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ЗБОРНИК РАДОВА

НОВА СЕРИЈА
Књ. 1 (9)

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Београд—Нови Сад, 18—21. 9. 1974.

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Академик *ТАТОМИР П. АНЂЕЛИЋ*

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A CLASS OF BALANCED LAWS ON QUASIGROUPS (I)

Branka P. Alimpić

Let $w_1 = w_2$ be a balanced law of the I kind [2], in the form

$$A(u_1, \ldots, u_m) = B(v_1 \ldots, v_n) \quad m \geqslant 2, \quad n \geqslant 2, \tag{1}$$

where u_i $(i = 1, \ldots, m)$ is either a variable or a term $A_i(x_{i_1}, \ldots, x_{i_\alpha})$, $x_{i_1}, \ldots, x_{i_\alpha}$ being variables, analogously v_j $(j = 1, \ldots, n)$ is either a variable or a term $B_j(x_{j_1}, \ldots, x_{j_\beta})$, $x_{j_1}, \ldots, x_{j_\beta}$ being variables. A, B, A_i and B_j are function letters. Let p be the number of different variables occuring in the law (1).

For (1) we suppose the following conditions hold:

(i) For any two terms u_i and v_j there is at most one variable occuring in each of them.

(ii) If there are terms u_i, u_{i+1} (or v_j, v_{j+1}) in each of them occurs exactly one variable, these variables occur in different terms v_j, v_{j+1} (or u_i, u_{i+1}) respectively.

For example, such is the law

$$A(A_1(x_1, x_2), x_3, x_4, x_5, A_5(x_6, x_7, x_8), x_9) = \tag{2}$$
$$= B(x_1, B_2(x_2, x_3), x_4, B_4(x_5, x_6), x_7, B_6(x_8, x_9)).$$

Let $|P|$ denote the length of function letter P. From (i) and (ii) it follows $|A_i| \leqslant 3$, $|B_j| \leqslant 3$. Indeed, if e. g. $u_i = A_i(x, y, z, u)$, the variables x, y, z, u occur in different terms $v_j, v_{j+1}, v_{j+2}, v_{j+3}$ respectively, and in each of terms v_{j+1}, v_{j+2} occurs exactly one variable. According to (ii) that is impossible, hence, $|A_i| \leqslant 3$.

We define a relation of the set $T = \{u_1, \ldots, u_m, v_1, \ldots, v_n\}$. For any two terms $t_1, t_2 \in T$ we say t is connected with t_2 iff tere are terms $s_1, \ldots, s_k \in T$ such that $t_1 = s_1$, $t_2 = s_k$ and s_i, s_{i+1} $(i = 1, \ldots, k-1)$ are terms occuring on opposite sides of (1) and having a common variable. This relation is an equivalence of the set T. Let r denote the number of equivalence classes.

For example, in (2) we have $r = 3$, and the classes are

$$C_1 = \{u_1, u_2, v_1, v_2\}, \quad C_2 = \{u_3, v_3\}, \quad C_3 = \{u_4, u_5, u_6, v_4, v_5, v_6\}.$$

In this paper we consider the law (1) as a functional equation of unknown functions A, B, A_i, B_j and we give its general solution, provided A, B, A_i, B_j are quasigroups defined on a nonempty set S.

We give some definitions.

If t is a term, let $[t]$ denote the set of variables occuring in t [2]. If t is a subterm either of w_1 or of w_2, $a \in S$ a fixed element, and $\tau \subset [t]$, let $t\big|_a^\tau$ denote the term obtained from t substituting all $x_i \in \tau$ by a. If e. g. $t = A_i(x, y, z)$, $\tau = \{x, z\}$, then $t\big|_a^\tau = A_i(a, y, a)$.

Let $t = P(t_1, \ldots, t_\alpha)$ be a subterm either of w_1 or of w_2 and $(t_\mu, \ldots, t_\nu)$ be some nonempty subsequence of the sequence $(t_1, \ldots, t_\alpha)$ of the length $k \leqslant \alpha$. We define an operation of the set S, derived from the quasigroup P

$$L^P_{\mu\ldots\nu} : S^k \to S,$$

putting

$$L^P_{\mu\ldots\nu}(t_\mu, \ldots, t_\nu) \stackrel{\text{def}}{=} t\,\Big|_a^{[t]\setminus([t_\mu]\cup\cdots\cup[t_\nu])}.$$

So, for the law (2) we have

$$L^A_{13}(u_1, u_3) = A(u_1, a, u_3, a, A_5(a, a, a), a),$$

$$L^A_{23}(x_7, x_8) = A_5(a, x_7, x_8),$$

$$L^B_{25}(v_2, v_5) = B(a, v_2, a, B_4(a, a), v_5, B_6(a, a)), \text{ etc.}$$

These operations depend on the choice of a and on the form of the law (1). Since A, B, A_i, B_j are quasigroups, the operations, derived from them, are quasigroups too.

Let $\mathcal{T}$ be the set of all quasigroups derived from A and B. We define the relation $\to$ of the set $\mathcal{T}$ on the following way: For quasigroups $L^P_{\alpha\ldots\beta}$ and $L^Q_{\mu\ldots\nu}$ $(P, Q \in \{A, B\}, P \neq Q)$ we put

$$L^P_{\alpha\ldots\beta} \to L^Q_{\mu\ldots\nu}$$

iff there are variables $x_\gamma, \ldots, x_\delta$ of the law (1) so that by substitution of all variables in (1), except $x_\gamma, \ldots, x_\delta$ we get

$$L^P_{\alpha\ldots\beta}(\varphi_\alpha x_\gamma, \ldots, \varphi_\beta x_\delta) = L^Q_{\mu\ldots\nu}(\psi_\mu(x_\gamma, \ldots, x_{\gamma'}), \ldots, \psi_\nu(x_{\delta'}, \ldots, x_\delta)),$$

where $\varphi_\alpha, \ldots, \varphi_\beta$ are bijections of the set S, $\psi_\mu, \ldots, \psi_\nu$ are either derived quasigroups or bijections of the set S. In this case we write also

$$L^P_{\alpha\ldots\beta} \xrightarrow{x_\gamma \ldots x_\delta} L^Q_{\mu\ldots\nu}.$$

For example, from the law (2) we have

$$L^A_{125}(L^{A_1}_2 x_2, x_2, x_5) = L^B_{23}(B_2(x_2, x_3), L^{B_4}_1 x_5),$$

namely

$$L^A_{125} \xrightarrow{x_2 x_3 x_5} L^B_{23}.$$

Let $\Rightarrow$ be the minimal transitive relation of the set $\mathcal{T}$ containing the relation $\to$. Relations $\Rightarrow$ and $\to$ of the set $\mathcal{T}_k$ of all derived quasigroups of the length k $(1 \leqslant k \leqslant \max(m, n))$ are symmetric and we denote them $\Leftrightarrow$ and $\leftrightarrow$ respectively. The relation $\Leftrightarrow$ of the set $\mathcal{T}_k$ is an equivalence.

We prove several lemmas.

Lemma 1. *If all terms u_i, v_j of the law* (1) *are connected, for any two binary quasigroups $L^P_{\alpha\beta}$ and $L^Q_{\mu\nu}$ $(P,Q \in \{A,B\})$ we have $L^P_{\alpha\beta} \Leftrightarrow L^Q_{\mu\nu}$.*

Proof. First we prove $L^A_{ij} \Leftrightarrow L^A_{ij+1}$, where $1 \leqslant i < j < m$. Let x be the first variable occuring in the term u_i, y the last variable occuring in the term u_j and z the first variable occuring in the term u_{j+1}. Since all terms are connected, the variables y and z occur in the same term v_k, and x occurs in a term v_h, $h<k$. Indeed, there is at least one variable x' occuring between x and y. If x occurs in v_k, in v_k occur at least four variables x, x', y, z, which is impossible. Therefore $L^A_{ij} \overset{xy}{\leftrightarrow} L^B_{hk} \overset{xz}{\leftrightarrow} L^A_{ij+1}$, namely $L^A_{ij} \Leftrightarrow L^A_{ij+1}$. Then we have

$$L^A_{ij} \Leftrightarrow L^A_{ik}, \quad 1 \leqslant i < m, \quad i<j<k \leqslant m. \tag{3}$$

Analogously it yields

$$L^A_{ki} \Leftrightarrow L^A_{ji}, \quad 1 \leqslant k<j<i, \quad 1<i \leqslant m. \tag{4}$$

Using (3) and (4), for any two $L^A_{\alpha\beta}$ and $L^A_{\gamma\delta}$ we get $L^A_{\alpha\beta} \Leftrightarrow L^A_{\gamma\beta} \Leftrightarrow L^A_{\gamma\delta}$.

For every $L^B_{\mu\nu}$ there are some L^A_{jk} so that $L^A_{jk} \leftrightarrow L^B_{\mu\nu}$. Using $L^A_{\alpha\beta} \Leftrightarrow L^A_{jk}$, we have $L^A_{\alpha\beta} \Leftrightarrow L^B_{\mu\nu}$. The lemma is proved.

Lemma 2. *If all terms u_i, v_j of the law* (1) *are connected, for every derived quasigroup $L^P_{\alpha\beta\ldots\gamma}$ $(P \in \{A, B\})$ of the length $\geqslant 3$ holds:*

$$L^P_{\alpha\beta\ldots\gamma} \Rightarrow L^A_{12}$$

and

$$L^P_{\alpha\beta\ldots\gamma} \Rightarrow L^A_{1}.$$

Proof. As all terms are connected, there is at least one term u_μ with two variables. The variables x and y of the term u_μ occur in two different terms v_j, v_k respectively, hence we get $L^B_{jk} \to L^A_\mu$. By lemma 1, we have $L^P_{\alpha\beta} \Leftrightarrow L^B_{jk}$ and therefore $L^P_{\alpha\beta} \Rightarrow L^A_\mu$. Hence, there is a quasigroup $L^A_{\mu\ldots\nu}$, so that $|L^P_{\mu\ldots\nu}|<|L^P_{\alpha\beta\ldots\gamma}|$ and $L^P_{\alpha\beta\ldots\gamma} \Rightarrow L^A_{\mu\ldots\nu}$. On this way we get a quasigroup $L^Q_{\lambda\rho}$ of the length 2 $(Q \in \{A, B\})$ such that $L^P_{\alpha\beta\ldots\gamma} \Rightarrow L^Q_{\lambda\rho}$, and according to Lemma 1 we have $L^P_{\alpha\beta\ldots\gamma} \Rightarrow L^A_{12}$. Since $L^A_{12} \Rightarrow L^A_1$, we have $L^P_{\alpha\beta\ldots\gamma} \Rightarrow L^A_1$.

For example, in the law

$$A\big(A_1(x_1, x_2), x_3, A_3(x_4, x_5, x_6), A_4(x_7, x_8)\big) = B\big(x_1, B_2(x_2, x_3, x_4), x_5, B_4(x_6, x_7), x_8\big)$$

all terms u_i, v_j are connected, and we have

$$A \xrightarrow{x_2x_3x_4x_7} L^B_{24} \xrightarrow{x_2x_6} L^A_{13} \xrightarrow{x_1x_4} L^B_{12} \xrightarrow{x_1x_2} L^A_1,$$

$$B \xrightarrow{x_1x_4x_5x_6x_8} L^A_{134} \xrightarrow{x_2x_4x_7} L^B_{24}, \text{ etc.}$$

Lemma 3. *For every* $L_{\alpha\beta}^{A_i}$ *and* $L_{\mu\nu}^{B_j}$ *there are quasigroups* $L_{\gamma\delta}^{B}$ *and* $L_{\lambda\rho}^{A}$ *so that* $L_i^A L_{\alpha\beta}^{A_i}(x, y) = L_{\gamma\delta}^{B}(\varphi_\gamma x, \varphi_\delta y)$, *and* $L_j^B L_{\mu\nu}^{B_j}(x, y) = L_{\lambda\rho}^{A}(\varphi_\lambda x, \varphi_\rho y)$, *where* φ_γ, φ_δ, φ_λ *and* φ_ρ *are bijections of the set* S.

Proof. These equalities are obtained by substitution of all variables of the law (1), except, x and y, by $a \in S$.

Theorem 1. *Let all terms* u_i, v_j *of the law* (1) *be connected. There exists a group* $(S, \circ)$ *so that the following equalities hold*:

$$A(x_1, \ldots, x_m) = L_1^A x_1 \circ \cdots \circ L_m^A x_m,$$

$$B(x_1, \ldots, x_n) = L_1^B x_1 \circ \cdots \circ L_n^B x_n,$$

$$L_i^A A_i(x_1, \ldots, x_\alpha) = L_i^A L_1^{A_i} x_1 \circ \cdots \circ L_i^A L_\alpha^{A_i} x_\alpha,$$

$$L_i^B B_i(x_1, \ldots, x_\alpha) = L_i^B L_1^{B_i} x_1 \circ \cdots \circ L_i^B L_\alpha^{B_i} x_\alpha, \qquad (1 \leqslant \alpha \leqslant 3).$$

Proof. We define a binary operation $\circ$ of the set S as follows:

$$L_{12}^A(x, y) = L_1^A x \circ L_2^A y. \tag{6}$$

Since L_{12}^A is a quasigroup. the operation $\circ$ is a loop [1], mainly isotopic with L_{12}^A. We prove the operation $\circ$ is isotopic with all binary quasigroups derived from A, B, A_i, B_j.

Let be $L_{\alpha\beta}^B \overset{xy}{\leftrightarrow} L_{12}^A$, namely

$$L_{\alpha\beta}^B(\varphi_\alpha x, \varphi_\beta y) = L_{12}^A(\psi_1 x, \psi_2 y). \tag{7}$$

If we put in (7) $y = a$, we get

$$L_\alpha^B \varphi_\alpha x = L_1^A \psi_1 x, \tag{8}$$

and, if we put $x = a$, we get

$$L_\beta^B \varphi_\beta y = L_2^A \psi_2 y. \tag{9}$$

Further, we have

$$\begin{aligned} L_{\alpha\beta}^B(\varphi_\alpha x, \varphi_\beta y) &= L_1^A \psi_1 x \circ L_2^A \psi_2 y && \text{(from (6) and (7))} \\ &= L_\alpha^B \varphi_\alpha x \circ L_\beta^B \varphi_\beta y && \text{(from (8) and (9))}. \end{aligned}$$

Since φ_α, φ_β are bijections, it follows

$$L_{\alpha\beta}^B(x, y) = L_\alpha^B x \circ L_\beta^B y. \tag{10}$$

Let $L_{\mu\nu}^P$ be any quasigroup derived from either A or B. According to lemma 1, we get $L_{\mu\nu}^P \Leftrightarrow L_{12}^A$, hence there is a chain of quasigroups $K_1, \ldots, K_s$ so that $L_{\mu\nu}^P = K_1$, $K_1 \leftrightarrow K_2 \leftrightarrow \cdots \leftrightarrow K_s$, $K_s = L_{12}^A$. By induction on s, we can prove

$$L_{\mu\nu}^P(x, y) = L_\mu^P x \circ L_\nu^P y.$$

With respect to lemma 3, for every quasigroup $L^{P_i}_{\mu\nu}$ derived from either A_i or B_i, it yields

(11) $$L^P_i L^{P_i}_{\mu\nu}(x, y) = L^P_i L^{P_i}_{\mu} x \circ L^P_i L^{P_i}_{\nu} y.$$

We prove the operation $\circ$ is associative. Substituting all variables of the law (1) except x_1, x_2, x_3, we get either

$$L^A_{12}(A_1(x_1, x_2), L^{A_2}_1 x_3) = L^B_{12}(B_1(x_1), L^{B_2}_{12}(x_2, x_3)),$$

or

$$L^A_{12}(A_1(x_1), L^{A_2}_{12}(x_2, x_3)) = L^B_{12}(B_1(x_1, x_2), L^{B_2}_1 x_3).$$

In both cases, using (10), and (11) and equalities of the form $L^A_i L^A_\alpha x_k = L^B_j L^B_\beta x_k$, $k = 1, \dots, p$, obtained from (1) by substitution of all variables except x_k, we have $(x \circ y) \circ z = x \circ (y \circ z)$, that is, $(S, \circ)$ is a group.

Finally, we prove for every quasigroup $L^P_{\alpha\dots\beta}$ $(P \in \{A, B\})$ of the length k holds

(12) $$L^P_{\alpha\dots\beta}(x_1, \dots, x_k) = L^P_\alpha x_1 \circ \dots \circ L^P_\beta x_k.$$

Let us remark the following. If for quasigroups derived from A and B holds (12), then for quasigroups derived from A_i and B_j hold analogous equalities, too. If, e. g. $|A_i| = 3$, there is a quasigroup $L^B_{\mu\nu\lambda}$ such that

$$L^A_i A_i(x, y, z) = L^B_{\mu\nu\lambda}(\varphi_\mu x, \varphi_\nu y, \varphi_\lambda z) = L^B_\mu \varphi_\mu x \circ L^B_\nu \varphi_\nu y \circ L^B_\lambda \varphi_\lambda z.$$

Using $L^A_i L^{A_i}_1 = L^B_\mu \varphi_\mu$, $L^A_i L^{A_i}_2 = L^B_\nu \varphi_\nu$, $L^A_i L^{A_i}_3 = L^B_\lambda \varphi_\lambda$, we get

$$L^A_i A_i(x, y, z) = L^A_i L^{A_i}_1 x \circ L^A_i L^{A_i}_2 y \circ L^A_i L^{A_i}_3 z.$$

By induction on k, we prove the equality (12).

If $|L^P_{\alpha\beta}| = 2$, the equality (12) holds with respect to lemma 1.

If $|L^P_{\alpha\dots\beta}| > 2$, there exists $L^Q_{\mu\dots\nu}$, such that $|L^Q_{\mu\dots\nu}| < |L^P_{\alpha\dots\beta}|$, and

$$L^P_{\alpha\dots\beta}(\varphi_\alpha x_1, \dots, \varphi_k x_k) = L^Q_{\mu\dots\nu}(\psi_{\mu\dots\mu'}(x_1, \dots, x_i), \dots, \psi_{\nu\dots\nu'}(x_j, \dots x_k)).$$

By induction's hypothesis, operations $\psi_{\mu\dots\mu'}$, $\psi_{\nu\dots\nu'}$, $L^Q_{\mu\dots\nu}$ are expressible by $\circ$, hence it yields

$$L^P_{\alpha\dots\beta}(\varphi_\alpha x_1, \dots, \varphi_\beta x_k) = L^Q_\mu \psi_{\mu\dots\mu'}(x_1, \dots x_i) \circ \dots \circ L^Q_\nu \psi_{\nu\dots\nu'}(x_j, \dots x_k)$$
$$= L^Q_\mu \psi_\mu x_1 \circ \dots \circ L^Q_\mu \psi_{\mu'} x_i \circ \dots \circ L^Q_\nu \psi_\nu x_j \circ \dots \circ L^Q_\nu \psi_{\nu'} x_k.$$

Using the equalities $L^P_\alpha \varphi_\alpha = L^Q_\mu \psi_\mu, \dots, L^P_\beta \varphi_\beta = L^Q_\nu \varphi_{\nu'}$, we get (12). The theorem is proved.

Now, we suppose for the relation of connectness of the set $T = \{u_1, \dots, u_m, v_1, \dots, v_n\}$ there is $r\,(r > 1)$ equivalence classes $C_i = \{u_{\alpha_i}, \dots, u_{\beta_i}, v_{\gamma_i}, \dots, v_{\delta_i}\}$, $i = 1, \dots, r$. In that case the law (1) has the form

(13) $$A(u_{\alpha_1}, \dots, u_{\beta_1}, \dots, u_{\alpha_r}, \dots, u_{\beta_r}) = B(v_{\gamma_1}, \dots, v_{\delta_1}, \dots, v_{\gamma_r}, \dots, v_{\delta_r}).$$

Let $\pi : S^r \rightarrow S$ be the operation of the set S defined by

$$\pi(L^A_{\alpha_1} x_1, \ldots, L^A_{\alpha_r} x_r) = L^A_{\alpha_1 \ldots \alpha_r}(x_1, \ldots, x_r).$$

The operation π is a loop, mainly isotopic to $L^A_{\alpha_1 \ldots \alpha_r}$.

Lemma 4. *For any two derived quasigroups $L^A_{\mu_1 \ldots \mu_r}$ and $L^B_{\nu_1 \ldots \nu_r}$ with $u_{\mu_i} \in C_i$, $v_{\nu_i} \in C_i$, $i = 1, \ldots, r$,*

(14) $$L^A_{\mu_1 \ldots \mu_r}(x_1, \ldots, x_r) = \pi(L^A_{\mu_1} x_1, \ldots, L^A_{\mu_r} x_r)$$

and

(15) $$L^B_{\nu_1 \ldots \nu_r}(x_1, \ldots, x_r) = \pi(L^B_{\nu_1} x_1, \ldots, L^B_{\nu_r} x_r).$$

Proof. Since all terms of the set C_i $(i = 1, \ldots, r)$ are connected, for any two derived quasigroups $L^P_{\mu_1 \ldots \mu_r}$ and $L^Q_{\nu_1 \ldots \nu_r}$ there is a finite sequence of derived quasigroups $K_1, \ldots, K_n$, such that

$$L^P_{\mu_1 \ldots \mu_r} = K_1 \leftrightarrow K_2 \cdots \leftrightarrow K_n = L^Q_{\nu_1 \ldots \nu_r},$$

that is

$$L^P_{\mu_1 \ldots \mu_r} \Leftrightarrow L^Q_{\nu_1 \ldots \nu_r} (P, Q \in \{A, B\}).$$

If, for example, $L^A_{\mu_1 \ldots \mu_r} \leftrightarrow L^B_{\nu_1 \ldots \nu_r}$ and

$$L^A_{\mu_1 \ldots \mu_r}(x_1, \ldots, x_r) = \pi(L^A_{\mu_1}(x_1, \ldots, L^A_{\mu_r} x_r),$$

then

(16) $$L^B_{\nu_1 \ldots \nu_r}(x_1, \ldots, x_r) = \pi(L^B_{\nu_1} x_1, \ldots, L^B_{\nu_r} x_r).$$

Indeed, from the law (13) it follows

$$L^A_{\mu_1 \ldots \mu_r}(\varphi_1, \ldots, \varphi_r x_r) = L^B_{\nu_1 \ldots \nu_r}(\psi_1 x_1, \ldots, \psi_r x_r)$$

and

$$L^A_{\mu_i} \varphi_i x_i = L^B_{\nu_i} \psi_i x_i \qquad i = 1, \ldots, r,$$

where φ_i and ψ_i are certain bijections of the set S, and so we have

$$L^B_{\nu_1 \ldots \nu_i}(\psi_1 x_1, \ldots, \psi_r x_r) = \pi(L^A_{\mu_1} \varphi_1 x_1, \ldots, L^A_{\mu_r} \varphi_r x_r) = \pi(L^B_{\nu_1} \psi_1 x_1, \ldots, L^B_{\nu_r} \psi_r x_r).$$

Since ψ_i are bijections, we get (16).

Analogously we can prove if $L^P_{\mu_1 \ldots \mu_r}$ is expressible by π, and

$$L^P_{\mu_1 \ldots \mu_r} \Leftrightarrow L^Q_{\nu_1 \ldots \nu_r}, \quad \text{then} \quad L^Q_{\nu_1 \ldots \nu_r}$$

is expressible by π, too

According to definition of π, it yields (14) and (15).

Lemma 5. *For quasigroups A and of the law* (13) *it holds*

$$A \Rightarrow L^A_{\alpha_1 \ldots \alpha_r} \quad \textit{and} \quad B \Rightarrow L^A_{\alpha_1 \ldots \alpha_r}.$$

P r o o f. By substitution of all variables, occuring in (13), except the variables occuring in the terms of C_i ($i \in \{1, \dots, r\}$), we get the law

$$L^A_{\alpha_i \dots \beta_i}(u_{\alpha_i}, \dots, u_{\beta_i}) = L^B_{\gamma_i \dots \delta_i}(v_{\gamma_i}, \dots, v_{\delta_i}),$$

in which all terms $u_{\alpha_i}, \dots, u_{\beta_i}, v_{\gamma_i}, \dots, v_{\delta_i}$ are connected.

By lemma 2 we have

$$L^A_{\alpha_i \dots \beta_i} \Rightarrow L^A_{\alpha_i},$$

and

$$L^B_{\gamma_i \dots \delta_i} \Rightarrow L^A_{\alpha_i}.$$

Hence,

$$A = L^A_{\alpha_1 \dots \beta_1 \dots \alpha_r \dots \beta_r} \Rightarrow L^A_{\alpha_1 \dots \alpha_r}$$

and

$$B = L^B_{\gamma_1 \dots \delta_1 \dots \gamma_r \dots \delta_r} \Rightarrow L^A_{\gamma_1 \dots \gamma_r}.$$

The lemma is proved.

T h e o r e m 2. *Let A, B, A_i, B_j are quasigroups satisfying the law* (13). *There exists a loop (S, π) of the length r, and for every class C_i with card $C_i > 2$, there exists a group $(S, \circ_i)$, $i = 1, \dots, r$, so that*

$$(17)\quad \begin{aligned} &A(x_{\alpha_1}, \dots, x_{\beta_1}, \dots, x_{\alpha_r}, \dots, x_{\beta_r}) \\ &\qquad = \pi(L^A_{\alpha_1} x_{\alpha_1} \circ_1 \cdots \circ_1 L^A_{\beta_1} x_{\beta_1}, \dots, L^A_{\alpha_r} x_{\alpha_r} \circ_r \cdots \circ_r L^A_{\beta_r} x_{\beta_r}). \\ &B(x_{\gamma_1}, \dots, x_{\delta_1}, \dots, x_{\gamma_r}, \dots, x_{\delta_r}) \\ &\qquad = \pi(L^B_{\gamma_1} x_{\gamma_1} \circ_1 \cdots \circ_1 L^B_{\delta_1} x_{\delta_1}, \dots, L^B_{\gamma_r} x_{\gamma_r} \circ_r \cdots \circ_r L^B_{\delta_r} x_{\delta_r}). \end{aligned}$$

If $u_\alpha \in C_i$, and $u_\alpha = A_\alpha(x_1, \dots, x_k)$, where $k = 2$ or $k = 3$, then

$$(18)\quad L^A_\alpha A_\alpha(x_1, \dots, x_k) = L^A_\alpha L^{A_\alpha}_1 x_1 \circ_i \cdots \circ_i L^A_\alpha L^{A_\alpha}_k x_k.$$

If $v_\beta \in C_i$, and $v_\beta = B_\beta(x_1, \dots, x_k)$, where $k = 2$ or $k = 3$, then

$$(19)\quad L^B_\beta B_\beta(x_1, \dots, x_k) = L^B_\beta L^{B_\beta}_1 x_1 \circ_i \cdots \circ_i L^B_\beta L^{B_\beta}_k x_k.$$

P r o o f. By theorem 1 there exist groups $(S, \circ_i)$ so that the equalities (18) and (19) hold.

Let $\mu_i, \dots, \nu_i$ be a subsequence of the sequence $\alpha_i, \dots, \beta_i$, and $\lambda_i, \dots, \rho_i$ a subsequence of the sequence $\gamma_i, \dots, \delta_i$, $i = 1, \dots, r$.

If $L^A_{\mu_1 \dots \nu_1 \dots \mu_r \dots \nu_r} \to L^B_{\lambda_1 \dots \rho_1 \dots \lambda_r \dots \rho_r}$, and if

$$L^B_{\lambda_1 \dots \rho_1 \dots \lambda_r \dots \rho_r}(x_{\lambda_1}, \dots, x_{\rho_r}) = \pi(L^B_{\lambda_1} x_{\lambda_1} \circ_1 \cdots \circ_1 L^B_{\rho_1} x_{\rho_1}, \dots, L^B_{\lambda_r} x_{\lambda_r} \circ_r \cdots \circ_r L^B_{\rho_r} x_{\rho_r}),$$

then, by definition of $\to$, it is easy to see that

$$L^A_{\mu_1 \dots \nu_1}(x_{\mu_1}, \dots, x_{\nu_r}) = \pi(L^A_{\mu_1} x_{\mu_1} \circ_1 \cdots \circ_1 L^A_{\nu_1} x_{\nu_1}, \dots, L^A_{\mu_1} x_{\mu_r} \circ_r \cdots \circ_r L^A_{\nu_r} x_{\nu_r}).$$

Using such equalities, by lemma 5 and by definition of π we prove the equalities (17). This completes the proof.

Theorem 3. *General solution of the functional equation* (13) *on unknown quasigroups* A, B, A_α, B_β *is given by*

$$(20)\quad \begin{cases} A(x_{\alpha_1}, \ldots, x_{\beta_1}, \ldots, x_{\alpha_r}, \ldots, x_{\beta_r}) \\ \qquad = \pi(\varphi_{\alpha_1} x_{\alpha_1} \circ_1 \cdots \circ_1 \varphi_{\beta_1} x_{\beta_1}, \ldots, \varphi_{\alpha_r} x_{\alpha_r} \circ_r \cdots \circ_r \varphi_{\beta_r} x_{\beta_r}), \\ B(x_{\gamma_1}, \ldots, x_{\delta_1}, \ldots, x_{\gamma_r}, \ldots, x_{\delta_r}) \\ \qquad = \pi(\psi_{\gamma_1} x_{\gamma_1} \circ_1 \cdots \circ_1 \psi_{\delta_1} x_{\delta_1}, \ldots, \psi_{\gamma_r} x_{\gamma_r} \circ_r \cdots \circ_r \psi_{\delta_r} x_{\delta_r}), \\ \varphi_\alpha A_\alpha(x_j, \ldots, x_{j+k}) = \varphi_\alpha \varepsilon_j x_j \circ_i \cdots \circ_i \varphi_\alpha \varepsilon_{j+k} x_{j+k}, \\ \qquad (u_\alpha \in C_i,\ u_\alpha = A_\alpha(x_j, \ldots, x_{j+k})), \qquad k \in \{0, 1, 2\}, \\ \psi_\beta B_\beta(x_j, \ldots, x_{j+k}) = \psi_\beta \eta_j x_j \circ_i \cdots \circ_i \psi_\beta \eta_{j+k} x_{j+k}, \\ \qquad (v_\beta \in C_i,\ v_\beta = B_\beta(x_j, \ldots, x_{j+k})), \qquad k \in \{0, 1, 2\}, \end{cases}$$

where (S, π) *is an arbitrary loop of the length* r, $(S, \circ_i)$ *are arbitrary groups, and* φ_α, ψ_γ, ε_i, η_i *are bijections of the set* S *satisfying the following conditions:*

For every $i \in \{1, \ldots, p\}$, *if the variable* x_i *occurs in the terms* u_α *and* v_γ, *then*

$$(21)\qquad \varphi_\alpha \varepsilon_i = \psi_\gamma \eta_i .$$

Proof. By theorem 2, for the law (13) there exist a loop (S, π), groups $(S, \circ_i)$ and bijections φ_α, ψ_γ, ε_i, η_i so that the equalities (20) and (21) hold.

Conversely, let for quasigroups A, B, A_α, B_β the equalities (20) and (21) hold. It is easy to verify in this case A, B, A_α, B_β satisfy the law (13).

We give an example.

Let us consider the functional equation

$$(22)\quad \begin{aligned} &A(A_1(x_1, x_2), x_3, x_4, x_5, A_5(x_6, x_7, x_8), A_6(x_9, x_{10})) \\ &\qquad = B(x_1, B_2(x_2, x_3), x_4, B_4(x_5, x_6), x_7, B_6(x_8, x_9), x_{10}). \end{aligned}$$

Here is $r = 3$, and

$$C_1 = \{u_1, u_2, v_1, v_2\},\ C_2 = \{u_3, v_3\},\ C_3 = \{u_4, u_5, u_6, v_4, v_5, v_6, v_7\}.$$

General solution of (22) is given by

$$\begin{aligned} &A(y_1, y_2, y_3, y_4, y_5, y_6) = \pi(\varphi_1 y_1 \circ_1 \varphi_2 y_2, \varphi_3 y_3, \varphi_4 y_4 \circ_3 \varphi_5 y_5 \circ_3 \varphi_6 y_6), \\ &B(y_1, \ldots, y_7) = \pi(\psi_1 y_1 \circ_1 \psi_2 y_2, \psi_3 y_3, \psi_4 y_4 \circ_3 \psi_5 y_5 \circ_3 \psi_6 y_6 \circ_3 \psi_7 y_7), \\ &\qquad \varphi_1 A_1(x_1, x_2) = \varphi_1 \varepsilon_1 x_1 \circ_1 \varphi_1 \varepsilon_2 x_2, \\ &\qquad \varphi_5 A_5(x_6, x_7, x_8) = \varphi_5 \varepsilon_6 x_6 \circ_3 \varphi_5 \varepsilon_7 x_7 \circ_3 \varphi_5 \varepsilon_8 x_8, \\ &\qquad \varphi_6 A_6(x_9, x_{10}) = \varphi_6 \varepsilon_9 x_9 \circ_3 \varphi_6 \varepsilon_{10} x_{10}, \\ &\qquad \psi_2 B_2(x_2, x_3) = \psi_2 \eta_2 x_2 \circ_1 \psi_2 \eta_3 x_3, \\ &\qquad \psi_4 B_4(x_5, x_6) = \psi_4 \eta_5 x_5 \circ_3 \psi_4 \eta_6 x_6, \\ &\qquad \psi_6 B_6(x_8, x_9) = \psi_6 \eta_8 x_8 \circ_3 \psi_6 \eta_9 x_9, \end{aligned}$$

where (S, π) is an arbitrary loop, $(S, \circ_1)$ and $(S, \circ_3)$ are arbitrary groups and $\varphi_\alpha, \psi_\mu, \varepsilon_i, \eta_j$ are bijections of the set S so that

$$\varphi_1\varepsilon_1=\psi_1,\ \varphi_1\varepsilon_2=\psi_2\eta_2,\ \varphi_2=\psi_2\eta_3,\ \varphi_3=\psi_3,\ \varphi_4=\psi_4\eta_5,\ \varphi_5\varepsilon_6=\psi_4\eta_6,$$

$$\varphi_5\varepsilon_7=\psi_5,\ \varphi_5\varepsilon_8=\psi_6\eta_8,\ \varphi_6\varepsilon_9=\psi_6\eta_9,\ \varphi_6\varepsilon_{10}=\psi_7.$$

Now, let us consider an arbitrary law of the I kind, in the form (1), on quasigroups. Such is, e. g. the law

$$A(x_1, A_2(x_2, x_3, x_4, x_5), A_3(x_6, x_7, x_8), x_9, x_{10})$$
$$=B(B_1(x_1, x_2, x_3), x_4, x_5, x_6, B_5(x_7, x_8), B_6(x_9, x_{10})).$$

We prove, that such a law can be transformed into a law satisfying conditions (i) and (ii), substituting some quasigroups by GD-groupoids [3].

Let for two terms u_i and v_j there are more than one variable occuring in both of them, e. g. $u_i=A_i(x_1, \ldots, x_k, \ldots, x_{k+s})$ and $v_j=B_j(x_k, \ldots, x_{k+s}, \ldots, x_{k+m})$. Then we define GD-groupoids

$$\bar{A}_i(x_1, \ldots, x_{k-1}, (x_k, \ldots, x_{k+s})) \overset{\text{def}}{=} A_i(x_1, \ldots, x_{k+s}),$$

$$\bar{B}_j((x_k, \ldots, x_{k+s}), x_{k+s+1}, \ldots, x_{k+m}) \overset{\text{def}}{=} B_j(x_k, \ldots, x_{k+m}).$$

So we get a law on GD-groupoids satisfying the condition (i).

If in such obtained law occurs a sequence of terms $u_k, \ldots, u_{k+s}$ (or $v_k, \ldots, v_{k+s}$) in each of them occurs exactly one variable $x_k, \ldots, x_{k+s}$, respectively, and if these variables occur in the same term v_j (or u_j) on the opposite side of law, we define GD-groupoids

$$\bar{A}(\ldots, (u_k, \ldots, u_{k+s}), \ldots) \overset{\text{def}}{=} A(\ldots, u_k, \ldots, u_{k+s}, \ldots),$$

and

$$\bar{B}_j(\ldots, (x_k, \ldots, k_{k+s}), \ldots) \overset{\text{def}}{=} B_j(\ldots, x_k, \ldots, x_{k+s}, \ldots),$$

or

$$\bar{B}(\ldots, (v_k, \ldots, v_{k+s}), \ldots) \overset{\text{def}}{=} B(\ldots, v_k, \ldots, v_{k+s}, \ldots),$$

and

$$\bar{A}_j(\ldots, (x_k, \ldots, x_{k+s}), \ldots) \overset{\text{def}}{=} A_j(\ldots, x_k, \ldots, x_{k+s}, \ldots).$$

In such a way we obtain the law (13) on GD-groupoids satisfying the conditions (i) and (ii).

We note that for so obtained law holds the condition:

(iii) If a term u_i (or v_j) is not a variable, then L_i^A (or L_j^B) is a bijection.

We prove in this case that the operation π is well defined.

If all functions $L_{\alpha_1}^A, \ldots, L_{\alpha_r}^A$ are bijections, the operation π is well defined.

If, for some i, function $L_{\alpha_i}^A$ is not a bijection, then u_{α_i} is a variable, but v_{γ_i} is not a variable, namely $L_{\gamma_i}^B$ is a bijection.

We assume, functions $L_{\alpha_1}^A, \ldots, L_{\alpha_s}^A$ $(1 \leqslant s < r)$ are bijections and $L_{\alpha_{s+1}}^A, \ldots, L_{\alpha_r}^A$ are surjections. The sequence $C_1, \ldots, C_r$ can be reordered so that it holds.

Let $z_1, \ldots, z_r$ be arbitrary choiced elements of the set S. Since $L_{\alpha_1}^A, \ldots, L_{\alpha_s}^A$ are bijections, for every $z_i (i=1, \ldots, s)$ there is exactly one x_i so that $L_{\alpha_1}^A x_1 = z_1, \ldots, L_{\alpha_s}^A x_s = z_s$.

Since functions $L^A_{\alpha_{s+1}}, \ldots, L^A_{\alpha_r}$ are surjections, for every z_i $(i=s+1, \ldots, r)$ there is some x_i so that $L^A_{\alpha_{s+1}} x_{s+1} = z_{s+1}, \ldots, L^A_{\alpha_r} x_r = z_r$.

We prove that from the equalities

$$L^A_{\alpha_j} x_j' = L^A_{\alpha_j} x_j'' \qquad (j=s+1, \ldots, r) \tag{23}$$

follows the equality

$$L^A_{\alpha_1 \ldots \alpha_r}(x_1, \ldots, x_s, x_{s+1}', \ldots, x_r') = L^A_{\alpha_1 \ldots \alpha_r}(x_1, \ldots, x_s, x_{s+1}'', \ldots, x_r'').$$

Substituting all variables except x_j $(j=s+1, \ldots, r)$ by $a \in S$, from (13) we get

$$L^A_{\alpha_j} x_j = L^B_{\gamma_j} \psi_j x_j. \tag{24}$$

From (23) and (24) it follows

$$L^B_{\gamma_j} \psi_j x_j' = L^B_{\gamma_j} \psi_j x_j'' \qquad (j=s+1, \ldots, r).$$

As $L^B_{\gamma_j}$ is bijection, we have

$$\psi_j x_j' = \psi_j x_j'', \qquad (j=s+1, \ldots, r). \tag{25}$$

Substituting all variables, except one in each of sets C_i, $i=1, \ldots, r$, from (13) we get

$$L^A_{\alpha_1 \ldots \alpha_r}(\varphi_1 y_1, \ldots, \varphi_s y_s, y_{s+1}, \ldots, y_r)$$
$$= L^B_{\gamma_1 \ldots \gamma_r}(\psi_1 y_1, \ldots, \psi_s y_s, \psi_{s+1} y_{s+1}, \ldots, \psi_r y_r).$$

Since φ_i are surjections, we can choose $y_1, \ldots, y_s$ so that

$$\varphi_1 y_1 = x_1, \ldots, \varphi_s y_s = x_s.$$

From (25) we have

$$L^B_{\gamma_1 \ldots \gamma_r}(\psi_1 y_1, \ldots, \psi_s y_s, \psi_{s+1} x_{s+1}', \ldots, \psi_r x_r')$$
$$= L^B_{\gamma_1 \ldots \gamma_r}(\psi_1 y_1, \ldots, \psi_s y_s, \psi_{s+1} x_{s+1}'', \ldots, \psi_r x_r'')$$

and therefore, by the choice of $y_1, \ldots, y_s$, it follows

$$L^A_{\alpha_1 \ldots \alpha_r}(x_1, \ldots, x_s, x_{s+1}', \ldots, x_r') = L^A_{\alpha_1 \ldots \alpha_r}(x_1, \ldots, x_s, x_{s+1}'', \ldots, x_r''),$$

that is

$$\pi(L^A_{\alpha_1} x_1, \ldots, L^A_{\alpha_s} x_s, L^A_{\alpha_{s+1}} x_{s+1}', \ldots, L^A_{\alpha_r} x_r')$$
$$= \pi(L^A_{\alpha_1} x_1, \ldots, L^A_{\alpha_s} x_s, L^A_{\alpha_{s+1}} x_{s+1}'', \ldots, L^A_{\alpha_r} x_r''),$$

hence $\pi(z_1, \ldots, z_r)$ is well defined. It is easy to see that the operation π is a loop.

Analogously we can prove that the groups $\circ_i$ are well defined. Hence, we can apply theorem 3, too.

We give an example.

Let

$$A(x,\ A_2(y,\ z),\ A_3(u,\ v)) = B(B_1(x,\ y,\ z,\ u),\ v)$$

be functional equation by unknown quasigroups A, B, A_2, A_3, B_1 of a set S. According to theorem 3, general solution of this equation is given by

$$A(x,\ y,\ z) = M(x,\ y) \circ \alpha z,$$

$$A_2(x,\ y) = P(x,\ y),$$

$$\alpha A_3(x,\ y) = \beta x \circ \gamma y,$$

$$\delta B_1(x,\ y,\ z,\ u) = M(x,\ P(y,\ z)) \circ \beta u,$$

$$B(x,\ y) = \delta x \circ \gamma y,$$

where M, P are arbitrary quasigroups, α, β, γ, δ are arbitrary bijections, and $\circ$ is an arbitrary group of the set S.

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A CLASS OF BALANCED LAWS ON QUASIGROUPS (II)

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In this paper we enlarge the results, obtained in [1] for a class of balanced laws of the I kind on an analogous class of balanced laws of the II kind. In both cases the operations, satisfying the laws, are quasigroups, defined on a nonempty set S.

Let $w_1 = w_2$ be a balanced law of the II kind in the form

$$(1) \qquad A(u_1, \ldots, u_m) = B(v_1, \ldots, v_n),\ m > 2,\ n > 2,$$

where u_i $(i = 1, \ldots, m)$ is either a variable or a term $A_i(x_{i_1}, \ldots, x_{i_\alpha})$, $x_{i_1}, \ldots, x_{i_\alpha}$ being variables, analogously v_j $(j = 1, \ldots, n)$ is either a variable or a term $B_j(x_{j_1}, \ldots, v_{j_\beta})$, $x_{j_1}, \ldots, x_{j_\beta}$ being variables. A, B, A_i, B_j are functions letters.

For (1) we suppose the following conditions hold:

(i) For any two terms u_i and v_j there is at most one variable occuring in each of them.

(ii) If in each of two terms u_i and u_k (v_j, v_h) occurs exactly one variable, these variables occur in different terms v_j, v_h (u_i, u_k) respectively.

(iii) The order of occurence of the variables in any term u_i (v_j) is equal to the order of occurence of these variables in the term w_2 (w_1).

For example, such is the law

$$A(A_1(x, y, z, u), A_2(v, w), t) = B(x, B_2(y, v), B_3(z, t), B_4(u, w)).$$

In the set of terms $T = \{u_1, \ldots, u_m, v_1, \ldots, v_n\}$ we introduce the relation of connectness defined in [1], and in the set of all quasigroups derived from A and B we introduce the relations $\Rightarrow$ and $\Leftrightarrow$, defined in [1], too.

For the laws of the II kind, hold the lemmas, analogous to the lemmas 1, 2, 3 from [1]. The proofs of the lemmas 2 and 3 rest unchanged, for the lemma 1 we need a new proof.

L e m m a 1. *If all terms u_i, v_j of the law* (1) *are connected, for any two binary quasigroups $L^P_{\alpha\beta}$ and $L^Q_{\mu\nu}$ $(P, Q \Leftarrow \{A, B\})$, derived from A and B, we have $L^P_{\alpha\beta} \Leftrightarrow L^Q_{\mu\nu}$.*

Proof. First, if $L^A_\alpha \leftrightarrow L^B_\mu$, and $\beta > \alpha$, there exists $\nu \neq \mu$ such that $L^A_{\alpha\beta} \leftrightarrow L^B_{\overline{\mu\nu}}$ ($L^B_{\overline{\mu\nu}} = L^B_{\mu\nu}$, for $\mu < \nu$, and $L^B_{\overline{\mu\nu}} = L^B_{\nu\mu}$, for $\nu < \mu$). Indeed, if $L^A_\alpha \overset{x}{\longleftrightarrow} L^B_\mu$, then x occurs in the terms u_α and v_μ. The term u_β either contains a variable y occuring in some term v_ν, $\nu \neq \mu$, or contains only one variable y occuring in the term v_μ. In the first case we get $L^A_{\alpha\beta} \overset{xy}{\longleftrightarrow} L^B_{\overline{\mu\nu}}$, and in the second case, in view of (ii), there exists a variable z occuring in the term u_α and in a term v_ν, $\nu \neq \mu$, and we get $L^A_{\alpha\beta} \overset{zy}{\longleftrightarrow} L^B_{\overline{\mu\nu}}$.

Since for every two L^P_α and L^Q_μ holds $L^P_\alpha \Leftrightarrow L^Q_\mu$, it follows for every $L^P_{\alpha\beta}$ and L^Q_μ ($P, Q \in \{A, B\}$) there exists ν so that $L^P_{\alpha\beta} \Leftrightarrow L^Q_{\overline{\mu\nu}}$.

Further, for every α, β, μ, $\alpha < \beta$, $\beta \neq \mu$ we get $L^P_{\alpha\beta} \Leftrightarrow L^P_{\overline{\mu\beta}}$ ($P \in \{A, B\}$). Let us consider a sequence

$$L^P_\alpha \leftrightarrow L^Q_\rho \leftrightarrow L^P_\sigma \leftrightarrow \cdots \leftrightarrow L^Q_\tau \leftrightarrow L^P_\mu$$

defining $L^P_\alpha \leftrightarrow L^P_\mu$. There exists an index ν so that $L^P_{\alpha\beta} \leftrightarrow L^Q_{\overline{\rho\nu}}$. If $\sigma \neq \beta$, then $L^Q_{\overline{\rho\nu}} \leftrightarrow L^P_{\overline{\sigma\beta}}$, if $\sigma = \beta$, then $L^Q_{\overline{\rho\nu}} \leftrightarrow L^P_{\alpha\sigma} = L^P_{\alpha\beta}$. On this way, we get finally $L^P_{\overline{\alpha\beta}} \leftrightarrow L^P_{\overline{\mu\beta}}$,

Let $L^P_{\alpha\beta}$ and $L^Q_{\mu\nu}$ be two arbitrary quasigroups. There exists an index λ so that $L^P_{\alpha\beta} \Leftrightarrow L^Q_{\overline{\mu\lambda}}$, and $L^Q_{\overline{\mu\lambda}} \Leftrightarrow L^Q_{\mu\nu}$. Hence we get $L^P_{\alpha\beta} \Leftrightarrow L^Q_{\mu\nu}$, and the lemma is proved.

Let $L^P_{\alpha\beta}$ and $L^P_{\mu\nu}$ ($P, Q \in \{A, B, A_i, B_j\}$) be two derived quasigroups so that $L^P_{\alpha\beta} \overset{xy}{\longleftrightarrow} L^Q_{\mu\nu}$. Since the law (1) is of the II kind, we have either

$$\varphi L^P_{\alpha\beta}(\varphi_1 x, \varphi_2 y) = \psi L^Q_{\mu\nu}(\psi_1 x, \psi_2 y), \text{ or}$$

$$\varphi L^P_{\alpha\beta}(\varphi_1 x, \varphi_2 y) = \psi L^Q_{\mu\nu}(\psi_2 y, \psi_1 x).$$

In the second case we say the relation $L^P_{\alpha\beta} \leftrightarrow L^Q_{\mu\nu}$ is an inversion.

Let $\approx$ be the following equivalence relation of the set I_2 of all quasigroups $L^P_{\alpha\beta}$ ($P \in \{A, B, A_i, B_j\}$). For $L^P_{\alpha\beta}$, $L^Q_{\mu\nu} \in I_2$ we put $L^P_{\alpha\beta} \approx L^Q_{\mu\nu}$, iff $L^P_{\alpha\beta} \Leftrightarrow L^Q_{\mu\nu}$ and there exists at least one sequence defining $L^P_{\alpha\beta} \Leftrightarrow L^Q_{\mu\nu}$ with an even number of inversions.

The relation $\approx$ is containing in the relation $\Leftrightarrow$, more precisely, each class $C_\Leftrightarrow$ of the relation $\Leftrightarrow$ is the union of at most two classes $C'_\approx$ and $C''_\approx$ of the relation $\approx$.

Let all terms u_i, v_j of the law (1) be connected. By the lemmas 1 and 3, for every two operations $L^P_{\alpha\beta}$ and $L^Q_{\mu\nu}$ ($P, Q \in \{A, B, A_i, B_j\}$) we have $L^P_{\alpha\beta} \Leftrightarrow L^Q_{\mu\nu}$. We distinguish two cases:

1. $C_\Leftrightarrow = C_\approx$,
2. $C_\Leftrightarrow = C'_\approx \cup C''_\approx$.

In the case 1. there are two possibilities:

1.' From the law (1) does not yield any inversion, that is the law (1) is of the I kind.

1.″ From the law (1) yields at least one inversion, that is, the law (1) is of the II kind.

In the case 1″ for every quasigroup $L^P_{\alpha\beta}$ there exists at least one sequence defining $L^P_{\alpha\beta} \Leftrightarrow L^P_{\alpha\beta}$ with an odd number of inversions. Let, for example, the relation $L^P_{\gamma\delta} \leftrightarrow L^Q_{\mu\nu}$ be an inversion. Since $L^P_{\alpha\beta} \approx L^P_{\gamma\delta}$ and $L^Q_{\mu\nu} \approx L^P_{\alpha\beta}$, there exists a sequence $L^P_{\alpha\beta} \Leftrightarrow L^P_{\gamma\delta} \leftrightarrow L^Q_{\mu\nu} \Leftrightarrow L^P_{\alpha\beta}$ with an odd number of inversions.

In the case 2 each sequence defining $L^P_{\alpha\beta} \Leftrightarrow L^P_{\alpha\beta}$ has an even number of inversions. Indeed, let, for example, be $L^P_{\alpha\beta} \in C'_{\approx}$, and let exist a sequence defining $L^P_{\alpha\beta} \Leftrightarrow L^P_{\alpha\beta}$ with an odd number of inversions. Let $L^P_{\mu\nu} \in C'_{\approx}$, and $L^Q_{\rho\lambda} \in C''_{\approx}$. Then there exists a sequence $L^P_{\mu\nu} \Leftrightarrow L^P_{\alpha\beta} \Leftrightarrow L^P_{\alpha\beta} \Leftrightarrow L^Q_{\lambda\rho}$ with an even number of inversions, that is, $L^P_{\mu\nu} \approx L^Q_{\lambda\rho}$, what is in contradiction with the assumption about $L^P_{\mu\nu}$ and $L^Q_{\lambda\rho}$.

Let for the law (1) hold 2. We change all operations $L^P_{\alpha\beta}$ of one of the classes, say $C''_{\approx}$, with the operations $L^{P*}_{\alpha\beta}$ $(L^{P*}_{\alpha\beta}(x, y) \overset{\text{def}}{=} L^P_{\alpha\beta}(y, x))$. The obtained law $w^*_1 = w^*_2$ is of the I kind. Indeed, from so obtained law $w^*_1 = w^*_2$ it yields $C_{\approx} = C_{\Leftrightarrow}$, and for every operation $L^P_{\alpha\beta}$ each sequence defining $L^P_{\alpha\beta} \Leftrightarrow L^P_{\alpha\beta}$ has an even number of inversions.

Hence, we can consider only the laws of the II kind for which the relation $\Leftrightarrow$ and $\approx$ on the set I_2 are the same.

Theorem 1. *Let all quasigroups derived from quasigroups satisfying the law (1) be in the relation $\approx$. Then there exist a commutative group (S, o) so that the following equalities hold:*

$$A(x_1, \ldots, x_m) = L^A_1 x_1 \circ \cdots \circ L^A_m x_m$$

$$B(x_1, \ldots, x_n) = L^B_1 x_1 \circ \cdots \circ L^B_n x_n$$

$$L^A_i A_i(x_1, \ldots, x_\alpha) = L^A_i L^{Ai}_1 x_1 \circ \cdots \circ L^A_i L^{Ai}_\alpha x_\alpha$$

$$L^B_j B_j(x_1, \ldots, x_\beta) = L^B_j L^{Bj}_1 x_1 \circ \cdots \circ L^B_j L^{Bj}_\beta x_\beta$$

Proof. In the set S we introduce the binary operation $\circ$ defined by

$$L^A_{12}(x, y) = L^A_1 x \circ L^A_2 y.$$

Let $L^P_{\alpha\beta}$ and $L^Q_{\mu\nu}$ $(P, Q \in \{A, B\})$ be two quasigroups derived from the law (1) so that $L^P_{\alpha\beta} \leftrightarrow L^Q_{\mu\nu}$ holds. If this relation is an inversion, and if $L^P_{\alpha\beta}(x, y) = L^P_\alpha x \circ L^P_\beta y$, we have $L^Q_{\mu\nu}(x, y) = L^Q_\nu y \circ L^Q_\mu x$.

Since all quasigroups $L^P_{\alpha\beta}$ are in the relation $\approx$, and for every $L^P_{\alpha\beta}$ there exists a sequence with an odd number of inversions, defining $L^P_{\alpha\beta} \Leftrightarrow L^P_{\alpha\beta}$, we get

$$L^P_{\alpha\beta}(x, y) = L^P_\alpha x \circ L^P_\beta y, \text{ and}$$

$$L^P_{\alpha\beta}(x, y) = L^P_\beta y \circ L^P_\alpha x.$$

Hence, we get for every $x, y \in S$

$$L^P_\alpha x \circ L^P_\beta y = L^P_\beta y \circ L^P_\alpha x.$$

Since L^p_α and L^p_β are bijections, we have

$$x \circ y = y \circ x,$$

that is, the operation $\circ$ is commutative.

The proof of the rest of the theorem is analogous to the proof of the theorem 1 in [1].

Now we suppose the relation of connectness of the set $T = \{u_1, \dots, u_m, v_1, \dots, v_n\}$ of the law (1) has r $(r > 1)$ equivalence classes $C_i = \{u_{\alpha_i}, \dots, u_{\beta_i}, v_{\gamma_i}, \dots, v_{\delta_i}\}$, $i = 1, \dots, r$. Introducing quasigroups A and B conjugated with A and B respectively, from (1) we obtain a law in the form

$$\text{(2)} \qquad A(u_{\alpha_1}, \dots, u_{\beta_1}, \dots, u_{\alpha_r}, \dots, u_{\beta_r}) = B(v_{\gamma_1}, \dots, v_{\delta_1}, \dots, v_{\gamma_r}, \dots, v_{\delta_r}).$$

For the law (2) hold all results obtained in [1] for the analogous law of the I kind.

Finally, let us consider an arbitrary law in the form

$$\text{(3)} \quad A(A_1(x_1, \dots, x_\alpha), \dots, A_m(x_\beta, \dots, x_p)) = B(B_1(y_1, \dots, y_\gamma), \dots, B_n(y_\delta, \dots, y_p)),$$

where the sequence $y_1, \dots, y_p$ is a permutation of $x_1, \dots, x_p$, and A, B, A_i and B_j are quasigroups on a set S. Such a law can be transformed into a law (1) satisfying conditions (i), (ii) and (iii) by substitution of some quasigroups by GD-groupoids.

We give an example.

Let

$$A(A_1(x, y), z, A_3(u, v), A_4(w, t)) = B(x, B_2(y, z), B_3(u, w), B_4(v, t))$$

be the functional equality by unknown quasigroups $A, B, A_1, \dots, B_4$ of a set S. A general solution of this equality is given by

$$A(x, y, z, u) = \pi(\alpha x \circ \beta y, \mu z + \nu u),$$

$$B(x, y, z, u) = \pi(\gamma x \circ \varepsilon y, \sigma z + \tau u),$$

$$\alpha A_1(x, y) = \gamma x \circ \delta y,$$

$$\varepsilon B_2(x, y) = \delta x \circ \beta y,$$

$$\mu A_3(x, y) = \lambda x + \rho y,$$

$$\nu A_4(x, y) = \omega x + \pi y,$$

$$\sigma B_3(x, y) = \mu x + \omega y,$$

$$\tau B_4(x, y) = \rho x + \pi y,$$

where π is an arbitrary loop, $\circ$ is an arbitrary group, $+$ is an arbitrary commutative group, and $\alpha, \beta, \dots, \pi$ are arbitrary bijections of the set S.

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ON EXTENDING OF SOLUTIONS OF FUNCTIONAL EQUATIONS IN A SINGLE VARIABLE

Karol Baron

In this talk I want to present two results regarding the problem of the unique extension of solutions of the functional equation

$$\varphi(x) = h\left(x, \underset{s \in S}{\triangle} \varphi \circ f_s(x)\right) \tag{1}$$

in which

$$h : X \times Y^S \to Y \text{ and } f_s : X \to X, \quad s \in S,$$

where X, Y and S are arbitrary sets, are given functions. Here and in the sequel Y^S denotes the set of all functions from S into Y with the Tychonoff topology in the case where Y is a topological space whereas $\underset{s \in S}{\triangle} g_s$ denotes the diagonal of a family of transformations $\{g_s : s \in S\}$ (i.e. if g_s map X into Y, $s \in S$, then $\underset{s \in S}{\triangle} g_s$ is a map from X into Y^S such that for the projection map p_s

$$p_s \circ \underset{s \in S}{\triangle} g_s = g_s, \qquad s \in S).$$

Theorem 1. *Let $U \subset X$ be an arbitrary set such that*

$$f_s(U) \subset U, \qquad s \in S. \tag{2}$$

If

(i) *for every $x \in X$ there exists a positive integer k such that for every $s_1, \dots, s_k \in S$*

$$f_{s_1} \circ \cdots \circ f_{s_k}(x) \in U,$$

then for every solution $\varphi_0 : U \to Y$ of the equation (1) *there exists exactly one solution $\varphi : X \to Y$ of it such that $\varphi|_U = \varphi_0$.*

Moreover, if X and Y are topological spaces, U is open, h, f_s, $s \in S$, and φ_0 are continuous functions and

(ii) *for every open set V such that $U \subset V \subset X$ we have $\bigcap\{f_s^{-1}(V) : s \in S\}$ open,*

then φ is also continuous.

The hypothesis (i) in this theorem cannot be replaced (an example may be given) by

(iii) for every $x \in X$ there exists a positive integrer k such that for every $s \in S$, $f_s^k(x) \in U$.

On the other hand the hypothesis

(iv) X is a closed subset of a finite dimensional Banach space and $\{f_s : s \in S\}$ is a locally equicontinuous family such that for a certain $\xi \in X$

$$\sup\{\|f_s(x) - \xi\| : s \in S\} < \|x - \xi\|, \qquad x \in X \setminus \{\xi\}, \tag{3}$$

implies (i) whenever U is open (in X) and $\xi \in U$.

Theorem 2. *Let X be a closed and convex subset of a finite dimensional Banach space, $U \subset X$ an open set (in X) such that condition* (2) *is satisfied. If $\{f_s : s \in S\}$ is a locally equicontinuous family such that* (3) *holds for a certain $\xi \in U$, then for every solution $\varphi_0 : U \to Y$ of* (1) *there exists exactly one solution $\varphi : X \to Y$ of it such that $\varphi|_U = \varphi_0$.*

Moreover, if Y is a topological space, h and φ_0 are continuous functions, then φ is also a continuous function.

In view of the above mentioned connexion between hypotheses (iv) and (i) the first part of Theorem 2 follows from Theorem 1. However, in the other part of Theorem 2 the restrictive hypothesis (ii) does not occur.

On the other hand the proof of Theorem 1 is effective contrary to the proof of Theorem 2 where the Kuratowski-Zorn Lemma is used.

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INFINITARY QUASIGROUPS

Belousov D. Valentin, Stojaković M. Zoran

In this paper we give a short survey of the communication on infinitary quasigroups which took place during the Symposium on quasigroups and functional epuations in Belgrade in September 1974. Since one part of the results of this work is published in [7] in extensive form, and the other part is going to be published in [8], here we shall restrict ourselves only to the main results, definitions and theorems without proofs.

The notion of infinitary operation in implicit form can be found in different fields of mathematics. There are some attempts to consider those operations but up to now the notion of infinitary quasigroup was not submitted to any investigations.

We shall use the following notations. The sequence $x_m, x_{m+1}, \ldots, x_n$ will be denoted by x_m^n or $\{x_i\}_{i=m}^n$. If $m>n$, x_m^n will be considered empty, and if $m=n$ then x_m^n is the element x_m. By $\overset{n}{x}$ we denote the sequence $x, x, \ldots, x$ where x is repeated n times. The symbol $\overset{0}{x}$ denotes the empty sequence.

The infinite sequence $x_m, x_{m+1}, \ldots, x_n, \ldots$ (of order type ω) will be denoted by x_m^∞ or $\{x_i\}_{i=m}^\infty$ (m finite natural number). The infinite sequence $x, x, \ldots, x, \ldots$ we denote by $\overset{\infty}{x}$.

If Q is a nonempty set and α any ordinal by Q^α we denote the set of all sequences of order type α, of elements form Q. A mapping $A: Q^\alpha \to Q$, $(\alpha \geqslant \omega)$ we call an infinitary operation of the type α defined on Q. The type of A we shall denote by $|A|$. The set Q together with the infinitary operation A we call an infinitary operative and denote by $Q(A)$. The infinitary operative $Q(A)$ such that the set Q contains only one element will be called trivial.

The notion of quasigroup in infinitary case can be introduced in a natural way.

Definition 1. A set Q together with an infinitary operation A of the type ω we call an infinitary quasigroup of type ω (briefly ω-quasigroup), if the equation

$$A(a_1^{i-1}, x, a_{i+1}^\infty) = b$$

has a unique solution x for all a_1^∞, $b \in Q$ and for every positive integer i.

Definition 2. The element e of the infinitary operative $Q(A)$ of the type ω is called a unity if

$$A(\overset{i-1}{e}, x, \overset{\infty}{e}) = x,$$

for all $x \in Q$ and every $i = 1, 2, \ldots, n, \ldots$.

If an infinitary quasigroup $Q(A)$ of the type ω contains at least one unity then $Q(A)$ is called an infinitary loop (ω-loop).

First we have shown the existence of nontrivial infinitary quasigroups and loops. In [7] are given the constructions, based on the axiom of choice, of infinitary quasigroups of the type ω defined on the set of all real numbers, on the set of all integers and on the set with n elements, n arbitrary natural number. In a similar manner is shown the existence of nontrivial ω-loops, defined on the set D of all real numbers, in which the set of all unities is an arbitrary subset $M \subseteq D$. These constructions can be extended to the infinitary quasigroups of the type $\omega + k$ (definition 6).

Now we shall consider (i, j)-associative infinitary quasigroups.

Definition 3. An infinitary operative $Q(A)$ of the type ω is called (i, j)-associative if it satisfies the identity

$$(1) \qquad A(x_1^{i-1}, A(x_i^{\infty}), y_1^{\infty}) = A(x_1^{j-1}, A(x_j^{\infty}), y_1^{\infty}),$$

for all $x_1^{\infty}, y_1^{\infty} \in Q$.

(Of course, we suppose $i \neq j$).

Definition 4. An infinitary operative $Q(A)$ of the type ω is called an infinitary semigroup of the type ω (ω-semigroup) if it satisfies the identity (1) for all i and j.

Definition 5. An ω-quasigroup which is ω-semigroup is called an infinitary group of the type ω (ω-group).

Examples of ω-semigroups are given in [1]. Also it is proved there that nontrivial ω-groups do not exist.

Theorem 1. There does not exist a nontrivial (i, j)-associative infinitary quasigroup.

Now we introduce infinitary quasigroup of the type $\omega + k$ and consider functional equation (2) of generalized associativity on infinitary quasigroups.

Definition 6. An infinitary operative of the type $\omega + k$ is called infinitary quasigroup of the type $\omega + k$ if the equations

$$A(a_1^{i+1}, x, a_{i+1}^{\infty}, b_1^k) = c, \quad A(a_1^{\infty}, b_1^{i-1}, x, b_{i+1}^k) = c,$$

have a unique solution x for all a_p, b_q, $c \in Q$ and for all positive integers i in the first equation and for all $i = 1, 2, \ldots, k$ in the second equation.

Theorem 2. All solutions of the equation

$$(2) \qquad A(x_1^{i-1}, B(x_i^{\infty}, y_1^r), y_{r+1}^{\infty}) = C(x_1^{j+1}, D(x_j^{\infty}, y_1^s), y_{s+1}^{\infty}),$$

where A, B, C, D are infinitary quasigroups of types $|A|=\omega$, $|B|=\omega+r$, $|C|=\omega$, $|D|=\omega+s$, defined on the same nonempty set Q, i, j, some fixed natural numbers and r, s non-negative integers, are given by the following relations:

I_1 $(i=j,\ \ r=s\geqslant 0)$:

$$D=\theta B, \qquad A(x_1^{i-1}, z, y_{r+1}^{\infty})=C(x_1^{i-1}, \theta z, y_{r+1}^{\infty}), \tag{3}$$

where θ is an arbitrary permutation of the set Q, B and C arbitrary infinitary quasigroups of types $|B|=\omega+r$, $|C|=\omega$ and D and A are determined by the equations (3).

I_2 $(i=j,\ \ s>r\geqslant 0)$:

$$D(x_i^{\infty}, y_1^s)=K(B(x_i^{\infty}, y_1^r), y_{r+1}^s), \tag{4}$$

$$A(x_1^{i-1}, z, y_{r+1}^{\infty})=C(x_1^{i-1}, K(z, y_{r+1}^s), y_{s+1}^{\infty}), \tag{5}$$

where B and C are arbitrary infinitrary quasigroups of types $|B|=\omega+r$, $|C|=\omega$, K arbitrary quasigroup of arity $s-r+1$ and A and D are determined by the equations (4) and (5).

II_1 $(i<j,\ \ r\geqslant s\geqslant 0)$:

$$B(x_{i+1}^{\infty}, y_1^r)=K(x_i^{j-1}, D(x_j^{\infty}, y_1^s), y_{s+1}^r), \tag{6}$$

$$C(x_1^{j-1}, z, y_{s+1}^{\infty})=A(x_1^{i-1}, K(x_i^{j-1}, z, y_{s+1}^r), y_{r+1}^{\infty}), \tag{7}$$

where A and D are arbitrary infinitary quasigroups of types $|A|=\omega$, $|D|=\omega+s$, K arbitrary quasigroup of arity $j-i+r-s+1$ and B and C are determined by the equations (6) and (7).

II_2 $(i<j,\ \ s>r\geqslant 0)$:

$$A(x_1^{i-1}, x, y_{r+1}^{\infty})=K(x_1^{i-1}, \alpha x\circ\beta(y_{r+1}^s), y_{s+1}^{\infty}),$$

$$B(x_i^{\infty}, y_1^r)=\alpha^{-1}(\gamma(x_i^{j-1})\circ\delta(x_j^{\infty}, y_1^r)),$$

$$C(x_1^{j-1}, y, y_{s+1}^{\infty})=K(x_1^{i-1}, \gamma(x_i^{j-1})\circ\varphi y, y_{s+1}^{\infty}),$$

$$D(x_j^{\infty}, y_1^s)=\varphi^{-1}(\delta(x_j^{\infty}, y_1^r)\circ\beta(y_{r+1}^s)),$$

where α, φ are permutations of the set Q, $Q(\circ)$ binary group, β and γ quasigroups of arities $|\beta|=s-r$, $|\gamma|=j-i$, and δ, K infinitary quasigroups of types $|\delta|=\omega+r$, $|K|=\omega$.

The isotopy of two quasigroups can be generalized to infinitary case.

Two infinitary quasigroups of the type ω defined on the same set Q are called isotopic if there exists a sequence $T=\alpha_0^{\infty}$ of permutations of Q such that $B(x_1^{\infty})=\alpha_0^{-1}A(\{\alpha_i x_i\}_{i=1}^{\infty})$ for all $x_1^{\infty}\in Q$. This we shall denote by $B=A^T$. The usual theorems for isotopy of finitary quasigroups are also valid for infinitary case.

Definition 7. Let Q be a nonempty set, $Q(+)$ abelian group and $Q(R)$ infinitary quasigroup of the type ω, such that

$$R(\{x_i+y_i\}_{i=1}^{\infty})=R(x_1^{\infty})+R(y_1^{\infty}),$$

for all x_1^{∞}, $y_1^{\infty}\in Q$, then ω-quasigroup $Q(R)$ is called additive over abelian group $Q(+)$.

The existence of such nontrivial ω-quasigroups additive over abelian group is proved in [8].

It is also proved there that every infinitary quasigroup $Q(R)$ of the type ω additive over abelian group $Q(+)$ can be represented in the form

$$R(z_1^{\infty})=\alpha_1 z_1+\alpha_2 z_2+\cdots+\alpha_n z_n+R_n(z_{n+1}^{\infty}), \tag{8}$$

where n is an arbitrary natural number, α_i, $i=1, 2, \ldots, n$ automorphisms of the group $Q(+)$ and $Q(R_n)$ an infinitary quasigroup of the type ω additive over $Q(+)$.

Theorem 3. All solutions of the functional equation

$$A(B_1(x_1^{\infty}),\ B_2(y_1^{\infty}))=C(\{D_i(x_i,\ y_i)\}_{i=1}^{\infty}),$$

where A, D_i, $i=1, 2, \ldots$ are binary quasigroups and B_1, B_2, C infinitary quasigroups of the type ω, all defined on the same nonempty set Q, are given by

$$A(x, y)=\alpha x+\beta y,$$

$$D_i(x, y)=\varphi_i^{-1}(\gamma_i x+\theta_i y),$$

$$B_1(x_1^{\infty})=\alpha^{-1}(R(\{\gamma_i x_i\}_{i=1}^{\infty})+b),$$

$$B_2(x_1^{\infty})=\beta^{-1}(R(\{\theta_i x_i\}_{i=1}^{\infty})+a),$$

$$C(x_1^{\infty})=R\{\varphi_i x_i\}_{i=1}^{\infty})+c,$$

where $Q(+)$ is an arbitrary abelian group, $Q(R)$ arbitrary infinitary quasigroup of the type ω additive over $Q(+)$, α, β, γ_i, θ_i, φ_i, $i=1, 2, \ldots$ arbitrary permutations of the set Q, a, b, c arbitrary elements from Q such that $a+b=c$.

At the end we shall make some remarks concerning further investigations on infinitary quasigroups.

The notion of infinitary quasigroup of an arbitrary type α can also be introduced. Let X_α be a linear ordered sequence of variables of the type α, and let x be an arbitrary variable from X_α. Let X_{α_1} be the set of all variables from X_α which are less than x and X_{α_2} the set of all elements from X_α which are greater than x. Both X_{α_1} and X_{α_2} are linear ordered and they have the order types α_1 and α_2 respectively. Hence α can be represented as $\alpha_1+1+\alpha_2$. The operative $Q(A)$ of the type α is an infinitary quasigroup if the equation

$$A(C_{\alpha_1},\ x,\ C_{\alpha_2})=b,$$

where C_{α_1} and C_{α_2} are arbitrary linear ordered sequences from Q^{α_1} and Q^{α_2} respectively, b is an arbitrary element from Q, has a unique solution.

The definitions of the infinitary quasigroups of the type ω and $\omega+k$ are particular cases of the definition given above.

The definition of the isotopy of ω-quasigroups is already given and the notions of principal isotopy, isomorphism, autotopy etc. can be introduced as in finitary case. The inverse operations of the infinitary quasigroup operation of the type α are also defined in [7]. The notion of parastrophy can be extended to finitary case in two ways — one which directly generalizes the parastrophy of finitary quasigroups and the other which is specific for infinitary case, in which the type of the quasigroup is changed.

The insertion algebra for finitary quasigroups, which one of the authors considered in [4], [5], can also be extended to infinitary case.

At the end some problems on infinitary quasigroups were given. The precise formulation of these problems can be found in [7].

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REMARKS ON THE ENTROPY EQUATION

Z. Daróczy

1. In the paper [4] A. Kamiński and J. Mikusiński have proved the following

Theorem. *If a function* $H(x, y, z)$ *is continuous, symmetric and positively homogeneous (of order* 1) *in the domain*

$$D=\{(x, y, z) \mid x, y, z \geq 0, \ xy+yz+zx>0\}$$

and satisfies in the interior of D the functional equation

$$H(x, y, z)=H(x+y, 0, z)+H(x, y, 0) \tag{1}$$

then

$$H(x, y, z)=c[(x+y+z) \ln (x+y+z)-x \ln x-y \ln y-z \ln z], \tag{2}$$

where c is a real constant and $0 \ln 0 \stackrel{\text{def.}}{=} 0$.

From this theorem the authors have deduced a proof for the Faddeev's theorem [2] on the Shannon entropy.

In this note I shall give some remarks to the above result. In section 2. I shall determine the *general* solution of the problem of A. Kamiński and J. Mikusiński. This is a simple remark by a theorem of B. Jessen – J. Karpf. – A. Thorup [3]. By help of the general solution of the problem, in section 3., I prove a generalization of [4]. In this proof I use a known result of N. G. de Bruijn [1] on the difference functions.

2. Let $\mathbf{R}_+$ be the set of nonnegative real numbers and let $\mathbf{R}$ be the set of real numbers. Let $\mathcal{H}$ denote the set of all functions $H^*: \mathbf{R}_+^3 \to \mathbf{R}$ with following properties:

(i) For all $x, y, z \in \mathbf{R}_+$

$$H^*(x, y, z)=H^*(x+y, 0, z)+H^*(x, y, 0);$$

(ii) For all $x, y, t \in \mathbf{R}_+$

$$H^*(tx, ty, 0)=t H^*(x, y, 0).$$

Theorem 1. *If $H^* \in \mathcal{H}$ is a symmetric function, then there exists a function $f: \mathbf{R}_+ \to \mathbf{R}$ such that*

$$f(xy) = xf(y) + yf(x) \tag{3}$$

for all $x, y \in \mathbf{R}_+$ and

$$H^*(x, y, z) = f(x+y+z) - f(x) - f(y) - f(z) \tag{4}$$

for all $x, y, z \in \mathbf{R}_+$.

Proof. Let the function $F: \mathbf{R}_+^2 \to \mathbf{R}$ be defined by

$$F(x, y) = H^*(x, y, 0) \qquad (x, y \in \mathbf{R}_+). \tag{5}$$

By the symmetry of H^*

$$F(x, y) = F(y, x) \qquad (x, y \in \mathbf{R}_+). \tag{A 1}$$

From (i), by the symmetry of H^*, we have

$$F(x+y, z) + F(x, y) = H(x, y, z) = H(y, z, x) = F(y+z, x) + F(y, z),$$

from which by (A 1) it follows

$$F(x, y, z) + F(x, y) = F(x, y+z) + F(y, z) \tag{A 2}$$

for all $x, y, z \in \mathbf{R}_+$. From (ii) it follows

$$F(tx, ty) = tF(x, y) \tag{A 3}$$

for all $x, y, t \in \mathbf{R}_+$. In the paper [3] of B. Jessen – J. Karpf – A. Thorup it can be found the following assertion: If $F: \mathbf{R}_+^2 \to \mathbf{R}$ satisfies the functional equations (A 1), (A 2) and (A 3) then there exists a function $f: \mathbf{R}_+ \to \mathbf{R}$ for which (3) is true and

$$F(x, y) = f(x+y) - f(x) - f(y) \tag{6}$$

for all $x, y \in \mathbf{R}_+$. From (6) and (i) we have

$$H^*(x, y, z) = F(x+y, z) + F(x, y) = f(x+y+z) - f(x+y) -$$

$$- f(z) + f(x+y) - f(x) - f(y) = f(x+y+z) - f(x) - f(y) - f(z)$$

for all $x, y, z \in \mathbf{R}_+$, where the function $f: \mathbf{R}_+ \to \mathbf{R}$ satisfies the equation (3). Thus the theorem is proved.

Theorem 2. *If a function $H: D \to \mathbf{R}$ is symmetric in D and satisfies the equation* (1) *in the interior of D and $H(x, y, 0)$ ($x > 0$, $y > 0$) is positively homogeneous (of order* 1), *then there exists a function $f: \mathbf{R}_+ \to \mathbf{R}$ such that* (3) *holds for all $x, y \in \mathbf{R}_+$ and*

$$H(x, y, z) = f(x+y+z) - f(x) - f(y) - f(z) \tag{7}$$

for all $(x, y, z) \in D$.

P r o o f. We define the function $H^*:\mathbf{R}_+^3\to\mathbf{R}$ by the equations $H^*(x, y, z)=H(x, y, z)$ for all $(x, y, z)\in D$, $H^*(x, 0, 0)=H^*(0, x, 0)=H^*(0, 0, x)=0$ for $x\in\mathbf{R}_+$. It is easy to see that the function H^* is an element of $\mathcal{H}$ and it is symmetric in $\mathbf{R}_+^3$. By Theorem **1.** it follows our theorem.

3. In this section I shall give a generalization of the theorem of A. Kamiński and J. Mikusiński [4].

T h e o r e m 3. *Let $H: D\to\mathbf{R}$ be a symmetric function fulfilling the equation* (1) *in the interior of D and let $H(x, y, 0)$ $(x>0, y>0)$ be a positively homogeneous* (*of order* 1) *function. If for each $y>0$ the function*

$$x\to H(x, y, 0) \qquad (x>0)$$

is continuous, then (2) *holds for all $(x, y, z)\in D$, where $c\in\mathbf{R}$ is a constant and* $0\ln 0 \overset{\text{def.}}{=} 0$.

P r o o f. By Theorem **2.** there exists a function $f:\mathbf{R}_+\to\mathbf{R}$ such that (3) holds for all $x, y\in\mathbf{R}_+$ (from which we obtain $f(0)=0$) and

$$H(x, y, 0)=f(x+y)-f(x)-f(y)$$

for all $x>0$, $y>0$. By our assumption for each $y>0$ the difference function

$$x\to f(x+y)-f(x)$$

is *continuous*. N. G. de Bruijn [1, Theorem 1.3.] has proved the following result: If $f(x)$ is defined in an interval I, and if $f(x)$ is such that, for each y, $x\to f(x+y)-f(x)$ is continuous for all $x\in I\cap(I-y)$, then we have $f(x)=g(x)+A(x)$, where $g(x)$ is continuous in I, and $A:\mathbf{R}\to\mathbf{R}$ is an additive function, i.e. $A(x+y)=A(x)+A(y)$ for all $x, y\in\mathbf{R}$. In our case we know, that $x\to f(x+y)-f(x)$ is continuous in $I=\{x\,|\,x>0\}$ for each $y\in I$. If $y\notin I$, then $-y\geq 0$ and $I\cap(I-y)=(-y, \infty)$. Let $x\in(-y, \infty)$ be arbitrary and let $x_n\in(-y, \infty)$ be a sequence with $x_n\to x$. Then we have

$$f(x_n+y)-f(x_n)=-[f(x_n+y-y)-f(x_n+y)]$$

from which by $x_n+y>0$ and $-y\geq 0$ we obtain

$$\lim_{n\to\infty}[f(x_n+y)-f(x_n)]=-[f(x+y-y)-f(x+y)]=$$
$$=f(x+y)-f(x),$$

i.e. $x\to f(x+y)-f(x)$ is continuous in $I\cap(I-y)$. From this result we have

$$f(x)=g(x)+A(x) \qquad (x>0), \tag{8}$$

where $g(x)$ is a continuous function in I and $A:\mathbf{R}\to\mathbf{R}$ is an additive function. By Theorem **2.** we know, that the function $f:\mathbf{R}_+\to\mathbf{R}$ satisfies the equation (3) for all $x, y\in\mathbf{R}_+$, from which by (8) we obtain

$$g(xy)+A(xy)=x[g(y)+A(y)]+y[g(x)+A(x)] \tag{9}$$

for all $x, y>0$. We put $x=r$ and $y=s$ in (9), where r and s are arbitrary positive rational numbers. Then by $A(r)=rA(1)$ we obtain from (9)

$$g(rs)=rg(s)+sg(r)+rsA(1) \tag{10}$$

from which by the continuity of g it follows by taking limits $r\to x$, $s\to y$ that

$$g(xy)=xg(y)+yg(x)+xyA(1) \tag{11}$$

holds for all $x, y>0$. From (11) we have that the function

$$m(x)=\frac{g(x)}{x}+A(1) \qquad (x>0) \tag{12}$$

satisfies the functional equation

$$m(xy)=m(x)+m(y)$$

for all $x>0$, $y>0$. It is clear that m is continuous, i.e.

$$m(x)=c\ln x \qquad (x>0),$$

where $c\in\mathbf{R}$ is a constant. From (12) we obtain

$$g(x)=cx\ln x-xA(1)$$

for all $x>0$, from which it follows by (8)

$$f(x)=cx\ln x-xA(1)+A(x) \qquad (x>0) \tag{13}$$

where $A:\mathbf{R}\to\mathbf{R}$ is an additive function. By (3) $f(0)=0$, from which we obtain that (13) is true for $x=0$, if $0\ln \overset{\text{def.}}{=} 0$. From Theorem **2.** by (13) it follows (2) for all $(x, y, z)\in D$. Thus the theorem is proved.

It is clear, that Theorem **3.** is a generalization of the result of A Kamiń-ski and J. Mikusiński [4].

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NORMENQUADRAT ÜBER GRUPPEN

M. Hosszu, M. Csikós

Beschäftigen wir uns mit der folgenden Funktionalgleichung:

$$f(xy)+f(xy^{-1})=2f(x)+2f(y) \quad x, y \in G; \quad f: G \to A \tag{1}$$

wo $G(\cdot)$ ist eine Gruppe, $A(+)$ ist eine Abelsche Gruppe. Das Einheitselement von G ist e, y^{-1} ist das Inverselement von y. Das Nullelement von A ist 0, und nehmen wir an, dass aus $f(x)+f(x)=0$ d. h. aus $2f(x)=0$ $f(x)=0$ folgt.

Die Eigenschaften der Funktion f:

Wenn in der Gleichung (1) $y=e$ gesetzt wird, bekommen wir

$$f(e)=0.$$

Aus $x=e$ folgt, dass

$$f(y)=f(y^{-1}).$$

Setzen wir y statt x und x statt y ein, und auf Grund von $f(xy^{-1})=$ $=f[(xy^{-1})^{-1}]=f(yx^{-1})$ bekommen wir

$$f(xy)=f(yx).$$

Definition: Es sei

$$F(x, y)=f(x)+f(y)-f(xy).$$

M. Hosszu hat den folgenden Satz bewiesen

Wenn G eine mit Elementen a und b erzeugbare Gruppe ist, dann die Lösung der Gleichung (1) hat den Gestalt:

$$f(x)=f(a^{l_1}b^{m_1}a^{l_2}b^{m_2}\ldots a^{l_p}b^{m_p})=f(a)\left(\sum_{i=1}^{p} l_i\right)^2+f(b)\left(\sum_{i=1}^{p} m_i\right)^2-$$
$$-F(a, b)\left(\sum_{i=1}^{p} l_i\right)\cdot\left(\sum_{i=1}^{p} m_i\right),$$

wo die Funktionenwerte $f(a)$, $f(b)$ und $f(ab)$ beliebige Elementen der Gruppe A sind.

Jetzt wird die Gruppe G durch drei Elemente a, b und c erzeugt.

Definition. Es sei

$$F(x, y, z) = f(x) + f(y) + f(z) - f(xy) - f(xz) - f(yz) + f(xyz).$$

Hilfsatz 1. *Für $x = a^l b^m c^n$ l, m, n sind ganze Zahlen $f(x)$ hat den Gestalt:*

$$(2) \quad \begin{aligned} f(x) = f(a^l b^m c^n) = l^2 f(a) + m^2 f(b) + n^2 f(c) - lmF(a, b) - lnF(a, c) - \\ - mnF(b, c) + lmnF(a, b, c). \end{aligned}$$

Wir beweisen die Gleichung (2) mit Induktion. Wenn die Werte von l, m, n 0 oder l sind, dann die Gleichung (2) gilt offensichtlich. Nehmen wir an, dass die Gleichung (2) für irgendwelche Werte der Exponenten m, n zusammen mit $l-1$ und auch mit l gilt. Wir beweisen, dass die Gleichung (2) auch für $l+1$ gilt.

Aus (1) folgt

$$\begin{aligned} f(a^{l+1} b^m c^n) = f(a \cdot a^l b^m c^n) = 2f(a) + 2f(a^l b^m c^n) - f(a c^{-n} b^{-m} a^{-l}) = \\ = 2f(a) + 2f(a^l b^m c^n) - f(a^{l-1} b^m c^n). \end{aligned}$$

Wir haben die Eigenschaften der Funktion f benützt. Auf Grund der Induktionsbedingung folgt daraus, dass die Gleichung (2) zusammen mit Exponentenwerten m und n auch für $l+1$ gilt.

Damit sind wir fertig für nichtnegative ganze Exponentenwerten l, m, n, denn wir können mit zyklischer Permutation der Veränderlichen die Exponenten der Elemente b und c vergrössern. Die Induktion ist ganz ähnlich für ganze negative Exponenten.

Wir brauchen mehrere Faktoren mit Form $a^l b^m c^n$ ein beliebiges Element der Gruppe G herzustellen. Für zwei Faktoren es gilt:

$$\begin{aligned} f(a^{l_1} b^{m_1} c^{n_1} a^{l_2} b^{m_2} c^{n_2}) = 2f(a^{l_1} b^{m_1} c^{n_1}) + 2f(a^{l_2} b^{m_2} c^{n_2}) - \\ - f(a^{l_1 - l_2} b^{m_1} c^{n_1 - n_2} b^{-m_2}) = 2f(a^{l_1} b^{m_1} c^{n_1}) + 2f(a^{l_2} b^{m_2} c^{n_2}) - \\ - 2f(a^{l_1 - l_2} b^{m_1}) - 2f(b^{m_2} c^{n_2 - n_1}) + f(a^{l_1 - l_2} b^{m_1 + m_2} c^{n_2 - n_1}). \end{aligned}$$

Hieraus bekommen wir auf Grund der Gleichung (2), dass das gleich mit

$$\begin{aligned} (l_1 + l_2)^2 f(a) + (m_1 + m_2)^2 f(b) + (n_1 + n_2)^2 f(c) - (l_1 + l_2)(m_1 + m_2) F(a, b) - \\ - (l_1 + l_2)(n_1 + n_2) F(a, c) - (m_1 + m_2)(n_1 + n_2) F(b, c) + \\ + [(l_1 + l_2)(m_1 + m_2)(n_1 + n_2) - 2 l_1 m_2 n_1 - 2 l_2 m_1 n_2] F(a, b, c) \quad \text{ist.} \end{aligned}$$

Man kann vom Koeffizient des Gliedes $F(a, b, c)$ sehen, dass die Analogie zur Gruppe mit zwei erzeugenden Elemente nicht vollkommen ist. Man kann auch sehen, dass im Allgemeinen $f(abc) \neq f(acb)$, also wir können die Faktoren im Argument der Funktion f ohne Änderung des Funktionenwertes beliebig nicht vertauschen.

Für $x = a^{l_1} b^{m_1} c^{n_1} a^{l_2} b^{m_2} c^{n_2} \dots a^{l_p} b^{m_p} c^{n_p}$ die Funktion $f(x)$ hat den Gestalt:

$$
\begin{aligned}
&\left(\sum_{i=1}^{p} l_i\right)^2 f(a) + \left(\sum_{i=1}^{p} m_i\right)^2 f(b) + \left(\sum_{i=1}^{p} n_i\right)^2 f(c) - \left(\sum_{i=1}^{p} l_i\right)\left(\sum_{i=1}^{p} m_i\right) F(a, b) - \\
(3)\quad &-\left(\sum_{i=1}^{p} l_i\right)\left(\sum_{i=1}^{p} n_i\right) F(a, c) - \left(\sum_{i=1}^{p} m_i\right)\left(\sum_{i=1}^{p} n_i\right) F(b, c) + \\
&+\left[\left(\sum_{i=1}^{p} l_i\right)\left(\sum_{i=1}^{p} m_i\right)\left(\sum_{i=1}^{p} n_i\right) - 2 S_p(l_1 m_1 n_1 l_2 m_2 n_2 \dots l_p m_p n_p)\right] F(a, b, c),
\end{aligned}
$$

wo

$$
S_p(l_1 m_1 n_1 l_2 m_2 n_2 \dots l_p m_p n_p) = \sum_{i=1}^{p} l_i \sum_{j=1}^{p-1} m_{i+j} \sum_{k=1}^{j} n_{i+k-1} \qquad p = 2, 3, \dots
$$

Hier soll man die Indexen mod p reduzieren.

Der Beweis geht mit Induktion für p, mit einfacher Rechnung.

Wir haben den Funktionenwert

$$
f(a^{l_1} b^{m_1} c^{n_1} a^{l_2} b^{m_2} c^{n_2} \dots a^{l_p} b^{m_p} c^{n_p}),
$$

also für beliebige $x \in G$ den Ausdruck des Funktionenwertes $f(x)$ mit Hilfe der sieben Funktionenwerte $f(a)$, $f(b)$, $f(c)$, $f(ab)$, $f(ac)$, $f(bc)$, $f(abc)$ bekommen. Wir beweisen, dass diese sieben Funktionenwerte voneinander unabhängig sind.

Es sei

$$
x = a^{l_1} b^{m_1} c^{n_1} \dots a^{l_p} b^{m_p} c^{n_p}
$$

und

$$
y = a^{r_1} b^{s_1} c^{t_1} \dots a^{r_q} b^{m_q} c^{n_q},
$$

wo die Exponenten sind ganze Zahlen, p und q sind positive ganze Zahlen.

Wir substituieren x und y in die Gleichung (1), und zuerst kontrollieren wir die Gleichheit der Koeffizienten von $f(a)$, $f(b)$, $f(c)$, $F(a, b)$, $F(a, c)$, $F(b, c)$ an der beiden Seiten der Gleichung (1). Wir bekommen:

$$
\begin{aligned}
&f(a^{l_1} b^{m_1} c^{n_1} \dots a^{l_p} b^{m_p} c^{n_p} a^{r_1} b^{s_1} c^{t_1} \dots a^{r_q} c^{m_q} c^{n_q}) + \\
&+ f(a^{l_1} b^{m_1} c^{n_1} \dots a^{l_p} b^{m_p} c^{n_p} c^{-t_q} b^{-s_q} a^{-r_q} \dots c^{-t_1} b^{-s_1} a^{-r_1}) = \\
&= 2 f(a^{l_1} b^{m_1} c^{n_1} \dots a^{l_p} b^{m_p} c^{n_p}) + 2 f(a^{r_1} b^{s_1} c^{t_1} \dots a^{r_q} b^{s_q} c^{t_q})
\end{aligned}
$$

und mit Hilfe der Gleichung (3) für die Glieder mit $f(a)$:

$$
f(a)\left[\left(\sum_{i=1}^{p} l_i + \sum_{j=1}^{q} r_j\right)^2 + \left(\sum_{i=1}^{p} l_i - \sum_{j=1}^{q} r_j\right)^2\right] = f(a)\left[2\left(\sum_{i=1}^{p} l_i\right)^2 + 2\left(\sum_{j=1}^{q} r_j\right)^2\right]
$$

Quadrieren wir an der linken Seite! Die Gleichung gilt offensichtlich. Daraus folgt mit den Vertauschungen

$$
l_i \to m_i \qquad (i = 1, 2, \dots, p) \quad \text{und} \quad r_j \to s_j \qquad (j, = 1, 2, \dots, q),
$$

dass die Koeffizienten des Funktionenwertes $f(b)$ an beiden Seiten der Gleichung (1) gleich sind. Mit ähnlichen Vertauschungen folgt, dass auch die Koeffizienten des Funktionenwertes $f(c)$ gleich sind. Noch einmal mit Hilfe der Gleichung (3) bekommen wir:

$$-F(a,b)\left[\left(\sum_{i=1}^{p} l_i+\sum_{j=1}^{q} r_j\right)\left(\sum_{i=1}^{p} m_i+\sum_{j=1}^{q} s_j\right)+\left(\sum_{i=1}^{p} l_i-\sum_{j=1}^{q} r_j\right)\left(\sum_{i=1}^{p} m_i-\sum_{j=1}^{q} s_j\right)\right]=$$

$$=-F(a,b)\left[2\left(\sum_{i=1}^{p} l_i\right)\left(\sum_{i=1}^{p} m_i\right)+2\left(\sum_{j=1}^{q} r_j\right)\left(\sum_{j=1}^{q} s_j\right)\right].$$

Multiplizieren wir an der linken Seite! Die Gleichung gilt. Daraus folgt die Gleichheit der Koeffizienten von $F(a, c)$ und $F(b, c)$ mit entsprechenden Vertauschungen an beiden Seiten der Gleichung (1). Die Gleichheit der Koeffizienten von $F(a, b, c)$ kann man auf Grund der Gleichung (3) mit einer Induktion für q kontrollieren.

Der Ausdruck (3) ist also die allgemeinste Lösung der Gleichung (1) mit sieben unabhängigen Konstanten.

Wir suchen jetzt die Lösung der Funktionalgleichung (1), wenn die Gruppe G durch n Elemente erzeugt mird ($u \geqq 4$).

Dazu brauchen wir drei Hilfsätze.

Hilfsatz 2.

(4) $$f(xyz)+f(xzy)=2f(xy)+2f(xz)+2f(yz)-2f(x)-2f(y)-2f(z)$$

gilt für alle $x, y, z \in G$.

Beweis. Wir haben

$$f(xy\cdot z)=2f(xy)+2f(z)-f(x\cdot yz^{-1})=2f(xy)+2f(z)-$$
$$-2f(x)-2f(y\cdot z^{-1})+f(xz\cdot y^{-1})=2f(xy)+2f(z)-2f(x)-4f(y)-4f(z)+$$
$$+2f(yz)+2f(xz)+2f(y)-f(xzy).$$

Daraus folgt der Hilfsatz mit Umordung.

Definition. Es sei

$$F(a,b,c,d)=f(a)+f(b)+f(c)+f(d)-f(ab)-f(ac)-f(ad)-f(bc)-$$
$$f(bd)-f(cd)+f(abc)+f(abd)+f(acd)+f(bcd)-f(abcd).$$

Hilfsatz 3. $F(a, b, c, d)=0$. *Die Verifikation des Hilfsatzes* (3) *geht mit einer ähnlichen Rechnung, wie die Verifikation des Hilfsatzes* (2).

Die Rechnung ist aber ein wenig kompliziert.

Definition. Es sei

$$F(a_1, a_2, \ldots, a_n)=\sum_{j=1}^{n}(-1)^{j+1}\sum_{1\leqq i_1<i_2<\cdots<i_j\leqq n} f(a_{i_1}a_{i_2}\ldots a_{i_j}).$$

Hilfsatz 4. *Für $n \geq 4$ gilt $F(a_1, a_2, \ldots, a_n) = 0$. Die Verifikation geht natürlich mit Induktion. Wir nehmen an, dass der Hilfsatz (4) gilt für n ($n \geq 4$). Daraus wird es folgen, dass er auch für $n+1$ gilt.*

Verwenden wir die Definition für $F(a_1, a_2, \ldots, a_n, a_{n+1})$, und schreiben wir drei Glieder der Summe ausführlich aus!

$$\begin{aligned} F(a_1, a_2, \ldots, a_n, a_{n+1}) &= \sum_{j=1}^{n-1} (-1)^{j+1} \sum_{1 \leq i_1 < \cdots < i_j \leq n+1} f(a_{i_1} a_{i_2} \ldots a_{i_j}) + \\ &+ (-1)^{n+1} f(a_1 a_2 \ldots a_{n-1} a_n) + (-1)^{n+1} f(a_1 a_2 \ldots a_{n-1} a_{n+1} + \cdots + \\ &+ (-1)^{n+2} f[a_1 a_2 \ldots a_{n-1} (a_n a_{n+1})]. \end{aligned} \tag{5}$$

Benützen wir für diese Glieder die Induktionsbedingung! Nach der Einsetzung und den Zusammenziehungen bekommen wir 0 an der rechten Seite der Gleichung (5). Es ist sichtbar, wenn wir die Glieder der Summe den Vorkommen von a_n, a_{n+1} und $a_n a_{n+1}$ entsprechend gruppieren.

Wenn $a_1, a_2, \ldots, a_n$ $n \geq 4$ die erzeugende Elemente der Gruppe G sind, und $x = a_1^{l_{11}} a_2^{l_{21}} \ldots a_n^{l_{n1}} a_1^{l_{12}} a_2^{l_{22}} \ldots a_n^{l_{n2}} \ldots a_1^{l_{1p}} a_2^{l_{2p}} \ldots a_n^{l_{np}}$ dann ist die Lösung der Gleichung (1)

$$\begin{aligned} f(x) = \sum_{j=1}^{n} f(a_j) \left(\sum_{i=1}^{p} l_{ji} \right)^2 - \sum_{1 \leq i < j \leq n} F(a_i, a_j) \left(\sum_{t=1}^{p} l_{it} \right) \left(\sum_{t=1}^{p} l_{jt} \right) + \\ + \sum_{1 \leq i < j < k \leq n} F(a_i, a_j, a_k) \cdot \left[\left(\sum_{t=1}^{p} l_{it} \right) \left(\sum_{t=1}^{p} l_{jt} \right) \left(\sum_{t=1}^{p} l_{kt} \right) - S_p(l_{i1} l_{j1} l_{k1} \ldots l_{ip} l_{jp} l_{kp}) \right], \end{aligned}$$

wo die Funktionenwerte

$$\begin{array}{ll} f(a_i) & 1 \leq i \leq n \\ f(a_i a_j) & 1 \leq i < j \leq n \\ f(a_i a_j a_k) & 1 \leq i < j < k \leq n \end{array} .$$

sind beliebige Elemente aus A.

Die Funktionalgleichung (1), und die Eigenschaften der Funktion f werden aus dem Werk von S. Kurepa genommen, aber er löst nicht diese Funktionalgleichung.

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LATIN SQUARES, *P*-QUASIGROUPS AND GRAPH DECOMPOSITIONS

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In this mainly expository paper, we discuss two particular types of quasigroup (latin square) which have connections with other branches of mathematics, notably with statistics, graph theory and coding theory.

A square $n \times n$ matrix L on n distinct symbols is called *row complete* if every pair of symbols of L occurs just once as an adjacent pair of elements in some row of L. It is called *row latin* if each symbol occurs exactly once in each row of the matrix. The concepts *column complete* and *column latin* are similarly defined. A square matrix which is both row latin and column latin is called a *latin square*.

We shall also find it convenient to call a rectangular matrix R of size $m \times n$ or $n \times m$ *row complete*, where $m \leqslant \left[\frac{1}{2} n\right]$, if each *unordered* pair of its symbols occurs just once as an adjacent pair of elements in some row of R. Here, [ ] denotes "integer part".

Row complete latin squares are used in statistics in connection with the design of experiments. They are of particular value for the design of sequential experiments but may also be useful for eliminating interactions between adjacent plots in field experiments. A detailed explanation of these applications is given in [2]. Here, we shall be content to give a single illustration. In an experiment on farm animals, it is desired to apply a number of different dietary treatments to a given animal in succession. The effect of a given treatment on the animal may be affected both by the number of treatments which that animal has already received and also by the nature of the immediately preceding treatment which it has had applied to it. If several animals are available for treatment, the first possibility can be allowed for statistically if it can be arranged that the number n of animals to be treated is equal to the number of treatments to be applied and if the order in which the treatments are to be applied to these n animals is allowed to be determined by the order of the entries in the n rows of an $n \times n$ latin square (whose n distinct elements denote the n treatments). Then any particular experiment has a different number of predecessors for each of the n different animals, since a given element of the latin square is preceded by a different number of other elements in each of the n rows of the square. The possibility of interaction between one experiment and the immediately preceding one can also be allowed for if the latin square chosen is row complete. The resulting experiment is then said to be statistically "balanced" both with respect to the effect of the immediately preceding experiment and also with respect to the number of preceding experiments.

Until very recently, the only row complete latin squares known were multiplication tables of groups (or quasigroups isotopic to groups). Also each of these known row complete latin squares had the property that it could be made column complete as well by a suitable reordering of its rows. In fact, it is not difficult to show:

Theorem 1. *Every row complete latin square which represents the multiplication table of a group can be made column complete as well as row complete by suitably reordering its rows.*

Proof: — Let the given square be the multiplication table of the group G where $h_1, h_2, \ldots, h_n$ and $g_1, g_2, \ldots, g_n$ are two orderings of the elements of G, as in Fig. 1.

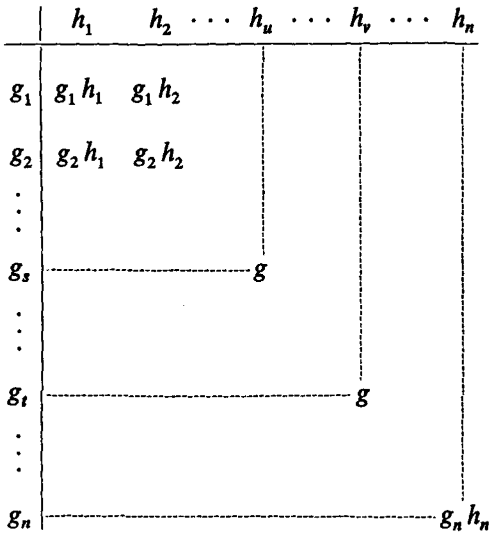

Fig. 1

Since the square is row complete the elements $h_1^{-1}h_2, h_2^{-1}h_3, \ldots, h_{n-1}^{-1}h_n$ are all distinct and are the non-identity elements of the group in a new order: for suppose that $h_u^{-1}h_{u+1} = h_v^{-1}h_{v+1} = k$ say. Let the arbitrary element g of G occur in the s^{th} row of column u and in the t^{th} row of column v. Then $g = g_s h_u = g_t h_v$. The entries in the $(u+1)^{\text{th}}$ column of row s and in the $(v+1)^{\text{th}}$ column of row t are $g_s h_{u+1} = (g_s h_u)(h_u^{-1} h_{u+1}) = gk$ and $g_t h_{v+1} = (g_t h_v)(h_v^{-1} h_{v+1}) = gk$ respectively. Hence, the ordered pair (g, gk) occur as adjacent elements in both the s^{th} and the t^{th} rows of the square, contrary to hypothesis.

Now let the rows be reordered according to the permutation

$$\begin{pmatrix} g_1 & g_2 & \cdots & g_n \\ h_1^{-1} & h_2^{-1} & \cdots & h_n^{-1} \end{pmatrix}$$

so that the reordered square takes the form shown in Fig. 2. This reordering will not affect the row completeness.

Moreover, in the new square each ordered pair of elements will occur at most once as a pair of adjacent elements in the columns : for, suppose that the entries of the $(s, u)^{\text{th}}$ and $(t, v)^{\text{th}}$ cells are the same, equal to g say. Then, $h_s^{-1} h_u = g = h_t^{-1} h_v$. The entries of the $(s+1, u)^{\text{th}}$ and $(t+1, v)^{\text{th}}$ cells must then be distinct, for $h_{s+1}^{-1} h_u = h_{t+1}^{-1} h_v$ would imply $(h_{s+1}^{-1} h_s)(h_s^{-1} h_u) = (h_{t+1}^{-1} h_t)(h_t^{-1} h_v)$ and so $(h_{s+1}^{-1} h_s)\, g = (h_{t+1}^{-1} h_t)\, g$. But then, $h_{s+1}^{-1} h_s = h_{t+1}^{-1} h_t$ whence $(h_{s+1}^{-1} h_s)^{-1} = (h_{t+1}^{-1} h_t)^{-1}$. Thus we would have $h_s^{-1} h_{s+1} = h_t^{-1} h_{t+1}$ which is contrary to hypothesis. This shows that the new square is column complete as well as row complete and so proves the theorem.

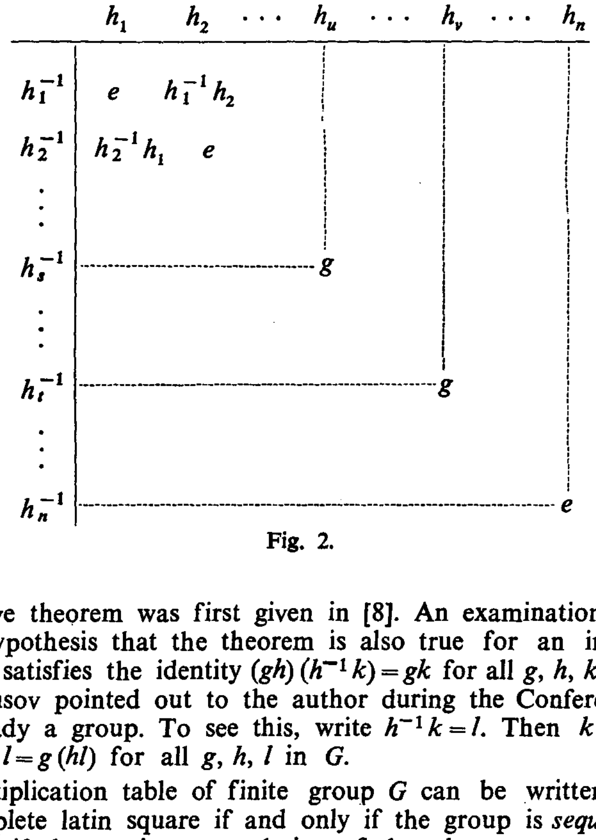

Fig. 2.

The above theorem was first given in [8]. An examination of the proof suggests the hypothesis that the theorem is also true for an inverse property loop G which satisfies the identity $(gh)(h^{-1}k) = gk$ for all g, h, k in G. However (as V.D. Belousov pointed out to the author during the Conference itself) such a loop is already a group. To see this, write $h^{-1}k = l$. Then $k = (eh)(h^{-1}k) = hl$ and so $(gh)\,l = g\,(hl)$ for all g, h, l in G.

The multiplication table of finite group G can be written in the form of a row complete latin square if and only if the group is *sequenceable* : that is, if and only if there exists an ordering of the elements $g_1, g_2, \ldots, g_n$ of G such that the partial products $p_s = \prod_{i=1}^{s} g_i$ for $s = 1, 2, \ldots n$ are all distinct. (For the orginal proof, see [4].). To see the necessity of this condition, let the row complete latin square $L = (g_{ij})$ be the multiplication table of G so that $g_{ij} = g_i g_j$. In that case $g_{ij}^{-1} g_{i,j+1} = g_j^{-1} g_i^{-1} g_i g_{j+1} = g_j^{-1} g_{j+1} = h_j$ say for all values of i. Suppose that $h_j = h_{j'}$ for $j' \neq j$. Then, because $g_{i'j'} = g_{ij}$ for some value of i'

(each element of G occurs exactly once in each column of L), we have $g_{i',j'+1} = g_{i'j'} h_{j'} = g_{ij} h_j = g_{i,j+1}$. However, this contradicts the row completeness of L. Thus $h_j \neq h_{j'}$ unless $j' = j$. Consider now the first row of L. Its j^{th} element is $g_{ij} = g_{11} (g_{11}^{-1} g_{12}) (g_{12}^{-1} g_{13}) \cdots (g_{1,j-1}^{-1} g_{1j}) = g_{11} h_1 h_2 \cdots h_{j-1}$. Since the elements of the first row of L are all different, it follows that the partial products $\prod_{k=1}^{j-1} h_k$ for $j = 2, 3, \ldots, n-1$ are all distinct, where the elements $h_1 h_2, \ldots, h_{n-1}$ are the non-identity elements of G. That is, the elements $e, h_1 h_2, \ldots, h_{n-1}$ form a sequencing for G. To see the sufficiency of the condition, consider the latin square $L = (g_{ij})$ where $g_{ij} = p_i^{-1} p_j$, p_i being one of the partial products defined above. We require to show that the ordered pair of elements (u, v) of G occur consecutively in some row of L. That is, we require to find integers i, j such that $p_i^{-1} p_j = u$ and $p_i^{-1} p_{j+1} = v$. From these two equations, $u g_{j+1} = v$. This determines j. Then $p_i = p_j u^{-1}$ and this fixes i. Thus, every pair of elements of G occurs exactly once and L is row complete.

Evidently, $p_1 = g_1 = e$ (where e denotes the group identity) is necessary for a group G to be sequenceable. If the group G is abelian, it is known that $p_n = e$ unless G has a unique element h of order two and that, in the latter case, $p_n = h$. (see [12]). Thus, a finite abelian group can be sequenceable only if it has a unique element of order two. B. Gordon [4] has proved that this condition is sufficient as well as necessary. Namely, a finite abelian group is sequenceable if and only if it is the direct product of two groups A and B such that A is a cyclic group of order 2^k, $k > 0$, and B is of odd order.

As regards the sequenceability of groups of odd order, little is known. It is clear from the preceding remarks that an abelian group of odd order cannot be sequenceable. The non-abelian group of smallest odd order is the (unique) non-abelian group of order 21 generated by two elements a and b with the defining relations $a^7 = b^3 = e$, $ab = ba^2$. This group has been shown to be sequenceable by N. S. Mendelsohn [10]. The non-abelian group of order 27 generated by two elements a and b with the defining relations $a^p = b^q = e$, $ab = ba^r$, where $p = 9$, $q = 3$ and $r = 4$, has been shown to be sequenceable by the present author [6] and very recently the groups on two generators with similar structure having orders 39 ($p = 13$, $q = 3$, $r = 3$), 55 ($p = 11$, $q = 5$, $r = 3$) and 57 ($p = 19$, $q = 3$, $r = 7$) have been shown to be sequenceable by L. L. Wang [13]. The present author has conjectured in [6] that all non-abelian groups on two generators are sequenceable and the recent results of L. L. Wang lend strength to this conjecture.

As regards non-abelian sequenceable groups of even order, B. Gordon [4] has shown that the dihedral groups D_3 and D_4 of orders 6 and 8 are not sequenceable, and J. Dénes and E. Török [3] have shown that the dihedral groups D_5, D_6, D_7 and D_8 of orders 10, 12, 14 and 16 are sequenceable but that the remaining non-abelian groups of orders less than or equal to 14 are not sequenceable.

It is known that there are no row complete latin squares of orders 2, 3, 5 or 7. This has been shown by D. Warwick [14] and by P. J. Owens [11]. Very recently, P. J. Owens has constructed the first examples of row complete latin squares which are not the multiplication tables of groups (that is, they do not satisfy the quadrangle condition, see [2]). These are of orders 8 and

10. He has also constructed a row complete latin square of order 14 which cannot be made column complete as well by any reordering of its rows.

Finally, we mention that a row complete latin square of order n defines a decomposition of the complete directed graph on n vertices into disjoint Hamiltonian paths. To see this, let the vertices of the graph be labelled by means of the symbols of the square. Then each row of the square defines a Hamiltonian path whose directed edges are given by the ordered pairs of adjacent symbols in that row. This fact was first observed by N. S. Mendelsohn [10] and by J. Dénes and E. Török [3]. If n is even, a row complete $\frac{1}{2}n\times n$ latin rectangle similarly defines a decomposition of the complete undirected graph on n vertices into disjoint Hamiltonian paths. Also, suitable row complete latin rectangles exist for all even values of n, as is shown in [7] and [2]. (Since each Hamiltonian path has $n-1$ edges and the complete undirected graph has $\frac{1}{2}n(n-1)$ edges, no decomposition of this kind can exist if n is odd).

Another type of quasigroup (latin square) which defines decompositions of the complete undirected graph is the so-called *P-quasigroup (or partition quasigroup)*.

Let us first define a P-groupoid.

D e f i n i t i o n. A groupoid $(Q, \cdot)$ is called a *P-groupoid* if it satisfies the following three properties: (i) $a\cdot a=a$ for all $a\in Q$; (ii) $a\neq b$ implies $a\neq a\cdot b$ and $b\neq a\cdot b$ for all $a, b\in Q$; (iii) $a\cdot b=c$ implies and is implied by $c\cdot b=a$ for all $a, b, c\in Q$.

A one-to-one correspondence between P-groupoids of n elements and decompositions of the complete undirected graph on n vertices into disjoint closed paths is easily established by labelling the vertices of the graph with the elements of the P-groupoid and prescribing that the edges (a, b) and (b, c) shall belong to the same closed path of the graph if and only if $a\cdot b=c$, $a\neq b$. We deduce at once that the number of elements of a P-groupoid is odd. *A* P-groupoid which is also a quasigroup is called a *P-quasigroup*. Thus, a P-quasigroup is an idempotent quasigroup with the additional property that whenever the relation $a\cdot b=c$ holds in $(Q,\cdot)$ so also does the relation $c\cdot b=a$.

The concepts of P-groupoid and P-quasigroup were introduced by A. Kotzig [9]. The following facts were first pointed out in [7], [5], [1] and [8] respectively.

O b s e r v a t i o n 1. *A decomposition of the complete undirected graph on n vertices* $v_1, v_2, \dots, v_n$ *into disjoint closed paths corresponds to a P-quasigroup* $(V,\cdot)$ *if and only if, for fixed values of i and k,* (v_i, v_j) *and* (v_j, v_k) *are adjacent edges of a closed path for one and only one value of j.*

P r o o f. If $(V,\cdot)$ is a P-quasigroup, the entry k occurs once and once only in the i^{th} row of the multiplication table of $(V,\cdot)$. Let the column in which this entry occurs be the j^{th}. Then we have $i\cdot j=k$ and (v_i, v_j), (v_j, v_k) are adjacent edges of a closed path of G_n for this value of j and no other.

O b s e r v a t i o n 2. *Commutative P-quasigroups of order n exist exactly when* $n\equiv 1$ *or 3 mod 6 and then and only then the complete undirected graph on n vertices can be decomposed into disjoint triangular circuits.*

Proof. The vertices of the triangles define the triads of a Steiner triple system.

Observation 3. A P-quasigroup of order n exists which defines a decomposition of the complete undirected graph on n vertices into disjoint Hamiltonian circuits whenever n is a prime.

Proof. We define the required P-quasigroup by taking the set $V=\{1, 2, \ldots n\}$ and observing that, if an operation $(\cdot)$ is defined on V by the statement $r \cdot s = 2s - r \pmod n$, we obtain a P-quasigroup $(V, \cdot)$ having the desired property.

For further details and a discussion of the connection between $(V, \cdot)$ and a certain row complete latin square, see [1].

Observation 4. The existence of a P-quasigroup of order $n=2r+1$ which defines a decomposition of the complete undirected graph on n vertices into a single Eulerian closed path is equivalent to the existence of a codeword on $2r+1$ symbols of length $r(2r+1)+1$ in which no pair of consecutive symbols and no pair of nearly consecutive symbols is repeated.

Proof. Two symbols of a codeword are said to be *nearly consecutive* if they are separated by a single symbol. We may establish a correspondence between Eulerian circuits of the complete undirected graph G_n on n vertices and codewords of length $\frac{1}{2}n(n-1)+1$ by regarding each pair of consecutive symbols of the codeword as representing an edge of the graph joining the vertices represented by those two symbols. The last symbol of the codeword is taken to be the same as the first in order that the path represented should be closed. Since each edge of the graph occurs exactly once in an Eulerian circuit, each pair of consecutive symbols must occur once and only once in the corresponding codeword. Also if the Eulerian circuit is to correspond to a P-quasigroup, each pair of nearly consecutive symbols must occur in the codeword at most once otherwise the property stated in observation 1 above would be violated.

In his original paper [9], A. Kotzig raised the question "For which values of n does a P-quasigroup exist which defines a decomposition of the complete undirected graph on n vertices into a single Eulerian closed path ?" He showed that such a P-quasigroup exists for the orders $n=3$ and 7 but not when $n=5$. Subsequent work on this topic has made use of the equivalence with the codeword existence problem which is stated in observation 4 above and has shown that suitable P-quasigroups exist whenever $n=4r+3$ except possibly when $r \equiv 127$ mod 595 and whenever $n=4r+1$ $(r \neq 1)$ except possibly when $r \equiv 5$ mod 7.

The main theorem required is as follows:

Theorem 2. *Let U denote a sequence of non-zero integers $u_1\, u_2, \ldots, u_r$ such that $-r \leqslant u_i \leqslant r$ and $|u_j| \neq |u_i|$ unless $j=i$ (so that $|u_1|, |u_2|, \ldots, |u_r|$ is a reordering of the natural numbers $1, 2, \ldots r$). Let $\sigma_s = \sum_{i=1}^{s} u_i$, mod $2r+1$. Also, let $u_i + u_{i+1} \equiv h_i$ mod $2r+1$ for $i=1, 2, \ldots, r-1$, where $-r \leqslant h_i \leqslant r$; and let h_r denote the smallest integer congruent to $u_r + u_1$ modulo $2r+1$. Then, if such a* *sequence U exists with the following additional properties: (a) the integers* $|h_i|$ *are all distinct for* $i=1, 2, \ldots, r$*; and (b)* $(\sigma_r, 2r+1)=1$*, there exists a codeword on* $2r+1$ *symbols of length* $r(2r+1)+1$ *in which no pair of consecutive symbols and no pair of nearly consecutive symbols is repeated.* (Equivalently, there exists an Eulerian circuit of the complete undirected graph on $2r+1$ vertices which corresponds to a P-quasigroup).

If such a sequence U exists with the following alternative additional properties: (a) the integers* $|h_i|$ *are all distinct for* $i=1, 2, \ldots, r-1$ *and no one of them is equal to* 1; *(b)** $u_1=1$ *or* 2; *and (c)* i f*$u_1=1$, $(-3+\sigma_r, 2r+1)=1$; *if* $u_1=2$, $(-2+\sigma_r, 2r+1)=1$, *then there exists a codeword on* $4r+3$ *symbols of length* $(2r+1)(4r+3)+1$ *in which no pair of consecutive symbols and no pair of nearly consecutive symbols is repeated.*

The second part of this theorem is due to the present author and a proof will be found in [8]. The first part is the joint work of the present author and A. J. W. Hilton. It is proved in [5].

Once the theorem has been established, it only remains to show the existence of suitable sequences U. By way of illustration we state the following theorem which is proved fully in [5].

Theorem 3. *The following sequences U satisfy the conditions* (*a*) *and* (*b*) *of theorem* 1: —

$$r=2t;\ u_1=-2t,\ u_2=2t-2,\ u_3=2t-4, \ldots, u_{t-1}=4,\ u_t=2,\ u_{t+1}=1,$$

$$u_{t+2}=3, \ldots, u_{2t-1}=2t-3,\ u_{2t}=2t-1,\ \textit{modulo}\ 4t+1;\ t\neq 1\ \textit{and}\ t\not\equiv 5\ \textit{mod}\ 7;$$

$$r=2t+1;\ u_1=-(2t+1),\ u_2=2t-1,\ u_3=2t-3, \ldots, u_t=3,\ u_{t+1}=1,$$

$$u_{t+2}=2,\ u_{t+3}=4, \ldots, u_{2t}=2t-2,\ u_{2t+1}=2t,\ \textit{modulo}\ 4t+3;\ t\not\equiv 1\ \textit{mod}\ 7.$$

Theorem 2 and the sequences obtained in [8] and [5] together solve Kotzig's problem for all values of n of the form $4r+1$ except $r=1$ and $r\equiv 5$ mod 7 and they also solve it for all values of n of the form $4r+3$ except when $r\equiv 127$ mod 595. It is likely that the construction of further sequences U which satisfy the conditions of theorem 2 (that is, sequences U additional to the several classes of such sequences obtained in [8] and [5]) would enable Kotzig's P-quasigroup problem and the equivalent codeword existence problem to be resolved completely.

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A FUNCTIONAL EQUATION WITH DIFFERENCES

Gyula Maksa

1. Introduction

Let $\mathbf{R}$ denote the set of real numbers. Let us determine all functions $f:\mathbf{R}\to\mathbf{R}$ such that

$$\frac{f(tx+ty)-f(tx)}{f(tx)-f(tx-ty)}=\frac{f(x+y)-f(x)}{f(x)-f(x-y)} \tag{1}$$

for all $x, y, t\in\mathbf{R}$, $yt\neq 0$. This problem is due to P. Drăgilă [1].

Without loss of generality we can assume that $f(0)=0$ and $f(1)=1$. In the following we employ the notation

$$\mathcal{F}=\left\{f \,\middle|\, \begin{array}{l} f:\mathbf{R}\to\mathbf{R};\ f \text{ satisfies the equation (1) for all } x, y, t\in\mathbf{R};\ yt\neq 0 \\ f(0)=0;\ f(1)=1 \end{array}\right\}$$

The purpose of this paper is to give a necessary and sufficient condition for $f\in\mathcal{F}$. A sufficient condition can easily be given: Let $f:\mathbf{R}\to\mathbf{R}$ be invertible with the properties $f(0)=0$ and $f(1)=1$, and additive or multiplicative. Then $f\in\mathcal{F}$. We shall show that these conditions are essentially also necessary.

2. Necessary conditions

The form of the equation (1) suggests a simple necessary condition for $f\in\mathcal{F}$. It can be formulated as follows:

Lemma 1. *Let $f\in\mathcal{F}$, then f is invertible and an odd function.*

Proof. If the function f were not invertible we could find real numbers $x_0, y_0, y_0\neq 0$ so that $f(x_0)=f(x_0-y_0)$, therefore the equation (1) could not hold for $x=x_0$, $y=y_0$, $t\neq 0$. In order to prove that f is odd, we must put $x=0$, $t=\frac{1}{y}$ in the equation (1). Thus we obtain $f(-y)=f(-1)f(y)$ for all $y\in\mathbf{R}$. Hence $f^2(-1)=1$, but f is invertible, thus $f(-1)=-1$. ∎

We guess that, if $f\in\mathcal{F}$ then either f is multiplicative or f is an additive function. The following lemma allows to distinguish these two cases.

Lemma 2. *Let $f\in\mathcal{F}$, then the set*

$$L=\{\alpha\mid\alpha\in\mathbf{R};\quad f(\alpha x)=f(\alpha)f(x)\ \text{for all}\ x\in\mathbf{R}\}$$

is a field.

Proof. It is obvious that $0, 1\in L$, furthermore if $\alpha\in L$ then $(-\alpha)\in L$, for f is odd by Lemma 1. Now, let $\alpha\in L$ and $\alpha\neq 0$, then the equation $f(x)=f(\alpha)f\left(\frac{x}{\alpha}\right)$, $(x\in\mathbf{R})$ implies that

$$f\left(\frac{1}{\alpha}x\right)=\frac{1}{f(\alpha)}f(x)=f\left(\frac{1}{\alpha}\right)f(x)\qquad(x\in\mathbf{R}),$$

therefore $\frac{1}{\alpha}\in L$. Putting $x=y$, $t=\frac{1}{x}$ in the equation (1), we have $2\in L$. Thus if we write $x+2y$ instead of y in the equation (1) we get

$$\text{(2)}\qquad \begin{aligned}&[f(2)f(tx+ty)-f(tx)][f(x)+f(2)f(y)]=\\&=[f(2)f(x+y)-f(x)][f(tx)+f(2)f(ty)]\end{aligned}$$

for all $x, y, t\in\mathbf{R}$, and therefore — with the substitution $x=\alpha_1\in L$, $y=\alpha_2\in L$ — we obtain that $\alpha_1+\alpha_2\in L$, provided that $\alpha_1\neq -2\alpha_2$. If $\alpha_1=-2\alpha_2$ then $\alpha_1+\alpha_2=-\alpha_2\in L$ too holds. Finally, if $\alpha_1, \alpha_2\in L$ then $f(\alpha_1\alpha_2x)=f(\alpha_1)f(\alpha_2x)=f(\alpha_1)f(\alpha_2)f(x)=f(\alpha_1\alpha_2)f(x)$ for all $x\in\mathbf{R}$ and therefore $\alpha_1\alpha_2\in L$. ∎

The following lemma yields an equation coming from the equation (2) which comfirms our above mentioned hypothesis.

Lemma 3. *Let $f\in\mathcal{F}$, then*

$$\text{(3)}\qquad [(f(2)-1)f(x+y)-f(x)-f(y)][f(tx)f(y)-f(ty)f(x)]=0$$

for all $x, y, t\in\mathbf{R}$.

Proof. By equation (2)

$$f(tx+ty)=\frac{f(tx)[f(x+y)+f(y)]+f(ty)[f(2)f(x+y)-f(x)]}{f(x)+f(2)f(y)}$$

for all $x, y, t\in\mathbf{R}$, $x\neq -2y$. Interchanging x and y, we have

$$\frac{f(tx)[f(x+y)+f(y)]+f(ty)[f(2)f(x+y)-f(x)]}{f(x)+f(2)f(y)}=\frac{f(ty)[f(x+y)+f(x)]+f(tx)[f(2)f(x+y)-f(y)]}{f(y)+f(2)f(x)}$$

for all $x, y, t\in\mathbf{R}$, $x\neq -2y$, $y\neq -2x$. Hence, after a simple calculation we obtain the equation (3).

3. The main result

Combining the statements of Lemma 1 and Lemma 2 we get the following theorem (in which we use the notations of the lemmas mentioned).

Theorem 1. *Let* $f\in\mathcal{F}$

a) *if* $L=\mathbf{R}$ *then* $f(xy)=f(x)f(y)$ *for all* $x, y\in\mathbf{R}$,

b) *if* $L\neq\mathbf{R}$ *then* $f(x+y)=f(x)+f(y)$ *and* $f(\alpha x)=f(\alpha)f(x)$ *for all* $x, y\in\mathbf{R}$ *and* $\alpha\in L$.

Proof. By definition of L the proof of part a) is trivial, furthermore in the case b) it will be sufficient to show that the equation

$$f(x+y)=f(x)+f(y) \tag{4}$$

holds for all $x, y\in\mathbf{R}$. Equation (3) implies that

$$[f(2)-1]f(x+y)=f(x)+f(y)$$

for all $x\in\mathbf{R}-L$, $y\in L$, $y\neq 0$. By Lemma 2 L is a field and therefore $x+y\in\mathbf{R}-L$, $(-y)\in L$. Since f is odd, it follows that

$$[f(2)-1]f(x)=f(x+y)-f(y)$$

for all $x\in\mathbf{R}-L$, $y\ni L$, $y\neq 0$. Thus we obtain

$$[f(2)-2]\,[f(x+y)+f(x)]=0$$

for all $x\in\mathbf{R}-L$, $y\in L$, $y\neq 0$. Since L is a field $x+y\neq -x$, furthermore — using that f is invertible — we get $f(2)=2$ and therefore the equation (4) holds for all $x\in\mathbf{R}-L$, $y\in L$. The equation (3) also implies that the equation (4) holds for all $x, y\in\mathbf{R}-L$. Finally, if $x, y\in L$ and $x_0\in\mathbf{R}-L$ then

$$f(x+y)=f(x+x_0+y-x_0)=f(x+x_0)+f(y-x_0)=$$
$$=f(x)+f(x_0)+f(y)-f(x_0)=f(x)+f(y). \blacksquare$$

We can summarize our results as follows:

Theorem 2. $f\in\mathcal{F}$ *if and only if* f *satisfies the following conditions:*

(i) $f:\mathbf{R}\to\mathbf{R}$, $f(0)=0$, $f(1)=1$

(ii) *the function* f *is invertible and odd*

(iii) *either*

$$f(xy)=f(x)f(y)$$

for all $x, y\in\mathbf{R}$ *or there exists a field* L, $L\subset\mathbf{R}$, $L\neq\mathbf{R}$ *such that*

$$f(x+y)=f(x)+f(y) \quad \text{and} \quad f(\alpha x)=f(\alpha)f(x)$$

for all $x, y\in\mathbf{R}$, $\alpha\in L$.

Finally, we make some simple remarks:

1. If L is a field of the rational numbers then the equation of additivity implies the equation

$$f(\alpha x)=f(\alpha)f(x) \quad (\alpha\in L,\ x\in\mathbf{R}).$$

2. If $L\neq\mathbf{R}$ but it has a measurable set of positive measure then $f(x)=x$ $(x\in\mathbf{R})$. In this case there exists a measurable set A of positive measure such that $f(x)\geqq 0$ for all $x\in A$.

3. The continuous (or measurable) solutions of the equations

$$f(xy)=f(x)f(y) \quad \text{and} \quad f(x+y)=f(x)+f(y)$$

are well-known thus the continuous (or measurable) solutions of the equation (1) can easily be given.

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ON MODULAR LAW FOR TERNARY *GD*-GROUPOIDS

Svetozar Milić

1. Let X_1, X_2, X_3, X_4 be four nonempty sets, and

$$A: X_1 \times X_2 \times X_3 \to X_4$$

then the ordered fivefold $(X_1, X_2, X_3, X_4; A)$ we call *G*-groupoid (generalized ternary groupoid).

Let us introduce following notations

$$L_1^A x = A(x, a_2, a_3), \quad L_2^A y = A(a_1, y, a_3), \quad L_3^A z = A(a_1, a_2, z)$$

where $a_i \in X_i$ $(i=1, 2, 3)$ are some fixed elements. The mapping L_k^A we call *G*-translation in relation to the fixed elements.

If L_k^A are surjections for arbitrary fixed elements, then the ternary *G*-groupoid we call *GD*-groupoid (*G*-groupoid with division).

For *G*-groupoids and *GD*-groupoids we introduce the notion of homotopy.

Definition: For ternary *G*-groupoid $(Y_1, Y_2, Y_3, Y_4; B)$ we say that it is the homotopic image of *G*-groupoid $(X_1, X_2, X_3, X_4; A)$ if there exist fourfold $H=[\alpha, \beta, \gamma]$ surjection

$$\alpha: X_1 \to Y_1 \quad \beta: X_2 \to Y_2 \quad \gamma: X_3 \to Y_3 \quad \delta: X_4 \to Y_4$$

such that is fulfilled

$$\delta A(x_1, x_2, x_3) = B(\alpha x_1, \beta x_2, \gamma x_3)$$

for arbitrary $x_i \in X_i$ $(i=1, 2, 3)$.

If $\alpha, \beta, \gamma, \delta$ are bijections, then the homotopy H we call an isotopy.

2. This paper is the enlargement (supplement) of the paper [2], therefore, we shall only formulate certain assertions without detailed proofs, as the proofs are similar to those in [2].

Theorem 1. *If four GD-groupoids A, B, C, D where*

$$
\begin{aligned}
&B: Y_1 \times Y_2 \times Y_3 \to Q_i \quad D: X_1 \times \cdots \times X_{i-1} \times Y_j \times X_{i+1} \times \cdots \times X_3 \to Q_j \\
(1)\quad &A: X_1 \times \cdots \times X_{i-1} \times Q_i \times X_{i+1} \times \cdots \times X_3 \to Q \\
&C: Y_1 \times \cdots \times Y_{j-1} \times Q_j \times Y_{j-1} \times \cdots \times Y_3 \to Q
\end{aligned}
$$

satisfy the equation

$$(2)\quad \begin{aligned} &A(x_1, \dots, x_{i-1}, B(y_1, y_2, y_3), x_{i+1}, \dots, x_3) \\ &= C(y_1, \dots, y_{j-1}, D(x_1, \dots, x_{i-1}, y_j, x_{i+1}, \dots, x_3), y_{j+1}, \dots, y_3) \end{aligned}$$

for every $x_k \in X_k$, $y_m \in Y_m$ $(k, m = 1, 2, 3)$ *and if*

$$L_i^A : Q_i \to Q, \quad L_j^C : Q_j \to Q$$

are bijections (for arbitrary elements) then there exists the group $(Q, \circ)$ *that*

$$(3)\quad \begin{aligned} &A(x_1, \dots, x_{i-1}, z, x_{i+1}, \dots, x_3) = L_i^A z \circ K(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_3) \\ &B(y_1, y_2, y_3) = (L_i^A)^{-1}(P(y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_3) \circ L_j^C L_i^D y_j) \\ &C(y_1, \dots, y_{j-1}, z, y_{j+1}, \dots, y_3) = P(y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_3) \circ L_j^C z \\ &D(x_1, \dots, x_{i-1}, y_j, x_{i+1}, \dots, x_3) \\ &= (L_j^C)^{-1}(L_j^C L_i^D y_j \circ K(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_3)) \end{aligned}$$

where $L_i^D : Y_j \to Q_j$ *and* K *and* P *are arbitrary binary GD-groupoids (Here, by definition for* $i, j \in \{1, 2, 3\}$ *we omit* X_0, Y_0, X_4, Y_4 *from* (1), *as well as* x_0, y_0, x_4, y_4 *from* (2).

By introducing the relation of equivalence in the set of surjection [1] [2] then is valid

Theorem 2. *If four ternary GD-groupoids* A, B, C, D *satisfy the conditions of Theorem 1, then the general solution of equation* (2)

$$\begin{aligned} &A(x_1, \dots, x_{i-1}, z, x_{i+1}, \dots, x_3) = \alpha z \circ K(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_3) \\ &B(y_1, y_2, y_3) = \alpha^{-1}(P(y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_3) \circ \beta y_j) \\ &C(y_1, \dots, y_{j-1}, z, y_{j+1}, \dots, y_3) = P(y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_3) \circ \gamma z \\ &D(x_1, \dots, x_{i-1}, y_j, x_{i+1}, \dots, x_3) = \gamma^{-1}(\beta y_j \circ K(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_3)) \end{aligned}$$

where $(Q, \circ)$ *is the group determined up to isomorphism and the mappings* α, β, γ *are determined up to on the equivalence and* K *and* P *are arbitrary binary GD-groupoids.*

For the proof of theorem 1. respectively of the theorem 2. is used the same method as in the paper [2], at it, we obtain the *GD*-groupoids S isotopic A, D and T arbitrary for B, C and they satisfy following modular laws for different i, j

1) (1, 1) — modular law

$$S(T(t, y_2, y_3), x_2, x_3) = T(S(t, x_2, x_3), y_2, y_3)$$

2) (1, 2) — modular law

$$S(T(y_1, t, y_2), x_2, x_3) = T(y_1, S(t, x_2, x_3), y_2)$$

3) (1, 3) — modular law

$$S(T(y_1, y_2, t), x_2, x_3) = T(y_1, y_2, S(t, x_2, x_3))$$

4) (2, 2) — modular law

$$S(x_1, T(y_1, t, y_3), x_3)=T(y_1, S(x_1, t, x_3), y_3)$$

5) (2, 3) — modular law

$$S(x_1, T(y_1, y_2, t), x_3)=T(y_1, y_2, S(x_1, t, x_3))$$

6) (3, 3) — modular law

$$S(x_1, x_2, T(y_1, y_2, t))=T(y_1, y_2, S(x_1, x_2, t)).$$

All these equations are solved similarly as the equation under 4) (see [2]).

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ÜBER EINIGE FORMALE ASPEKTE DER DYNAMIK (DIVIDIERBARKEIT UND ANALYTISCHE ITERATION KONTRAHIERENDER BIHOLOMORPHER ABBILDUNGEN)

Ludwig Reich

§ 1. Einleitung. Zusammenstellung bekannter Ergebnisse

In meiner Arbeit [1] habe ich einen Zusammenhang zwischen der Existenz m-ter Wurzeln einer kontrahierenden biholomorphen Abbildung F (vgl. die Definitionen in [1] oder [2]) und der Existenz einer analytischen Iteration von F hergestellt. Dieser Zusammenhang konnte gewonnen werden mittels der Methode der sogenannten vollständigen Linearisierung von F (vgl. [1], [3]). m-te Wurzel von F war dabei eine formal-biholomorphe Abbildung G (— die sich als eo ipso lokal-biholomorph herausstellt —) für deren m-te Iterierte G^m gilt $F=G^m$. Für diese *Funktionalgleichung* und ihre lange Geschichte vgl. etwa [4]. Unter einer analytischen Iteration von F versteht man die Einbettung von F in einen einparametrigen Lie'schen Gruppenkeim

$$F_t = x \to x(t) = A(t)\,x + \mathfrak{R}(t, x)$$

mit dem additiven Gruppenparameter t, der Substitution als Gruppenoperation und den Randbedingungen

$$F_0 = id, \quad F_1 = F.$$

Der in [1] ausgesprochene Zusammenhang zwischen der Existenz m-ter Wurzeln und der Existenz analytischer Iteration von F lautet nun;

Satz: (i) *Es mögen für unendlich viele $m \in \mathrm{N}$ m-te Wurzeln $F^{\frac{1}{m}}$ von F existieren.*

(ii) *Die vollständigen Linearisierungen von $F^{\frac{1}{m}}$, $L(F^{\frac{1}{m}})$ mögen alle ein und derselben analytischen Iteration $L(F)$ angehören.*

Behauptung: Dann existiert eine analytische Iteration $F(t)$ von F, für die außerdem gilt $F\left(\frac{1}{m}\right) = F^{\frac{1}{m}}$.

Es ist nun zunächst das Ziel der vorliegenden Arbeit, in einer Reihe von Sätzen zu untersuchen, wie weit die Voraussetzungen dieses Satzes abgeschwächt oder durch andere ersetzt werden können. Im einzelnen: Es wird z.B. in Satz 1 (§ 2) gezeigt, daß man u.U. die Voraussetzung der Existenz m-ter Wurzeln durch die Voraussetzung der Existenz „$^l/_m$-ter Wurzeln“, d.h. von Abbildung $F^{l/m}$ für die $(F^{l/m})^m = F^l$, $l \in \mathbf{N}$, gilt, ersetzen kann. Hingegen geht es in Satz 2 (§ 1) darum, die Voraussetzung (ii) abzuändern. Hier kann z.B. die mehr algebraische Voraussetzung getroffen werden, daß für jedes U aus dem Kommutator von $L(F)$ (in der Gruppe der linearen Automorphismen) jedes $U^{-1} \circ L(F^{1/m}) \circ U$ die Varietät der vollständigen Linearisierung, $L(\mathbf{C}^n)$, als ganzes invariant läßt.

In einigen weiteren Sätzen (§ 3) wird das Verfahren der vollständigen Linearisierung dazu benutzt, zu untersuchen, was die Existenz stetiger Flüsse oder stetig differenzierbarer Flüsse, in die F eingebettet ist, für die Existenz analytischer Iterationen impliziert, (Satz 3, Satz 4). Z.B. impliziert die Einbettbarkeit von F in einen einparametrigen, stetig differenzierbaren Fluß die Existenz einer analytischen Iteration (Satz 3).

Die Einbettbarkeit in eine kontinuierliche einparametrige Gruppe F_τ hat die Existenz einer analytischen Iteration zur Konsequenz, wenn z.B. noch gefordert wird, daß für eine in $\mathbf{R}$ dicht liegende Menge von ρ die Linearisierungen $L(F_\rho)$ alle ein und derselben analytischen Iteration von $L(F)$ angehören (Satz 4).

In § 4 werden wir uns schließlich der Frage zuwenden, wann es m-te Wurzeln $F^{1/m}$ von F gibt zu vorgegebenen Bestimmungen $\rho_i^{1/m}$ der Eigenwerte des Linearteils von $F^{1/m}$ (Satz 5). Auch dies geht über das Verfahren der vollständigen Linearisierung, und man erhält ein algebraisches Existenzkriterium. Schließlich werden wir zeigen, daß es genügt, die Existenz von hinreichend, aber *endlich vielen* m-ten Wurzeln von F vorauszusetzen, wenn es um die Frage geht, ob eine vorgegebene analytische Iteration $\mathfrak{F}_t$ von $L(F)$ aus linearen Abbildungen das Bild einer analytischen Iteration F_t von F ist:

$$\mathfrak{F}_t = L \circ F_t \circ L^{-1}, \quad \text{ob also } \mathfrak{F}_t$$

eine analytische Iteration von F „fortsetzt“. Das wird in Satz 8, Satz 9 behandelt.

Wir beenden diesen Paragraphen mit einer Zusammenstellung der wichtigsten Tatsachen über die vollständige Linearisierung.

Die vollständige Linearisierung einer kontrahierenden Abbildung F bedeutet (vgl. [3]): Es existiert eine biholomorphe, sogar birationale, überall bireguläre Abbildung L des $\mathbf{C}^n$ auf eine analytische (sogar algebraische Varietät $L(\mathbf{C}^n) = \mathfrak{N} \subset \mathbf{C}^N$ mit folgenden Eigenschaften:

1) $L \circ F \circ L^{-1}$ ist die Einschränkung genau eines linearen Automorphismus $L(F)$ von $\mathbf{C}^N$ auf $\mathfrak{N}$, bei dem $\mathfrak{N}$ invariant bleibt.

2) Ist G eine mit F kommutierende formal-biholomorphe Abbildung mit Fixpunkt $x=0$, dann ist auch $L \circ G \circ L^{-1}$ die Einschränkung genau eines linearen Automorphismus $L(G)$ des $\mathbf{C}^N$ auf $\mathfrak{N}$, bei dem $\mathfrak{N}$ invariant bleibt.

3) Die Koeffizienten der Matrix $M(L(G))$, falls dabei die gegeben kontrahierende Abbildung G in halbkanonischer vorliegt, sind Polynome in den Koeffizienten der Potenzreihe G, d.h. ihrer polynomialen halbkanonischen Formen; ebenso sind die Koeffizienten der Potenzreihen in G Polynome in den Elementen von $M(L(G))$. Die Eigenwerten von F und $L(F)$ stimmen überein, abgesehen von den Multiplizitäten.

4) Einer analytischen Iteration von F entspricht durch den Übergang $F_t \to L(F_t)$ eine eineindeutig bestimmte analytische Iteration der linearen Abbildung $L(F)$, dabei sind die $L(F_t)$ lineare Abbildungen, sie lassen $\mathfrak{R}$ fest (F_t, vgl. [1], ist nämlich mit F vertauschbar, also vollständig linearisierbar).

Wir zeigen noch, daß jede formale m-te Wurzel $F^{1/m}$ eines lokalbiholomorphen kontrahierenden F lokal biholomorph ist:

Jede formale m-te Wurzel eines lokal-biholomorphen kontrahierenden F ist eo ipso konvergent. Jede rationale Iterierte $F^{l/m}$, d.h. eine formal-biholomorphe Abbildung $F^{l/m}$ mit $(F^{l/m})^m = F^l$, $l \in \mathbf{Z}$, ist eo ipso konvergent.

Beweis: Es gilt $F^{-1/m} \circ F \circ {}^{1/m} = F^{-1/m} \circ (F^{1/m})^m \circ F^{1/m} = F$, also ist $F^{1/m}$ mit F vertauschbar. Es sei nun T eine konvergente Transformation von F auf halbkanonische Form. Dann ist $T^{-1} \circ F^{1/m} \circ T$ m-te Wurzel von $T^{-1} \circ F \circ T$, also nach [5], bzw. [1], selbst polynomial, also konvergent. Da T konvergent ist, trifft dies auch für $T \circ (T^{-1} \circ F^{1/m} \circ T) \circ T^{-1} \doteq F^{1/m}$ zu. Da nun F^l ebenfalls mit F vertauschbar ist, so wird es durch das obige T auf polynomisch vereinfachte Gestalt transformiert, und es folgt auch für $F^{l/m}$ analog, da es mit F^l vertauschbar ist, daß es durch T auf polynomische Gestalt transformiert wird. Somit ist auch $F^{1/m}$ konvergent.

Im übrigen sei ausdrücklich darauf verwiesen, daß wir hier den Inhalt und die Methoden der Arbeit [1] als bekannt voraussetzen werden und sets auf die von uns benötigten Stellen verweisen werden.

§ 2. Dividierbarkeit uud analytische Iteration

Als Verallgemeinerung des in der Einleitung zitierten Satzes aus der Arbeit [1] beweisen wir zunächst:

Satz 1: a) *Es mögen für unendlich viele rationale mod 1 inkongruente l/m ($l>0$, $m>0$) rationale Iterierte $F^{l/m}$ existeren.*

b) *Für $l>1$ seien dieses $F^{l/m}$ mit F vertauschbar, und die dann eo ipso existierenden $L(F^{1/m})$ mögen einer und derselben analytischen Iteration von $L(F)$ angehören.*

Dann existiert eine analytische Iteration $F(t)$ von F und es ist $F(l/m) = F^{l/m}$.

Beweis: Es existiere für ein $l/m\, F^{l/m}$, und es sei $l/m = [l/m] + k/m$, $0 \leqslant k/m < 1$. Nach Voraussetzung bilden die k/m eine unendliche Menge. Wir setzen $F^{k/m} = F^{l/m} \circ F^{-[l/m]}$. Es gilt mit der Bezeichnung $F^{-k/m} = (F^{k/m})^{-1}$:

$$F^{-k/m} \circ F \circ F^{k/m} = F^{-l/m} \circ F^{[l/m]} \circ F \circ F^{-[l/m]} \circ F^{l/m} = F^{-l/m} \circ F \circ F^{l/m} = F,$$

also existiert auch $L(F^{k/m})$, und diese $L(F^{k/m})$ gehören ebenfalls der analytischen Iteration an, zu der die $L(F^{1/m})$ gehören, wegen

$$L(F^{k/m}) = L(F^{l/m - [l/m]}) = L(F^{l/m}) \circ$$
$$L(F^{-[l/m]}) = L(F^{l/m}) \circ \left(L(F^{[l/m]})^{-1}\right).$$

Wir gehen von einer halbkanonischen Form von F aus, und nehmen an, die Elemente der analytischen Iteration von $L(F)$, $\mathfrak{F}_t$, seien gegeben durch $\xi \to M(t)\xi$. Dabei sind die Elemente von $M(t)$, wie aus [1] bekannt, ganze Funktionen von t. Es seien $\Phi_1 = 0, \ldots, \Phi_s = 0$ die Gleichungen der algebraischen

Varietät $\mathfrak{R}$, ξ sei ein allgemeiner Punkt von $\mathfrak{R}$. Auf Grund von [2] bedeutet dies, daß die Koordinaten ξ_i von ξ Potenzprodukte der Unbestimmten $x_1, \ldots, x_n$ sind. Nun hat $L(F^{k/m})$ als Element der analytischen Iteration die Darstellung $\xi \to M(k/m)\xi$, wie man leicht nachrechnet, $L(F^{k/m})$ läßt $\mathfrak{R}$ invariant, und wir haben wie in [1], für den allgemeinen Punkt ξ von $\mathfrak{R}: \Phi_1(M(k/m)\xi) = 0, \ldots, \Phi_s(M(k/m)\xi) = 0$ für die unendlich vielen k/m mit $0 \leqslant k/m < 1$. Wie in [1], p. 13—19, schließen wir nun daraus mit Hilfe des Identitätssatzes für analytische Funktionen, daß die Varietät $\mathfrak{R}$ auch bei der allgemeinen Abbildung $\xi \to M(t)\xi$ der analytischen Iteration von $L(F)$ festbleibt. Die Anwendung des Identitätsprinzips beruht dabei auf der Existenz eines Häufungspunktes der Menge der k/m, in dessen Umgebung die Elemente von $M(t)$ holomorph sind. Wie in [1] wird nun geschlossen, daß F in angegebener Weise eine analytische Iteration besitzt.

Die Bedingung, daß alle $F^{l/m}$ einer und derselben analytischen Iteration von $L(F)$ angehören müssen, wird in folgendem Satz durch eine mehr algebraische Bedingung ersetzt werden.

Satz 2: a) *Für unendlich viele* $1/m$ *existieren m-te Wurzeln* $F^{1/m}$ *von* F. b) *Für jedes* U *aus dem Kommutator von* $L(F)$ *lasse der lineare Automorphismus* $U^{-1} \circ L(F^{1/m}) \circ U \mathfrak{R}$ *invariant.* c) *Die* $L(F^{1/m})$ *mögen analytischen Iterationen von* $L(F)$ *zu den gleichen von* m *unabhängigen Bestimmungen der Logarithmen* $\ln \rho_i$ *der Eigenwerte von* ρ_i *angehören.*

Dann existiert eine analytische Iteration von F, $F(t)$, *und es gilt* $F(1/m) = F^{1/m}$.

Beweis: Wie in [1] gezeigt wurde, werden sämtliche analytische Iterationen $\mathfrak{F}_t$ von $\mathfrak{F} = L(F)$ zu festen Bestimmungen der $\ln \rho_i$ gegeben durch $U^{-1} \circ \mathfrak{F}_t \circ U$, wobei U die Automorphismen mit $U^{-1} \circ \mathfrak{F} \circ U$, det $U = 1$, durchläuft, und $\mathfrak{F}_t$ eine beliebige feste analytische Iteration zu diesen Bestimmungen $\ln \rho_i$ bezeichnet. Es sei also ein solches $\mathfrak{F}_t$ fest gewählt. Nach Voraussetzung gehört $L(F^{1/m})$ einer analytischen Iteration ${}_{1/m}\mathfrak{F}_t$ von $L(F)$ zu den fest gewählten Bestimmungen $\ln \rho_i$ an. Somit existiert ein ${}_{1/m}U$ aus dem Kommutator mit $({}_{1/m}U)^{-1} \circ {}_{1/m}\mathfrak{F}_t \circ {}_{1/m}U = \mathfrak{F}_t$. Dies bedeutet, daß $({}_{1/m}U^{-1}) \circ L(F^{1/m}) \circ {}_{1/m}U$ der fest gewählten analytischen Iteration $\mathfrak{F}_t$ angehört. Außerdem läßt nach Voraussetzung $({}_{1/m}U^{-1}) \circ L(F^{1/m}) \circ {}_{1/m}U$ invariant. Mit der schon in [1] verwendeten Schlußweise, ergibt sich also die Existenz einer analytischen Iteration $\mathfrak{F}_t$ von $L(F)$, die $\mathfrak{R}$ invariant läßt, somit existiert auch die angegebene analytische Iteration von F.

§ 3. Kontinuierliche Gruppen und analytische Iteration

Dieser § ist der Frage gewidmet, was die Einbettbarkeit von F in einparametrige stetig differenzierbare oder gar nur kontinuierliche Gruppen für die Existenz analytischer Iterationen impliziert.

Satz 3: $F = F_1$ *besitze eine Einbettung in eine einparametrige reelle additive stetig differenzierbare Gruppe* F_r $(r \in \mathbf{R})$ *von lokalbiholomorphen Abbildungen mit Fixpunkt* $x = 0$.

Dann existiert eine analytische Iteration $F(t)$ *von* F, *sodaß* F_r *Untergruppe derselben mit* $F(r) = F_r$ *ist.*

Beweis: Da die Gruppe additiv ist, gilt $F \circ F_r = F_{1+r} = F_{r+1} = F_r \circ F$, also existiert $L(F_r)$. Da die Elemente der Matrizen der $L(F_r)$ Polynome in den

Koeffizienten der F_r sind, handelt es sich bei $L(F_r)$ um eine stetig differenzierbare Gruppe von Matrizen. Bezeichnet $\cdot$ die Ableitung nach dem Gruppenparameter r, so folgt

$$\dot{L}(F_r) = \dot{L}(F_r)\Big|_{r=0} L(F_r), \quad r \in \mathbf{R}.$$

Aus dem Existenz- und Eindeutigkeitssatz für Differentialsysteme folgt aber, daß es sich bei $L(F_r)$ sogar um eine einparametrige Liesche Gruppe handelt, da die Lösungen eo ipso holomorph ausfallen. Ihre Elemente seien mit $\mathfrak{F}_t$, $t \in \mathbf{C}$, bezeichnet. Speziell ist $\mathfrak{F}_r = L(F_r)$ für $r \in \mathbf{R}$. Für unendlich viele $t = r$, die sich etwa in $r = 1$ häufen, gilt diese obige Relation. Verwendet man nun wieder die Beweismethode von [1], p. 14—19, d.h. die Betrachtung gewisser die Invarianz von $\mathfrak{R}$ ausdrückender analytischer Gleichungssysteme, so ergibt sich, im wesentlichen mittels des Identitätssatzes für analytische Funktionen, die Existenz einer analytischen Iteration F_t von F, der $\{F_r \mid r \in \mathbf{R}\}$ als Untergruppe angehört.

Hinsichtlich des Zusammenhanges zwischen kontinuierlichen einparametrigen Gruppen (stetigen Flüssen) und analytischen Iterationen zeigen wir:

Satz 4: a) *Es besitze F eine Einbettung in einen stetigen reellen Fluß, d.h. eine einparametrige, additive, reelle, kontinuierliche Gruppe $\{F_r, r \in \mathbf{R}\}$. Es sei* P *eine Teilmenge von* $\mathbf{R}$, *sodaß es unendlich viele* mod 1 *inkongruente* $\rho \in$ P *gibt und sodaß* $\{l\rho \mid \rho \in \mathrm{P}, l \in \mathbf{Z}\}$ *dicht in* $\mathbf{R}$ *ist.*

b) *Für unendlich viele* mod 1 *inkongruente* ρ *mögen die* $L(F_\rho)$ *ein und derselben analytischen Iteration* $\mathfrak{F}_t$ *von* $L(F)$ *angehören.*

Dann besitzt F eine analytische Iteration $F(t)$, mit F_r als Untergruppe von $F(t)$.

Beweis: $L(F_r)$ existiert. Denn es ist $F_{1+r} = F_{r+1} = F \circ F_r = F_r \circ F$. Es sei $\{\rho\} = \rho - [\rho]$. Nach Voraussetzung besitzen diese $\{\rho\}$, $\rho \in$ P, einen Häufungspunkt in $[0, 1]$, und es gehören die $L(F_{\{\rho\}}) = L(F_\rho) \circ L(F_{-[\rho]})$ alle der analytischen Iteration $\mathfrak{F}_t$ an. Nach der aus [1] übernommenen Beweismethode, das analytische Gleichungssystem zu betrachten, das die Invarianz von $\mathfrak{R}$ ausdrückt, und den Identitätssatz anzuwenden, ergibt sich wieder, daß eine analytische Iteration $F(t)$ von F existiert, mit $F(\rho) = F_\rho$, $\rho \in$ P, somit auch $F(\{\rho\}) = F_{\{\rho\}}$, $F(l\rho) = F_{l\rho}$. Es sei $r \in \mathbf{R}$ beliebig. Nach Voraussetzung gibt es dann eine Folge ρ_k, $\rho_k \in$ P, und zugehörige $l_k \in \mathbf{Z}$, sodaß $r = \lim_{k \to \infty} l_k \rho_k$. Da die Gruppe der F_r kontinuierlich ist, folgt

$$\lim_{k \to \infty} F(l_k \rho_k) = F(r).$$

Da aber $F_{r_{k\rho k}} = F(l_k \rho_k)$, so folgt $F(r) = F_r$. W.z.bw.

Wir halten noch fest:

Folgerung: a) *Es besitze F eine Einbettung in einen stetigen Fluß F_r.* b) *Für ein irrationales ρ möge $L(F\rho)$ einer analytischen Iteration $\mathfrak{F}_t$ von $L(F)$ angehören.*

Dann existiert eine analytische Iteration $F(t)$ von F, und es gilt $F(\rho) = F_\rho$.

Beweis: Für irrationales ρ sind die Zahlen $\{m\rho\}$ dicht in $[0, 1]$, (vgl. [6]), und es gehören auch alle $L(F_{\{m\rho\}})=L(F_{m\rho})\circ L(F_{-[m\rho]})$ der analytischen Iteration $\mathfrak{F}_t$ an. Außerdem ist $\{l\{m\rho\}\,|\,l\in\mathbf{Z},\ m\in\mathbf{Z}\}$ dicht in $\mathbf{R}$. Somit sind auch die Voraussetzungen von Satz 4 erfüllt, und die Folgerung ist bewiesen.

§ 4. Existenz m-ter Wurzeln

Wir behandeln hier die in der Theorie der Funktionalgleichungen seit langem studierte Funktionalgleichung von BABBAGE für kontrahierende biholomorphe Abbildung F. Gesucht ist eine lokal biholomorphe Abbildung G mit

$$G^m = F.$$

Es ist nicht bekannt, ob jedes F für jedes m eine m-te Wurzel besitzt. Die Eigenwerte des Linearteils von $F^{1/m}$ sind $\rho_1^{1/m}, \ldots, \rho_n^{1/m}$, mit gewissen Bestimmungen der m-ten Wurzeln der Eigenwerte $\rho_1, \ldots, \rho_n$ von F. Es zeigt nun ein einfaches Beispiel, auf dessen Detaildiskussion wir verzichten dürfen, daß es Abbildungen F gibt, daß nicht für jedes m und jede Wahl der $\rho_1^{1/m}, \ldots, \rho_n^{1/m}$ eine m-te Wurzel $F^{1/m}$ existiert. Sind nämlich μ_1, μ_2 komplexe Zahlen mit $0<|\mu_i|<1$ und der Relation $\mu_2=-\mu_1^{\nu}$, und setzen wir $\rho_1=\mu_1^2$, $\rho_2=\mu_2^2$, so liegt

$$F:\quad \begin{aligned} x_1^{(1)} &= \rho_1 x_1 \\ x_1^{(2)} &= \rho_2 x_2 + a x_1^{\nu}, \quad a\neq 0 \end{aligned}$$

in Normalform vor, und es zeigt sich, daß keine Abbildung $F^{1/2}$ existiert, deren Linearteil die Eigenwerte $\rho_1^{1/2}=\mu_1$, $\rho_2^{1/2}=\mu_2$ aufweist.

Wir beginnen mit folgenden Bemerkungen, deren Beweis auf Grund von [1], § 3, fast trivial ist, und daher ausgelassen werde.

1) Die Eigenwerte von $F^{1/m}$ sind Bestimmungen $\rho_1^{1/m}, \ldots, \rho_n^{1/m}$.

2) Jede lineare Abbildung $A\colon x\to Ax$, $\det A\neq 0$, besitzt für jedes $m\in\mathrm{N}$, m-te Wurzeln und diese ergeben sich, indem man in einer geeigneten analytischen Iteration A_t von A für $t=1/m$ das Element $A_{1/m}$ herausfaßt.

Es sei nun $L(F)=\mathfrak{F}$ die vollständige Linearisierung von F. $\mathfrak{F}_t$ sei eine analytische Iteration von $L(F)$, zu den Bestimmungen $(\ln\rho_1)_0, \ldots, (\ln\rho_n)_0$ der Logarithmen $\rho_1, \ldots, \rho_n$ und zu einem Parameter η der irreduziblen Parametermannigfaltigkeit (vgl. [1], § 3). Falls F die m-te Wurzel $F^{1/m}$ besitzt, dann ist, da L ein Isomorphismus ist, $L(F^{1/m})$ m-te Wurzel von $L(F)$, und es ist nach Bemerkung 2.) in einer analytischen Iteration $\mathfrak{F}_t$ von $\mathfrak{F}$, etwa der gegebenen, als $\mathfrak{F}_{1/m}$ enthalten. Wir werden dann sagen: $F_{1/m}$ gehöre zu $(\ln\rho_1)_0, \ldots, (\ln\rho_n)_0$; η. Es gilt nun aber $\mathfrak{F}_{1/m}=L(F^{1/m})$ für ein $F^{1/m}$ genau dann, falls $\mathfrak{F}_{1/m}$ die Varietät $\mathfrak{R}$ der Linearisierung als ganzes invariant läßt. Es sei nun ξ ein allgemeiner Punkt von $\mathfrak{R}$. Drückt man es durch Gleichungen aus, daß auch $\mathfrak{F}_{1/m}\,\xi$ ein Punkt von $\mathfrak{R}$ ist, so erhält wie in [1], p. 14—19, ein System von ganz rationalen Relationen

$$\psi_1(\varphi)=0, \ldots, \psi_\sigma(\varphi)=0,$$

wo bei $\varphi_{\alpha\beta}$ die Elemente der Matrix $\mathfrak{F}_{1/m}$ bezeichnet. Aus [1], § 3, folgt daß die $\varphi_{\alpha\beta}$ ihrerseits Polynome in den Elementen von $L(F)$, also auch den Koeffizienten von F und außerdem in Parametern $(\eta_1, \ldots, \eta_l)$ aus der Parametermannigfaltigkeit $\mathfrak{M}$ sind. Die

$$(1) \qquad \psi_1(p, \eta)=0, \ \ldots, \ \psi_N(p, \eta)=0$$

und $\mathfrak{M}$ hängen dabei von $(\ln \rho_1)_0, \ldots, (\ln \rho_n)_0$ allein ab. Für die Existenz eines $F^{1/m}$ zu $(\ln \rho_1)_0, \ldots,) \ln \rho_n$, η erhalten wir daher als notwendige und hinreichende Bedingung das Bestehen ganz-rationaler Relationen, wobei p die Koeffizienten von F (bis zur maximalen Ordnung des Zusatzmonoms) bezeichnet und $\eta \in \mathfrak{M}$ ist. Somit

Satz 5: *Notwending und hinreichend für die Existenz eines $F^{1/m}$ zu $1/m(\ln \rho_1)_0, \ldots, 1/m(\ln \rho_n)_0$ ist das Bestehen der Relationen* (1) *mit einem geeigneten $\eta \in \mathfrak{M}$. Die $\psi_1, \ldots, \psi_N$ und $\mathfrak{M}$ hängen nur ab von den gewählten Bestimmungen der Logarithmen von $L(F)$.*

Daraus ergibt sich unmittelbar.

Satz 6: *Es gibt kontrahierende Abbildungen F mit folgender Eigenschaft: Zu jeder Wahl von $(\ln \rho_1)_0, \ldots, (\ln \rho_n)_0$ und zu jedem $\mathring{\eta} \in \mathfrak{M}$ existieren höchstens für endlich viele $m \in \mathbb{N}$ m-te Wurzeln $F^{1/m}$ zu $(\ln \rho_1)_0, \ldots, (\ln \rho_n)$, $\mathring{\eta}$.*

Satz 7: *Es gibt Normalformen $\mathcal{G}$ linearer Abbildungen und natürliche Zahlen m, daß die Polynome* (1) *nicht identisch verschwinden, u.zw. für jede Wahl von $(\ln \rho_1)^0, \ldots, (\ln \rho_n)^0$.*

Der Beweis dieser Sätze ergibt sich unmittelbar aus dem Einleitung zitierten Satz über den Zusammenhang der unendlichen Dividierbarkeit mit der analytischen Iteration.

Wir kommen nun nochmals auf diesen Zusammenhang zurück, insbesondere auf den zitierten Satz. Die Voraussetzung (ii) dieses Satzes besagt in der Sprechweise dieses Paragraphen, daß alle $F^{1/m}$ zu einer und derselben Bestimmung $(\ln \rho_1)^0, \ldots, (\ln \rho_n)^0$ und einem η gehören, eben zu diesen, die die analytische Iteration von $L(F)$ gemäß [1] festlegen. Es ist also das in diesem § konstruierte Gleichungssystem, das ja auch von $1/m$ abhängt, für unendlich viele $m_1, m_2 \ldots$ richtig. Mit $\mathfrak{a}_{m_i}$ bezeichne ich nun das von den Polynomen von (1) für $1/m_i$ erzeugte Ideal und betrachte die aufsteigende Idealkette

$$\mathfrak{a}_{m_1} \subseteq (\mathfrak{a}_{m_1}, \mathfrak{a}_{m_2}) \subseteq (\mathfrak{a}_{m_1}, \mathfrak{a}_{m_2}, \mathfrak{a}_{m_3}) \subseteq \ldots .$$

Diese ist stationär, d.h. es existiert ein k_0, sodaß

$$\mathfrak{a}_{m_1}, \ldots, \mathfrak{a}_{m_{k_0}}) = (\mathfrak{a}_{m_1}, \ldots, \mathfrak{a}_{m_{k_0}+l})$$

für alle $l>0$. Das bedeutet aber, dass die notwendigen und hinreichenden Bedingungsgleichungen für die Existenz der m_{k+l_0}-ten Wurzeln F^{1/m_0} erfüllt sind, sobald $F^{1/m_1}, \ldots, F^{1/m_{k_0}}$ existieren. m_{k_0} ist dabei abhängig von $\mathcal{G}$, $(\ln \rho_1)_0, \ldots, (\ln \rho_n)_0$, η, d.h. von der analytischen Iteration von $L(F)$, $\mathfrak{F}_t$, und von der Folge $m=(m_1<m_2<\cdots)$. Wir können daher formulieren:

Satz 8: *Es sei* $\mathfrak{F}_t$ *eine gegebene analytische Iteration von* $L(F)$, $m=(m_1<m_2<\cdots)$ *eine unendliche Folge natürlicher Zahlen. Dann existiert eine natürliche Zahl* $n_0(\mathfrak{F}_t, m)$, *sodaß* $\mathfrak{F}_t$ *das L-Bild einer analytischen Iteration von F ist* $\mathfrak{F}_t=L\circ F_t\circ L^{-1}$, *falls nur Wurzeln* $F_{1/m_1}, \ldots, F_{1/m_k}$ *mit* $k>n_0$ *existieren, deren* $L(F_{1/m_j})$ *zu* $\mathfrak{F}_t$ *gehört.*

Speziell:

Satz 9: *Es sei* $\mathfrak{F}_t$ *eine gegebene analytische Iteration von* $L(F)$. *Dann existiert eine natürliche Zahl* $n_0(\mathfrak{F}_t)$ *so daß* $\mathfrak{F}_t$ *das L-Bild einer analytischen Iteration von F,* $F(t)$ *ist, sofern nur Wurzeln* $F_{1/2}, \ldots, F_{1/k}$ *für* $k>n_0$ *existieren, deren* $L(_{1/m})$ *zu* $\mathfrak{F}_t$ *gehört.*

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ON AN EXTENSION OF PEXIDER'S EQUATION

János Rimán

1. Introduction

Some results of Z. Daróczy and L. Losonczi [2] on the extensions of additive functions seem to have important applications in the theory of functional equations (cp. e. g. K. Lajkó [3], L. Székelyhidi [4]).

In connection with the above-mentioned results, in this paper we shall deal with the extensions of the equation

$$f(x+y)=g(x)+h(y).$$

2. Definitions and notations

We shall use the following notations and definitions.

Let $D \subset \mathbf{R}^2$ be an arbitrary non-empty set ($\mathbf{R}$ is the set of real numbers) and

$$D_x=\{x \mid \exists y, \quad (x, y) \in D\},$$

$$D_y=\{y \mid \exists x, \quad (x, y) \in D\},$$

$$D_{x+y}=\{x+y \mid (x, y) \in D\}$$

Throughout the paper E denotes an Abelian group (written additively).

Definition 1. Let $D \subset \mathbf{R}^2$ $(D \neq \varnothing)$, $f: D_{x+y} \to E$, $g: D_x \to E$, $h: D_y \to E$ be functions such that

$$f(x+y)=g(x)+h(y), \quad (x, y) \in D.$$

If there exists an ordered triple of functions (F, G, H) such that

(i) $F, G, H: \mathbf{R} \to E$,

(ii) $F(x+y)=G(x)+H(y)$ for all $(x, y) \in \mathbf{R}^2$

and

(iii)
$$F(x)=f(x) \text{ for all } x \in D_{x+y},$$
$$G(x)=g(x) \text{ for all } x \in D_x,$$
$$H(x)=h(x) \text{ for all } x \in D_y,$$

then (F, G, H) is called an *extension* of (f, g, h) from the set D.

Definition 2. Let $D \subset \mathbf{R}^2$ $(D \neq \emptyset)$, $f: D_{x+y} \to E$, $g: D_x \to E$, $h: D_y \to E$ be functions such that

$$f(x+y) = g(x) + h(y), \quad (x, y) \in D.$$

If there exists an ordered triple of functions (F^*, G^*, H^*) and a point $(u, v) \in D$ such that

1° $F^*, G^*, H^*: \mathbf{R} \to E$,

2° $F^*(x+y) = G^*(x) + H^*(y)$ for all $(x, y) \in \mathbf{R}^2$

and

3°
$$F^*(x) - F^*(u+v) = f(x) - f(u+v) \text{ for all } x \in D_{x+y},$$
$$G^*(x) - G^*(u) = g(x) - g(u) \text{ for all } x \in D_x,$$
$$H^*(x) - H^*(v) = h(x) - h(v) \text{ for all } x \in D_y,$$

then (F^*, G^*, H^*) is called a *quasi-extension* of (f, g, h) from the set D.

3. Results

Let $D = K_r = \{(x, y) \mid x^2 + y^2 < r^2\}$ $(r > 0$ is a constant) be an open disk. Then we have

Theorem 1. (*cp.* [2], *Satz* 2.) *Let* $f: (K_r)_{x+y} \to E$, $g: (K_r)_x \to E$ *and* $h: (K_r)_y \to E$ *be functions such that*

$$f(x+y) = g(x) + h(y), \quad (x, y) \in K_r.$$

Then (f, g, h) *has one and only one extension* (F, G, H) *from the set* K_r.

Proof. Clearly $(K_r)_x = (K_r)_y = (-r, r)$ and $(K_r)_{x+y} = (-r\sqrt{2}, r\sqrt{2})$. Every $x \in \mathbf{R}$ can be written in one and only one way in the form

$$x = n\frac{r}{2} + t,$$

where $n \in \mathbf{Z} = \{0, \pm 1, \pm 2, \ldots\}$ and $t \in \left[0, \frac{r}{2}\right)$. Let us define the functions $F: \mathbf{R} \to E$, $G: \mathbf{R} \to E$, $H: \mathbf{R} \to E$ as follows:

$$F(x) = n f\left(\frac{r}{2}\right) + f(t) - n(a+b),$$

$$G(x) = n g\left(\frac{r}{2}\right) + g(t) - na,$$

$$H(x) = n h\left(\frac{r}{2}\right) + h(t) - nb,$$

where $a = g(0)$ and $b = h(0)$. We show that (F, G, H) is the unique extension of (f, g, h) from K_r.

A) First we prove that (ii) holds. If $(x, y)\in \mathbf{R}^2$, then

$$x=n\frac{r}{2}+t_1,\quad y=m\frac{r}{2}+t_2,\qquad \left(n, m\in \mathbf{Z},\ t_1, t_2\in\left[0, \frac{r}{2}\right)\right).$$

1) If $t_1+t_2\in\left[0, \frac{r}{2}\right)$, then we have

$$F(x+y)=F\left((n+m)\frac{r}{2}+t_1+t_2\right)=$$

$$=(n+m)f\left(\frac{r}{2}\right)+f(t_1+t_2)-(n+m)(a+b)=$$

$$=n\left(g\left(\frac{r}{2}\right)+b\right)+m\left(a+h\left(\frac{r}{2}\right)\right)+g(t_1)+h(t_2)-(m+n)(a+b)=$$

$$=ng\left(\frac{r}{2}\right)+g(t_1)-na+mh\left(\frac{r}{2}\right)+h(t_2)-mb=G(x)+H(y).$$

2) If $t_1+t_2\in\left[\frac{r}{2}, r\right)$, then we can write $t_1+t_2=\frac{r}{2}+t$ with $t\in\left[0, \frac{r}{2}\right)$ and

$$F(x+y)=F\left((n+m+1)\frac{r}{2}+t\right)=(n+m+1)f\left(\frac{r}{2}\right)+f(t)-(n+m+1)(a+b)=$$

$$=n\left(g\left(\frac{r}{2}\right)+b\right)+m\left(a+h\left(\frac{r}{2}\right)\right)+g\left(\frac{r}{2}\right)+b+a+h(t)-$$

$$-(n+m+1)(a+b)=ng\left(\frac{r}{2}\right)-na+mh\left(\frac{r}{2}\right)-mb+f\left(\frac{r}{2}+t\right)=$$

$$=ng\left(\frac{r}{2}\right)-na+mh\left(\frac{r}{2}\right)-mb+f(t_1+t_2)=$$

$$=ng\left(\frac{r}{2}\right)+g(t_1)-na+mh\left(\frac{r}{2}\right)+h(t_2)-mb=G(x)+H(y).$$

Thus $F(x+y)=G(x)+H(y)$ for all $(x, y)\in\mathbf{R}^2$.

B) Now we show that (iii) also holds.

1) If $x\in\left[0, \frac{r}{2}\right)$, then $x=0\cdot\frac{r}{2}+t\left(t\in\left[0, \frac{r}{2}\right)\right)$ and

$$F(x)=0\cdot f\left(\frac{r}{2}\right)+f(t)-0\cdot(a+b)=f(t)=f(x)\text{ and}$$

similarly $G(x)=g(x)$, $H(x)=h(x)$.

2) If $x\in\left[\frac{r}{2}, r\right)$, then $x=\frac{r}{2}+t$ and thus

$$F(x)=f\left(\frac{r}{2}\right)+f(t)-a-b=g\left(\frac{r}{2}\right)+b+a+h(t)-a-b=$$

$$=g\left(\frac{r}{2}\right)+h(t)=f\left(\frac{r}{2}+t\right)=f(x),$$

$$G(x)=g\left(\frac{r}{2}\right)+g(t)-a=g\left(\frac{r}{2}\right)+h(t)-(a+h(t))+g(t)=$$

$$=f\left(\frac{r}{2}+t\right)-f(t)+g(t)=f(x)-f(t)+g(t)=$$

$$=g(x)+b-g(t)-b+g(t)=g(x) \text{ and similarly } H(x)=h(x).$$

3) If $x\in[r, r\sqrt{2})$, then $\frac{x}{2}\in[0, r)$ and so by (ii), B./1. and B./2. we have

$$F(x)=F\left(\frac{x}{2}+\frac{x}{2}\right)=G\left(\frac{x}{2}\right)+H\left(\frac{x}{2}\right)=g\left(\frac{x}{2}\right)+h\left(\frac{x}{2}\right)=f(x).$$

4) It is easy to see that

$$\begin{aligned} f(-x)&=-f(x)+2(a+b), && x\in D_{x+y},\\ g(-x)&=-g(x)+2a, && x\in D_x,\\ h(-x)&=-h(x)+2b, && x\in D_y, \end{aligned}$$

and similarly for functions F, G, H for all $x\in\mathbf{R}$. On the basis of the above

$$F(x)=f(x) \quad \text{for all } x\in(-r\sqrt{2}, 0),$$

$$\left.\begin{aligned} G(x)&=g(x)\\ H(x)&=h(x) \end{aligned}\right\} \text{ for all } x\in(-r, 0)$$

and thus (iii) is proved.

C) Finally we show that (F, G, H) is the unique extension of (f, g, h) from K_r.

Namely if (F_1, G_1, H_1) is also an extension of (f, g, h) from K_r, then by (iii)

$$\begin{aligned} F(0)&=F_1(0)=f(0)=a+b,\\ G(0)&=G_1(0)=g(0)=a, \qquad (1)\\ H(0)&=H_1(0)=h(0)=b. \end{aligned}$$

Let t be an arbitrary real number. There exist $x \in \left[0, \frac{r}{2}\right)$ and $n \in \mathbf{Z}$ such that $t = nx$, furthermore one easily proves that for all $x \in \mathbf{R}$ and for all $n \in \mathbf{Z}$

$$\begin{aligned} F(nx) &= nF(x) - (n-1)(a+b), \\ G(nx) &= nG(x) - (n-1)a, \\ H(nx) &= nH(x) - (n-1)b, \end{aligned} \tag{2}$$

and similarly for functions F_1, G_1, H_1.
By virtue of (1), (2) and (iii) we have

$$F(t) = F(nx) = nF(x) - (n-1)(a+b) = nf(x) - (n-1)(a+b) =$$
$$= nF_1(x) - (n-1)(a+b) = F_1(nx) = F_1(t)$$

and $G(t) = G_1(t)$, $H(t) = H_1(t)$ for all $t \in \mathbf{R}$, q.e.d.

Before formulating Theorem 2. we note the following:

Let $D \subset \mathbf{R}^2$ $(D \neq \emptyset)$, $f: D_{x+y} \to E$, $g: D_x \to E$, $h: D_y \to E$ be functions such that

$$f(x+y) = g(x) + h(y), \quad (x, y) \in D.$$

If (F^*, G^*, H^*) is a quasi-extension of (f, g, h) from the set D and

$$F_1^*(x) \equiv F^*(x) + c_1, \quad G_1^*(x) \equiv G^*(x) + c_2, \quad H_1^*(x) \equiv H^*(x) + c_3 \ (x \in \mathbf{R}),$$

where $c_1, c_2, c_3 \in E$ and $c_1 - c_2 - c_3 = 0$, then (F_1^*, G_1^*, H_1^*) is also a quasi-extension of (f, g, h) from D.

In the sequel the quasi-extensions of the two above types of (f, g, h) will be regarded as equivalent.

Define the set $K_r(u, v) \subset \mathbf{R}^2$ as follows:

$K_r(u, v) = \{(x, y) \mid (x-u)^2 + (y-v)^2 < r^2\}$ ($r > 0$ is a constant and $(u, v) \in \mathbf{R}^2$).
Then we have

Theorem 2. (cp. [2], Satz 3) *Let $f: (K_r(u, v))_{x+y} \to E$, $g: (K_r(u, v))_x \to E$, $h: (K_r(u, v))_y \to E$ be functions such that*

$$f(x+y) = g(x) + h(y), \ (x, y \in K_r(u, v)).$$

Then (f, g, h) has a quasi-extension (F^*, G^*, H^*) from the set $K_r(u, v)$ which is unique up to equivalence.

Proof. Put $x = X + u$ and $y = Y + v$, where $(X, Y) \in K_r$. Then $(x, y) \in K_r(u, v)$ and

$$f(X + Y + u + v) = g(X + u) + h(Y + v), \ (X, Y) \in K_r. \tag{3}$$

Setting $Y = 0$ and $X = 0$ in (3) we obtain

$$f(X + u + v) = g(X + u) + h(v), \quad X \in (K_r)_x \tag{4}$$

and

(5) $$f(Y+u+v)=g(u)+h(Y+v), \quad Y\in(K_r)_y$$

respectively.

Define functions f^*, g^*, h^* by

$$f^*(X)=f(X+u+v), \qquad X\in(K_r)_{x+y},$$

$$g^*(X)=f(X+u+v)-g(u), \quad X\in(K_r)_x \text{ and}$$

$$h^*(X)=f(X+u+v)-h(v), \quad X\in(K_r)_y.$$

By virtue of equations (3), (4) and (5) we have

$$f^*(X+Y)=g^*(X)+h^*(Y) \text{ for all } (X,Y)\in K_r.$$

By virtue of Theorem 1. (f^*, g^*, h^*) has one and only one extension (F^*, G^*, H^*) from the set K_r.

Obviously,

$$F^*(x+y)=G^*(x)+H^*(y) \text{ for all } (x,y)\in\mathbf{R}^2.$$

Now we prove that 3° also holds. First choose $x\in(K_r(u,v))_x$. Then

$$G^*(x)-G^*(u)=G^*(X+u)-G^*(u)=F^*(X+u+v)-H^*(v)-G^*(u)=$$

$$=G^*(X)+H^*(u+v)-H^*(v)-G^*(u)=G^*(X)-G^*(0)=$$

$$=g^*(X)-g^*(0)=f(X+u+v)-g(u)-f(u+v)+g(u)=$$

$$=f(X+u+v)-g(u)-h(v)=g(X+u)-g(u)=g(x)-g(u).$$

In a similar manner we can prove that

$$H^*(x)-H^*(v)=h(x)-h(v) \text{ for all } x\in(K_r(u,v))_y.$$

Finally if $t\in(K_r(u,v))_{x+y}$, then we can write $t=x+y$, where $x\in(K_r(u,v))_x$ and $y\in(K_r(u,v))_y$. Thus

$$F^*(t)-F^*(u+v)=F^*(x+y)-F^*(u+v)=$$

$$=G^*(x)-G^*(u)+H^*(y)-H^*(v)=g(x)-g(u)+$$

$$+h(y)-h(v)=f(x+y)-f(u+v)=f(t)-f(u+v).$$

By a simple calculation it can be shown that (F^*, G^*, H) is the unique quasi-extension of (f, g, h) from $K_r(u,v)$, apart from equivalence, q. e. d.

The following lemma has fundamental importance for the proof of the main result of the present paper:

Lemma. (cp. [2], Hilfssatz) *Let $D\subset\mathbf{R}^2$ be a set, $D=D^1\cup D^2$, where D^1, D^2 are open sets and $D^1\cap D^2\neq\emptyset$. Furthermore let $f: D_{x+y}\to E$, $g: D_x\to E$, $h: D_y\to E$ be functions such that*

$$f(x+y)=g(x)+h(y), \quad (x,y)\in D.$$

Assume that (f, g, h) has a quasi-extension (F_i, G_i, H_i) unique up to equivalence from the set D^i $(i=1,2)$. Then

$$F_1(x)\equiv F_2(x)+c_1, \quad G_1(x)\equiv G_2(x)+c_2, \quad H_1(x)\equiv H_2(x)+c_3 \quad (x\in\mathbf{R})$$

and $c_1-c_2-c_3=0$ $(c_1, c_2, c_3\in E)$ *and with the notations* $F=F_1$, $G=G_1$, $H=H_1$ (F, G, H) *is a quasi-extension of* (f, g, h) *from the set* D, *which is unique up to equivalence.*

Proof. First we note that the point $(u, v)\in D$ in Definition 2. can be replaced by an arbitrary point $(c, d)\in D$.

It is known (see e. g., [1]) that there exist additive functions $\varphi_1:\mathbf{R}\to E$ and $\varphi_2:\mathbf{R}\to E$ such that

$$(6)\qquad F_i(x)=\varphi_i(x)+a_i+b_i,\quad G_i(x)=\varphi_i(x)+a_i,\quad H_i(x)=\varphi_i(x)+b_i\ (x\in\mathbf{R})$$

$$(i=1, 2),\ \text{where } a_i=G_i(0),\ b_i=H_i(0).$$

Let $(c, d)\in D^1\cap D^2$ be an arbitrary point. Since D^1 and D^2 are open, $D_x^1\cap D_x^2$ contains an open interval I_x and by our conditions we obtain

$$G_1(x)-G_1(c)=g(x)-g(c)\ \text{and}\ G_2(x)-G_2(c)=g(x)-g(c),\ x\in I_x.$$

From this we have

$$G_1(x)-G_1(c)=G_2(x)-G_2(c)\ \text{for all } x\in I_x \text{ and by (6)}$$

$$\varphi_1(x)+a_1-\varphi_1(c)-a_1=\varphi_2(x)+a_2-\varphi_2(c)-a_2,\ x\in I_x,$$

i. e.

$$\varphi_1(x-c)=\varphi_2(x-c)\ \text{for all } x\in I_x.$$

Thus $\varphi_1(x)\equiv\varphi_2(x)$ $(x\in\mathbf{R})$ and with the notation $\varphi(x)=\varphi_1(x)$ we obtain

$$G_1(x)=\varphi(x)+a_1\ \text{and}\ G_2(x)=\varphi(x)+a_2\ \text{for all } x\in\mathbf{R},$$

i. e.

$$G_1(x)=G_2(x)+a_1-a_2\ \text{for all } x\in\mathbf{R}.$$

Similarly we can prove that

$$H_1(x)=H_2(x)+b_1-b_2\ \text{and}$$

$$F_1(x)=F_2(x)+a_1+b_1-a_2-b_2\ \text{for all } x\in\mathbf{R}.$$

With the notations $c_1=a_1+b_1-a_2-b_2$, $c_2=a_1-a_2$, $c_3=b_1-b_2$ one indeed has $c_1-c_2-c_3=0$.

By a simple calculation we obtain that $(F=F_1, G=G_1, H=H_1)$ is a quasi-extension of (f, g, h) from D unique up to equivalence.

By the lemma and by theorem 2. one can prove the following

Theorem 3. (cp. [2], Satz 4.) *Let* $D\subset\mathbf{R}^2$ $(D\neq\emptyset)$ *be an arbitrary open connected set and* $f:D_{x+y}\to E$, $g:D_x\to E$, $h:D_y\to E$ *be functions such that*

$$f(x+y)=g(x)+h(y),\quad (x, y)\in D.$$

Then (f, g, h) *has a quasi-extension* (F, G, H) *from the set* D, *which is unique up to equivalence.*

Proof. Since D is open and connected, there exist open disks $K^1, K^2,\ldots, K^n,\ldots$ such that $D=\bigcup_{i=1}^{\infty} K^i$ and $(K^1\cup K^2\cup\ldots\cup K^n)\cap K^{n+1}\neq\emptyset$ $(n=1, 2,\ldots)$.

By virtue of Theorem 2. (f, g, h) has a quasi-extension (F_n, G_n, H_n) from the set K^n $(n=1, 2, \ldots)$, which is unique up to equivalence. From this with the aid of the Lemma, the statement of Theorem 3. already follows.

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НЕКОТОРЫЕ СИСТЕМЫ ФУНКЦИОНАЛЬНЫХ УРАВНЕНИЙ ОБЩЕЙ АССОЦИАТИВНОСТИ И ИХ СВЯЗ С ФУНКЦИОНАЛЬНЫМИ УРАВНЕНИЯМИ ОБЩЕЙ АССОЦИАТИВНОСТИ НА АЛГЕБРЕ КВАЗИГРУПП

Янез Ушан

1. Введение

Речь идет о некоторых классах систем функциональных уравнений

$$\bigwedge_{j\in I} X_1[X_2(a_1^{|X_2|}),\ a_{|X_2|+1}^p] = X_{2j-1}[a_1^{j-1},\ X_{2j}(a_j^{j+|X_{2j}|-1}),\ a_{j+|X_{2j}|}^p], \tag{1}$$

и их отношений к функциональным уравнениям общей ассоциативности

$$X_{2s-1}[a_1^{s-1},\ X_{2s}(a_s^{s+|X_{2s}|-1}),\ a_{s+|X_{2s}|}^p] = \tag{2}$$

$$= X_{2j-1}[a_1^{j-1},\ X_{2j}(a_j^{j+|X_{2j}|-1}),\ a_{j+|X_{2j}|}^p]$$

на алгебре. квазигрупп;

$$(\forall j\in N)\,(2\leqslant|X_{2j}|<p) \quad \text{и} \quad (\forall s\in N)\,(2\leqslant|X_{2s}|<p).$$

D: $I=\{2, \dots, q\}\wedge|X_{2q}|=p-(q-1)$

A: $(\forall j\in N)\,(|X_{2j}|=|X_{2j-1}|=n)\wedge D$; $p=2n-1$, $q=n$

B: $(\forall j\in N)\,(|X_{2j-1}|=n\wedge|X_{2j}|=n+d)\wedge D$

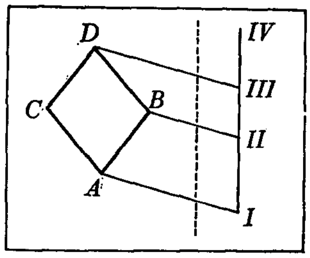
C: $(\forall j\in N)\,(|X_{2j}|=m\wedge|X_{2j-1}|=m+d)\wedge D$

I: $X_1[X_2(a_1^2),\ a_3^3]=X_3[a_1^1,\ X_4(a_2^3)]$

II: $X_1[X_2(a_1^n),\ a_{n+1}^{n+1}]=X_3[a_1^1,\ X_4(a_2^{n+1})]$

III: на пример $X_1[X_2(a_1^2),\ a_3^4]=X_3[a_1,\ X_4(a_2^4)]$

IV: уравнения общей ассоциативности (2).



Диагр. 1.

Диаграма 1 — диаграма систем функциональных уравнений, которые авторы работ [5] — [9] изучали или еще изучают, и их отношение к функциональным уравнениям общей ассоциативности (2), изученные В. Д. Белоусовым ([3], [4]). Самый общий случай систем функциональных уравнений на Диагр. 1 является система (D). Определена седующим образом: Это система (1) в которой $I=\{2, \ldots, q\}$ — множество натуральных чисел, начиная с 2, закончивается с q, а последняя операция с четным индексом имеет арность $|X_{2q}|=p-(q-1)$. Таким образом система (D) обладает некоторым свойством максимальности: j проходит все натуральные числа, начиная с 2, закончивая с q и каждая переменная из последовательности

$$a_1, \ldots, a_p$$

является переменной хотя бы в одной из операций с четным индексом. Система (A) — это система (D) в которой все операции имеют одну и ту же арность $n \in N \setminus \{1\}$. Система (B) — это система в которой все операции с нечетным индексом имеют арность $n \in N \setminus \{1\}$, а операции с четным индексом арность $n+d$. В системе (C) все операции с четным индексом имеют арность m, а все операции с нечетным индексом арность $m+d$; m, $m+d \in N \setminus \{1\}$.

I являются функциональным уравнением общей ассоциативности для бинарного случая, а *IV* самый общий случай функциональных уравнений общей ассоциативности, т. е. для любых арностей. *II* и *III* — это частные случай класса *IV*. В *II* X_1 и X_3 бинарные, а X_2 и X_4 n-арные операции; $n \in N \setminus \{1\}$. В *III* операции с четным индексом — отдельно — не имеют одну и ту же арность, но $q=2$, как и в случаях *I* и *II*. Они сразу являются и системами функциональных уравнений класса (D), как показывает настоящая диаграмма 1. Это и является отношением функциональных уравнений общей ассоциативности (2) к системам функциональных уравнений общей ассоциативности класса (D).

Сейчас переидем к подробнейшем описании систем (A), (B) и (C).

2. Случай *A* ([6])

Система (A), как уже упомянуто, это система (D) в которой все операции имеют одну и ту же арность $n \in N \setminus \{1\}$. Задача решить эту систему. В этом направлении доказываются две теоремы. Первой теоремой является следующее положение.

Т. 1. Если n-квазигруппы A_i, $i \in N_{2n}$, удовлетворяют системе

$$(A) \qquad \bigwedge_{j \in \{2, \ldots, n\}} X_1[X_2(a_1^n), a_{n+1}^{2n-1}] = X_{2j-1}[a_1^{j-1}, X_{2j}(a_j^{j+n-1}), a_{j+n}^{2n-1}],$$

то справедливи равенства

$$(a) \qquad \begin{cases} A_{2j-1}(a_1^n) = A(\{T_{2i-1}^{(i)} T_{2i}^{(1)} a_i\}_{i=1}^{j-1}, T_{2j-1}^{(j)} a_j, \{T_{2i-1}^{(i)} T_{2i}^{(n)} a_i\}_{i=j+1}^{n}) \\ A_{2j}(a_1^n) = T_{2j-1}^{(j)^{-1}} A(\{T_{2i-1}^{(i)} T_{2i}^{(1)} a_{i-j+1}\}_{i=j}^{n}, \{T_{2i-1}^{(i)} T_{2i}^{(n)} a_{n+(i-j)}\}_{i=2}^{j}), \end{cases}$$

где

$$A(a_1^n) = B[B(\ldots(B(a_1, a_2), a_3), \ldots), a_n],$$

B — группа, и

$$T_i^{(t)}x = A_i(\overset{t-1}{k}, x, \overset{n-t}{k});$$

$k \in Q$ фиксированный элемент.

Т. 1. — один из полиадических аналогонов теоремы Белоусова о четырех квазигруппах.

Опись доказательства

Шаг 1. Определяются ретракции операции A_1 и A_2:

$$(c) \quad \begin{cases} A_1^{(L,d)}(a_1^d) = A_1(a_1, \overset{n-d}{k}, a_2^d), \; A_1^{(R,d)}(a_1^d) = A_1(a_1^d, \overset{n-d}{k}), \\ A_2^{(L,d)}(a_1^d) = A_2(\overset{n-d}{k}, a_1^d), \; A_2^{(R,d)}(a_1^d) = A_2(a_1^d, \overset{n-d}{k}). \end{cases}$$

(Оказалось полезным, что „левую" ретракцию операции $A_1 - A_1^{(L,d)}$ — надо определить фиксированием переменных слева, начиная с второй переменной, а для операции A_2 начиная с первой переменной.)

Шаг 2. Утверждается что

$$(d) \quad (\forall i \in \{2, \ldots, n\}) \, (A_2^{(R,i)}(a_1^i) = T_1^{(1)^{-1}} A_1^{(R,i)(L,2)}[A_2^{(R,i-1)}(a_1^{i-1}), T_{2i}^{(n)^{-1}} T_{2i}^{(1)} a_i] \wedge$$
$$\wedge A_1^{(R,i)(L,2)}(x, y) = B(T_1^{(1)}x, T_{2i-1}^{(i)} T_{2i}^{(n)} y)),$$

где B одна и та же группа для всех $i \in \{2, \ldots, n\}$, построена через формулы теоремы Белоусова о четирех квазигруппах, на пример из равенства

$$A_1^{(L,2)}[A_2^{(L,2)}(x, y), z] = A_{2n-1}[\overset{n-2}{k}, x, A_{2n}(y, \overset{n-2}{k}, z)].$$

Шаг 3. Учитывая (d), впервые, получаем (a) для $\underline{A_2}$, A_{2n-1}, $\underline{A_1}$ и A_{2n} (в том же порядке).

Шаг 4. Доказывается что справедливо

$$(\forall i \in N_n) \, (T_{2i-1}^{(i)} T_{2i}^{(n)} k = T_{2i-1}^{(i)} T_{2i}^{(1)} k = e);$$

e — единица группы B из (d), $k \in Q$ — в доказательстве Т. 1. используется один и тот же k.

Шаг 5. Из формул для A_1 и A_2 (полученне в 3. шаге), учитывая результат из 4. шага, впервые получаем формулы для

$$A_1^{(L,d)}, \; A_1^{(R,d)}, \; A_2^{(L,d)}, \; A_2^{(R,d)},$$

а затем, так как

$$A_{2j-1}(a_1^n) = A_1^{(L,n-j+1)}[A_2^{(R,j)}(a_1^{j-1}, T_{2j}^{(1)^{-1}} a_j), a_{j+1}^n]$$

и подобно для A_{2j}, получаем формулы (a).

Учитывая Т. 1,, находим, что справедлива и

Т. 2. Координаты любого решения системы (A) получаются через формулы

$$A_{2j-1}(a_1^n)=A(\alpha_1^{j-1}a_1^{j-1},\ \beta_j a_j,\ \alpha_{j+n}^{2n-1}a_{j+1}^n),$$
$$A_{2j}(a_1^n)=\beta_j^{-1}A(\alpha_j^{j+n-1}a_1^n);$$

α_i, $\beta_j\in Q$! A — n-группа обладающая единицей, определена через некоторую группу B образом

$$A(a_1^n)=B[B(\ldots(B(a_1,a_2),a_3),\ldots),a_n].$$

3. Случай B*) ([8])

Система (B), как уже упомянуто, это система (D) в которой все операции с четным индексом имеют арность

$$n+d\in N\setminus\{1\},\quad d\in N,$$

а операции с нечетным индексом имеот арность $n\in N\setminus\{1\}$. Задача решить эту систему. В этом направлении доказываются две теоремы. Первой теоремой является Т. 1′, т. е. следующее положение:

Т. 1′. Если квазигруппы A_i, $i\in N_{2n}$, удовлетворяют системе

$$(B)\qquad \bigwedge_{j\in\{2,\ldots,n\}} X_1[X_2(a_1^{n+d}),\ a_{n+d+1}^{2n+d-1}]=X_{2j-1}[a_1^{j-1},\ X_{2j}(a_j^{j+n+d-1}),\ a_{j+n+d}^{2n+d-1}],$$

то

$$(\overline{a_1})\qquad A_{2j-1}(a_1^n)=A(\{T_{2i-1}^{(i)}T_{2i}^{(1)}a_1\}_{i=1}^{j-1},\ T_{2j-1}^{(j)}a_j,\ \{T_{2i-1}^{(i)}T_{2i}^{(n+d)}a_i\}_{i=j+1}^{n}),$$

$$(\overline{a_2})\qquad A_{2j}(a_1^{(n+d)})=T_{2j-1}^{(j)-1}A(\{T_{2i-1}^{(i)}T_{2i}^{(1)}a_{i-j+1}\}_{i=j}^{n-1},$$
$$T_1^{(1)}D(a_{n-j+1}^{n-j+1+d}),\ \{T_{2i-1}^{(i)}T_{2i}^{(n+d)}a_{n+(i-j)+d}\}_{i=2}^{j}),$$

где A построена через группу B как в Т. 1., B построена как в Т. 1.**), а

$$D(a_1^{d+1})=A_2(\overset{n-1}{k},\ a_1^{d+1}).$$

(Т. 1′., как и Т. 1., является одним из полиадических аналогонов теоремы Белоусова о четырех квазигруппах.)

Примечание 1.

Если в (B) положим $d=0$, то (B) превращается в (A) и ($\overline{a_1}$) в первую из формул (a) — непосредственно. Факт, что при $d=0$ ($\overline{a_2}$) превращается в вторую из формул (a), получим следующим образом. Для $d=0$

$$D(a_1^{d+1})=A_2(\overset{n-1}{k},\ a_1^{d+1})$$

превращается в $T_2^{(n)}a_1$, Отсюда, так как

$$T_1^{(1)}T_2^{(n)}=T_{2n-1}^{(n)}T_{2n}^{(1)},$$

*) Самое первое внимание на систему (B) и систему (C) мне обращено от Г. Чупона.

**) См. опись доказательства — шаг 2.

последовательность

$$\{T_{2i-1}^{(i)} T_{2i}^{(1)} a_{i-j+1}\}_{j=1}^{n-1}$$

продолжается с членом $T_{2n-1}^{(n)} T_{2n}^{(1)} a_n$, т. е. превращается в первую последовательность из второй формулы (a). Последняя последовательность в $(\overline{a_2})$ при $d=0$ превращается в последнюю (вторую) последовательность второй формулы (a).

Опись доказательства

Шаг 1. Если в $(\overline{B})$ — (B) в которую положены A_i — положим $a_{n+1} = \cdots = a_{n+d} = k$, учитывая, что, в только что построеной системе $(\bar{A})$, $T_{2j}^{(t)}$ является $T_{2j}^{(t+d)}$ если $t > n-(j-1)$, находим формулы $(\overline{a_1})$; на оснований Т. 1.

Шаг 2. Если в $(\overline{B})$ положим

$$a_1 = \cdots = a_{j-1} = a_{j+n+d} = \cdots = a_{2n+d-1} = k,$$

учитывая формулу для A_1 и результат 4. шага из доказательства Т. 1., получаем

$$(\bar{c}) \qquad A_{2j}(a_1^{n+d}) = T_{2j-1}^{(j)-1} \overset{j-1}{B} [T_1^{(1)} A_2^{(L,n+d-j+1)}(a_1^{n+d-j+1}), \{T_{2i-1}^{(i)} T_{2i}^{(n+d)} a_{n+d+(i-j)}\}_{i=2}^{j}],$$

где

$$\overset{j-1}{B}(a_1^j) = B[B(\ldots(B(a_1^2), a_3), \ldots), a_j];$$

B — группа, построена способом из доказательства Т. 1.

Шаг 3. Если положим $(\bar{a}_1)$ и $(\bar{c})$ в $(\overline{B})$, а затем

$$a_{n+d+1} = \cdots = a_{2n+d-1} = k,$$

учитывая результат 4. шага из доказательства Т. 1., получаем

$$(\bar{\bar{c}}) \qquad T_1^{(1)} A_2(a_1^{n+d}) = \overset{j-1}{B} [\{T_{2i-1}^{(i)} T_{2i}^{(1)} a_i\}_{i=1}^{j-1}, T_1^{(1)} A_2^{(L,n+d-j+1)}(a_j^{n+d})].$$

Если в $(\bar{\bar{c}})$ положим $j=s$ и $j=s+1$, можно получить

$$(d) \qquad A_2^{(L,n+d-j+1)}(a_1^{n+d-j+1}) =$$

$$= T_1^{(1)-1} \overset{n-j}{B} [\{T_{2i-1}^{(i)} T_{2i}^{(1)} a_{i-j+1}\}_{i=j}^{n-1}, T_1^{(1)} A_2^{(L,d+1)}(a_{n-j+1}^{n+d-j+1})].$$

Шаг 4. Если (d) положим в $(\bar{c})$, находим формулы для A_{2j}, $j \in \{1, \ldots, n\}$. Учитывая Т. 1'., находим, что справедливо и положение:

Т. 2'. Координаты любого решения системы (B) получаем из формул

$$A_{2j-1}(a_1^n) = A(\alpha_1^{j-1} a_1^{j-1}, \beta_j a_j, \alpha_{j+n}^{2n-1} a_{j+1}^n),$$

$$A_{2j}(a_1^{n+d}) = \beta_j^{-1} A[\alpha_j^{n-1} a_1^{n-j}, D(a_{n-j+1}^{n-j+1+d}), \alpha_{n+1}^{j+n-1} a_{n-j+d+2}^{n+d}];$$

$\alpha_i, \beta_j \in Q!$, A—любая n-арная группа обладающая единицей (определена через некоторую группу B), D—любая квазигруппа арности $d+1$.

4. Случай C ([9])

Система (C), как уже упомянуто, это система (D)-в которой все операции с нечетным индексом имеют арность

$$n+d\in N\setminus\{1\},\ d\in N,$$

а операции с четным индексом имеют арность $n\in N\setminus\{1\}$. Задача решить эту систему. В этом направлении доказываются две теоремы. Первой теоремой является Т. 1.″, т. е. следующее положение:

Т. 1″. Если квазигруппы A_i, $i\in N_{2(n+d)}$, удовлетворяют

$$\text{(C)}\qquad \bigwedge_{j\in\{2,\dots,n+d\}} X_1[X_2(a_1^n),\ a_{n+1}^{2n+d-1}] = X_{2j-1}[a_1^{j-1},\ X_{2j}(a_j^{j+n-1}),\ a_{j+n}^{2n+d-1}],$$

то

$$(\bar{\bar{a}}_1)\qquad A_{2j-1}(a_1^{n+d}) = \overset{n+d-1}{B}(\{T_{2i-1}^{(i)}\,T_{2i}^{(1)}\,a_i\}_{i=1}^{j-1},\ T_{2j-1}^{(j)}\,a_j,\ \{T_{2i-1}^{(i)}\,T_{2i}^{(n)}\,a_i\}_{i=j+1}^{n+d}),$$

$$(\bar{\bar{a}}_2)\qquad A_{2j}(a_1^n) = T_{2j-1}^{(j)\,-1}\,\overset{n-1}{B}(\{T_{2i-1}^{(i)}\,T_{2i}^{(1)}\,a_{i-j+1}\}_{i=1}^{n},\ \{T_{2i-1}^{(i)}\,T_{2i}^{(n)}\,a_{n+(i-j)}\}_{i=2}^{j}),$$

где B онда и та же группа, построена как в Т. 1*.

(Т. 1., Т. 1.′ и Т. 1.″ являются, отдельно, полиадическими аналогонами теоремы Белоусова о четырех квазигруппах).

Примечание 2.

Если в (C) положим $d=0$, то (C) непосредственно превращается в (A), а $(\bar{\bar{a}}_1)$ и $(\bar{\bar{a}}_2)$, таким же образом, в формулы (a).

Опись доказательства

Ш а г 1. Построены s-ретракции системы равенств $(\bar{C})$ (—это (C) в которую положени A_i):

$$(\bar{b})\qquad \bigwedge_{i\in N/\{1\}} A_{2s-1}[\overset{s-1}{k},\ A_{2s}(a_s^{s+n-1}),\ a_{s+n}^{s+2n-2},\ \overset{d-s+1}{k}] = A_{2(s+i-1)-1}[\overset{s-1}{k},\ a_s^{s+i-2},\ A_{2(s+i-1)}\,a_{s+i-1}^{s+i+n-2},\ a_{s+i+n-1}^{s+2n-2},\ \overset{d-s+1}{k}],$$

s-фиксированный элемент множества $\{1,\dots,d+1\}$.

Системы $(\bar{b})$ являются системами случая (A). Учитывая Т. 1., доказывается, что

$$(\forall s\in\{1,\dots,d+1\})\ ({}^{(s)}\overset{n-1}{B} = \overset{n-1}{B}),$$

и отсюда, учитывая связ между главноизотопними группами и результат из 4. шага доказательства Т. 1., что

$$(\forall s\in\{1,\dots,d+1\})\ ({}^{(s)}B = B);$$

${}^{(s)}B$ — группа, построена образом из 2. шага описи доказательства Т. 1., принадлежащая к s-ретракции $(\bar{b})$.

(Этим создана возможность получения формул $(\bar{\bar{a}}_2)$.)

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*) См. опись доказательства — шаг 2.

Шаг 2. Утверждается, что все A_{2j-1} изотопны между собой.

Шаг 3. Определяется ретракция

$$A_{2i-1}^{(R,i)}(\overset{i}{a_1}) = A_{2i-1}(\overset{i}{a_1}, \overset{n+d-i}{k}).$$

Шаг 4. Из $(\bar{b})$ для $s=1$, на основании Т. 1., получаем формулы (a) для $A_{2n-1}^{(R,n)}$.

Шаг 5. В последовательности

$$(\bar{\bar{d}}) \qquad A_{2n-1}^{(R,n)}, A_{2(n+1)-1}^{(R,n+1)}, \ldots, A_{2(n+d)-1}^{(R,n+d)} = A_{2(n+d)-1},$$

начиняя с $A_{2n-1}^{(R,n)}$ (формула полученная в 4. шаге), впервые получается формула для $A_{2(n+1)-1}^{(R,n+1)}$, а затем, таким же образом получаются (по очереди) формулы для всех операций из $(\bar{\bar{d}})$.

Шаг 6. Так как в 5. шаге получена формула для

$$A_{2(n+d)-1},$$

учитывая 2. шаг, получаются и формулы $(\bar{\bar{a}}_1)$.

Вот на примере.

I

$$A_1[A_2(x_1, x_2, x_3), x_4, x_5, x_6, x_7] =$$

II

$$A_3[x_1, A_4(x_2, x_3, x_4), x_5, x_6, x_7] =$$

III

$$A_5[x_1, x_2, A_6(x_3, x_4, x_5), x_6, x_7] =$$

$$A_7[x_1, x_2, x_3, A_8(x_4, x_5), x_6), x_7] =$$

$$A_9[x_1, x_2, x_3, x_4, A_{10}(x_5, x_6) x_7)] =$$

1—ретракция получается из I рамки, для $x_6 = x_7 = k$;

2—ретракция получается из II рамки, для $x_1 = x_7 = k$;

3—ретракция получается из III рамки, для $x_1 = x_2 = k$.

Формула для $A_5^{(R,3)}$ получается из 1—ретракции (на основании Т. 1.), а для $A_7^{(R,4)}$ из равенства

$$A_7^{(R,4)}[x_1, x_2, x_3, A_8(x_4, x_5, k)] = A_5^{(R,3)}[x_1, x_2, A_6(x_3, x_4, x_5)].$$

Так как формулы для A_6, A_8 и $A_5^{(R,3)}$ уже получены, отсюда, поставляя по очереди формулы для A_6, A_8 и $A_5^{(R,3)}$ и $x_5 = k$, учитывая и 4. шаг из описи доказательства Т. 1, получаем формулу для $A_9^{(R,5)} = A_9$.

Учитывая Т. 1''., находим, что справедлива и

Т. 2''. Координаты любого решения системы (C) получается из формул

$$A_{2j-1}(a_1^{n+d}) = \overset{n+d-1}{B}(\{\alpha_i a_i\}_{i=1}^{j-1}, \beta_j a_j, \{\alpha_{i+n} a_{i+1}\}_{i=j}^{n+d}),$$

$$A_{2j}(a_1^n) = \beta_j^{-1} \overset{n-1}{B}(\{\alpha_i a_{i-j+1}\}_{i=j}^{j+n-1});$$

$\alpha_i, \beta_j \in Q!$, B—любая группа.

5. О общих формулах для (a), ($\overline{a}$) и ($\overline{\overline{a}}$)

Интересным является следующий факт: формулы ($\overline{a}_1$) и ($\overline{a}_2$) могуть стать формулами ($\overline{\overline{a}}_1$) и ($\overline{\overline{a}}_2$), если положим:

$$(n) \qquad E[a_1^{j-1}, F(a_j^{j+d}), a_{j+d+1}^{p+d}] \overset{\text{def}}{=\!=} \overline{E}(a_1^{j+d}, a_{j+d+1}^{p+d}),$$

когда $d<0$; E и $\overline{E}$, по очереди, арности p и $p+d$, являются суперпозициями одной и той же группе B. В самом деле:

1°. из $m=n+d<n\in N$ следует, что система (B) превращается в систему (C), а ($\overline{a}_1$) превращается в ($\overline{\overline{a}}_1$); и

2°. Учитывая (n) и $m=n+d<n\in N$, из ($\overline{a}_2$) получаем ($\overline{\overline{a}}_2$) – $T_1^{(1)} D\,(a_{n-j+1}^{n-j+1+d})$ превращается в пустую последовательность, а первая последовательность из ($\overline{a}_2$) превращается в первую последовательность из ($\overline{\overline{a}}_2$).

Что формулы ($\overline{a}$)—могут стать формулами (a)—это уже рассмотрено в Примечании 1.

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$$\underset{j\in\widehat{\{2,\ldots,n+d\}}}{} X_1[X_2(a_1^n), a_{n+1}^{2n+d-1}] = X_{2j-1}[a_1^{j-1}, X_{2j}(a_j^{j+n-1}), a_{j+n}^{2n+d-1}].$$

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SOLUTIONS OF BOUNDED VARIATION OF A LINEAR HOMOGENEOUS FUNCTIONAL EQUATION

Marek Cezary Zdun

This paper aims at giving a survey of some results related to the functional equation

(1) $$\varphi(f(x)) = g(x)\,\varphi(x)$$

where, f and g are given functions and φ is unknown function.

Let J be an interval in R. Let us put

$$\operatorname{Var} g \,|\, J = \sup \{\operatorname{Var} g \,|\, \langle c, d\rangle \,|\, \langle c, d\rangle \subset J\}.$$

Let us denote by $BV[J]$ the class of real-valued functions of bounded variation on the interval J, i.e.

$$BV[J] = \{g : J \to R \,|\, \operatorname{Var} g \,|\, J < \infty\}.$$

We assume the following general hypothesis:

(H) f is continuous and strictly increasing in $J = (a, b\rangle$ (we admit $a = -\infty$) and inequality $a < f(x) < x$ holds for $x \in J$ moreover $g \in BV[J]$, $\lim\limits_{x \to a+} g(x| = 1$ and $\inf \langle \{g(x) \,|\, x \in J\} \rangle 0$.

Let us consider the following sequence of functions

$$G_n(x) = \prod_{i=0}^{n-1} g(f^i(x)) \qquad n \geqslant 1, \quad x \in J,$$

where f^i denotes the i-th iterate of the function f.

There are three posibilities regarding the behaviour of the sequence G_n.

(A) there exists an $x_0 \in J$ such that there exists a finite limit $\lim\limits_{n \to \infty} G_n(x_0) \neq 0$.

(B) there exists an $x_0 \in J$ such that $\lim\limits_{n \to \infty} G_n(x_0) = 0$.

(C) neither of the cases (A) and (B) occurs.

Theorem 1. *Let hypothesis* (H) *be fulfilled and suppose that case* (A) *occurs. Then equation* (1) *has at most one-parameter family of solutions in the class* $BV[J]$. *If a function* $\varphi\in BV[J]$ *satisfies equation* (1) *then* φ *is given by the formula*

$$\varphi(x)=\eta/\prod_{i=0}^{\infty} g(f^i(x)), \quad x\in J.$$

The product $\prod_{i=0}^{\infty} g(f^i(x))$ *converges uniformly on any compact* $K\subset J$.

In this case there need not exist solution $\varphi\in BV[J]$ and $\varphi\not\equiv 0$.

In the case (B) we have the following theorems.

Theorem 2. *Let hypothesis* (H) *be fulfilled. If* $\sum_{n=1}^{\infty} G_n(n)<\infty$ *for an* $x\in J$, *then equation* (1) *has a solution in the class* $BV[J]$ *depending on an arbitrary function. More exactly: for any* $y\in J$ *and for any function* $\varphi_0\in BV[f(y), y\rangle]$ *there exists exactly one function* $\varphi\in BV[J]$ *satisfying equation* (1) *such that* $\varphi(x)=\varphi_0(x)$ *for* $x\in(f(y), y\rangle$.

Theorem 3. *If hypothesis* (H) *is fulfilled and case* (B) *occurs and* $\sum_{n=1}^{\infty} G_n(x)=\infty$ *for an* $x\in J$, *then equation* (1) *has at most one-parameter family of solutions in the class* $BV[J]$. *If a function* $\varphi\in BV[J]$ *satisfies equation* (1), *then* φ *is given by the formula*

$$\varphi(x)=\eta\lim_{n\to\infty} G_n(x)/G_n(\bar{x}), \quad \text{for an } \bar{x}\in J. \tag{2}$$

Sequence (2) *converges uniformly on any compact* $K\subset J$.

Under the above assumption there need not exists a solution $\varphi\in BV[J]$ and $\varphi\not\equiv 0$. The example is given in paper [1].

In the case (C) the only solution in the class $BV[J]$ is a function $\varphi\equiv 0$.

We give some conditions of the existance of solutions in the class $BV[J]$ non identically equal to zero.

Theorem 4. *Let hypothesis* (H) *be fulfilled and suppose that case* (A) *or* (B) *occurs and* $\sum_{n=1}^{\infty} G_n(x)\operatorname{Var} g|(a, f^n(x)\rangle<\infty$ *for an* $x\in J$. *Then equation* (1) *has a one-parameter family of solutions in the class* $BV[J]$.

Theorem 5. *Let hypothesis* (H) *be fulfilled. If case* (A) *or* (B) *occurs and* g *is monotonic in a neighbourhood of* a, *then equation* (1) *has a one-parameter family of solutions in the class* $BV[J]$.

Finally we shall consider the case where $g(x)<0$ for $x\in J$ and $\lim_{x\to a+} g(x)=-1$.

We have the following.

Theorem 6. *If the functions* f *and* $-g$ *satisfy hypothesis* (H), *then equation* (1) *has a solution* $\varphi\in BV[J]$ *and* $\varphi\not\equiv 0$ *if and only if* $\sum_{n=1}^{\infty}|G_n(x)|<\infty$ *for an* $x\in J$. *The solution* $\varphi\in BV[J]$ *and* $\varphi\not\equiv 0$ *if it exists then* φ *depends on an arbitrary function.*

The proofs of the above theorems are presented in paper [1]. The solutions of bounded variation of a general linear equation

$$\varphi(f(x)) = g(x)\varphi(x) + h(x)$$

are considered in papers [2], [3]. In particular, the case where $\lim_{x\to a+} |g(x)| \neq 1$ is investigated in paper [3].

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