

$$\sigma^{\mathcal{A}}_{ea'}(F) = \{\alpha \in \mathcal{A} \mid (F-\alpha I) \notin \mathcal{M}\Phi^{-'}_{+}(H_{\mathcal{A}})\},$$

$$\sigma^{\mathcal{A}}_{eb'}(F) = \{\alpha \in \mathcal{A} \mid (F-\alpha I) \notin \mathcal{M}\Phi^{+'}_{-}(H_{\mathcal{A}})\}.$$

By Remark 3.4.13 we have that

$$\mathcal{M}\Phi^{-'}_{+}(H_{\mathcal{A}}) \subseteq \mathcal{M}\Phi^{-}_{+}(H_{\mathcal{A}}) \text{ and } \mathcal{M}\Phi^{+'}_{-}(H_{\mathcal{A}}) \subseteq \mathcal{M}\Phi^{+}_{-}(H_{\mathcal{A}}).$$

Hence, we get $\sigma^{\mathcal{A}}_{ea}(F) \subseteq \sigma^{\mathcal{A}}_{ea'}(F) \subseteq \sigma^{\mathcal{A}}_{e\tilde{a}}(F)$ and $\sigma^{\mathcal{A}}_{eb}(F) \subseteq \sigma^{\mathcal{A}}_{eb'}(F) \subseteq \sigma^{\mathcal{A}}_{e\tilde{b}}(F)$.
We present the following proposition.

Proposition 7.2.13. *Let $F \in B^a(H_{\mathcal{A}})$. Then*

$$\partial\sigma_{e\tilde{a}}^{\mathcal{A}}(F) \subseteq \partial\sigma_{ea'}^{\mathcal{A}}(F) \subseteq \partial\sigma_{ea}^{\mathcal{A}}(F),$$

$$\partial\sigma_{e\tilde{b}}^{\mathcal{A}}(F) \subseteq \partial\sigma_{eb'}^{\mathcal{A}}(F) \subseteq \partial\sigma_{eb}^{\mathcal{A}}(F).$$

Proof. It suffices to show

$$\partial\sigma_{e\tilde{a}}^{\mathcal{A}}(F) \subseteq \sigma_{ea'}^{\mathcal{A}}(F),\ \partial\sigma_{ea'}^{\mathcal{A}}(F) \subseteq \sigma_{ea}^{\mathcal{A}}(F),$$

$$\partial\sigma_{e\tilde{b}}^{\mathcal{A}}(F) \subseteq \sigma_{eb'}^{\mathcal{A}}(F),\ \partial\sigma_{eb'}^{\mathcal{A}}(F) \subseteq \sigma_{eb}^{\mathcal{A}}(F).$$

Suppose that $\alpha \in \partial\sigma_{e\tilde{a}}^{\mathcal{A}}(F) \setminus \sigma_{ea'}^{\mathcal{A}}(F)$. Then

$$F - \alpha I \in {\mathcal{M}\Phi_+^-}'(H_{\mathcal{A}}) \setminus \tilde{\mathcal{M}}\Phi_+^-(H_{\mathcal{A}})$$

$$= {\mathcal{M}\Phi_+^-}'(H_{\mathcal{A}}) \setminus ({\mathcal{M}\Phi_+^-}'(H_{\mathcal{A}}) \cap \mathcal{M}\Phi(H_{\mathcal{A}}))$$

$$= {\mathcal{M}\Phi_+^-}'(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$$

$$= {\mathcal{M}\Phi_+^-}'(H_{\mathcal{A}}) \cap (\mathcal{M}\Phi_+(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})),$$

where in the first equality we apply Proposition 3.4.12 and in the last equality we apply the fact that ${\mathcal{M}\Phi_+^-}'(H_{\mathcal{A}}) \subseteq \mathcal{M}\Phi_+(H_{\mathcal{A}})$ by definition. Now, by Theorem 3.3.1 and Lemma 3.4.14, we obtain that ${\mathcal{M}\Phi_+^-}'(H_{\mathcal{A}}) \setminus \tilde{\mathcal{M}}\Phi_+^-(H_{\mathcal{A}})$ is open in the norm topology. As $F - \alpha I$ is in ${\mathcal{M}\Phi_+^-}'(H_{\mathcal{A}}) \setminus \tilde{\mathcal{M}}\Phi_+^-(H_{\mathcal{A}})$, it follows that $\alpha \notin \partial\sigma_{e\tilde{a}}^{\mathcal{A}}(F)$, which is a contradiction. Thus we must have that $\partial\sigma_{e\tilde{a}}^{\mathcal{A}}(F) \subseteq \sigma_{ea'}^{\mathcal{A}}(F)$.

Next suppose that $\alpha \in \partial\sigma_{ea'}^{\mathcal{A}}(F) \setminus \sigma_{ea}^{\mathcal{A}}(F)$. Since ${\mathcal{M}\Phi_+^-}'(H_{\mathcal{A}})$ is open by Lemma 3.4.14, we must have that $\sigma_{ea'}^{\mathcal{A}}(F)$ is closed, hence $F - \alpha I \in \mathcal{M}\Phi_+^-(H_{\mathcal{A}}) \setminus {\mathcal{M}\Phi_+^-}'(H_{\mathcal{A}})$.
Now, as ${\mathcal{M}\Phi_+^-}'(H_{\mathcal{A}}) \subseteq \mathcal{M}\Phi_+^-(H_{\mathcal{A}}) \subseteq \mathcal{M}\Phi_+(H_{\mathcal{A}})$, we get that

$$\mathcal{M}\Phi_+^-(H_{\mathcal{A}}) \setminus {\mathcal{M}\Phi_+^-}'(H_{\mathcal{A}}) = \mathcal{M}\Phi_+^-(H_{\mathcal{A}}) \cap (\mathcal{M}\Phi_+(H_{\mathcal{A}}) \setminus {\mathcal{M}\Phi_+^-}'(H_{\mathcal{A}})),$$

so by Corollary 3.4.10 and Lemma 3.4.16 we deduce that $\mathcal{M}\Phi_+^-(H_{\mathcal{A}}) \setminus {\mathcal{M}\Phi_+^-}'(H_{\mathcal{A}})$ is open in the norm topology. It follows that $\alpha \notin \partial\sigma_{ea'}^{\mathcal{A}}(F)$, which is a contradiction. We conclude then that $\partial\sigma_{ea'}^{\mathcal{A}}(F) \subseteq \sigma_{ea}^{\mathcal{A}}(F)$.
Similarly we can prove that $\partial\sigma_{e\tilde{b}}^{\mathcal{A}}(F) \subseteq \sigma_{eb'}^{\mathcal{A}}(F)$ and $\partial\sigma_{eb'}^{\mathcal{A}}(F) \subseteq \sigma_{eb}^{\mathcal{A}}(F)$. $\square$

Corollary 7.2.14. *The sets*

$${\mathcal{M}\Phi_+^-}'(H_{\mathcal{A}}) \setminus \tilde{\mathcal{M}}\Phi_+^-(H_{\mathcal{A}}),\ \mathcal{M}\Phi_+^-(H_{\mathcal{A}}) \setminus {\mathcal{M}\Phi_+^-}'(H_{\mathcal{A}}),\ {\mathcal{M}\Phi_-^+}'(H_{\mathcal{A}}) \setminus {\tilde{\mathcal{M}}\Phi_-^+}{}'(H_{\mathcal{A}})$$

and $\mathcal{M}\Phi_-^+(H_{\mathcal{A}}) \setminus {\mathcal{M}\Phi_-^+}'(H_{\mathcal{A}})$ are open.

Proposition 7.2.15. *Let M be a Hilbert C^*-module over a unital C^*-algebra $\mathcal{A}$ and $F \in B(M)$. If $K_0(\mathcal{A})$ satisfies the cancellation property, then*

$$\partial\sigma_{ea}^{\mathcal{A}}(F) \subseteq \partial\sigma_{e\,uf}^{\mathcal{A}}(F) \text{ and } \partial\sigma_{eb}^{\mathcal{A}}(F) \subseteq \partial\sigma_{e\,lf}^{\mathcal{A}}(F).$$

Proof. It suffices to show that $\partial\sigma_{ea}^{\mathcal{A}}(F) \subseteq \sigma_{e\,uf}^{\mathcal{A}}(F)$. To this end, assume that $\tilde{\alpha} \in \partial\sigma_{ea}^{\mathcal{A}}(F)$ and that $\tilde{\alpha} \notin \sigma_{e\,uf}^{\mathcal{A}}(F)$. Since Theorem 3.3.1 and Lemma 3.4.9 hold in the case of arbitrary Hilbert C^*-modules, it follows that Corollary 3.4.10 also remains valid in the case of arbitrary Hilbert C^*-modules, hence $\mathcal{M}\Phi_+^-(M)$ is open. Therefore we must have $(F - \tilde{\alpha}I) \in \mathcal{M}\Phi_+(M)$

and $(F-\tilde{\alpha}I) \notin \mathcal{M}\Phi_+^-(M)$. This means by the definitions of $\mathcal{M}\Phi_+^-(M)$ and $\mathcal{M}\Phi_+(M)$ that $(F-\tilde{\alpha}I) \in \mathcal{M}\Phi(M)$ and that given any decomposition

$$M = M_1 \tilde{\oplus} N_1 \xrightarrow{F-\tilde{\alpha}I} M_2 \tilde{\oplus} N_2 = M$$

with respect to which $(F-\tilde{\alpha}I)$ has the matrix $\begin{bmatrix} (F-\alpha I)_1 & 0 \\ 0 & (F-\tilde{\alpha}I)_4 \end{bmatrix}$, where $(F-\tilde{\alpha}I)_1$ is an isomorphism and N_1, N_2 are finitely generated, then N_1 is not isomorphic to a closed submodule of N_2 . By the proof of Lemma 2.0.42 there exists an $\epsilon > 0$ such that if $\tilde{\alpha}' \in \mathcal{A}$ and $||\tilde{\alpha}-\tilde{\alpha}'|| < \epsilon$, then $(F-\tilde{\alpha}'I) \in \mathcal{M}\Phi(M)$ and $(F-\tilde{\alpha}'I)$ has the matrix $\begin{bmatrix} (F-\tilde{\alpha}'I)_1 & 0 \\ 0 & (F-\tilde{\alpha}'I)_4 \end{bmatrix}$ with respect to the decomposition

$$M = M_1 \tilde{\oplus} U(N_1) \xrightarrow{F-\tilde{\alpha}'I} V^{-1}(M_2) \tilde{\oplus} N_2 = M,$$

where $(F-\tilde{\alpha}'I)_1, U, V$ are isomorphisms. As N_1 is not isomorphic to a closed submodule od N_2 and U is an isomorphism from M onto M , it follows that $U(N_1)$ is not isomorphic to a closed submodule of N_2 . Now, if $(F-\tilde{\alpha}'I) \in \mathcal{M}\Phi_+^-(M)$, then we must have $(F-\tilde{\alpha}'I) \in \tilde{\mathcal{M}}\Phi_+^-(M)$, as $(F-\tilde{\alpha}'I) \in \mathcal{M}\Phi(M)$ and $\tilde{\mathcal{M}}\Phi_+^-(M) = \mathcal{M}\Phi_+^-(M) \cap \mathcal{M}\Phi(M)$ by definition. By Lemma 3.4.3, as $K_0(\mathcal{A})$ satisfies the cancellation property, we must then have that $U(N_1) \preceq N_2$, which is a contradiction. So $\partial\sigma_{ea}^{\mathcal{A}}(F) \subseteq \sigma_{e\,uf}^{\mathcal{A}}(F)$.

Similarly we can prove that $\partial\sigma_{eb}^{\mathcal{A}}(F) \subseteq \partial\sigma_{e\,lf}^{\mathcal{A}}(F)$. □

Remark 7.2.16. The proof of Proposition 7.2.15 is similar to the proof of [20, Theorem 5].

Remark 7.2.17. Notice that, except Lemma 7.2.7, the results presented in this section hold also in the setting of non-adjointable semi-$\mathcal{A}$-Fredholm operators.

Example 7.2.18. Consider the Hilbert space $L^2((0,1),\mu)$. For every $f \in C([0,1])$ or $f \in L^\infty((0,1),\mu)$ we consider the multiplication operator M_f on $L^2((0,1),\mu)$, i.e. $M_f(g) = gf$ for all $g \in L^2((0,1),\mu)$. Then M_f is well defined, bounded linear operator on $L^2((0,1),\mu)$, $\| M_f \| \leq \| f \|_\infty$, and $M_f^* = M_{\overline{f}}$. If $F \in B(L^2(0,1),\mu)$, then the operators $F - M_f$, when f runs through $C([0,1])$ or $L^\infty((0,1),\mu)$, give rise to another kind of generalized spectra of F in $C([0,1])$ or in $L^\infty((0,1),\mu)$, respectively. Many of the results presented in this chapter have their natural analogue in this setting here. However, we should notice that, since $L^2((0,1),\mu)$ is an ordinary Hilbert space, we consider now generalized spectra in $C([0,1])$ or in $L^\infty((0,1),\mu)$ induced by the corresponding subclasses of the classical semi-Fredholm operators on $L^2((0,1),\mu)$.

Chapter 8

Perturbations of generalized spectra of operators over C^*-algebras

In this chapter we are going to study perturbations of generalized spectra of operators over C^*-algebras. The first section of this chapter gives an overview of the basic results concerning perturbations of generalized spectra, that are an analogue in this setting of the results in [56], whereas in the second section of this chapter we study perturbations of generalized spectra of upper triangular operator 2×2 matrices acting on $H_{\mathcal{A}} \oplus H_{\mathcal{A}}$ and provide a generalization in this setting of the results from [7].

8.1 Basic results

First we recall the following definitions concerning perturbation classes and the radical of a Banach algebra.

Definition 8.1.1. [56, Definition 1.8.1] Let S be a subset of a Banach space $\mathcal{A}$. The perturbation class of S, denoted by $P(S)$, is the set

$$P(S) = \{a \in \mathcal{A} : a + s \in S \text{ for every } s \in S\}.$$

We assume that S satisfies the additional condition $\lambda S \subset S$ for every scalar $\lambda \neq 0$.

Definition 8.1.2. [56, Definition 1.8.7] Let $\mathcal{A}$ be a Banach algebra with the unit 1. The *(Jacobson) radical* of $\mathcal{A}$, denoted by Rad $(\mathcal{A})$, is defined as

$$\text{Rad } (\mathcal{A}) = \{x \in \mathcal{A} : r(ax) = 0 \text{ for every } a \in \mathcal{A}\}$$

$$= \{x \in \mathcal{A} : r(xa) = 0 \text{ for every } a \in \mathcal{A}\}.$$

For a Banach space X, we denote the closed ideal of compact operators on X by $C(X)$ and we let $\pi : B(X) \to B(X)/C(X)$ be the quotient map. We recall the following.

Definition 8.1.3. [56, Definition 1.8.18] The set of all operators $T \in B(X)$ satisfying $\pi(T) \in Rad(C(X))$, is the set of *inessential* operators, denoted by $I(X)$, i.e. $I(X) = \pi^{-1}(Rad(C(X)))$.

Then we set $\mathcal{M}I(H_{\mathcal{A}}) = \pi^{-1}(Rad(B^a(H_{\mathcal{A}})/\mathcal{K}^*(H_{\mathcal{A}}))$, where π stands now for the quotient map from $B^a(H_{\mathcal{A}})$ onto $B^a(H_{\mathcal{A}})/\mathcal{K}^*(H_{\mathcal{A}})$. Since

$$\mathcal{M}\Phi(H_{\mathcal{A}}) = \pi^{-1}(G(B^a(H_{\mathcal{A}})/\mathcal{K}^*(H_{\mathcal{A}})),$$

we easily obtain the following lemma as an analogue of [56, Lemma 1.8.19].

Lemma 8.1.4. *$\mathcal{M}I(H_{\mathcal{A}})$ is a closed two sided ideal in $B^a(H_{\mathcal{A}})$ and*

$$\mathcal{M}I(H_{\mathcal{A}}) = \{D \in B^a(H_{\mathcal{A}}) \mid I + DF \in \mathcal{M}\Phi(H_{\mathcal{A}})\ \forall F \in B^a(H_{\mathcal{A}})\}$$

$$= \{D \in B^a(H_{\mathcal{A}}) \mid I + DF \in \mathcal{M}\Phi(H_{\mathcal{A}})\ \forall F \in \mathcal{M}\Phi(H_{\mathcal{A}})\}$$
$$= \{D \in B^a(H_{\mathcal{A}}) \mid I + FD \in \mathcal{M}\Phi(H_{\mathcal{A}})\ \forall F \in B^a(H_{\mathcal{A}})\}$$
$$= \{D \in B^a(H_{\mathcal{A}}) \mid I + FD \in \mathcal{M}\Phi(H_{\mathcal{A}})\ \forall F \in \mathcal{M}\Phi(H_{\mathcal{A}})\}.$$

Recall that we have

$$\mathcal{M}\Phi_+(H_{\mathcal{A}}) = \pi^{-1}(G_l(B^a(H_{\mathcal{A}})/\mathcal{K}^*(H_{\mathcal{A}}))) \text{ and } \mathcal{M}\Phi_-(H_{\mathcal{A}}) = \pi^{-1}(G_r(B^a(H_{\mathcal{A}})/\mathcal{K}^*(H_{\mathcal{A}}))).$$

Hence, by the similar arguments as in the proof of [56, Lemma 1.8.20], we deduce that

$$\mathcal{M}I(H_{\mathcal{A}}) = P(\mathcal{M}\Phi(H_{\mathcal{A}})) = P(\mathcal{M}\Phi_+(H_{\mathcal{A}})) = P(\mathcal{M}\Phi_-(H_{\mathcal{A}})).$$

Therefore, we get $\mathcal{K}^*(H_{\mathcal{A}}) \subseteq \mathcal{M}I(H_{\mathcal{A}})$, as $\mathcal{M}\Phi(H_{\mathcal{A}})$, $\mathcal{M}\Phi_+(H_{\mathcal{A}})$ and $\mathcal{M}\Phi_-(H_{\mathcal{A}})$ are invariant under compact perturbations. Since $\mathcal{M}\Phi_+(H_{\mathcal{A}})$ is an open subset of $\mathcal{M}\Phi_\pm(H_{\mathcal{A}})$ and, by Remark 3.3.4, $\mathcal{M}\Phi_\pm(H_{\mathcal{A}})$ does not contain boundary points of $\mathcal{M}\Phi_+(H_{\mathcal{A}})$, from [56, Lemma 1.8.3] we deduce that $P(\mathcal{M}\Phi_\pm(H_{\mathcal{A}})) \subseteq P(\mathcal{M}\Phi_+(H_{\mathcal{A}}))$. Similarly, $P(\mathcal{M}\Phi_\pm(H_{\mathcal{A}})) \subseteq P(\mathcal{M}\Phi_-(H_{\mathcal{A}}))$. On the other hand, we obviously have that $P(\mathcal{M}\Phi_+(H_{\mathcal{A}})) \cap P(\mathcal{M}\Phi_-(H_{\mathcal{A}}))$ is included in $P(\mathcal{M}\Phi_\pm(H_{\mathcal{A}}))$. Thus, $P(\mathcal{M}\Phi_\pm(H_{\mathcal{A}})) = P(\mathcal{M}\Phi_+(H_{\mathcal{A}})) \cap P(\mathcal{M}\Phi_-(H_{\mathcal{A}}))$. Those arguments are essentially the same as in the proof of [56, Corollary 1.8.21].

Next we have the following generalization of [56, Lemma 1.8.22].

Lemma 8.1.5. *a) If $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$ and $D \in P(\mathcal{M}\Phi(H_{\mathcal{A}}))$, then*

$$F + D \in \mathcal{M}\Phi_+(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}}).$$

b) If $F \in \mathcal{M}\Phi_-(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$ and $D \in P(\mathcal{M}\Phi(H_{\mathcal{A}}))$, then

$$F + D \in \mathcal{M}\Phi_-(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}}).$$

c) If $F \in \mathcal{M}\Phi(H_{\mathcal{A}})$ and $D \in P(\mathcal{M}\Phi(H_{\mathcal{A}}))$, then $D + F \in \mathcal{M}\Phi(H_{\mathcal{A}})$ and

$$\text{index } (D + F) = \text{index } F.$$

d) If $F \in \mathcal{M}\Phi_0(H_{\mathcal{A}})$ and $D \in P(\mathcal{M}\Phi(H_{\mathcal{A}}))$, then $F + D \in \mathcal{M}\Phi_0(H_{\mathcal{A}})$.

Proof. The proof is essentially the same as the proof of [56, Lemma 1.8.22]. Indeed, from [56, Lemma 1.8.2] it follows that $\lambda D \in P(\mathcal{M}\Phi(H_{\mathcal{A}}))$ for all $\lambda \in [0, 1]$. We have already noticed that

$$P(\mathcal{M}\Phi_+(H_{\mathcal{A}})) = P(\mathcal{M}\Phi_-(H_{\mathcal{A}})) = P(\mathcal{M}\Phi(H_{\mathcal{A}}))$$

Hence, by considering the map $f : [0, 1] \to \mathcal{M}\Phi_\pm(H_{\mathcal{A}})$ given by $f(t) = F + tD$ and applying Corollary 3.3.5 we deduce a), b), and c) in the lemma, whereas for the part d) we apply Corollary 3.4.17 part 6).

□

Lemma 8.1.6. *We have $P(\mathcal{M}\Phi_0(H_{\mathcal{A}})) = P(\mathcal{M}\Phi(H_{\mathcal{A}}))$.*

Proof. The proof of this lemma is exactly the same as the proof of [56, Lemma 1.8.23]. □

Since $\mathcal{K}^*(H_{\mathcal{A}}) \subseteq P(\mathcal{M}\Phi_+^{-'}(H_{\mathcal{A}})) \cap P(\mathcal{M}\Phi_-^{+'}(H_{\mathcal{A}})) \cap P(\mathcal{M}\Phi(H_{\mathcal{A}}))$, (which follows from Lemma 2.0.45 and Lemma 3.4.14), from Theorem 3.4.18, Corollary 3.4.22 and Theorem 3.4.25 we deduce the following lemma.

Lemma 8.1.7. *Let $F \in B^a(H_{\mathcal{A}})$. Then*
(1) $F \in \mathcal{M}\Phi_+^{-'}(H_{\mathcal{A}})$ if and only if there exist some $D \in M^a(H_{\mathcal{A}})$ and
$K \in P(\mathcal{M}\Phi_+^{-'}(H_{\mathcal{A}}))$ such that $F = D + K$,
(2) $F \in \mathcal{M}\Phi_-^{+'}(H_{\mathcal{A}})$ if and only if there exist some $G \in Q^a(H_{\mathcal{A}})$ and
$K \in P(\mathcal{M}\Phi_-^{+'}(H_{\mathcal{A}}))$ such that $F = G + K$,
(3)$F \in \mathcal{M}\Phi_0(H_{\mathcal{A}})$ if and only if there exists some invertible $T \in B^a(H_{\mathcal{A}})$ and some $K \in P(\mathcal{M}\Phi(H_{\mathcal{A}}))$ such that $F = T + K$.

Finally we are ready to prove the first result in this chapter regarding perturbations of generalized spectra.

Proposition 8.1.8. *Let $F \in B^a(H_{\mathcal{A}})$. Then*

$$\sigma_{ew}^{\mathcal{A}}(F) = \bigcap_{D \in \mathcal{K}^*(H_{\mathcal{A}})} \sigma^{\mathcal{A}}(F + D) = \bigcap_{D \in \mathcal{M}I(H_{\mathcal{A}})} \sigma^{\mathcal{A}}(F + D).$$

Proof. The proof is similar to the proof of [56, Theorem 2.1.3]. Indeed, from Lemma 8.1.7 it follows that $F - \alpha I \in \mathcal{M}\Phi_0(H_{\mathcal{A}})$ if and only if there exists some $K \in P(\mathcal{M}\Phi(H_{\mathcal{A}})) = \mathcal{M}I(H_{\mathcal{A}})$ such that $F - \alpha I + K$ is invertible or, equivalently, if and only if there exists some $K \in \mathcal{K}^*(H_{\mathcal{A}})$ such that $F - \alpha I + K$ is invertible, which follows from Theorem 3.4.25. □

Next we have the following results as an analogue of the results in [56, Chapter 2.4] in the setting of the generalized spectra in $\mathcal{A}$.

Lemma 8.1.9. *The operator $D \in B^a(H_{\mathcal{A}})$ satisfies the condition*

$$\sigma_{ek}^{\mathcal{A}}(F + D) = \sigma_{ek}^{\mathcal{A}}(F)$$

for every $F \in B^a(H_{\mathcal{A}})$ if and only if

$$D \in P(\mathcal{M}\Phi_+(H_{\mathcal{A}})) \cap P(\mathcal{M}\Phi_-(H_{\mathcal{A}})) = P(\mathcal{M}\Phi(H_{\mathcal{A}})).$$

Proof. The proof is similar to the proof of [56, Theorem 2.4.1]. Indeed, since $P(\mathcal{M}\Phi(H_{\mathcal{A}}))$ is a subspace by [56, Lemma 1.8.2], it follows that $-D \in P(\mathcal{M}\Phi(H_{\mathcal{A}}))$ when $D \in P(\mathcal{M}\Phi(H_{\mathcal{A}}))$. Therefore, if $\alpha \in \mathcal{A}$ and $D \in P(\mathcal{M}\Phi(H_{\mathcal{A}}))$, then $F - \alpha I \in \mathcal{M}\Phi_{\pm}(H_{\mathcal{A}})$ if and only if we have that $F + D - \alpha I \in \mathcal{M}\Phi_{\pm}(H_{\mathcal{A}})$, so $\alpha \notin \sigma_{ek}^{\mathcal{A}}(F)$ if and only if $\alpha \notin \sigma_{ek}^{\mathcal{A}}(F + D)$. On the other hand, if $\sigma_{ek}^{\mathcal{A}}(F + D) = \sigma_{ek}^{\mathcal{A}}(F)$ for every $F \in B^a(H_{\mathcal{A}})$, then, if we choose an $F \in \mathcal{M}\Phi_{\pm}(H_{\mathcal{A}})$, we get that $0 \notin \sigma_{ek}^{\mathcal{A}}(F) = \sigma_{ek}^{\mathcal{A}}(F + D)$, so $F + D \in \mathcal{M}\Phi_{\pm}(H_{\mathcal{A}})$. Therefore, $D \in P(\mathcal{M}\Phi_{\pm}(H_{\mathcal{A}}))$. Since we have from above that $P(\mathcal{M}\Phi_{\pm}(H_{\mathcal{A}})) = P(\mathcal{M}\Phi_+(H_{\mathcal{A}}) \cap P(\mathcal{M}\Phi_-(H_{\mathcal{A}}))$, we deduce the desired result. □

Similarly we can prove the following results.

Lemma 8.1.10. *The operator $D \in B^a(H_{\mathcal{A}})$ satisfies the condition*
$\sigma_{e\,uf}^{\mathcal{A}}(F + D) = \sigma_{e\,uf}^{\mathcal{A}}(F)$ for every $F \in B^a(H_{\mathcal{A}})$ if and only if $D \in P(\mathcal{M}\Phi(H_{\mathcal{A}}))$.

Lemma 8.1.11. *The operator $D \in B^a(H_{\mathcal{A}})$ satisfies the condition*
$\sigma_{e\,lf}^{\mathcal{A}}(F + D) = \sigma_{e\,lf}^{\mathcal{A}}(F)$ for every $F \in B^a(H_{\mathcal{A}})$ if and only if $D \in P(\mathcal{M}\Phi(H_{\mathcal{A}}))$.

Lemma 8.1.12. *The operator $D \in B^a(H_{\mathcal{A}})$ satisfies the condition*
$\sigma_{ef}^{\mathcal{A}}(F + D) = \sigma_{ef}^{\mathcal{A}}(F)$ for every $F \in B^a(H_{\mathcal{A}})$ if and only if $D \in P(\mathcal{M}\Phi(H_{\mathcal{A}}))$.

Lemma 8.1.13. *The operator $D \in B^a(H_{\mathcal{A}})$ satisfies the condition $\sigma_{ew}^{\mathcal{A}}(F+D) = \sigma_{ew}^{\mathcal{A}}(F)$ for every $F \in B^a(H_{\mathcal{A}})$ if and only if $D \in P(\mathcal{M}\Phi(H_{\mathcal{A}}))$.*

Corollary 8.1.14. *Let $F \in B^a(H_{\mathcal{A}})$. Then*
(1) $ms_+(F+D) = ms_+(F)$ for every $D \in P(\mathcal{M}\Phi(H_{\mathcal{A}}))$,
(2) $ms_-(F+D) = ms_-(F)$ for every $D \in P(\mathcal{M}\Phi(H_{\mathcal{A}}))$,
(3) $ms(F+D) = ms(F)$ for every $D \in P(\mathcal{M}\Phi(H_{\mathcal{A}}))$,
(4) $ms_\Phi(F+D) = ms_\Phi(F)$ for every $D \in P(\mathcal{M}\Phi(H_{\mathcal{A}}))$.

Lemma 8.1.15. *$\mathcal{M}\Phi_+(H_{\mathcal{A}})$ (respectively $\mathcal{M}\Phi_-(H_{\mathcal{A}})$) does not contain the boundary points of $\mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}})$ (respectively $\mathcal{M}\Phi_-^{+\prime}(H_{\mathcal{A}})$).*

Proof. Using Lemma 3.4.16 we can proceed in the same way as in the proof of Corollary 3.3.3. □

Corollary 8.1.16. *It holds that*

$$P(\mathcal{M}\Phi_+(H_{\mathcal{A}})) \subseteq P(\mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}})) \text{ and } P(\mathcal{M}\Phi_-(H_{\mathcal{A}})) \subseteq P(\mathcal{M}\Phi_-^{-\prime}(H_{\mathcal{A}})).$$

Proof. Since $\mathcal{M}\Phi_+(H_{\mathcal{A}})$ does not contain the boundary points of $\mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}})$, from [56, Lemma 1.8.3] the first statement of corollary follows. The proof of the second statement is similar. □

We recall the definition of $\sigma_{ea'}^{\mathcal{A}}(F)$ and $\sigma_{eb'}^{\mathcal{A}}(F)$ from Section 7.2. Moreover, for $F \in B^a(H_{\mathcal{A}})$ we set

$$\sigma_d^{\mathcal{A}}(F) := \{\alpha \in \mathcal{A} \mid F - \alpha I \text{ is not surjective }\}.$$

The next lemma follows from Theorem 3.4.18, Corollary 3.4.22 and Lemma 8.1.7.

Lemma 8.1.17. *Let $F \in B^a(H_{\mathcal{A}})$. Then*

$$\sigma_{ea'}^{\mathcal{A}}(F) = \bigcap_{D \in \mathcal{K}^*(H_{\mathcal{A}})} \sigma_a^{\mathcal{A}}(F+D) = \bigcap_{D \in P(\mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}}))} \sigma_a^{\mathcal{A}}(F+D),$$

$$\sigma_{eb'}^{\mathcal{A}}(F) = \bigcap_{D \in \mathcal{K}^*(H_{\mathcal{A}})} \sigma_d^{\mathcal{A}}(F+D) = \bigcap_{D \in P(\mathcal{M}\Phi_-^{+\prime}(H_{\mathcal{A}}))} \sigma_d^{\mathcal{A}}(F+D).$$

Lemma 8.1.18. *Let $F \in B^a(H_{\mathcal{A}})$. Then*
1) We have $\sigma_{ea'}^{\mathcal{A}}(F+D) = \sigma_{ea'}^{\mathcal{A}}(D)$ for every $D \in B^a(H_{\mathcal{A}})$ if and only if $F \in P(\mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}}))$.
2) We have $\sigma_{eb'}^{\mathcal{A}}(F+D) = \sigma_{eb'}^{\mathcal{A}}(D)$ for every $D \in B^a(H_{\mathcal{A}})$ if and only $F \in P(\mathcal{M}\Phi_-^{+\prime}(H_{\mathcal{A}}))$.

Proof. The proof is similar to the proof of Lemma 8.1.9. □

Remark 8.1.19. Observe that all the results from this section are valid also in the setting of non-adjointable operators. However, in Lemma 8.1.7 we should replace $M^a(H_{\mathcal{A}})$ by the class of bounded below operators with complementable image, whereas $Q^a(H_{\mathcal{A}})$ should be replaced by the class of surjective operators with complementable kernel. (Moreover, obviously we should replace $\mathcal{K}^*(H_{\mathcal{A}})$ by $\mathcal{K}(H_{\mathcal{A}})$).

8.2 Perturbations of generalized spectra of operator 2×2 matrices over C^*-algebras

The aim of this section is to provide a generalization of the results in [7] in the setting of semi-$\mathcal{A}$-Fredholm operators and generalized spectra in $\mathcal{A}$. Moreover, by applying our results from this section in the special case of operators on Hilbert spaces, we show that [7, Theorem 4.4] and [7, Theorem 4.6] can be simplified when Hilbert spaces (and not arbitrary Banach spaces) are considered.

In this section, for $F, C, D \in B^a(H_{\mathcal{A}})$, we will consider the operator

$$\mathbf{M}_C^{\mathcal{A}}(F, D) : H_{\mathcal{A}} \oplus H_{\mathcal{A}} \to H_{\mathcal{A}} \oplus H_{\mathcal{A}}$$

given as 2×2 operator matrix $\begin{bmatrix} F & C \\ 0 & D \end{bmatrix}$. To simplify notation, throughout this section, we will only write $\mathbf{M}_C^{\mathcal{A}}$ instead of $\mathbf{M}_C^{\mathcal{A}}(F, D)$ when $F, D \in B^a(H_{\mathcal{A}})$ are given.
Let

$$\sigma_e^{\mathcal{A}}(\mathbf{M}_C^{\mathcal{A}}) = \{\alpha \in \mathcal{A} \mid \mathbf{M}_C^{\mathcal{A}} - \alpha I \text{ is not } \mathcal{A}\text{-Fredholm } \}.$$

(We notice that this notation is different from the notation in the previous section and previous chapter. However, since the results in this section generalize the results from [7], we introduce another notation in this section which is more similar to the notation in [7]).

We have the following proposition.

Proposition 8.2.1. *[22, Proposition 3.1] For given $F, C, D \in B^a(H_{\mathcal{A}})$, one has*

$$\sigma_e^{\mathcal{A}}(\mathbf{M}_C^{\mathcal{A}}) \subseteq (\sigma_e^{\mathcal{A}}(F) \cup \sigma_e^{\mathcal{A}}(D)).$$

Proof. Observe first that

$$\mathbf{M}_C^{\mathcal{A}} - \alpha I = \begin{bmatrix} 1 & 0 \\ 0 & D - \alpha 1 \end{bmatrix} \begin{bmatrix} 1 & C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} F - \alpha 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Now, $\begin{bmatrix} 1 & C \\ 0 & 1 \end{bmatrix}$ is clearly invertible in $B^a(H_{\mathcal{A}} \oplus H_{\mathcal{A}})$ with inverse $\begin{bmatrix} 1 & -C \\ 0 & 1 \end{bmatrix}$, so it follows that $\begin{bmatrix} 1 & C \\ 0 & 1 \end{bmatrix}$ is $\mathcal{A}$-Fredholm. If in addition both $\begin{bmatrix} F - \alpha 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & D - \alpha 1 \end{bmatrix}$ are $\mathcal{A}$-Fredholm, then $\mathbf{M}_C^{\mathcal{A}} - \alpha I$ is $\mathcal{A}$-Fredholm being a composition of $\mathcal{A}$-Fredholm operators. This holds because $H_{\mathcal{A}} \oplus H_{\mathcal{A}} \cong H_{\mathcal{A}}$ by the Kasparov stabilization Theorem 2.0.13, so that we can apply Lemma 2.0.43. However, if $F - \alpha 1$ is $\mathcal{A}$-Fredholm, then, clearly, $\begin{bmatrix} F - \alpha 1 & 0 \\ 0 & 1 \end{bmatrix}$ is $\mathcal{A}$-Fredholm, and similarly, if $D - \alpha 1$ is $\mathcal{A}$-Fredholm, then $\begin{bmatrix} 1 & 0 \\ 0 & D - \alpha 1 \end{bmatrix}$ is $\mathcal{A}$-Fredholm. Thus, if both $F - \alpha 1$ and $D - \alpha 1$ are $\mathcal{A}$-Fredholm, then $\mathbf{M}_C^{\mathcal{A}} - \alpha I$ is $\mathcal{A}$-Fredholm. The proposition follows. $\square$

Theorem 8.2.2. *[22, Theorem 3.2] Let $F, D \in B^a(H_{\mathcal{A}})$. If $\mathbf{M}_C^{\mathcal{A}} \in \mathcal{M}\Phi(H_{\mathcal{A}} \oplus H_{\mathcal{A}})$ for some $C \in B^a(H_{\mathcal{A}})$, then $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}}), D \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$ and for all decompositions*

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}},$$

$$H_{\mathcal{A}} = M_1' \tilde{\oplus} N_1' \xrightarrow{D} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}},$$

with respect to which F, D *have matrices* $\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}$, $\begin{bmatrix} D_1 & 0 \\ 0 & D_4 \end{bmatrix}$, *respectively, where* F_1, D_1 *are isomorphisms, and* N_1, N_2' *are finitely generated, there exist closed finitely generated submodules* P *and* P' *of* $H_{\mathcal{A}}$ *such that* $N_2 \oplus P \cong N_1' \oplus P'$.

Proof. Again write $\mathbf{M}_C^{\mathcal{A}}$ as $\mathbf{M}_C^{\mathcal{A}} = D'C'F'$ where

$$F' = \begin{bmatrix} F & 0 \\ 0 & 1 \end{bmatrix}, C' = \begin{bmatrix} 1 & C \\ 0 & 1 \end{bmatrix}, D' = \begin{bmatrix} 1 & 0 \\ 0 & D \end{bmatrix}.$$

Since $\mathbf{M}_C^{\mathcal{A}}$ is $\mathcal{A}$-Fredholm, if

$$H_{\mathcal{A}} \oplus H_{\mathcal{A}} = M \tilde{\oplus} N \xrightarrow{\mathbf{M}_C^{\mathcal{A}}} M' \tilde{\oplus} N' = H_{\mathcal{A}} \oplus H_{\mathcal{A}}$$

is a decomposition with respect to which $\mathbf{M}_C^{\mathcal{A}}$ has the matrix $\begin{bmatrix} (\mathbf{M}_C^{\mathcal{A}})_1 & 0 \\ 0 & (\mathbf{M}_C^{\mathcal{A}})_4 \end{bmatrix}$, where $(\mathbf{M}_C^{\mathcal{A}})_1$ is an isomorphism and N, N' are finitely generated, then, by Lemma 3.5.6 and also using that C' is invertible, one may easily deduce that there exists a chain of decompositions

$$H_{\mathcal{A}} \oplus H_{\mathcal{A}} = M \tilde{\oplus} N \xrightarrow{F'} R_1 \tilde{\oplus} R_2 \xrightarrow{C'} C'(R_1) \tilde{\oplus} C'(R_2) \xrightarrow{D'} M' \tilde{\oplus} N' = H_{\mathcal{A}} \oplus H_{\mathcal{A}}$$

with respect to which F', C', D' have matrices

$$\begin{bmatrix} F_1' & 0 \\ 0 & F_4' \end{bmatrix}, \begin{bmatrix} C_1' & 0 \\ 0 & C_4' \end{bmatrix}, \begin{bmatrix} D_1' & D_2' \\ 0 & D_4' \end{bmatrix},$$

respectively, where F_1', C_1', C_4', D_1' are isomorphisms. So D' has the matrix $\begin{bmatrix} D_1' & 0 \\ 0 & D_4' \end{bmatrix}$ with respect to the decomposition

$$H_{\mathcal{A}} \oplus H_{\mathcal{A}} = WC'(R_1) \tilde{\oplus} WC'(R_2) \xrightarrow{D'} M' \tilde{\oplus} N' = H_{\mathcal{A}} \oplus H_{\mathcal{A}},$$

where W is an isomorphism. It follows from this that

$$F' \in \mathcal{M}\Phi_+(H_{\mathcal{A}} \oplus H_{\mathcal{A}}), D' \in \mathcal{M}\Phi_-(H_{\mathcal{A}} \oplus H_{\mathcal{A}}),$$

as N and N' are finitely generated Hilbert submodules of $H_{\mathcal{A}} \oplus H_{\mathcal{A}}$. Moreover, $R_2 \cong WC'(R_2)$, because WC' is an isomorphism. Since there exists an adjointable isomorphism between $H_{\mathcal{A}}$ and $H_{\mathcal{A}} \oplus H_{\mathcal{A}}$, by applying Theorem 3.1.2 and Theorem 3.1.4 it is easy to deduce that F' is left invertible and D' is right invertible in the Calkin algebra $B^a(H_{\mathcal{A}} \oplus H_{\mathcal{A}}) / \mathcal{K}^*(H_{\mathcal{A}} \oplus H_{\mathcal{A}})$. It follows from this that F is left invertible and D is right invertible in the Calkin algebra $B^a(H_{\mathcal{A}})/\mathcal{K}^*(H_{\mathcal{A}})$, hence $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$ and $D \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$ again by Theorem 3.1.2 and Theorem 3.1.4, respectively. Choose arbitrary $\mathcal{M}\Phi_+$ and $\mathcal{M}\Phi_-$-decompositions for F and D, respectively, i.e.

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}},$$

$$H_{\mathcal{A}} = M_1' \tilde{\oplus} N_1' \xrightarrow{D} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}}.$$

Then

$$H_{\mathcal{A}} \oplus H_{\mathcal{A}} = (M_1 \oplus H_{\mathcal{A}}) \tilde{\oplus} (N_1 \oplus \{0\})$$

$$\downarrow F'$$

$$H_{\mathcal{A}} \oplus H_{\mathcal{A}} = (M_2 \oplus H_{\mathcal{A}}) \tilde{\oplus} (N_2 \oplus \{0\})$$

and

$$\begin{array}{c} H_{\mathcal{A}} \oplus H_{\mathcal{A}} = (H_{\mathcal{A}} \oplus M_1')\tilde{\oplus}(\{0\} \oplus N_1') \\ \downarrow D' \\ H_{\mathcal{A}} \oplus H_{\mathcal{A}} = (H_{\mathcal{A}} \oplus M_2')\tilde{\oplus}(\{0\} \oplus N_2') \end{array}$$

are an $\mathcal{M}\Phi_+$ and an $\mathcal{M}\Phi_-$-decomposition for F' and D', respectively. Hence the decomposition

$$H_{\mathcal{A}} \oplus H_{\mathcal{A}} = M\tilde{\oplus}N \xrightarrow{F'} R_1\tilde{\oplus}R_2 = H_{\mathcal{A}} \oplus H_{\mathcal{A}}$$

and the $\mathcal{M}\Phi_+$-decomposition given above for F' are two $\mathcal{M}\Phi_+$-decompositions for F'. Again, since there exists an adjointable isomorphism between $H_{\mathcal{A}} \oplus H_{\mathcal{A}}$ and $H_{\mathcal{A}}$, we may apply Lemma 3.1.23 on the operator F' to deduce that

$$((N_2 \oplus \{0\})\tilde{\oplus}P) \cong (R_2\tilde{\oplus}\tilde{P})$$

for some finitely generated Hilbert submodules $P, \tilde{P}$ of $H_{\mathcal{A}} \oplus H_{\mathcal{A}}$. Similarly, since

$$H_{\mathcal{A}} \oplus H_{\mathcal{A}} = WC'(R_1)\tilde{\oplus}WC'(R_2) \xrightarrow{D'} M'\tilde{\oplus}N' = H_{\mathcal{A}} \oplus H_{\mathcal{A}}$$

and

$$\begin{array}{c} H_{\mathcal{A}} \oplus H_{\mathcal{A}} = (H_{\mathcal{A}} \oplus M_1')\tilde{\oplus}(\{0\} \oplus N_1') \\ \downarrow D' \\ H_{\mathcal{A}} \oplus H_{\mathcal{A}} = (H_{\mathcal{A}} \oplus M_2')\tilde{\oplus}(\{0\} \oplus N_2') \end{array}$$

are two $\mathcal{M}\Phi_-$-decompositions for D', we may by the same arguments apply Lemma 3.5.22 on the operator D' to deduce that

$$((\{0\} \oplus N_1')\tilde{\oplus}P') \cong (WC'(R_2)\tilde{\oplus}\tilde{P}')$$

for some finitely generated Hilbert submodules $P', \tilde{P}'$ of $H_{\mathcal{A}}\oplus H_{\mathcal{A}}$. Since WC' is an isomorphism, we get

$$((\{0\} \oplus N_1')\tilde{\oplus}P' \oplus \tilde{P}) \cong (WC'(R_2) \oplus \tilde{P}' \oplus \tilde{P}) \cong (R_2 \oplus \tilde{P} \oplus \tilde{P}').$$

Hence

$$((N_2 \oplus \{0\})\tilde{\oplus}P \oplus \tilde{P}') \cong (R_2 \oplus \tilde{P} \oplus \tilde{P}') \cong ((\{0\} \oplus N_1')\tilde{\oplus}P' \oplus \tilde{P}).$$

This gives $(N_2 \oplus P\tilde{\oplus}\tilde{P}') \cong (N_1' \oplus P' \oplus \tilde{P})$. (Here $\oplus$ always denotes the direct sum of modules in the sense of Example 2.0.7.) □

Remark 8.2.3. [22, Remark 3.3] We have that [7, Theorem 3.2], part (i) implies (ii) follows actually as a corollary from Theorem 8.2.2 in the case when $X = Y = H$, where H is a Hilbert space. Indeed, by Theorem 8.2.2, if $\mathbf{M}_C \in \Phi(H \oplus H)$, then $F \in \Phi_+(H)$ and $D \in \Phi_-(H)$. Hence ImF and ImD are closed, $\dim\ker F, \dim ImD^{\perp} < \infty$. Moreover, with respect to the decompositions

$$H = \ker F^{\perp} \oplus \ker F \xrightarrow{F} ImF \oplus ImF^{\perp} = H,$$

$$H = \ker D^{\perp} \oplus \ker D \xrightarrow{D} ImD \oplus ImD^{\perp} = H,$$

the operators F, D have matrices $\begin{bmatrix} F_1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix}$, respectively, where F_1, D_1 are isomorphisms.

From Theorem 8.2.2 it follows that there exist finite dimensional subspaces P and P' such that $P \oplus ImF^{\perp} \cong \ker D \oplus P'$. However, this just means that $ImF^{\perp}$ and $\ker D$ are isomorphic up to a finite dimensional subspace in the sense of [7, Definition 2.2] because in this case either both $ImF^{\perp}$ and $\ker D$ are infinite-dimensional or they are both finite dimensional.

Proposition 8.2.4. *[22, Proposition 3.4] Suppose that there exists some $C \in B^a(H_{\mathcal{A}})$ such that the inclusion $\sigma_e^{\mathcal{A}}(\mathbf{M}_C^{\mathcal{A}}) \subset \sigma_e^{\mathcal{A}}(F) \cup \sigma_e^{\mathcal{A}}(D)$ is proper. Then for any*

$$\alpha \in [\sigma_e^{\mathcal{A}}(F) \cup \sigma_e^{\mathcal{A}}(D)] \setminus \sigma_e^{\mathcal{A}}(\mathbf{M}_C^{\mathcal{A}})$$

we have that

$$\alpha \in \sigma_e^{\mathcal{A}}(F) \cap \sigma_e^{\mathcal{A}}(D).$$

Proof. Assume that

$$\alpha \in [\sigma_e^{\mathcal{A}}(F) \setminus \sigma_e^{\mathcal{A}}(D)] \setminus \sigma_e^{\mathcal{A}}(\mathbf{M}_C^{\mathcal{A}}).$$

Then $(F - \alpha 1) \notin \mathcal{M}\Phi(H_{\mathcal{A}})$ and $(D - \alpha 1) \in \mathcal{M}\Phi(H_{\mathcal{A}})$. Moreover, since $\alpha \notin \sigma_e^{\mathcal{A}}(\mathbf{M}_C^{\mathcal{A}})$, then $(\mathbf{M}_C^{\mathcal{A}} - \alpha I)$ is $\mathcal{A}$-Fredholm. From Theorem 8.2.2 it follows that $(F - \alpha 1) \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$. Since $(F - \alpha 1) \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$ and $(D - \alpha 1) \in \mathcal{M}\Phi(H_{\mathcal{A}})$, we can find decompositions

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F-\alpha 1} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}},$$

$$H_{\mathcal{A}} = M_1' \tilde{\oplus} N_1' \xrightarrow{D-\alpha 1} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}},$$

with respect to which $F - \alpha 1, D - \alpha 1$ have matrices

$$\begin{bmatrix} (F-\alpha 1)_1 & 0 \\ 0 & (F-\alpha 1)_4 \end{bmatrix}, \begin{bmatrix} (D-\alpha 1)_1 & 0 \\ 0 & (D-\alpha 1)_4 \end{bmatrix},$$

respectively, where $(F - \alpha 1)_1, (D - \alpha 1)_1$ are isomorphisms, N_1, N_1' and N_2' are finitely generated. By Theorem 8.2.2 there exist then closed finitely generated submodules P and P' such that $N_2 \oplus P \cong N_1' \oplus P'$. Since $N_1' \oplus P'$ is finitely generated, it follows that N_2 is finitely generated as well. Hence $F - \alpha 1$ is in $\mathcal{M}\Phi(H_{\mathcal{A}})$. This is a contradiction. Thus, we must have that

$$[\sigma_e^{\mathcal{A}}(F) \setminus \sigma_e^{\mathcal{A}}(D)] \setminus \sigma_e^{\mathcal{A}}(\mathbf{M}_C^{\mathcal{A}}) = \varnothing.$$

Analogously, we can prove that

$$[\sigma_e^{\mathcal{A}}(D) \setminus \sigma_e^{\mathcal{A}}(F)] \setminus \sigma_e^{\mathcal{A}}(\mathbf{M}_C^{\mathcal{A}}) = \varnothing.$$

The proposition follows. □

Theorem 8.2.5. *[22, Theorem 3.6] Let $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}}), D \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$ and suppose that there exist decompositions*

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} N_2^{\perp} \oplus N_2 = H_{\mathcal{A}},$$

$$H_{\mathcal{A}} = N_1'^{\perp} \oplus N_1' \xrightarrow{D} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}},$$

with respect to which F, D have matrices

$$\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}, \begin{bmatrix} D_1 & 0 \\ 0 & D_4 \end{bmatrix},$$

respectively, where F_1, D_1 are isomorphims and N_1, N_2' are finitely generated.
Assume also that one of the following statements hold.
a) There exists some $J \in B^a(N_2, N_1')$ such that J is an isomorphism of N_2 onto ImJ and $ImJ^{\perp}$ is finitely generated.
b) There exists some $J' \in B^a(N_1', N_2)$ such that J' is an isomorphism of N_1' onto ImJ' and $(ImJ')^{\perp}$ is finitely generated.

Then $\mathbf{M}_C^{\mathcal{A}} \in \mathcal{M}\Phi(H_{\mathcal{A}} \oplus H_{\mathcal{A}})$ for some $C \in B^a(H_{\mathcal{A}})$.

Proof. We remark that $ImJ^{\perp}$ in part a) denotes the orthogonal complement of ImJ in N_1' and $ImJ'^{\perp}$ denotes the orthogonal complement of ImJ' in N_2.
By Theorem 2.0.20, if ImJ is closed, then ImJ is indeed orthogonally complementable, so, since by the assumption a) we have $ImJ \cong N_2$, it follows that $N_1' = ImJ \oplus ImJ^{\perp}$. Similarly, by the assumption b) we get $N_2 = ImJ' \oplus ImJ'^{\perp}$.

Suppose that b) holds and consider the operator $\tilde{J}' := J'P_{N_1'}$ where $P_{N_1'}$ denotes the orthogonal projection onto N_1'. Then $\tilde{J}'$ can be considered as a bounded adjointable operator on $H_{\mathcal{A}}$ (as N_2 is orthogonally complementable in $H_{\mathcal{A}}$, so the inclusion of N_2 into $H_{\mathcal{A}}$ is adjointable). To simplify notation, we let $M_2 = N_2^{\perp}, M_1' = N_1'^{\perp}$ and $\mathbf{M}^{\mathcal{A}}_{\tilde{J}'} = \mathbf{M}_{\tilde{J}'}$. We claim then that with respect to the decomposition

$$H_{\mathcal{A}} \oplus H_{\mathcal{A}} = (M_1 \oplus H_{\mathcal{A}}) \tilde{\oplus} (N_1 \oplus \{0\})$$

$$\downarrow \mathbf{M}_{\tilde{J}'}$$

$$H_{\mathcal{A}} \oplus H_{\mathcal{A}} = ((M_2 \tilde{\oplus} ImJ') \oplus M_2') \tilde{\oplus} (ImJ'^{\perp} \oplus N_2'),$$

$\mathbf{M}_{\tilde{J}'}$ has the matrix

$$\left[\begin{array}{cc} (\mathbf{M}_{\tilde{J}'})_1 & (\mathbf{M}_{\tilde{J}'})_2 \\ (\mathbf{M}_{\tilde{J}'})_3 & (\mathbf{M}_{\tilde{J}'})_4 \end{array}\right],$$

where $(\mathbf{M}_{\tilde{J}'})_1$ is an isomorphism. To see this, observe first that

$$(\mathbf{M}_{\tilde{J}'})_1 = \sqcap_{(M_2 \tilde{\oplus} ImJ') \oplus M_2'} \mathbf{M}_{\tilde{J}'}{}_{|_{M_1 \oplus H_{\mathcal{A}}}} = \left[\begin{array}{cc} F_{|_{M_1}} & \tilde{J}' \\ 0 & D\sqcap_{M_1'} \end{array}\right],$$

where $\sqcap_{(M_2 \tilde{\oplus} ImJ') \oplus M_2'}$ denotes the projection onto $(M_2 \tilde{\oplus} ImJ') \oplus M_2'$ along $ImJ'^{\perp} \oplus N_2'$, $\sqcap_{M_1'}$ denotes the projection onto M_1' along N_1'. Here we use that $F(M_1) = M_2$ and $\sqcap_{M_2'} D = D\sqcap_{M_1'}$ where $\sqcap_{M_2'}$ stands for the projection onto M_2' along N_2'. Clearly, $(\mathbf{M}_{\tilde{J}'})_1$ is onto $(M_2 \tilde{\oplus} ImJ') \oplus M_2'$. Indeed, given $(m_2 + n_2, m_2') \in (M_2 \tilde{\oplus} ImJ') \tilde{\oplus} M_2'$ (where $m_2 \in M_2, n_2 \in ImJ' \subseteq N_2$ and $m_2' \in M_2'$), there exist some $m_1 \in M_1, n_1' \in N_1'$ and $m_1' \in M_1'$ such that $Fm_1 = m_2, J'n_1' = n_2$ and $Dm_1' = m_2'$, as $F_{|_{M_1}}$ and $D_{|_{M_1'}}$ are isomorphisms onto M_2 and M_2', respectively. Since $D \sqcap_{M_1'} (m_1' + n_1') = Dm_1' = m_2'$ and $\tilde{J}'(m_1' + n_1') = J'P_{N_1'}(m_1' + n_1') = J'n_1' = n_2$ (recall that $M_1' = N_1'^{\perp}$, so $P_{N_1'} m_1' = 0$), we get that

$$\left[\begin{array}{cc} F_{|_{M_1}} & \tilde{J}' \\ 0 & D\sqcap_{M_1'} \end{array}\right] \left[\begin{array}{c} m_1 \\ m_1' + n_1' \end{array}\right] = \left[\begin{array}{c} m_2 + n_2 \\ m_2' \end{array}\right].$$

Now, if $(\mathbf{M}_{\tilde{J}'})_1 \left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$ for some $x \in M_1, y \in H_{\mathcal{A}}$, then $D \sqcap_{M_1'} y = 0$, so $y \in N_1'$ as $D_{|_{M_1'}}$ is bounded below. Also, $Fx + \tilde{J}'y = 0$. However, since $y \in N_1'$, then $\tilde{J}'y = J'y$, so we get $Fx + J'y = 0$. Since $Fx \in M_2, J'y = N_2$ and $M_2 \cap N_2 = \{0\}$, we have $Fx = J'y = 0$. As $F_{|_{M_1}}$ and J' are bounded below, we get $x = y = 0$. So $(\mathbf{M}_{\tilde{J}'})_1$ is injective as well, thus an isomorphism.
Recall next that $N_1 \oplus \{0\}$ and $ImJ'^{\perp} \oplus N_2'$ are finitely generated. By using the procedure of diagonalization of the matrix of $\mathbf{M}_{\tilde{J}'}$ as done in the proof of Lemma 2.0.42, we obtain that $\mathbf{M}_{\tilde{J}'} \in \mathcal{M}\Phi(H_{\mathcal{A}} \oplus H_{\mathcal{A}})$.

Assume now that a) holds. Then, by the Banach open mapping theorem and Remark 2.0.21, there exists some $\iota \in B^a(ImJ, N_2)$ such that $\iota J = id_{N_2}$. Let $\widehat{\iota} = \iota P_{ImJ}$ where P_{ImJ} denotes the orthogonal projection onto ImJ (notice that ImJ is orthogonally complementable in $H_{\mathcal{A}}$ since it is orthogonally complementable in N_1' and $H_{\mathcal{A}} = N_1' \oplus N_1'^{\perp}$). Thus, we have $\widehat{\iota} \in B^a(H_{\mathcal{A}})$

and $Im\widehat{\iota} = Im(\widehat{\iota}_{|_{ImJ}}) = N_2$. Consider $\mathbf{M}_{\widehat{\iota}} = \begin{bmatrix} F & \widehat{\iota} \\ 0 & D \end{bmatrix}$. We claim that with respect to the decomposition

$$H_{\mathcal{A}} \oplus H_{\mathcal{A}} = (M_1 \oplus (M_1' \oplus ImJ))\tilde{\oplus}(N_1 \oplus ImJ^{\perp}))$$
$$\downarrow \mathbf{M}_{\widehat{\iota}}$$
$$H_{\mathcal{A}} \oplus H_{\mathcal{A}} = (H_{\mathcal{A}} \oplus M_2')\tilde{\oplus}(\{0\} \oplus N_2'),$$

$\mathbf{M}_{\widehat{\iota}}$ has the matrix $\begin{bmatrix} (\mathbf{M}_{\widehat{\iota}})_1 & (\mathbf{M}_{\widehat{\iota}})_2 \\ (\mathbf{M}_{\widehat{\iota}})_3 & (\mathbf{M}_{\widehat{\iota}})_4 \end{bmatrix}$, where $(\mathbf{M}_{\widehat{\iota}})_1$ is an isomorphism. To see this, observe again that

$(\mathbf{M}_{\widehat{\iota}})_1 = \sqcap_{(H_{\mathcal{A}} \oplus M_2')} \mathbf{M}_{\widehat{\iota}|_{M_1 \oplus (M_1' \oplus ImJ)}} = \begin{bmatrix} F_{|_{M_1}} & \widehat{\iota}_{|_{(M_1' \oplus ImJ)}} \\ 0 & D\sqcap_{M_1'} \end{bmatrix}$, so $(\mathbf{M}_{\widehat{\iota}})_1$ is obviously onto $H_{\mathcal{A}} \oplus M_2'$. Indeed, given $(x, m_2') \in H_{\mathcal{A}} \oplus M_2'$, there exist some $m_2 \in M_2$ and $n_2 \in N_2$ such that $x = m_2 + n_2$. Since $\iota J = id_{N_2}$, there exists an $n_1' \in ImJ \subseteq N_1'$ such that $\widehat{\iota} n_1' = n_2$. Moreover, we can find some $m_1 \in M_1$ and $m_1' \in M_1'$ such that $Fm_1 = m_2$ and $Dm_1' = m_2'$. Hence

$$\begin{bmatrix} F_{|_{M_1}} & \widehat{\iota}_{|_{(M_1' \oplus ImJ)}} \\ 0 & D\sqcap_{M_1'} \end{bmatrix} \begin{bmatrix} m_1 \\ m_1' + n_1' \end{bmatrix} = \begin{bmatrix} m_2 + n_2 \\ m_2' \end{bmatrix},$$

since $\widehat{\iota} m_1' = \iota P_{ImJ} m_1' = 0$.

Next, if $(\mathbf{M}_{\widehat{\iota}})_1 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for some $x \in M_1$ and $y \in M_1' \oplus ImJ$, we get that $D \sqcap_{M_1'} y = 0$, so $y \in ImJ$. Hence $\widehat{\iota} y = \iota y$, so $Fx + \widehat{\iota} y = Fx + \iota y = 0$. Since $Fx \in M_2, \iota y \in N_2, M_2 \cap N_2 = \{0\}$, we get $Fx = \iota y = 0$. As $F_{|_{M_1}}$ and ι are bounded below, we deduce that $x = y = 0$. So $(\mathbf{M}_{\widehat{\iota}})_1$ is also injective, hence an isomorphism.

Finally, we recall that $N_1 \oplus ImJ^{\perp}$ and $\{0\} \oplus N_2'$ are finitely generated, so, by the same arguments as before, we deduce that $\mathbf{M}_{\widehat{\iota}} \in \mathcal{M}\Phi(H_{\mathcal{A}} \oplus H_{\mathcal{A}})$. □

Remark 8.2.6. [22, Remark 3.8] We know from the proofs of Theorem 3.1.2 and Theorem 3.1.4, part 1) implies 2), that, since $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}}), D \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$, we can find the decompositions

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} N_2^{\perp} \oplus N_2 = H_{\mathcal{A}},$$

$$H_{\mathcal{A}} = N_1'^{\perp} \oplus N_1' \xrightarrow{D} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}},$$

with respect to which F, D have matrices

$$\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}, \begin{bmatrix} D_1 & 0 \\ 0 & D_4 \end{bmatrix},$$

respectively, where F_1, D_1 are isomorphisms and N_1, N_2' are finitely generated. However, in this theorem we have also the additional assumptions a) and b).

Remark 8.2.7. [22, Remark 3.9] We have that [7, Theorem 3.2], part (ii) implies (i) follows as a direct consequence of Theorem 8.2.5 in the case when $X = Y = H$, where H is a Hilbert space. Indeed, if $F \in \Phi_+(H), D \in \Phi_-(H)$ and in addition $\ker D$ and $ImF^{\perp}$ are isomorphic up to a finite dimensional subspace, then we may let

$$M_1 = \ker F^{\perp}, N_1 = \ker F, N_2 = ImF^{\perp}, N_1' = \ker D, M_2' = ImD, N_2' = ImD^{\perp}.$$

If $\ker D$ and $ImF^{\perp}$ are isomorphic up to a finite dimensional subspace, by [7, Definition 2.2] this means that either the condition a) or the condition b) in Theorem 8.2.5 holds. By Theorem 8.2.5 it follows then that $\mathbf{M}_C \in \Phi(H \oplus H)$.

For $F, D \in B^a(H_{\mathcal{A}})$, let $\tilde{W}(F,D)$ be the set of all $\alpha \in \mathcal{A}$ such that there exist decompositions

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F-\alpha 1} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}},$$

$$H_{\mathcal{A}} = M_1' \tilde{\oplus} N_1' \xrightarrow{D-\alpha 1} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}},$$

with respect to which $F - \alpha 1, D - \alpha 1$ have matrices

$$\begin{bmatrix} (F-\alpha 1)_1 & 0 \\ 0 & (F-\alpha 1)_4 \end{bmatrix}, \begin{bmatrix} (D-\alpha 1)_1 & 0 \\ 0 & (D-\alpha 1)_4 \end{bmatrix}, \text{ respectively,}$$

where $(F-\alpha 1)_1, (D-\alpha 1)_1$ are isomorphisms, N_1, N_2' are finitely generated submodules and such that there are no closed finitely generated submodules P and P' satisfying $N_2 \oplus P \cong N_1' \oplus P'$.

Put $W(F,D)$ to be the set of all $\alpha \in \mathcal{A}$ such that there are no decompositions

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F-\alpha 1} N_2^{\perp} \oplus N_2 = H_{\mathcal{A}},$$

$$H_{\mathcal{A}} = N_1'^{\perp} \oplus N_1' \xrightarrow{D-\alpha 1} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}},$$

with respect to which $F - \alpha 1, D - \alpha 1$ have matrices

$$\begin{bmatrix} (F-\alpha 1)_1 & 0 \\ 0 & (F-\alpha 1)_4 \end{bmatrix}, \begin{bmatrix} (D-\alpha 1)_1 & 0 \\ 0 & (D-\alpha 1)_4 \end{bmatrix}, \text{ respectively,}$$

where $(F-\alpha 1)_1, (D-\alpha 1)_1$ are isomorphisms, N_1, N_2' are finitely generated and with the property that a) or b) in the Theorem 8.2.5 hold.
Moreover, for $F \in B^a(H_{\mathcal{A}})$ we set

$$\sigma_{re}^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid (F - \alpha I) \notin \mathcal{M}\Phi_-(H_{\mathcal{A}})\},$$

$$\sigma_{le}^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid (F - \alpha I) \notin \mathcal{M}\Phi_+(H_{\mathcal{A}})\}.$$

Then we have the following corollary.

Corollary 8.2.8. *[22, Corollary 3.10] For given $F \in B^a(H_{\mathcal{A}})$ and $D \in B^a(H_{\mathcal{A}})$, we have*

$$\sigma_{le}^{\mathcal{A}}(F) \cup \sigma_{re}^{\mathcal{A}}(D) \cup \tilde{W}(F,D) \subseteq \bigcap_{C \in B^a(H_{\mathcal{A}})} \sigma_e^{\mathcal{A}}(\mathbf{M}_C^{\mathcal{A}}) \subseteq W(F,D) \cup \sigma_{le}^{\mathcal{A}}(F) \cup \sigma_{re}^{\mathcal{A}}(D).$$

Next, we shall give a description of the right generalized Fredholm spectra of $\mathbf{M}_C^{\mathcal{A}}$ in terms of the right generalized Fredholm spectra of F and D. To this end, we present the following theorem.

Theorem 8.2.9. *[22, Theorem 3.11] Suppose that $\mathbf{M}_C^{\mathcal{A}} \in \mathcal{M}\Phi_-(H_{\mathcal{A}} \oplus H_{\mathcal{A}})$ for some $C \in B^a(H_{\mathcal{A}})$. Then $D \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$ and in addition the following statement holds:*
Either $F \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$ or there exists decompositions

$$H_{\mathcal{A}} \oplus H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F'} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}} \oplus H_{\mathcal{A}},$$

$$H_{\mathcal{A}} \oplus H_{\mathcal{A}} = M_1' \tilde{\oplus} N_1' \xrightarrow{D'} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}} \oplus H_{\mathcal{A}},$$

with respect to which F', D' have the matrices $\begin{bmatrix} F_1' & 0 \\ 0 & F_4' \end{bmatrix}, \begin{bmatrix} D_1' & 0 \\ 0 & D_4' \end{bmatrix}$, respectively, where F_1', D_1' are isomorphisms, N_2' is finitely generated, N_1' is <u>not</u> finitely generated, and in addition $M_2 \cong M_1', N_2 \cong N_1'$.

Proof. If $\mathbf{M}_C^{\mathcal{A}} \in \mathcal{M}\Phi_-(H_{\mathcal{A}} \oplus H_{\mathcal{A}})$, then there exists a decomposition

$$H_{\mathcal{A}} \oplus H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{\mathbf{M}_C^{\mathcal{A}}} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}} \oplus H_{\mathcal{A}},$$

with respect to which $\mathbf{M}_C^{\mathcal{A}}$ has the matrix $\begin{bmatrix} (\mathbf{M}_C^{\mathcal{A}})_1 & 0 \\ 0 & (\mathbf{M}_C^{\mathcal{A}})_4 \end{bmatrix}$, where $(\mathbf{M}_C^{\mathcal{A}})_1$ is an isomorphism and N_2 is finitely generated. By the proof of Theorem 3.1.4 part 1) implies 2), we may assume that $M_1 = N_1^{\perp}$. Hence $F'_{|_{M_1}}$ is adjointable. Moreover, as $\mathbf{M}_C^{\mathcal{A}} = D'C'F'$, it follows that $F'(M_1) \subseteq (D'C')^{-1}(M_2)$. Since $F'_{|_{M_1}}$ can be viewed as an operator in $B^a(M_1, (D'C')^{-1}(M_2))$ because M_1 is orthogonally complementable, by Theorem 2.0.20 we have that $F'(M_1)$ is orthogonally complementable in $(D'C')^{-1}(M_2)$. By the same arguments as in Lemma 3.5.6 we deduce that there exists a chain of decompositions

$$M_1 \oplus N_1 \xrightarrow{F'} R_1 \tilde{\oplus} R_2 \xrightarrow{C'} C'(R_1) \tilde{\oplus} C'(R_2) \xrightarrow{D'} M_2 \tilde{\oplus} N_2$$

with respect to which F', C', D' have matrices $\begin{bmatrix} F'_1 & 0 \\ 0 & F'_4 \end{bmatrix}, \begin{bmatrix} C'_1 & 0 \\ 0 & C'_4 \end{bmatrix}, \begin{bmatrix} D'_1 & D'_2 \\ 0 & D'_4 \end{bmatrix}$, respectively, where F'_1, C'_1, C'_4, D'_1 are isomorphisms. Hence D' has the matrix $\begin{bmatrix} D'_1 & 0 \\ 0 & \tilde{D}'_4 \end{bmatrix}$ with respect to the decomposition

$$H_{\mathcal{A}} \oplus H_{\mathcal{A}} = WC'(R_1) \tilde{\oplus} WC'(R_2) \xrightarrow{D'} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}} \oplus H_{\mathcal{A}},$$

where W is an isomorphism. It follows that $D' \in \mathcal{M}\Phi_-(H_{\mathcal{A}} \tilde{\oplus} H_{\mathcal{A}})$ since N_2 is finitely generated. Hence $D \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$ (by the same arguments as in the proof of Theorem 8.2.2).

Next, assume that $F \notin \mathcal{M}\Phi_-(H_{\mathcal{A}})$, then $F' \notin \mathcal{M}\Phi_-(H_{\mathcal{A}} \oplus H_{\mathcal{A}})$ since any $\mathcal{M}\Phi_-$-decomposition for F induce in a natural way an $\mathcal{M}\Phi_-$-decomposition for F'. Therefore, R_2 can not be finitely generated. Now, $R_1 \cong WC'(R_1)$ and $R_2 = WC'(R_2)$. □

Remark 8.2.10. [22, Remark 3.12] In the case of ordinary Hilbert spaces, [7, Theorem 4.4] part (ii) implies (iii) follows as a corollary of Theorem 8.2.9. Indeed, suppose that $D, F \in B(H)$ (where H is a Hilbert space). If $\ker D \prec ImF^{\perp}$, this means by [7, Remark 4.4] that $\ker D$ is finite dimensional. Now, if (ii) in [7, Theorem 4.4] holds, that is $\mathbf{M}_C \in \Phi_-(H \oplus H)$ for some $C \in B(H)$, then by Theorem 8.2.9 we have that $D \in \Phi_-(H)$ and either $F \in \Phi_-(H)$ or there exist decompositions

$$H \oplus H = M_1 \tilde{\oplus} N_1 \xrightarrow{F'} M_2 \tilde{\oplus} N_2 = H \oplus H,$$

$$H \oplus H = M'_1 \tilde{\oplus} N'_1 \xrightarrow{D'} M'_2 \tilde{\oplus} N'_2 = H \oplus H,$$

which satisfy the conditions described in Theorem 8.2.9. In particular, N_2, N'_1 are infinite-dimensional, whereas N'_2 is finite dimensional. Suppose that $F \notin \Phi_-(H)$ and that the above decompositions exist. Observe that $\ker D' = \{0\} \oplus \ker D$. Hence, if $\dim \ker D < \infty$, then $\ker D'$ is finite dimensional. Since $D'_{|_{M'_1}}$ is an isomorphism, by Lemma 3.1.3 one can deduce that $\ker D' \subseteq N'_1$. Assume that $\dim \ker D (= \dim \ker D') < \infty$ and let $\tilde{N_1}'$ be the orthogonal complement of $\ker D'$ in $N_1{}'$, that is $N'_1 = \ker D' \oplus \tilde{N}'_1$. Now, since ImD' is closed as D' is in $\mathcal{M}\Phi_-(H \oplus H)$, then $D'_{|_{\tilde{N}'_1}}$ is an isomorphism by the Banach open mapping theorem. Since $\dim N'_1 = \infty$ and $\dim \ker D' < \infty$, we must have $\dim \tilde{N}'_1 = \infty$. Hence $D'(\tilde{N}'_1)$ is infinite-dimensional subspace of N'_2. This is a contradiction since $\dim N'_2$ is finite. Thus, if $F \notin \Phi_-(H)$, we must have that $\ker D$ is infinite-dimensional. Hence, we deduce, as a corollary, [7, Theorem 4.4] in the case when $X = Y = H$, where H is a Hilbert space. In this case, part $(iii)(b)$ in [7, Theorem 4.4] could be reduced to the following statement: Either $F \in \Phi_-(H)$ or $\dim \ker D = \infty$.

Theorem 8.2.11. *[22, Theorem 3.13] Let $F, D \in B^a(H_{\mathcal{A}})$. Suppose that $D \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$ and either $F \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$ or that there exist decompositions*

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} N_2^{\perp} \oplus N_2 = H_{\mathcal{A}},$$

$$H_{\mathcal{A}} = N_1'^{\perp} \oplus N_1' \xrightarrow{D} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}},$$

with respect to which F, D have the matrices $\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}$, $\begin{bmatrix} D_1 & 0 \\ 0 & D_4 \end{bmatrix}$, respectively, where F_1, D_1 are isomorphisms and N_2' is finitely generated. Assume in addition that in this case there exists also some $\iota \in B^a(N_2, N_1')$ such that ι is an isomorphism onto its image in N_1'. Then we have $\mathbf{M}_C^{\mathcal{A}} \in \mathcal{M}\Phi_-(H_{\mathcal{A}} \oplus H_{\mathcal{A}})$ for some $C \in B^a(H_{\mathcal{A}})$.

Proof. If $F \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$, then $F' \in \mathcal{M}\Phi_-(H_{\mathcal{A}} \oplus H_{\mathcal{A}})$. Also, as $D \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$, we have $D' \in \mathcal{M}\Phi_-(H_{\mathcal{A}} \oplus H_{\mathcal{A}})$, hence $\mathbf{M}_C^{\mathcal{A}} = D'C'F'$ belongs to $\mathcal{M}\Phi_-(H_{\mathcal{A}} \oplus H_{\mathcal{A}})$. All this follows by the similar arguments as in the proof of Theorem 8.2.2.

Suppose now that $F \notin \mathcal{M}\Phi_-(H_{\mathcal{A}})$ and that the second part of the assumptions in Theorem 8.2.11 holds. Then, since $Im\iota$ is closed and $\iota \in B^a(N_2, N_1')$, we have that $Im\iota$ is orthogonally complementable in N_1' by Theorem 2.0.20, that is $N_1' = Im\ \iota \oplus \tilde{N}_1'$ for some closed submodule $\tilde{N}_1'$. Hence $H_{\mathcal{A}} = Im\iota \oplus \tilde{N}_1' \oplus N_1'^{\perp}$, that is $Im\iota$ is orthogonally complementable in $H_{\mathcal{A}}$. Also, there exists some $J \in B^a(Im\iota, N_2)$ such that $J\iota = id_{N_2}, \iota J = id_{Im\iota}$. Let $P_{Im\iota}$ be the orthogonal projection onto $Im\iota$ and set $C = JP_{Im\iota}$. Then $C \in B^a(H_{\mathcal{A}})$. Indeed, since N_2 is orthogonally complementable, the inclusion of N_2 into $H_{\mathcal{A}}$ is adjointable, hence C can be viewed as an adjointable operator on $H_{\mathcal{A}}$. Moreover, with respect to the decomposition

$$H_{\mathcal{A}} \oplus H_{\mathcal{A}} = (M_1 \oplus (N_1'^{\perp} \oplus Im\iota))\tilde{\oplus}(N_1 \oplus \tilde{N}_1')$$

$$\downarrow \mathbf{M}_C^{\mathcal{A}}$$

$$H_{\mathcal{A}} \oplus H_{\mathcal{A}} = (H_{\mathcal{A}} \oplus M_2')\tilde{\oplus}(\{0\} \oplus N_2'),$$

$\mathbf{M}_C^{\mathcal{A}}$ has the matrix $\begin{bmatrix} (\mathbf{M}_C^{\mathcal{A}})_1 & (\mathbf{M}_C^{\mathcal{A}})_2 \\ (\mathbf{M}_C^{\mathcal{A}})_3 & (\mathbf{M}_C^{\mathcal{A}})_4 \end{bmatrix}$, where $(\mathbf{M}_C^{\mathcal{A}})_1$ is an isomorphism. This follows by the same arguments as in the proof of Theorem 8.2.5. Using that N_2' is finitely generated and proceeding further as in the proof of Theorem 8.2.5, we reach the desired conclusion. □

Remark 8.2.12. [22, Remark 3.14] In the case of ordinary Hilbert spaces, [7, Theorem 4.4] part (i) implies (ii) can be deduced as a corollary of Theorem 8.2.11. Indeed, if ImF is closed and $D \in \Phi_-(H)$, (which also gives that ImD is closed), then the pair of decompositions

$$H = \ker F^{\perp} \oplus \ker F \xrightarrow{F} ImF \oplus ImF^{\perp} = H,$$

$$H = \ker D^{\perp} \oplus \ker D \xrightarrow{D} ImD \oplus ImD^{\perp} = H,$$

for F and D, respectively, is one particular pair of the decompositions that satisfy the hypotheses of Theorem 8.2.11 as long we assume that $ImF^{\perp} \preceq \ker D$.

For $F, D \in B^a(H_{\mathcal{A}})$ let $R(F, D)$ be the set of all $\alpha \in \mathcal{A}$ such that there exist no decompositions

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F-\alpha 1} N_2{}^{\perp} \oplus N_2 = H_{\mathcal{A}},$$

$$H_{\mathcal{A}} = N_1'^{\perp} \oplus N_1' \xrightarrow{D-\alpha 1} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}}$$

that satisfy the hypotheses of the Theorem 8.2.11. Put $R'(F, D)$ to be the set of all $\alpha \in \mathcal{A}$ such that there exist no decompositions

$$H_{\mathcal{A}} \oplus H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F'-\alpha I} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}} \oplus H_{\mathcal{A}},$$

$$H_{\mathcal{A}} \oplus H_{\mathcal{A}} = M_1' \tilde{\oplus} N_1' \xrightarrow{D'-\alpha I} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}} \oplus H_{\mathcal{A}}$$

that satisfy the hypotheses of the Theorem 8.2.9. Then we have the following corollary.

Corollary 8.2.13. *[22, Corollary 3.15] Let $F, D \in B^a(H_{\mathcal{A}})$. Then*

$$\sigma_{re}^{\mathcal{A}}(D) \cup (\sigma_{re}^{\mathcal{A}}(F) \cap R'(F,D)) \subseteq \bigcap_{C \in B^a(H_{\mathcal{A}})} \sigma_{re}^{\mathcal{A}}(\mathbf{M}_C^{\mathcal{A}}) \subseteq \sigma_{re}^{\mathcal{A}}(D) \cup (\sigma_{re}^{\mathcal{A}}(F) \cap R(F,D)).$$

Finally, we give a description of the left generalized Fredholm spectra of $\mathbf{M}_C^{\mathcal{A}}$ in terms of the left generalized Fredholm spectra of F and D. To this end, we present the following theorem.

Theorem 8.2.14. *[22, Theorem 3.16] Suppose that $\mathbf{M}_C^{\mathcal{A}} \in \mathcal{M}\Phi_+(H_{\mathcal{A}} \oplus H_{\mathcal{A}})$ for some $C \in B^a(H_{\mathcal{A}})$. Then $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$ and either $D \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$ or there exist decompositions*

$$H_{\mathcal{A}} \oplus H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F'} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}} \oplus H_{\mathcal{A}},$$

$$H_{\mathcal{A}} \oplus H_{\mathcal{A}} = M_1' \tilde{\oplus} N_1' \xrightarrow{D'} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}} \oplus H_{\mathcal{A}},$$

with respect to which F', D' have matrices $\begin{bmatrix} F_1' & 0 \\ 0 & F_4' \end{bmatrix}$, $\begin{bmatrix} D_1' & 0 \\ 0 & D_4' \end{bmatrix}$, respectively, where F_1', D_1' are isomorphisms, $M_2 \cong M_1'$ and $N_2 \cong N_1'$, N_1 is finitely generated and N_2, N_1' are closed, but not finitely generated.

Proof. Since $\mathbf{M}_C^{\mathcal{A}} \in \mathcal{M}\Phi_+(H_{\mathcal{A}} \oplus H_{\mathcal{A}})$, there exists an $\mathcal{M}\Phi_+$-decomposition for $\mathbf{M}_C^{\mathcal{A}}$,

$$H_{\mathcal{A}} \oplus H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{\mathbf{M}_C^{\mathcal{A}}} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}} \oplus H_{\mathcal{A}},$$

so N_1 is finitely generated. By the proof of Theorem 3.1.2 part 1) implies 2), we may assume that $M_1 = N_1^{\perp}$. Hence $F'_{|_{M_1}}$ is adjointable. As in the proof of Lemma 3.5.6 and Theorem 8.2.2, we may consider a chain of decompositions

$$H_{\mathcal{A}} \oplus H_{\mathcal{A}} = M_1 \oplus N_1 \xrightarrow{F'} R_1 \tilde{\oplus} R_2 \xrightarrow{C'} C'(R_1) \tilde{\oplus} C'(R_2) \xrightarrow{D'} M_2' \tilde{\oplus} M_2' = H_{\mathcal{A}} \oplus H_{\mathcal{A}}$$

with respect to which F', C', D' have matrices $\begin{bmatrix} F_1' & 0 \\ 0 & F_4' \end{bmatrix}$, $\begin{bmatrix} C_1' & 0 \\ 0 & C_4' \end{bmatrix}$, $\begin{bmatrix} D_1' & D_2' \\ 0 & D_4' \end{bmatrix}$, respectively, where F_1', C_1', C_4', D_1' are isomorphisms. Then we can proceed in the same way as in the proof of Theorem 8.2.9. $\square$

Remark 8.2.15. [22, Remark 3.17] In the case of Hilbert spaces, the implication (ii) implies (iii) in [7, Theorem 4.6] follows as a corollary of Theorem 8.2.14. Indeed, for the implication (ii) implies $(iii)(b)$, we may proceed as follows: Since $ImF^{\circ} \cong ImF^{\perp}$ when one considers Hilbert spaces and $\ker D' \cong \ker D$, then, by [7, Remark 4.3], the relation $ImF^{\circ} \prec \ker D'$ simply means that $\dim ImF^{\perp} < \infty$ whereas $\dim \ker D = \infty$. Now, if $\dim ImF^{\perp} < \infty$, then $F \in \Phi(H)$, since $F \in \Phi_+(H)$ and $\dim ImF^{\perp} < \infty$. Hence $F' \in \Phi(H \oplus H)$, so, by Corollary 3.1.12, N_2 must be finitely generated. Thus, N_1' must be finitely generated being isomorphic to N_2. If in addition $D \notin \Phi_+(H)$, then $D' \notin \Phi_+(H \oplus H)$. By the same arguments as earlier, we have that $\ker D' \subseteq N_1'$. Since we consider Hilbert spaces now, the fact that N_1' is finitely generated means actually that N_1' is finite dimensional. Hence $\ker D'$ must be finite dimensional, so $\dim \ker D = \dim \ker D' < \infty$. This is in a contradiction to $ImF^{\perp} \prec \ker D$. So, in the case of Hilbert spaces, if $\mathbf{M}_C \in \Phi_+(H \oplus H)$, then from Theorem 8.2.14 it follows that $F \in \Phi_+(H)$ and either $D \in \Phi_+(H)$ or $ImF^{\perp}$ is infinite-dimensional.

Theorem 8.2.16. *[22, Theorem 3.18] Let $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$ and suppose that either $D \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$ or that there exist decompositions*

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} N_2^{\perp} \oplus N_2 = H_{\mathcal{A}},$$

$$H_{\mathcal{A}} = N_1'^{\perp} \oplus N_1' \xrightarrow{D} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}},$$

with respect to which F, D have matrices $\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}$, $\begin{bmatrix} D_1 & 0 \\ 0 & D_4 \end{bmatrix}$, respectively, where F_1, D_1 are isomorphisms, N_1 is finitely generated and in addition there exists some $\iota \in B^a(N_1', N_2)$ such that ι is an isomorphism onto its image. Then

$$\mathbf{M}_C^{\mathcal{A}} \in \mathcal{M}\Phi_+(H_{\mathcal{A}} \oplus H_{\mathcal{A}}),$$

for some $C \in B^a(H_{\mathcal{A}})$.

Proof. Let $C = \iota P_{N_1'}$ where $P_{N_1'}$ denotes the orthogonal projection onto N_1', then apply similar arguments as in the proof of Theorem 8.2.5 and Theorem 8.2.11. In this case, with respect to the decomposition

$$H_{\mathcal{A}} \oplus H_{\mathcal{A}} = H(M_1 \oplus H_{\mathcal{A}}) \tilde{\oplus} (N_1 \oplus \{0\})$$
$$\downarrow \mathbf{M}_C^{\mathcal{A}}$$
$$H_{\mathcal{A}} \oplus H_{\mathcal{A}} = ((N_2^{\perp} \oplus Im\iota) \oplus M_2') \tilde{\oplus} (Im\iota^{\perp} \oplus N_2'),$$

the operator $\mathbf{M}_C^{\mathcal{A}}$ has the matrix $\begin{bmatrix} (\mathbf{M}_C^{\mathcal{A}})_1 & (\mathbf{M}_C^{\mathcal{A}})_2 \\ (\mathbf{M}_C^{\mathcal{A}})_3 & (\mathbf{M}_C^{\mathcal{A}})_4 \end{bmatrix}$, where $(\mathbf{M}_C^{\mathcal{A}})_1$ is an isomorphism. □

Remark 8.2.17. [22, Remark 3.19] The implication (i) implies (ii) in [7, Theorem 4.6] in the case of Hilbert spaces could also be deduced as a corollary of Theorem 8.2.16. Indeed, if ImD is closed, then $D_{|_{\ker D^{\perp}}}$ is an isomorphism from $\ker D^{\perp}$ onto ImD. Moreover, if $F \in \Phi_+(H)$, then $F_{|_{\ker F^{\perp}}}$ is also an isomorphism from $\ker F^{\perp}$ onto ImF and $\dim \ker F < \infty$. If in addition $\ker D \preceq ImF^{\perp}$, then the pair of decompositions

$$H = \ker F^{\perp} \oplus \ker F \xrightarrow{F} ImF \oplus ImF^{\perp} = H,$$

$$H = \ker D^{\perp} \oplus \ker D \xrightarrow{D} ImD \oplus ImD^{\perp} = H,$$

is one particular pair of the decompositions that satisfy the hypotheses of Theorem 8.2.16.

For $F, D \in B^a(H_{\mathcal{A}})$ let $L'(F, D)$ be the set of all $\alpha \in \mathcal{A}$ such that there exist no decompositions

$$H_{\mathcal{A}} \oplus H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F' - \alpha I} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}} \oplus H_{\mathcal{A}},$$

$$H_{\mathcal{A}} \oplus H_{\mathcal{A}} = M_1' \tilde{\oplus} N_1' \xrightarrow{D' - \alpha I} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}} \oplus H_{\mathcal{A}},$$

for $F' - \alpha I, D' - \alpha I$, respectively, which satisfy the hypotheses of Theorem 8.2.14.
Put $L(F, D)$ to be the set of all $\alpha \in \mathcal{A}$ such that there exist no decompositions

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F - \alpha 1} N_2^{\perp} \oplus N_2 = H_{\mathcal{A}},$$

$$H_{\mathcal{A}} = N_1'^{\perp} \oplus N_1' \xrightarrow{D - \alpha 1} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}},$$

for $F - \alpha 1, D - \alpha 1$, respectively, which satisfy the hypotheses of Theorem 8.2.16.
Then we have the following corollary.

Corollary 8.2.18. *[22, Corollary 3.20] Let $F, D \in B^a(H_{\mathcal{A}})$. Then*

$$\sigma_{le}^{\mathcal{A}}(F) \cup (\sigma_{le}^{\mathcal{A}}(D) \cap L'(F,D)) \subseteq \bigcap_{C \in B^a(H_{\mathcal{A}})} \sigma_{le}^{\mathcal{A}}(\mathbf{M}_C^{\mathcal{A}}) \subseteq \sigma_{le}^{\mathcal{A}}(F) \cup (\sigma_{le}^{\mathcal{A}}(D) \cap L(F,D)).$$

Remark 8.2.19. Notice first that Lemma 3.1.23 also holds for non-adjointable operators. Next, by applying Proposition 3.5.4 instead of Theorem 3.1.2 and Theorem 3.1.4, we obtain an analogue of the results in this section in the setting of non-adjointable operators.
However, in Theorem 8.2.5 part a), if $J \in B(N_2, N_1')$ and J is not adjointable, then we should require in addition that ImJ is complementable and that the complement of ImJ is finitely generated. Similar requirement should be added in part b) in Theorem 8.2.5 in the case when the operator J' is not adjointable. In Theorem 8.2.11 and Theorem 8.2.16 in the case when ι is not adjointable, we should require then in addition that the image of ι is complementable.

Chapter 9

Compressions and generalized spectra of operators over C^*-algebras

9.1 Relations between generalized spectra of operator and its compressions

If $\alpha \in \mathcal{A}$ and $(x_1, x_2, \dots) \in N \subseteq H_{\mathcal{A}}$, where N is Hilbert submodule of $H_{\mathcal{A}}$, then we do not have in general that $(\alpha x_1, \alpha x_2, \dots) \in N$. However, if $\alpha \in Z(\mathcal{A})$ (recall that $Z(\mathcal{A})$ denotes center of $\mathcal{A}$), then $(\alpha x_1, \alpha x_2, \dots) = (x_1, x_2, \dots) \cdot \alpha \in N$. Since we are going to deal with closed submodules of $H_{\mathcal{A}}$ and the compressions of operators on $H_{\mathcal{A}}$ with respect to these submodules, we are now going to consider generalized spectra in $Z(\mathcal{A})$ instead of $\mathcal{A}$. The aim of this section is to provide a generalizations in this settings of the results in [54] and [56, Section 2.10] by Zemanek regarding the relationship between the spectra of an operator and the spectra of its compressions.

Let $\tilde{\mathcal{M}\Phi}_0(H_{\mathcal{A}})$ be the set of all $F \in B(H_{\mathcal{A}})$ satisfying that there exists a decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

with respect to which F has the matrix $\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}$, where F_1 is an isomorphism, N_1, N_2 are finitely generated and

$$N \tilde{\oplus} N_1 = N \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

for some closed submodule $N \subseteq H_{\mathcal{A}}$.
Notice that this implies that $F \in \mathcal{M}\Phi(H_{\mathcal{A}})$ and $N_1 \cong N_2$, hence $\tilde{\mathcal{M}\Phi}_0(H_{\mathcal{A}}) \subseteq \mathcal{M}\Phi_0(H_{\mathcal{A}})$.
Let $\mathcal{P}(H_{\mathcal{A}}) = \{P \in B(H_{\mathcal{A}}) \mid P \text{ is a projection and } \ker P \text{ is finitely generated}\}$ and for $F \in B(H_{\mathcal{A}})$ we put

$$\sigma_{eW}^{\mathcal{A}}(F) = \{\alpha \in Z(\mathcal{A}) \mid (F - \alpha I) \notin \tilde{\mathcal{M}\Phi}_0(H_{\mathcal{A}})\}.$$

Then we have the following theorem.

Theorem 9.1.1. *[20, Theorem 1] Let $F \in B(H_{\mathcal{A}})$. Then*

$$\sigma_{eW}^{\mathcal{A}}(F) = \cap \{\sigma^{\mathcal{A}}(PF_{|_{ImP}}) \mid P \in \mathcal{P}(H_{\mathcal{A}})\},$$

where
$\sigma^{\mathcal{A}}(PF_{|_{ImP}}) = \{\alpha \in Z(\mathcal{A}) \mid (PF - \alpha I)_{|_{ImP}}$ *is not invertible in* $B(ImP)\}$.

Proof. Let $\alpha \in Z(\mathcal{A}) \setminus (\cap\{\sigma^{\mathcal{A}}(PF_{|_{ImP}}) \mid P \in \mathcal{P}(H_{\mathcal{A}})\})$. Then there exists some $P \in \mathcal{P}(H_{\mathcal{A}})$ such that $(PF - \alpha I)_{|_{ImP}}$ is invertible in $B(ImP)$. Hence $(PF - \alpha I)_{|_{ImP}}$ is an isomorphism from ImP onto ImP, so with respect to the decomposition

$$H_{\mathcal{A}} = ImP\tilde{\oplus}\ker P \stackrel{F-\alpha I}{\longrightarrow} ImP\tilde{\oplus}\ker P = H_{\mathcal{A}},$$

$F - \alpha I$ has the matrix $\left[\begin{array}{cc}(F-\alpha I)_1 & (F-\alpha I)_2 \\ (F-\alpha I)_3 & (F-\alpha I)_4\end{array}\right]$, where $(F - \alpha I)_1 = (PF - \alpha I)_{|_{ImP}}$ is an isomorphism. Then, with respect to the decomposition

$$H_{\mathcal{A}} = ImP\tilde{\oplus}U(\ker P) \stackrel{F-\alpha I}{\longrightarrow} V^{-1}(ImP)\tilde{\oplus}\ker P = H_{\mathcal{A}},$$

$F-\alpha I$ has the matrix $\left[\begin{array}{cc}\overbrace{(F-\alpha I)_1} & 0 \\ 0 & \overbrace{(F-\alpha I)_4}\end{array}\right]$, where U, V are isomorphisms and $\overbrace{(F-\alpha I)}_1$ is an isomorphism. This follows from the proof of Lemma 2.0.42 given in [38].
Set $M_1 = ImP$, $N_1 = U(\ker P)$, $M_2 = V^{-1}(ImP), N_2 = \ker P$ and $N = ImP$. It follows that $(F - \alpha I) \in \tilde{\mathcal{M}}\Phi_0(H_{\mathcal{A}})$, so $\alpha \notin \sigma^{\mathcal{A}}_{eW}(F)$.

Conversely, suppose that $\alpha \in Z(\mathcal{A}) \backslash \sigma^{\mathcal{A}}_{eW}(F)$. Then, by definition of $\sigma^{\mathcal{A}}_{eW}(F)$ and $\tilde{\mathcal{M}}\Phi_0(H_{\mathcal{A}})$, there exists an $\widehat{\mathcal{M}\Phi}$-decomposition

$$H_{\mathcal{A}} = M_1\tilde{\oplus}N_1 \stackrel{F-\alpha I}{\longrightarrow} M_2\tilde{\oplus}N_2 = H_{\mathcal{A}}$$

for $F - \alpha I$, where $N\tilde{\oplus}N_1 = N\tilde{\oplus}N_2 = H_{\mathcal{A}}$ for some closed submodule N. In particular, N_1 and N_2 are finitely generated.
Let $\sqcap_{M_1}, \sqcap_{M_2}$ denote the projections onto M_1 along N_1 and onto M_2 along N_2, respectively. Since $F - \alpha I$ has the matrix $\left[\begin{array}{cc}(F-\alpha I)_1 & 0 \\ 0 & (F-\alpha I)_4\end{array}\right]$ with respect to the decomposition

$$H_{\mathcal{A}} = M_1\tilde{\oplus}N_1 \stackrel{F-\alpha I}{\longrightarrow} M_2\tilde{\oplus}N_2 = H_{\mathcal{A}},$$

it follows that

$$\sqcap_{M_2}(F - \alpha I)_{|_N} = (F - \alpha I)\sqcap_{M_{1|_N}} .$$

As $H_{\mathcal{A}} = N\tilde{\oplus}N_1 = M_1\tilde{\oplus}N_1$, we have that $\sqcap_{M_{1|_N}}$ is an isomorphism from N onto M_1 . Using this together with the fact that $(F-\alpha I)_{|_{M_1}}$ is an isomorphism from M_1 onto M_2, one gets that $\sqcap_{M_2}(F - \alpha I)_{|_N} = (F - \alpha I)\sqcap_{M_{1|_N}}$ is an isomorphism from N onto M_2 . Therefore, with respect to the decomposition

$$H_{\mathcal{A}} = N\tilde{\oplus}N_1 \stackrel{F-\alpha I}{\longrightarrow} M_2\tilde{\oplus}N_2 = H_{\mathcal{A}},$$

$F - \alpha I$ has the matrix $\left[\begin{array}{cc}(F-\alpha I)_1 & 0 \\ (F-\alpha I)_3 & (F-\alpha I)_4\end{array}\right]$, where $(F - \alpha I)_1$ is an isomorphism, since $(F - \alpha I)_1 = \sqcap_{M_2}(F - \alpha I)_{|_N}$. Hence $F - \alpha I$ has the matrix $\left[\begin{array}{cc}\overbrace{(F-\alpha I)_1} & 0 \\ 0 & \overbrace{(F-\alpha I)_4}\end{array}\right]$ with respect to the decomposition

$$H_{\mathcal{A}} = N\tilde{\oplus}N_1 \stackrel{F-\alpha I}{\longrightarrow} V^{-1}(M_2)\tilde{\oplus}N_2 = H_{\mathcal{A}},$$

where V and $\overbrace{(F-\alpha I)}_1$ are isomorphisms. It follows that $(F - \alpha I)_{|_N}$ is an isomorphism from N onto $V^{-1}(M_2)$. Next, since

$$H_{\mathcal{A}} = N\tilde{\oplus}N_2 = V^{-1}(M_2)\tilde{\oplus}N_2,$$

we obtain that $P_{|_{V^{-1}(M_2)}}$ is an isomorphism from $V^{-1}(M_2)$ onto N, where P denotes the projection onto N along N_2 . Hence $P(F-\alpha I)_{|_N}$ is an isomorphism from N onto N, so we have that

$$\alpha \notin \cap\{\sigma^{\mathcal{A}}(PF_{|_{ImP}}) \mid P \in \mathcal{P}(H_{\mathcal{A}})\}.$$

□

Lemma 9.1.2. *[20, Lemma 2] $\tilde{\mathcal{M}}\Phi_0(H_{\mathcal{A}})$ is open in $B(H_{\mathcal{A}})$.*

Proof. If $F \in \tilde{\mathcal{M}}\Phi_0(H_{\mathcal{A}})$, then there exists a decomposition $H_{\mathcal{A}} = M_1\tilde{\oplus}N_1 \xrightarrow{F} M_2\tilde{\oplus}N_2 = H_{\mathcal{A}}$ with respect to which F has the matrix $\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}$, where F_1 is an isomorphism, N_1, N_2 are finitely generated and $H_{\mathcal{A}} = N\tilde{\oplus}N_1 = N\tilde{\oplus}N_2$ for some closed submodule N. We may without loss of generality assume that $M_1 = N$. Indeed, as we have seen in the proof of the Theorem 9.1.1, we have that $PF_{|_N}$ is invertible in $B(N)$, where P is the projection onto N along N_2 . Then, with respect to the decomposition

$$H_{\mathcal{A}} = N\tilde{\oplus}N_1 \xrightarrow{F} N\tilde{\oplus}N_2 = H_{\mathcal{A}},$$

F has the matrix $\begin{bmatrix} \tilde{F}_1 & 0 \\ \tilde{F}_2 & F_4 \end{bmatrix}$, where $\tilde{F}_1$ is an isomorphism, so F has the matrix $\begin{bmatrix} \tilde{\tilde{F}}_1 & 0 \\ 0 & \tilde{F}_4 \end{bmatrix}$ with respect to the decomposition

$$H_{\mathcal{A}} = N\tilde{\oplus}N_1 \xrightarrow{F} \tilde{V}^{-1}(N)\tilde{\oplus}N_2 = H_{\mathcal{A}},$$

where $\tilde{\tilde{F}}_1, \tilde{V}$ are isomorphisms. Therefore, we may without loss of generality assume that $N = M_1$.
Now, by the proof of Lemma 2.0.42, there exists some $\epsilon > 0$ such that if $D \in B(H_{\mathcal{A}})$ and $||D-F|| < \epsilon$, then D has the matrix $\begin{bmatrix} D_1 & 0 \\ 0 & D_4 \end{bmatrix}$ with respect to the decomposition

$$H_{\mathcal{A}} = N\tilde{\oplus}U(N_1) \xrightarrow{D} V^{-1}\tilde{V}^{-1}(N)\tilde{\oplus}N_2 = H_{\mathcal{A}},$$

where U, V and D_1 are isomorphisms. Since $H_{\mathcal{A}} = N\tilde{\oplus}U(N_1) = N\tilde{\oplus}N_2$, it follows that $D \in \tilde{\mathcal{M}}\Phi_0(H_{\mathcal{A}})$. □

Definition 9.1.3. We put $\widehat{\mathcal{M}\Phi}_+^-(H_{\mathcal{A}})$ to be the set of all $F \in B^a(H_{\mathcal{A}})$ such that there exists a decomposition $H_{\mathcal{A}} = M_1\tilde{\oplus}N_1 \xrightarrow{F} M_2\tilde{\oplus}N_2 = H_{\mathcal{A}}$ with respect to which F has the matrix $\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}$, where F_1 is an isomorphism, N_1 is finitely generated and such that there exist closed submodules N_2', N, where $N_2' \cong N_1$, $H_{\mathcal{A}} = N\tilde{\oplus}N_1 = N\tilde{\oplus}N_2'$ and the projection onto N along N_2' is adjointable.

Then we set

$$\sigma_{e\hat{a}}^{\mathcal{A}}(F) := \{\alpha \in Z(\mathcal{A}) \mid (F-\alpha I) \notin \widehat{\mathcal{M}\Phi}_+^-(H_{\mathcal{A}})\}.$$

Theorem 9.1.4. *[20, Theorem 2] Let $F \in B^a(H_{\mathcal{A}})$. Then*

$$\sigma_{e\hat{a}}^{\mathcal{A}}(F) = \cap\{\sigma_a^{\mathcal{A}}(PF_{|_{ImP}}) \mid P \in \mathcal{P}^a(H_{\mathcal{A}})\}$$

where $\sigma_a^{\mathcal{A}}(PF_{|_{ImP}})$ is the set of all $\alpha \in Z(\mathcal{A})$ such that $(PF-\alpha I)_{|_{ImP}}$ is not bounded below on ImP and $\mathcal{P}^a(H_{\mathcal{A}}) = \mathcal{P}(H_{\mathcal{A}}) \cap B^a(H_{\mathcal{A}})$.

Proof. Suppose that $\alpha \in Z(\mathcal{A}) \setminus \sigma_a^{\mathcal{A}}(PF_{|_{ImP}})$ for some $P \in \mathcal{P}^a(H_{\mathcal{A}}), \alpha \in Z(\mathcal{A})$. Then we have that the operator $(PF-\alpha I)_{|_{ImP}}$ is bounded below on ImP, hence its image is closed. However, we also have $Im((PF-\alpha I)_{|_{ImP}}) = Im(PFP-\alpha P)$. Since $(PFP-\alpha P)$ can be viewed as an adjointable operator from $H_{\mathcal{A}}$ into ImP , from Theorem 2.0.20 it follows that $Im(PF-\alpha I)_{|_{ImP}} = Im(PFP-\alpha P)$ is orthogonally complementable in ImP. So if we let $M = Im(PFP-\alpha P)$, we get that $ImP = M \oplus M'$ for some Hilbert submodule M'. Hence $H_{\mathcal{A}} = M \tilde{\oplus} M' \tilde{\oplus} \ker P$ and $(PF-\alpha I)_{|_{ImP}}$ is an isomorphism from ImP onto M. It follows that with respect to the decomposition

$$H_{\mathcal{A}} = ImP \tilde{\oplus} \ker P \stackrel{F-\alpha I}{\longrightarrow} M \tilde{\oplus} (M' \tilde{\oplus} \ker P) = H_{\mathcal{A}},$$

$F-\alpha I$ has the matrix $\begin{bmatrix} (F-\alpha I)_1 & (F-\alpha I)_2 \\ (F-\alpha I)_3 & (F-\alpha I)_4 \end{bmatrix}$, where $(F-\alpha I)_1$ is an isomorphism. Hence, with respect to the decomposition

$$H_{\mathcal{A}} = ImP \tilde{\oplus} U(\ker P) \stackrel{F-\alpha I}{\longrightarrow} V^{-1}(M) \tilde{\oplus} (M' \tilde{\oplus} \ker P) = H_{\mathcal{A}},$$

$F-\alpha I$ has the matrix $\begin{bmatrix} \overbrace{(F-\alpha I)_1} & 0 \\ 0 & \overbrace{(F-\alpha I)_4} \end{bmatrix}$, where $\overbrace{(F-\alpha I)_1}, U, V$ are isomorphisms. Set $N = M_1 = ImP, N_1 = U(\ker P), M_2 = V^{-1}(M)$, $N_2 = M' \tilde{\oplus} \ker P$ and $N_2' = \ker P$. It follows that

$$H_{\mathcal{A}} = N \tilde{\oplus} N_1 = N \tilde{\oplus} N_2', N_2' \subseteq N_2$$

and $F-\alpha I$ has the matrix $\begin{bmatrix} \overbrace{(F-\alpha I)_1} & 0 \\ 0 & \overbrace{(F-\alpha I)_4} \end{bmatrix}$ with respect to the decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \stackrel{F-\alpha I}{\longrightarrow} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}},$$

where $\overbrace{(F-\alpha I)_1}$ is an isomorphism and $N_1 = U(\ker P)$ is finitely generated. Thus, $\alpha \notin \sigma_{e\hat{a}}^{\mathcal{A}}(F)$.

Conversely, suppose that $\alpha \in Z(\mathcal{A}) \setminus \sigma_{e\hat{a}}^{\mathcal{A}}(F)$. Then there exists a decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \stackrel{F-\alpha I}{\longrightarrow} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

with respect to which $F-\alpha I$ has the matrix $\begin{bmatrix} (F-\alpha I)_1 & 0 \\ 0 & (F-\alpha I)_4 \end{bmatrix}$, where $(F-\alpha I)_1$ is an isomorphism, N_1 is finitely generated and there exists some closed submodules N, N_2' such that $N_2' \subseteq N_2$, $N \tilde{\oplus} N_1 = N \tilde{\oplus} N_2' = H_{\mathcal{A}}$ and the projection onto N along N_2' is adjointable. As we have seen in the proof of Theorem 9.1.1 , the operator $\sqcap_{M_2}(F-\alpha I)_{|_N}$ is then an isomorphism onto M_2, where $\sqcap_{M_2}$ denotes the projection onto M_2 along N_2 . Therefore, with respect to the decomposition

$$H_{\mathcal{A}} = N \tilde{\oplus} U(N_1) \stackrel{F-\alpha I}{\longrightarrow} V^{-1}(M_2) \tilde{\oplus} N_2 = H_{\mathcal{A}},$$

$F-\alpha I$ has the matrix $\begin{bmatrix} \overbrace{(F-\alpha I)_1} & 0 \\ 0 & \overbrace{(F-\alpha I)_4} \end{bmatrix}$, where $\overbrace{(F-\alpha I)_1}, U, V$ are isomorphisms. Hence $(F-\alpha I)_{|_N}$ maps N isomorphically onto $V^{-1}(M_2)$. Since $N_2' \cong N_1$ as $N \tilde{\oplus} N_1 = N \tilde{\oplus} N_2'$, it follows that N_2' is finitely generated (as N_1 is so), hence, by Lemma 2.0.25, as N_2' is a closed submodule of N_2, we get that $N_2 = N_2' \oplus N_2''$ for some closed submodule N_2'' of N_2. So

$$H_{\mathcal{A}} = V^{-1}(M_2) \tilde{\oplus} N_2 = V^{-1}(M_2) \tilde{\oplus} N_2'' \tilde{\oplus} N_2' = N \tilde{\oplus} N_2'.$$

It follows that if P is the projection onto N along N_2', then $P_{|_{V^{-1}(M_2)\tilde{\oplus}N_2''}}$ is an isomorphism from $V^{-1}(M_2)\tilde{\oplus}N_2''$ onto N. Hence $P_{|_{V^{-1}(M_2)}}$ maps $V^{-1}(M_2)$ isomorphically onto some closed submodule of N. By using this together with the fact that $(F-\alpha I)_{|_N}$ is an isomorphism from N onto $V^{-1}(M_2)$, we obtain that $P(F-\alpha I)_{|_N}$ is bounded below. Thus, $\alpha \notin \sigma_a^{\mathcal{A}}(PF_{|_{ImP}})$. $\square$

Remark 9.1.5. [20, Remark 3] In the similar way as for $\mathcal{M}\tilde{\Phi}_0(H_{\mathcal{A}})$, one can show that $\widehat{\mathcal{M}\Phi}_+^-(H_{\mathcal{A}})$ is open in $B^a(H_{\mathcal{A}})$. Indeed, let $F \in \widehat{\mathcal{M}\Phi}_+^-(H_{\mathcal{A}})$ and choose a decomposition

$$H_{\mathcal{A}} = M_1\tilde{\oplus}N_1 \xrightarrow{F} M_2\tilde{\oplus}N_2 = H_{\mathcal{A}}$$

with respect to which F has the matrix $\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}$, where F_1 is an isomorphism. Let N be a closed submodule of $H_{\mathcal{A}}$ such that $H_{\mathcal{A}} = N\tilde{\oplus}N_1 = N\tilde{\oplus}N_2'$, $N_2' \subseteq N_2$, and the projection onto N along N_2' is adjointable. Such decomposition exists since $F \in \widehat{\mathcal{M}\Phi}_+^-(H_{\mathcal{A}})$. It is easy to see that if we let $\sqcap_{M_1}, \sqcap_{M_2}$ denote the projections onto M_1 along N_1 and onto M_2 along N_2, respectively, then $\sqcap_{M_2}F_{|_N} = F\sqcap_{M_1|_N}$ is an isomorphism onto M_2. Hence, with respect to the decomposition $H_{\mathcal{A}} = N\tilde{\oplus}N_1 \xrightarrow{F} M_2\tilde{\oplus}N_2 = H_{\mathcal{A}}$, F has the matrix $\begin{bmatrix} \tilde{F}_1 & 0 \\ \tilde{F}_2 & F_4 \end{bmatrix}$, where $\tilde{F}_1$ is an isomorphism. Then, using the technique of diagonalization as in the proof of Lemma 2.0.42, we get that F has the matrix $\begin{bmatrix} \tilde{\tilde{F}}_1 & 0 \\ 0 & F_4 \end{bmatrix}$ with respect to the decomposition

$$H_{\mathcal{A}} = N\tilde{\oplus}N_1 \xrightarrow{F} V^{-1}(M_2)\tilde{\oplus}N_2 = H_{\mathcal{A}},$$

where V and $\tilde{\tilde{F}}_1$ are isomorphisms. By the proof of Lemma 2.0.42, there exists an $\epsilon > 0$ such that if $\| F - D \| < \epsilon$, then D has the matrix $\begin{bmatrix} D_1 & 0 \\ 0 & D_4 \end{bmatrix}$ with respect to the decomposition

$$H_{\mathcal{A}} = N\tilde{\oplus}U(N_1) \xrightarrow{D} \tilde{V}^{-1}V^{-1}(M_2)\tilde{\oplus}N_2 = H_{\mathcal{A}},$$

where $U, \tilde{V}, D_1$ are isomorphisms. Since $H_{\mathcal{A}} = N\tilde{\oplus}U(N_1) = N\tilde{\oplus}N_2'$, $N_2' \subseteq N_2$ and the projection onto N along N_2' is adjointable, it follows that $D \in \widehat{\mathcal{M}\Phi}_+^-(H_{\mathcal{A}})$.

Definition 9.1.6. [20, Definition 4] We set $\widehat{\mathcal{M}\Phi}_-^+(H_{\mathcal{A}})$ to be the set of all $D \in B^a(H_{\mathcal{A}})$ such that there exists a decomposition

$$H_{\mathcal{A}} = M_1'\tilde{\oplus}N_1' \xrightarrow{D} M_2'\tilde{\oplus}N_2' = H_{\mathcal{A}}$$

with respect to which D has the matrix $\begin{bmatrix} D_1 & 0 \\ 0 & D_4 \end{bmatrix}$, where D_1 is an isomorphism, N_2' is finitely generated and such that $H_{\mathcal{A}} = M_1'\tilde{\oplus}N\tilde{\oplus}N_2'$ for some closed submodule N, where the projection onto $M_1'\tilde{\oplus}N$ along N_2' is adjointable.

Then we set

$$\sigma_{e\tilde{d}}^{\mathcal{A}}(D) = \{\alpha \in Z(\mathcal{A}) \mid (D-\alpha I) \notin \widehat{\mathcal{M}\Phi}_-^+(H_{\mathcal{A}})\}$$

and for $P \in \mathcal{P}^a(H_{\mathcal{A}})$ we put

$$\sigma_d^{\mathcal{A}}(PD_{|_{ImP}}) = \{\alpha \in Z(\mathcal{A}) \mid (PD-\alpha I)_{|_{ImP}} \text{ is not onto } ImP\}.$$

We have the following theorem.

Theorem 9.1.7. *[20, Theorem 3] Let $D \in B^a(H_{\mathcal{A}})$. Then*

$$\sigma_{e\tilde{d}}^{\mathcal{A}}(D) = \bigcap \{\sigma_d^{\mathcal{A}}(PD_{|_{ImP}}) \mid P \in \mathcal{P}^a(H_{\mathcal{A}})\}.$$

Proof. Suppose first that $\alpha \in Z(\mathcal{A}) \setminus (\cap\{\sigma_d^{\mathcal{A}}(PD_{|_{ImP}}) \mid P \in \mathcal{P}^a(H_{\mathcal{A}})\})$. Then $(PD - \alpha I)_{|_{ImP}}$ is onto ImP for some $P \in \mathcal{P}^a(H_{\mathcal{A}})$. Since P is adjointable and ImP is closed, by Theorem 2.0.20 ImP is orthogonally complementable in $H_{\mathcal{A}}$, hence $(PD - \alpha I)_{|_{ImP}}$ can be viewed as an adjointable operator from ImP onto ImP. Then, again by Theorem 2.0.20, $\ker(PD - \alpha I)_{|_{ImP}}$ is orthogonally complementable in ImP, that is $ImP = (\ker(PD - \alpha I)_{|_{ImP}}) \oplus \tilde{N}$ for some closed submodule $\tilde{N}$. The operator $PD - \alpha I$ is an isomorphism from $\tilde{N}$ onto ImP. Hence, with respect to the decomposition

$$H_{\mathcal{A}} = \tilde{N}\tilde{\oplus}((\ker(PD - \alpha I)_{|_{ImP}})\tilde{\oplus} \ker P) \stackrel{D-\alpha I}{\longrightarrow} ImP\tilde{\oplus} \ker P = H_{\mathcal{A}},$$

$D-\alpha I$ has the matrix $\left[\begin{array}{cc} (D-\alpha I)_1 & (D-\alpha I)_2 \\ (D-\alpha I)_3 & (D-\alpha I)_4 \end{array}\right]$, where $(D-\alpha I)_1$ is an isomorphism. It follows that $D - \alpha I$ has the matrix $\left[\begin{array}{cc} \overbrace{(D-\alpha I)_1} & 0 \\ 0 & \overbrace{(D-\alpha I)_4} \end{array}\right]$ with respect to the decomposition

$$H_{\mathcal{A}} = \tilde{N}\tilde{\oplus}U((\ker(PD - \alpha I)_{|_{ImP}})\tilde{\oplus} \ker P) \stackrel{D-\alpha I}{\longrightarrow} V^{-1}(ImP)\tilde{\oplus} \ker P = H_{\mathcal{A}},$$

where U, V and $\overbrace{(D-\alpha I)_1}$ are isomorphisms.
Set $N = \ker((PD - \alpha I)_{|_{ImP}})$, $M_1' = \tilde{N}, M_2' = V^{-1}(ImP), N_1' = U((\ker(PD - \alpha I)_{|_{ImP}})\tilde{\oplus} \ker P)$ and $N_2' = \ker P$. Since $ImP = N \oplus \tilde{N}$ and $\ker P$ is finitely generated, it follows that $(D-\alpha I) \in \widehat{\mathcal{M}\Phi}_-^+(H_{\mathcal{A}})$.

Conversely, suppose that $\alpha \in Z(\mathcal{A}) \setminus \sigma_{e\tilde{d}}^{\mathcal{A}}(D)$ and let

$$H_{\mathcal{A}} = M_1'\tilde{\oplus}N_1' \stackrel{D-\alpha I}{\longrightarrow} M_2'\tilde{\oplus}N_2' = H_{\mathcal{A}}$$

be decomposition with respect to which $D - \alpha I$ has the matrix

$$\left[\begin{array}{cc} (D-\alpha I)_1 & 0 \\ 0 & (D-\alpha I)_4 \end{array}\right],$$

where $(D-\alpha I)_1$ is an isomorphism, N_2' is finitely generated and such that $H_{\mathcal{A}} = M_1'\tilde{\oplus}N\tilde{\oplus}N_2'$ for some closed submodule N, where the projection onto $M_1'\tilde{\oplus}N$ along N_2' is adjointable. It follows that $P_{|_{M_2'}}$ is an isomorphism onto $M_1'\tilde{\oplus}N$, where P is the projection onto $M_1'\tilde{\oplus}N$ along N_2'. Hence $P(D-\alpha I)_{|_{M_1'}}$ is an isomorphism onto $M_1'\tilde{\oplus}N$ (since $(D-\alpha I)_{|_{M_1'}}$ is an isomorphism onto M_2'). Therefore, $P(D-\alpha I)_{|_{M_1'\tilde{\oplus}N}}$ is onto $M_1'\tilde{\oplus}N$. Now, $ImP = M_1'\tilde{\oplus}N$, so $\alpha \notin \sigma_d^{\mathcal{A}}(PD_{|_{ImP}})$. □

Remark 9.1.8. As explained in [20], similarly as for $\mathcal{M}\tilde{\Phi}_0(H_{\mathcal{A}})$ and $\widehat{\mathcal{M}\Phi}_+^-(H_{\mathcal{A}})$, one can show that $\widehat{\mathcal{M}\Phi}_-^+(H_{\mathcal{A}})$ is open. Indeed, let $D \in \widehat{\mathcal{M}\Phi}_-^+(H_{\mathcal{A}})$, choose an $\widehat{\mathcal{M}\Phi}_-^+$-decomposition for D,

$$H_{\mathcal{A}} = M_1'\tilde{\oplus}N_1' \stackrel{D}{\longrightarrow} M_2'\tilde{\oplus}N_2' = H_{\mathcal{A}}.$$

Let N be a closed submodule such that $H_{\mathcal{A}} = M_1'\tilde{\oplus}N\tilde{\oplus}N_2'$ and such that the projection onto $M_1'\tilde{\oplus}N$ along N_2' is adjointable. By the proof of Lemma 2.0.42, there exists an $\epsilon > 0$ such that if $\| G - D \| < \epsilon$ for an operator $G \in B^a(H_{\mathcal{A}})$, then G has the matrix $\begin{bmatrix} G_1 & 0 \\ 0 & G_4 \end{bmatrix}$ with respect to the decomposition

$$H_{\mathcal{A}} = M_1' \tilde{\oplus} U(N_1') \xrightarrow{G} V^{-1}(M_2') \tilde{\oplus} N_2' = H_{\mathcal{A}},$$

where U, V, G_1 are isomorphisms. It follows that $G \in \widehat{\mathcal{M}\Phi}_{-}^{+}(H_{\mathcal{A}})$.

Remark 9.1.9. If $\mathcal{A} = \mathbb{C}$, that is if $H_{\mathcal{A}} = H$ is an ordinary Hilbert space, then

$$\mathcal{M}\tilde{\Phi}_0(H) = \Phi_0(H),\ \widehat{\mathcal{M}\Phi}_{+}^{-}(H) = \Phi_{+}^{-}(H) \text{ and } \widehat{\mathcal{M}\Phi}_{-}^{+}(H) = \Phi_{-}^{+}(H).$$

In addition, observe that $\widehat{\mathcal{M}\Phi}_{+}^{-}(H_{\mathcal{A}}) \subseteq \mathcal{M}\Phi_{+}^{-\prime}(H_{\mathcal{A}})$ and $\widehat{\mathcal{M}\Phi}_{-}^{+}(H_{\mathcal{A}}) \subseteq \mathcal{M}\Phi_{-}^{+\prime}(H_{\mathcal{A}})$.

Next we consider non-adjointable operators and give a modified version of the above results in the setting of non-adjointable operators. We start with the following definition.

Definition 9.1.10. [21, Definition 14] We let $\widehat{\widehat{\mathcal{M}\Phi}}_{+}^{-}(H_{\mathcal{A}})$ be the set of all $F \in B(H_{\mathcal{A}})$ such that there exists an $\widehat{\mathcal{M}\Phi}_l$-decomposition for F

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}},$$

and closed submodules N, N_2' with the property $N_2' \subseteq N_2$ and

$$H_{\mathcal{A}} = N \tilde{\oplus} N_1 = N \tilde{\oplus} N_2'.$$

Recall that $\mathcal{P}(H_{\mathcal{A}})$ denotes the set of all projections on $H_{\mathcal{A}}$, not necessarily adjointable, with finitely generated kernel. Put

$$\sigma_{e\tilde{a}0}^{\mathcal{A}}(F) = \{\alpha \in Z(\mathcal{A}) \mid (F - \alpha I) \notin \widehat{\widehat{\mathcal{M}\Phi}}_{+}^{-}(H_{\mathcal{A}})\}.$$

Then we have the following version of Theorem 9.1.4 in the setting of non-adjointable operators on Hilbert C^*-modules.

Theorem 9.1.11. *[21, Theorem 6] For $F \in B(H_{\mathcal{A}})$ we have*

$$\sigma_{e\tilde{a}0}^{\mathcal{A}}(F) = \cap \{\sigma_{a0}^{\mathcal{A}}(PF_{|_{ImP}}) \mid P \in \mathcal{P}(H_{\mathcal{A}})\},$$

where

$$\sigma_{a0}^{\mathcal{A}}(PF_{|_{ImP}}) = \{\alpha \in Z(\mathcal{A}) \mid (PF - \alpha I)_{|ImP} \text{ is not bounded below on } ImP\}$$

$$\cup \{\alpha \in Z(\mathcal{A}) \mid Im(PFP - \alpha P) \text{ is not complementable in } ImP\}.$$

Proof. If $\alpha \in Z(\mathcal{A}) \setminus \sigma_{a0}^{\mathcal{A}}(PF_{|_{ImP}})$ for some $P \in \mathcal{P}(H_{\mathcal{A}})$, then $(PF - \alpha I)_{|_{ImP}}$ is bounded below and $Im(PFP - \alpha P)$ is complementable in ImP. Hence, we may proceed as in the proof of the Theorem 9.1.4 to deduce that $F - \alpha I \in \widehat{\widehat{\mathcal{M}\Phi}}_{+}^{-}(H_{\mathcal{A}})$.

Conversely, if $\alpha \in Z(\mathcal{A}) \setminus \sigma_{e\tilde{a}0}^{\mathcal{A}}(F)$, then we recall from the proof of Theorem 9.1.4 that we obtain the decomposition

$$H_{\mathcal{A}} = V^{-1}(M_2) \tilde{\oplus} N_2 = V^{-1}(M_2) \tilde{\oplus} N_2'' \tilde{\oplus} N_2' = N \tilde{\oplus} N_2' = N \tilde{\oplus} N_1,$$

where $N_2 = N_2' \tilde{\oplus} N_2''$, V is an isomorphism, N_1 is finitely generated and $(F - \alpha I)_{|_N}$ maps N isomorphically onto $V^{-1}(M_2)$. If we let, as in that proof, P be the projection onto N along N_2', then $P_{|_{V^{-1}(M_2) \tilde{\oplus} N_2''}}$ is an isomorphism onto N. Set $\tilde{N} = P(V^{-1}(M_2))$, $\tilde{\tilde{N}} = P(N_2'')$. Then we have that $N = \tilde{N} \tilde{\oplus} \tilde{\tilde{N}}$. Hence, $P(F - \alpha I)_{|_N}$ is an isomorphism onto $\tilde{N}$, which is complementable in $N = ImP$, so $\alpha \notin \sigma_{a0}^{\mathcal{A}}(PF_{|_{ImP}})$. $\square$

Remark 9.1.12. [21, Remark 5] It can be shown that $\widehat{\widehat{\mathcal{M}\Phi}}_+^-(H_{\mathcal{A}})$ is open.

Definition 9.1.13. [21, Definition 15] We set $\widehat{\widehat{\mathcal{M}\Phi}}_-^+(H_{\mathcal{A}})$ to be the set of all $G \in B(H_{\mathcal{A}})$ such that there exists an $\widehat{\mathcal{M}\Phi}_r$-decomposition for G

$$H_{\mathcal{A}} = M_1'\tilde{\oplus}{N_1}' \xrightarrow{G} M_2'\tilde{\oplus}N_2' = H_{\mathcal{A}}$$

and a closed submodule N with the property that $H_{\mathcal{A}} = M_1'\tilde{\oplus}N\tilde{\oplus}{N_2}'$.

Then we put

$$\sigma_{e\tilde{d}0}^{\mathcal{A}}(G) = \{\alpha \in Z(\mathcal{A}) \mid (G-\alpha I) \notin \widehat{\widehat{\mathcal{M}\Phi}}_-^+(H_{\mathcal{A}})\}$$

and obtain the following analogue of Theorem 9.1.7 in the setting of non-adjointable operators.

Theorem 9.1.14. *[21, Theorem 7] For $G \in B(H_{\mathcal{A}})$ we have*

$$\sigma_{e\tilde{d}0}^{\mathcal{A}}(G) = \cap\{\sigma_{d0}^{\mathcal{A}}(PG_{|_{ImP}}) \mid P \in \mathcal{P}(H_{\mathcal{A}})\},$$

where $\sigma_{d0}^{\mathcal{A}}(PG_{|_{ImP}}) = \{\alpha \in Z(\mathcal{A}) \mid ImP$ does not split into the decomposition $ImP = \tilde{N}\tilde{\oplus}N$ with the property that $(PG-\alpha I)_{|_{\tilde{N}}}$ is an isomorphism onto $ImP\}$.

Proof. If $\alpha \in Z(\mathcal{A}) \setminus \sigma_{d0}^{\mathcal{A}}(PG_{|_{ImP}})$ for some $P \in \mathcal{P}(H_{\mathcal{A}})$, then $ImP = \tilde{N}\tilde{\oplus}N$ for some closed submodules $\tilde{N}, N$ of ImP such that $(PG-\alpha I)_{|_{\tilde{N}}}$ is an isomorphism onto ImP. Letting N play the role of $\ker(PD-\alpha I)$ in the proof of Theorem 9.1.7, we may proceed in the same way as in that proof to conclude that $G-\alpha I \in \widehat{\widehat{\mathcal{M}\Phi}}_-^+(H_{\mathcal{A}})$.

On the other hand, if $\alpha \in Z(\mathcal{A}) \setminus \sigma_{e\tilde{d}0}^{\mathcal{A}}(G)$, then $G-\alpha I \in \widehat{\widehat{\mathcal{M}\Phi}}_-^+(H_{\mathcal{A}})$. As in the proof of Theorem 9.1.7 (and using the same notation), we may consider the projection P onto $M_1'\tilde{\oplus}N$ along N_2' and obtain that $P(G-\alpha I)_{|_{M_1'}}$ is an isomorphism onto $M_1'\tilde{\oplus}N$. □

Remark 9.1.15. [21, Remark 6] In a similar way as for $\widehat{\widehat{\mathcal{M}\Phi}}_+^-(H_{\mathcal{A}})$, one can show that $\widehat{\widehat{\mathcal{M}\Phi}}_-^+(H_{\mathcal{A}})$ is open.

9.2 Examples of semi-C^*-Weyl operators

We observe first that $\widehat{\widehat{\mathcal{M}\Phi}}_+^- \subseteq \mathcal{M}\Phi_+^{-\prime}$ and $\widehat{\widehat{\mathcal{M}\Phi}}_-^+ \subseteq \mathcal{M}\Phi_-^{+\prime}$, so $\widehat{\widehat{\mathcal{M}\Phi}}_+^-$ and $\widehat{\widehat{\mathcal{M}\Phi}}_-^+$ operators are also upper and lower semi-C^*-Weyl operators, respectively. In this section we are going to present some examples of $\widehat{\widehat{\mathcal{M}\Phi}}_+^-$ and $\widehat{\widehat{\mathcal{M}\Phi}}_-^+$ operators. In order to construct such examples we are first going to give some examples of Hilbert submodules N, N_1 and N_2 satisfying that $H_A = N\tilde{\oplus}N_1 = N\tilde{\oplus}N_2 = H_A$ and $N_1 \neq N_2$.

Example 9.2.1. Let $x = (\alpha_1, \dots, \alpha_n, \dots) \in H_{\mathcal{A}}$ and suppose that α_1 is invertible. Set $N_1 = Span_{\mathcal{A}}\{x\}$, then N_1 is closed. It is not difficult to see that $N_1 \cong L_1$ via the orthogonal projection onto L_1. Hence $H_{\mathcal{A}} = L_1^{\perp}\tilde{\oplus}N_1$. Indeed, in order to see that N_1 is closed, suppose that $\{g_m\} \subseteq N_1$ such that $g_m \to y$ for some $y \in H_{\mathcal{A}}$. Then $g_m = x \cdot \beta_m$ for each $m \in \mathbb{N}$ and some sequence $\{\beta_m\} \subseteq \mathcal{A}$. It follows that $\alpha_1\beta_m \to y_1$ in $\mathcal{A}$ as $m \to \infty$. Hence $\beta_m \to \alpha_1^{-1}y_1$ in $\mathcal{A}$ as $m \to \infty$, so we deduce that $x \cdot \beta_m \overset{m\to\infty}{\longrightarrow} x \cdot \alpha_1^{-1}y_1$, which is an element of N_1. Next, if p_1 denotes the orthogonal projection onto L_1, then, for $\beta \in \mathcal{A}$, we have $x \cdot \alpha_1^{-1}\beta \in N_1$ and $p_1(x \cdot \alpha_1^{-1}\beta) = (\beta, 0, 0, \dots)$. Furthermore, if $p_1(x \cdot \beta) = 0$ for some $\beta \in \mathcal{A}$, then $\alpha_1\beta = 0$, hence $\beta = \alpha_1^{-1}\alpha_1\beta = 0$. Thus, $x \cdot \beta = 0$. Hence $p_{1_{|_{N_1}}}$ is indeed an isomorphism onto L_1.

Example 9.2.2. Let $\mathcal{A} = L^\infty((0,1),\mu)$, choose an $x = (f_1, f_2, \dots)$ in $H_{\mathcal{A}}$ and set

$$M_n = \mid f_1 \mid^{-1} ((\frac{1}{n}, \infty)) \text{ for each } n \in \mathbb{N}.$$

We can choose $x \in H_{\mathcal{A}}$ in a such way that $\mu(M_n) \neq 0$ for some $n \in \mathbb{N}$ and that $f_{k|_{M_n^c}} = 0$ μ-a.e. for all $k \in \mathbb{N}$. If we set $N_1 = Span_{\mathcal{A}}\{x\}$ and $N_2 = Span_{\mathcal{A}}\{(\chi_{M_n,0,0,0,\dots})\}$, then N_1 and N_2 are closed. Indeed, $\mathcal{X}_{M_n}\frac{1}{f_1} \in L^\infty((0,1),\mu)$ and $\| \mathcal{X}_{M_n}\frac{1}{f_1} \|_\infty \leq n$ where $\mathcal{X}_{M_n}\frac{1}{f_1}$ denotes the function given by $\mathcal{X}_{M_n}\frac{1}{f_1}(t) = \begin{cases} \dfrac{1}{f_1(t)}, & \text{if } t \in M_n, \\ 0, & \text{else.} \end{cases}$

Now, if $x \cdot g_m \to y$ as $m \to \infty$ for some sequence $\{g_m\} \subseteq \mathcal{A}$ and $y \in H_{\mathcal{A}}$, then we must have that $f_k g_m \overset{m\to\infty}{\longrightarrow} y_k$ in $L^\infty((0,1),\mu)$ for all k. In particular, $f_1 g_m \to y_1$ as $m \to \infty$. Hence $\mathcal{X}_{M_n} g_m \to \mathcal{X}_{M_n}\frac{1}{f_1} y_1$ as $m \to \infty$. Moreover, since $f_{k|_{M_n^c}} = 0$ for all $k \geq 2$, we get that $f_k g_m = f_k \mathcal{X}_{M_n} g_m$. Therefore, $f_k g_m \overset{m\to\infty}{\longrightarrow} f_k \mathcal{X}_{M_n}\frac{1}{f_1} y_1$. Thus, $y_k = f_k \mathcal{X}_{M_n}\frac{1}{f_1} y_1$ for all k, so we get $y = x \cdot \mathcal{X}_{M_n}\frac{1}{f_1} y_1$, which is an element of N_1. Hence, N_1 is closed and it is easy to verify that also N_2 is closed.

Moreover, $N_2 \subseteq L_1$ and it is not difficult to see that $p_{1|_{N_1}}$ is an isomorphism onto N_2, where p_1 denotes the orthogonal projection onto L_1. Indeed, if $g = (g_1, g_2, \dots) \in Span_{\mathcal{A}}\{x\}$, then $g = x \cdot \alpha$ for some $\alpha \in \mathcal{A}$, so in particular $g_1 = \alpha f_1$. It follows that if $g_1 = 0$, then $\alpha_{|_{M_n}} = 0$ $\mu - a.e$, hence $\alpha f_k = 0$ for all $k \geq 2$ because $f_{k|_{M_n^c}} = 0$ for all k. Consequently, $g_k = 0$ for all k, so $g = 0$. Thus, $p_{|_{N_1}}$ is injective. Next, since $f_{k|_{M_n^c}} = 0$ for all $k \in \mathbb{N}$, then, in particular, $f_{1|_{M_n^c}} = 0$, so $Imp_{|_{N_1}} \subseteq N_2$. Finally, $p_{|_{N_1}}$ is onto N_2, because f_1 invertible on M_n since $|f_1|\chi_{M_n} \geq \frac{1}{n}\chi_{M_n}$.

Put $N = N_2^\perp$. Then $H_{\mathcal{A}} = N \tilde{\oplus} N_1 = N \oplus N_2$. Moreover, it follows from the above arguments that if $y = (g_1, g_2, \dots) \in H_{\mathcal{A}}$ such that $M_n = |g_1|^{-1}((\frac{1}{m}, \infty))$ for some $m \in \mathbb{N}$ and such that $g_{k|_{M_n^c}} = 0$ μ-a.e. for all $k \in \mathbb{N}$, then $H_{\mathcal{A}} = N \tilde{\oplus} N_1 = N \tilde{\oplus} \tilde{N}_1$, where $\tilde{N}_1 = Span_{\mathcal{A}}\{y\}$.

Example 9.2.3. Let $\mathcal{A} = B(H)$ and $x = (T_1, T_2, T_3, \dots) \in H_{\mathcal{A}}$. Suppose that ImT_1 is closed and $\ker T_1 \subseteq \ker T_k$ for all k. Let P_{ImT_1}, $P_{\ker T_1^\perp}$ denote the orthogonal projections onto ImT_1 and $\ker T_1^\perp$, respectively. Set $N_1 = Span_{\mathcal{A}}\{x\}$ and $N_2 = Span_{\mathcal{A}}\{(P_{ImT_1}, 0, 0, 0, \dots)\}$. Once again, we wish to argue that N_1 is closed. Since ImT_1 is closed by assumption, there exists an operator $T' \in B(H)$ such that $T'T_1 = P_{\ker T_1^\perp}$ and $T_1 T' = P_{ImT_1}$. If there is a sequence $\{S_n\}$ in $B(H)$ such that $x \cdot S_n \overset{n\to\infty}{\longrightarrow} y$ for some $y = (E_1, E_2, ...) \in H_{\mathcal{A}}$, then $T_k S_n \to E_k$ in $B(H)$ for all k as $n \to \infty$. Hence $P_{\ker T_1^\perp} S_n = T'T_1 S_n \overset{n\to\infty}{\longrightarrow} T'E_1$. Now, since we have $\ker T_1 \subseteq \ker T_k$ for all $k \geq 2$, then $T_k S_n = T_k P_{\ker T_k^\perp} S_n = T_k P_{\ker T_1^\perp} S_n$ for all k, n. So, for all k we get that $T_k S_n = T_k P_{\ker T_1^\perp} S_n \to T_k T' E_1$ as $n \to \infty$. Thus, $y = x \cdot T'E_1 \in Span_{\mathcal{A}}\{x\} = N_1$, hence N_1 is closed. Moreover, N_2 is closed, which is easy to verify.

Also, $p_{1|_{N_1}}$ is an isomorphism onto N_2 by the same arguments as in Example 9.2.2. (since $T_1 T' = P_{ImT_1}$ and $\ker T_1 \subseteq \ker T_k$ for all $k \in \mathbb{N}$). Set

$$N' = Span_{\mathcal{A}}\{(I - P_{ImT_1}, 0, 0, \dots)\}, \ N = L_1^\perp \oplus N'.$$

Then $H_{\mathcal{A}} = N \tilde{\oplus} N_1 = N \oplus N_2$. Moreover, it follows from the above arguments that if

$$y = (S_1, S_2, \dots) \in H_{\mathcal{A}}, \ ImS_1 = ImT_1 \text{ and } \ker S_1 \subseteq \ker S_k \text{ for all } k,$$

then $H_{\mathcal{A}} = N \tilde{\oplus} \tilde{N}_1$, where $\tilde{N}_1 = Span_{\mathcal{A}}\{y\}$.

Example 9.2.4. In general, let N' be any finitely generated Hilbert submodule of $H_{\mathcal{A}}$. Then by Theorem 2.0.34, there exists some $n \in \mathbb{N}$ and a finitely generated Hilbert sumodule P such that $H_{\mathcal{A}} = L_n^\perp \tilde{\oplus} P \tilde{\oplus} p_n(N') = L_n^\perp \tilde{\oplus} P \tilde{\oplus} N'$ (where p_n is the orthogonal projection onto L_n).

Example 9.2.5. Once we have constructed closed submodules N, N_1, N_2 such that

$$H_{\mathcal{A}} = N\tilde{\oplus}N_1 = N\tilde{\oplus}N_2$$

where N_1, N_2 are finitely generated, it is then easy to construct $\tilde{\mathcal{M}}\Phi_0$, $\widehat{\widehat{\mathcal{M}\Phi}}_+^-$, $\widehat{\widehat{\mathcal{M}\Phi}}_-^+$ operators using the previous examples of isomorphisms of $H_{\mathcal{A}}$ and examples of $\mathcal{M}\Phi_+$ and $\mathcal{M}\Phi_-$ operators. Namely, by the Dupre-Filmore Theorem 2.0.15 we have that $N \cong H_{\mathcal{A}}$.
In fact, as regards the above examples, we can construct concrete isomorphisms between N and $H_{\mathcal{A}}$. Let S denote the unilateral shift operator on $H_{\mathcal{A}}$ as given in Section 7. In Example 9.2.1 $N = L_1^{\perp}$, hence we can let S be the isomorphism of $H_{\mathcal{A}}$ onto N. Next, it is not hard to see that $N = H_{\mathcal{A}} \cdot \chi_{M_n^c} \oplus L_1^{\perp} \cdot \chi_{M_n}$ in Example 9.2.2. Let W be the operator with the matrix $\begin{bmatrix} 1 & 0 \\ 0 & S \end{bmatrix}$ with respect to the decomposition

$$H_{\mathcal{A}} = H_{\mathcal{A}} \cdot \chi_{M_n^c} \oplus H_{\mathcal{A}} \cdot \chi_{M_n} \xrightarrow{W} H_{\mathcal{A}} \cdot \chi_{M_n^c} \oplus L_1^{\perp} \cdot \chi_{M_n} = N.$$

Then W is an isomorphism of $H_{\mathcal{A}}$ onto N. Finally, in Example 9.2.3 we have $N = \tilde{N} \oplus \tilde{\tilde{N}}$, where

$$\tilde{N} = \{(P_{ImT_1^{\perp}} F_1, P_{ImT_1^{\perp}} F_2, \dots) \mid (F_1, F_2, \dots) \in H_{\mathcal{A}}\},$$

$$\tilde{\tilde{N}} = \{(0, P_{ImT_1} F_1, P_{ImT_1} F_2, \dots) \mid (F_1, F_2, \dots) \in H_{\mathcal{A}}\}.$$

Set $M = \{(P_{ImT_1} F_1, P_{ImT_1} F_2, \dots) \mid (F_1, F_2, \dots) \in H_{\mathcal{A}}\}$, then $H_{\mathcal{A}} = M \oplus \tilde{N}$. Put W to be the operator with the matrix $\begin{bmatrix} 1 & 0 \\ 0 & S \end{bmatrix}$ with respect to the decomposition

$$H_{\mathcal{A}} = \tilde{N} \oplus M \xrightarrow{W} \tilde{N} \oplus \tilde{\tilde{N}} = N$$

Then W is an isomorphism of $H_{\mathcal{A}}$ onto N.
Let now $\sqcap_1$ denote the projection onto N along N_1 and $\sqcap_2$ be the projection onto N_2 along N. If $H_{\mathcal{A}} = N\tilde{\oplus}N_1 = N\tilde{\oplus}N_2$, then ${\sqcap_2}_{|_{N_1}}$ is an isomorphism onto N_2. Hence it follows that if $\phi \in B(N_1)$ and $\varphi \in B(N_2)$, then we have that

$$WFW^{-1} \sqcap_1 + \varphi \sqcap_2 \phi(I - \sqcap_1) \in \widehat{\widehat{\mathcal{M}\Phi}}_+^-(H_{\mathcal{A}}) \text{ and } WGW^{-1} \sqcap_1 + \varphi \sqcap_2 \phi(I - \sqcap_1) \in \widehat{\widehat{\mathcal{M}\Phi}}_-^+(H_{\mathcal{A}}),$$

when $F \in M^a(H_{\mathcal{A}})$ and $G \in Q^a(H_{\mathcal{A}})$. (Recall that semi-$\mathcal{A}$-Fredholm operators presented in examples 3.7.1 - 3.7.6 are examples of operators that are either bounded below or surjective). Moreover, if U is an isomorphism of $H_{\mathcal{A}}$ (recall examples of isomorphisms from Section 3.7), then $WUW^{-1} \sqcap_1 + \varphi \sqcap_2 \phi(I - \sqcap_1)$ is an $\tilde{\mathcal{M}}\Phi_0$-operator. In order to construct some $\phi \in B(N_1)$ and $\varphi \in B(N_2)$, we just need to observe that if $T \in B(H_{\mathcal{A}})$, then $(I - \sqcap_1)T_{|_{N_1}} \in B(N_1)$ and $\sqcap_2 T_{|_{N_2}} \in B(N_2)$.

Of course, there are many other examples of $\widehat{\widehat{\mathcal{M}\Phi}}_+^-$ and $\widehat{\widehat{\mathcal{M}\Phi}}_-^+$ operators. The most simple examples are the following.

Example 9.2.6. Let S, S' be subsets of $\mathbb{N}$ such that S is finite, S' and $\mathbb{N} \setminus S'$ infinite and $S \subseteq S'$. Choose a bijection $\iota : \mathbb{N} \setminus S \to \mathbb{N} \setminus S'$ and let

$$F(e_k) = \begin{cases} e_{\iota(k)}, & \text{for } k \in \mathbb{N} \setminus S, \\ e_k, & \text{for } k \in S. \end{cases}.$$

Then $F \in \widehat{\widehat{\mathcal{M}\Phi}}_+^-(H_{\mathcal{A}})$. Similarly, if S' is finite, $S' \subseteq S$ and $S, \mathbb{N} \setminus S$ are infinite, and if we set

$$G(e_k) = \begin{cases} e_k & \text{for } k \in S', \\ e_{\iota(k)} & \text{for } k \in \mathbb{N} \setminus S, \\ 0 & \text{else,} \end{cases}$$

then $G \in \widehat{\widehat{\mathcal{M}\Phi}}_{-}^{+}(H_{\mathcal{A}})$.

At the end of this chapter we also introduce an example which shows how the proofs from Section 9.1 can be used to extend Zemarek's result in [54] in the special case of operators on infinite-dimensional Hilbert spaces.

Example 9.2.7. Let H be an infinite-dimensional Hilbert space and put $g\mathcal{M}\Phi_0(H)$ to be the set of all $F \in B(H)$ such that there exists a decomposition

$$H = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H$$

with respect to which F has the matrix $\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}$, where F_1 is an isomorphism and such that $N \tilde{\oplus} N_1 = N \tilde{\oplus} N_2 = H$ for some closed, infinite-dimensional subspace N. Put

$$g\mathcal{P}(H) = \{P \in B(H) | P \text{ is a projection and } \dim\, ImP = \infty\},$$

$$\sigma_{egW}(F) = \{\lambda \in \mathbb{C} | (F - \lambda T) \notin g\mathcal{M}\Phi_0(H_{\mathcal{A}})\}.$$

Then we have

$$\sigma_{egW}(F) = \cap \sigma \left\{ PF_{|_{ImP}} | P \in g\mathcal{P}(H) \right\}.$$

Moreover, $g\mathcal{M}\Phi_0(H)$ is open in the norm topology of $B(H)$.
Next, put $g\mathcal{M}\Phi_+^-(H)$ to be the set of all $F \in B(H)$ satisfying that there exists a decomposition

$$H = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H$$

with respect to which F has the matrix $\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}$, where F_1 is an isomorphism and such that there exists an infinite-dimensional closed subspace N and a closed subspace N_2' of N_2 with the property $H = N \tilde{\oplus} N_1 = N \tilde{\oplus} N_2'$. Then $g\mathcal{M}\Phi_+^-(H)$ is open.
If we set

$$\sigma_{ega}(F) := \{\lambda \in \mathbb{C} \mid F - \lambda I \notin g\mathcal{M}\Phi_+^-(H)\},$$

then we get

$$\sigma_{ega}(F) = \cap \sigma_a \{PF_{|_{ImP}} \mid P \in g\mathcal{P}(H)\}.$$

Finally, put $g\mathcal{M}\Phi_-^+(H)$ to be the set of all $D \in B(H)$ satisfying that there exists a decomposition

$$H = M_1' \tilde{\oplus} N_1' \xrightarrow{D} M_2' \tilde{\oplus} N_2' = H$$

with respect to which D has the matrix $\begin{bmatrix} D_1 & 0 \\ 0 & D_4 \end{bmatrix}$, where D_1 is an isomorphism, M_1' and M_2' are infinite-dimensional, and such that there exists a closed subspace N with the property that $H = M_1' \tilde{\oplus} N \tilde{\oplus} N_1'$. Then $g\mathcal{M}\Phi_-^+(H)$ is open.
If we set

$$\sigma_{egd}(D) := \{\lambda \in \mathbb{C} \mid D - \lambda I \notin g\mathcal{M}\Phi_-^+(H)\},$$

we get that

$$\sigma_{egd}(D) = \cap \sigma_d \{PD_{|_{ImP}} \mid P \in g\mathcal{P}(H)\}.$$

Chapter 10

Final remarks

The unpublished results in Section 7.1 are available on arXiv in [24], whereas the other unpublished results in the thesis are available on arXiv in [23].

At the end of this thesis we will now give an overview of the results that can be generalized from the standard module to arbitrary Hilbert C^*-modules.

Chapter 3

Since many of the results from this chapter can be generalized to arbitrary Hilbert C^*-modules, we will here just specify which of the results are valid only for the standard module case. Below is the list of those results.

Theorem 3.1.2 and Theorem 3.1.4, part 2) $\Rightarrow$ 1), (we notice that part 1) $\Rightarrow$ 2) holds for arbitrary countably generated Hilbert C^*-modules in both these theorems), Lemma 3.1.13, Corollary 3.1.14, Corollary 3.1.19, Lemma 3.1.23 and Corollary 3.2.4. The analogue of Lemma 3.1.13, Corollary 3.1.14 and Corollary 3.1.19 hold in the case of arbitrary self-dual Hilbert W^*-modules, which has been proved in several results at the end of Chapter 4. All the other results from Section 3.1 that have not been mentioned here hold in the case of arbitrary Hilbert C^*-modules.

As regards Section 3.2, except Lemma 3.2.1, all the other results in this section are constructed for the standard module case.

As regards the results from Section 3.3, all these results are valid also in the case of arbitrary Hilbert C^*-modules.

As regards Section 3.4, most of the results here have been constructed for the standard module case, so we will just mention now the results from this section which can be generalized to arbitrary Hilbert C^*- modules. These are Lemma 3.4.3, Lemma 3.4.9, Corollary 3.4.10, Proposition 3.4.12, Remark 3.4.13, Proposition 3.4.19 and Lemma 3.4.21. The first statement in Lemma 3.4.14 concerning the openess of the classes of semi-$\mathcal{A}$-Weyl operators is also valid in the case of arbitrary Hilbert C^*-modules. Lemma 3.4.7, Lemma 3.4.8, Lemma 3.4.16 and Corollary 3.4.23 hold in the case of self-dual Hilbert W^*-modules, as explained at the end of Chapter 4.

As regards Section 3.5, except Proposition 3.5.4, Corollary 3.5.5, Lemma 3.5.16, Lemma 3.5.22 and Remark 3.5.24 all the other results in this section hold in the case of arbitrary Hilbert C^*-modules, whereas most of the results in Section 3.6 are valid only in the case of the standard module.

Chapter 4

Except Corollary 4.0.2 and Proposition 4.0.3 that are valid only in the standard module case and except the results at the end of this chapter where we consider self-dual Hilbert W^*-modules, all the other results in this chapter hold in the case of arbitrary Hilbert W^*-modules.

Chapter 5

As regards Section 5.1, all the results except Proposition 5.1.18, Corollary 5.1.28 and Lemma 5.1.29 hold in the case of arbitrary Hilbert C^*-modules. As regards Section 5.2, most of the results in this section are constructed only for the standard module case. However, the exceptions are Proposition 5.2.3, Proposition 5.2.4 and Corollary 5.2.12. Moreover, Proposition 5.1.18 ca be reformulated to hold in the case of arbitrary Hilbert W^*-modules as stated in Corollary 5.2.12.

Chapter 6

All the results from this chapter are valid in the case of arbitrary Hilbert C^*-modules except Lemma 6.0.11. The reformulated version of this lemma given in Corollary 6.0.12 holds for arbitrary Hilbert W^*-modules.

Chapter 7

As explained in the beginning of Chapter 9, if we wish to extend the notion of the operator αI from the standard module to arbitrary modules over C^*-algebras, then we should only consider Hilbert modules over commutative C^*-algebras. Except from the results concerning shift operators, all the other results from Section 7.1 are therefore valid in the case of arbitrary Hilbert C^*-modules over commutative C^*-algebras.

As regards Section 7.2, all the results can be transferred to the case of arbitrary Hilbert C^*-modules over commutative C^*-algebras since the key arguments in the proofs here are actually the results from Section 3.3 and those results remain valid also in the case of arbitrary Hilbert C^*-modules. However, in some of the proofs in this section we apply also Lemma 3.4.14 which has so far only been proved for the standard module case and for the case of self-dual Hilbert W^*-modules as explained in Lemma 4.0.15. Author believes that this result can be generalized to arbitrary Hilbert C^*-modules, but this still remains as an open question for further research. Therefore, we also need slight modifications in the formulation of Lemma 7.2.9, Theorem 7.2.10 and Theorem 7.2.11 in order to hold for arbitrary Hilbert C^*-modules over commutative C^*-algebras. More precisely, if we let

$$\mathcal{M}\Phi_{0,ind}(M) = \{F \in \mathcal{M}\Phi(M) | \text{ index } F = 0\},$$

where M is an arbitrary Hilbert C^*-module, and set

$$\mathcal{M}\Phi_{0,ind}(F) = \{\alpha \in \mathcal{A} | F - \alpha I \in \mathcal{M}\Phi_{0,ind}(M)\} \text{ and } \sigma_{ew,ind}^{\mathcal{A}}(F) = \mathcal{A} \setminus \mathcal{M}\Phi_{0,ind}(F),$$

then, replacing $\mathcal{M}\Phi_0(F)$ by $\mathcal{M}\Phi_{0,ind}(F)$ in Lemma 7.2.9 and replacing $\sigma_{ew}^{\mathcal{A}}(F)$ by $\sigma_{ew,ind}^{\mathcal{A}}(F)$ in Theorem 7.2.10 and Theorem 7.2.11, we obtain the results that are valid in the case of arbitrary Hilbert C^*-modules over commutative C^*-algebras. However, the last inclusion in Theorem 7.2.11 (which is $\partial\sigma_{ea}^{\mathcal{A}}(F) \subseteq \partial\sigma_{e\ uf}^{\mathcal{A}}(F)$) holds only in the case of arbitrary Hilbert C^*-modules over commutative C^*-algebras whose K-group satisfies the cancellation property, as explained in Proposition 7.2.15. In addition, the inclusions $\partial\sigma_{ea'}^{\mathcal{A}}(F) \subseteq \partial\sigma_{ea}^{\mathcal{A}}(F)$ and $\partial\sigma_{eb'}^{\mathcal{A}}(F) \subseteq \partial\sigma_{eb}^{\mathcal{A}}(F)$ given in Proposition 7.2.13 hold only in the case of the standard module and in the case of self-dual Hilbert W^*-modules. Similarly, as regards Corollary 7.2.12 and Corollary 7.2.14, the sets $\mathcal{M}\Phi \setminus \tilde{\mathcal{M}\Phi}_{+}^{-}$, $\mathcal{M}\Phi \setminus \tilde{\mathcal{M}\Phi}_{-}^{+}$, $\mathcal{M}\Phi_{+}^{-} \setminus \mathcal{M}\Phi_{+}^{-'}$ and $\mathcal{M}\Phi_{-}^{+} \setminus \mathcal{M}\Phi_{-}^{+'}$ are open only in the standard module case and in the case of self-dual Hilbert W^*-modules.

Chapter 8

Although Theorem 8.2.5, Theorem 8.2.11 and Theorem 8.2.16 hold in the case of arbitrary Hilbert C^*-modules, this chapter deals mainly with the standard module case.

Chapter 9

All the results in this chapter are valid in the case of arbitrary Hilbert C^*-modules.

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2. HARMONIC AND SPECTRAL ANALYSIS International Zoom Conference, 2020 and 2021

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