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Statistical tests based on Laplace and Hankel transforms, and their application in change point detection

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Statistički testovi zasnovani na Laplasovim i Hankelovim transformacijama, i njihova primena u otkrivanju promena režima

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Dissertation title: Statistical tests based on Laplace and Hankel transforms, and their application in change point detection

Abstract: The main goal of this dissertation is twofold. In the first part, two novel two-sample tests for matrix data are presented. The theoretical properties of these novel tests are investigated in the context of testing orthogonal invariance in distribution, while the empirical values are presented in other cases. The tests are not distribution-free under H_0. Therefore, their quality is investigated through a power study by implementing the warp-speed bootstrap algorithm. The novel tests are applied to multiple cases of real data, primarily originating in the field of finance. These tests are the first of their kind for two-sample tests of positive definite symmetric matrix distributions and are based on Laplace and Hankel transforms.

The second part of this dissertation addresses problems related to data segmentation (or change point detection). Two novel classes of univariate tests for offline data segmentation are outlined, and their theoretical properties are studied. The powers are estimated using the permutation bootstrap algorithm, and the novel tests are shown to have higher test powers than the well-known tests based on the characteristic function. The location of the change point is estimated, and the novel tests are empirically demonstrated to possess greater precision. These tests are applied to two distinct datasets from meteorology and macroeconomics, further emphasizing their applicability in real-case scenarios.

Moreover, the two-sample test based on the Hankel transform is modified to address change point problems. The asymptotic properties of this novel test are derived. A power study is presented, demonstrating the quality of the novel test in small-sample scenarios. The novel test is applied to financial data, emphasizing the practical applicability of this approach. This represents the first test for change point inference based on integral transforms for matrix data.

Keywords: Hankel transform, Laplace transform, matrix distributions, Wishart distribution, noncentral Wishart distribution, cryptocurrency data, stability of financial markets, two-sample tests, change point inference.

Scientific Area: Mathematics

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Sažetak: Glavni cilj ove disertacije je dvojak. U prvom delu su predstavljeni dvouzorački testovi za matrične raspodele. Teorijska svojstva novih testova su predstavljena u slučaju testiranja ortogonalne invarijatnosti u raspodeli, dok su u ostalim slučajevima date empirijske vrednosti. Novi testovi nisu slobodni od raspodele pri nultoj hipotezi. Zbog toga su empirijske moći testova dobijene uz pomoć ubrzanog butstrepa. Novi testovi su primenjeni na stvarne podatke, uglavnom iz oblasti finansija. Ovi testovi su prvi takvi dvouzorački testovi za simetrične pozitivno definitne matrice zasnovani na Laplasovim i Hankelovim transformacijama.

Drugi deo ove disertacije posvećen je problemu otkrivanja tačke promene režima (ili segmentiranja podataka). Nove klase jednodimenzionih testova za otkrivanje aposteriori promene režima su predstavljene i njihova teorijska svojstva su istražena. Empirijske moći testova su ocenjene korištenjem permutacijskog butstrepa. Novi testovi imaju veće moći testova u odnosu na konkurentske testove. Prikazan je i empirijski kvalitet ocene tačke promene i primećeno je da novi testovi imaju veću preciznost u odnosu na konkurentske testove. Ovi testovi su primenjeni na različite podatke iz meteorologije i makroekonomije, čime je pokazana njihova praktična primena.

Dodatno, dovuzorački test zasnovan na Hankelovoj transformaciji je modifikovan da bi otkrivao tačke promene režima. Asimptotska svojstva novog testa su izvedena. Određene su i empirijske moći testova, čime je pokazan kvalitet novog testa u slučaju uzoraka malog obima. Novi test je primenjen na finansijske podatke, što je dodatno prikazalo praktičnu primenu ovoga testa. Ovaj test je prvi takav test promene režima zasnovan na integralnim transformacijama za matrične podatke.

Ključne reči: Hankelova transformacija, Laplasova transformacija, matrične raspodele, Višartova raspodela, necentralna Višartova raspodela, podaci vezani za kriptovalute, stabilnost finansijskih tržišta, dvouzorački testovi, otkrivanje promena režima.

Naučna oblast: Matematika

Uža naučna oblast: Verovatnoća i statistika

Contents

Symbol	Description
$\mathbb{R}$	Set of real numbers
$\mathbb{N}$	Set of natural numbers
$\mathbb{N}_0$	$\mathbb{N} \cup 0$
$\mathbb{C}$	Set of complex numbers
$\Re(z)$	Real part of a complex number z
$\Gamma(z)$	Gamma function
$M_l(A)$	l-th principal minor of a matrix A
$\det(A)$	Determinant of a matrix A
$A > 0$	Matrix A is positive definite
$A > B$	$A - B$ is a positive definite matrix
$A \geqslant 0$	A is a positive semidefinite matrix
$\mathscr{P}_+^{n\times n}$	Cone of symmetric $n \times n$ positive definite matrices
$\mathscr{P}_+$	Ditto, dimension known
$\bar{\mathscr{P}}_+$	Cone of positive semidefinite matrices
$B[0,\infty)$	Space of bounded functions on $[0,\infty)$
$\mathrm{Trace}(A)$	Trace of a matrix A
$\mathrm{etr}(A)$	$\exp(\mathrm{Trace}(A))$
I_n	$n \times n$ identity matrix
$O(n)$	Orthogonal group of $n \times n$ matrices
$\mathrm{Ind}(A)$	Indicator function of an event A
$A_\nu(T)$	Bessel polynomial of one matrix argument [54]
$A_\nu(T,X)$	Bessel polynomial of two matrix arguments [54]

Table 1: Notation used throughout the text

Commonly used univariate probability distributions:

- Exponential distribution $\mathscr{E}(\lambda)$, with a rate parameter $\lambda > 0$ and a density

$$f(x,\lambda) = \lambda \exp(-\lambda x),\ x > 0.$$

- Gamma distribution $\Gamma(\alpha,\beta)$, with a shape parameter $\alpha > 0$ and a rate parameter $\beta > 0$ and a density

$$f(x,\alpha,\beta) = \frac{x^{\alpha-1}\beta^{\alpha}\exp(-\beta x)}{\Gamma(\alpha)},\ x > 0.$$

- Uniform distribution $U[a,b]$, with a density

$$f(x,a,b) = \frac{1}{b-a}, -\infty < a < x < b < +\infty.$$

- Lévy distribution $L(\lambda)$, with a shape parameter $\lambda > 0$ and a density

$$f(x,\lambda) = \sqrt{\frac{\lambda}{2\pi}}x^{-\frac{3}{2}}\exp\left(-\frac{\lambda}{2x}\right),\ x > 0.$$

- Inverse Gaussian $IG(\mu,\lambda)$ distribution, with a location parameter $\mu > 0$ and a scale parameter $\lambda > 0$ and a density

$$f(x,\mu,\lambda) = \sqrt{\frac{\lambda}{2\pi}}x^{-\frac{3}{2}}\exp\left(-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right),\ x > 0.$$

- Normal $\mathcal{N}(\mu, \sigma^2)$ distribution, with a location parameter $\mu \in \mathbb{R}$ and a scale parameter $\sigma^2 > 0$ and a density

$$f(x, \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \; x \in \mathbb{R}.$$

Preface

Prava svakom veličina teče,
a od nogu pa do vrha glave
u pametnih mjeri se visina.
Nije moje, što stekao nisam,
i što pusta podade mi sreća,
već je moje što sam dohvatio
trateć svoju od njedara krvcu.

Ivan Mažuranić, Javor

Matrix methods have been used for an extensive period of time. The earliest results are from the first half of the 20th century, where the main focus was on distributional properties. The Wishart distribution was introduced in 1928 in [138]. The noncentral Wishart distribution was introduced by Anderson in 1946 in [7]. An alternative derivation of the Wishart distribution was presented in 1948 in [115]. Other derivations followed in 1956 by Olkin and Roy [106] and James [65]. The matrix beta distribution was developed in 1939 in [61]. The noncentrality parameter was introduced in 1961 in [74]. In the same year, the same author developed a matrix t-distribution, studying regression-type problems [75]. However, the practical application of matrix distributions occurred much later. We point the reader to the works from 1982 by Muirhead [102] and 1984 by Farrell [37] for a textbook overview of matrix theory.

The first part of this dissertation will develop novel two-sample tests for matrix data, mainly focusing on the case of small dimensions.

Modern research employs symmetric positive definite matrix distributions in the context of cluster analysis [43, 130], classification [128], and regression [34].

Positive semidefinite matrices have many applications, including medical imaging [99] and finance [32]. Positive definite matrix-variate distributions have recently been studied in [133] and in [107], but works on goodness-of-fit testing in this context are sparse. The work of Hadjicosta and Richards from 2020 (see [54]) is the only known goodness-of-fit (GOF) test that uses integral transform methods in this context to date. The work of Alfelt et al. from 2020 (see [4]) considers the Bartlett decomposition to construct a goodness-of-fit test for the centralized Wishart process. Furthermore, there has been recent interest in covariance matrix testing in high dimensions, such as in 2012 (see [66]), in 2020 (see [48]), and in 2023 (see [36]).

On the other hand, the need for data segmentation occurred much earlier. It usually does not require significant computational resources. The classical works dealing with the problem of change point[1] inference are numerous. For example, one may refer to the textbook approach by Csörgö and Horváth from 1997 (see [28]). The area is still developing. In recent years, scientific interest has been present due to numerous applications in finance [8, 9, 127], genetics [22], medicine [27], ecology [1], climatology [81], and other fields. Many of the methods outlined in the statistical literature focus on so-called parametric change point analysis, where a

[1] *Change-point* is the historically accurate way to write this term; however, *change point* and *changepoint* have also appeared in the literature. We will use *change point* for the remainder of the text.

certain model is assumed and the existence of the change point corresponds to a change in the parameters of that model. This problem has been addressed in numerous works. We mention only a few examples, such as Chakar et al. from 2017 (see [21]), Davis et al. from 2006 (see [30]), Fryzlewicz from 2014 (see [41]), and Tsukuda and Nishiyama from 2014 (see [132]).

Moreover, there are nonparametric change point methods that do not rely on any underlying assumptions regarding the model. In these cases, the primary focus is typically on detecting change points in the distribution of sample elements. Numerous papers address this issue, for example Brodsky and Darkhovsky from 1993 (see [18]), Dehling et al. from 2013 (see [31]), and Hawkins and Deng from 2010 (see [57]). One popular approach for nonparametric change point inference involves the use of two-sample test statistics. Specifically, a test statistic is formed using the following nonparametric analogue of a CUSUM-type statistic, which consists of the following steps:

1) The sample of size n is divided into two subsamples of lengths k and $n-k$, respectively;

2) The two-sample statistic is calculated for selected subsamples, and multiplied by a suitable constant;

3) The value of the change point test statistic is taken to be the maximum of such values over k ranging from 1 to n.

This idea, specifically designed to detect change points, was introduced for the first time in 2006 in [62]. Hušková and Meintanis used a two-sample test based on characteristic functions to implement this approach. This approach will serve as a general motivation for our work in modifying the existing two-sample tests to address the change point type of problems in this dissertation, filling the gap for matrix data.

Chapter 1

Introduction

In this chapter, we introduce the preliminary notions that are important building blocks of the ensuing theoretical outline. We cannot aim to cover every aspect of contemporary theory, as that task is currently beyond our reach. We present a broad outline of the current state-of-the-art, aiming to motivate our upcoming presentation and to point the reader to the broader areas from which the inspiration for our research stems. In Section 1.1, we focus on real random variables. The univariate case is usually technically simpler and has facilitated research prior to the development of tools for more complex cases. We focus on the tests constructed using Laplace and Hankel transforms. Following that part, we present the basic outline of matrix theory required for a better understanding of the results in the following chapters. In Section 1.2, the basic outline of matrix integration is presented. In Section 1.4, we introduce the noncentral Wishart distribution and derive its properties. It is an important matrix distribution for understanding the subsequent chapters. In Section 1.5, we introduce the Hankel transform of matrix arguments. We finish the chapter by presenting a short review of integral transform-type tests in the matrix case in Section 1.6.

1.1 Integral transform-type tests

Due to their favorable properties, integral transforms have been used extensively in contemporary statistical literature. They uniquely determine the distribution and have consistent, simple estimators. Therefore, statistical tests are typically constructed as a weighted integral of the difference between the empirical transforms in the case of two-sample tests, and the difference between the theoretical and empirical transforms in the case of GOF tests. We start with the Laplace transform, which has a simpler form than the corresponding Hankel transform, which we discuss later. Laplace transforms are named after the French mathematician Pierre-Simon, Marquis de Laplace; however, the exact historical origin is unclear and can be traced back to Euler [10]. Hankel transforms incorporate special functions, and consequently, the results are sparser than in the case of Laplace transforms. Let us start with the Laplace transform and its basic properties.

Let X be a nonnegative random variable with density f. Then the Laplace transform of a random variable X is defined as:

$$\psi_X(s) = E(\exp(-sX)) = \int_0^\infty \exp(-sx) f(x)\,\mathrm{d}x. \tag{1.1}$$

It has many favorable properties, such as:

- $\psi_X(0) = 1$;

- $0 \leqslant \psi_X(s) \leqslant 1$ for every $s \geqslant 0$;
- ψ_X uniquely determines the distribution of the random variable X;
- There exists a so-called Post-Widder inversion formula, and it is usually possible to recover the density function from its Laplace transform, at least numerically [39].

Its empirical counterpart, the so-called empirical Laplace transform, is defined as:

$$\hat{\psi}_X(t) = \int_0^\infty \exp(-tx) \mathrm{d}\hat{F}_n(x) = \frac{1}{n}\sum_{k=1}^{n} \exp(-tX_k),$$

where $\hat{F}_n$ is the empirical cumulative distribution function of the sample $X_1, \ldots, X_n$, namely

$$\hat{F}_n = \frac{1}{n}\sum_{k=1}^{n} \mathrm{Ind}(X_i \leqslant t).$$

It is possible to examine empirical Laplace transforms through the prism of empirical point processes. However, we do not delve into the theory of empirical point processes, since it is not the central focus of our study. For more information on this topic, we refer the reader to the corresponding literature [70].

The empirical Laplace transform is a consistent estimator of the Laplace transform, and consequently, it has been the basis of many statistical tests. This consistency directly follows from the Law of Large Numbers (see, e.g., [38]). We examine two distinct types of statistical tests. The general form of the so-called L^2-type test statistic is

$$T = C \int_D (\varphi_1(t) - \varphi_2(t))^2 w(t) \mathrm{d}t, \tag{1.2}$$

where the difference $\varphi_1(t) - \varphi_2(t)$ is constructed to be small under the null hypothesis, while $w(t)$ is a weight function that ensures the integral converges and C is a constant depending on the sample. D is an interval on which the functions in question are well-defined. The weight function needs to be integrable, i.e.,

$$\int_D w(t) \mathrm{d}t < \infty.$$

Usually, one takes the difference between the consistent estimator of the transform, and the transform itself in the GOF setting. The difference between two consistent estimators of the transform is commonly used in two-sample tests of equivalence. Another common technique is to use the characterizations of distributions to build GOF tests.

On the other hand, the general form of the so-called L^∞ (or supremum-type) tests is

$$L = C \sup_{0 \leqslant t \leqslant \infty} |(\varphi_1(t) - \varphi_2(t)) w(t)|, \tag{1.3}$$

where C is a constant that depends on the sample size and plays a role in ensuring the convergence in distribution of the novel test.

There were many tests based on the V-empirical Laplace transforms for the exponential distribution. We mention just a few.

Given the sample $X_1, X_2, \ldots, X_n$, the authors want to test the null hypothesis whether the sample comes from the exponential $\mathcal{E}(\lambda)$ distribution, against the alternative that the data do not come from the exponential $\mathcal{E}(\lambda)$ distribution.

Henze and Meintanis considered the test statistic of the form (1.2) in [59]. They considered $\varphi_1(t)$ to be the $\psi_n(t)$, for the scaled sample, $Y_1 = \frac{X_1}{\bar{X}_n}, Y_2 = \frac{X_2}{\bar{X}_n}, \dots, Y_n = \frac{X_n}{\bar{X}_n}$, $\varphi_2(t)$ was taken to be the Laplace transform of the exponential distribution, i.e., $\varphi_2(t) = \frac{1}{1+t}$. The constant was set to the sample size $C = n$, and they considered several weight functions, $w_1(t) = \exp(-at)$ and $w_2(t) = (1+t)^2\exp(-at)$.

Note that equidistribution-type tests have gained in popularity in recent years [104]. The test statistics are usually of the form (1.2), and the functions $\varphi_1(t)$ and $\varphi_2(t)$ are chosen so that the difference $\varphi_1(t) - \varphi_2(t)$ is small under the null hypothesis. For example, in [98], Milošević and Obradović considered the test of exponentiality, where

$$\varphi_1(t) = \frac{1}{n^2}\sum_{l,m=1}^{n} \mathrm{Ind}(\max(X_l, X_m) < t)$$

and

$$\varphi_2(t) = \frac{1}{n^3}\sum_{k,l,m=1}^{n} \mathrm{Ind}\Big(X_k + \min(X_l, X_m) < t\Big),$$

where $\varphi_1(t)$ and $\varphi_2(t)$ are V-empirical processes. One might consider the U-empirical version of the test, i.e.,

$$\varphi_1(t) = \frac{2!}{\binom{n}{2}}\sum_{l \neq m \leqslant n} \mathrm{Ind}(\max(X_l, X_m) < t)$$

and

$$\varphi_2(t) = \frac{3!}{\binom{n}{3}}\sum_{k \neq l \neq m \leqslant n} \mathrm{Ind}\Big(X_k + \min(X_l, X_m) < t\Big),$$

There are many papers that explore the same idea as outlined in [98]. See, for example, [14, 56, 135].

Cuparić et al. considered the test statistics of type (1.2) and (1.3) in [29] for testing exponentiality for the scaled sample, $Y_1 = \frac{X_1}{\bar{X}_n}, Y_2 = \frac{X_2}{\bar{X}_n}, \dots, Y_n = \frac{X_n}{\bar{X}_n}$, where

$$\varphi_1(t) = \frac{1}{n}\sum_{i=1}^{n} \exp(-t\,Y_i),$$
$$\varphi_2(t) = -\frac{1}{n^2}\sum_{i,j=1}^{n} \exp(-t \cdot 2\min(Y_i, Y_j))$$

and the weight function has the form $w(t) = \exp(-at)$.

Henze and Klar constructed a GOF test of the inverse Gaussian distribution in [58]. Given the sample $X_1, X_2, \dots, X_n$, they defined

$$\hat{\theta} = \frac{1}{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{X_i} - \frac{1}{\bar{X}_n}\right)},$$

and

$$\varphi_1(t) = \sqrt{\bar{X}_n}\,\psi_n(t),$$
$$\varphi_2(t) = -\frac{1}{\sqrt{\bar{X}_n}}\left(1 + 2\bar{X}_n^2\frac{t}{\hat{\theta}}\right)^{\frac{1}{2}}\psi_n'(t).$$

The weight function they implemented is sample-dependent and has the form $w(t) = \exp(-a\bar{X}_n t)$, while the constant equals $C = n$.

We point the reader to the work [93, Equation 1.1], in which Meintanis and Iliopoulos considered a similar GOF test for the Rayleigh distribution.

The unfavorable properties of the distributions, such as the infinite expectation, make the construction of the GOF test more challenging. In the following part, we present the GOF test for the Lévy distribution. The Lévy distribution is one of only three stable distributions possessing a density in closed form. Lukić and Milošević constructed GOF tests for the Lévy distribution in [83] and the test statistics are of the form

$$J_{n,a} = \sup_{t>0}\Big|\Big(\frac{1}{n^2}\sum_{i_1,i_2=1}^{n}\exp\Big(-\frac{t(Y_{i_1}+Y_{i_2})}{4}\Big) - \frac{1}{n}\sum_{i=1}^{n}\exp(-tY_i)\Big)\exp(-at)t^{\frac{3}{2}}\Big|,$$

$$R_{n,a} = \int_0^\infty \Big(\frac{1}{n^2}\sum_{i_1,i_2=1}^{n}\exp\Big(-\frac{t(Y_{i_1}+Y_{i_2})}{4}\Big) - \frac{1}{n}\sum_{i=1}^{n}\exp(-tY_i)\Big)\exp(-at)t^{\frac{3}{2}}\,\mathrm{d}t,$$

where $Y_k = \dfrac{X_k\sum_{j=1}^{n}\frac{1}{X_j}}{n}$ is a scaled sample. The weight function was modified slightly to ensure convergence. The test statistic $R_{n,a}$ is an integral-type test statistic, which is simpler to construct than an L^2-type test. Integral-type tests are usually easier to construct, but often do not possess some properties of L^2 tests, such as consistency against fixed alternatives. The tests are distribution-free. Establishing the asymptotic results was more challenging than in the case of the distributions having every moment. More recently, some other works concerning the GOF tests for the Lévy distribution have emerged [77, 108].

Let us now introduce the Hankel transform and present some of its basic properties.

There are many classical works that explore the concept of Hankel transform. We follow the definition from [124], and introduce the Hankel transform $J_0:\mathbb{R}\to\mathbb{R}$, as

$$J_0(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!^2}\Big(\frac{x}{2}\Big)^{2k} = \frac{1}{\pi}\int_0^\pi \cos(x\sin(\theta))\,\mathrm{d}\theta = \frac{1}{\pi}\int_0^\pi \cos(x\cos(\theta))\,\mathrm{d}\theta.$$

Baringhaus and Taherizadeh defined the modified Hankel transform of the arbitrary function $f:\mathbb{R}\to\mathbb{R}_+$ as a functional $\mathcal{H}: B[0,\infty)\to\mathbb{R}$ in [12], as

$$\mathcal{H}(f) = \int_0^\infty J_0(2\sqrt{xt})f(x)\,\mathrm{d}x,\ t\geqslant 0.$$

Since $|J_0(x)|\leqslant 1$ holds for each $x\geqslant 0$, the modified Hankel transform is defined for each nonnegative real random variable X, having a density function f, and is equal to $E(J_0(2\sqrt{tX}))$. They proved the following theorem, which establishes that the Hankel transform uniquely determines the distribution. This result is very useful in the construction of statistical tests.

Theorem 1.1. *[12, Theorem 2.1] Let A and B be two nonnegative independent random variables with corresponding Hankel transforms $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Then $\mathcal{H}_A = \mathcal{H}_B$ if and only if A and B are equally distributed.*

Baringhaus and Taherizadeh presented a series of examples. The first example provides the value of the modified Hankel transform for the Erlang distribution. Let the Laguerre polynomial L_n of order n be defined as

$$L_n(t) = \frac{e^t}{t!}\frac{\mathrm{d}^n}{\mathrm{d}t}(e^{-t}t^n).$$

EXAMPLE 1.1. [12, Example 2.1] Let Y have an Erlang distribution, i.e., a Gamma distribution $\Gamma(n+1,\lambda)$, where $0<\lambda=\frac{n+1}{EY}$ and n is a nonnegative integer. Then the Hankel transform of Y is equal to

$$\mathscr{H}_Y(t)=L_n\Big(\frac{t}{\lambda}\Big)\exp\Big(-\frac{t}{\lambda}\Big),\, t\geqslant 0,$$

where L_n denotes the Laguerre polynomial of order n. Taking $n=0$, one obtains the Hankel transform of the exponential distribution as

$$\mathscr{H}_Y(t)=\lambda\int_0^\infty J_0(2\sqrt{tx})\exp(-\lambda x)\,\mathrm{d}x=\exp\Big(-\frac{t}{\lambda}\Big),\, t\geqslant 0.$$

Laplace and Hankel transforms are related, as is shown by the following example:

EXAMPLE 1.2. [12, Example 2.2] Let A be an exponentially distributed random variable with parameter 1, and B be another nonnegative random variable independent of A. The Hankel transform of AB is given by

$$\mathscr{H}_{AB}(t)=E(J_0(2\sqrt{tAB}))=E(\exp(-tB))=\psi_B(t),\, t\geqslant 0,$$

where $\psi_B(t)$ denotes the Laplace transform of the random variable B.

Baringhaus and Taherizadeh proved the following characterization of the exponential distribution. It was a basis of their GOF test of exponentiality, since the cornerstone of their test was the difference between the empirical Hankel transform and the Hankel transform.

THEOREM 1.2. *[12, Theorem 2.2] Let A be a nonnegative random variable whose Hankel transform is of the form $\mathscr{H}_A(t)=\exp(-t), 0\leqslant t\leqslant\varepsilon$, for some $\varepsilon>0$. Then $A\in\mathscr{E}(1)$.*

They then wanted to construct statistical tests which test the null hypothesis H_0 : $X_1,X_2,\dots,X_n\in\mathscr{E}(\lambda)$ against the alternative H_1 : $X_1,X_2,\dots,X_n\notin\mathscr{E}(\lambda)$. The first test [12] is of the form (1.2), where $C=n$, and

$$\varphi_1(t)=\mathscr{H}_n(t)=\frac{1}{n}\sum_{j=1}^{n}J_0(2\sqrt{tY_j})$$

is the empirical Hankel transform of the empirically standardized variables, i.e., $Y_j=0$ if $X_j=0$ and $Y_1=\frac{X_1}{\bar{X}_n}$, $Y_2=\frac{X_2}{\bar{X}_n},\dots,Y_n=\frac{X_n}{\bar{X}_n}$ and $\bar{X}_n=\frac{1}{n}\sum\limits_{j=1}^{n}X_j$. Let $\varphi_2(t)=\exp(-t)$ be the theoretical Hankel transform of the exponential $\mathscr{E}(1)$ distribution. The weight function is $w(t)=\exp(-t)$. Since $\frac{1}{\bar{X}_n}$ is a consistent estimator of the parameter λ, they established that the test statistics are free of the parameter λ. Baringhaus and Taherizadeh developed a test of type (1.3), taking $\varphi_1(t)$ and $\varphi_2(t)$ as above and $C=\sqrt{n}$ and $w(t)=1$ in [13].

At this point, there are no other GOF tests for other families of distributions. It is possible to construct the GOF tests using the $\varphi_1(t)$ given above and varying the form of $\varphi_2(t)$ to fit the theoretical Hankel transform of the distribution in question. If the computation of the integral in (1.2) seems challenging, it may be possible to consider the test statistic of the form (1.3).

Another possibility is to construct the two-sample test of the type (1.2). Let $X=X_1,X_2,\dots,X_{n_1}$ and $Y=Y_1,Y_2,\dots,Y_{n_2}$ be two independent random samples with empirical Hankel transforms $\mathscr{H}_{n_1,X}$ and $\mathscr{H}_{n_2,Y}$, respectively. Baringhaus and Taherizadeh wanted to test $H_0: X$ and Y are equally distributed, against the alternative $H_1: X$ and Y are not equally distributed [11]. They set $C=\frac{n_1n_2}{n_1+n_2}$, $\varphi_1(t)=\mathscr{H}_{n_1,X}$, $\varphi_2(t)=\mathscr{H}_{n_2,Y}$ and $w(t)=\exp(-t)$.

Based on previous research, Hadjicosta and Richards worked with more general Hankel transforms in [53] . They used the following objects:

- the Bessel function of the first kind of order ν, where $-\nu \notin \mathbb{N}$ and $\nu \in \mathbb{R}$:

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\nu+1+k)} \left(\frac{z}{2}\right)^{2k+\nu}, \ z \in \mathbb{C};$$

- the modified Bessel function of the first kind of order ν, $\nu \in \mathbb{R}$ and $-\nu \notin \mathbb{N}$ and $x \in \mathbb{R}$:

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(\nu+1+k)} \left(\frac{x}{2}\right)^{2k+\nu};$$

- Let $s, t \in \mathbb{R}$, where $-t \notin \mathbb{N}_0$. Then the confluent hypergeometric function is defined as

$${}_1F_1(s;t;x) = \sum_{k=0}^{\infty} \frac{(s)_k}{(t)_k} \frac{x^k}{k!},$$

  where $(x)_k = x(x+1)\ldots(x+k-1)$ is a so-called rising factorial.

- For $\nu \geqslant -\frac{1}{2}$ and $t \geqslant 0$, the Hankel transform of order ν of a nonnegative random variable X with a density $f(x)$ is defined as

$$\mathscr{H}_{X,\nu}(t) = \Gamma(\nu+1) \int_0^{\infty} (tx)^{-\frac{\nu}{2}} J_\nu(2\sqrt{tx}) f(x)\, \mathrm{d}x.$$

- Having the random sample $X_1, \ldots, X_n$, the empirical Hankel transform of order ν of the scaled sample $Y_1 = \frac{X_1}{\bar{X}_n}, Y_2 = \frac{X_2}{\bar{X}_n}, \ldots, Y_n = \frac{X_n}{\bar{X}_n}$ is given as

$$\mathscr{H}_{n,\nu}(t) = \frac{\Gamma(\nu+1)}{n} \sum_{k=1}^{n} (tY_k)^{-\nu/2} J_\nu(2\sqrt{tY_k}), \ t \geqslant 0.$$

Let us have the sample $X = X_1, X_2, \ldots, X_n$. Hadjicosta and Richards wanted to test the null hypothesis $H_0 : X \in \Gamma(\alpha, \lambda)$, where α is known, against the alternative $H_1 : X \notin \Gamma(\alpha, \lambda)$. They defined the test statistic in [53] of the type (1.2) taking $C = n$, $\varphi_1(t) = \mathscr{H}_{n,\nu}(t)$, $\varphi_2(t) = {}_1F_1(\alpha; \nu+1; -\frac{t}{\alpha})$ and $w(t) = \exp(-\alpha t)$, where $\varphi_1(t)$ is the empirical Hankel transform of the scaled sample.

To the best of our knowledge, this is the only test utilizing the empirical Hankel transform of the form given above. Of course, there are many possible generalizations, given that the expression in the integral (1.2) is tractable. Other distributions can be exploited as well, hence filling the existing gap in the literature.

In the subsequent section, we focus on the fundamental theory of matrix integration. This step is necessary to generalize some univariate concepts to the matrix case, which is one of the main goals of this dissertation.

1.2 Matrix integration

In this section, the notion of the integral $\int_A f(X)\mathrm{d}X$ of a function $f : \mathscr{M}_{n\times n} \to \mathrm{R}$ over a set $A \subseteq \mathscr{M}_{n\times n}$ is studied. Usually, A is a subset of the cone of positive semidefinite matrices, or the orthogonal group. This concept is central in the construction of novel statistical tests, since these tests usually utilize objects such as the integral difference of suitably chosen functions with respect to a suitably chosen weight function.

The theory presented below can be found in many classical works, such as the book by Muirhead [102] or Farrell [37]. We follow the approach in [102], since it is the simplest one and it does not delve deep into the algebraic theory. The following results are taken from [102], and that fact will not be explicitly mentioned below.

We use differential forms as a means of computing integrals. Differential forms have been studied extensively in the scientific literature [19, 35].

DEFINITION 1.1. An exterior differential form of degree k in $\mathbb{R}^n$ is a function

$$\sum_{j_1<j_2<\ldots<j_k} f_{j_1,j_2,\ldots,j_k}(x_1,x_2,\ldots,x_n)\,\mathrm{d}x_{j_1}\wedge \mathrm{d}x_2\wedge\cdots\wedge \mathrm{d}x_{j_k},$$

where $f_{j_1,j_2,\ldots,j_r}$ are analytical functions of $x_1,x_2,\ldots,x_n$ and $\wedge$ denotes the wedge product.

For example, exterior differential forms of degree zero are simply analytical functions, forms of degree one are differentials, and so on. Moreover, since we will not be interested in determining the specific sign of the differential form, as is common practice in probability theory, we may observe that in $\mathbb{R}^n$ there exists only one exterior differential form of degree n, namely $f(x_1,x_2,\ldots,x_n)\mathrm{d}x_1\wedge \mathrm{d}x_2\wedge\cdots\wedge \mathrm{d}x_n$. Consequently, since $\mathrm{d}x_i\wedge \mathrm{d}x_i=-\mathrm{d}x_i\wedge \mathrm{d}x_i=0$, there exist no exterior differential forms of degree greater than n in $\mathbb{R}^n$.

Let X be an $n\times m$ matrix, and let us denote by $(\mathrm{d}X)=\bigwedge_{j=1}^{m}\bigwedge_{l=1}^{n}\mathrm{d}x_{ij}$ the wedge product of the distinct matrix elements. This is not to be confused with the matrix differential, denoted by $\mathrm{d}X=(\mathrm{d}x_{ij})_{i,j=1}^{m,n}$. If the matrix X is a symmetric $n\times n$ matrix, then $(\mathrm{d}X)=\bigwedge_{1\leqslant i\leqslant j\leqslant n}\mathrm{d}x_{ij}$. Similarly, if X is an $m\times n$ upper triangular matrix, one can establish that $(\mathrm{d}X)=\bigwedge_{i\leqslant j}\mathrm{d}x_{ij}$.

Denote by X' the matrix transpose of a matrix X. For a positive definite $n\times n$ matrix X, a unique decomposition $X=T'T$ exists, where T is an $n\times n$ upper triangular matrix, having positive diagonal elements. This helps us to establish the following theorem, which will prove useful in computing matrix integrals.

THEOREM 1.3. *If X is an $n\times n$ positive definite matrix and $X=T'T$, where T is upper triangular with positive definite elements, then*

$$(\mathrm{d}X)=2^n\prod_{i=1}^{n}t_{ii}^{n+1-i}(\mathrm{d}T). \tag{1.4}$$

PROOF. Since

$$\begin{pmatrix} x_{11} & x_{12} & \ldots & x_{1n}\\ x_{21} & x_{22} & \ldots & x_{2n}\\ & & \vdots & \\ x_{n1} & x_{n2} & \ldots & x_{nn}\end{pmatrix}=\begin{pmatrix} t_{11} & 0 & \ldots & 0\\ t_{21} & t_{22} & \ldots & 0\\ & & \vdots & \\ t_{n1} & t_{n2} & \ldots & t_{nn}\end{pmatrix}\begin{pmatrix} t_{11} & t_{12} & \ldots & t_{1n}\\ 0 & t_{22} & \ldots & t_{2n}\\ & & \vdots & \\ 0 & 0 & \ldots & t_{nn}\end{pmatrix},$$

we need to express every element on or above the main diagonal of X in terms of elements of T and then take differentials. Now, we get

$$\begin{aligned} &x_{11}=t_{11}^2, \text{ hence } \mathrm{d}x_{11}=2t_{11}\,\mathrm{d}t_{11}\\ &x_{12}=t_{11}t_{12}, \qquad \mathrm{d}x_{12}=t_{11}\,\mathrm{d}t_{12}+\ldots\\ &\vdots\\ &x_{1n}=t_{11}t_{1n}, \qquad \mathrm{d}x_{1n}=t_{11}\,\mathrm{d}t_{1n}+\ldots\end{aligned}$$

$$
\begin{aligned}
&x_{22} = t_{11}^2 + t_{22}^2, \qquad \mathrm{d}x_{22} = 2t_{22}\,\mathrm{d}t_{22} + \dots \\
&\vdots \\
&x_{2n} = t_{12}t_{1n} + t_{22}t_{2n},\ \mathrm{d}x_{2n} = t_{22}\,\mathrm{d}t_{2n} + \dots \\
&\vdots \\
&x_{nn} = t_{11}^2 + \dots + t_{nn}^2,\ \mathrm{d}x_{nn} = 2t_{nn}\,\mathrm{d}t_{nn} + \dots
\end{aligned}
$$

We did not write the terms that will cancel out. Now, taking the exterior products finishes the proof, since

$$(\mathrm{d}X) = \bigwedge_{i\leqslant j}^{n} \mathrm{d}x_{ij} = 2^n t_{11}^n t_{22}^{n-1} \dots t_{nn} \bigwedge_{i\leqslant j}^{n} \mathrm{d}t_{ij} = 2^n \prod_{i=1}^{n} t_{ii}^{n+1-i} (\mathrm{d}T).$$

□

The result stated above can be used to evaluate the multivariate gamma function. Let $X > 0$ denote that an $n \times n$ matrix is positive definite, and denote by $\mathrm{etr}(X) = \exp(\mathrm{Trace}(X))$.

DEFINITION 1.2. The multivariate gamma function $\Gamma_n : \mathbb{C} \to \mathbb{C}$ is defined as

$$\Gamma_n(a) = \int_{X>0} \mathrm{etr}(-X)(\det X)^{a-(n+1)/2}(\mathrm{d}X),$$

where the integral has been taken over all positive definite $n \times n$ matrices. The function is well defined for $\Re(a) > \frac{1}{2}(n-1)$.

The integral stated above is simply an integral over the open cone of positive definite matrices, i.e., the one which satisfies the following system of inequalities:

$$X > 0 \iff x_{11} > 0,\ \det\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} > 0, \dots, \det X > 0$$

with respect to the Lebesgue measure

$$(\mathrm{d}X) = \mathrm{d}x_{11} \wedge \mathrm{d}x_{12} \wedge \dots \wedge \mathrm{d}x_{nn} = \mathrm{d}x_{11}\,\mathrm{d}x_{22} \dots \mathrm{d}x_{nn}.$$

Now, we are ready to compute the multivariate gamma function, which is shown to be the product of univariate gamma functions:

THEOREM 1.4. *The exact form of the multivariate Gamma function is*

$$\Gamma_n(a) = \int_{X>0} \mathrm{etr}(-X)(\det X)^{a-(n+1)/2}(\mathrm{d}X) = \pi^{n(n-1)/4} \prod_{i=1}^{n} \Gamma\left(a - \frac{1}{2}(i-1)\right).$$

PROOF. Let $X = B'B$, where B is an upper triangular matrix with positive diagonal elements. Then we have the basic identities:

$$
\begin{aligned}
\mathrm{Trace}(X) &= \mathrm{Trace}(B'B) = \sum_{i\leqslant j}^{n} b_{ij}^2, \\
\det X &= \det(B'B) = (\det B)^2 = \prod_{i=1}^{n} b_{ii}^2.
\end{aligned}
$$

Therefore, using (1.4), we obtain

$$(\mathrm{d}X) = 2^n \prod_{i=1}^{n} b_{ii}^{n+1-i} \bigwedge_{i \leqslant j}^{n} \mathrm{d}b_{ij}.$$

Therefore,

$$\Gamma_n(a) = \int \cdots \int \exp\left(-\sum_{i \leqslant j}^{n} b_{ij}^2\right) 2^n \prod_{i=1}^{n} b_{ii}^{2a-i} \bigwedge_{i \leqslant j}^{n} \mathrm{d}b_{ij}$$

$$= \prod_{i<j}^{n} \left(\int_{-\infty}^{\infty} \exp(-b_{ij}^2)\, \mathrm{d}b_{ij} \right) \prod_{i=1}^{n} \left(\int_{0}^{\infty} \exp(-b_{ii}^2)(b_{ii}^2)^{a-(i+1)/2}\, \mathrm{d}b_{ii}^2 \right).$$

The direct computation yields

$$\int_{-\infty}^{\infty} \exp(-b_{ij}^2)\, \mathrm{d}b_{ij} = \sqrt{\pi}$$

and

$$\int_{0}^{\infty} \exp(-b_{ii}^2)(b_{ii}^2)^{a-(i+1)/2}\, \mathrm{d}b_{ii}^2 = \Gamma\left(a - \frac{1}{2}(i-1)\right)$$

and the result of the theorem follows. □

The computation of the following matrix integral gives the density of the Wishart distribution, which will play a significant role in constructing the statistical test.

THEOREM 1.5. *Let* $\Re(a) > \frac{1}{2}(n-1)$ *and let* Σ *be an* $n \times n$ *matrix and* $\Sigma > 0$. *Then*

$$\int_{X>0} \operatorname{etr}\left(-\frac{1}{2}\Sigma^{-1} X\right) (\det X)^{a-(n+1)/2} (\mathrm{d}X) = \Gamma_n(a) (\det \Sigma)^a 2^{na}.$$

PROOF. We make the change of variables in the integral, namely $Y = 2\Sigma^{\frac{1}{2}} X \Sigma^{\frac{1}{2}}$, where $\Sigma^{\frac{1}{2}}$ denotes the positive definite square root of Σ. It follows that $(\mathrm{d}Y) = 2^{n(n+1)/2} (\det \Sigma)^{(n+1)/2} (\mathrm{d}X)$ and the integral becomes

$$\int_{Y>0} \operatorname{etr}(-Y) (\det Y)^{a-(n+1)/2} (\mathrm{d}Y) 2^{na} (\det \Sigma)^a = \Gamma_n(a) 2^{na} (\det \Sigma)^a.$$

□

Putting $a = \frac{m}{2}$, where $m > n-1$ is a real number, and if $\Sigma > 0$, the function

$$f(X) = \frac{1}{2^{mn/2} \Gamma_n(\frac{m}{2}) (\det \Sigma)^{m/2}} \operatorname{etr}\left(-\frac{1}{2}\Sigma^{-1} X\right) (\det X)^{(m-n-1)/2},\ X > 0 \tag{1.5}$$

is a density function, since it is nonnegative and integrates to 1. It determines the Wishart distribution. If m is an integer, then it is the density function of $(m-1)S$, where S is a sample covariance matrix of the random sample of size m taken from the multivariate normal $\mathscr{N}_n(\mu, \Sigma)$ distribution.

In the next section, we outline a measure which plays an important role in matrix statistics due to its orthogonal invariance.

1.3 Haar measure

The following results are taken from [102], and that fact will not be explicitly mentioned below.

Let H be an $n \times m$ matrix, where $n \geqslant m$, and $H'H = I_m$, i.e., its columns are orthonormal. The space of such matrices H, usually denoted by $V_{m,n}$, is called a Stiefel manifold. The equation $H'H = I_m$ provides $\frac{1}{2}m(m+1)$ independent conditions on the matrix elements; therefore, the Stiefel manifold can be regarded as an $mn - \frac{1}{2}m(m+1)$ surface in $\mathbb{R}^{mn}$. Moreover, if the matrix elements satisfy the condition $\sum_{i=1}^{n}\sum_{j=1}^{m} h_{ij}^2 = m$, then this is a surface which is a subset of the sphere of radius $\sqrt{m}$ in $\mathbb{R}^{mn}$.

When $m = n$, the Stiefel manifold is equal to the orthogonal group, because

$$V_{m,m} = \{H : H \in \mathcal{M}_{m\times m} \wedge H'H = I_m\} = O(m).$$

When $m = 1$, the Stiefel manifold is a sphere, namely

$$V_{1,n} = \{h : h \in \mathbb{R}^n \wedge h'h = 1\} = \mathbb{S}_n,$$

which is an $n-1$-dimensional surface in $\mathbb{R}^n$.

Let us focus on the case $m = n$, since it is significant in the further part of the text. Let $H \in O(m)$. The differential form

$$(H'\,\mathrm{d}H) = \bigwedge_{i<j}^{m} h_j'\,\mathrm{d}H_i$$

is the wedge product of the subdiagonal elements of $H'\,\mathrm{d}H$. However, it is significant that this differential form is invariant under orthogonal transforms. It is invariant under left orthogonal transforms, because if $H \to GH$ for some $G \in O(m)$, then $H'\,\mathrm{d}H \to H'G'G\,\mathrm{d}H = H'I_m\,\mathrm{d}H = H'\,\mathrm{d}H$. Therefore,$(H'\,\mathrm{d}H) \to (H'\,\mathrm{d}H)$. It is invariant under right orthogonal transforms as well, since if $H \to HG$ for some $G \in O(m)$, then $H'\,\mathrm{d}H \to GH'\,\mathrm{d}HG'$ and then $(H'\,\mathrm{d}H) \to (\det G)^{m-1}(H'\,\mathrm{d}H) = (H'\,\mathrm{d}H)$ [102, Theorem 2.1.7], because we ignore the sign of the differential form.

This differential form defines an invariant measure μ on $O(m)$, as follows:

$$\mu(A) = \int_A (H'\,\mathrm{d}H),\ A \subseteq O(m).$$

The invariance of the measure directly follows from the invariance of the differential form $(H'\,\mathrm{d}H)$; namely

$$\mu(QA) = \mu(AQ) = \mu(A)$$

for every $Q \in O(m)$. This measure is named after Alfréd Haar, the Hungarian mathematician who established the existence of the unique invariant measure on any locally compact topological group [51, 103].

This measure is finite. The following theorem establishes that fact.

THEOREM 1.6. *The volume of the orthogonal group is given by*

$$\mu(O(m)) = \int_{O(m)} (H'\,\mathrm{d}H) = \frac{2^m \pi^{m^2/2}}{\Gamma_m(m/2)}.$$

PROOF. Let A be a random matrix which contains m^2 independent normal $\mathcal{N}(0,1)$ random variables. Then the density function of such a matrix is equal to

$$f(A) = (2\pi)^{-m^2/2}\operatorname{etr}(-A'A/2),$$

and now, it follows that

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \operatorname{etr}(-A'A/2)(\mathrm{d}A) = (2\pi)^{m^2/2}. \tag{1.6}$$

There exists a unique decomposition $A = HT$, where $H \in O(m)$ and T is an upper triangular $m \times m$ matrix. Therefore,

$$\operatorname{Trace}(A'A) = \operatorname{Trace}(T'T) = \sum_{i \leqslant j}^{m} t_{ij}^2$$

and from [102, Theorem 2.1.13], it follows that

$$(\mathrm{d}A) = \prod_{i=1}^{m} t_{ii}^{m-1} (\mathrm{d}T)(H\,\mathrm{d}H).$$

Now (1.6) becomes

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\sum_{i \leqslant j}^{m} t_{ij}^2\right) \prod_{i=1}^{m} t_{ii}^{m-1} (\mathrm{d}T) \int_{O(m)} (H\,\mathrm{d}H) = (2\pi)^{m^2/2}.$$

and the integral containing $(\mathrm{d}T)$ can be computed as

$$\prod_{i<j}^{m} \int_{-\infty}^{\infty} \exp(-t_{ij}^2/2)\,\mathrm{d}t_{ij} \prod_{i=1}^{m} \int_{0}^{\infty} \exp(-t_{ii}^2/2) t_{ii}^{m-1}\,\mathrm{d}t_{ii} = \prod_{i<j}^{m} \sqrt{2\pi} \prod_{i=1}^{m} \Gamma((m-i+1)/2)$$
$$= \pi^{m(m-1)/4} \prod_{i=1}^{m} \Gamma((m-i+1)/2) 2^{m^2/2-m} = \Gamma_m(m/2) 2^{m^2/2-m}.$$

Finally, we have

$$\int_{O(m)} (H\,\mathrm{d}H) = \frac{(2\pi)^{m^2/2}}{\Gamma_m(m/2) 2^{m^2/2-m}} = \frac{2^m \pi^{m^2/2}}{\Gamma_m(m/2)}$$

and this finishes the proof. □

One may define the following differential form on $O(m)$:

$$(\mathrm{d}H) = \frac{1}{\mu(O(m))}(H'\,\mathrm{d}H) = \frac{\Gamma_m(m/2)}{2^m \pi^{m^2/2}}(H'\,\mathrm{d}H),$$

and then $\int_{O(m)}(\mathrm{d}H) = 1$. Therefore, the measure given by

$$\xi(A) = \int_A (\mathrm{d}H) = \frac{1}{\mu(O(m))} \int_A (H'\,\mathrm{d}H) = \frac{\Gamma_m(m/2)}{2^m \pi^{m^2/2}} \int_A (H'\,\mathrm{d}H),\ A \subseteq O(m) \tag{1.7}$$

is an orthogonally invariant probability distribution (Haar distribution) on $O(m)$.

For the case of $m = 1$, it is well known that the only orthogonally invariant distribution on $\mathbb{S}_n$ is the von Mises distribution (uniform spherical distribution), which is a popular topic in the contemporary statistical literature [69, 117].

1.4 Noncentral Wishart distribution

This section presents the basic notions concerning the noncentral Wishart distribution, following the outline given in [79]. Let us focus on the standard noncentral Wishart distribution. Denote by E_r the set of all partitions υ of length r. Whenever we do not explicitly state the dimension of the matrices, we assume that we are working with $n\times n$ matrices. The important question of the *existence* of such a measure will be addressed later.

PROPOSITION 1.4.1. *[79, Proposition 3.1] Let $p>\frac{n-1}{2}$ and let $A\geqslant 0$. Then*

$$\gamma(p,A)(\mathrm{d}T)=\frac{e^{-\mathrm{Trace}(T+A)}}{\Gamma_n(p)}(\det T)^{p-\frac{n+1}{2}}\left(\sum_{k=0}^{\infty}\sum_{|\upsilon|=k}\frac{C_\upsilon(A^{\frac{1}{2}}TA^{\frac{1}{2}})}{k!(p)_\upsilon}\right)\mathrm{Ind}(T\in\mathscr{P}_+)(\mathrm{d}T)$$

is a probability measure on $\mathscr{P}_+$ satisfying the Laplace transform identity for $I_n+S\in\mathscr{P}_+$:

$$\int\limits_{T>0} e^{-\mathrm{Trace}(ST)}\gamma(p,A)(\mathrm{d}T)=\frac{1}{(\det(I_n+S))^p}e^{-\mathrm{Trace}((I_n+S)^{-1}SA)}.$$

PROOF. We need to establish that $\gamma(p,A)(\mathrm{d}T)$ is a positive measure. However, one readily sees that every minor of a positive definite matrix is positive, therefore C_υ is positive, from which we conclude that γ must be a positive measure.

We need to compute the Laplace transform, since putting $S=0$ in that case establishes that $\gamma(p,A)$ integrates to one and is a density.

We return to the integral

$$I_\upsilon(A)=\frac{e^{-\mathrm{Trace}(A)}}{\Gamma_n(p)}\int_{T>0} e^{-\mathrm{Trace}((S+I_n)T)}(\det T)^{p-\frac{n+1}{2}}\frac{C_\upsilon(A^{\frac{1}{2}}TA^{\frac{1}{2}})}{k!(p)_\upsilon}(\mathrm{d}T).$$

We make the change of variables $X=A^{\frac{1}{2}}TA^{\frac{1}{2}}$. We know that $(\mathrm{d}X)=(\det A)^{(k+1)/2}(\mathrm{d}T)$ from the basic properties of differential forms [102, Theorem 2.1.7]. Now we have that our integral takes the following form:

$$\frac{e^{-\mathrm{Trace}(A)}}{\Gamma_n(p)}\int_{X>0} e^{-\mathrm{Trace}((S+I_n)A^{-\frac{1}{2}}XA^{-\frac{1}{2}})}(\det A^{-\frac{1}{2}}XA^{-\frac{1}{2}})^{p-\frac{n+1}{2}}\frac{C_\upsilon(X)}{k!(p)_\upsilon}(\det A)^{(k+1)/2}(\mathrm{d}X).$$

The following property of the zonal polynomials is used to directly compute the integral itself [79, Eq. (12)]:

$$(p)_\upsilon\det(S)^{-p}C_\upsilon(S^{-1})=\frac{1}{\Gamma_n(p)}\int_{X>0} e^{-\mathrm{Trace}(SX)}C_\upsilon(X)(\det X)^{p-\frac{k+1}{2}}(\mathrm{d}X).$$

Directly applying this result, we obtain

$$I_\upsilon(A)=e^{-\mathrm{Trace}(A)}\det(I_n+S)^{-p}\frac{C_\upsilon(A^{\frac{1}{2}}(I_n+S)^{-1}A^{\frac{1}{2}})}{n!}.$$

This establishes the result for the nonsingular A. However, one needs to treat the case of the singular A as well. If A is singular, then $\det(A_k)=\det(A+\frac{1}{k}I_n)>0$ for every $k\in\mathbb{N}$. The following identity holds [79, Equation 10]

$$\mathrm{etr}(X)=\sum_{n=0}^{\infty}\sum_{\upsilon\in E_n}\frac{1}{n!}C_\upsilon(X)$$

and C_υ is a function of eigenvalues, i.e., for every $H \in O(n)$ $C_\upsilon(H'XH) = C_\upsilon(X)$. This is a well-known result; see for example [116]. Subsequently, we conclude that

$$(\mathrm{Trace}(X))^n = \sum_{\upsilon \in E_n} C_\upsilon(X).$$

Now since $A_1 > A_k$, we conclude that the following set of inequalities hold:

$$0 \leqslant C_\upsilon(A_k^{\frac{1}{2}} X A_k^{\frac{1}{2}}) \leqslant (\mathrm{Trace}(A_k))^n \leqslant (\mathrm{Trace}(A_1 X))^n.$$

We are now able to apply Lebesgue's dominated convergence theorem, since

$$f(A) = e^{-\mathrm{Trace}((S+I_n)T)} (\det T)^{p-\frac{n+1}{2}} \frac{\mathrm{Trace}(A_1 X)^n}{k!(p)_\upsilon} \frac{1}{\Gamma_n(p)}$$

is an integrable dominant. Letting $k \to \infty$, one obtains $\lim_{k\to\infty} I_\upsilon(A_k) = I_\upsilon(A)$. This establishes that the result holds even for the singular A.

Using another very well-known equality for the zonal polynomials, namely [79, Eq. (10)]

$$\mathrm{etr}(X) = \sum_{m=1}^{\infty} \sum_{\upsilon \in E_m} \frac{C_\upsilon(X)}{m!},$$

we obtain the result from the theorem. □

We now present the 2×2 case for methodological reasons. In the case of 2×2 matrices, the results stated earlier are simpler and allow for easier computation. We present the main results from Section 4 in [79].

The zonal polynomials can be expressed in terms of Legendre polynomials. Denote by $(P_k)_{k \geqslant 0}$ the Legendre polynomials. The most common definition is by using their generating formula.

$$\frac{1}{\sqrt{1 - 2ab - b^2}} = \sum_{k=0}^{\infty} b^k P_k(a).$$

One may also define the Legendre polynomials using the so-called Rodrigues formula (see, e.g. [125]) as:

$$P_k(a) = \frac{1}{k! 2^k} \frac{\mathrm{d}^k}{\mathrm{d}a^k} (a^2 - 1)^k, k = 0, 1, \ldots$$

Let A be an $n \times n$ matrix, whose l-th principal minor is denoted by $M_l(A)$. Assume that $\upsilon = (k_1, k_2, \ldots, k_n)$ is a partition. Then $\Delta_\upsilon(A)$ will denote the following expression:

$$\Delta_\upsilon(A) = (M_1(A))^{k_1 - k_2} \times (M_2(A))^{k_2 - k_3} \times \cdots \times (M_n(A))^{k_n}.$$

Denote by $\mathbb{SO}(n)$ the special orthogonal group (rotation group) of order n, i.e., $\mathbb{SO}(n) = \{A : A \in O(n) \wedge \det(A) = 1\}$. This helps us formulate the following result:

PROPOSITION 1.4.2. *[79, Proposition 4.1] Let k be a non-negative integer and let $\upsilon = (k_1, k_2)$ with $k_1 + k_2 = k$ and $k_1, k_2 \geqslant 0$. Then for a symmetric non-singular matrix T of order 2, one has*

$$\int_{\mathbb{SO}(2)} \Delta_\upsilon(A^{-1} T A) \mathrm{d}A = (\det T)^{\frac{k}{2}} P_{k_1 - k_2}\Big(\frac{\mathrm{Trace}(T)}{2(\det(T))^{\frac{1}{2}}}\Big).$$

In the case of a non-zero singular matrix and a nontrivial partition (i.e., not of a form $(k, 0)$), one has $\int_{\mathbb{SO}(2)} \Delta_\upsilon(A^{-1} T A) \mathrm{d}A = 0$, while in the case $\upsilon = (k, 0)$, the integral takes the form

$$\int_{\mathbb{SO}(2)} \Delta_\upsilon(A^{-1} T A) \mathrm{d}A = \frac{(2k)!}{2^{2k} (k!)^2}.$$

Proof. Assume $\alpha \in [0,\pi]$ represents an angle. Then, a typical element of $\mathbb{SO}(2)$ can be represented as

$$r(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}.$$

Now, assuming $T = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, one may see that $\Delta_v(T) = \lambda_1^{k_1-k_2}(\lambda_1\lambda_2)^{k_2}$, and consequently

$$\int_{\mathbb{SO}(2)} \Delta_v(A^{-1}TA)\,\mathrm{d}A = \frac{1}{2\pi}\int_0^{2\pi} \Delta_v(r(-\alpha)\mathrm{Trace}(-\alpha))\,\mathrm{d}\alpha = (\lambda_1\lambda_2)^{k_2}\frac{1}{2\pi}\int_0^{2\pi}(\lambda_1\cos^2\alpha+\lambda_2\sin^2\alpha)^{k_1-k_2}\,\mathrm{d}\alpha.$$

We compute the generating function:

$$\frac{1}{2\pi}\sum_{l=0}^{\infty} t^l \int_0^{2\pi}(\lambda_1\cos^2\alpha+\lambda_2\sin^2\alpha)^l\,\mathrm{d}\alpha = \frac{1}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}\alpha}{1-t(\lambda_1\cos^2\alpha+\lambda_2\sin^2\alpha)}$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\alpha}{1-t(\lambda_1\cos^2\alpha+\lambda_2\sin^2\alpha)} = \frac{1}{\sqrt{(1-t\lambda_1)(1-t\lambda_2)}}.$$

The last integral is computed using elementary methods (the change of variables $u = \tan\alpha$). Therefore, in the non-singular case, we have established that

$$\frac{1}{\sqrt{(1-t\lambda_1)(1-t\lambda_2)}} = \sum_{l=0}^{\infty} t^l(\lambda_1\lambda_2)^{\frac{l}{2}} P_l\Big(\frac{\lambda_1+\lambda_2}{2\sqrt{\lambda_1\lambda_2}}\Big).$$

This finishes the proof in the non-singular case. In the singular case, the proof is analogous to the one given above, and we omit it here. □

The zonal polynomials in case of 2×2 matrices have tractable expressions. It is well known that [102, p. 237]:

$$C_{(k,0)} = \frac{2^{2k}(m!)^2}{(2m)!},$$

$$C_{(k_1,k_2)} = 2^{2k}k!k_1!\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\dots\cdot\frac{2k_2-1}{2}\cdot\frac{2(k_1-k_2)+1}{(2k_1+1)!(2k_2)!},\ \text{for } k_2\neq 0 \text{ and } k = k_1+k_2.$$

We state the form of the density of a 2×2 noncentral Wishart distribution. The noncentral Wishart density in the case $p > \frac{1}{2}$ and A non-singular is given by:

$$\gamma(p,A)(\mathrm{d}T) = \exp(-\mathrm{Trace}(T+A))(\det T)^{p-\frac{3}{2}}$$
$$\times\Big(\sum_{k=0}^{\infty}\frac{\det(AT)^{\frac{k}{2}}}{k!}\sum_{v\in E_k} C_{(k_1,k_2)}\frac{P_{k_1-k_2}\big(\frac{\mathrm{Trace}(AT)}{2\sqrt{\det(AT)}}\big)}{\Gamma(k_1+p)\Gamma(k_2+p+\frac{1}{2})}\Big)\mathrm{Ind}(T\in\mathscr{P}_2)(\mathrm{d}T).$$

If the matrix A is a zero matrix, then the density does not exist. If the matrix A is singular and non-zero, then the density in the case $p > \frac{1}{2}$ is given by

$$\gamma(p,A)(\mathrm{d}T) = \frac{\exp(-\mathrm{Trace}(T+A))}{\Gamma(p+\frac{1}{2})}(\det(T))^{p-\frac{3}{2}}\sum_{k=0}^{\infty}\frac{\mathrm{Trace}(AT)^k}{k!\Gamma(k+p)}\mathrm{Ind}(T\in\mathscr{P}_2)(\mathrm{d}T).$$

In the non-zero singular case, the matrix A is such that $\mathrm{Rank}(A) = 1$ and admits the representation $A = \lambda k\otimes k$, where $k \in R^2 - \{0\}$ and $\lambda \in R - \{0\}$. Now, we see that $\mathrm{Trace}(AT) = \lambda\mathrm{Trace}(k\otimes kT) = \lambda kTk^T$. The special case when λ is an integer larger than 2 is useful, since the following lemma holds.

Lemma 1.1. *[79, Proposition 4.2] If $X_1, X_2, \ldots, X_\lambda$ are standard normal $\mathcal{N}(0, I_2)$ random vectors and let $k \in \mathbb{R} - \{0\}$ be arbitrary. Then, the symmetric random matrix*

$$T = \frac{1}{2}[(X_1 - k)'(X_1 - k) + (X_2 - k)'(X_2 - k) + \cdots + (X_\lambda - k)'(X_\lambda - k)]$$

has the density of

$$\gamma\Big(\frac{\lambda}{2}, \lambda k \otimes k\Big)(\mathrm{d}T) = \frac{\exp(-\mathrm{Trace}(T) - \lambda||k||^2)}{\Gamma\big(\frac{\lambda+1}{2}\big)}(\det T)^{\frac{\lambda-3}{2}}\Big(\sum_{l=0}^{\infty} \frac{\lambda^l (kTk^T)^l}{l!\Gamma(l + \frac{\lambda}{2})}\Big)\mathrm{Ind}(T \in \mathscr{P}_2)(\mathrm{d}T).$$

It was initially believed that the $k \times k$ noncentral Wishart distribution is defined for the parameter p belonging to the so-called Gindikin set Λ of order k, i.e.,

$$\Lambda = \Big\{\frac{1}{2}, 1, \ldots, \frac{k-1}{2}\Big\} \cup \Big(\frac{k-1}{2}, \infty\Big),$$

where no other conditions should be imposed on the rank of the scale matrix Σ or the noncentrality matrix ω. The rationale behind this assumption, formalized in [79, Proposition 2.3], was that the analogous result holds to the central Wishart case. The condition in the central Wishart case was investigated in papers by Gindikin [47], Shanbhag [120], and Peddada and Richards [111]. This initial belief was proven to be false by Meierhofer [91], in a paper that utilized the techniques of affine Markov spaces. The corrected version was established via the techniques of linear algebra in [80], providing a definite answer to the question of the existence of the noncentral Wishart distribution. Therefore, we will use the results from [80] whenever we refer to the problem of the existence of a noncentral Wishart distribution.

1.5 Hankel transforms of matrix argument

In this section, we aim to introduce the Bessel functions of one and two matrix arguments, as well as the (orthogonally invariant) Hankel transforms.

The following is taken from [54]. Let κ be a vector of p nonnegative integers $\kappa = (k_1, \ldots, k_p)$. We call κ a partition, whose length $l(\kappa)$ is the number of nonzero entries and its weight, denoted by $|\kappa|$, is the sum $k_1 + \ldots + k_p$. Let $z \in \mathbb{C}$ and $r > 0$. Denote the shifted factorial by $(z)_r = z(z+1)(z+2)\ldots(z+r-1)$. If $\kappa = (k_1, k_2, \ldots, k_p)$ is a partition, the partitional shifted factorial can be defined as

$$[z]_\kappa = \prod_{i=1}^{p}\Big(z - \frac{1}{2}(i-1)\Big)_{k_i}.$$

Moreover, denote by $S^{n\times n}$ the space of symmetric $n \times n$ matrices. The definition of zonal polynomials, which are significant for the construction of novel tests, is given below.

Definition 1.3. If κ is an arbitrary partition, we define the zonal polynomial C_κ as:

$$C_\kappa(I_n) = 2^{2|\kappa|}|\kappa!|\Big[\frac{m}{2}\Big]_\kappa \frac{\prod_{r<s}^{l(\kappa)}(2k_r - 2k_s - r + s)}{\prod_{r=1}^{l(\kappa)}(2k_r + l(\kappa) - r)!}.$$

and if $Y \in S^{n\times n}$ is an arbitrary symmetric matrix

$$C_\kappa(Y) = C_\kappa(I_n)(\det Y)^{k_m} \int_{O(n)} \prod_{i=1}^{n-1} (M_i(HYH^{-1}))^{k_i - k_{i+1}} \, \mathrm{d}\mu(H),$$

where $\mathrm{d}\mu(H)$ denotes the normalized Haar measure (1.7) and $M_i(X)$ denotes the i-th principal minor of the matrix X.

From the orthogonal invariance of the measure μ, it follows directly that C_κ is an orthogonally invariant function, namely $C_\kappa(HYH') = C_\kappa(Y)$ for all $H \in O(n)$. Moreover, $C_\kappa(Y)$ is a function of eigenvalues of Y. If $A \in S^{n\times n}$ and $B^{\frac{1}{2}}$ denotes the positive definite square root of a positive definite matrix $B \in \mathscr{P}_+^{n\times n}$, then the matrices $B^{\frac{1}{2}}AB^{\frac{1}{2}}$, AB and BA have the same eigenvalues, and consequently $C_\kappa(B^{\frac{1}{2}}AB^{\frac{1}{2}}) = C_\kappa(AB) = C_\kappa(BA)$. This holds for any orthogonally invariant function.

Let $Y > 0$ and $X \in S^{n\times n}$. The following mean value property is established in [102, p. 243]:

$$\int_{O(n)} C_\kappa(HYH'X)\mathrm{d}H = \frac{C_\kappa(Y)C_\kappa(X)}{C_\kappa(I_n)}. \tag{1.8}$$

The properties of zonal polynomials are studied in [52, 54, 67, 102].

We are ready to introduce the Bessel polynomials of one matrix argument. This topic was first entertained in [60] and there exist several approaches; however, we conform to that outlined in [102, Chapter 7] and the subsequent work of Hadjicosta and Richards [54].

Let ν be a complex number such that $-\nu + \frac{1}{2}(i-n) \notin \mathbb{N}$ for all $i = 1, 2, \ldots, n$. This condition is necessary to ensure $[\nu + \frac{1}{2}(n+1)]_\kappa \neq 0$ for any partition κ.

Definition 1.4. Let $Y \in S^{n\times n}$ and $\nu \in \mathbb{C}$ be defined as above. The Bessel function of the first kind of order ν is defined as

$$A_\nu(Y) = \frac{1}{\Gamma_n(\nu + \frac{1}{2}(n+1))}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{|\kappa|=r}\frac{C_\kappa(Y)}{[\nu + \frac{1}{2}(n+1)]_\kappa}. \tag{1.9}$$

Let $X \in \mathscr{M}_{n\times n}$ and let $A = \{X'X \in \mathscr{M}_{n\times n} : X'X < I_n\}$, where with $X < Y$ we denoted $Y - X \in \mathscr{P}_+^{n\times n}$. Herz [60] established the following generalization of the Poisson integral [60, Equation (3.6')]:

$$A_\nu(X'X) = \frac{1}{\pi^{n^2/2}\Gamma_n(\nu + \frac{1}{2})}\int_A \operatorname{etr}(2\mathrm{i}X'Q)(\det(I_n - X'X))^{\nu - \frac{1}{2}n}(\mathrm{d}X),\ \Re(\nu) > \frac{1}{2}(n-2).$$

The integral is with respect to the Lebesgue measure on A [54, Equation 12]. This can be used to prove the following inequality:

Lemma 1.2. *[52, Lemma 6.2.1] For $\Re(\nu) > \frac{1}{2}(n-2)$ and any $n\times n$ matrix X, the following inequality holds:*

$$|A_\nu(X'X)| \leqslant \frac{1}{\Gamma_n(\nu + \frac{1}{2}(n+1))}.$$

Proof. Since $|\operatorname{etr}(2\mathrm{i}X'Q)| = 1$, we have that

$$|A_\nu(X'X)| \leqslant \frac{1}{\pi^{n^2/2}\Gamma_n(\nu + \frac{1}{2})}\int_A (\det(I_n - X'X))^{\nu - \frac{1}{2}n}(\mathrm{d}X) = A_\nu(0) = \frac{1}{\Gamma_n(\nu + \frac{1}{2}(n+1))}. \tag{1.10}$$

□

Let us introduce the Bessel function of two matrix arguments.

Definition 1.5. Let ν be a complex number such that $-\nu + \frac{1}{2}(i-n) \notin \mathbb{N}$ for all $i = 1, 2, \ldots, n$. Let $X, Y \in S^{n\times n}$. The Bessel function of the first kind of order ν of two matrix arguments is defined as

$$A_\nu(X, Y) = \frac{1}{\Gamma_n(\nu + \frac{1}{2}(n+1))}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{|\kappa|=r}\frac{C_\kappa(X)C_\kappa(Y)}{[\nu + \frac{1}{2}(n+1)]_\kappa C_\kappa(I_n)}. \tag{1.11}$$

From (1.9) and (1.8), we can establish that [102, p. 260]:

$$A_\nu(X,Y) = \int_{O(n)} A_\nu(HXH'Y)\mathrm{d}H,\ X>0, Y\in S^{n\times n}. \tag{1.12}$$

and using (1.12) and (1.10), we can see that [54, Equation 26]:

$$|A_\nu(X,Y)| \leqslant \frac{1}{\Gamma_n(\nu+\frac{1}{2}(n+1))}. \tag{1.13}$$

Now we are ready to introduce the Hankel transform of the matrix argument.

Definition 1.6. [54, p. 1329] Let $X>0$ be a symmetric random matrix with probability density function $f(X)$. For $\Re(\nu) > \frac{1}{2}(m-2)$ the Hankel transform of order ν is defined as the function

$$\mathscr{H}_{X,\nu}(T) = E_X\Big[\Gamma_m\Big(\nu+\frac{1}{2}(m+1)\Big)A_\nu(TX)\Big],$$

where $T>0$ is a symmetric matrix, Γ_m denotes the multivariate Gamma function and $A_\nu(T)$ denotes the Bessel function of the first kind of order ν.

It is important to note that the Bessel function in the definition above has only one argument.

The Hankel transform has many attractive properties [52, Lemma 6.3.1]. It is a continuous function of T, because $A_\nu(TX)$ is a continuous function of T for every fixed $X>0$. Let g be an arbitrary Lebesgue integrable function. Function $\Gamma_n(\nu+\frac{1}{2}(n+1))A_\nu(TX)g(X)$ is bounded by the Lebesgue integrable function $g(X)$ for all $X,T>0$. This follows from (1.10). Therefore, the continuity of the Hankel transform follows from the Dominated Convergence theorem.

Its norm is bounded from above by 1. This follows from (1.10) and the triangle inequality [52, Lemma 6.3.1].

There exists an inversion formula as well:

Theorem 1.7. *[52, Theorem 6.3.5] Let $X>0$ be an $n\times n$ random matrix with Hankel transform $\mathscr{H}_{X,\nu}$, and probability density function $f\in L^2_\nu$. Then,*

$$f(X) = \frac{1}{\Gamma_n(\nu+\frac{1}{2}(n+1))}\int_{T>0} A_\nu(TX)(\det(TX))^\nu \mathscr{H}_{X,\nu}(T)\mathrm{d}T.$$

Denote by $X_k \xrightarrow{D} X$ the weak convergence of X_k to X when $k\to\infty$. The Hankel transform is continuous with respect to X. The following theorem establishes that fact:

Theorem 1.8. *[52, Theorem 6.3.6] Let $\{X_k, k\in\mathbb{N}\}$ be a sequence of $n\times n$ positive definite random matrices whose Hankel transforms are denoted by $\{\mathscr{H}_k, k\in\mathbb{N}\}$. If there exists a positive $n\times n$ semidefinite real random matrix X with Hankel transform $\mathscr{H}_X$ such that $X_k\xrightarrow{D} X$, then for each $T>0$*

$$\lim_{k\to\infty}\mathscr{H}_k(T) = \mathscr{H}_X(T). \tag{1.14}$$

Conversely, if there exists a function $\mathscr{H}:\mathscr{P}_+^{n\times n}\to\mathbb{R}$ such that $\mathscr{H}(0)=1$, it is continuous at 0 and (1.14) holds, then $\mathscr{H}$ is the Hankel transform of an $n\times n$ positive definite random matrix X, and $X_k\xrightarrow{D} X$.

Denote by $W_d(a,\Sigma)$ a $d \times d$ random matrix that has the Wishart distribution with the shape parameter a and the scale matrix Σ. It is a random variable with a density given by (1.5).

The Hankel transform uniquely determines the distribution of the random variable X, i.e., the following theorem holds.

THEOREM 1.9. *[54, p. 1330] Let X and Y be $m \times m$ positive definite random matrices with Hankel transforms $\mathscr{H}_{X,\nu}$ and $\mathscr{H}_{Y,\nu}$ respectively. If $\mathscr{H}_{X,\nu} = \mathscr{H}_{Y,\nu}$, then $X \stackrel{D}{=} Y$.*

PROOF. The Hankel transform $\mathscr{H}_{Z,\nu}$ of a random variable $Z \in W_m(a,\Sigma)$ is equal to $\mathrm{etr}(-T\Sigma^{-1})$ when $\nu = a - \frac{1}{2}(m+1)$. Furthermore, if $X > 0$ is an $m \times m$ random matrix with an arbitrary distribution that is independent of a random variable $Z \in W_m(a, I_m)$, then the following relation holds for every $T > 0$:

$$E_Z(\mathscr{H}_{X,\nu}(T^{\frac{1}{2}} Z T^{\frac{1}{2}})) = E_X(\mathscr{H}_{Z,\nu}(T^{\frac{1}{2}} X T^{\frac{1}{2}})). \tag{1.15}$$

Taking a distribution $Z_1 \in W_m(a, I_m)$ independent of X, where $a = \nu + \frac{1}{2}(m+1)$, the relation (1.15) becomes

$$E_{Z_1}(\mathscr{H}_{X,\nu}(T^{\frac{1}{2}} Z_1 T^{\frac{1}{2}})) = E_X(\mathscr{H}_{Z_1,\nu}(T^{\frac{1}{2}} X T^{\frac{1}{2}})) = E_X(\mathrm{etr}(-TX)) = \mathscr{L}_X(T). \tag{1.16}$$

The expression on the right side is a Laplace transform of a random variable X. The equality $E_X(\mathscr{H}_{Z_1,\nu}(T^{\frac{1}{2}} X T^{\frac{1}{2}})) = E_X(\mathrm{etr}(-TX))$ is justified by the use of the Kummer formula [60]. We extend the uniqueness of the Laplace transform to the uniqueness of the Hankel transform.

Assume $\mathscr{H}_{X,\nu} = \mathscr{H}_{Y,\nu}$. Furthermore, assume that $Z_2 \in W_m(a, I_m)$ is independent of X and Y and assume $a = \nu + \frac{1}{2}(m+1)$. Then we have, by applying the relation (1.16) twice:

$$\mathscr{L}_X(T) = E_{Z_2}(\mathscr{H}_{X,\nu}(T^{\frac{1}{2}} Z_2 T^{\frac{1}{2}})) = E_{Z_2}(\mathscr{H}_{Y,\nu}(T^{\frac{1}{2}} Z_2 T^{\frac{1}{2}})) = \mathscr{L}_Y(T).$$

By establishing the equality of the Laplace transforms of X and Y for every $T > 0$, and applying [37, p. 16], the result of the theorem follows. □

However, the Hankel transform is not an orthogonally invariant function. The orthogonally invariant Hankel transform can be defined as

DEFINITION 1.7. [54, p. 1329] Let $X > 0$ be a random matrix with probability density function $f(X)$. For $\Re(\nu) > \frac{1}{2}(m-2)$ we define the orthogonally invariant Hankel transform of order ν as the function

$$\tilde{\mathscr{H}}_{X,\nu}(T) = E_X\Big[\Gamma_m\Big(\nu + \frac{1}{2}(m+1)\Big) A_\nu(T,X)\Big],$$

where $T > 0$, $A_\nu(T,X)$ denotes the Bessel function of the first kind of order ν with two matrix arguments.

One may readily see that from (1.12) and Definition 1.7, the following relation holds [52, Remark 6.4.2]:

$$\tilde{\mathscr{H}}_{X,\nu}(T) = \int_{O(m)} \mathscr{H}_{X,\nu}(T) = (HTH')\mathrm{d}H, \tag{1.17}$$

from which we conclude that the properties stated in [52, Lemma 6.3.1] hold for the orthogonally invariant Hankel transforms as well. Moreover, the uniqueness theorem can be established analogously:

THEOREM 1.10. *[52, Theorem 6.4.4] Let $\tilde{X}$ and $\tilde{Y}$ be $m \times m$ positive definite random matrices with orthogonally invariant distributions and corresponding orthogonally invariant Hankel transforms $\tilde{\mathscr{H}}_{\tilde{X},\nu}$ and $\tilde{\mathscr{H}}_{\tilde{Y},\nu}$ respectively. If $\tilde{\mathscr{H}}_{\tilde{X},\nu} = \tilde{\mathscr{H}}_{\tilde{Y},\nu}$, then $\tilde{X} \stackrel{D}{=} \tilde{Y}$.*

The empirical orthogonally invariant Hankel transform of $X_1, X_2, \ldots, X_n$ of order ν is given by

$$\tilde{\mathscr{H}}_{n,\nu}(T) = \frac{\Gamma_m\left(\nu + \frac{1}{2}(m+1)\right)}{n_1} \sum_{j=1}^{n} A_\nu(T, X_j). \tag{1.18}$$

It is a consistent estimator of the orthogonally invariant Hankel transform. Since the orthogonally invariant Hankel transform uniquely determines the distribution, it forms the foundation of statistical tests in the space of symmetric positive definite matrix distributions. The idea behind the construction of these statistical tests can be traced back to the univariate case.

1.6 Matrix-variate statistical tests

In this section, we present contemporary matrix-variate statistical tests, which motivate subsequent chapters of this dissertation.

In [4], Alfelt et al. considered the GOF tests for the Wishart process. The family of random matrices $(A_t)_{t=1}^T$ with the accompanying filtration of σ-algebras $(\mathscr{F}_t)_{t=1}^T$ is said to be a Wishart process if $A_t|\mathscr{F}_t \in W_n(a, \Sigma_t)$, where the dimension n and the scale parameter a are fixed, while the covariance matrix Σ_t varies over the index set. They reduced the problem of testing the GOF of the Wishart process to performing the GOF tests of normality on the residual matrix. The second part of the paper tests for autocovariance, but that is out of scope for this dissertation.

The Wishart process, as a time-dependent object is of interest in finance, since the Wishart models are commonly used in portfolio modeling. For some singular autoregressive Wishart models, refer to [3].

We now present the GOF test for the Wishart distribution, i.e., the case of independent equally distributed random variables, as discussed in [54], and present the calculations that will be important in the following part of the text.

Hadjicosta and Richards proposed the following test statistic:

$$T_n^2 = n \int\limits_{T>0} \left(\tilde{\mathscr{H}}_{n,\nu} - \operatorname{etr}(-T/\alpha)\right)^2 \mathrm{d}P_0(T), \tag{1.19}$$

where $\tilde{\mathscr{H}}_{n,\nu}$ is given by (1.18), $\mathrm{d}P_0$ corresponds to the Wishart $W_m(\alpha, I_m)$ measure (1.5). $\operatorname{etr}(-T/\alpha)$ is the almost sure limit of $\tilde{\mathscr{H}}_{n,\nu}$ when the sample comes from the Wishart distribution [52, p. 161 - 162]. It is important to note they assume α is known.

They set $\nu = \alpha - \frac{1}{2}(m+1)$ and explicitly calculate the value of the test statistic.

The authors also derived the asymptotic properties of the proposed test. The space they considered is the space $L^2(\mathrm{d}W)$ of all Borel-measurable functions $f : \mathscr{P}_+ \to \mathbb{C}$ such that $\int\limits_{T>0} |f(T)|^2 \mathrm{d}W(T)$, where $\mathrm{d}W(T)$ denotes a standard Wishart measure. This space is a separable Hilbert space, when equipped with an inner product

$$\langle f, g\rangle = \int\limits_{T>0} f(T)\overline{g(T)} \mathrm{d}W(T).$$

The norm is of the form $\|f\| = \sqrt{\langle f, g\rangle}$.

For more results on the test by Hadjicosta and Richards, we refer the reader to [52] and [54]. This work served as an inspiration in deriving the two-sample test, which we discuss in the following section.

Chapter 2

Two-sample tests of equivalence for matrix distributions

In this chapter, we present two novel two-sample tests of equivalence in the space of positive semidefinite matrix distributions. In the first part, we introduce the notion of orthogonal invariance in distribution (see [84]), as it is important for the construction of one of the tests. Following that, we present two first-of-their-kind tests.

The test outlined in [84] is the first-of-its-kind test of orthogonal invariance in distribution in the space of symmetric positive definite matrix distributions (see Section 1.6). The choice to employ orthogonally invariant Hankel transforms in the construction of the test had dual motivation. The theoretical structure was already established in [54], and the algorithmic strategies detailed in [72] facilitated the numerical computation of the test statistic. In practical scenarios, data are typically subjected to orthogonal transformations through PCA or by eliminating the reliance on the scale parameter. The use of orthogonal transformations is also prevalent in the field of finance [119, 141].

The test in [86] fills an important gap in the literature, as it is the first two-sample test of equivalence in distribution within the class of arbitrary (i.e., not necessarily orthogonally invariant) matrix distributions. The test statistic is presented in Section 2.2.1. The test statistic itself is not orthogonally invariant, hence the favorable ability of the test to distinguish between distributions belonging to the same family of distributions.

We proceed with the large-sample behavior of the novel tests. Following that, we present the empirical test powers of the tests and conclude the section with real data applications in finance and insurance. All results of this section can be found in [84, 86].

2.1 Notion of orthogonal invariance in distribution

The concept of orthogonal invariance in distribution (OID) is introduced as follows:

Definition 2.1. We call two matrix distributions X and Y *orthogonally invariant in distribution* if there exists an orthogonal matrix P such that $X \overset{D}{=} P'YP$.

The following result is fundamental in the construction of the test. It extends the uniqueness of the orthogonal Hankel transforms to the class of distributions orthogonally invariant in distribution, compared to Theorem 1.10, which asserts the uniqueness in the case of orthogonally invariant distributions.

We use the following notation throughout the chapter, for reasons of brevity:

$$J_\nu(T) = \Gamma\big(\nu + \tfrac{1}{2}(m+1)\big) A_\nu(T)$$

and

$$J_\nu(T,X)=\Gamma\big(\nu+\tfrac{1}{2}(m+1)\big)A_\nu(T,X).$$

THEOREM 2.1. *[84, Theorem 2] Let X and Y be $m\times m$ random symmetric positive definite matrices and $\mathscr{H}_{X,\nu}$ and $\mathscr{H}_{Y,\nu}$ their orthogonally invariant Hankel transforms respectively. Then $\mathscr{H}_{X,\nu}=\mathscr{H}_{Y,\nu}$ if and only if X and Y are orthogonally invariant in distribution.*

PROOF. From (1.12), we have that for every positive definite matrix X and every symmetric matrix T, the following equality holds:

$$A_\nu(T,X)=\int\limits_{O(m)} A_\nu(HTH'X)\mathrm{d}H=\int\limits_{O(m)} A_\nu(H'HTH'XH)\mathrm{d}H=\int\limits_{O(m)} A_\nu(TH'XH)\mathrm{d}H. \qquad (2.1)$$

The middle equality emerges from the fact that matrices X and $H'XH$ have the same eigenvalues if H is an orthogonal matrix and A_ν is a function of eigenvalues.

Assume now that $P'YP=X$. We obtain

$$A_\nu(T,X)=\int\limits_{O(m)} A_\nu(TH'XH)\mathrm{d}H=\int\limits_{O(m)} A_\nu(TH'P'YPH)\mathrm{d}H=\int\limits_{O(m)} A_\nu(T(PH)'Y(PH))\mathrm{d}H.$$

Note that the Haar measure is orthogonally invariant, therefore if we denote by H_1 the matrix PH, we obtain

$$\begin{aligned}A_\nu(T,X)&=\int\limits_{O(m)} A_\nu(T(PH)'Y(PH))\mathrm{d}H=\int\limits_{O(m)} A_\nu(TH_1'YH_1)\mathrm{d}H_1\\&=\int\limits_{O(m)} A_\nu(HTH'Y)\mathrm{d}H=A_\nu(T,Y).\end{aligned}$$

Therefore, directly from the definition of the orthogonal Hankel transform and using the fact that the differential form is orthogonally invariant, we get

$$\tilde{\mathscr{H}}_{X,\nu}(T)=E_X\big(J_\nu(T,X)\big)=E_Y\big(J_\nu(T,P'YP)\big)=E_Y\big(J_\nu(T,Y)\big)=\tilde{\mathscr{H}}_{Y,\nu}(T).$$

Let us assume the equality of orthogonal Hankel transforms. Using (1.17), and the definition of the orthogonally invariant Hankel transform, we have that

$$\tilde{\mathscr{H}}_{X,\nu}(T)=E_XE_H\big(J_\nu(HTH'X)\big).$$

Now we use the fact that $A_\nu(\cdot)$ depends only on the eigenvalues of its arguments to obtain

$$\tilde{\mathscr{H}}_{X,\nu}(T)=E_XE_H\big(J_\nu(THXH')\big),$$

but since $THXH'$ and TZ_1 for some Z_1 orthogonally invariant to X are symmetric matrices, which have the same eigenvalues, and the differential form is orthogonally invariant, we have:

$$\tilde{\mathscr{H}}_{X,\nu}(T)=E_{Z_1}\big(J_\nu(TZ_1)\big)=\mathscr{H}_{Z_1,\nu}(T).$$

We obtain a similar result for Y, i.e., $\tilde{\mathscr{H}}_{Y,\nu}(T)=\mathscr{H}_{Z_2,\nu}(T)$, and the equality of distributions Z_1 and Z_2 follows from Theorem 1.9. Now, since $Z_1=P_1'XP_1$ and $Z_2=P_2'YP_2$, where $P_1,P_2\in O(m)$, we get that X is orthogonally invariant to Y. □

2.2 Test statistics

We present the construction of the test statistic from [86], followed by the construction of the test statistic from [84].

2.2.1 Laplace transform case

In this subsection, we present our two-sample test statistic based on the Laplace transform. Based on the samples X and Y, we want to test the null hypothesis

$$H_0 : X \text{ and } Y \text{ are equally distributed,}$$
$$\text{against the alternative}$$
$$H_1 : X \text{ and } Y \text{ are not equally distributed.}$$

Since the Laplace transform uniquely determines the distribution of the random matrices [37, Theorem 2.1.9], the null hypothesis can be stated in the terms of equality of the Laplace transforms as:

$$H_0 : \mathscr{L}_X(T) = \mathscr{L}_Y(T), \text{ for all } T > 0,$$

where $\mathscr{L}_X(T) = E\exp(-\text{Trace}(TX))$ denotes the Laplace transform of the random variable X. Note that the argument of the Laplace transform is a matrix T. A test statistic is constructed using the difference of the empirical Laplace transforms. Let $\hat{L}_{n_1}(T)$ represent the empirical Laplace transform of X, defined as:

$$\hat{L}_{n_1}(T) = \frac{1}{n_1}\sum_{k=1}^{n_1}\exp(-\text{Trace}(TX_k)).$$

Similarly, let $\hat{L}_{n_2}(T)$ represent the empirical Laplace transform of Y, defined as:

$$\hat{L}_{n_2}(T) = \frac{1}{n_2}\sum_{k=1}^{n_2}\exp(-\text{Trace}(TY_k)).$$

The test statistic is of the form [86, Equation 2]:

$$\begin{aligned} L_{n_1,n_2,\nu,\Sigma,\omega} &= \int_{T>0} (\hat{L}_{n_1}(T) - \hat{L}_{n_2}(T))^2 \, \mathrm{d}NCW(T) \\ &= \int_{T>0} \Big(\frac{1}{n_1}\sum_{k=1}^{n_1}\exp(-tr(TX_k)) - \frac{1}{n_2}\sum_{k=1}^{n_2}\exp(-tr(TY_k))\Big)^2 \mathrm{d}NCW(T), \end{aligned} \tag{2.2}$$

where $\mathrm{d}NCW(T)$ is a corresponding noncentral Wishart (ν, Σ, ω) measure.

The noncentral Wishart $d \times d$ distribution $NCW(2\nu, \Sigma, \omega)$, where $\nu > 0$ and is not necessarily an integer, exists if $\nu \geqslant \frac{d-1}{2}$ or if $\nu \in \{0.5, 1, 1.5, \ldots, (d-2)/2\}$ and $\text{Rank}(\omega) \geqslant 2\nu$ [80]. For the explicit form of the density function, we refer the reader to Section 1.4.

The Laplace transform of the $NCW(2\nu, \Sigma, \omega)$ distribution is given by [80, Equation (2)]:

$$\mathscr{L}_X(s) = E(\exp(-\text{Trace}(sX))) = \frac{\exp(-\text{Trace}(2s(I_d + 2\Sigma s)^{-1}\omega))}{\det(I_d + 2\Sigma s)^{\nu}}. \tag{2.3}$$

The direct computation, taking into account the basic identity $\text{Trace}(AB) = \text{Trace}(BA)$, yields

$$L_{n_1,n_2,\nu,\omega} = \frac{1}{n_1^2 n_2^2} \sum_{i=1,j=1}^{n_1} \sum_{l=1,k=1}^{n_2} \Psi(X_i, X_j; Y_k, Y_l),$$

where $\Psi(X_i, X_j; Y_k, Y_l)$ is a symmetric kernel of the form:

$$\Psi(X_i, X_j; Y_k, Y_l) = \Big(\mathscr{L}(X_i + X_j) + \mathscr{L}(Y_k + Y_l) - \frac{1}{2}\mathscr{L}(X_i + Y_k) - \frac{1}{2}\mathscr{L}(X_i + Y_l) - \frac{1}{2}\mathscr{L}(X_j + Y_k) - \frac{1}{2}\mathscr{L}(X_j + Y_l)\Big),$$

where the function $\mathscr{L}(\cdot)$ is the Laplace transform of the $NCW(\nu, \Sigma, \omega)$ distribution, and is of the form given in equation (2.3).

2.2.2 Hankel transform case

We now present the computation of the test statistic from [84], which is based on empirical Hankel transforms, but follows the same idea presented above.

Let $X = X_1, X_2, \ldots, X_{n_1}$ and $Y = Y_1, Y_2, \ldots, Y_{n_2}$ be two independent random samples identically distributed as X and Y, where X and Y are symmetric positive definite random matrices, respectively. We wish to test the null hypothesis

$H_0 : X$ and Y are orthogonally invariant in distribution,
against the alternative
$H_0 : X$ and Y are not orthogonally invariant in distribution.

Following Theorem 2.1, the null hypothesis can be stated as:

$$H_0 : \tilde{\mathscr{H}}_{X,\nu}(T) = \tilde{\mathscr{H}}_{Y,\nu}(T), \text{ for all } T > 0.$$

Given that the concept of orthogonal invariance in distribution is characterized by the equivalence of corresponding orthogonal Hankel transforms, a logical approach to devising a test is to base it on the disparity between suitable empirical equivalents, specifically, empirical orthogonal Hankel transforms (1.18). The indices n_1 and n_2 will be associated with the samples X and Y, respectively.

The test statistic is constructed as follows [84, Equation 4]:

$$I_{n_1,n_2,\nu} = \int_{T>0} \Big(\tilde{\mathscr{H}}_{n_1,\nu}(T) - \tilde{\mathscr{H}}_{n_2,\nu}(T)\Big)^2 \, \mathrm{d}W(T), \tag{2.4}$$

where $\tilde{\mathscr{H}}_{n_1,\nu}(T)$ and $\tilde{\mathscr{H}}_{n_2,\nu}(T)$ are empirical orthogonal Hankel transform of X and Y, respectively, defined in (1.18) and $\mathrm{d}W(T)$ is a standard Wishart measure.

Let us compute the test statistic by following the steps outlined in [52, p. 163]:

$$\int_{T>0} (\tilde{\mathscr{H}}_{n_1,\nu}(T))^2 \, \mathrm{d}W(T) = \frac{1}{n^2} \int_{T>0} \left(\sum_{i=1}^{n} \Gamma_m(\alpha) A_\nu(T, Y_i) \right)^2 \mathrm{d}W(T)$$
$$= \frac{\Gamma_m(\alpha)}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{T>0} A_\nu(T, Y_i) A_\nu(T, Y_j) (\det T)^\nu \operatorname{etr}(-T) \, \mathrm{d}T.$$

Using Fubini's theorem and (1.12), we get that

$$\int\limits_{T>0} (\tilde{\mathscr{H}}_{n_1,\nu}(T))^2 \,\mathrm{d}W(T)$$
$$= \frac{\Gamma_m(\alpha)}{n^2} \sum_{i=1}^{n}\sum_{j=1}^{n} \int\limits_{O(m)} \int\limits_{O(m)} \left(\int\limits_{T>0} A_\nu(HTH'Y_i) A_\nu(KTK'Y_j) (\det T)^\nu \operatorname{etr}(-T)\,\mathrm{d}T \right) \mathrm{d}H\,\mathrm{d}K.$$

Having

$$A_\nu(HTH'Y_i) = A_\nu(H'Y_iHT) \;\; \text{and} \;\; A_\nu(KTK'Y_j) = A_\nu(K'Y_jKT),$$

the inside integral (with respect to $\mathrm{d}T$) is a special case of Weber's second exponential integral. This was obtained in [60, Equation (5.8)].

For $\Re(\nu) > -1$, $n \times n$ symmetric matrices X and Y, and $Z > 0$,

$$\int\limits_{T>0} \operatorname{etr}(-TZ) A_\nu(XT) A_\nu(YT) (\det T)^\nu \,\mathrm{d}T = (\det Z)^{-\nu-\frac{1}{2}(n+1)} \operatorname{etr}(-(X+Y)Z^{-1}) A_\nu(-XZ^{-1}YZ^{-1}).$$

Applying Weber's second exponential integral, we obtain

$$\int\limits_{T>0} (\tilde{\mathscr{H}}_{n_1,\nu}(T))^2 \,\mathrm{d}W(T) = \frac{\Gamma_m(\alpha)}{n^2} \sum_{i=1}^{n}\sum_{j=1}^{n} \int\limits_{O(m)} \int\limits_{O(m)} \operatorname{etr}(-H'Y_iH - K'Y_jK) A_\nu(-H'Y_iHKY_jK)\,\mathrm{d}H\,\mathrm{d}K.$$

Using the fact that

$$\operatorname{etr}(-H'Y_iH) = \operatorname{etr}(-Y_iHH') = \operatorname{etr}(-Y_i),$$

and (1.12) gives us:

$$\int\limits_{O(m)} \int\limits_{O(m)} A_\nu(-H'Y_iHKY_jK)\,\mathrm{d}H\,\mathrm{d}K = \int\limits_{O(m)} \int\limits_{O(m)} A_\nu(-H(K'Y_jK)H'Y_i)\,\mathrm{d}H\,\mathrm{d}K$$
$$= \int\limits_{O(m)} A_\nu(-K'Y_jK', Y_i)\,\mathrm{d}K = A_\nu(-Y_j, Y_i) \int\limits_{O(m)} \mathrm{d}K = A_\nu(-Y_j, Y_i).$$

We have used the fact that $A_\nu(-K'Y_jK, Y_i) = A_\nu(-Y_j, Y_i)$ in the last row.

Combining the results, we get

$$\int\limits_{T>0} (\tilde{\mathscr{H}}_{n_1,\nu}(T))^2 \,\mathrm{d}W(T) = \frac{\Gamma_m(\alpha)}{n^2} \sum_{i=1}^{n}\sum_{j=1}^{n} \int\limits_{O(m)} \int\limits_{O(m)} \operatorname{etr}(-H'Y_iH - K'Y_jK) A_\nu(-H'Y_iHKY_jK)\,\mathrm{d}H\,\mathrm{d}K$$
$$= \operatorname{etr}(-Y_i - Y_j) \frac{\Gamma_m(\alpha)}{n^2} \sum_{i=1}^{n}\sum_{j=1}^{n} \int\limits_{O(m)} \int\limits_{O(m)} A_\nu(-H'Y_iHKY_jK)\,\mathrm{d}H\,\mathrm{d}K$$
$$= \frac{\Gamma_m(\alpha)}{n^2} \sum_{i=1}^{n}\sum_{j=1}^{n} \operatorname{etr}(-Y_i - Y_j) A_\nu(-Y_j, Y_i). \tag{2.5}$$

Applying (2.5) twice, we establish

$$\int\limits_{T>0} (\tilde{\mathscr{H}}_{n_1,\nu}(T))^2 \,\mathrm{d}W(T) = \frac{1}{n_1^2}\sum_{i=1}^{n_1}\sum_{j=1}^{n_1} \mathrm{etr}(-X_i - X_j) J_\nu(-X_i, X_j), \tag{2.6}$$

$$\int\limits_{T>0} (\tilde{\mathscr{H}}_{n_2,\nu}(T))^2 \,\mathrm{d}W(T) = \frac{1}{n_2^2}\sum_{i=1}^{n_2}\sum_{j=1}^{n_2} \mathrm{etr}(-Y_i - Y_j) J_\nu(-Y_i, Y_j). \tag{2.7}$$

Following exactly the same steps as in the derivation of (2.5), we get

$$\int\limits_{T>0} \tilde{\mathscr{H}}_{n_1,\nu}(T)\tilde{\mathscr{H}}_{n_2,\nu}(T)\,\mathrm{d}W(T) =$$

$$\frac{\Gamma_m(\nu + \frac{1}{2}(m+1))}{n_1 n_2}\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\int\limits_{T>0} A_\nu(T, X_i) A_\nu(T, Y_j)(\det T)^\nu \mathrm{etr}(-T)\,\mathrm{d}T =$$

$$\frac{1}{n_1 n_2}\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} \mathrm{etr}(-X_i - Y_j) J_\nu(-X_i, Y_j),$$

and similarly

$$\int\limits_{T>0} \tilde{\mathscr{H}}_{n_2,\nu}(T)\tilde{\mathscr{H}}_{n_1,\nu}(T)\,\mathrm{d}W(T) = \frac{1}{n_1 n_2}\sum_{i=1}^{n_2}\sum_{j=1}^{n_1} \mathrm{etr}(-Y_i - X_j) J_\nu(-Y_i, X_j).$$

Finally, we obtain that (2.4) has the following form:

$$I_{n_1,n_2} = \frac{1}{n_1^2 n_2^2}\sum_{i=1,j=1}^{n_1}\sum_{l=1,k=1}^{n_2} \Phi_\nu(X_i, X_j; Y_k, Y_l),$$

where $\Phi_\nu(X_i, X_j; Y_k, Y_l)$ is of the form

$$\begin{aligned}\Phi_\nu(X_i, X_j; Y_k, Y_l) = &\ \mathrm{etr}(-X_i - X_j) J_\nu(-X_i, X_j) + \mathrm{etr}(-Y_k - Y_l) J_\nu(-Y_k, Y_l)\\ &- \mathrm{etr}(-Y_k - X_i) J_\nu(-Y_k, X_i) - \mathrm{etr}(-X_i - Y_k) J_\nu(-X_i, Y_k).\end{aligned}$$

The computation of the test statistic is demanding due to the intricate nature of the functions that need to be computed. This complexity escalates with an increase in the dimensionality of the problem. This is a common challenge in high-dimensional data analysis.

2.3 Large sample properties of the novel tests

In this section, we present the theoretical result that determines the asymptotic behavior of the test statistic (2.4), as well as a numerical study of the behavior of the test statistic (2.2).

The space $L^2 = L^2(W)$ of orthogonally invariant Borel measurable functions $f : \mathscr{P}_+^{m\times m} \to \mathbb{C}$ such that $\int\limits_{X>0} |f(X)|^2 \,\mathrm{d}W(X) < \infty$ forms a separable Hilbert space when equipped with the inner product

$$\langle f, g\rangle = \int\limits_{X>0} f(X)\overline{g(X)}\,\mathrm{d}W(X).$$

The norm in this space is defined as $\|f\| = \sqrt{\langle f, f\rangle}$. These facts are established in [54] and are instrumental in establishing the asymptotic behavior of the test statistic (2.4). We now state the main theoretical result of this section.

THEOREM 2.2. *[84, Theorem 3] Let $X_1, X_2, \ldots, X_{n_1}$ and $Y_1, Y_2, \ldots, Y_{n_2}$ be two sequences of independent orthogonally invariant random matrices having the same orthogonal Hankel transform $\tilde{\mathscr{H}}_\nu(T)$. Assume $N = n_1 + n_2$ and $\frac{n_1}{N} \to \eta \in (0,1)$ when $n_1, n_2 \to \infty$. Then*

$$\frac{n_1 n_2}{N} I_{n_1,n_2} \xrightarrow{D} \|\mathscr{Z}\|^2,$$

where $\{\mathscr{Z}(T), T > 0\}$ is a centred Gaussian process on L^2 with a covariance kernel

$$\rho(S,T) = E[J_\nu(S,X)J_\nu(T,X)] - \tilde{\mathscr{H}}_\nu(S)\tilde{\mathscr{H}}_\nu(T).$$

PROOF. The proof will follow the one outlined in [2]. Assume T is a symmetric positive definite $m \times m$ matrix. Let us define the stochastic process

$$\mathscr{Z}_{n_1,n_2,\nu}(T) = \frac{1}{n_1}\sum_{j=1}^{n_1} J_\nu(T, X_j) - \frac{1}{n_2}\sum_{j=1}^{n_2} J_\nu(T, Y_j).$$

Note that although the process $\mathscr{Z}_{n_1,n_2,\nu}$ depends on ν, we drop the index ν for the sake of brevity. The same applies to the test statistic $I_{n_1,n_2,\nu}$.

From (1.13), we get

$$|J_\nu(X,Y)| = \Gamma_m\Big(\nu + \frac{1}{2}(m+1)\Big)|A_\nu(X,Y)| \leqslant \Gamma_m\Big(\nu + \frac{1}{2}(m+1)\Big)\Big(\Gamma_m\Big(\nu + \frac{1}{2}(m+1)\Big)\Big)^{-1} = 1. \tag{2.8}$$

By applying (2.8) and using the triangle inequality, we obtain that

$$|\mathscr{Z}_{n_1,n_2}(T)| \leqslant \frac{1}{n_1}\sum_{j=1}^{n_1}|J_\nu(T,X_j)| + \frac{1}{n_2}\sum_{j=1}^{n_2}|J_\nu(T,Y_j)| \leqslant 2.$$

Now, it follows that

$$\left\|\mathscr{Z}_{n_1,n_2}(T)\right\|^2 = \int\limits_{T>0} (\mathscr{Z}_{n_1,n_2}(T))^2 \,\mathrm{d}W(X) \leqslant \int\limits_{T>0} 4\,\mathrm{d}W(T) = 4. \tag{2.9}$$

Therefore, the random field $\{\mathscr{Z}_{n_1,n_2}(T), T > 0\}$ is a random element of L^2. The test statistic I_{n_1,n_2} can be represented as $I_{n_1,n_2} = \left\|\mathscr{Z}_{n_1,n_2}(T)\right\|^2$.

Let $\mathscr{Z}_{n_1,X}$ and $\mathscr{Z}_{n_1,Y}$ denote the following random processes: $\mathscr{Z}_{n_1,X}(T) = \sqrt{n_1}(\tilde{\mathscr{H}}_{n_1,\nu}(T) - \tilde{\mathscr{H}}_\nu(T))$ and $\mathscr{Z}_{n_2,Y}(T) = \sqrt{n_2}(\tilde{\mathscr{H}}_{n_2,\nu}(T) - \tilde{\mathscr{H}}_\nu(T))$ respectively.

Note that under H_0 we can write

$$\frac{n_1 n_2}{N} I_{n_1,n_2} = \left\|\sqrt{\frac{n_2}{N}}\mathscr{Z}_{n_1,X} - \sqrt{\frac{n_1}{N}}\mathscr{Z}_{n_2,Y}\right\|^2.$$

It is worth mentioning that if the constants p and q satisfy $p^2 + q^2 = 1$, the process $\mathscr{Z} = p\mathscr{Z}_1 + q\mathscr{Z}_2$ has the covariance structure ρ. Since $\sqrt{\frac{n_1}{N}}^2 + \sqrt{\frac{n_2}{N}}^2 = 1$ and $\frac{n_1}{N} \to \eta \in (0,1)$, it follows that $\{\sqrt{\frac{n_2}{N}}\mathscr{Z}_{n_1,X} - \sqrt{\frac{n_1}{N}}\mathscr{Z}_{n_2,Y}\}$ converges in distribution to the Gaussian process $\mathscr{Z}(T)$ with the covariance kernel ρ. The result of the theorem follows from the continuous mapping theorem.

Noting that for every $S > 0$, $E(\Gamma_m(\nu + \frac{1}{2}(m+1))A_\nu(S,X) - \tilde{\mathscr{H}}_\nu(S)) = 0$, direct computation yields

$$\begin{aligned}\rho(S,T) =& \operatorname{Cov}(\tilde{\mathscr{H}}_{n_1,\nu}(S) - \tilde{\mathscr{H}}_\nu(S), \tilde{\mathscr{H}}_{n_1,\nu}(T) - \tilde{\mathscr{H}}_\nu(T))\\ &= E[(J_\nu(S,X) - \tilde{\mathscr{H}}_\nu(S)) \times (J_\nu(T,X) - \tilde{\mathscr{H}}_\nu(T))] = E[J_\nu(S,X)J_\nu(T,X)] - \tilde{\mathscr{H}}_\nu(S)\tilde{\mathscr{H}}_\nu(T).\end{aligned}$$

□

It is important to emphasize that the null distribution of the test statistic (2.4) is not free of the underlying distributions of X and Y. Consequently, the utilization of specific approximation methods becomes necessary for practical testing.

Currently, it is not possible to establish the asymptotic results for the test statistic (2.2) by similar methods as given above, since the separability of the Hilbert space $\tilde{L}^2$ of all (not necessarily orthogonally invariant) Borel measurable functions $f : \mathcal{P}_+^{m\times m} \to \mathbb{C}$, such that $\int_{X>0} |f(X)|^2 \,\mathrm{d}W(X) < \infty$ is still an open question. In order to illustrate the large sample behavior of the test statistic (2.2), we present the 95-th empirical percentiles of the empirical distributions of the scaled test statistic $\frac{n_1 n_2}{n_1+n_2} L_{n_1,n_2,\nu,\Sigma,\omega}$. The $N = 1000$ values of the statistic $\frac{n_1 n_2}{n_1+n_2} L_{n_1,n_2,\nu,\Sigma,\omega}$ have been obtained. The constant $c_{n_1,n_2} = \frac{n_1+n_2}{n_1 n_2}$ has been used in assessing asymptotic behavior, as in Theorem 2.2. We aim to tabulate the values if the null hypothesis holds. In the case of dimension 2, we assume that both samples come from the $W_2(2.5, I_2)$ Wishart distribution, while in the case of dimension 3, we assume that both samples come from the $W_3(3, I_3)$ Wishart distribution. We fix the parameter $\Sigma = I_d$ for simplicity.

Table 2.1: Empirical 95-th percentiles of the distribution of the scaled statistics. The case of 2×2 matrices.

	$n_1=n_2=100$	$n_1=n_2=200$	$n_1=n_2=500$	$n_1=n_2=750$	$n_1=n_2=1000$
$\nu=1,\ \omega=I_2$	0.0495	0.0524	0.0549	0.0497	0.0518
$\nu=1,\ \omega=2I_2$	0.0196	0.0208	0.0218	0.0196	0.0212
$\nu=2,\ \omega=I_2$	0.0203	0.0216	0.0223	0.0203	0.0219
$\nu=2,\ \omega=2I_2$	0.0084	0.0090	0.0089	0.0082	0.0086
$\nu=5,\ \omega=I_2$	0.0030	0.0028	0.0029	0.0027	0.0027
$\nu=5,\ \omega=2I_2$	0.0017	0.0014	0.0014	0.0014	0.0013

Table 2.2: Empirical 95-th percentiles of the distribution of the scaled statistics. The case of 3×3 matrices.

	$n_1=n_2=100$	$n_1=n_2=200$	$n_1=n_2=500$	$n_1=n_2=750$	$n_1=n_2=1000$
$\nu=1,\ \omega=I_3$	0.0109	0.0108	0.0115	0.0107	0.0108
$\nu=1,\ \omega=2I_3$	0.0021	0.0021	0.0022	0.0020	0.0020
$\nu=2,\ \omega=I_3$	0.0016	0.0016	0.0017	0.0015	0.0016
$\nu=2,\ \omega=2I_3$	3.410×10^{-4}	3.290×10^{-4}	3.404×10^{-4}	3.122×10^{-4}	3.083×10^{-4}
$\nu=5,\ \omega=I_3$	2.614×10^{-5}	2.315×10^{-5}	2.177×10^{-5}	2.093×10^{-5}	2.183×10^{-5}
$\nu=5,\ \omega=2I_3$	8.125×10^{-6}	7.132×10^{-6}	6.729×10^{-6}	6.990×10^{-6}	7.327×10^{-6}

The results are presented in Tables 2.1 and 2.2. It can be observed that the values of the empirical percentiles of the appropriately scaled test statistic tend to stabilize with the increase of n_1 and n_2. However, the parameters ν and ω greatly influence the values of the percentiles of the empirical distribution.

2.4 Empirical test powers of novel tests

In this section, we present the results of the power study for test statistics (2.4) and (2.2). Empirical powers are obtained using a warp-speed bootstrap algorithm (see [46]) with $N = 10000$ replications.

For the statistic (2.4), for dimensions $d=2$ and $d=3$, the parameter ν is fixed to $\nu=1$, as is common practice in problems of this nature [12]. For the statistic (2.2), we fix the parameter Σ to I_d for simplicity. To assess the impact of the parameters on the test power of the test statistic (2.2), we consider three values of ν: $\nu=1$, $\nu=2$, and $\nu=5$, and two values of ω: $\omega=I_d$, and $\omega=2I_d$.

We provide the meta-level pseudocode for the warp-speed bootstrap algorithms, assuming the test statistic is denoted by T. This meta-algorithm is applied directly to the test statistic (2.4) and the test statistic (2.2). The computations are performed using MATLAB [64].

Algorithm 1 Warp-speed bootstrap algorithm

1: Sample $\mathbf{x}=(x_1,\dots,x_{n_1})$ from F_X and $\mathbf{y}=(y_1,\dots,y_n)$ from F_Y;
2: Compute the value of the test statistic $T(\mathbf{x},\mathbf{y})$;
3: Generate bootstrap samples $\mathbf{x}^*=(x_1^*,\dots,x_{n_1}^*)$ and $\mathbf{y}^*=(y_1^*,\dots,y_{n_2}^*)$ from $F_{n_1+n_2}$- sampling distribution based on the joint sample $(\mathbf{x},\mathbf{y})$;
4: Compute $T(\mathbf{x}^*,\mathbf{y}^*)$;
5: Repeat steps 1-4 N times and obtain two sequences of statistics $\{T^{(j)}\}$ and $\{T^{*(j)}\}$, $j=1,\dots,N$;
6: Reject the null hypothesis for the j–sample ($j=1,\dots,N$), if $T^{(j)}>c_\alpha$, where c_α denotes the $(1-\alpha)\%$ quantile of the empirical distribution of the bootstrap test statistics ($T^{*(j)}$, $j=1,\dots,N$).

The level of significance is set to $\alpha=0.05$, and large values of the test statistics are considered significant. The algorithm developed in [72] was implemented to evaluate the Bessel functions of two matrix arguments for the test (2.4). In all cases, we assume that d denotes the dimension of the respective matrices. When estimating sample covariance matrices, samples of dimension d have been considered.The following distributions are considered:

1. Wishart distributions with the shape parameter $a>\frac{1}{2}(d-1)$ and the scale matrix $\Sigma>0$, denoted by $W_d(a,\Sigma)$, with a density

$$f_{W,a,\Sigma}(X)=\frac{1}{\Gamma_d(a)}(\det\Sigma)^a(\det X)^{a-\frac{1}{2}(d+1)}\mathrm{etr}(-\Sigma X),\ X>0. \tag{2.10}$$

2. Inverse Wishart distributions with the shape parameter $a>\frac{1}{2}(d-1)$ and the scale matrix $\Sigma>0$, denoted by $IW_d(a,\Sigma)$, with a density

$$f_{IW,a,\Sigma}(X)=\frac{(\det\Sigma)^{\frac{a}{2}}\mathrm{etr}(-\frac{1}{2}\Sigma X^{-1})}{2^{\frac{ad}{2}}\Gamma_d(\frac{a}{2})(\det X)^{\frac{a+d+1}{2}}},X>0. \tag{2.11}$$

3. Sample covariance matrix distributions obtained from the uniform vectors $(U_1,\dots,U_d)$, where $U_i\in\mathcal{U}[0,1]$, denoted by CMU_d, with a density

$$f_{(U_1,\dots,U_d)}((x_1,\dots,x_d))=1,\ x_i\in[0,1],\ 1\leqslant i\leqslant d. \tag{2.12}$$

4. Sample covariance matrix distributions obtained from the random vectors having the multivariate t-distribution with $a>0$ degrees of freedom, denoted by $CMT_d(a,\Sigma)$, and matrix parameter $\Sigma>0$, with a density

$$f_{t,a}(\mathbf{x})=\frac{1}{(\det(\Sigma))^{\frac{1}{2}}}\frac{\Gamma(\frac{a+d}{2})}{\Gamma(\frac{d}{2})(a\pi)^{\frac{d}{2}}}\left(1+\frac{x'\Sigma^{-1}x}{a}\right)^{-\frac{a+d}{2}},\ \mathbf{x}\in\mathbb{R}^d. \tag{2.13}$$

Denote by K_2 the following covariance matrix: $K_2 = \begin{bmatrix} \cos(0.7) & \sin(0.7) \\ \sin(0.7) & \cos(0.7) \end{bmatrix}$ and denote by K_3 the following covariance matrix: $K_3 = \begin{bmatrix} 1 & -1 & 0.95 \\ -1 & 5 & 0.01 \\ 0.95 & 0.01 & 7 \end{bmatrix}$. The covariance matrices given above are utilized as distributional parameters in the simulation study.

The results of the power study for the test statistic (2.4) are presented in Tables 2.3 and 2.4. The results of the power study for the test (2.4) are presented in Tables 2.5 - 2.10. Whenever a matrix in Tables 2.3 - 2.10 is symmetric, we leave the lower part of the table empty.

The importance of the test (2.4) lies in its ability to distinguish between different distributions that share the same expectation ($W_2(2.5, I_2)$ versus $IW_2(4, 2.5I_2)$ and $W_3(3, I_3)$ versus $IW_3(5, 3I_3)$) with a reasonable level of precision. Generally, the test (2.4) tends to have a lower power when testing against distributions from the same family. However, when dealing with different families of distributions, the test typically demonstrates higher power.

The power of the test (2.4) tends to decrease as the dimension of the matrix increases. Despite this, the test seems to be well calibrated, and the bootstrap approximation is not prone to size distortions. Given that the test statistic (2.4) is a function of eigenvalues, high test powers are expected when the underlying distributions have significantly different eigenvalues. This is apparent when one of the samples originates from the CMU distribution. In contrast, when the distributions have very similar eigenvalues, lower test powers are expected, as seen when one of the samples comes from $CMT(3, I)$ and the other from $CMT(5, I)$.

Although calculating the inverse of the matrix is the most resource-intensive step in evaluating the test statistic (2.2), evaluating the test statistic is not as computationally challenging as the evaluation of the test statistic (2.4), as it does not require the use of special functions. However, the computational time needed increases with the dimensionality of the problem in this scenario as well.

It is important to note that the test powers are not evaluated on the same samples, and consequently, some discrepancies may be present. Based on Tables 2.5 - 2.10, it can be inferred that the empirical test powers are significantly influenced by the selection of parameters. In most scenarios, when the parameter Σ is set to I_d, the empirical test powers are superior compared to when $\Sigma = 2I_d$. In addition, smaller values of the parameter ν are associated with higher empirical test powers. The empirical test powers for $\nu = 5$ are typically the lowest. Based on these findings, we suggest using the test for $\nu = 1$ and $\Sigma = I_d$ for optimal results. Additionally, it is noticeable that the empirical test powers diminish as the dimensionality increases. Furthermore, no substantial size distortions are detected for the new test.

Given that the test in (2.2) is the first of its kind to test for the equality in the distribution of the symmetric positive definite matrix distributions, it is not feasible to make any further comparisons of empirical test powers.

In the subsequent discussion, we juxtapose the tests (2.4) and (2.2). It is crucial to highlight that the test (2.4) tests for orthogonal invariance in distribution, while the test (2.2) tests for equivalence in distribution. Consequently, these two tests are not directly comparable. Nevertheless, one can compare the empirical test powers of these tests. It is important to note that the test powers are not obtained on the same samples. Moreover, it is essential to take into account the theoretical characteristics and constraints of each test to make meaningful comparisons.

When juxtaposing the test (2.2) with the test (2.4), it is noticeable that the test (2.2) typically surpasses the test (2.4), especially when $\nu = 1$ and $\Sigma = I_d$. This superior performance is credited to the fact that the test (2.4) focuses on orthogonal invariance in distribution rather than equivalence in distribution. However, as the parameter ν increases, the disparity in empirical test powers between the test (2.2) and the test (2.4) becomes less pronounced. It is important to note that the test (2.2) does not consistently outperform the test (2.4) in all scenarios. For instance, when testing the inverse Wishart against the Wishart distribution, the test (2.4) demonstrates the best performance.

We examine the behavior of the tests in a well-known theoretical scenario. Note that if $x_1, x_2, \ldots, x_n \in \mathcal{N}_p(0, \Sigma)$, then $X = (x_1, x_2, \ldots, x_n)$ follows a matrix-variate normal distribution. Furthermore, if $n \geqslant p$, then $X'X > 0$ with probability 1 and $X'X \in W_p(n, \Sigma)$ [49, p. 88]. We determined the test powers of the Wishart $\frac{1}{499} W_d(500, I_d)$ distribution against the $CMT_d(df, I_d)$, where we calculated the sample covariance matrices on $NCov = 500$ random vectors. The parameter df is selected from the set $\{1, 21, 41, \ldots, 501\}$. The results for $d = 2$ and $d = 3$ are presented in Figures 2.1 and 2.2. The test powers are expressed in percentage. The test powers are computed on the same samples. It is interesting to note that the drop in the test powers is the same for both dimensions and both sample sizes. The behavior of the tests is as expected since as the degrees of freedom increase, the t-distribution becomes closer to the normal distribution. Consequently, the distribution of its covariance matrix becomes closer to the appropriately scaled Wishart distribution, leading to a decrease in test powers and reaching test size for large enough degrees of freedom for both cases.

Figure 2.1: The case of 2×2 matrices.

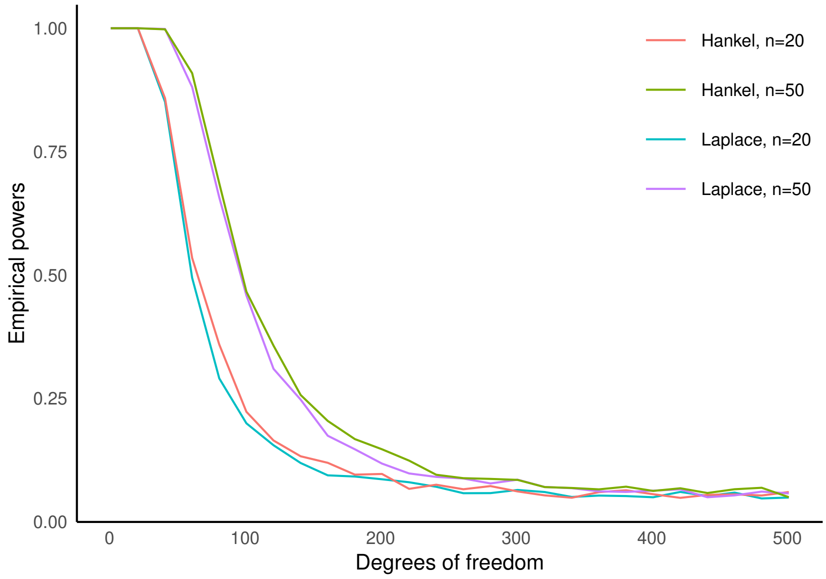


Figure 2.2: The case of 3×3 matrices.

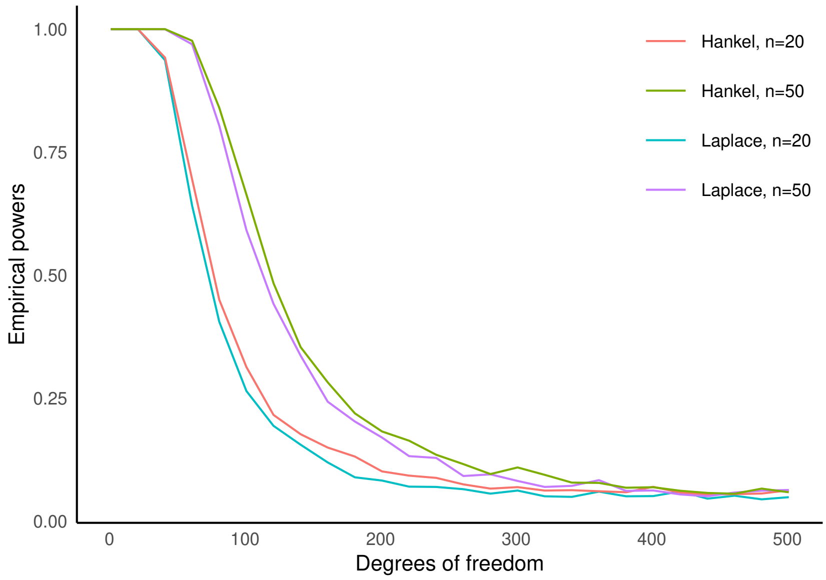


Table 2.3: The percentage of rejected null hypotheses for different sample sizes for 2×2 matrices - statistic (2.4).

$n_1 = 20, n_2 = 20$	$W_2(2.5, I_2)$	$IW_2(2.5, I_2)$	$CMT_2(1, I_2)$	CMU_2	$W_2(2.5, 2I_2)$	$IW_2(4, 2.5I_2)$	$W_2(2.5, K_2)$	$CMT_2(3, K_2)$	$CMT_2(5, K_2)$	$CMT_2(3, I_2)$	$CMT_2(5, I_2)$
$W_2(2.5, I_2)$	5	39	12	100	33	41	45	87	92	67	87
$IW_2(2.5, I_2)$		5	10	100	82	7	8	44	60	24	33
$CMT_2(1, I_2)$			5	100	36	7	14	62	74	37	49
CMU_2				5	100	100	100	100	99	100	100
$W_2(2.5, 2I_2)$					4	96	89	97	99	91	95
$IW_2(4, 2.5I_2)$						5	8	57	73	33	48
$W_2(2.5, K_2)$							5	49	61	28	41
$CMT_2(3, K_2)$								5	6	10	16
$CMT_2(5, K_2)$									5	7	11
$CMT_2(3, I_2)$										5	6
$CMT_2(5, I_2)$											5

$n_1 = 30, n_2 = 20$	$W_2(2.5, I_2)$	$IW_2(2.5, I_2)$	$CMT_2(1, I_2)$	CMU_2	$W_2(2.5, 2I_2)$	$IW_2(4, 2.5I_2)$	$W_2(2.5, K_2)$	$CMT_2(3, K_2)$	$CMT_2(5, K_2)$	$CMT_2(3, I_2)$	$CMT_2(5, I_2)$
$W_2(2.5, I_2)$	5	49	19	100	38	46	57	93	97	79	89
$IW_2(2.5, I_2)$	44	5	12	100	90	6	9	55	70	29	43
$CMT_2(1, I_2)$	9	11	5	100	34	6	17	71	81	45	59
CMU_2	100	100	100	5	100	100	100	100	100	100	100
$W_2(2.5, 2I_2)$	45	91	48	100	4	97	96	99	100	96	98
$IW_2(4, 2.5I_2)$	46	10	11	100	96	5	11	70	82	44	62
$W_2(2.5, K_2)$	50	8	19	100	94	7	5	59	71	37	50
$CMT_2(3, K_2)$	92	48	66	100	99	63	53	6	7	10	6
$CMT_2(5, K_2)$	97	66	80	100	100	79	68	6	5	19	11
$CMT_2(3, I_2)$	75	22	42	100	95	35	31	12	21	5	7
$CMT_2(5, I_2)$	87	36	58	100	98	51	42	8	13	6	5

$n_1 = 50, n_2 = 20$	$W_2(2.5, I_2)$	$IW_2(2.5, I_2)$	$CMT_2(1, I_2)$	CMU_2	$W_2(2.5, 2I_2)$	$IW_2(4, 2.5I_2)$	$W_2(2.5, K_2)$	$CMT_2(3, K_2)$	$CMT_2(5, K_2)$	$CMT_2(3, I_2)$	$CMT_2(5, I_2)$
$W_2(2.5, I_2)$	5	61	24	100	43	55	68	97	99	90	94
$IW_2(2.5, I_2)$	50	5	16	100	95	5	10	66	81	37	53
$CMT_2(1, I_2)$	10	11	5	100	37	5	19	80	90	55	68
CMU_2	100	100	100	5	100	100	100	100	100	100	100
$W_2(2.5, 2I_2)$	55	96	62	100	4	99	99	100	100	99	100
$IW_2(4, 2.5I_2)$	53	11	20	100	98	5	14	80	90	54	70
$W_2(2.5, K_2)$	56	8	26	100	98	7	5	70	82	46	60
$CMT_2(3, K_2)$	96	56	75	100	100	68	58	5	7	11	6
$CMT_2(5, K_2)$	99	73	87	100	100	85	77	7	5	22	12
$CMT_2(3, I_2)$	83	23	47	100	99	37	22	13	23	5	8
$CMT_2(5, I_2)$	93	39	64	100	100	55	50	8	14	6	5

$n_1 = 50, n_2 = 30$	$W_2(2.5, I_2)$	$IW_2(2.5, I_2)$	$CMT_2(1, I_2)$	CMU_2	$W_2(2.5, 2I_2)$	$IW_2(4, 2.5I_2)$	$W_2(2.5, K_2)$	$CMT_2(3, K_2)$	$CMT_2(5, K_2)$	$CMT_2(3, I_2)$	$CMT_2(5, I_2)$
$W_2(2.5, I_2)$	5	68	25	100	59	68	80	99	100	94	98
$IW_2(2.5, I_2)$	64	5	19	100	99	7	12	76	89	41	60
$CMT_2(1, I_2)$	15	14	5	100	58	8	28	89	95	62	78
CMU_2	100	100	100	5	100	100	100	100	100	100	100
$W_2(2.5, 2I_2)$	66	99	68	100	5	100	100	100	100	100	100
$IW_2(4, 2.5I_2)$	64	10	17	100	100	5	14	87	95	61	77
$W_2(2.5, K_2)$	73	10	36	100	100	8	5	78	90	54	69
$CMT_2(3, K_2)$	99	69	85	100	100	84	75	5	7	14	7
$CMT_2(5, K_2)$	100	86	95	100	100	94	88	7	5	29	16
$CMT_2(3, I_2)$	93	33	61	100	100	51	45	17	30	5	8
$CMT_2(5, I_2)$	98	55	76	100	100	73	66	8	17	8	5

$n_1 = 50, n_2 = 50$	$W_2(2.5, I_2)$	$IW_2(2.5, I_2)$	$CMT_2(1, I_2)$	CMU_2	$W_2(2.5, 2I_2)$	$IW_2(4, 2.5I_2)$	$W_2(2.5, K_2)$	$CMT_2(3, K_2)$	$CMT_2(5, K_2)$	$CMT_2(3, I_2)$	$CMT_2(5, I_2)$
$W_2(2.5, I_2)$	5	79	26	100	77	78	89	100	100	98	100
$IW_2(2.5, I_2)$		5	19	100	100	9	15	84	95	48	71
$CMT_2(1, I_2)$			6	100	78	16	49	95	99	74	89
CMU_2				5	100	100	100	100	100	100	100
$W_2(2.5, 2I_2)$					4	100	100	100	100	100	100
$IW_2(4, 2.5I_2)$						5	15	94	99	69	86
$W_2(2.5, K_2)$							5	87	95	64	80
$CMT_2(3, K_2)$								5	8	19	37
$CMT_2(5, K_2)$									5	9	20
$CMT_2(3, I_2)$										5	9
$CMT_2(5, I_2)$											5

Table 2.4: The percentage of rejected null hypotheses for different sample sizes for 3×3 matrices - statistic (2.4).

$n_1 = 20, n_2 = 20$	$W_3(3,I_3)$	$IW_3(3,I_3)$	$CMT_3(1,I_3)$	CMU_3	$W_3(3,2I_3)$	$IW_3(5,3I_3)$	$W_3(3,K_3)$	$CMT_3(3,K_3)$	$CMT_3(5,K_3)$	$CMT_3(3,I_3)$	$CMT_3(5,I_3)$
$W_3(3,I_3)$	5	25	6	100	16	51	19	57	71	52	69
$IW_3(3,I_3)$		5	20	100	39	5	42	22	35	17	29
$CMT_3(1,I_3)$			5	100	10	33	11	51	67	47	61
CMU_3				5	100	100	100	100	100	100	100
$W_3(3,2I_3)$					6	80	8	63	80	63	78
$IW_3(5,3I_3)$						5	80	23	36	19	29
$W_3(3,K_3)$							6	66	80	63	78
$CMT_3(3,K_3)$								4	6	5	8
$CMT_3(5,K_3)$									5	5	6
$CMT_3(3,I_3)$										5	6
$CMT_3(5,I_3)$											5

$n_1 = 30, n_2 = 20$	$W_3(3,I_3)$	$IW_3(3,I_3)$	$CMT_3(1,I_3)$	CMU_3	$W_3(3,2I_3)$	$IW_3(5,3I_3)$	$W_3(3,K_3)$	$CMT_3(3,K_3)$	$CMT_3(5,K_3)$	$CMT_3(3,I_3)$	$CMT_3(5,I_3)$
$W_3(3,I_3)$	4	36	7	100	14	65	17	70	85	66	80
$IW_3(3,I_3)$	25	4	21	100	41	3	45	29	43	23	38
$CMT_3(1,I_3)$	4	27	5	100	8	43	8	64	78	60	75
CMU_3	100	100	100	5	100	100	100	100	100	100	100
$W_3(3,2I_3)$	23	54	16	100	6	89	7	78	90	77	88
$IW_3(5,3I_3)$	56	7	44	100	84	5	88	33	48	26	43
$W_3(3,K_3)$	30	57	17	100	10	90	6	79	92	79	89
$CMT_3(3,K_3)$	62	21	56	100	71	20	74	5	7	5	10
$CMT_3(5,K_3)$	80	36	73	100	88	38	88	6	5	5	6
$CMT_3(3,I_3)$	57	17	53	100	67	18	71	5	5	5	7
$CMT_3(5,I_3)$	74	31	70	100	87	31	87	8	6	7	5

$n_1 = 50, n_2 = 20$	$W_3(3,I_3)$	$IW_3(3,I_3)$	$CMT_3(1,I_3)$	CMU_3	$W_3(3,2I_3)$	$IW_3(5,3I_3)$	$W_3(3,K_3)$	$CMT_3(3,K_3)$	$CMT_3(5,K_3)$	$CMT_3(3,I_3)$	$CMT_3(5,I_3)$
$W_3(3,I_3)$	5	52	10	100	12	75	14	83	93	81	91
$IW_3(3,I_3)$	26	4	21	100	46	3	48	36	55	31	48
$CMT_3(1,I_3)$	2	37	5	100	4	50	6	78	89	73	88
CMU_3	100	100	100	5	100	100	100	100	100	100	100
$W_3(3,2I_3)$	36	72	23	100	5	96	4	90	97	90	96
$IW_3(5,3I_3)$	66	9	53	100	93	5	95	44	62	36	56
$W_3(3,K_3)$	45	73	25	100	13	97	6	92	97	90	96
$CMT_3(3,K_3)$	70	22	60	100	83	20	83	5	8	5	6
$CMT_3(5,K_3)$	86	41	82	100	95	40	95	6	5	8	4
$CMT_3(3,I_3)$	65	17	61	100	81	15	81	6	12	5	8
$CMT_3(5,I_3)$	84	33	78	100	94	32	96	5	7	7	5

$n_1 = 50, n_2 = 30$	$W_3(3,I_3)$	$IW_3(3,I_3)$	$CMT_3(1,I_3)$	CMU_3	$W_3(3,2I_3)$	$IW_3(5,3I_3)$	$W_3(3,K_3)$	$CMT_3(3,K_3)$	$CMT_3(5,K_3)$	$CMT_3(3,I_3)$	$CMT_3(5,I_3)$
$W_3(3,I_3)$	4	55	8	100	21	84	29	90	97	87	97
$IW_3(3,I_3)$	44	5	33	100	70	4	70	43	62	35	54
$CMT_3(1,I_3)$	3	41	5	100	10	61	11	85	95	82	93
CMU_3	100	100	100	5	100	100	100	100	100	100	100
$W_3(3,2I_3)$	37	77	22	100	5	99	5	96	99	95	99
$IW_3(5,3I_3)$	79	7	63	100	99	5	99	47	67	40	60
$W_3(3,K_3)$	47	80	24	100	11	99	6	96	99	96	99
$CMT_3(3,K_3)$	86	33	80	100	94	32	94	5	8	5	5
$CMT_3(5,K_3)$	96	55	93	100	99	60	99	7	5	11	6
$CMT_3(3,I_3)$	83	26	76	100	94	25	94	6	12	5	9
$CMT_3(5,I_3)$	95	48	91	100	99	51	99	6	7	7	6

$n_1 = 50, n_2 = 50$	$W_3(3,I_3)$	$IW_3(3,I_3)$	$CMT_3(1,I_3)$	CMU_3	$W_3(3,2I_3)$	$IW_3(5,3I_3)$	$W_3(3,K_3)$	$CMT_3(3,K_3)$	$CMT_3(5,K_3)$	$CMT_3(3,I_3)$	$CMT_3(5,I_3)$
$W_3(3,I_3)$	5	63	6	100	39	92	50	96	99	95	99
$IW_3(3,I_3)$		6	48	100	86	5	88	47	73	39	64
$CMT_3(1,I_3)$			4	100	19	76	21	93	98	90	98
CMU_3				5	100	100	100	100	100	100	100
$W_3(3,2I_3)$					5	100	9	99	100	99	100
$IW_3(5,3I_3)$						5	100	54	78	43	68
$W_3(3,K_3)$							6	99	100	99	100
$CMT_3(3,K_3)$								5	9	6	15
$CMT_3(5,K_3)$									5	5	6
$CMT_3(3,I_3)$										5	9
$CMT_3(5,I_3)$											6

Table 2.5: The percentage of rejected H_0 for different sample sizes for 2×2 matrices, $\nu=1, \omega=I_d(\nu=2, \omega=I_d)$ - statistic (2.2).

$n_1=20, n_2=20$	$W_2(2.5,I_2)$	$IW_2(2.5,I_2)$	$CMT_2(1,I_2)$	CMU_2	$W_2(2.5,2I_2)$	$IW_2(4,2.5I_2)$	$W_2(2.5,K_2)$	$CMT_2(3,K_2)$	$CMT_2(5,K_2)$	$CMT_2(3,I_2)$	$CMT_2(5,I_2)$
$W_2(2.5,I_2)$	4 (4)	22 (24)	29 (21)	100 (100)	51 (39)	20 (21)	99 (86)	91 (76)	96 (85)	99 (91)	100 (97)
$IW_2(2.5,I_2)$		5 (5)	16 (9)	100 (100)	87 (81)	9 (10)	97 (63)	71 (54)	83 (66)	97 (80)	99 (88)
$CMT_2(1,I_2)$			5 (5)	100 (100)	60 (37)	47 (16)	92 (30)	50 (34)	64 (45)	86 (60)	93 (73)
CMU_2				5 (5)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
$W_2(2.5,2I_2)$					4 (4)	95 (92)	100 (99)	99 (91)	100 (96)	100 (98)	100 (99)
$IW_2(4,2.5I_2)$						5 (5)	89 (89)	94 (68)	89 (82)	99 (88)	100 (95)
$W_2(2.5,K_2)$							4 (4)	83 (52)	90 (63)	67 (57)	76 (71)
$CMT_2(3,K_2)$								5 (5)	6 (6)	29 (13)	41 (18)
$CMT_2(5,K_2)$									5 (5)	24 (9)	31 (14)
$CMT_2(3,I_2)$										5 (5)	6 (6)
$CMT_2(5,I_2)$											5 (5)

$n_1=30, n_2=20$	$W_2(2.5,I_2)$	$IW_2(2.5,I_2)$	$CMT_2(1,I_2)$	CMU_2	$W_2(2.5,2I_2)$	$IW_2(4,2.5I_2)$	$W_2(2.5,K_2)$	$CMT_2(3,K_2)$	$CMT_2(5,K_2)$	$CMT_2(3,I_2)$	$CMT_2(5,I_2)$
$W_2(2.5,I_2)$	5 (4)	30 (32)	44 (30)	100 (100)	58 (42)	22 (21)	100 (96)	97 (89)	99 (94)	100 (97)	100 (99)
$IW_2(2.5,I_2)$	28 (29)	5 (5)	32 (15)	100 (100)	92 (89)	7 (8)	99 (82)	85 (70)	92 (81)	100 (91)	100 (96)
$CMT_2(1,I_2)$	27 (16)	12 (7)	4 (4)	100 (100)	64 (39)	48 (13)	96 (35)	59 (41)	73 (55)	94 (73)	98 (83)
CMU_2	100 (100)	100 (100)	100 (100)	5 (5)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
$W_2(2.5,2I_2)$	65 (52)	94 (89)	77 (55)	100 (100)	3 (4)	98 (96)	100 (100)	100 (98)	100 (99)	100 (100)	100 (100)
$IW_2(4,2.5I_2)$	30 (29)	13 (14)	82 (32)	100 (100)	98 (97)	5 (5)	100 (97)	96 (83)	99 (92)	100 (96)	100 (99)
$W_2(2.5,K_2)$	100 (93)	99 (76)	98 (54)	100 (100)	100 (100)	100 (95)	4 (5)	94 (67)	96 (78)	78 (72)	88 (81)
$CMT_2(3,K_2)$	95 (84)	77 (60)	56 (38)	100 (100)	100 (97)	94 (76)	89 (56)	5 (5)	7 (7)	37 (15)	52 (25)
$CMT_2(5,K_2)$	98 (92)	87 (73)	72 (50)	100 (100)	100 (99)	98 (88)	93 (68)	7 (6)	4 (6)	32 (12)	40 (18)
$CMT_2(3,I_2)$	100 (96)	99 (87)	92 (67)	100 (100)	100 (100)	100 (95)	70 (63)	37 (13)	31 (10)	5 (5)	6 (7)
$CMT_2(5,I_2)$	100 (99)	100 (95)	96 (79)	100 (100)	100 (100)	100 (98)	83 (76)	49 (24)	40 (16)	7 (6)	5 (4)

$n_1=50, n_2=20$	$W_2(2.5,I_2)$	$IW_2(2.5,I_2)$	$CMT_2(1,I_2)$	CMU_2	$W_2(2.5,2I_2)$	$IW_2(4,2.5I_2)$	$W_2(2.5,K_2)$	$CMT_2(3,K_2)$	$CMT_2(5,K_2)$	$CMT_2(3,I_2)$	$CMT_2(5,I_2)$
$W_2(2.5,I_2)$	4 (4)	37 (41)	70 (44)	100 (100)	66 (49)	26 (25)	100 (99)	99 (96)	100 (99)	100 (100)	100 (100)
$IW_2(2.5,I_2)$	29 (33)	5 (6)	62 (29)	100 (100)	96 (93)	5 (7)	100 (94)	94 (84)	98 (92)	100 (98)	100 (99)
$CMT_2(1,I_2)$	22 (14)	10 (5)	5 (5)	100 (100)	70 (42)	37 (10)	99 (37)	71 (52)	83 (67)	99 (84)	100 (92)
CMU_2	100 (100)	100 (100)	100 (100)	5 (5)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
$W_2(2.5,2I_2)$	77 (67)	97 (95)	92 (73)	100 (100)	4 (4)	99 (99)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
$IW_2(4,2.5I_2)$	42 (37)	20 (20)	100 (57)	100 (100)	100 (99)	5 (5)	100 (100)	100 (95)	100 (98)	100 (100)	100 (100)
$W_2(2.5,K_2)$	100 (97)	100 (85)	100 (78)	100 (100)	100 (100)	100 (98)	5 (4)	98 (85)	99 (91)	88 (83)	94 (90)
$CMT_2(3,K_2)$	98 (91)	83 (64)	65 (43)	100 (100)	100 (99)	97 (86)	93 (62)	5 (5)	7 (7)	46 (20)	62 (32)
$CMT_2(5,K_2)$	100 (96)	93 (82)	78 (59)	100 (100)	100 (100)	100 (95)	97 (79)	6 (6)	5 (5)	41 (15)	54 (22)
$CMT_2(3,I_2)$	100 (99)	100 (93)	96 (77)	100 (100)	100 (100)	100 (99)	79 (69)	43 (14)	36 (11)	5 (5)	7 (6)
$CMT_2(5,I_2)$	100 (100)	100 (98)	98 (87)	100 (100)	100 (100)	100 (100)	91 (85)	58 (26)	48 (17)	7 (5)	5 (5)

$n_1=50, n_2=30$	$W_2(2.5,I_2)$	$IW_2(2.5,I_2)$	$CMT_2(1,I_2)$	CMU_2	$W_2(2.5,2I_2)$	$IW_2(4,2.5I_2)$	$W_2(2.5,K_2)$	$CMT_2(3,K_2)$	$CMT_2(5,K_2)$	$CMT_2(3,I_2)$	$CMT_2(5,I_2)$
$W_2(2.5,I_2)$	4 (4)	45 (49)	78 (45)	100 (100)	84 (69)	36 (35)	100 (100)	100 (99)	100 (100)	100 (100)	100 (100)
$IW_2(2.5,I_2)$	42 (44)	5 (5)	67 (27)	100 (100)	99 (98)	9 (11)	100 (99)	98 (90)	99 (96)	100 (99)	100 (100)
$CMT_2(1,I_2)$	53 (26)	29 (10)	5 (4)	100 (100)	91 (67)	91 (29)	100 (73)	81 (62)	92 (78)	100 (93)	100 (97)
CMU_2	100 (100)	100 (100)	100 (100)	5 (5)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
$W_2(2.5,2I_2)$	87 (77)	99 (99)	96 (78)	100 (100)	4 (4)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
$IW_2(4,2.5I_2)$	49 (46)	17 (20)	100 (63)	100 (100)	100 (100)	5 (5)	100 (100)	100 (98)	100 (99)	100 (100)	100 (100)
$W_2(2.5,K_2)$	100 (100)	100 (98)	100 (91)	100 (100)	100 (100)	100 (100)	4 (4)	100 (91)	100 (95)	93 (89)	98 (96)
$CMT_2(3,K_2)$	100 (98)	95 (85)	79 (59)	100 (100)	100 (100)	100 (96)	99 (82)	5 (5)	8 (6)	63 (24)	78 (41)
$CMT_2(5,K_2)$	100 (99)	99 (93)	91 (75)	100 (100)	100 (100)	100 (99)	100 (91)	8 (6)	5 (5)	57 (17)	69 (28)
$CMT_2(3,I_2)$	100 (100)	100 (99)	99 (89)	100 (100)	100 (100)	100 (100)	91 (86)	62 (21)	53 (14)	4 (4)	7 (7)
$CMT_2(5,I_2)$	100 (100)	100 (100)	100 (96)	100 (100)	100 (100)	100 (100)	97 (95)	74 (37)	66 (26)	7 (7)	5 (5)

$n_1=50, n_2=50$	$W_2(2.5,I_2)$	$IW_2(2.5,I_2)$	$CMT_2(1,I_2)$	CMU_2	$W_2(2.5,2I_2)$	$IW_2(4,2.5I_2)$	$W_2(2.5,K_2)$	$CMT_2(3,K_2)$	$CMT_2(5,K_2)$	$CMT_2(3,I_2)$	$CMT_2(5,I_2)$
$W_2(2.5,I_2)$	4 (4)	54 (61)	86 (50)	100 (100)	94 (88)	56 (55)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
$IW_2(2.5,I_2)$		5 (5)	79 (26)	100 (100)	100 (100)	19 (20)	100 (100)	99 (95)	100 (99)	100 (100)	100 (100)
$CMT_2(1,I_2)$			6 (4)	100 (100)	100 (85)	100 (68)	100 (98)	100 (73)	100 (87)	100 (97)	100 (100)
CMU_2				5 (5)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
$W_2(2.5,2I_2)$					4 (4)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
$IW_2(4,2.5I_2)$						5 (4)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
$W_2(2.5,K_2)$							4 (5)	100 (96)	100 (99)	98 (95)	100 (99)
$CMT_2(3,K_2)$								6 (5)	8 (7)	80 (30)	92 (52)
$CMT_2(5,K_2)$									4 (4)	74 (21)	85 (37)
$CMT_2(3,I_2)$										5 (6)	8 (8)
$CMT_2(5,I_2)$											5 (5)

Table 2.6: The percentage of rejected H_0 for different sample sizes for 3×3 matrices, $\nu=1, \omega=I_d (\nu=2, \omega=I_d)$ - statistic (2.2).

$n_1=20, n_2=20$	$W_3(3,I_3)$	$IW_3(3,I_3)$	$CMT_3(1,I_3)$	CMU_3	$W_3(3,2I_3)$	$IW_3(5,3I_3)$	$W_3(3,K_3)$	$CMT_3(3,K_3)$	$CMT_3(5,K_3)$	$CMT_3(3,I_3)$	$CMT_3(5,I_3)$
$W_3(3,I_3)$	3 (3)	14 (18)	12 (4)	100 (100)	59 (29)	40 (36)	99 (63)	83 (58)	95 (74)	92 (67)	99 (81)
$IW_3(3,I_3)$		5 (4)	27 (13)	100 (100)	73 (56)	11 (6)	95 (67)	58 (35)	79 (50)	73 (43)	90 (60)
$CMT_3(1,I_3)$			3 (3)	100 (100)	14 (9)	82 (23)	37 (13)	88 (53)	96 (71)	94 (62)	99 (78)
CMU_3				5 (5)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
$W_3(3,2I_3)$					3 (3)	100 (98)	67 (13)	99 (76)	100 (90)	100 (80)	100 (94)
$IW_3(5,3I_3)$						5 (4)	100 (100)	67 (43)	85 (59)	82 (52)	94 (69)
$W_3(3,K_3)$							3 (3)	100 (82)	100 (93)	100 (85)	100 (95)
$CMT_3(3,K_3)$								4 (4)	6 (5)	9 (6)	16 (9)
$CMT_3(5,K_3)$									4 (3)	9 (4)	10 (5)
$CMT_3(3,I_3)$										4 (4)	6 (5)
$CMT_3(5,I_3)$											4 (5)

$n_1=30, n_2=20$	$W_3(3,I_3)$	$IW_3(3,I_3)$	$CMT_3(1,I_3)$	CMU_3	$W_3(3,2I_3)$	$IW_3(5,3I_3)$	$W_3(3,K_3)$	$CMT_3(3,K_3)$	$CMT_3(5,K_3)$	$CMT_3(3,I_3)$	$CMT_3(5,I_3)$
$W_3(3,I_3)$	3 (3)	21 (27)	27 (7)	100 (100)	68 (29)	44 (43)	100 (68)	94 (77)	99 (89)	98 (82)	100 (94)
$IW_3(3,I_3)$	15 (17)	4 (4)	38 (16)	100 (100)	82 (61)	7 (4)	98 (77)	73 (51)	90 (70)	88 (60)	97 (78)
$CMT_3(1,I_3)$	13 (3)	33 (16)	3 (3)	100 (100)	10 (6)	87 (24)	32 (10)	95 (72)	99 (87)	98 (82)	100 (93)
CMU_3	100 (100)	100 (100)	100 (100)	4 (5)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
$W_3(3,2I_3)$	78 (52)	85 (73)	30 (19)	100 (100)	3 (3)	100 (99)	77 (9)	100 (92)	100 (98)	100 (95)	100 (99)
$IW_3(5,3I_3)$	55 (48)	19 (11)	97 (50)	100 (100)	100 (99)	4 (5)	100 (100)	85 (62)	95 (78)	95 (71)	99 (86)
$W_3(3,K_3)$	100 (87)	99 (84)	59 (24)	100 (100)	87 (24)	100 (100)	3 (4)	100 (94)	100 (99)	100 (96)	100 (100)
$CMT_3(3,K_3)$	90 (62)	60 (35)	93 (59)	100 (100)	100 (85)	70 (42)	100 (89)	3 (4)	8 (6)	10 (6)	22 (12)
$CMT_3(5,K_3)$	98 (83)	82 (55)	98 (79)	100 (100)	100 (88)	90 (53)	100 (94)	6 (5)	4 (5)	11 (4)	15 (6)
$CMT_3(3,I_3)$	97 (70)	78 (45)	97 (71)	100 (100)	100 (88)	87 (53)	100 (94)	10 (5)	12 (5)	4 (4)	8 (6)
$CMT_3(5,I_3)$	100 (89)	94 (66)	99 (87)	100 (100)	100 (97)	98 (74)	100 (99)	19 (8)	12 (5)	6 (5)	4 (4)

$n_1=50, n_2=20$	$W_3(3,I_3)$	$IW_3(3,I_3)$	$CMT_3(1,I_3)$	CMU_3	$W_3(3,2I_3)$	$IW_3(5,3I_3)$	$W_3(3,K_3)$	$CMT_3(3,K_3)$	$CMT_3(5,K_3)$	$CMT_3(3,I_3)$	$CMT_3(5,I_3)$
$W_3(3,I_3)$	4 (4)	30 (37)	56 (13)	100 (100)	77 (32)	51 (50)	100 (83)	99 (91)	100 (98)	100 (96)	100 (99)
$IW_3(3,I_3)$	14 (15)	5 (5)	51 (22)	100 (100)	89 (69)	6 (4)	100 (84)	89 (71)	97 (85)	97 (78)	100 (92)
$CMT_3(1,I_3)$	12 (3)	39 (19)	3 (4)	100 (100)	7 (3)	92 (26)	32 (6)	99 (87)	100 (96)	100 (93)	100 (99)
CMU_3	100 (100)	100 (100)	100 (100)	5 (5)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
$W_3(3,2I_3)$	91 (76)	94 (87)	54 (33)	100 (100)	3 (4)	100 (100)	89 (8)	100 (98)	100 (100)	100 (99)	100 (100)
$IW_3(5,3I_3)$	70 (63)	35 (16)	100 (90)	100 (100)	100 (100)	5 (4)	100 (100)	96 (82)	99 (93)	99 (88)	100 (97)
$W_3(3,K_3)$	100 (98)	100 (95)	83 (43)	100 (100)	96 (44)	100 (100)	3 (4)	100 (99)	100 (100)	100 (100)	100 (100)
$CMT_3(3,K_3)$	94 (68)	68 (33)	97 (70)	100 (100)	100 (92)	73 (41)	100 (96)	4 (4)	9 (8)	15 (8)	31 (17)
$CMT_3(5,K_3)$	100 (89)	90 (61)	99 (88)	100 (100)	100 (96)	93 (67)	100 (100)	6 (5)	5 (4)	14 (5)	18 (9)
$CMT_3(3,I_3)$	99 (82)	86 (47)	99 (78)	100 (100)	100 (96)	94 (59)	100 (97)	12 (5)	15 (6)	4 (4)	10 (8)
$CMT_3(5,I_3)$	100 (95)	97 (75)	100 (93)	100 (100)	100 (100)	99 (84)	100 (100)	26 (9)	16 (6)	7 (4)	5 (5)

$n_1=50, n_2=30$	$W_3(3,I_3)$	$IW_3(3,I_3)$	$CMT_3(1,I_3)$	CMU_3	$W_3(3,2I_3)$	$IW_3(5,3I_3)$	$W_3(3,K_3)$	$CMT_3(3,K_3)$	$CMT_3(5,K_3)$	$CMT_3(3,I_3)$	$CMT_3(5,I_3)$
$W_3(3,I_3)$	3 (3)	36 (42)	69 (10)	100 (100)	93 (61)	68 (71)	100 (98)	100 (96)	100 (99)	100 (98)	100 (100)
$IW_3(3,I_3)$	24 (28)	5 (3)	63 (30)	100 (100)	97 (86)	14 (4)	100 (96)	93 (75)	99 (91)	99 (85)	100 (96)
$CMT_3(1,I_3)$	34 (4)	55 (26)	4 (3)	100 (100)	24 (9)	99 (60)	65 (16)	100 (94)	100 (99)	100 (97)	100 (99)
CMU_3	100 (100)	100 (100)	100 (100)	5 (5)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
$W_3(3,2I_3)$	96 (82)	98 (92)	50 (30)	100 (100)	3 (3)	100 (100)	99 (24)	100 (100)	100 (100)	100 (100)	100 (100)
$IW_3(5,3I_3)$	81 (77)	34 (14)	100 (93)	100 (100)	100 (100)	5 (4)	100 (100)	98 (86)	100 (96)	100 (93)	100 (99)
$W_3(3,K_3)$	100 (99)	100 (98)	87 (41)	100 (100)	100 (53)	100 (100)	3 (3)	100 (100)	100 (100)	100 (100)	100 (100)
$CMT_3(3,K_3)$	99 (90)	86 (57)	99 (88)	100 (100)	100 (99)	93 (72)	100 (100)	4 (4)	9 (8)	18 (9)	40 (20)
$CMT_3(5,K_3)$	100 (99)	97 (80)	100 (97)	100 (100)	100 (100)	99 (90)	100 (100)	8 (6)	4 (4)	20 (7)	27 (11)
$CMT_3(3,I_3)$	100 (95)	97 (74)	100 (93)	100 (100)	100 (100)	99 (84)	100 (100)	19 (6)	20 (7)	4 (4)	10 (8)
$CMT_3(5,I_3)$	100 (99)	100 (92)	100 (99)	100 (100)	100 (100)	100 (97)	100 (100)	35 (14)	24 (8)	8 (6)	4 (4)

$n_1=50, n_2=50$	$W_3(3,I_3)$	$IW_3(3,I_3)$	$CMT_3(1,I_3)$	CMU_3	$W_3(3,2I_3)$	$IW_3(5,3I_3)$	$W_3(3,K_3)$	$CMT_3(3,K_3)$	$CMT_3(5,K_3)$	$CMT_3(3,I_3)$	$CMT_3(5,I_3)$
$W_3(3,I_3)$	3 (4)	41 (47)	82 (6)	100 (100)	99 (91)	88 (87)	100 (100)	100 (99)	100 (100)	100 (100)	100 (100)
$IW_3(3,I_3)$		4 (4)	76 (42)	100 (100)	100 (97)	38 (11)	100 (100)	97 (83)	100 (96)	100 (92)	100 (99)
$CMT_3(1,I_3)$			3 (3)	100 (100)	55 (25)	100 (95)	94 (42)	100 (98)	100 (100)	100 (99)	100 (100)
CMU_3				6 (5)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
$W_3(3,2I_3)$					3 (3)	100 (100)	100 (60)	100 (100)	100 (100)	100 (100)	100 (100)
$IW_3(5,3I_3)$						5 (5)	100 (100)	100 (93)	100 (99)	100 (98)	100 (100)
$W_3(3,K_3)$							3 (3)	100 (100)	100 (100)	100 (100)	100 (100)
$CMT_3(3,K_3)$								5 (4)	10 (8)	28 (9)	55 (24)
$CMT_3(5,K_3)$									5 (4)	30 (9)	39 (13)
$CMT_3(3,I_3)$										5 (4)	11 (9)
$CMT_3(5,I_3)$											6 (4)

Table 2.7: The percentage of rejected H_0 for different sample sizes for 2×2 matrices, $\nu = 1, \omega = 2I_d (\nu = 2, \omega = 2I_d)$ - statistic (2.2).

$n_1=20, n_2=20$	$W_2(2.5,I_2)$	$IW_2(2.5,I_2)$	$CMT_2(1,I_2)$	CMU_2	$W_2(2.5,2I_2)$	$IW_2(4,2.5I_2)$	$W_2(2.5,K_2)$	$CMT_2(3,K_2)$	$CMT_2(5,K_2)$	$CMT_2(3,I_2)$	$CMT_2(5,I_2)$
$W_2(2.5,I_2)$	4 (4)	19 (18)	28 (19)	100 (100)	35 (23)	13 (11)	96 (67)	83 (62)	90 (73)	96 (79)	99 (88)
$IW_2(2.5,I_2)$		5 (5)	16 (11)	100 (100)	81 (73)	11 (13)	92 (44)	69 (48)	79 (59)	91 (69)	96 (81)
$CMT_2(1,I_2)$			4 (4)	100 (100)	47 (28)	30 (17)	60 (11)	33 (22)	44 (30)	67 (45)	76 (54)
CMU_2				5 (5)	100 (100)	100 (100)	100 (100)	100 (100)	100 (99)	100 (99)	100 (98)
$W_2(2.5,2I_2)$					4 (4)	92 (86)	100 (95)	95 (73)	99 (84)	100 (89)	100 (93)
$IW_2(4,2.5I_2)$						5 (4)	99 (66)	82 (61)	91 (69)	97 (78)	99 (89)
$W_2(2.5,K_2)$							4 (5)	64 (40)	76 (49)	62 (55)	72 (63)
$CMT_2(3,K_2)$								5 (4)	6 (5)	16 (11)	25 (15)
$CMT_2(5,K_2)$									5 (4)	13 (8)	18 (12)
$CMT_2(3,I_2)$										5 (5)	6 (5)
$CMT_2(5,I_2)$											5 (5)

$n_1=30, n_2=20$	$W_2(2.5,I_2)$	$IW_2(2.5,I_2)$	$CMT_2(1,I_2)$	CMU_2	$W_2(2.5,2I_2)$	$IW_2(4,2.5I_2)$	$W_2(2.5,K_2)$	$CMT_2(3,K_2)$	$CMT_2(5,K_2)$	$CMT_2(3,I_2)$	$CMT_2(5,I_2)$
$W_2(2.5,I_2)$ 4 (4)	27 (27)	41 (29)	100 (100)	38 (24)	14 (12)	99 (84)	93 (79)	97 (87)	99 (92)	100 (96)	
$IW_2(2.5,I_2)$	25 (22)	5 (5)	29 (18)	100 (100)	87 (81)	9 (10)	98 (69)	83 (64)	91 (75)	98 (84)	100 (91)
$CMT_2(1,I_2)$	27 (15)	12 (8)	4 (4)	100 (100)	52 (26)	26 (16)	66 (10)	42 (29)	56 (39)	80 (56)	90 (68)
CMU_2	100 (100)	100 (100)	100 (100)	5 (5)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
$W_2(2.5,2I_2)$	50 (34)	90 (83)	68 (42)	100 (100)	3 (4)	95 (93)	100 (99)	99 (90)	100 (95)	100 (96)	100 (99)
$IW_2(4,2.5I_2)$	23 (20)	19 (18)	51 (30)	100 (100)	97 (94)	5 (5)	100 (90)	93 (77)	97 (85)	100 (91)	100 (95)
$W_2(2.5,K_2)$	99 (79)	97 (52)	83 (21)	100 (100)	100 (98)	100 (75)	5 (5)	83 (57)	89 (66)	75 (68)	85 (78)
$CMT_2(3,K_2)$	88 (69)	73 (52)	37 (22)	100 (100)	99 (84)	89 (66)	74 (42)	5 (5)	6 (6)	22 (13)	32 (21)
$CMT_2(5,K_2)$	95 (77)	84 (63)	48 (34)	100 (100)	100 (91)	95 (76)	82 (55)	5 (5)	5 (5)	17 (9)	25 (15)
$CMT_2(3,I_2)$	99 (86)	96 (78)	74 (50)	100 (100)	100 (94)	99 (86)	69 (58)	19 (11)	14 (7)	5 (5)	6 (6)
$CMT_2(5,I_2)$	100 (94)	99 (87)	85 (63)	100 (100)	100 (97)	100 (94)	81 (69)	29 (18)	22 (11)	5 (6)	5 (5)

$n_1=50, n_2=20$	$W_2(2.5,I_2)$	$IW_2(2.5,I_2)$	$CMT_2(1,I_2)$	CMU_2	$W_2(2.5,2I_2)$	$IW_2(4,2.5I_2)$	$W_2(2.5,K_2)$	$CMT_2(3,K_2)$	$CMT_2(5,K_2)$	$CMT_2(3,I_2)$	$CMT_2(5,I_2)$
$W_2(2.5,I_2)$	4 (4)	32 (34)	59 (43)	100 (100)	44 (24)	13 (12)	100 (96)	99 (91)	100 (96)	100 (98)	100 (99)
$IW_2(2.5,I_2)$	30 (28)	5 (5)	52 (32)	100 (100)	95 (87)	7 (9)	100 (86)	94 (82)	98 (87)	100 (95)	100 (98)
$CMT_2(1,I_2)$	22 (12)	9 (6)	5 (4)	100 (100)	55 (26)	25 (12)	78 (8)	52 (37)	68 (49)	91 (70)	97 (82)
CMU_2	100 (100)	100 (100)	100 (100)	5 (5)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
$W_2(2.5,2I_2)$	63 (52)	96 (91)	86 (62)	100 (100)	4 (4)	98 (97)	100 (100)	100 (97)	100 (99)	100 (99)	100 (100)
$IW_2(4,2.5I_2)$	40 (32)	25 (26)	88 (49)	100 (100)	99 (98)	5 (5)	100 (99)	99 (90)	100 (95)	100 (98)	100 (99)
$W_2(2.5,K_2)$	100 (86)	99 (57)	96 (45)	100 (100)	100 (100)	100 (87)	4 (5)	94 (74)	96 (82)	88 (81)	94 (88)
$CMT_2(3,K_2)$	95 (79)	79 (56)	43 (27)	100 (100)	100 (92)	94 (74)	79 (46)	5 (5)	6 (5)	28 (17)	42 (26)
$CMT_2(5,K_2)$	98 (88)	91 (71)	57 (39)	100 (100)	100 (97)	98 (86)	88 (60)	5 (4)	5 (5)	23 (13)	32 (18)
$CMT_2(3,I_2)$	100 (94)	99 (87)	83 (55)	100 (100)	100 (99)	100 (94)	76 (64)	23 (10)	16 (8)	5 (5)	6 (7)
$CMT_2(5,I_2)$	100 (98)	100 (94)	92 (71)	100 (100)	100 (100)	100 (98)	85 (77)	32 (17)	27 (12)	5 (6)	5 (5)

$n_1=50, n_2=30$	$W_2(2.5,I_2)$	$IW_2(2.5,I_2)$	$CMT_2(1,I_2)$	CMU_2	$W_2(2.5,2I_2)$	$IW_2(4,2.5I_2)$	$W_2(2.5,K_2)$	$CMT_2(3,K_2)$	$CMT_2(5,K_2)$	$CMT_2(3,I_2)$	$CMT_2(5,I_2)$
$W_2(2.5,I_2)$	4 (4)	40 (43)	65 (45)	100 (100)	64 (45)	23 (20)	100 (99)	100 (95)	100 (98)	100 (100)	100 (100)
$IW_2(2.5,I_2)$	38 (39)	5 (5)	56 (29)	100 (100)	99 (97)	15 (15)	100 (95)	97 (87)	99 (94)	100 (98)	100 (99)
$CMT_2(1,I_2)$	43 (23)	23 (14)	5 (4)	100 (100)	83 (47)	55 (26)	95 (19)	63 (43)	76 (59)	97 (80)	99 (90)
CMU_2	100 (100)	100 (100)	100 (100)	5 (4)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
$W_2(2.5,2I_2)$	74 (58)	98 (97)	91 (64)	100 (100)	4 (4)	100 (99)	100 (100)	100 (99)	100 (100)	100 (100)	100 (100)
$IW_2(4,2.5I_2)$	43 (35)	26 (26)	93 (52)	100 (100)	100 (100)	5 (6)	100 (100)	100 (96)	100 (98)	100 (99)	100 (100)
$W_2(2.5,K_2)$	100 (98)	100 (89)	99 (45)	100 (100)	100 (100)	100 (99)	4 (4)	97 (79)	99 (89)	93 (88)	97 (94)
$CMT_2(3,K_2)$	99 (92)	94 (78)	58 (37)	100 (100)	100 (99)	99 (91)	95 (69)	5 (5)	6 (6)	39 (19)	57 (31)
$CMT_2(5,K_2)$	100 (97)	98 (88)	72 (51)	100 (100)	100 (100)	100 (97)	98 (80)	6 (5)	5 (5)	30 (13)	43 (21)
$CMT_2(3,I_2)$	100 (99)	100 (97)	95 (75)	100 (100)	100 (100)	100 (99)	89 (83)	33 (16)	28 (11)	5 (5)	7 (6)
$CMT_2(5,I_2)$	100 (100)	100 (99)	98 (87)	100 (100)	100 (100)	100 (100)	96 (91)	49 (26)	38 (18)	6 (6)	5 (5)

$n_1=50, n_2=50$	$W_2(2.5,I_2)$	$IW_2(2.5,I_2)$	$CMT_2(1,I_2)$	CMU_2	$W_2(2.5,2I_2)$	$IW_2(4,2.5I_2)$	$W_2(2.5,K_2)$	$CMT_2(3,K_2)$	$CMT_2(5,K_2)$	$CMT_2(3,I_2)$	$CMT_2(5,I_2)$
$W_2(2.5,I_2)$	4 (4)	55 (54)	72 (46)	100 (100)	86 (70)	46 (40)	100 (100)	100 (99)	100 (100)	100 (100)	100 (100)
$IW_2(2.5,I_2)$		5 (4)	56 (29)	100 (100)	100 (99)	27 (26)	100 (99)	99 (94)	100 (98)	100 (100)	100 (100)
$CMT_2(1,I_2)$			5 (5)	100 (100)	96 (75)	96 (53)	100 (58)	73 (56)	88 (69)	99 (90)	100 (96)
CMU_2				5 (5)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
$W_2(2.5,2I_2)$					4 (4)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
$IW_2(4,2.5I_2)$						5 (5)	100 (100)	100 (99)	100 (100)	100 (100)	100 (100)
$W_2(2.5,K_2)$							5 (5)	99 (89)	100 (94)	98 (94)	99 (98)
$CMT_2(3,K_2)$								5 (5)	7 (6)	50 (23)	67 (41)
$CMT_2(5,K_2)$									5 (5)	43 (16)	58 (27)
$CMT_2(3,I_2)$										5 (4)	7 (6)
$CMT_2(5,I_2)$											4 (5)

Table 2.8: The percentage of rejected H_0 for different sample sizes for 3×3 matrices, $\nu=1,\omega=2I_d(\nu=2,\omega=2I_d)$ - statistic (2.2).

$n_1=20,n_2=20$	$W_3(3,I_3)$	$IW_3(3,I_3)$	$CMT_3(1,I_3)$	CMU_3	$W_3(3,2I_3)$	$IW_3(5,3I_3)$	$W_3(3,K_3)$	$CMT_3(3,K_3)$	$CMT_3(5,K_3)$	$CMT_3(3,I_3)$	$CMT_3(5,I_3)$
$W_3(3,I_3)$	3 (3)	14 (14)	4 (5)	100 (100)	38 (13)	30 (16)	74 (25)	62 (34)	78 (44)	74 (39)	88 (50)
$IW_3(3,I_3)$		4 (4)	9 (5)	100 (100)	60 (39)	7 (7)	84 (46)	41 (22)	57 (31)	54 (28)	73 (39)
$CMT_3(1,I_3)$			3 (4)	100 (100)	11 (8)	17 (5)	17 (8)	56 (27)	75 (40)	68 (34)	85 (44)
CMU_3				5 (5)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
$W_3(3,2I_3)$					3 (3)	98 (89)	31 (6)	84 (41)	94 (54)	89 (46)	98 (61)
$IW_3(5,3I_3)$						4 (4)	100 (98)	51 (28)	68 (37)	64 (31)	81 (44)
$W_3(3,K_3)$							3 (4)	89 (42)	96 (55)	91 (48)	98 (60)
$CMT_3(3,K_3)$								3 (4)	4 (4)	5 (4)	10 (7)
$CMT_3(5,K_3)$									3 (4)	6 (4)	6 (4)
$CMT_3(3,I_3)$										3 (4)	4 (4)
$CMT_3(5,I_3)$											3 (4)

$n_1=30,n_2=20$	$W_3(3,I_3)$	$IW_3(3,I_3)$	$CMT_3(1,I_3)$	CMU_3	$W_3(3,2I_3)$	$IW_3(5,3I_3)$	$W_3(3,K_3)$	$CMT_3(3,K_3)$	$CMT_3(5,K_3)$	$CMT_3(3,I_3)$	$CMT_3(5,I_3)$
$W_3(3,I_3)$	3 (3)	21 (20)	9 (8)	100 (100)	27 (9)	21 (20)	82 (19)	82 (54)	93 (66)	91 (59)	97 (73)
$IW_3(3,I_3)$	15 (12)	4 (4)	16 (8)	100 (100)	71 (41)	5 (4)	89 (49)	61 (40)	80 (49)	77 (45)	90 (59)
$CMT_3(1,I_3)$	3 (3)	10 (5)	3 (4)	100 (100)	7 (4)	14 (3)	12 (5)	77 (45)	90 (59)	85 (51)	96 (66)
CMU_3	100	100 (100)	100 (100)	5 (5)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
$W_3(3,2I_3)$	51 (27)	78 (58)	22 (13)	100 (100)	3 (3)	99 (96)	25 (4)	96 (63)	99 (79)	98 (69)	100 (84)
$IW_3(5,3I_3)$	36 (29)	14 (12)	55 (10)	100 (100)	100 (96)	4 (4)	100 (100)	74 (43)	88 (58)	86 (54)	94 (70)
$W_3(3,K_3)$	93 (44)	94 (66)	34 (15)	100 (100)	44 (11)	100 (100)	3 (3)	98 (71)	100 (82)	99 (74)	100 (85)
$CMT_3(3,K_3)$	67 (29)	42 (20)	63 (25)	100 (100)	89 (42)	54 (26)	94 (47)	3 (4)	6 (5)	6 (6)	14 (8)
$CMT_3(5,K_3)$	83 (46)	59 (31)	81 (41)	100 (100)	98 (58)	73 (37)	99 (61)	4 (4)	3 (3)	6 (4)	8 (6)
$CMT_3(3,I_3)$	78 (38)	58 (25)	77 (34)	100 (100)	95 (46)	69 (32)	97 (51)	6 (4)	6 (4)	3 (4)	6 (5)
$CMT_3(5,I_3)$	92 (52)	79 (39)	91 (45)	100 (100)	99 (65)	87 (50)	100 (68)	9 (6)	7 (4)	4 (4)	3 (4)

$n_1=50,n_2=20$	$W_3(3,I_3)$	$IW_3(3,I_3)$	$CMT_3(1,I_3)$	CMU_3	$W_3(3,2I_3)$	$IW_3(5,3I_3)$	$W_3(3,K_3)$	$CMT_3(3,K_3)$	$CMT_3(5,K_3)$	$CMT_3(3,I_3)$	$CMT_3(5,I_3)$
$W_3(3,I_3)$	3 (4)	26 (30)	17 (14)	100 (100)	28 (5)	24 (23)	91 (14)	95 (76)	99 (89)	98 (83)	100 (92)
$IW_3(3,I_3)$	15 (11)	4 (4)	30 (12)	100 (100)	77 (43)	3 (3)	97 (58)	82 (60)	94 (73)	92 (67)	98 (82)
$CMT_3(1,I_3)$	2 (3)	11 (4)	4 (4)	100 (100)	3 (3)	13 (3)	7 (3)	91 (64)	98 (80)	97 (75)	99 (86)
CMU_3	100 (100)	100 (100)	100 (100)	5 (5)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
$W_3(3,2I_3)$	74 (44)	91 (79)	43 (26)	100 (100)	3 (4)	100 (99)	25 (4)	100 (87)	100 (95)	100 (90)	100 (97)
$IW_3(5,3I_3)$	66 (49)	27 (20)	96 (27)	100 (100)	100 (99)	4 (4)	100 (100)	93 (70)	98 (82)	97 (78)	99 (90)
$W_3(3,K_3)$	99 (75)	99 (85)	61 (30)	100 (100)	68 (21)	100 (100)	3 (4)	100 (88)	100 (97)	100 (91)	100 (97)
$CMT_3(3,K_3)$	73 (29)	42 (15)	72 (27)	100 (100)	96 (43)	56 (22)	99 (49)	4 (4)	7 (6)	11 (7)	20 (11)
$CMT_3(5,K_3)$	93 (49)	65 (27)	90 (43)	100 (100)	100 (64)	81 (37)	100 (68)	4 (3)	4 (4)	8 (5)	12 (8)
$CMT_3(3,I_3)$	87 (39)	64 (21)	85 (34)	100 (100)	99 (53)	76 (31)	100 (56)	6 (4)	9 (5)	4 (4)	7 (7)
$CMT_3(5,I_3)$	98 (60)	82 (41)	96 (59)	100 (100)	100 (74)	94 (46)	100 (75)	11 (5)	7 (4)	4 (4)	5 (4)

$n_1=50,n_2=30$	$W_3(3,I_3)$	$IW_3(3,I_3)$	$CMT_3(1,I_3)$	CMU_3	$W_3(3,2I_3)$	$IW_3(5,3I_3)$	$W_3(3,K_3)$	$CMT_3(3,K_3)$	$CMT_3(5,K_3)$	$CMT_3(3,I_3)$	$CMT_3(5,I_3)$
$W_3(3,I_3)$	3 (3)	34 (34)	17 (11)	100 (100)	62 (20)	43 (39)	100 (48)	98 (81)	100 (93)	99 (89)	100 (96)
$IW_3(3,I_3)$	27 (23)	5 (4)	35 (13)	100 (100)	93 (69)	7 (6)	100 (83)	87 (63)	96 (78)	96 (71)	100 (85)
$CMT_3(1,I_3)$	3 (3)	21 (8)	3 (4)	100 (100)	10 (4)	50 (4)	25 (7)	96 (70)	99 (88)	99 (81)	100 (92)
CMU_3	100 (100)	100 (100)	100 (100)	5 (5)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
$W_3(3,2I_3)$	82 (49)	95 (82)	40 (21)	100 (100)	3 (3)	100 (100)	66 (5)	100 (92)	100 (98)	100 (94)	100 (99)
$IW_3(5,3I_3)$	73 (55)	25 (17)	96 (25)	100 (100)	100 (100)	5 (5)	100 (100)	95 (74)	99 (88)	99 (84)	100 (93)
$W_3(3,K_3)$	100 (84)	100 (91)	60 (25)	100 (100)	86 (20)	100 (100)	3 (4)	100 (93)	100 (98)	100 (96)	100 (100)
$CMT_3(3,K_3)$	94 (58)	70 (38)	91 (52)	100 (100)	100 (78)	85 (48)	100 (82)	4 (4)	7 (6)	12 (7)	22 (12)
$CMT_3(5,K_3)$	99 (81)	90 (57)	98 (73)	100 (100)	100 (92)	97 (71)	100 (95)	5 (4)	4 (4)	9 (5)	14 (7)
$CMT_3(3,I_3)$	98 (70)	87 (46)	97 (62)	100 (100)	100 (84)	95 (60)	100 (86)	9 (4)	10 (5)	4 (4)	8 (6)
$CMT_3(5,I_3)$	100 (88)	97 (70)	100 (83)	100 (100)	100 (95)	99 (81)	100 (97)	18 (8)	12 (5)	5 (4)	4 (4)

$n_1=50,n_2=50$	$W_3(3,I_3)$	$IW_3(3,I_3)$	$CMT_3(1,I_3)$	CMU_3	$W_3(3,2I_3)$	$IW_3(5,3I_3)$	$W_3(3,K_3)$	$CMT_3(3,K_3)$	$CMT_3(5,K_3)$	$CMT_3(3,I_3)$	$CMT_3(5,I_3)$
$W_3(3,I_3)$	3 (3)	42 (37)	11 (6)	100 (100)	91 (52)	76 (67)	100 (89)	100 (90)	100 (97)	100 (94)	100 (99)
$IW_3(3,I_3)$		4 (4)	41 (15)	100 (100)	99 (91)	20 (14)	100 (97)	93 (66)	99 (85)	99 (79)	100 (94)
$CMT_3(1,I_3)$			3 (4)	100 (100)	34 (15)	98 (18)	63 (19)	99 (81)	100 (94)	100 (89)	100 (98)
CMU_3				5 (5)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
$W_3(3,2I_3)$					3 (3)	100 (100)	95 (17)	100 (97)	100 (100)	100 (98)	100 (100)
$IW_3(5,3I_3)$						5 (5)	100 (100)	98 (80)	100 (93)	100 (89)	100 (97)
$W_3(3,K_3)$							3 (3)	100 (98)	100 (100)	100 (99)	100 (100)
$CMT_3(3,K_3)$								4 (4)	8 (6)	15 (6)	32 (14)
$CMT_3(5,K_3)$									4 (4)	13 (6)	20 (7)
$CMT_3(3,I_3)$										4 (3)	8 (6)
$CMT_3(5,I_3)$											4 (3)

Table 2.9: The percentage of rejected H_0 for different sample sizes for 2×2 matrices, $\nu=5,\omega=I_d(\nu=5,\omega=2I_d)$ - statistic (2.2).

$n_1=20,n_2=20$	$W_2(2.5,I_2)$	$IW_2(2.5,I_2)$	$CMT_2(1,I_2)$	CMU_2	$W_2(2.5,2I_2)$	$IW_2(4,2.5I_2)$	$W_2(2.5,K_2)$	$CMT_2(3,K_2)$	$CMT_2(5,K_2)$	$CMT_2(3,I_2)$	$CMT_2(5,I_2)$
$W_2(2.5,I_2)$	5 (6)	16 (12)	12 (13)	100 (100)	10 (8)	6 (5)	17 (12)	36 (28)	41 (33)	56 (41)	60 (50)
$IW_2(2.5,I_2)$		5 (4)	11 (11)	100 (100)	45 (31)	14 (14)	8 (8)	30 (23)	37 (28)	49 (40)	55 (47)
$CMT_2(1,I_2)$			4 (4)	100 (99)	16 (15)	14 (14)	7 (8)	13 (10)	17 (13)	25 (22)	36 (28)
CMU_2				5 (5)	100 (100)	100 (100)	100 (100)	96 (90)	94 (86)	87 (74)	82 (66)
$W_2(2.5,2I_2)$					6 (6)	49 (38)	29 (20)	39 (31)	47 (35)	58 (48)	66 (52)
$IW_2(4,2.5I_2)$						5 (5)	16 (14)	35 (28)	43 (33)	53 (44)	60 (50)
$W_2(2.5,K_2)$							5 (5)	26 (20)	30 (24)	42 (35)	51 (41)
$CMT_2(3,K_2)$								5 (4)	5 (5)	8 (8)	12 (11)
$CMT_2(5,K_2)$									5 (4)	6 (7)	9 (8)
$CMT_2(3,I_2)$										5 (5)	5 (5)
$CMT_2(5,I_2)$											5 (4)

$n_1=30,n_2=20$	$W_2(2.5,I_2)$	$IW_2(2.5,I_2)$	$CMT_2(1,I_2)$	CMU_2	$W_2(2.5,2I_2)$	$IW_2(4,2.5I_2)$	$W_2(2.5,K_2)$	$CMT_2(3,K_2)$	$CMT_2(5,K_2)$	$CMT_2(3,I_2)$	$CMT_2(5,I_2)$
$W_2(2.5,I_2)$	5 (6)	19 (14)	21 (18)	100 (100)	8 (6)	5 (4)	24 (19)	52 (41)	58 (47)	69 (60)	76 (65)
$IW_2(2.5,I_2)$	18 (12)	4 (4)	16 (17)	100 (100)	49 (33)	12 (11)	12 (12)	45 (40)	50 (45)	64 (54)	70 (63)
$CMT_2(1,I_2)$	10 (9)	7 (8)	4 (4)	100 (100)	14 (11)	11 (9)	5 (6)	19 (15)	23 (18)	37 (30)	45 (37)
CMU_2	100 (100)	100 (100)	100 (100)	5 (5)	100 (100)	100 (100)	100 (100)	99 (93)	97 (92)	92 (82)	90 (78)
$W_2(2.5,2I_2)$	15 (12)	57 (47)	25 (21)	100 (100)	6 (6)	62 (46)	40 (31)	57 (47)	64 (49)	73 (62)	80 (70)
$IW_2(4,2.5I_2)$	10 (7)	20 (21)	23 (20)	100 (100)	60 (45)	4 (5)	26 (23)	52 (41)	59 (47)	70 (60)	77 (67)
$W_2(2.5,K_2)$	14 (10)	5 (5)	11 (12)	100 (100)	25 (15)	13 (10)	4 (4)	37 (32)	43 (36)	56 (51)	64 (56)
$CMT_2(3,K_2)$	35 (29)	30 (24)	13 (10)	99 (95)	44 (31)	37 (29)	23 (20)	4 (5)	5 (5)	11 (11)	15 (13)
$CMT_2(5,K_2)$	45 (34)	35 (27)	17 (13)	98 (94)	53 (37)	44 (33)	30 (24)	5 (4)	5 (5)	9 (8)	12 (12)
$CMT_2(3,I_2)$	59 (48)	51 (40)	31 (24)	94 (83)	65 (46)	60 (45)	44 (36)	8 (7)	7 (5)	4 (5)	6 (5)
$CMT_2(5,I_2)$	67 (54)	61 (50)	36 (28)	89 (78)	73 (57)	69 (54)	55 (45)	11 (10)	9 (8)	5 (5)	5 (5)

$n_1=50,n_2=20$	$W_2(2.5,I_2)$	$IW_2(2.5,I_2)$	$CMT_2(1,I_2)$	CMU_2	$W_2(2.5,2I_2)$	$IW_2(4,2.5I_2)$	$W_2(2.5,K_2)$	$CMT_2(3,K_2)$	$CMT_2(5,K_2)$	$CMT_2(3,I_2)$	$CMT_2(5,I_2)$
$W_2(2.5,I_2)$	5 (5)	26 (17)	32 (29)	100 (100)	5 (4)	4 (3)	37 (27)	69 (60)	77 (65)	85 (77)	89 (81)
$IW_2(2.5,I_2)$	19 (13)	4 (5)	25 (26)	100 (100)	52 (37)	8 (9)	17 (18)	61 (54)	69 (62)	80 (73)	86 (78)
$CMT_2(1,I_2)$	8 (5)	5 (4)	4 (4)	100 (100)	10 (16)	6 (7)	4 (5)	23 (20)	31 (24)	47 (41)	59 (50)
CMU_2	100 (100)	100 (100)	100 (100)	5 (5)	100 (100)	100 (100)	100 (100)	100 (100)	99 (96)	97 (89)	95 (84)
$W_2(2.5,2I_2)$	20 (15)	73 (62)	38 (33)	100 (100)	5 (6)	73 (51)	65 (48)	76 (65)	83 (70)	88 (79)	92 (83)
$IW_2(4,2.5I_2)$	13 (11)	29 (31)	34 (31)	100 (100)	72 (57)	5 (5)	43 (35)	72 (60)	75 (66)	85 (77)	89 (82)
$W_2(2.5,K_2)$	13 (8)	4 (5)	17 (18)	100 (100)	26 (14)	10 (7)	4 (5)	51 (46)	58 (52)	71 (66)	79 (72)
$CMT_2(3,K_2)$	37 (28)	29 (22)	12 (8)	100 (99)	46 (30)	39 (27)	23 (18)	5 (4)	5 (6)	14 (14)	19 (18)
$CMT_2(5,K_2)$	47 (32)	36 (28)	17 (12)	100 (98)	56 (36)	48 (33)	31 (23)	4 (4)	4 (5)	11 (10)	16 (15)
$CMT_2(3,I_2)$	68 (48)	57 (46)	34 (23)	99 (92)	72 (57)	66 (53)	49 (38)	8 (8)	6 (5)	5 (5)	6 (6)
$CMT_2(5,I_2)$	79 (60)	70 (57)	43 (32)	97 (87)	84 (64)	79 (62)	59 (47)	11 (11)	9 (7)	5 (4)	5 (5)

$n_1=50,n_2=30$	$W_2(2.5,I_2)$	$IW_2(2.5,I_2)$	$CMT_2(1,I_2)$	CMU_2	$W_2(2.5,2I_2)$	$IW_2(4,2.5I_2)$	$W_2(2.5,K_2)$	$CMT_2(3,K_2)$	$CMT_2(5,K_2)$	$CMT_2(3,I_2)$	$CMT_2(5,I_2)$
$W_2(2.5,I_2)$	5 (5)	30 (21)	32 (29)	100 (100)	10 (7)	5 (4)	44 (29)	76 (64)	82 (72)	91 (83)	95 (89)
$IW_2(2.5,I_2)$	25 (19)	5 (5)	23 (23)	100 (100)	73 (60)	18 (16)	19 (17)	65 (58)	74 (65)	85 (79)	92 (86)
$CMT_2(1,I_2)$	16 (12)	9 (9)	4 (4)	100 (100)	23 (16)	15 (14)	6 (5)	29 (21)	34 (27)	58 (46)	67 (58)
CMU_2	100 (100)	100 (100)	100 (100)	5 (5)	100 (100)	100 (100)	100 (100)	100 (100)	100 (99)	100 (97)	99 (95)
$W_2(2.5,2I_2)$	20 (14)	82 (71)	39 (32)	100 (100)	6 (6)	85 (69)	73 (49)	81 (71)	86 (74)	93 (85)	97 (90)
$IW_2(4,2.5I_2)$	14 (8)	29 (29)	34 (29)	100 (100)	87 (75)	5 (5)	44 (35)	76 (64)	83 (73)	92 (84)	95 (89)
$W_2(2.5,K_2)$	28 (17)	8 (7)	14 (16)	100 (100)	49 (28)	22 (17)	4 (4)	54 (49)	63 (57)	77 (71)	85 (79)
$CMT_2(3,K_2)$	63 (46)	51 (40)	20 (17)	100 (100)	68 (53)	62 (45)	42 (30)	5 (4)	6 (5)	16 (14)	23 (19)
$CMT_2(5,K_2)$	72 (58)	63 (49)	28 (20)	100 (100)	80 (62)	73 (58)	50 (38)	5 (4)	5 (4)	12 (10)	17 (16)
$CMT_2(3,I_2)$	86 (74)	80 (68)	46 (37)	100 (98)	90 (77)	86 (75)	70 (60)	10 (10)	9 (8)	5 (5)	6 (5)
$CMT_2(5,I_2)$	93 (81)	87 (78)	60 (47)	100 (96)	95 (85)	93 (83)	79 (68)	17 (14)	14 (11)	5 (5)	5 (5)

$n_1=50,n_2=50$	$W_2(2.5,I_2)$	$IW_2(2.5,I_2)$	$CMT_2(1,I_2)$	CMU_2	$W_2(2.5,2I_2)$	$IW_2(4,2.5I_2)$	$W_2(2.5,K_2)$	$CMT_2(3,K_2)$	$CMT_2(5,K_2)$	$CMT_2(3,I_2)$	$CMT_2(5,I_2)$
$W_2(2.5,I_2)$	5 (5)	41 (29)	30 (25)	100 (100)	19 (12)	11 (7)	48 (28)	85 (74)	91 (79)	97 (91)	99 (95)
$IW_2(2.5,I_2)$		4 (5)	21 (20)	100 (10)	91 (82)	30 (32)	16 (14)	74 (65)	84 (74)	93 (89)	96 (93)
$CMT_2(1,I_2)$			5 (4)	100 (100)	43 (29)	32 (27)	14 (13)	33 (25)	41 (30)	65 (54)	79 (67)
CMU_2				5 (5)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (99)
$W_2(2.5,2I_2)$					5 (6)	96 (89)	80 (54)	89 (77)	94 (85)	98 (92)	99 (96)
$IW_2(4,2.5I_2)$						5 (5)	46 (32)	84 (72)	90 (82)	97 (93)	99 (95)
$W_2(2.5,K_2)$							5 (4)	60 (52)	74 (64)	87 (80)	92 (86)
$CMT_2(3,K_2)$								5 (5)	5 (5)	17 (15)	24 (22)
$CMT_2(5,K_2)$									5 (5)	11 (11)	18 (17)
$CMT_2(3,I_2)$										5 (4)	6 (5)
$CMT_2(5,I_2)$											5 (4)

Table 2.10: The percentage of rejected H_0 for different sample sizes for 3×3 matrices, $\nu = 5, \omega = I_d (\nu = 5, \omega = 2I_d)$.

$n_1=20, n_2=20$	$W_3(3,I_3)$	$IW_3(3,I_3)$	$CMT_3(1,I_3)$	CMU_3	$W_3(3,2I_3)$	$IW_3(5,3I_3)$	$W_3(3,K_3)$	$CMT_3(3,K_3)$	$CMT_3(5,K_3)$	$CMT_3(3,I_3)$	$CMT_3(5,I_3)$
$W_3(3,I_3)$	5 (6)	11 (9)	6 (8)	100 (100)	7 (7)	11 (8)	8 (7)	13 (10)	14 (12)	14 (9)	15 (12)
$IW_3(3,I_3)$		4 (5)	6 (6)	100 (100)	15 (12)	7 (8)	14 (12)	10 (9)	11 (10)	12 (9)	13 (11)
$CMT_3(1,I_3)$			6 (6)	100 (100)	7 (7)	7 (6)	8 (7)	10 (9)	13 (10)	12 (9)	13 (10)
CMU_3				6 (5)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
$W_3(3,2I_3)$					6 (6)	36 (25)	6 (7)	13 (10)	16 (11)	14 (11)	17 (12)
$IW_3(5,3I_3)$						4 (4)	41 (28)	12 (9)	13 (10)	13 (10)	15 (11)
$W_3(3,K_3)$							6 (6)	13 (10)	16 (10)	14 (10)	17 (12)
$CMT_3(3,K_3)$								5 (5)	5 (5)	4 (5)	5 (5)
$CMT_3(5,K_3)$									5 (5)	5 (5)	5 (5)
$CMT_3(3,I_3)$										4 (5)	5 (5)
$CMT_3(5,I_3)$											4 (5)

$n_1=30, n_2=20$	$W_3(3,I_3)$	$IW_3(3,I_3)$	$CMT_3(1,I_3)$	CMU_3	$W_3(3,2I_3)$	$IW_3(5,3I_3)$	$W_3(3,K_3)$	$CMT_3(3,K_3)$	$CMT_3(5,K_3)$	$CMT_3(3,I_3)$	$CMT_3(5,I_3)$
$W_3(3,I_3)$	5 (6)	16 (13)	9 (8)	100 (100)	5 (4)	10 (5)	5 (5)	20 (16)	24 (18)	22 (16)	24 (18)
$IW_3(3,I_3)$	8 (6)	4 (5)	6 (7)	100 (100)	12 (8)	5 (5)	12 (9)	16 (14)	21 (16)	19 (14)	22 (17)
$CMT_3(1,I_3)$	5 (5)	5 (4)	6 (6)	100 (100)	5 (5)	4 (4)	5 (5)	17 (13)	20 (14)	17 (14)	21 (16)
CMU_3	100 (100)	100 (100)	100 (100)	5 (5)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
$W_3(3,2I_3)$	11 (10)	23 (18)	11 (10)	100 (100)	6 (6)	51 (36)	5 (5)	21 (16)	25 (18)	22 (15)	28 (21)
$IW_3(5,3I_3)$	13 (9)	12 (13)	9 (9)	100 (100)	37 (24)	5 (5)	44 (26)	20 (15)	22 (17)	20 (16)	25 (17)
$W_3(3,K_3)$	12 (11)	24 (18)	11 (10)	100 (100)	8 (8)	59 (46)	6 (6)	21 (16)	26 (17)	23 (17)	25 (18)
$CMT_3(3,K_3)$	10 (6)	7 (7)	8 (6)	100 (100)	12 (7)	10 (7)	11 (8)	4 (5)	5 (5)	5 (6)	6 (7)
$CMT_3(5,K_3)$	12 (7)	10 (7)	9 (7)	100 (100)	12 (9)	10 (8)	11 (8)	4 (5)	5 (5)	6 (5)	5 (6)
$CMT_3(3,I_3)$	9 (6)	8 (6)	9 (7)	100 (100)	10 (7)	10 (7)	11 (8)	5 (5)	4 (4)	5 (5)	5 (6)
$CMT_3(5,I_3)$	14 (8)	10 (7)	11 (7)	100 (100)	13 (9)	12 (9)	13 (8)	5 (5)	4 (5)	4 (5)	5 (5)

$n_1=50, n_2=20$	$W_3(3,I_3)$	$IW_3(3,I_3)$	$CMT_3(1,I_3)$	CMU_3	$W_3(3,2I_3)$	$IW_3(5,3I_3)$	$W_3(3,K_3)$	$CMT_3(3,K_3)$	$CMT_3(5,K_3)$	$CMT_3(3,I_3)$	$CMT_3(5,I_3)$
$W_3(3,I_3)$	5 (5)	25 (19)	13 (12)	100 (100)	4 (4)	12 (5)	4 (4)	32 (25)	40 (30)	35 (27)	42 (31)
$IW_3(3,I_3)$	5 (4)	4 (4)	7 (8)	100 (100)	8 (5)	3 (3)	8 (5)	26 (23)	34 (26)	30 (25)	36 (27)
$CMT_3(1,I_3)$	3 (3)	4 (3)	5 (5)	100 (100)	3 (3)	3 (3)	3 (3)	26 (18)	32 (22)	29 (22)	36 (25)
CMU_3	100 (100)	100 (100)	100 (100)	5 (5)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
$W_3(3,2I_3)$	15 (14)	36 (31)	15 (14)	100 (100)	5 (5)	71 (55)	3 (3)	35 (26)	42 (29)	39 (28)	46 (31)
$IW_3(5,3I_3)$	13 (9)	20 (20)	13 (13)	100 (100)	40 (26)	5 (4)	43 (28)	32 (25)	40 (28)	33 (27)	42 (31)
$W_3(3,K_3)$	18 (18)	38 (31)	16 (13)	100 (100)	10 (11)	78 (66)	5 (5)	35 (26)	43 (28)	38 (27)	43 (31)
$CMT_3(3,K_3)$	6 (4)	4 (3)	5 (4)	100 (100)	6 (4)	5 (4)	7 (4)	5 (5)	5 (6)	6 (7)	7 (7)
$CMT_3(5,K_3)$	7 (4)	6 (3)	7 (4)	100 (100)	7 (5)	6 (5)	7 (4)	4 (4)	4 (5)	4 (6)	5 (6)
$CMT_3(3,I_3)$	6 (4)	5 (4)	6 (4)	100 (100)	7 (4)	6 (5)	7 (5)	4 (4)	4 (4)	4 (4)	5 (5)
$CMT_3(5,I_3)$	9 (5)	7 (4)	7 (5)	100 (100)	9 (5)	8 (4)	10 (6)	4 (3)	4 (3)	4 (4)	4 (5)

$n_1=50, n_2=30$	$W_3(3,I_3)$	$IW_3(3,I_3)$	$CMT_3(1,I_3)$	CMU_3	$W_3(3,2I_3)$	$IW_3(5,3I_3)$	$W_3(3,K_3)$	$CMT_3(3,K_3)$	$CMT_3(5,K_3)$	$CMT_3(3,I_3)$	$CMT_3(5,I_3)$
$W_3(3,I_3)$	5 (5)	24 (18)	11 (10)	100 (100)	5 (4)	15 (6)	5 (5)	32 (21)	39 (26)	34 (23)	41 (29)
$IW_3(3,I_3)$	11 (9)	4 (5)	7 (8)	100 (100)	19 (12)	6 (5)	18 (10)	25 (19)	30 (22)	28 (20)	34 (26)
$CMT_3(1,I_3)$	4 (4)	4 (4)	5 (6)	100 (100)	4 (5)	4 (5)	5 (5)	24 (17)	32 (20)	28 (20)	35 (25)
CMU_3	100 (100)	100 (100)	100 (100)	5 (5)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
$W_3(3,2I_3)$	14 (11)	38 (26)	12 (10)	100 (100)	5 (6)	80 (63)	5 (5)	33 (22)	38 (27)	37 (24)	45 (27)
$IW_3(5,3I_3)$	19 (10)	18 (16)	10 (10)	100 (100)	67 (44)	4 (4)	72 (53)	30 (22)	37 (26)	31 (22)	40 (27)
$W_3(3,K_3)$	16 (11)	36 (29)	13 (11)	100 (100)	9 (8)	85 (74)	6 (6)	32 (22)	41 (25)	35 (24)	44 (26)
$CMT_3(3,K_3)$	12 (8)	9 (6)	11 (7)	100 (100)	15 (8)	12 (8)	14 (8)	4 (5)	5 (5)	6 (6)	7 (7)
$CMT_3(5,K_3)$	16 (9)	13 (9)	14 (9)	100 (100)	19 (10)	14 (9)	17 (9)	4 (4)	4 (5)	5 (5)	7 (6)
$CMT_3(3,I_3)$	15 (8)	10 (8)	12 (8)	100 (100)	15 (8)	15 (9)	15 (9)	4 (4)	4 (4)	4 (5)	6 (5)
$CMT_3(5,I_3)$	21 (11)	16 (10)	16 (10)	100 (100)	22 (11)	20 (13)	22 (12)	4 (4)	4 (4)	4 (4)	4 (5)

$n_1=50, n_2=50$	$W_3(3,I_3)$	$IW_3(3,I_3)$	$CMT_3(1,I_3)$	CMU_3	$W_3(3,2I_3)$	$IW_3(5,3I_3)$	$W_3(3,K_3)$	$CMT_3(3,K_3)$	$CMT_3(5,K_3)$	$CMT_3(3,I_3)$	$CMT_3(5,I_3)$
$W_3(3,I_3)$	5 (5)	22 (17)	7 (7)	100 (100)	8 (7)	24 (11)	10 (7)	28 (18)	38 (21)	33 (18)	42 (25)
$IW_3(3,I_3)$		4 (4)	6 (6)	100 (100)	37 (26)	12 (13)	39 (27)	21 (15)	26 (19)	25 (17)	33 (20)
$CMT_3(1,I_3)$			5 (6)	100 (100)	9 (8)	6 (7)	9 (8)	23 (14)	30 (18)	27 (17)	35 (21)
CMU_3				5 (5)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
$W_3(3,2I_3)$					6 (5)	89 (72)	7 (6)	31 (17)	45 (23)	35 (22)	48 (26)
$IW_3(5,3I_3)$						4 (5)	95 (83)	28 (17)	38 (21)	30 (18)	43 (22)
$W_3(3,K_3)$							6 (6)	30 (16)	40 (22)	35 (20)	48 (27)
$CMT_3(3,K_3)$								5 (4)	5 (4)	5 (5)	7 (6)
$CMT_3(5,K_3)$									4 (5)	4 (4)	5 (5)
$CMT_3(3,I_3)$										4 (4)	5 (4)
$CMT_3(5,I_3)$											4 (4)

2.5 Application of novel tests to real data

In this section, we explore the applicability of tests (2.4) and (2.2) to real data, facilitating their real-world application. The test (2.4) is applied to several financial datasets, while the test (2.2) is applied to the insurance dataset as well.

2.5.1 Financial application

Recent studies have used matrix-valued distributions to model stock market data [54, 55], with the first application to cryptocurrency data detailed in [84]. The extreme volatility of cryptocurrency markets [82] requires that traders be aware of significant changes in asset statistical properties to mitigate risks.

The characteristics of logarithmic returns are particularly intriguing when analyzing the cryptocurrency market [16, 26, 113]. The correlation structure of major cryptocurrency prices was examined in [76].

We analyzed hourly data for Bitcoin (BTC) and Ethereum (ETH), sourced from Gemini (`http://www.gemini.com`). Two periods were selected: January 1, 2019, to March 1, 2019 (no significant market changes; see Figures 2.3 and 2.4), and April 1, 2021, to June 1, 2021 (positive bubble followed by a negative bubble [122]). The first period comprises $n_1 = 1416$ data points, while the second has $n_2 = 1464$.

The hourly closing prices X_t were partitioned into daily periods of 24 observations. We calculated hourly logarithmic returns $\log(\frac{X_t}{X_{t-1}})$ and computed a 2×2 unnormalized covariance matrix for each day.

For the first period, 59 covariance matrices were computed (31 for January, 28 for February). The p-value of the statistic $K_{31,28}$ was 0.9756, indicating that there was no significant change in the covariance structure of hourly logarithmic returns from January to March 2019.

For the second period, 61 2×2 matrices were obtained (31 for April, 30 for May). Using bootstrap with $N = 10000$ replications, the p-value of the statistic $K_{31,30}$ was 0.0003, indicating a significant change in the covariance structure from April to May 2021.

To explore more practical scenarios, we studied the covariance structure of hourly logarithmic returns 15 days before and after well-documented Bitcoin price drops coinciding with significant historical events [40]. Each period consists of $n = 720$ data points, with the results presented in Table 2.11. In most cases, the test failed to detect changes in the covariance structure, likely due to the high volatility in the cryptocurrency market and the 15-day period being too large.

We then investigated smaller time scales, analyzing 1-minute BTC [146] and ETH [73] data over two-day periods. We computed covariance matrices for each hour, resulting in $n = 48$ matrices (24 per day). The results in Table 2.12 show that the test (2.4) detected significant differences in the distribution of the covariance matrix on the day and the day after the event, but not on the day before or on the day of the event.

We also used the test (2.2) with $\nu = 1$ and $\Sigma = I_2$, as it is the most powerful for these parameters. The results in Table 2.13 show that this test consistently identified significant differences in the distribution of the covariance matrix the day before and the day of the event in most cases.

In real financial applications, these tests could be performed on new hourly data, allowing for portfolio adjustments if significant changes are detected. It is expected for both families of tests to detect similar events. False detection is expected, but the number of falsely detected changes is expected to be relatively low. This approach could be particularly valuable for implementing stop losses [68], potentially protecting traders' capital during significant market crashes.

Table 2.11: p-values of testing the change in covariance structure before and after the Bitcoin important events - 1 hour data, statistic (2.4)

Period I start date	Period II start date	Date of event (T_0)	Period II end date	Event description	$p_{[T_0-30D,T_0]}$	$p_{[T_0-15D,T_0+15D]}$
9 October 2017	24 October 2017	8 November 2017	23 November 2017	Developers cancel splitting of Bitcoin.	0.31	0.05
28 November 2017	13 December 2017	28 December 2017	12 January 2018	South Korea announces strong measures to regulate trading of cryptocurrencies.	0.32	0.27
14 December 2017	28 December 2017	13 January 2018	28 January 2018	Announcement that 80% of Bitcoin has been mined.	0.21	0.22
31 December 2017	15 January 2018	30 January 2018	14 February 2018	Facebook bans advertisements promoting, cryptocurrencies.	0.17	0.82
5 February 2018	20 February 2018	7 March 2018	22 March 2018	The US Securities and Exchange Commission says it is necessary for crypto trading platforms to register.	0.03	0.01
12 February 2018	28 February 2018	14 March 2018	29 March 2018	Google bans advertisements promoting cryptocurrencies.	0.50	0.34

Table 2.12: p-values of testing the change in covariance structure before and after the Bitcoin important events - 1 minute data, statistic (2.4)

Date of event (T_0)	Event description	$p_{[T_0-2D,T_0-1D]}$	$p_{[T_0-1D,T_0]}$	$p_{[T_0,T_0+1D]}$
8 November 2017	Developers cancel splitting of Bitcoin.	0.0523	0.0832	0.2049
28 December 2017	South Korea announces strong measures to regulate the trading of cryptocurrencies.	0.5889	0.0027	0.0080
13 January 2018	Announcement that 80% of Bitcoin has been mined.	0.0300	0.0035	0.0493
30 January 2018	Facebook bans advertisements promoting cryptocurrencies.	0.8224	0.0352	0.7774
7 March 2018	The US Securities and Exchange Commission says it is necessary for crypto trading platforms to register.	0.0029	0.0225	0.6398
14 March 2018	Google bans advertisements promoting cryptocurrencies.	0.403	0.3019	0.0453

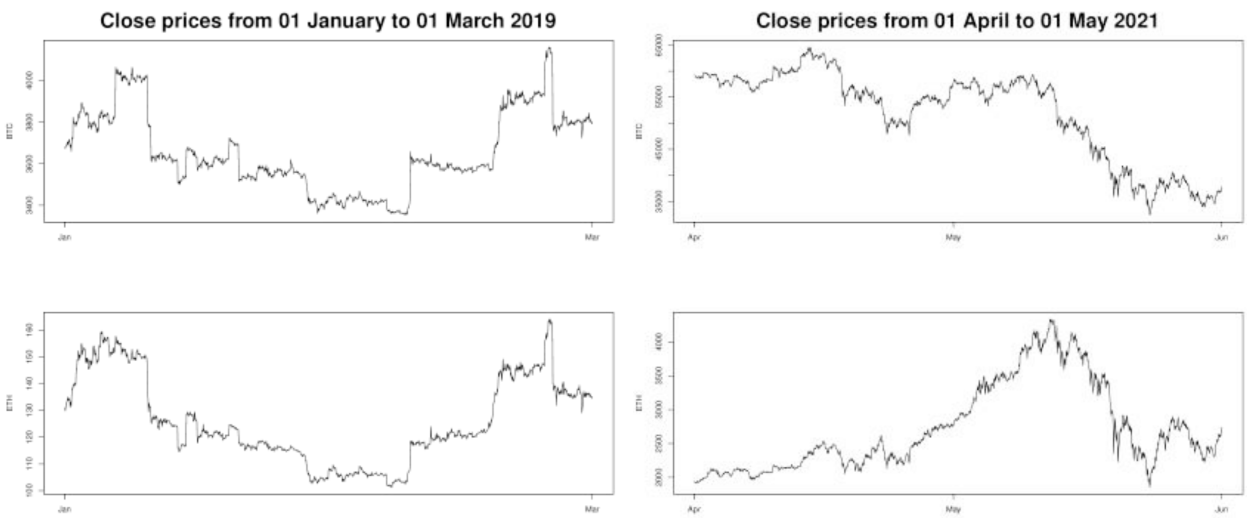


Figure 2.3: (a) Period 1 (b) Period 2

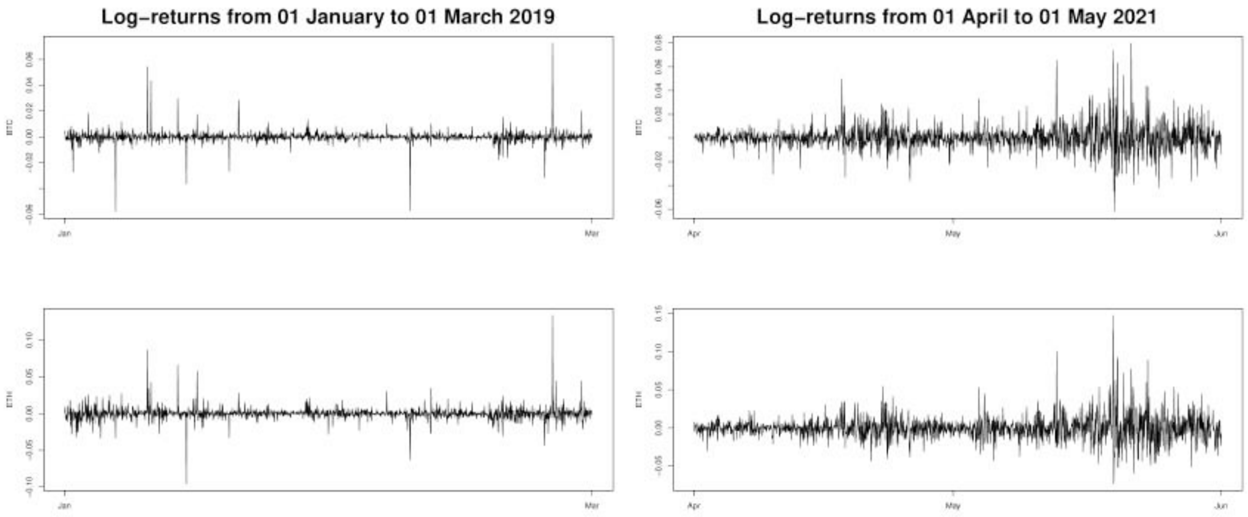


Figure 2.4: (a) Period 1 (b) Period 2

Table 2.13: p-values of testing the change in covariance structure before and after the Bitcoin important events - 1 minute data, statistic (2.2)

Date of event (T_0)	Event description	$p_{[T_0-2D,T_0-1D]}$	$p_{[T_0-1D,T_0]}$	$p_{[T_0,T_0+1D]}$
8 November 2017	Developers cancel splitting of Bitcoin.	0.04	0.09	0.20
28 December 2017	South Korea announces strong measures to regulate the trading of cryptocurrencies.	0.48	0.02	0.09
13 January 2018	Announcement that 80% of Bitcoin has been mined.	0.04	0.01	0.06
30 January 2018	Facebook bans advertisements promoting cryptocurrencies.	0.73	0.03	0.79
7 March 2018	The US Securities and Exchange Commission says it is necessary for crypto trading platforms to register.	0	0.02	0.61
14 March 2018	Google bans advertisements promoting cryptocurrencies.	0.43	0.24	0.03

The second dataset with which we were working comprises the stock data of the three largest S&P 500 companies at the moment. We calculated the daily log-returns for the closing prices of Apple Inc. (AAPL), Microsoft (MSFT), and Amazon (AMZN) from January 1, 2021, to January 1, 2023, covering 503 trading days.

The data were sourced from Yahoo Finance (`https://finance.yahoo.com`). We were partitioning the data into blocks of 7 trading days and computing a covariance matrix for each block. We then tested whether there is a significant change in the covariance structure between the first 36 blocks (January 1, 2021, to January 1, 2022) and the remaining 36 blocks (January 2, 2022, to January 1, 2023) using the test (2.4), obtaining a p-value of 0.001. The outcome suggests that there is a significant change in the covariance structure of the three largest companies in the S&P 500 [112].

This example is intended to demonstrate the use of the test (2.4) on less volatile data. However, selecting the three largest companies at the moment of analysis is not reasonable in the case of real algorithmic trading, as no algorithmic trading system evaluation on the data of successful companies in the future can provide useful estimates. The information that these companies were going to be successful was simply not available at the time of simulated trading.

2.5.2 Non-life insurance data

The dataset 'Insurance' from the R package **splm** [97] was considered for the first time in [96]. The data comprises insurance consumption data across Italian provinces during a 5-year period from 1998 to 2002. Following the approach in [129], we computed the empirical covariance matrices per province for the following covariates:

1. Real per-capita non-life premiums in 2000 Euros (PPCD);
2. Density of insurance agencies per 1000 inhabitants (AGEN);
3. Real per-capita GDP (RGDP);

We divided the data into two groups based on geographic location. The first sample consisted of $n_1 = 67$ empirical 3×3 covariance matrices corresponding to the Northern provinces, while the second sample consisted of $n_2 = 36$ empirical 3×3 covariance matrices corresponding to the Southern and Island provinces. The results in [129] corroborate the findings of [96]. The results suggest that there is a clear separation between Central-Northern Italy and Southern-Insular Italy data exists. Therefore, we tested for the difference in distribution of the covariance matrices between Northern and Southern-Insular data. The p-values of the tests are presented in Table 2.14. Since the test (2.2) for every parameter value reports a p-value lower than $\alpha = 0.05$, we reject the null hypothesis that the covariance matrices of the selected covariates for Northern and Southern-Insular data are equally distributed, providing further evidence that the regional differences explored in [96] and [129] exist.

Table 2.14: p-values of the novel tests for Italy insurance data

Parameters	$\nu = 1, \Sigma = I_5$	$\nu = 1, \Sigma = 2I_5$	$\nu = 2, \Sigma = I_5$	$\nu = 2, \Sigma = 2I_5$	$\nu = 5, \Sigma = I_5$	$\nu = 5, \Sigma = 2I_5$
p-value	0	0	0	0	0.0025	0.0020

Chapter 3

One-dimensional change point inference

3.1 Introduction to change point analysis

In this paragraph, we aim to review the state-of-the-art approaches in change point analysis. The topic has been widely researched in the literature. In the subsequent part, we will outline the main references related to change point analysis and contemporary detection algorithms. We focus heavily on modifications of two-sample tests for the purpose of change point detection, as that is our main goal in this part of the dissertation.

Change point analysis (or data segmentation) algorithms can be naturally divided into two groups. The first group consists of algorithms for offline (or a posteriori) change point detection. The goal is to sequence the data, i.e., estimate the change point locations, where the entire data is present at the start of the analysis. The second group consists of algorithms for online (or sequential) change point detection, where the goal is to sequentially partition the data as they arrive. We will focus on the offline methods for the remainder of the text.

Let us assume we are presented with the sequence $X_1, X_2, \dots, X_n$, where $X_i \in \mathcal{N}(\mu_i, \sigma)$. We are interested in testing

$$\begin{aligned} &H_0 : \mu_1 = \mu_2 = \cdots = \mu_n \\ &\qquad\qquad\text{against the alternative} \\ &H_1 : \text{there exists some } k \text{ such that } \mu_1 = \mu_2 = \cdots = \mu_k \neq \mu_{k+1} = \cdots = \mu_n. \end{aligned}$$

This setting can be modified to test for the change in variance [15].

If one wishes to detect the change point without knowing whether it has occurred, methods of direct optimization might be useful, such as those present in the seminal works of Yao. These works deal with AIC minimization [143] and least squares estimation [144]. For a detailed exposition, one may refer to [105, 131].

For the purpose of testing whether a change point has occurred, one of the classical ways to construct the test is the cumulative sum (CUSUM) approach. The CUSUM approach derives its name from the fact that the function constructed is a cumulative sum of suitably transformed sample elements.

More specifically, the test statistic is constructed as follows:

$$\max_{1 \leqslant k \leqslant n} |CS_k|,$$

where

$$CS_k = \sqrt{\frac{k(n-k)}{n}} \Big(\frac{1}{k} \sum_{j=1}^{k} X_j - \frac{1}{n-k} \sum_{j=k+1}^{n} X_j \Big).$$

Many modifications of the CUSUM statistic are present in contemporary literature; see, for example, [100, 101]. Note that the CUSUM statistic can be used for both parametric and nonparametric inference. The case presented earlier is historically the first to be studied, and, generally, the CUSUM approach does not require the data to be normally distributed.

There exist other approaches as well. The likelihood ratio (LR) test is one of the first tests modified for change point inference. The problem of the change in the mean is well-known [28]. Let $X_1, X_2, \ldots, X_n$ be independent random vectors in $\mathbb{R}^d$, having distribution functions $F(x,\theta_1), F(x,\theta_2), \ldots, F(x,\theta_n)$ respectively, where $\theta_i \in \Theta \subseteq \mathbb{R}^p$. We want to test the null hypothesis

$$H_0 : \theta_1 = \theta_2 = \cdots = \theta_n = \theta^{\text{null}} \tag{3.1}$$

against the alternative

$$H_1: \text{ there exists an integer } k \text{ such that } \theta_1 = \theta_2 = \cdots = \theta_k = \theta^{(1)} \neq \theta^{(2)} = \theta_{k+1} = \cdots = \theta_n.$$

Assume that the location of the change point is not known, as well as every parameter defined above. If the location of the change point is known, we should reject H_0 if the value of the test statistic

$$L_k = \frac{\sup\limits_{\lambda \in \Theta} \prod\limits_{i=1}^{n} f(X_i, \lambda)}{\sup\limits_{(\lambda^{(1)},\lambda^{(2)}) \in \Theta \times \Theta} \prod\limits_{i=1}^{k} f(X_i, \lambda^{(1)}) \prod\limits_{i=k+1}^{n} f(X_i, \lambda^{(2)})}$$

is small. The rationale behind this construction is that if H_0 holds, the data will have a single mean, and L_k will be equal to 1, since the values in the numerator and denominator should be equal. Conversely, if H_1 holds, the data will have two distinct means, resulting in the denominator being larger than the numerator, and consequently, L_k will be strictly less than 1. Since the location of the change point is usually unknown, one may wish to use a maximum likelihood ratio, namely

$$\Lambda_k = \max_{1 \leqslant k < n} (-2 \log L_k),$$

and reject H_0 if Λ_k is large. The LR test for change point inference has been a subject of many different publications over the years, such as in the context of linear regression [71], online detection [33, 145] and so on.

However, the approaches outlined above are not suitable for multiple change point detection. For that purpose, one needs to implement the suitable algorithm. The fastest approach is to use the so-called binary segmentation algorithm [136]. Whenever a change point is detected, say at a location k, the data $X_1, X_2, \ldots, X_n$ is split into $X_1, X_2, \ldots, X_k$ and $X_{k+1}, \ldots, X_n$, and the testing is repeated on each part until no splits occur. The obvious issue with this approach is that the interval can have any number of change points and, therefore, the performance is considered suboptimal. However, whenever computational aspects and simplicity are of the essence, one may prefer to choose the binary segmentation algorithm over more demanding alternative approaches. The most prominent modification is the so-called wild binary segmentation algorithm [41], which randomly selects intervals, allowing for better detection on narrower intervals. This process was refined in [42], by introducing the so-called wild binary segmentation 2 algorithm, which draws novel intervals whenever a detection has occurred. The pitfall of this approach in real settings, where the speed of detection is of the essence, is the computational time involved. One may prefer to use the seeded binary segmentation algorithm, a modification of binary segmentation with deterministically selected points [25]. The algorithms are still being developed. For a recent data-adaptive detection algorithm, refer to [6]. For a recent adaptation of the wild binary segmentation algorithm to high-dimensional linear models, see [142].

3.2 Modification of existing integral-type tests for change point inference

In this section, we briefly introduce the methods used to modify the existing test statistics of two-sample tests and adapt them for change point inference. This approach is used to modify our two-sample tests presented in previous chapters, therefore obtaining the change point tests.

The integral-type tests have been modified to address the change point problem as well. Having a sample of independent real random variables $X_1, X_2, \dots, X_n$ with respective cumulative distribution functions $F_1, F_2, \dots, F_n$, we are interested in testing

$$H_0: F_1 = F_2 = \dots = F_n,$$

against the alternative

$$H_1: \text{there exists } k \text{ such that } F_1 = F_2 = \dots = F_k \neq F_{k+1} = F_{k+2} = \dots = F_n.$$

In [62, Equation (3)], Hušková and Meintanis considered the modification of the two-sample test based on characteristic functions, considering the test statistic of the form

$$T_{n,\gamma} = \max_{1\leqslant k\leqslant n}\Big[\Big(\frac{k(n-k)}{n^2}\Big)^{\gamma}\frac{k(n-k)}{n}\int_{-\infty}^{\infty}|\phi_k(t)-\phi_k^0(t)|^2 w(t)\mathrm{d}t\Big], \tag{3.2}$$

where $w(t)$ denotes the integrable weight function, i.e., such that

$$0 < \int_{-\infty}^{\infty} w(t)\mathrm{d}t < \infty,$$

and $\phi_k(t)$ is the empirical characteristic function of the first k elements of the sample, namely

$$\phi_k(t) = \frac{1}{k}\sum_{i=1}^{k}\exp(itX_i),$$

and $\phi_k^0(t)$ denotes the empirical characteristic function of the last $n-k$ elements of the sample, namely

$$\phi_k^0(t) = \frac{1}{n-k}\sum_{i=k+1}^{n}\exp(itX_i).$$

Hušková and Meintanis considered several weight functions, $w_1 = \exp(-a|t|)$ and $w_2 = \exp(-at^2)$, which produce the test statistics

$$\begin{aligned} T_{n,\gamma}^{(1)}(a) = 2a\max_{1\leqslant k\leqslant n}\Big[\Big(\frac{k(n-k)}{n^2}\Big)^{\gamma}\frac{k(n-k)}{n}\Big(&\frac{1}{k^2}\sum_{l=1}^{k}\sum_{m=1}^{k}\frac{1}{a^2+(X_l-X_m)^2} \\ &+\frac{1}{(n-k)^2}\sum_{l=k+1}^{n}\sum_{m=k+1}^{n}\frac{1}{a^2+(X_l-X_m)^2} \\ &-\frac{2}{k(n-k)}\sum_{l=1}^{k}\sum_{m=k+1}^{n}\frac{1}{a^2+(X_l-X_m)^2}\Big)\Big]. \end{aligned} \tag{3.3}$$

$$T^{(2)}_{n,\gamma}(a)=\sqrt{\frac{\pi}{2}}\max_{1\leqslant k\leqslant n}\Big[\Big(\frac{k(n-k)}{n^2}\Big)^{\gamma}\frac{k(n-k)}{n}\Big(\frac{1}{k^2}\sum_{l=1}^{k}\sum_{m=1}^{k}\exp\Big(-\frac{(X_l-X_m)^2)}{4a}\Big) \tag{3.4}$$
$$+\frac{1}{(n-k)^2}\sum_{l=k+1}^{n}\sum_{m=k+1}^{n}\exp\Big(-\frac{(X_l-X_m)^2)}{4a}\Big)$$
$$-\frac{2}{k(n-k)}\sum_{l=1}^{k}\sum_{m=k+1}^{n}\exp\Big(-\frac{(X_l-X_m)^2)}{4a}\Big)\Big)\Big].$$

Note that under the null hypothesis, a small value of the test statistic is expected. Therefore, large values of the test statistic are considered significant. The parameter $\gamma\in(0,1]$ acts as a tuning parameter, ensuring the convergence of the test statistic in probability whenever $\gamma\neq 0$. In the literature, γ does not significantly influence the test power (see, e.g., [62]).

The same authors modify the test based on the empirical characteristic function of ranks, thereby obtaining the distribution-free test under the null hypothesis [63]. They considered the change point detection in the multivariate setting [94] as well.

3.3 Novel test statistics

In this section, we introduce the test statistics used for univariate change point inference. One class of test statistics is based on the modified Hankel transform, while the other class of test statistics is based on the empirical Laplace transform. Whenever referring to the Hankel transform, we use the Hankel transform, as introduced in the work of Baringhaus and Taherizadeh [12] (see Section 1.1).

Let $X_1,X_2,\dots,X_n$ be independent random variables, X_j having a distribution function F_j, for $j\in\{1,2,\dots,n\}$. We consider testing the null hypothesis of the type (3.1), i.e.,

$$H_0: F_1=F_2=\cdots=F_n$$

against the alternative

$$H_1: F_1=F_2=\cdots=F_{k-1}\neq F_k=F_{k+1}=\cdots=F_n,$$

where k, F_1 and F_n are unknown. We consider the following class of test statistics of the form (3.2), motivated by the two-sample test proposed in [11]:

$$\mathscr{I}_{n,\gamma}=\max_{1\leqslant k\leqslant n}\Big[\Big(\frac{k(n-k)}{n^2}\Big)^{\gamma}\frac{k(n-k)}{n}\int_0^{\infty}(\mathscr{H}_X(t)-\mathscr{H}^0_X(t))^2\exp(-t)\,\mathrm{d}t\Big],$$

where $\mathscr{H}_X(t)$ denotes the empirical Hankel transform of $X_1,X_2,\dots,X_k$ and $\mathscr{H}^0_X(t)$ denotes the empirical Hankel transform of $X_{k+1},X_{k+2},\dots,X_n$. Using results from [11], we obtain the following formula for our test statistic:

$$\mathscr{I}_{n,\gamma}=\max_{1\leqslant k\leqslant n}\Big[\Big(\frac{k(n-k)}{n}\Big)^{\gamma}\frac{k(n-k)}{n}\Big(\frac{1}{k^2}\sum_{l=1}^{k}\sum_{m=1}^{k}I_0(2\sqrt{X_lX_m})\exp(-(X_l+X_m))$$
$$+\frac{1}{(n-k)^2}\sum_{l=k+1}^{n}\sum_{m=k+1}^{n}I_0(2\sqrt{X_lX_m})\exp(-(X_l+X_m))$$
$$-\frac{2}{k(n-k)}\sum_{l=1}^{k}\sum_{m=k+1}^{n}I_0(2\sqrt{X_lX_m})\exp(-(X_l+X_m))\Big)\Big],$$

where I_0 is a modified Bessel function of the first kind of order 0.

Analogously to the construction above, we consider the class of change point test statistics based on the Laplace transform. The test statistics are of the form (3.2):

$$\mathscr{L}_{n,\gamma,a}=\max_{1\leqslant k\leqslant n}\Big[\Big(\frac{k(n-k)}{n^2}\Big)^{\gamma}\frac{k(n-k)}{n}\int\limits_0^{\infty}(\mathscr{L}_X(t)-\mathscr{L}^0_X(t))^2\exp(-a\,t)\,\mathrm{d}t\Big],$$

where $\mathscr{L}_X(t)$ denotes the empirical Laplace transform of the first k elements of the sample, while $\mathscr{L}^0_X(t)$ denotes the empirical Laplace transform of the last $n-k$ elements of the sample (see (1.1)), and $a>0$. Direct computation yields the following form of the Laplace-based test statistic:

$$\begin{aligned}\mathscr{L}_{n,\gamma,a}=\max_{1\leqslant k\leqslant n}\Big[\Big(\frac{k(n-k)}{n^2}\Big)^{\gamma}\frac{k(n-k)}{n}\Big(&\frac{1}{k^2}\sum_{l=1}^{k}\sum_{m=1}^{k}\frac{1}{X_l+X_m+a}\\&+\frac{1}{(n-k)^2}\sum_{l=k+1}^{n}\sum_{m=k+1}^{n}\frac{1}{X_l+X_m+a}\\&-\frac{2}{k(n-k)}\sum_{l=1}^{k}\sum_{m=k+1}^{n}\frac{1}{X_l+X_m+a}\Big)\Big].\end{aligned}$$

Remark. It is possible to consider other weight functions, besides $w(t)=\exp(-a\,t)$. However, it is important to note that the weight function does not have a significant influence in the construction of this type of tests. That is why we have focused only on the simplest one.

3.4 Asymptotic results

In this section, we present the results regarding the null limiting distribution of the proposed test statistics. The proofs in this context follow the same methodology as outlined in [62] and [85], and are omitted. However, the proofs are given in the subsequent chapter for a more complicated matrix case.

To obtain the null limit distribution of the test statistic, we first note that the test statistic $\mathscr{J}_{n,\gamma}$ can be represented as $\mathscr{J}_{n,\gamma}=\max\limits_{1\leqslant k\leqslant n}[c_{n,k}(\gamma)J_{k,n-k}]$, where $J_{k,n-k}$ is a two-sample test statistic introduced in [11].

Denote by $h^1(x,y)=I_0(2\sqrt{xy})\exp(-(x+y))$ and define

$$\tilde{h}^1(x,y)=h^1(x,y)-Eh^1(x,X_s)-Eh^1(X_r,y)+Eh^1(X_r,X_s),$$

for $r\neq s$. Let $(\lambda^1_j)_{j=1}^{\infty}$ be a sequence of eigenvalues of the integral operator

$$\int\limits_{-\infty}^{\infty}(\tilde{h}^1(x,y))^2\,\mathrm{d}\hat{F}_n(x)\,\mathrm{d}\hat{F}_n(y).$$

Theorem 3.1. *Let $X_1,X_2,\dots X_n$ be independent random variables, where $X_1\in F$. Let $\gamma\in(0,1]$ and let $(\lambda_j)_{j=1}^{\infty}$ be the descending sequence of eigenvalues defined above. Then the asymptotic distribution of $\mathscr{J}_{n,\gamma}$ is the same as that of*

$$\sup_{t\in(0,1)}\Big[(t(1-t))^{\gamma}\Big|(EI_0(2X_1)\exp(-2X_1))-EI_0(2\sqrt{X_1X_2})\exp(-(X_1+X_2)))+\sum_{j=1}^{\infty}\lambda^1_j\Big(\frac{B_j^2(t)}{t(1-t)}-1\Big)\Big|\Big],$$

where $\{B_j(t),\ t\in(0,1)\}$, $j=1,2,\dots$ are independent Brownian bridges.

The main idea in proving the theorem is to decompose $J_{k,n-k}$ in the following way:

$$J_{k,n-k} = C_{k1} + C_{k2} + C_{k3} + C_{k4},$$

where

$$\begin{aligned}
C_{k1} &= \frac{n}{k(n-k)}\Big(\frac{1}{k}\sum_{v=1}^{k}\sum_{\substack{s=1\\ s\neq v}}^{k}\tilde{h}^{1}(X_v,X_s) + \frac{1}{n-k}\sum_{v=k+1}^{n}\sum_{\substack{s=k+1\\ s\neq v}}^{n}\tilde{h}^{1}(X_v,X_s) - \frac{1}{n}\sum_{v=1}^{n}\sum_{\substack{s=1\\ s\neq v}}^{n}\tilde{h}^{1}(X_v,X_s)\Big),\\
C_{k2} &= \frac{n}{k(n-k)}(E I_0(2X_1)\exp(-2X_1) - E h^1(X_1,X_2)),\\
C_{k3} &= -\frac{2}{k^2}\sum_{r=1}^{k}(E h^1((X_r,X_s)|X_r) - E h^1(X_1,X_2)) - \frac{2}{(n-k)^2}\sum_{r=k+1}^{n}(E h^1((X_r,X_s)|X_r) - E h^1(X_1,X_2)),\\
C_{k4} &= \frac{1}{k^2}\sum_{i=1}^{k}(I_0(2X_r)\exp(-2X_r) - E I_0(2X_r)\exp(-2X_r))\\
&\quad + \frac{1}{(n-k)^2}\sum_{i=k+1}^{n}(I_0(2X_i)\exp(-2X_i) - E I_0(2X_i)\exp(-2X_i)).
\end{aligned}$$

Following exactly the same steps as outlined in [62], the proof can be completed. The only difference here is the summand C_{k_4}, which can be shown to be negligible by employing the approach outlined in [85].

Denote by $h^{(2)}(x,y) = \frac{1}{x+y+a}$ and with

$$\tilde{h}^{(2)}(x,y) = h^{(2)}(x,y) - E h^{(2)}(x,X_s) - E h^{(2)}(X_r,y) + E h^{(2)}(X_r,X_s),$$

for $r \neq s$. Let $(\lambda_j^{(2)})_{j=1}^{\infty}$ be a sequence of eigenvalues of the integral operator

$$\int_{-\infty}^{\infty} (\tilde{h}^{(2)}(x,y))^2 \,\mathrm{d}\hat{F}_n(x)\,\mathrm{d}\hat{F}_n(y).$$

By following the same steps and making slight modifications to the decomposition, one can obtain the asymptotics of the Laplace-based test, as stated in the following theorem.

Theorem 3.2. *Let $X_1, X_2, \dots X_n$ be independent random variables, where $X_1 \in F$. Let $\gamma \in (0,1]$ and let $(\lambda_j^{(2)})_{j=1}^{\infty}$ be the descending sequence of eigenvalues defined above. Then the asymptotic distribution of $\mathscr{L}_{n,\gamma}$ is the same as that of*

$$\sup_{t\in(0,1)}\Big[(t(1-t))^{\gamma}\Big|(E\Big(\frac{1}{X_1+X_2+a}\Big) - E\Big(\frac{1}{2X_1+a}\Big) + \sum_{j=1}^{\infty}\lambda_j^{(2)}\Big(\frac{B_j^2(t)}{t(1-t)} - 1\Big)\Big|\Big],$$

where $\{B_j(t),\ t \in (0,1)\}$, $j = 1,2,\dots$ are independent Brownian bridges.

Since the limiting null distribution of the novel statistics is not free of the distribution of X_1, for the derivation of p-values, we suggest the usage of the permutation bootstrap algorithm from [62]. This approach can be theoretically justified in a similar manner for our tests.

3.5 A power study

In this section, we investigate the finite sample properties of the novel tests. As a benchmark, we consider the test statistics (3.3) and (3.4).

Note that under H_0, none of the presented test statistics, whether novel statistics or competitors, are free of the underlying distribution, therefore, there must exist a procedure to estimate the critical values of the aforementioned tests. The procedure implemented to estimate the distribution under H_0 is the permutation without replacement (PWOR) with $N = 2000$ replacements and $B = 500$ replicates. For the meta-algorithm for the arbitrary test statistic M, refer to Algorithm 2. For every test presented here, we take large values to be significant.

Algorithm 2 Permutation bootstrap algorithm

1: Sample $\mathbf{x} = (x_1, \ldots, x_{n_1})$ from F_X and $y = (y_1, \ldots, y_{n_2})$ from F_Y and let $n = n_1 + n_2$, and create a pooled sample $\mathbf{z} = (z_1, z_2, \ldots, z_{n_1+n_2}) = (x_1, \ldots, x_{n_1}, y_1, y_2, \ldots, y_{n_2})$.;
2: Compute $M_{n,\gamma,a} := M_{n,\gamma,a}(\mathbf{z})$;
3: Generate random permutations $\pi : \{1,2,\ldots,n_1+n_2\} \to \{1,2,\ldots,n_1+n_2\}$ and the corresponding bootstrap permutation sample $\mathbf{z}^* = (z_{\pi(1)}, \ldots, z_{\pi(n_1)}, z_{\pi(n_1+1)}, \ldots, z_{\pi(n_1+n_2)})$.
4: Compute $M^*_{n,\gamma,a} := M_{n,\gamma,a}(\mathbf{z}^*)$;
5: Repeat steps 3-4 N times and obtain the sequence of statistics $\{M^{*(j)}_{n,\gamma,a}\}$, $j = 1, \ldots, N$;
6: Reject the null hypothesis, if $M_{n,\gamma,a} > c_\alpha$, where c_α denotes the $(1-\alpha)$% quantile of the empirical distribution of the bootstrap test statistics $(M^{*(j)}_{n,\gamma,a},\ j = 1, \ldots, N)$;
7: Repeat steps 1-6 B times.

We consider sample sizes $n = 40$ and $n = 100$ and the level of significance $\alpha = 0.05$. The samples $Y_1, Y_2, \ldots, Y_m$ and $Y_{m+1}, Y_{m+2}, \ldots, Y_n$ are generated from distributions F_1 and F_2, where $m = \frac{n}{2}$ or $m = \frac{n}{4}$ and $Y_n = b\,Y_1 + 1$ is considered, where $b = 1$ or $b = \sqrt{2}$. For the distributions F_1 and F_2 we consider Uniform $U[0,1]$ (U), Gamma $\Gamma(1,1)$ (G1) and Gamma $\Gamma(2,1)$ (G2) distributions. Whenever * is present in the table, it denotes a test power of 100 percent.

From Tables 3.1 and 3.2, all tests seem to be well calibrated. The tables for the Uniform distribution have been omitted, as all tests achieve a test power of 100 percent. It is clear that both novel tests, namely $\mathcal{J}_{n,\gamma}$ and $\mathcal{L}_{n,\gamma,a}$, have more favorable test powers than the benchmark tests. It seems that the parameter γ does not significantly influence the test powers in the case of the test $\mathcal{J}_{n,\gamma}$, while for the test $\mathcal{L}_{n,\gamma,a}$, the sensitivity increases with an increase in a. Lower values of a usually correspond to higher test powers. The test $\mathcal{L}_{n,\gamma,a}$ usually has slightly larger test powers than the test $\mathcal{J}_{n,\gamma}$. Moreover, the actual location of the change point significantly influences the test powers, which peak in the symmetric case.

In order to test the quality of the change point estimators, we have generated $N = 2000$ replications of the sample and estimated the change point of every statistic on each of the samples. We present the mean and the standard deviation of the position estimate in Table 3.3. The boxplots are presented in Figure 3.1. The alternative U has been omitted since in almost every case there is a perfect detection and they are uninformative. From the results presented, it is clear that the novel tests give more precise estimates when the real position of a change point is at $\frac{1}{4}$ of the sample, while for $\frac{1}{2}$ of the sample, the novel tests are comparable to the benchmark, having smaller standard deviation for G_1 and G_2. The number of outliers is usually smaller for the novel tests. It can be seen that the tests detect the true position of the change point in the uniform case with great precision. Usually, the test based on the Hankel transform gives more precise estimates than the test based on the Laplace transform. However, it is important to note that the parameter a has a significant influence on the change point precision. It is also observable that the higher values of the parameter a usually result in estimates which are more precise. Therefore, if one wishes to estimate the change point location, it would be preferable to use the test based on the Hankel transform. If one wishes to have a test with the highest test power, and disregards the actual precision of the change point estimate, then it would be advisable to consider the Laplace transform-based test with the parameter $a = 0.5$. Alternatively, one may wish to use the battery of the tests whenever the precision is of the utmost importance, for example in high-frequency trading. On the other hand, whenever one is looking into macroeconomic data, the exact change point moment might not be of the utmost importance, but the information on whether there is a change point is more important, therefore, one may opt for the tests which have larger test powers in general, even if they are not the most precise ones.

Table 3.1: Test powers for different alternatives and $n = 40$ using the PWOR method and B=500.

	$G1$				
	$m=40$	$m=20, b=1$	$m=20, b=\sqrt{2}$	$m=10, b=1$	$m=10, b=\sqrt{2}$
$\mathscr{L}_{40,0.5,0.5}$	5	98	99	96	98
$\mathscr{L}_{40,1,0.5}$	5	99	*	96	98
$\mathscr{L}_{40,0.5,1}$	5	98	99	94	97
$\mathscr{L}_{40,1,1}$	5	99	99	92	97
$\mathscr{L}_{40,0.5,2}$	5	97	99	90	95
$\mathscr{L}_{40,1,2}$	5	97	99	88	95
$\mathscr{I}_{40,0.5}$	4	98	99	94	96
$\mathscr{I}_{40,1}$	5	98	*	93	95
$T^{(1)}_{40,0.5}(1)$	5	96	97	78	78
$T^{(1)}_{40,0.5}(1.5)$	5	94	96	75	77
$T^{(1)}_{40,0.5}(2)$	5	92	95	73	76
$T^{(1)}_{40,0.5}(3)$	5	89	94	69	74
$T^{(1)}_{40,0.5}(4)$	5	87	93	66	72
$T^{(1)}_{40,1}(1)$	4	97	98	74	73
$T^{(1)}_{40,1}(1.5)$	5	96	97	72	73
$T^{(1)}_{40,1}(2)$	5	94	96	70	72
$T^{(1)}_{40,1}(3)$	5	91	96	66	71
$T^{(1)}_{40,1}(4)$	5	89	95	64	70
$T^{(2)}_{40,0.5}(2)$	5	91	94	72	75
$T^{(2)}_{40,0.5}(1.5)$	5	90	94	69	74
$T^{(2)}_{40,0.5}(2)$	5	88	93	68	72
$T^{(2)}_{40,0.5}(3)$	5	86	93	66	71
$T^{(2)}_{40,0.5}(4)$	5	85	92	64	70
$T^{(2)}_{40,1}(2)$	5	93	96	68	71
$T^{(2)}_{40,1}(1.5)$	5	91	95	66	70
$T^{(2)}_{40,1}(2)$	5	90	95	65	70
$T^{(2)}_{40,1}(3)$	5	89	95	63	69
$T^{(2)}_{40,1}(4)$	5	88	94	62	69

Table 3.1: Test powers for different alternatives and $n = 40$ using the PWOR method and B=500.

	$G2$				
	$m=40$	$m=20, b=1$	$m=20, b=\sqrt{2}$	$m=10, b=1$	$m=10, b=\sqrt{2}$
$\mathscr{L}_{40,0.5,0.5}$	4	71	92	64	85
$\mathscr{L}_{40,1,0.5}$	5	78	95	62	83
$\mathscr{L}_{40,0.5,1}$	4	72	92	62	84
$\mathscr{L}_{40,1,1}$	5	76	94	59	82
$\mathscr{L}_{40,0.5,2}$	5	69	92	58	82
$\mathscr{L}_{40,1,2}$	5	74	94	54	79
$\mathscr{J}_{40,0.5}$	5	71	91	57	77
$\mathscr{J}_{40,1}$	5	74	93	52	75
$T^{(1)}_{40,0.5}(1)$	5	52	74	30	45
$T^{(1)}_{40,0.5}(1.5)$	5	54	79	32	50
$T^{(1)}_{40,0.5}(2)$	5	54	80	32	53
$T^{(1)}_{40,0.5}(3)$	5	54	82	32	55
$T^{(1)}_{40,0.5}(4)$	5	53	83	31	56
$T^{(1)}_{40,1}(1)$	5	55	77	26	40
$T^{(1)}_{40,1}(1.5)$	5	57	80	28	45
$T^{(1)}_{40,1}(2)$	5	56	83	28	49
$T^{(1)}_{40,1}(3)$	5	56	85	29	52
$T^{(1)}_{40,1}(4)$	5	56	85	29	52
$T^{(2)}_{40,0.5}(2)$	5	53	78	31	50
$T^{(2)}_{40,0.5}(1.5)$	6	54	80	31	52
$T^{(2)}_{40,0.5}(2)$	5	53	81	31	54
$T^{(2)}_{40,0.5}(3)$	5	53	82	31	55
$T^{(2)}_{40,0.5}(4)$	5	52	82	31	55
$T^{(2)}_{40,1}(2)$	5	56	80	28	46
$T^{(2)}_{40,1}(1.5)$	5	56	82	28	49
$T^{(2)}_{40,1}(2)$	5	56	84	29	50
$T^{(2)}_{40,1}(3)$	5	55	85	29	51
$T^{(2)}_{40,1}(4)$	5	55	85	30	53

Table 3.2: Test powers for different alternatives and $n = 100$ using the PWOR method and B=500.

	$G1$				
	$m = 100$	$m = 50, b = 1$	$m = 50, b = \sqrt{2}$	$m = 25, b = 1$	$m = 25, b = \sqrt{2}$
$\mathscr{L}_{100,0.5,0.5}$	5	*	*	*	*
$\mathscr{L}_{100,1,0.5}$	5	*	*	*	*
$\mathscr{L}_{100,0.5,1}$	5	*	*	*	*
$\mathscr{L}_{100,1,1}$	5	*	*	*	*
$\mathscr{L}_{100,0.5,2}$	5	*	*	*	*
$\mathscr{L}_{100,1,2}$	5	*	*	*	*
$\mathscr{I}_{100,0.5}$	4	*	*	*	*
$\mathscr{I}_{100,1}$	4	*	*	*	*
$T^{(1)}_{100,0.5}(1)$	5	*	*	*	99
$T^{(1)}_{100,0.5}(1.5)$	4	*	*	*	99
$T^{(1)}_{100,0.5}(2)$	4	*	*	99	99
$T^{(1)}_{100,0.5}(3)$	5	*	*	99	99
$T^{(1)}_{100,0.5}(4)$	5	*	*	98	99
$T^{(1)}_{100,1}(1)$	5	*	*	*	99
$T^{(1)}_{100,1}(1.5)$	5	*	*	99	99
$T^{(1)}_{100,1}(2)$	5	*	*	99	99
$T^{(1)}_{100,1}(3)$	5	*	*	98	99
$T^{(1)}_{100,1}(4)$	5	*	*	97	99
$T^{(2)}_{100,0.5}(2)$	5	*	*	99	99
$T^{(2)}_{100,0.5}(1.5)$	5	*	*	99	99
$T^{(2)}_{100,0.5}(2)$	5	*	*	98	99
$T^{(2)}_{100,0.5}(3)$	5	*	*	98	99
$T^{(2)}_{100,0.5}(4)$	5	*	*	97	99
$T^{(2)}_{100,1}(2)$	5	*	*	99	99
$T^{(2)}_{100,1}(1.5)$	5	*	*	98	99
$T^{(2)}_{100,1}(2)$	5	*	*	98	99
$T^{(2)}_{100,1}(3)$	5	*	*	97	99
$T^{(2)}_{100,1}(4)$	5	*	*	97	99

Table 3.2: Test powers for different alternatives and $n = 100$ using the PWOR method and B=500.

	$G2$				
	$m=100$	$m=50, b=1$	$m=20, b=\sqrt{2}$	$m=25, b=1$	$m=25, b=\sqrt{2}$
$\mathscr{L}_{100,0.5,0.5}$	5	99	*	97	*
$\mathscr{L}_{100,1,0.5}$	4	*	*	96	*
$\mathscr{L}_{100,0.5,1}$	5	99	*	96	*
$\mathscr{L}_{100,1,1}$	5	*	*	95	*
$\mathscr{L}_{100,0.5,2}$	5	99	*	95	*
$\mathscr{L}_{100,1,2}$	5	99	*	93	*
$\mathscr{J}_{100,0.5}$	6	99	*	94	*
$\mathscr{J}_{100,1}$	6	99	*	93	*
$T^{(1)}_{100,0.5}(1)$	5	94	99	74	92
$T^{(1)}_{100,0.5}(1.5)$	5	94	99	76	94
$T^{(1)}_{100,0.5}(2)$	6	94	99	77	95
$T^{(1)}_{100,0.5}(3)$	6	93	99	76	96
$T^{(1)}_{100,0.5}(4)$	6	92	99	75	97
$T^{(1)}_{100,1}(1)$	5	95	99	69	90
$T^{(1)}_{100,1}(1.5)$	5	95	99	71	93
$T^{(1)}_{100,1}(2)$	5	95	99	72	94
$T^{(1)}_{100,1}(3)$	5	95	99	73	95
$T^{(1)}_{100,1}(4)$	5	94	*	72	96
$T^{(2)}_{100,0.5}(2)$	5	93	99	76	94
$T^{(2)}_{100,0.5}(1.5)$	6	93	99	76	95
$T^{(2)}_{100,0.5}(2)$	6	93	99	76	96
$T^{(2)}_{100,0.5}(3)$	6	92	99	75	96
$T^{(2)}_{100,0.5}(4)$	6	92	*	74	97
$T^{(2)}_{100,1}(2)$	5	95	99	72	93
$T^{(2)}_{100,1}(1.5)$	5	95	99	72	94
$T^{(2)}_{100,1}(2)$	5	94	99	72	95
$T^{(2)}_{100,1}(3)$	6	94	99	72	96
$T^{(2)}_{100,1}(4)$	5	93	*	71	96

Table 3.3: Estimates and standard deviations of change point locations.

	$n=40, b=1, k=20$		
	G1	G2	U
$\mathscr{L}_{40,0.5,0.5}$	18.59 (2.405)	17.59 (4.344)	19.55 (0.914)
$\mathscr{L}_{40,1,0.5}$	18.85 (1.924)	18.23 (3.495)	19.6 (0.803)
$\mathscr{L}_{40,0.5,1}$	18.77 (2.305)	17.91 (4.374)	19.71 (0.672)
$\mathscr{L}_{40,1,1}$	18.99 (1.871)	18.43 (3.508)	19.74 (0.601)
$\mathscr{L}_{40,0.5,2}$	18.98 (2.355)	18.25 (4.476)	19.83 (0.476)
$\mathscr{L}_{40,1,2}$	19.17 (1.938)	18.64 (3.616)	19.85 (0.434)
$\mathscr{J}_{40,0.5}$	18.87 (2.204)	18.12 (4.442)	19.78 (0.567)
$\mathscr{J}_{40,1}$	19.06 (1.756)	18.57 (3.61)	19.8 (0.501)
$T^{(1)}_{40,0.5}(1)$	19.27 (2.13)	19.3 (5.083)	20.01 (0.224)
$T^{(1)}_{40,0.5}(1.5)$	19.36 (2.356)	19.39 (5.187)	20.01 (0.223)
$T^{(1)}_{40,0.5}(2)$	19.48 (2.524)	19.47 (5.174)	20.01 (0.215)
$T^{(1)}_{40,1}(1)$	19.38 (1.729)	19.34 (4.037)	20.01 (0.206)
$T^{(1)}_{40,1}(1.5)$	19.48 (1.867)	19.38 (4.019)	20.01 (0.195)
$T^{(1)}_{40,1}(2)$	19.52 (2.061)	19.44 (4.141)	20.01 (0.196)
$T^{(2)}_{40,0.5}(1)$	19.52 (2.65)	19.42 (5.358)	20.01 (0.217)
$T^{(2)}_{40,0.5}(1.5)$	19.57 (2.841)	19.45 (5.366)	20.01 (0.212)
$T^{(2)}_{40,0.5}(2)$	19.61 (2.962)	19.5 (5.473)	20.01 (0.211)
$T^{(2)}_{40,1}(1)$	19.57 (2.134)	19.51 (4.205)	20.01 (0.186)
$T^{(2)}_{40,1}(1.5)$	19.65 (2.301)	19.55 (4.362)	20.01 (0.186)
$T^{(2)}_{40,1}(2)$	19.67 (2.485)	19.53 (4.343)	20.01 (0.184)
	$n=40, b=1, k=10$		
$\mathscr{L}_{40,0.5,0.5}$	9.4 (1.932)	10.23 (5.123)	9.85 (0.485)
$\mathscr{L}_{40,1,0.5}$	10.04 (1.987)	11.48 (4.882)	10.03 (0.256)
$\mathscr{L}_{40,0.5,1}$	9.68 (2.214)	10.7 (5.471)	9.95 (0.289)
$\mathscr{L}_{40,1,1}$	10.39 (2.35)	11.77 (5.015)	10.09 (0.357)
$\mathscr{L}_{40,0.5,2}$	10.1 (2.648)	11.18 (5.779)	9.99 (0.221)
$\mathscr{L}_{40,1,2}$	10.92 (2.873)	12.28 (5.331)	10.19 (0.529)
$\mathscr{J}_{40,0.5}$	9.75 (2.169)	11.29 (5.925)	9.97 (0.25)
$\mathscr{J}_{40,1}$	10.39 (2.377)	12.29 (5.37)	10.12 (0.43)
$T^{(1)}_{40,0.5}(1)$	10.97 (4.085)	14.38 (7.485)	10.13 (0.513)
$T^{(1)}_{40,0.5}(1.5)$	11.2 (4.395)	14.03 (7.345)	10.15 (0.538)
$T^{(1)}_{40,0.5}(2)$	11.42 (4.583)	13.98 (7.434)	10.16 (0.545)
$T^{(1)}_{40,1}(1)$	11.85 (4.229)	15.25 (6.404)	10.33 (0.827)
$T^{(1)}_{40,1}(1.5)$	11.99 (4.259)	14.89 (6.401)	10.38 (0.86)
$T^{(1)}_{40,1}(2)$	12.23 (4.407)	14.74 (6.428)	10.41 (0.879)
$T^{(2)}_{40,0.5}(1)$	11.56 (4.688)	14.12 (7.527)	10.17 (0.556)
$T^{(2)}_{40,0.5}(1.5)$	11.87 (5.031)	14 (7.543)	10.18 (0.577)
$T^{(2)}_{40,0.5}(2)$	12.07 (5.307)	14.03 (7.623)	10.19 (0.586)
$T^{(2)}_{40,1}(1)$	12.39 (4.497)	14.82 (6.529)	10.42 (0.893)
$T^{(2)}_{40,1}(1.5)$	12.57 (4.661)	14.73 (6.491)	10.44 (0.905)
$T^{(2)}_{40,1}(2)$	12.73 (4.867)	14.69 (6.494)	10.45 (0.906)

Table 3.3: Estimates and standard deviations of change point locations.

	G1	G2	U
	$n=40, b=\sqrt{2}, k=20$		
$\mathscr{L}_{40,0.5,0.5}$	18.7 (2.31)	18.57 (3.141)	19.81 (0.572)
$\mathscr{L}_{40,1,0.5}$	18.94 (1.841)	18.92 (2.458)	19.84 (0.484)
$\mathscr{L}_{40,0.5,1}$	18.93 (2.206)	18.87 (3.037)	19.94 (0.296)
$\mathscr{L}_{40,1,1}$	19.12 (1.748)	19.11 (2.424)	19.95 (0.246)
$\mathscr{L}_{40,0.5,2}$	19.16 (2.128)	19.15 (2.947)	19.99 (0.109)
$\mathscr{L}_{40,1,2}$	19.32 (1.785)	19.36 (2.389)	19.99 (0.08)
$\mathscr{J}_{40,0.5}$	18.97 (2.211)	19.05 (3.172)	19.98 (0.172)
$\mathscr{J}_{40,1}$	19.18 (1.728)	19.29 (2.542)	19.98 (0.149)
$T^{(1)}_{40,0.5}(1)$	19.39 (2.321)	19.73 (3.746)	20 (0)
$T^{(1)}_{40,0.5}(1.5)$	19.56 (2.44)	19.81 (3.68)	20 (0.022)
$T^{(1)}_{40,0.5}(2)$	19.67 (2.529)	19.97 (3.716)	20 (0.022)
$T^{(1)}_{40,1}(1)$	19.46 (1.875)	19.76 (2.991)	20 (0)
$T^{(1)}_{40,1}(1.5)$	19.57 (2.054)	19.81 (2.88)	20 (0.022)
$T^{(1)}_{40,1}(2)$	19.69 (2.144)	19.95 (2.869)	20 (0.022)
$T^{(2)}_{40,0.5}(1)$	19.72 (2.664)	19.91 (3.731)	20 (0.022)
$T^{(2)}_{40,0.5}(1.5)$	19.8 (2.838)	20.05 (3.738)	20 (0.022)
$T^{(2)}_{40,0.5}(2)$	19.9 (2.885)	20.15 (3.68)	20 (0.022)
$T^{(2)}_{40,1}(1)$	19.71 (2.211)	19.92 (3.019)	20 (0.022)
$T^{(2)}_{40,1}(1.5)$	19.81 (2.29)	20.02 (2.933)	20 (0.022)
$T^{(2)}_{40,1}(2)$	19.87 (2.296)	20.04 (2.933)	20 (0.022)
	$n=40, b=\sqrt{2}, k=10$		
$\mathscr{L}_{40,0.5,0.5}$	9.47 (1.619)	10.08 (3.457)	9.98 (0.158)
$\mathscr{L}_{40,1,0.5}$	10.1 (1.765)	11.04 (3.347)	10 (0.05)
$\mathscr{L}_{40,0.5,1}$	9.79 (1.895)	10.34 (3.522)	10 (0.022)
$\mathscr{L}_{40,1,1}$	10.44 (2.096)	11.35 (3.527)	10.02 (0.144)
$\mathscr{L}_{40,0.5,2}$	10.16 (2.25)	10.73 (3.696)	10 (0)
$\mathscr{L}_{40,1,2}$	10.88 (2.581)	11.75 (3.776)	10.05 (0.23)
$\mathscr{J}_{40,0.5}$	9.86 (2.046)	10.9 (4.31)	10 (0)
$\mathscr{J}_{40,1}$	10.46 (2.291)	11.9 (4.145)	10.03 (0.168)
$T^{(1)}_{40,0.5}(1)$	11.04 (3.793)	13.51 (6.22)	10.02 (0.158)
$T^{(1)}_{40,0.5}(1.5)$	11.27 (4.065)	13.3 (6.202)	10.04 (0.197)
$T^{(1)}_{40,0.5}(2)$	11.52 (4.367)	13.28 (6.156)	10.05 (0.256)
$T^{(1)}_{40,1}(1)$	12.01 (4.072)	14.54 (5.742)	10.12 (0.438)
$T^{(1)}_{40,1}(1.5)$	12.15 (4.198)	14.25 (5.689)	10.17 (0.529)
$T^{(1)}_{40,1}(2)$	12.22 (4.154)	13.96 (5.509)	10.22 (0.608)
$T^{(2)}_{40,0.5}(1)$	11.72 (4.594)	13.29 (6.214)	10.06 (0.278)
$T^{(2)}_{40,0.5}(1.5)$	11.83 (4.687)	13.28 (6.151)	10.07 (0.297)
$T^{(2)}_{40,0.5}(2)$	12.02 (4.859)	13.3 (6.164)	10.07 (0.304)
$T^{(2)}_{40,1}(1)$	12.38 (4.332)	14.18 (5.702)	10.25 (0.639)
$T^{(2)}_{40,1}(1.5)$	12.54 (4.421)	14.05 (5.635)	10.28 (0.668)
$T^{(2)}_{40,1}(2)$	12.59 (4.427)	14.03 (5.592)	10.32 (0.714)

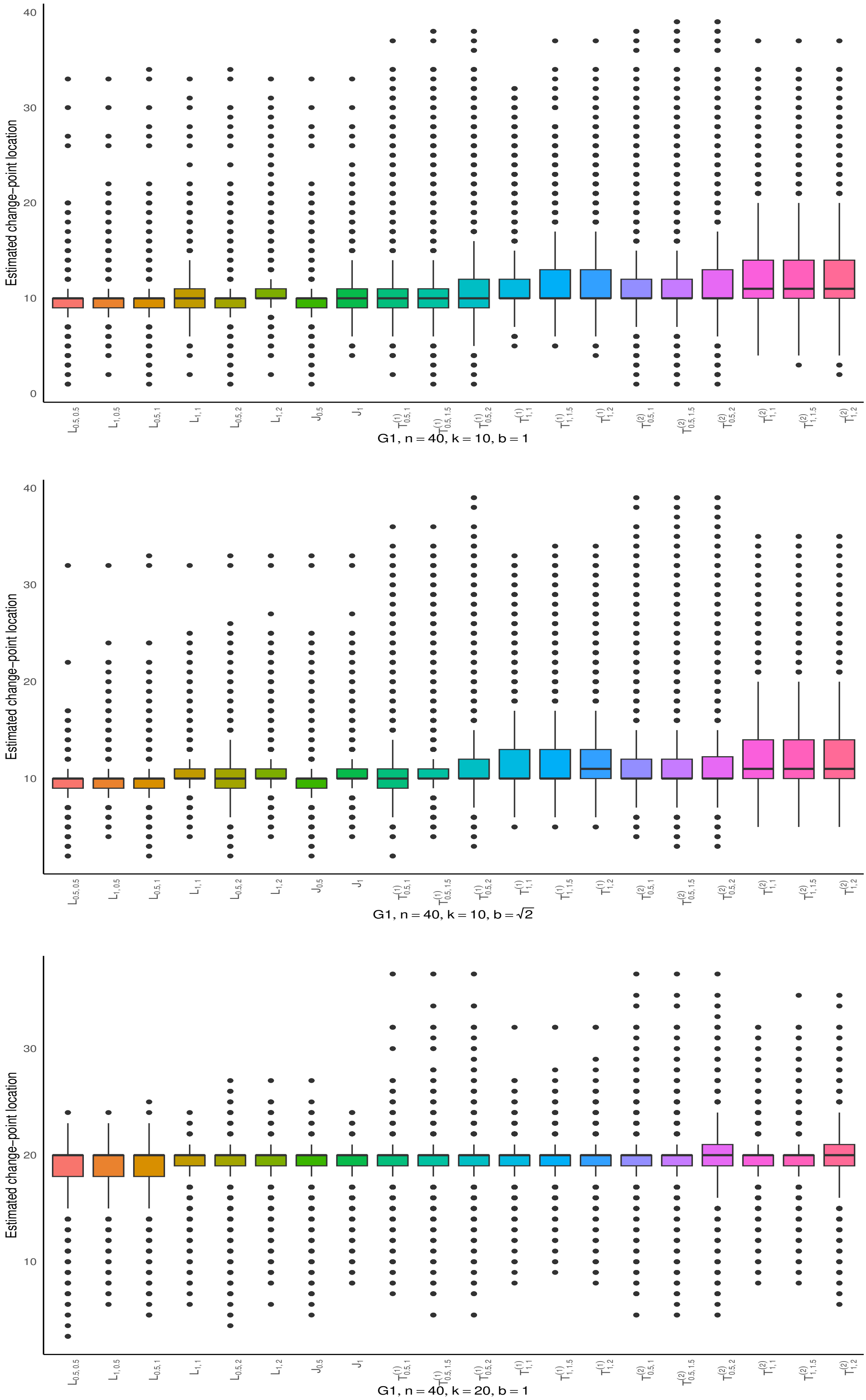
Table 3.3: Estimates and standard deviations of change point locations.

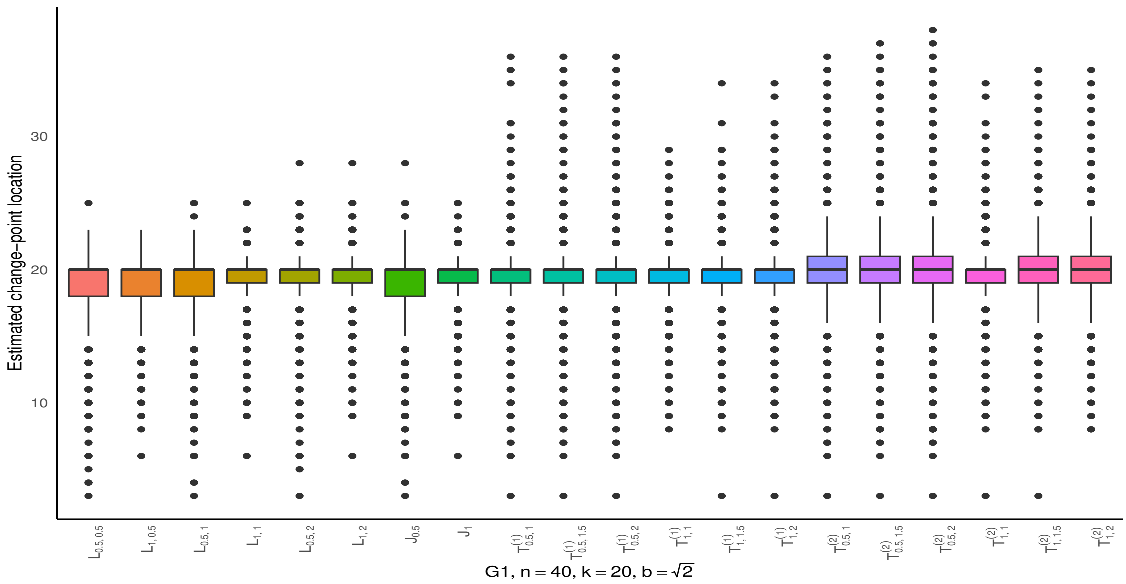
	$n=100, b=1, k=50$		
	G1	G2	U
$\mathscr{L}_{100,0.5,0.5}$	48.42 (2.792)	46.98 (5.494)	49.54 (0.977)
$\mathscr{L}_{100,1,0.5}$	48.6 (2.414)	47.46 (4.476)	49.57 (0.896)
$\mathscr{L}_{100,0.5,1}$	48.62 (2.644)	47.31 (5.561)	49.72 (0.674)
$\mathscr{L}_{100,1,1}$	48.75 (2.336)	47.77 (4.47)	49.73 (0.643)
$\mathscr{L}_{100,0.5,2}$	48.84 (2.584)	47.62 (5.706)	49.83 (0.49)
$\mathscr{L}_{100,1,2}$	48.91 (2.404)	48.17 (4.546)	49.83 (0.474)
$\mathscr{J}_{100,0.5}$	48.7 (2.518)	47.47 (5.671)	49.78 (0.584)
$\mathscr{J}_{100,1}$	48.83 (2.234)	47.98 (4.552)	49.79 (0.543)
$T^{(1)}_{100,0.5}(1)$	49.07 (2.208)	48.41 (6.485)	50.01 (0.156)
$T^{(1)}_{100,0.5}(1.5)$	49.16 (2.401)	48.63 (6.707)	50.01 (0.141)
$T^{(1)}_{100,0.5}(2)$	49.25 (2.631)	48.72 (6.733)	50.01 (0.139)
$T^{(1)}_{100,1}(1)$	49.14 (2.028)	48.61 (5.066)	50.01 (0.145)
$T^{(1)}_{100,1}(1.5)$	49.21 (2.214)	48.7 (5.125)	50.01 (0.138)
$T^{(1)}_{100,1}(2)$	49.3 (2.42)	48.87 (5.218)	50.01 (0.138)
$T^{(2)}_{100,0.5}(1)$	49.27 (2.843)	48.77 (7.04)	50.01 (0.139)
$T^{(2)}_{100,0.5}(1.5)$	49.32 (3.039)	48.87 (7.045)	50.01 (0.139)
$T^{(2)}_{100,0.5}(2)$	49.41 (3.21)	49.01 (7.284)	50.01 (0.139)
$T^{(2)}_{100,1}(1)$	49.35 (2.525)	48.87 (5.391)	50.01 (0.13)
$T^{(2)}_{100,1}(1.5)$	49.4 (2.743)	48.95 (5.697)	50.01 (0.128)
$T^{(2)}_{100,1}(2)$	49.46 (2.837)	49.08 (5.778)	50.01 (0.128)
	$n=100, b=1, k=25$		
$\mathscr{L}_{100,0.5,0.5}$	24.17 (1.909)	24.33 (5.44)	24.83 (0.499)
$\mathscr{L}_{100,1,0.5}$	24.83 (1.88)	26.44 (5.736)	25.03 (0.236)
$\mathscr{L}_{100,0.5,1}$	24.41 (2.077)	24.8 (5.728)	24.92 (0.299)
$\mathscr{L}_{100,1,1}$	25.21 (2.247)	27.12 (6.456)	25.09 (0.337)
$\mathscr{L}_{100,0.5,2}$	24.76 (2.464)	25.74 (6.886)	24.99 (0.202)
$\mathscr{L}_{100,1,2}$	25.85 (3.11)	27.9 (7.185)	25.17 (0.47)
$\mathscr{J}_{100,0.5}$	24.46 (1.969)	25.34 (6.668)	24.96 (0.25)
$\mathscr{J}_{100,1}$	25.17 (2.134)	27.72 (7.331)	25.12 (0.388)
$T^{(1)}_{100,0.5}(1)$	25.02 (3.155)	29.35 (11.845)	25.11 (0.384)
$T^{(1)}_{100,0.5}(1.5)$	25.32 (3.694)	29.26 (11.634)	25.12 (0.41)
$T^{(1)}_{100,0.5}(2)$	25.61 (4.057)	29.12 (11.325)	25.13 (0.422)
$T^{(1)}_{100,1}(1)$	25.88 (3.619)	31.82 (11.452)	25.32 (0.808)
$T^{(1)}_{100,1}(1.5)$	26.46 (4.365)	31.43 (11.113)	25.36 (0.862)
$T^{(1)}_{100,1}(2)$	26.88 (4.941)	31.57 (11.11)	25.42 (0.952)
$T^{(2)}_{100,0.5}(1)$	25.75 (4.266)	29.28 (11.505)	25.14 (0.433)
$T^{(2)}_{100,0.5}(1.5)$	26.1 (4.871)	29.58 (11.741)	25.15 (0.436)
$T^{(2)}_{100,0.5}(2)$	26.3 (5.218)	29.73 (11.839)	25.15 (0.445)
$T^{(2)}_{100,1}(1)$	27.21 (5.338)	31.62 (11.235)	25.45 (0.979)
$T^{(2)}_{100,1}(1.5)$	27.66 (5.817)	31.74 (11.401)	25.47 (1.008)
$T^{(2)}_{100,1}(2)$	27.92 (6.128)	31.83 (11.592)	25.48 (1.024)

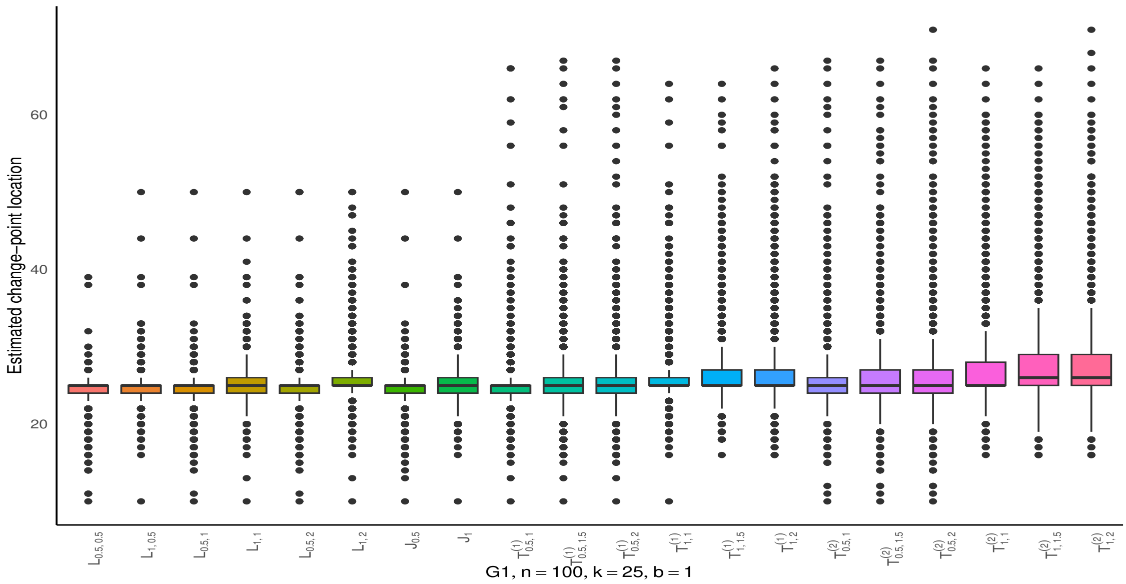
Table 3.3: Estimates and standard deviations of change point locations.

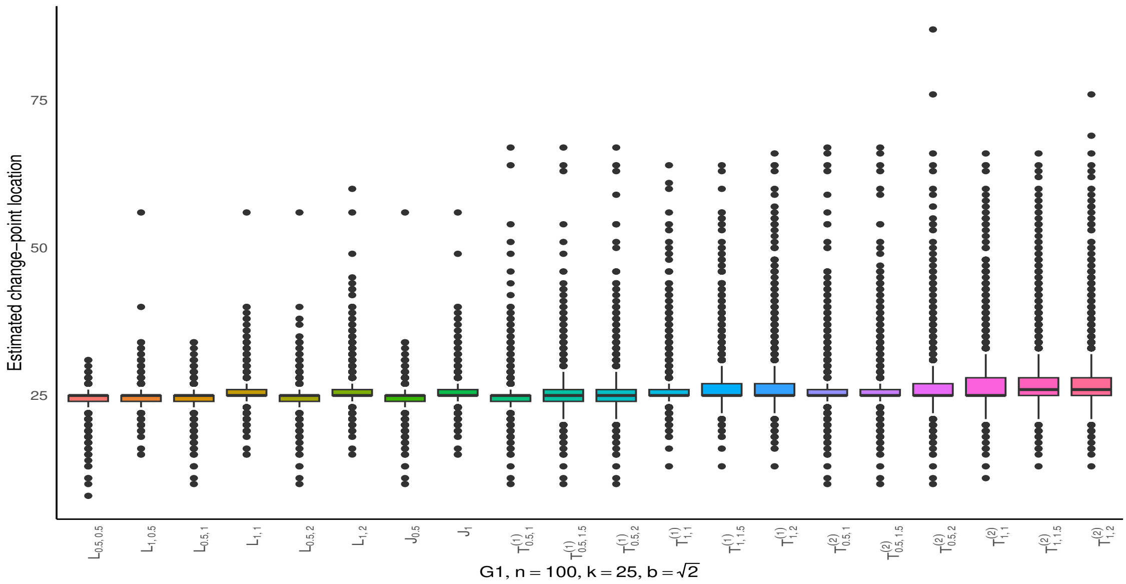
	G1	G2	U
	$n=100, b=\sqrt{2}, k=50$		
$\mathscr{L}_{100,0.5,0.5}$	48.55 (2.586)	48.25 (3.725)	49.84 (0.496)
$\mathscr{L}_{100,1,0.5}$	48.69 (2.259)	48.48 (3.124)	49.85 (0.435)
$\mathscr{L}_{100,0.5,1}$	48.83 (2.373)	48.56 (3.531)	49.95 (0.257)
$\mathscr{L}_{100,1,1}$	48.97 (2.115)	48.75 (3.034)	49.96 (0.24)
$\mathscr{L}_{100,0.5,2}$	49.1 (2.299)	48.91 (3.414)	50 (0.045)
$\mathscr{L}_{100,1,2}$	49.19 (2.109)	49.03 (2.952)	50 (0.039)
$\mathscr{J}_{100,0.5}$	48.89 (2.322)	48.76 (3.532)	49.99 (0.074)
$\mathscr{J}_{100,1}$	48.97 (2.143)	48.9 (3.21)	49.99 (0.074)
$T^{(1)}_{100,0.5}(1)$	49.05 (2.427)	49.35 (4.224)	50 (0)
$T^{(1)}_{100,0.5}(1.5)$	49.24 (2.484)	49.56 (4.103)	50 (0)
$T^{(1)}_{100,0.5}(2)$	49.37 (2.598)	49.64 (4.17)	50 (0)
$T^{(1)}_{100,1}(1)$	49.11 (2.266)	49.42 (3.529)	50 (0)
$T^{(1)}_{100,1}(1.5)$	49.32 (2.244)	49.55 (3.517)	50 (0)
$T^{(1)}_{100,1}(2)$	49.45 (2.273)	49.67 (3.502)	50 (0)
$T^{(2)}_{100,0.5}(1)$	49.39 (2.778)	49.65 (4.346)	50 (0)
$T^{(2)}_{100,0.5}(1.5)$	49.56 (2.774)	49.75 (4.25)	50 (0)
$T^{(2)}_{100,0.5}(2)$	49.68 (2.729)	49.83 (4.129)	50 (0)
$T^{(2)}_{100,1}(1)$	49.48 (2.387)	49.64 (3.593)	50 (0)
$T^{(2)}_{100,1}(1.5)$	49.66 (2.414)	49.77 (3.615)	50 (0)
$T^{(2)}_{100,1}(2)$	49.73 (2.482)	49.82 (3.716)	50 (0)
	$n=100, b=\sqrt{2}, k=50$		
$\mathscr{L}_{100,0.5,0.5}$	24.27 (1.825)	24.64 (3.197)	24.99 (0.077)
$\mathscr{L}_{100,1,0.5}$	24.92 (1.768)	26.12 (3.815)	25 (0.032)
$\mathscr{L}_{100,0.5,1}$	24.56 (1.853)	24.99 (3.298)	25 (0)
$\mathscr{L}_{100,1,1}$	25.3 (2.193)	26.53 (4.288)	25.01 (0.095)
$\mathscr{L}_{100,0.5,2}$	24.87 (2.136)	25.53 (3.781)	25 (0)
$\mathscr{L}_{100,1,2}$	25.84 (2.858)	27.07 (4.773)	25.05 (0.229)
$\mathscr{J}_{100,0.5}$	24.58 (1.963)	25.42 (3.877)	25 (0)
$\mathscr{J}_{100,1}$	25.24 (2.187)	26.91 (4.799)	25.01 (0.118)
$T^{(1)}_{100,0.5}(1)$	25.04 (2.855)	27.44 (7.333)	25.01 (0.111)
$T^{(1)}_{100,0.5}(1.5)$	25.31 (3.261)	27.47 (7.043)	25.02 (0.158)
$T^{(1)}_{100,0.5}(2)$	25.6 (3.671)	27.61 (7.106)	25.04 (0.2)
$T^{(1)}_{100,1}(1)$	26.05 (3.784)	29.67 (8.305)	25.12 (0.403)
$T^{(1)}_{100,1}(1.5)$	26.59 (4.502)	29.52 (7.876)	25.17 (0.492)
$T^{(1)}_{100,1}(2)$	27.05 (5.048)	29.46 (7.623)	25.2 (0.552)
$T^{(2)}_{100,0.5}(1)$	25.77 (4.02)	27.7 (7.422)	25.05 (0.227)
$T^{(2)}_{100,0.5}(1.5)$	25.99 (4.162)	27.86 (7.365)	25.06 (0.263)
$T^{(2)}_{100,0.5}(2)$	26.23 (4.739)	27.81 (7.194)	25.06 (0.272)
$T^{(2)}_{100,1}(1)$	27.25 (5.273)	29.7 (8.063)	25.23 (0.611)
$T^{(2)}_{100,1}(1.5)$	27.48 (5.419)	29.56 (7.684)	25.27 (0.677)
$T^{(2)}_{100,1}(2)$	27.69 (5.781)	29.69 (7.708)	25.29 (0.696)

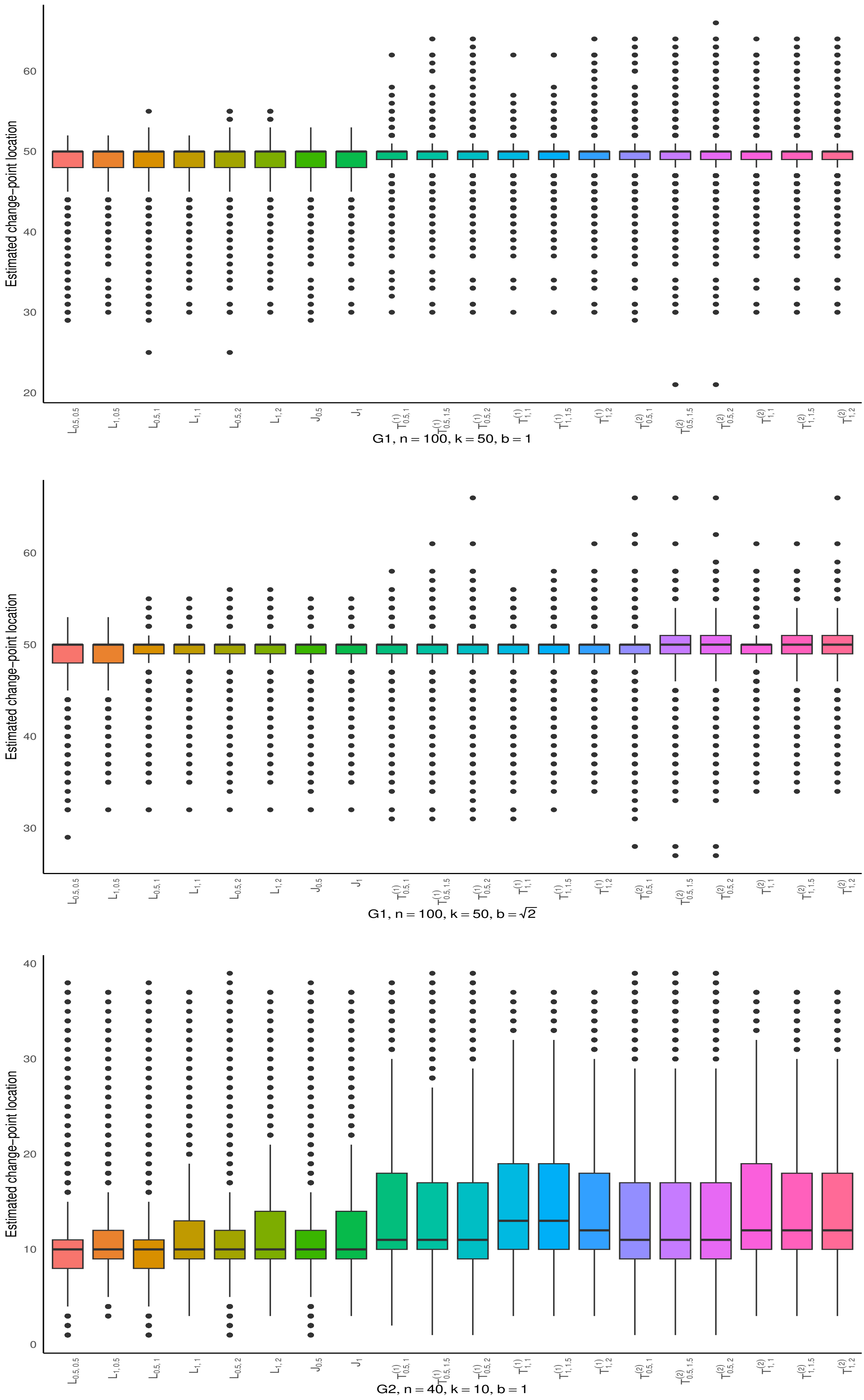
Figure 3.1: Estimated change point location - $N = 2000$ replications

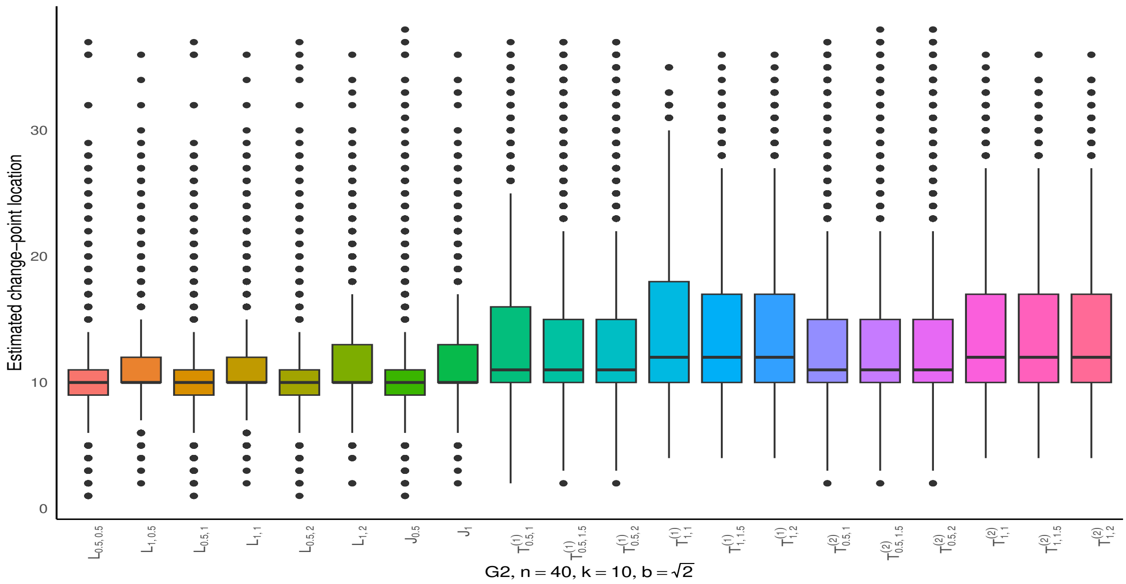


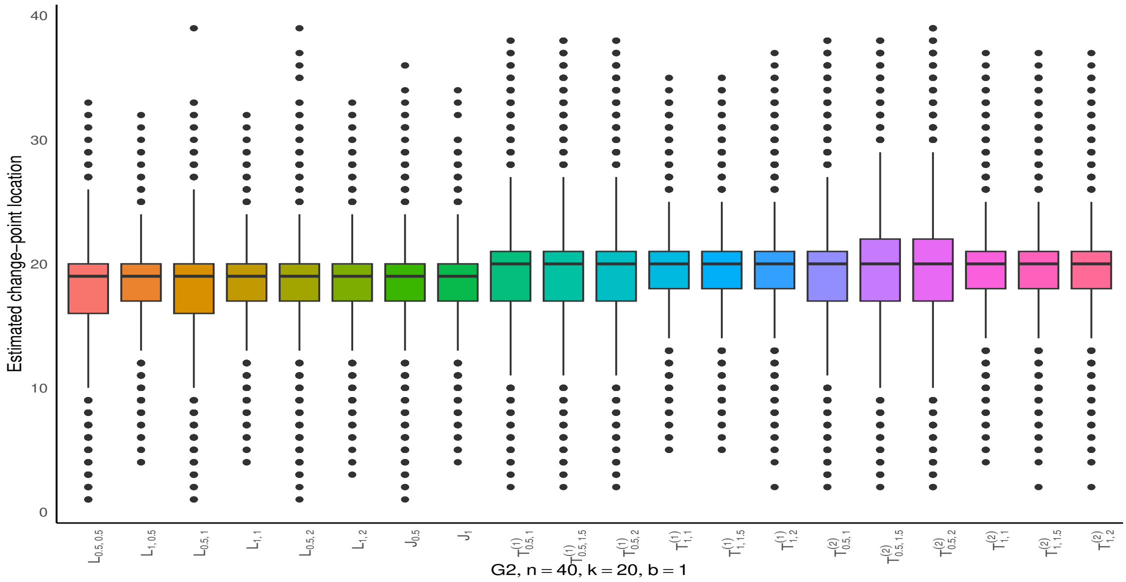

Estimated change−point location
G1, n = 40, k = 20, b = √2

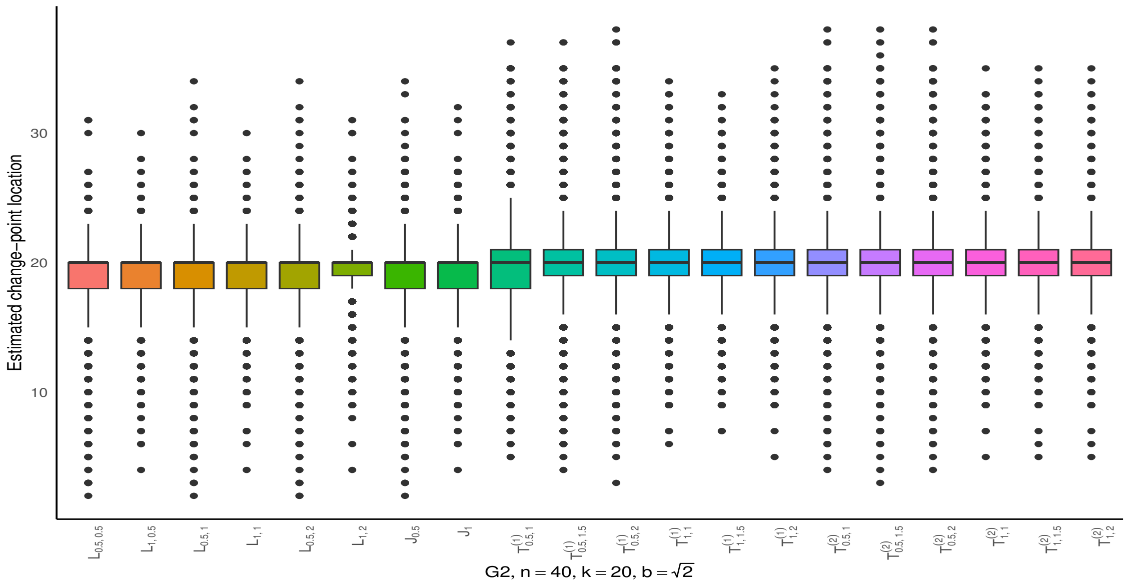

Estimated change−point location
G1, n = 100, k = 25, b = 1

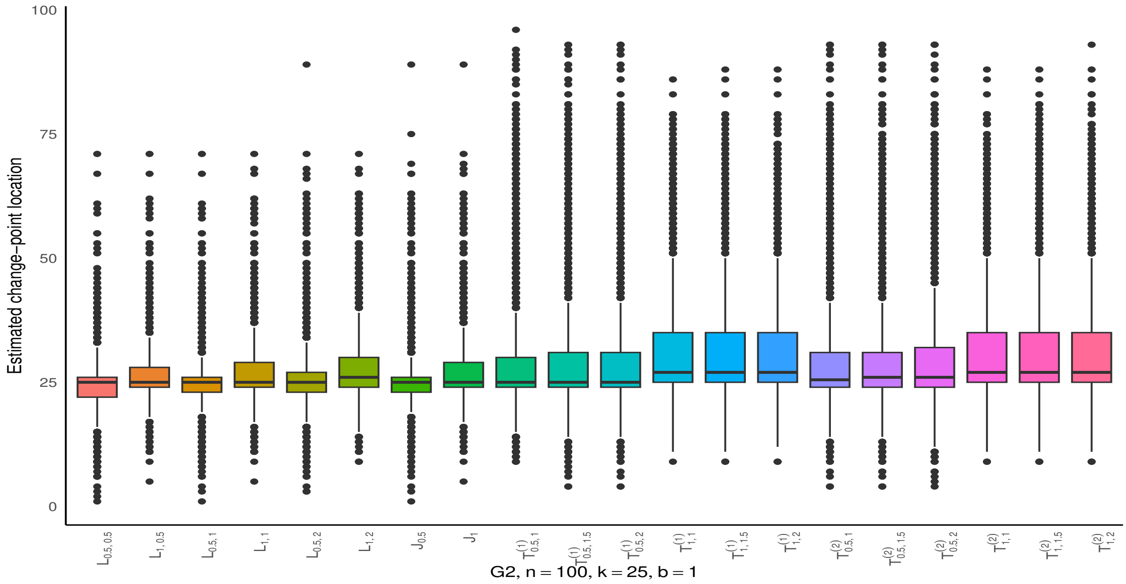

Estimated change−point location
G1, n = 100, k = 25, b = √2

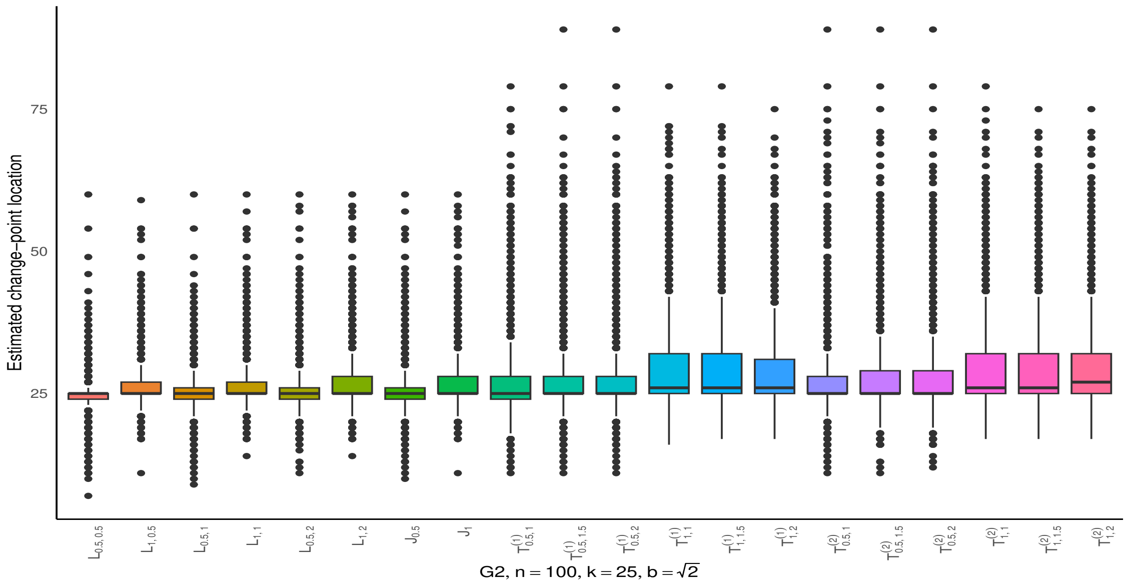

Estimated change−point location
G1, n = 100, k = 50, b = 1
Estimated change−point location
G1, n = 100, k = 50, b = √2
Estimated change−point location
G2, n = 40, k = 10, b = 1

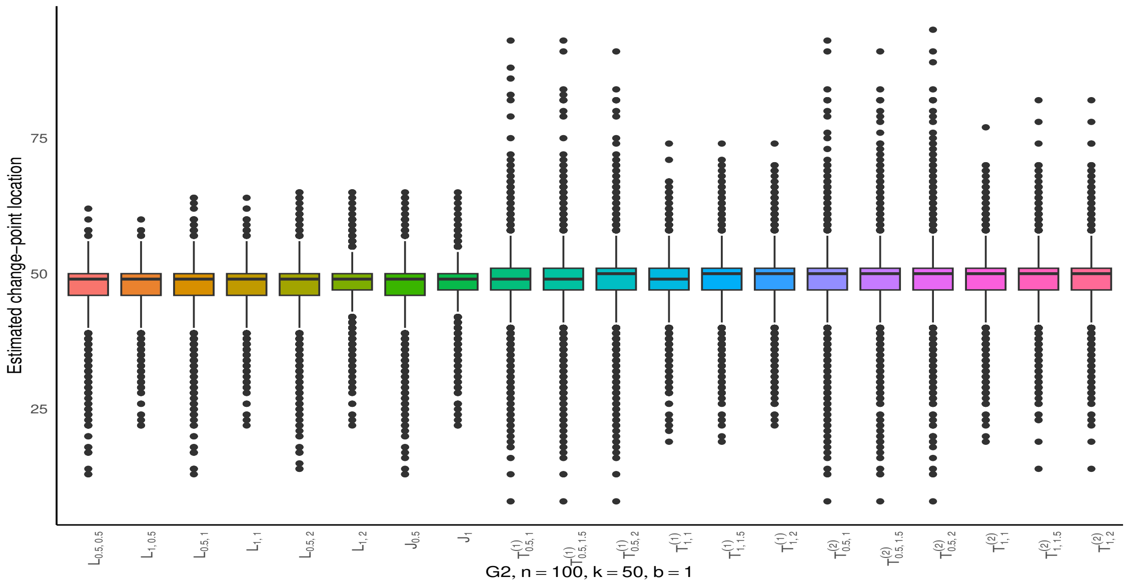

Estimated change−point location
G2, n = 40, k = 10, b = √2

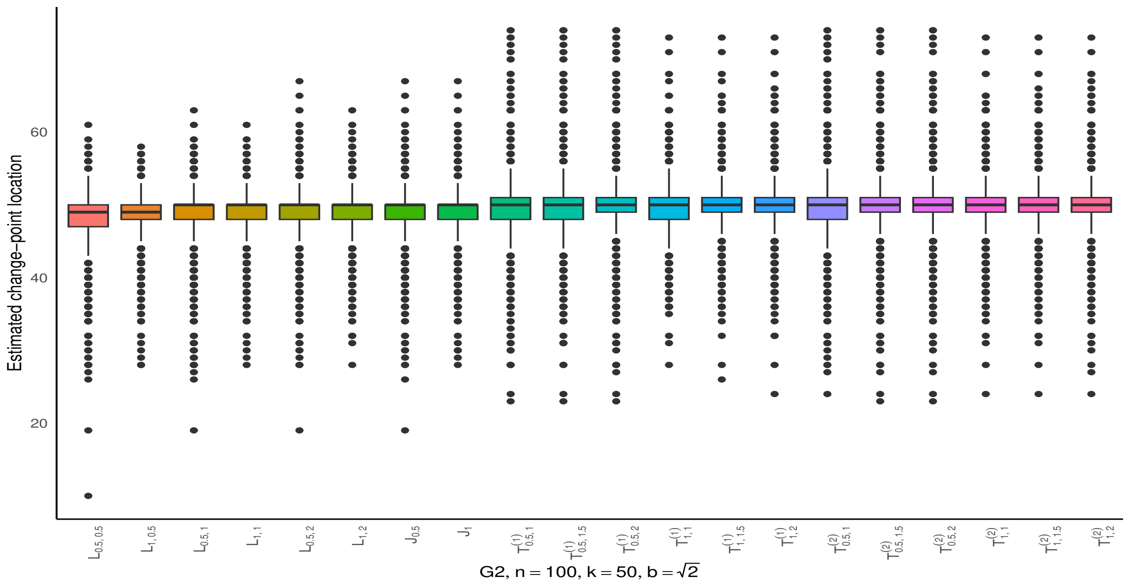

Estimated change−point location
G2, n = 40, k = 20, b = 1


Estimated change−point location
G2, n = 40, k = 20, b = √2


Estimated change−point location
100
75
50
25
0
G2, n = 100, k = 25, b = 1


Estimated change−point location
75
50
25
G2, n = 100, k = 25, b = √2


Estimated change−point location
75
50
25
G2, n = 100, k = 50, b = 1



3.6 Real data examples

In this section, we apply the novel tests to several types of data, highlighting their applicability in macroeconomics and meteorology. Both cases are well-documented in the literature, and our tests provide further insight into the observed phenomena.

3.6.1 US GNP data

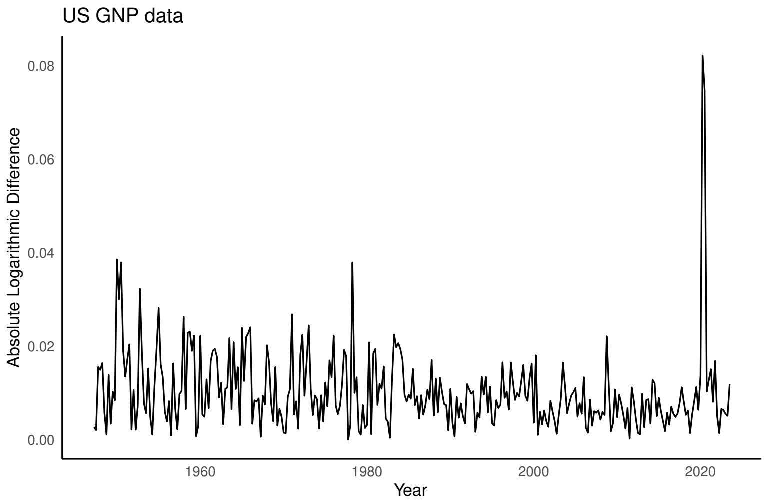
In this part, we analyze the GNP (Gross National Product) data. The data consist of the quarterly U.S. GNP in billions of chained 2017 dollars from 1947(1) to 2023(3), and they have been seasonally adjusted [134]. Chained dollars were introduced in 1996 by the U.S. Bureau of Economic Analysis as a method to better adjust the dollar currency for inflation over time [110]. The difference in the logarithm of GNP can be naturally interpreted as the growth rate of GNP [121]. In [123], Shumway and Stoffer considered a shorter time frame, 1947(1) to 2002(3), in chained 1996 dollars, also seasonally adjusted. They examined the difference in the logarithm of the GNP and concluded that there might be a structural break (i.e., a change-point) in the data in the year 1985. Shao and Zhang reached the same conclusion in [121], using the same dataset as in [123].

Note that our tests are designed for nonnegative data. Therefore, we considered the absolute difference in the logarithm of the GNP. The data can be seen in Figure 3.2. There is an obvious spike in the data in the year 2020, which may be attributed to a combination of factors, including increased government spending to alleviate the effects of the COVID-19 pandemic, the rebound effect after the pandemic subsided, and direct monetary measures taken by the U.S. Federal Reserve Bank, such as lowering interest rates and other monetary expansion measures. The change-point in the volatility of GNP growth in the mid-1980s can be attributed to the reduction in the volatility of durable goods production [92]. There might be similar reasons for the change observed in Figure 3.2. We have performed single change point detection using the permutation bootstrap algorithm (Algorithm 2) with $N = 2000$ replications to obtain p-values. The results are presented in Table 3.4. The novel tests report a p-value smaller than 0.05. Our results for both novel tests are consistent with those in [121, 123] and provide further evidence that the change point occurred in the year 1985. Moreover, it is clear that the competition tests detect the change point existence, albeit the location is slightly earlier, which can be explained by their lower precision, as demonstrated in Figure 3.1.

Table 3.4: p-values of novel tests - US GNP data

Statistic	$\mathscr{I}_{0.5}$	$\mathscr{I}_1$	$\mathscr{L}_{0.5,0.5}$	$\mathscr{L}_{1,0.5}$	$\mathscr{L}_{0.5,1}$	$\mathscr{L}_{1,1}$	$\mathscr{L}_{0.5,2}$	$\mathscr{L}_{1,2}$
p-values	0.0050	0.0025	0.0025	0.0010	0.0035	0.0020	0.0065	0.0025
Position	1985 (3)	1985 (3)	1985 (3)	1985 (3)	1985 (3)	1985 (3)	1985 (3)	1985 (3)
Statistic	$T^{(1)}_{0.5,1}$	$T^{(1)}_{0.5,1.5}$	$T^{(1)}_{0.5,2}$	$T^{(1)}_{0.5,3}$	$T^{(1)}_{1,1}$	$T^{(1)}_{1,1.5}$	$T^{(1)}_{1,2}$	$T^{(1)}_{1,3}$
p-values	0.0070	0.0085	0.0060	0.0075	0.0030	0.0020	0.0040	0.0025
Position	1984 (2)	1984 (2)	1984 (2)	1984 (2)	1984 (2)	1984 (2)	1984 (2)	1985 (3)
Statistic	$T^{(2)}_{0.5,1}$	$T^{(2)}_{0.5,1.5}$	$T^{(2)}_{0.5,2}$	$T^{(2)}_{0.5,3}$	$T^{(2)}_{1,1}$	$T^{(2)}_{1,1.5}$	$T^{(2)}_{1,2}$	$T^{(2)}_{1,3}$
p-values	0.0060	0.0095	0.0080	0.0085	0.0025	0.0025	0.0040	0.0035
Position	1984 (2)	1984 (2)	1984 (2)	1984 (2)	1984 (2)	1984 (2)	1984 (2)	1985 (3)

Figure 3.2: Quarterly U.S. GNP absolute growth rate from 1947(1) to 2023(3)



3.6.2 Argentinian rainfall data

We applied our test to detect a change in the distribution of Argentinian yearly rainfall data. The rainfall data contain yearly rainfall in millimeters in Argentina from 1884 to 1996. The dataset originally came from [139], where Wu et al. proposed a test statistic based on isotonic regression. Shao and Zhang were interested in detecting the change point in the mean of the data in [121]. Wu et al. stated that the data provider believes that there is a change in the mean, which corresponds to the construction of a dam during 1952–1962, and the results from [121] supported this hypothesis.

Since we were unable to locate the tabulated data, we used Figure 5 from [121] to reconstruct the exact values from the dataset; see Figure 3.3. The results are presented in Table 3.5. The novel tests reported p-values of less than 0.05, and the position of the change point is the year 1956 for every novel test. The competitors' tests do not detect the change point, and the estimate they provide is imprecise. The results of the novel tests are shown to be consistent with those given in [121, 139].

Figure 3.3: Argentinian rainfall data: yearly rainfall (milimeters) in Argentina from 1884 to 1996.

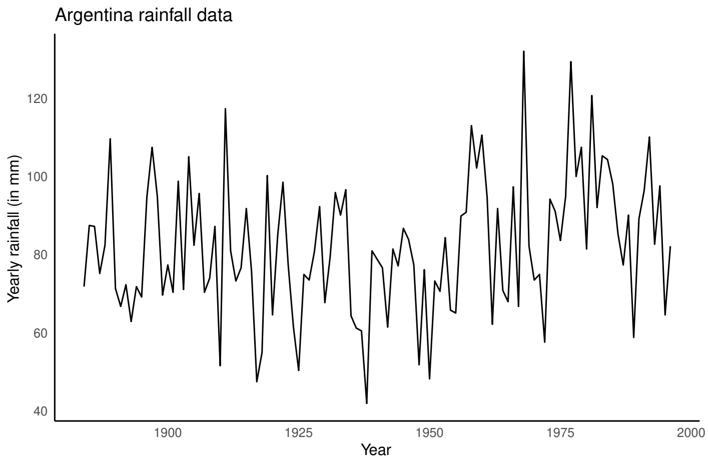


Table 3.5: p-values - Argentinian rainfall data

Statistic	$\mathscr{I}_{0.5}$	$\mathscr{I}_{1}$	$\mathscr{L}_{0.5,0.5}$	$\mathscr{L}_{1,0.5}$	$\mathscr{L}_{0.5,1}$	$\mathscr{L}_{1,1}$	$\mathscr{L}_{0.5,2}$	$\mathscr{L}_{1,2}$
p-values	0.0040	0.0080	0	0.0010	0.0035	0.0015	0.0020	0.0010
Position	1956	1956	1956	1956	1956	1956	1956	1956
Statistic	$T^{(1)}_{0.5,1}$	$T^{(1)}_{0.5,1.5}$	$T^{(1)}_{0.5,2}$	$T^{(1)}_{0.5,3}$	$T^{(1)}_{1,1}$	$T^{(1)}_{1,1.5}$	$T^{(1)}_{1,2}$	$T^{(1)}_{1,3}$
p-values	0.0780	0.0790	0.0890	0.0575	0.2175	0.1185	0.0750	0.0049
Position	1910	1910	1953	1956	1953	1953	1953	1956
Statistic	$T^{(2)}_{0.5,1}$	$T^{(2)}_{0.5,1.5}$	$T^{(2)}_{0.5,2}$	$T^{(2)}_{0.5,3}$	$T^{(2)}_{1,1}$	$T^{(2)}_{1,1.5}$	$T^{(2)}_{1,2}$	$T^{(2)}_{1,3}$
p-values	0.0670	0.0840	0.0850	0.0900	0.1380	0.1285	0.1100	0.0775
Position	1910	1914	1914	1953	1914	1953	1953	1953

Chapter 4

Change point analysis for matrix data

4.1 Introduction

In this chapter, we present the modification of the test statistic (2.4) to address change point problems. The results of this chapter can be found in [85]. The modification is constructed using a similar idea as in the univariate case, producing a statistic of the form (3.2). This is the first test of its kind for matrix data. Many applications of covariance matrices have driven research in this area and motivated us to modify the two-sample test. Since similar equities are usually correlated, and changes in the structure of such correlations are important to traders, one may wish to examine samples of covariance matrices. However, it is important to mention the caveat of independence. The data need to be independent; therefore, one needs to consider the source data that are independent.

4.2 The test statistic

In this part, we follow the construction (3.2) adapted for matrix data. We modify the test statistic (2.4) to address the change point-type problems. Let $X_1, X_2, \ldots, X_n$ be the sample of independent symmetric positive definite random matrices, where X_j has a cumulative distribution function F_j.

We want to test the hypothesis

$$\begin{gathered} H_0 : X_1 \overset{OID}{=} X_2 \overset{OID}{=} \ldots \overset{OID}{=} X_n \\ \text{against the alternative} \\ H_1 : X_1 \overset{OID}{=} X_2 \overset{OID}{=} \ldots \neq X_k \overset{OID}{=} X_{k+1} \overset{OID}{=} \ldots \overset{OID}{=} X_n, \end{gathered} \tag{4.1}$$

where the corresponding distributions of X_1 and X_n are unknown and the index k is also unknown. For the notion of OID random variables, refer to Definition 2.1. To achieve this, we use the orthogonally invariant Hankel transform (see Definition 1.6).

Remark. Hadjicosta and Richards demonstrated that the orthogonally invariant Hankel transform uniquely determines the distribution within the class of orthogonally invariant distributions in [54]. Theorem 2.1 extends this result, establishing that the orthogonally invariant Hankel transform uniquely determines the distribution within the broader class of OID distributions.

More precisely, when the distributions in question are orthogonally invariant, the result in Theorem 2.1 aligns with that presented in [54]. As a consequence, change point inference in the (class of) OID distributions becomes equivalent to change point inference in distribution.

For the remainder of this section, we will operate under the assumption that we are working with OID distributions.

For testing hypotheses (4.1), we propose the following test statistic (see [85]):

$$\mathscr{I}_{n,\gamma,\nu} = \max_{1\leqslant k<n}\Big[\Big(\frac{k(n-k)}{n^2}\Big)^{\gamma}\frac{k(n-k)}{n}\int_{T>0}\Big(\tilde{\mathscr{H}}_{k,\nu}(T)-\tilde{\mathscr{H}}^{0}_{n-k,\nu}(T)\Big)^2\mathrm{d}W(T)\Big],$$

where $\mathrm{d}W(T)$ is a standard Wishart measure, $\tilde{\mathscr{H}}_{k,\nu}$ denotes the empirical orthogonally invariant Hankel transform of the first k elements of the sample, while $\tilde{\mathscr{H}}^{0}_{n-k,\nu}$ denotes the empirical orthogonally invariant Hankel transform of the last $n-k$ elements of the sample. Using the same arguments as in Section 2.2.2, the test statistic can be expressed as:

$$\begin{aligned}\mathscr{I}_{n,\gamma,\nu} = \max_{1\leqslant k<n}\Big[\Big(\frac{k(n-k)}{n^2}\Big)^{\gamma}\frac{k(n-k)}{n}\Big(\frac{1}{k^2}\sum_{i=1}^{k}\sum_{j=1}^{k}\operatorname{etr}(-X_i-X_j)J_\nu(-X_i,X_j)\\
+\frac{1}{(n-k)^2}\sum_{i=k+1}^{n}\sum_{j=k+1}^{n}\operatorname{etr}(-X_i-X_j)J_\nu(-X_i,X_j)\\
-\frac{2}{k(n-k)}\sum_{i=1}^{k}\sum_{j=k+1}^{n}\operatorname{etr}(-X_i-X_j)J_\nu(-X_i,X_j)\Big)\Big].\end{aligned}$$

Note that under the null hypothesis, a small value of the test statistic is expected. Therefore, large values of the test statistic are considered significant. The parameter $\gamma\in(0,1]$ acts as a tuning parameter, ensuring the convergence of the test statistic in probability whenever $\gamma\neq 0$. In the literature, γ does not significantly influence the test power (see, e.g., [62]). In the next section, we present the asymptotic results of the novel test.

4.3 Asymptotic results

In order to obtain the limiting null distribution of the test statistic, we first note that the test statistic $\mathscr{I}$ can be represented as $\mathscr{I}_{n,\gamma,\nu} = \max_{1\leqslant k\leqslant n}[c_{n,k}(\gamma)I_{k,n-k,\nu}]$, where $I_{k,n-k,\nu}$ is a two-sample test statistic (2.4), and $c_{n,k} = \big(\frac{k(n-k)}{n^2}\big)^{\gamma}\frac{k(n-k)}{n}$. For the sake of brevity, we drop the parameter ν in the subsequent text and simply denote $I_{k,n-k,\nu}$ as $I_{k,n-k}$.

Using the notation: $q(x,y)=\operatorname{etr}(-x-y)J_\nu(-x,y)$ and

$$\tilde{q}(x,y) = q(x,y)-E(q(X,X_s)|X=x)-E(q(X_r,X)|X=y)+Eq(X_r,X_s),\ r\neq s,$$

the following equalities are obtained:

$$E(\tilde{q}(X_r,X_s)|X_r) = E(\tilde{q}(X_r,X_s)|X_s) = E\tilde{q}(X_r,X_s)=0, \tag{4.2}$$

under H_0. It is worth noting that the function $q(x,y)$ is symmetric in its arguments. Moreover, we can easily establish that under H_0

$$E\tilde{q}^2(X_1,X_2)=\int_{X_1>0,X_2>0}\tilde{q}^2(X_1,X_2)\mathrm{d}F(X_1)\mathrm{d}F(X_2)<\infty.$$

From this, we conclude that there exist orthogonal eigenfunctions $f_s(t)$, $s=1,2,\ldots$ and corresponding eigenvalues λ_s, $s=1,2,\ldots$, such that the following spectral approximation holds (see [17, 118]):

$$\lim_{K\to\infty}\int_{X_1>0,X_2>0}\Big(\tilde{q}^2(X_1,X_2)-\sum_{s=1}^{K}\lambda_s f_s(X_1)f_s(X_2)\Big)^2\mathrm{d}F(X_1)\mathrm{d}F(X_2)=0; \tag{4.3}$$

$$\int_{X_1>0} f_s^2(X_1)\,\mathrm{d}F(X_1) = 1,\ s = 1,2,\ldots;$$
$$\int_{X_1>0} f_r(X_1) f_s(X_1)\,\mathrm{d}F(X_1) = 0,\ r \neq s = 1,2,\ldots,$$

from which we obtain

$$E\tilde{q}^2(X_1,X_2) = \int_{X_1>0,X_2>0} \tilde{q}^2(X_1,X_2)\,\mathrm{d}F(X_1)\,\mathrm{d}F(X_2) = \sum_{j=1}^{\infty}\lambda_j^2.$$

In the proof of the following theorem, we make use of the Hájek-Rényi inequality. Its most general form is as follows:

THEOREM 4.1. *[95, Theorem 2.3] Let $\{X_n, n \geqslant 1\}$ be an associated sequence of random variables defined on a Hilbert space H. Let $EX_n = 0$ and $E\|X_n\|^2 < \infty$ for every $n \geqslant 1$. Let $\{b_n, n \geqslant 1\}$ be a nondecreasing sequence of positive real numbers. Then, for any $\varepsilon > 0$ and any $\alpha > 1$, we have*

$$P\left(\max_{1\leqslant k\leqslant n}\left\|\frac{1}{b_k}\sum_{j=1}^{k} X_j\right\| \geqslant \varepsilon\right) \leqslant \frac{2}{\varepsilon^2}\frac{\alpha^2}{1-\frac{1}{\alpha^2}}\left\{\sum_{j=1}^{n}\frac{E\|X_j\|^2}{b_j^2} + 2\sum_{j=1}^{n}\frac{E\langle X_j, \sum_{k=1}^{j-1} X_k\rangle}{b_j^2}\right\}.$$

We state the main result of this section:

THEOREM 4.2. *[85, Theorem 1] Let $X_1, X_2, \ldots X_n$ be independent random matrices orthogonally invariant in distribution, where $X_1 \in F$. Let $\gamma \in (0,1]$ and let $(\lambda_j)_{j=1}^{\infty}$ be the descending sequence of eigenvalues defined in* (4.3). *Then the asymptotic distribution of $\mathscr{I}_{n,\gamma,\nu}$ is the same as that of*

$$\sup_{t\in(0,1)}\left[(t(1-t))^{\gamma}\left|(EJ_\nu(-X_1,X_1) - E\,\mathrm{etr}(-X_1-X_2)J_\nu(-X_1,X_2)) + \sum_{j=1}^{\infty}\lambda_j\Big(\frac{B_j^2(t)}{t(1-t)} - 1\Big)\right|\right],$$

where $\{B_j(t),\ t \in (0,1)\}$, $j = 1,2,\ldots$ are independent Brownian bridges.

Proof. We break the proof into several steps.

In Step (I), we decompose the test statistic into four parts, and show that two of them do not influence the limiting distribution. We assert this by making repeated use of the Hájek-Rényi inequality to majorize the terms.

In Step (II), we determine the finite-dimensional distributions of the limiting process, by establishing the approximation formula, and making use of the Donsker theorem.

In Step (III), using the properties of the Wiener process, we determine the asymptotic distribution in question and finalize the proof.

In the following, we denote $E(q(X,Y) \mid Y = X_s)$, where X is independent of X_s and has the same cumulative distribution function F, as $E(q(X,X_s) \mid X_s)$. Similarly, we denote $E(q(X,Y) \mid X = X_r)$, where X is independent of X_r and has the same cumulative distribution function F, as $E(q(X_r,X) \mid X_r)$.

Step (I): The following decomposition holds:

$$I_{k,n-k} = C_{k1} + C_{k2} + C_{k3} + C_{k4},$$

where

$$C_{k1} = \frac{n}{k(n-k)}\Bigg(\frac{1}{k}\sum_{r=1}^{k}\sum_{\substack{s=1\\ s\neq r}}^{k}\tilde{q}(X_r,X_s) + \frac{1}{n-k}\sum_{r=k+1}^{n}\sum_{\substack{s=k+1\\ s\neq r}}^{n}\tilde{q}(X_r,X_s)$$

$$-\frac{1}{n}\sum_{r=1}^{n}\sum_{\substack{s=1\\s\neq r}}^{n}\tilde{q}(X_r,X_s)\Bigg),$$

$$C_{k2}=\frac{n}{k(n-k)}(E\,J_\nu(-X_1,X_1)-E\,q(X_1,X_2)),$$

$$C_{k3}=-\frac{2}{k^2}\sum_{r=1}^{k}(E(q(X_r,X_s)|X_r)-E\,q(X_1,X_2))-\frac{2}{(n-k)^2}\sum_{r=k+1}^{n}(E(q(X_r,X_s)|X_r)$$
$$-E\,q(X_1,X_2)),$$

$$C_{k4}=\frac{1}{k^2}\sum_{r=1}^{k}(J_\nu(-X_r,X_r)-E\,J_\nu(-X_r,X_r))+\frac{1}{(n-k)^2}\sum_{r=k+1}^{n}(J_\nu(-X_r,X_r)$$
$$-E\,J_\nu(-X_r,X_r)).$$

We prove this decomposition holds.

$$C_{k1}+C_{k2}+C_{k3}+C_{k4}=\frac{n}{k^2(n-k)}\sum_{r=1}^{k}\sum_{\substack{s=1\\s\neq r}}^{k}q(X_r,X_s)-\frac{n}{k^2(n-k)}\sum_{r=1}^{k}\sum_{\substack{s=1\\s\neq r}}^{k}E(q(X,X_s)|X_s)$$
$$-\frac{n}{k^2(n-k)}\sum_{r=1}^{k}\sum_{\substack{s=1\\s\neq r}}^{k}E(q(X_r,X)|X_r)+\frac{n}{k^2(n-k)}\sum_{r=1}^{k}\sum_{\substack{s=1\\s\neq r}}^{k}E\,q(X_1,X_2)+\frac{n}{k(n-k)^2}\sum_{r=k+1}^{n}\sum_{\substack{s=k+1\\s\neq r}}^{n}q(X_r,X_s)$$
$$-\frac{n}{k(n-k)^2}\sum_{r=k+1}^{n}\sum_{\substack{s=k+1\\s\neq r}}^{n}E(q(X,X_s)|X_s)-\frac{n}{k(n-k)^2}\sum_{r=k+1}^{n}\sum_{\substack{s=k+1\\s\neq r}}^{n}E(q(X_r,X)|X_r)$$
$$+\frac{n}{k(n-k)^2}\sum_{r=k+1}^{n}\sum_{\substack{s=k+1\\s\neq r}}^{n}E\,q(X_1,X_2)-\frac{1}{k(n-k)}\sum_{r=1}^{n}\sum_{\substack{s=1\\s\neq r}}^{n}q(X_r,X_s)+\frac{1}{k(n-k)}\sum_{r=1}^{n}\sum_{\substack{s=1\\s\neq r}}^{n}E(q(X,X_s)|X_s)$$
$$+\frac{1}{k(n-k)}\sum_{r=1}^{n}\sum_{\substack{s=1\\s\neq r}}^{n}E(q(X_r,X)|X_r)-\frac{1}{k(n-k)}\sum_{r=1}^{n}\sum_{\substack{s=1\\s\neq r}}^{n}E\,q(X_1,X_2)+\frac{n}{k(n-k)}E\,J_\nu(-X_1,X_1)$$
$$-\frac{n}{k(n-k)}E\,q(X_1,X_2)-\frac{2}{k^2}\sum_{r=1}^{k}E(q(X_r,X)|X_r)+\frac{2}{k^2}\sum_{r=1}^{k}E\,q(X_1,X_2)-\frac{2}{(n-k)^2}\sum_{r=k+1}^{n}E(q(X_r,X)|X_r)$$
$$+\frac{2}{(n-k)^2}\sum_{r=k+1}^{n}E\,q(X_1,X_2)+\frac{1}{k^2}\sum_{r=1}^{k}J_\nu(-X_r,X_r)-\frac{1}{k}E\,J_\nu(-X_1,X_1)+\frac{1}{(n-k)^2}\sum_{r=k+1}^{n}J_\nu(-X_r,X_r)$$
$$-\frac{1}{n-k}E\,J_\nu(-X_1,X_1).$$

Simplifying the blue part, we obtain:

$$\frac{n}{k^2(n-k)}\sum_{r=1}^{k}\sum_{\substack{s=1\\s\neq r}}^{k}q(X_r,X_s)+\frac{n}{k(n-k)^2}\sum_{r=k+1}^{n}\sum_{\substack{s=k+1\\s\neq r}}^{n}q(X_r,X_s)-\frac{1}{k(n-k)}\sum_{r=1}^{n}\sum_{\substack{s=1\\s\neq r}}^{n}q(X_r,X_s)$$
$$+\frac{1}{k^2}\sum_{r=1}^{k}J_\nu(-X_r,X_r)+\frac{1}{(n-k)^2}\sum_{r=k+1}^{n}J_\nu(-X_r,X_r)$$
$$=\frac{n}{k^2(n-k)}\sum_{r=1}^{k}\sum_{\substack{s=1\\s\neq r}}^{k}q(X_r,X_s)+\frac{n}{k(n-k)^2}\sum_{r=k+1}^{n}\sum_{\substack{s=k+1\\s\neq r}}^{n}q(X_r,X_s)-\frac{1}{k(n-k)}\sum_{r=1}^{k}\sum_{\substack{s=1\\s\neq r}}^{k}q(X_r,X_s)$$

$$-\frac{1}{k(n-k)}\sum_{r=k+1}^{n}\sum_{s=1}^{k}q(X_r,X_s)-\frac{1}{k(n-k)}\sum_{r=1}^{k}\sum_{s=k+1}^{n}q(X_r,X_s)-\frac{1}{k(n-k)}\sum_{r=k+1}^{n}\sum_{\substack{s=k+1\\ s\neq r}}^{n}q(X_r,X_s)$$
$$+\frac{1}{k^2}\sum_{r=1}^{k}J_\nu(-X_r,X_r)+\frac{1}{(n-k)^2}\sum_{r=k+1}^{n}J_\nu(-X_r,X_r)=\frac{1}{k^2}\sum_{r=1}^{k}\sum_{s=1}^{k}q(X_r,X_s)-\frac{1}{k^2}\sum_{r=1}^{k}J_\nu(-X_r,X_r)$$
$$+\frac{1}{(n-k)^2}\sum_{r=k+1}^{n}\sum_{s=k+1}^{n}q(X_r,X_s)-\frac{1}{(n-k)^2}\sum_{r=k+1}^{n}J_\nu(-X_r,X_r)-\frac{2}{k(n-k)}\sum_{r=1}^{k}\sum_{s=k+1}^{n}q(X_r,X_s)$$
$$+\frac{1}{k^2}\sum_{r=1}^{k}J_\nu(-X_r,X_r)+\frac{1}{(n-k)^2}\sum_{r=k+1}^{n}J_\nu(-X_r,X_r)=\frac{1}{k^2}\sum_{r=1}^{k}\sum_{s=1}^{k}q(X_r,X_s)$$
$$+\frac{1}{(n-k)^2}\sum_{r=k+1}^{n}\sum_{s=k+1}^{n}q(X_r,X_s)-\frac{2}{k(n-k)}\sum_{r=1}^{k}\sum_{s=k+1}^{n}q(X_r,X_s),$$

which is exactly the form of the statistic $I_{k,n}$. We used the symmetry of $q(x,y)$ and the fact that finite sums commute.

Let us focus on the red part now. We have:

$$\frac{n}{k^2(n-k)}\sum_{r=1}^{k}\sum_{\substack{s=1\\ s\neq r}}^{k}Eq(X_1,X_2)+\frac{n}{k(n-k)^2}\sum_{r=k+1}^{n}\sum_{\substack{s=k+1\\ s\neq r}}^{n}Eq(X_1,X_2)-\frac{1}{k(n-k)}\sum_{r=1}^{n}\sum_{\substack{s=1\\ s\neq r}}^{n}Eq(X_1,X_2)$$
$$-\frac{n}{k(n-k)}Eq(X_1,X_2)+\frac{2}{k^2}\sum_{r=1}^{k}Eq(X_1,X_2)+\frac{2}{(n-k)^2}\sum_{r=k+1}^{n}Eq(X_1,X_2)+\frac{n}{k(n-k)}EJ_\nu(-X_1,X_1)$$
$$-\frac{1}{k}EJ_\nu(-X_1,X_1)-\frac{1}{n-k}EJ_\nu(-X_1,X_1)=\frac{nk-n}{k(n-k)}Eq(X_1,X_2)+\frac{n^2-nk-n}{k(n-k)}Eq(X_1,X_2)$$
$$+\frac{n-n^2}{k(n-k)}Eq(X_1,X_2)-\frac{n}{k(n-k)}Eq(X_1,X_2)+\frac{2n-2k}{k(n-k)}Eq(X_1,X_2)+\frac{2k}{k(n-k)}Eq(X_1,X_2)$$
$$=\frac{nk-n+n^2-nk-n-n^2+n-n+2n-2k+2k}{k(n-k)}Eq(X_1,X_2)=0.$$

For the violet part, note that:

$$-\frac{n}{k^2(n-k)}\sum_{r=1}^{k}\sum_{\substack{s=1\\ s\neq r}}^{k}E(q(X,X_s)|X_s)-\frac{n}{k(n-k)^2}\sum_{r=k+1}^{n}\sum_{\substack{s=k+1\\ s\neq r}}^{n}E(q(X,X_s)|X_s)$$
$$+\frac{1}{k(n-k)}\sum_{r=1}^{n}\sum_{\substack{s=1\\ s\neq r}}^{n}E(q(X,X_s)|X_s)=\frac{-n(k-1)}{k^2(n-k)}\sum_{s=1}^{k}E(q(X,X_s)|X_s)$$
$$+\frac{-n(n-k-1)}{k(n-k)^2}\sum_{s=k+1}^{n}E(q(X,X_s)|X_s)+\frac{n-1}{k(n-k)}\sum_{s=1}^{n}E(q(X,X_s)|X_s)$$
$$=\frac{-nk+n+nk-k}{k^2(n-k)}\sum_{s=1}^{k}E(q(X,X_s)|X_s)+\frac{-n^2+nk+n^2-nk-n+k}{k(n-k)^2}\sum_{s=k+1}^{n}E(q(X,X_s)|X_s)$$
$$=\frac{1}{k^2}\sum_{s=1}^{k}E(q(X,X_s)|X_s)+\frac{1}{(n-k)^2}\sum_{s=k+1}^{n}E(q(X,X_s)|X_s).$$

Since $q(x,y)=q(y,x)$, we can conclude the violet part equals

$$\frac{1}{k^2}\sum_{s=1}^{k}E(q(X_s,X)|X_s)+\frac{1}{(n-k)^2}\sum_{s=k+1}^{n}E(q(X_s,X)|X_s).$$

For the brown part, we repeat the same steps used above and obtain:

$$-\frac{n}{k^2(n-k)}\sum_{r=1}^{k}\sum_{\substack{s=1\\ s\neq r}}^{k} E(q(X_r,X)|X_r)-\frac{n}{k(n-k)^2}\sum_{r=k+1}^{n}\sum_{\substack{s=k+1\\ s\neq r}}^{n} E(q(X_r,X)|X_r)$$
$$+\frac{1}{k(n-k)}\sum_{r=1}^{n}\sum_{\substack{s=1\\ s\neq r}}^{n} E(q(X_r,X)|X_r)-\frac{2}{k^2}\sum_{r=1}^{k} E(q(X_r,X)|X_r)-\frac{2}{(n-k)^2}\sum_{r=k+1}^{n} E(q(X_r,X)|X_r)$$
$$=\frac{-n(k-1)-2(n-k)}{k^2(n-k)}\sum_{s=1}^{k} E(q(X_s,X)|X_s)+\frac{-n(n-k-1)-2k}{k(n-k)^2}\sum_{s=k+1}^{n} E(q(X_s,X)|X_s)$$
$$+\frac{n-1}{k(n-k)}\sum_{s=1}^{n} E(q(X_s,X)|X_s)=\frac{-nk+n+nk-k-2n-2k}{k^2(n-k)}\sum_{s=1}^{k} E(q(X_s,X)|X_s)$$
$$+\frac{-n^2+nk+n^2-nk-n+k-2k}{k(n-k)^2}\sum_{s=k+1}^{n} E(q(X_s,X)|X_s)=-\frac{1}{k^2}\sum_{s=1}^{k} E(q(X_s,X)|X_s)$$
$$-\frac{1}{(n-k)^2}\sum_{s=k+1}^{n} E(q(X_s,X|X_s).$$

From the calculation above, we conclude that the violet and brown parts cancel out. Moreover, the proposed decomposition holds.

The main idea is to demonstrate that C_{k3} and C_{k4} have no impact on the limiting distribution, while the non-random term C_{k2} and the random term C_{k1} do. Then, we establish the limiting distribution of C_{k1}, which is not a straightforward task and depends on the ability to determine the distribution of an arbitrarily close approximation. The final part of Step (I) consists of demonstrating that the asymptotic distributions of C_{k1} and the approximation are identical.

We begin by showing that C_{k3} and C_{k4} do not influence the limiting distribution.

We can see that if X is a random variable independent of X_r having the same distribution function F, random variables

$$L_r = E(q(X_r,X)|X_r) - E\,q(X_1,X_2),\, r = 1,2,\dots,n$$

are independent and identically distributed with zero mean and finite variance. Note that while X_i are random matrices, random variables L_r are real-valued. By applying the Hájek-Rényi inequality, we obtain that for every positive real constant ξ and every $\gamma \in [0,1]$, the following inequality holds:

$$P_{H_0}\Big(\max_{1\leqslant k<n}\Big[c_{k,n}(\gamma)\frac{2}{k^2}\Big|\sum_{r=1}^{k} L_r\Big|\Big]\geqslant \xi\Big)\leqslant \frac{4\mathbf{Var}(L_1)}{\xi^2 n^{2\gamma}}\sum_{k=1}^{n}\frac{1}{k^{2(1-\gamma)}}.$$

The last term can be upper-bounded by $D_3 n^{-\min(2\gamma,1)}(1+\mathrm{Ind}(\gamma\geqslant \frac{1}{2})\log(n))$, where D_3 is a constant greater than zero. Similarly, we also have

$$P_{H_0}\Big(\max_{1\leqslant k<n}\Big[c_{k,n}(\gamma)\frac{2}{(n-k)^2}\Big|\sum_{r=k+1}^{n} L_r\Big|\Big]\geqslant \xi\Big)\leqslant D_3 n^{-\min(2\gamma,1)}(1+\mathrm{Ind}\Big(\gamma\geqslant \frac{1}{2}\Big)\log(n)).$$

Therefore, given that the above inequalities hold:

$$\max_{1\leqslant k<n}\Big[c_{k,n}(\gamma)|C_{k3}|\Big]=o_P(1),\, n\to\infty,$$

and the term $|C_{k3}|$ does not impact the limiting distribution.

Next, we demonstrate that C_{k4} does not impact the limiting behavior. We note that the random variables

$$K_r = J_\nu(-X_r, X_r) - E\,J_\nu(-X_r, X_r),\ r = 1,2,\ldots,n$$

are independent and identically distributed with zero mean and finite variance. By applying the Hájek-Rényi inequality, we obtain that for every real constants $\xi > 0$ and $\gamma \in [0,1]$, the following inequality holds:

$$P_{H_0}\Big(\max_{1\leqslant k<n}\Big[c_{k,n}(\gamma)\frac{1}{k^2}\Big|\sum_{r=1}^{k}K_r\Big|\Big] \geqslant \xi\Big) \leqslant \frac{\mathbf{Var}(K_1)}{\xi^2 n^{2\gamma}}\sum_{k=1}^{n}\frac{1}{k^{2(1-\gamma)}}.$$

The last term can be upper-bounded by $D_4 n^{-\min(2\gamma,1)}(1+\mathrm{Ind}(\gamma \geqslant \frac{1}{2})\log(n))$, where D_4 is a positive constant. We obtain

$$P_{H_0}\Big(\max_{1\leqslant k<n}\Big[c_{k,n}(\gamma)\frac{1}{(n-k)^2}\Big|\sum_{r=k+1}^{n}K_r\Big|\Big] \geqslant \xi\Big) \leqslant D_4 n^{-\min(2\gamma,1)}(1+\mathrm{Ind}\Big(\gamma \geqslant \frac{1}{2}\Big)\log(n)).$$

Hence, it follows that

$$\max_{1\leqslant k<n}[c_{k,n}(\gamma)|C_{k4}|] = o_P(1),\ n \to \infty,$$

and the term $|C_{k4}|$ does not affect the limiting distribution. Furthermore, it is worth noting that

$$\mathbf{Var}(C_{k1}) = O\Big(2\Big(\frac{n}{k(n-k)}\Big)^2 E\,\tilde{q}^2(X_1,X_2)\Big).$$

Since $\sqrt{\mathbf{Var}(C_{k1})}$ and C_{k2} are of the same order, it remains to determine the limiting distribution of C_{k1} in Step (II).

Step (II): Consider the auxiliary statistic:

$$S_k(\tilde{q}) = \sum_{1\leqslant i<j\leqslant k}\tilde{q}(X_i,X_j),\ k = 1,2,\ldots,n.$$

Under H_0, we have:

$$\begin{aligned} E(S_{k+1}(\tilde{q})|X_1,X_2,\ldots,X_k) &= S_k(\tilde{q}) + \sum_{j=1}^{k}E(\tilde{q}(X_i,X_{k+1})|X_1,X_2,\ldots,X_k) \\ &= S_k(\tilde{q}),\ k = 1,2,\ldots,n-1. \end{aligned} \tag{4.4}$$

We establish that $\{(S_k(\tilde{q}),\sigma(X_1,X_2,\ldots,X_k)), k = 1,2,\ldots,n\}$ is a martingale. $\sigma(X_1,X_2,\ldots,X_k)$ denotes the σ-algebra generated by $X_1,X_2,\ldots,X_k$. Applying the Háyek-Rényi inequality yields the following result:

$$\begin{aligned} &P\Big(\max_{1\leqslant k<n}\Big[n\Big(\frac{k(n-k)}{n^2}\Big)^{\gamma+1}\frac{1}{k^2}|S_k(\tilde{q})|\Big] \geqslant \xi\Big) \leqslant P\Big(\max_{1\leqslant k<n}\Big[n\Big(\frac{k}{n}\Big)^{\gamma+1}\frac{1}{k^2}|S_k(\tilde{q})|\Big] \geqslant \xi\Big) \leqslant \\ &\frac{1}{n^{2\gamma+2}}\frac{1}{\xi^2}\sum_{k=1}^{n-1}\frac{1}{k^{2\gamma-2}}E(S_k(\tilde{q})-S_{k-1}(\tilde{q}))^2 \leqslant \frac{1}{n^{2\gamma}\xi^2}\sum_{k=1}^{n-1}\frac{1}{k^{2-2\gamma}}E(S_k(\tilde{q})-S_{k-1}(\tilde{q}))^2 \\ &\leqslant \frac{D_0}{\xi^2}E\,\tilde{q}^2(X_1,X_2), \end{aligned}$$

for some positive constant D_0. Repeating the computation for the function $\tilde{S}_k(\tilde{q}) = \sum_{k+1\leqslant i<j\leqslant n} \tilde{q}(X_i, X_j)$, which satisfies (4.4), and combining the results, we obtain:

$$P(\max_{1\leqslant k<n}[c_{k,n}(\gamma)|C_{k1}|] \geqslant \xi) \leqslant \frac{D_1}{\xi^2} E\tilde{q}^2(X_1, X_2),$$

for some positive constant D_1. This equality holds true if we replace $\tilde{q}$ with any function that satisfies (4.2). We utilize the function $\tilde{q} - \tilde{q}_K$, such that $\tilde{q}_K$ is defined by:

$$\tilde{q}_K(x,y) = \sum_{s=1}^{K} \lambda_s f_s(x) f_s(y),$$

where (λ_s, f_s) are the (sorted) eigenvalues and eigenfunctions defined in (4.3) and K denotes the arbitrary natural number.

Moreover, by utilizing the fact that

$$E\tilde{q}^2(X_1, X_2) = \sum_{k=1}^{\infty} \lambda_k^2,$$

and applying the Háyek-Rényi inequality once more, we establish that

$$P\Big(\max_{1\leqslant k<n}[c_{k,n}(\gamma)|C_{k1} - C_{k1}(K)|] \geqslant \xi\Big) \leqslant \frac{D_5}{\xi^2} E(\tilde{q}(X_1, X_2) - \tilde{q}_K(X_1, X_2))^2 = \frac{D_5}{\xi^2} \sum_{j=K+1}^{\infty} \lambda_j^2, \tag{4.5}$$

where D_5 is a positive constant and $n \geqslant 2$ and $K \in \mathbb{N}$ and $C_{k1}(K)$ denotes C_{k1} where $\tilde{q}$ is replaced with $\tilde{q}_K$. We can now conclude that

$$\frac{k(n-k)}{n} C_{k1}(K) = \sum_{s=1}^{K} \lambda_s \Big(\frac{n}{k(n-k)} \Big(\sum_{j=1}^{k} f_s(X_j) - \frac{k}{n} \sum_{j=k+1}^{n} f_s(X_j) \Big)^2$$
$$- \frac{1}{n} \Big(\frac{n-k}{k} \sum_{j=1}^{k} f_s^2(X_j) + \frac{k}{n-k} \sum_{j=1}^{n} f_s^2(X_j) \Big) \Big).$$

Using the Donsker theorem [126], we obtain:

$$\Big\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \boldsymbol{f_K}(X_i), t \in (0,1) \Big\} \xrightarrow[n\to\infty]{D((0,1))} \boldsymbol{W_K}(t), t \in (0,1),$$

where $\boldsymbol{W_K} = (W_1, \ldots, W_K)$ is a K-dimensional Wiener process with independent components in $\mathbb{R}^K$ and $\boldsymbol{f_K} = (f_1, f_2, f_3, \ldots, f_K)$ is a vector of eigenfunctions up to the position K. Therefore, we establish that the asymptotic distribution of $\max_{1\leqslant k<n}[c_{n,k}(\gamma) C_{k1}(K)]$ is given by:

$$\max_{1\leqslant k<n} \Big[\Big(\frac{k(n-k)}{n^2} \Big)^{\gamma} \sum_{s=1}^{K} \lambda_s \Big(\frac{n^2}{k(n-k)} (W_s(k/n) - \frac{k}{n} W_s(1))^2 - 1 \Big) \Big].$$

Step (III): Note that the random processes $\lambda_s\Big(\frac{n^2}{k(n-k)}(W_s(k/n) - \frac{k}{n}W_s(1))^2 - 1\Big)$ are independent for $s = K+1, K+2, \ldots$. Additionally, these random processes have a zero mean.

Furthermore, as their variances are finite, we can apply the Hájek-Rényi inequality, yielding the following:

$$P\Big(\max_{1\leqslant k<n} \Big[\Big(\frac{k(n-k)}{n^2} \Big)^{\gamma} \sum_{s=K+1}^{\infty} \lambda_s \Big(\frac{n^2}{k(n-k)} \Big(W_s(k/n) - \frac{k}{n} W_s(1) \Big)^2 - 1 \Big) \Big] \geqslant \xi \Big) \leqslant \frac{D_5}{\xi^2} \sum_{s=K+1}^{\infty} \lambda_s^2.$$

From (4.5), we can observe that for a sufficiently large value of K, the asymptotic distributions of C_{k1} and $C_{k1}(K)$ are close to each other. Consequently, by taking the limit as K tends to infinity, we conclude that the asymptotic distribution of $\max_{1\leqslant k<n}[c_{n,k}(\gamma)C_{k1}] = \max_{1\leqslant k<n}\Big[\Big(\frac{k(n-k)}{n^2}\Big)^{\gamma}\frac{k(n-k)}{n}C_{k1}\Big]$ is identical to that of

$$\max_{t\in(0,1)}\Big[\Big(t(1-t)\Big)^{\gamma}\sum_{s=1}^{\infty}\lambda_s\Big(\frac{(W_s(t)-tW_s(1))^2}{t(1-t)}-1\Big)\Big].$$

This concludes the proof. □

Since the limiting null distribution of novel statistics is not free of distribution of X_1, for the derivation of p-values, we suggest the usage of the permutation bootstrap algorithm from [62]. This approach can be theoretically justified in a similar manner for our test.

4.4 A power study

In this section, we present the results of a power study. All computations were performed using MATLAB [64]. The algorithm developed in [72] was implemented to evaluate the Bessel functions of two matrix arguments.

In our study, we fix the value of parameter ν to $\nu = 1$, as is common practice in problems of this nature (see [12, 84]).

The empirical powers in various settings, for the level of significance $\alpha = 0.05$, are obtained using a warp-speed modification of the permutation bootstrap algorithm - Algorithm 3 (see, e.g., [46]), with $N = 500$ replications.

The following distributions are considered:

1. Wishart distributions with the shape parameter $a > \frac{1}{2}(d-1)$ and the scale matrix $\Sigma > 0$, denoted by $W_d(a,\Sigma)$, with a density (2.10).

2. Inverse Wishart distributions with the shape parameter $a > \frac{1}{2}(d-1)$ and the scale matrix $\Sigma > 0$, denoted by $IW_d(a,\Sigma)$, with a density (2.11).

3. Sample covariance matrix distributions obtained from the uniform vectors $(U_1,\dots,U_d)$, where $U_i \in \mathscr{U}[0,1]$, denoted by CMU_d, with a density (2.12).

4. Sample covariance matrix distributions obtained from the random vectors having the multivariate t-distribution with $a > 0$ degrees of freedom, denoted by $CMT_d(a,\Sigma)$, and matrix parameter $\Sigma > 0$, with a density (2.13).

Throughout the above cases, d denotes the dimension of the respective matrices. When estimating sample covariance matrices, we considered samples of dimension d.

Tables 4.1 and 4.2 present the empirical powers for $n = 40$ and $n = 100$, with varying change point positions (k) and tuning parameter (γ) values, for $d = 2$ and $d = 3$, respectively. The results indicate that the empirical sizes closely approximate the significance level, with only a few notable deviations that diminish as the sample size increases. It is also noteworthy that the tests exhibit low sensitivity to the tuning parameter γ - in most instances, discrepancies do not exceed 10%. This behavior aligns with expectations, as similar test statistics have shown robustness to variations in the parameter γ [62].

Conversely, the change point location appears to significantly influence its detection. Generally, central change points are more easily detected compared to those near the boundaries. In asymmetric scenarios, optimizing the parameter γ could potentially improve test power.

Importantly, change point detection occurred even in cases where distributions shared the same expectation.

As anticipated, there is a decrease in power as the dimension increases. Nevertheless, the ratio of rejection rates among different alternatives remains relatively consistent.

Table 4.1: Powers of the test for different sample sizes and change point locations: 2×2 matrices, $\gamma = 0.5$ ($\gamma = 1$).

						$n=40, k=20$					
Distributions	$W_2(2.5, I_2)$	$IW_2(2.5, I_2)$	$CMT_2(1, I_2)$	CMU_2	$W_2(2.5, 2I_2)$	$IW_2(4, 2.5I_2)$	$W_2(2.5, K_2)$	$CMT_2(3, K_2)$	$CMT_2(5, K_2)$	$CMT_2(3, I_2)$	$CMT_2(5, I_2)$
$W_2(2.5, I_2)$	7 (4)	29 (29)	12 (11)	100 (100)	16 (23)	27 (32)	18 (33)	51 (52)	57 (62)	68 (71)	80 (85)
$IW_2(2.5, I_2)$		4 (3)	9 (12)	100 (100)	62 (68)	6 (8)	5 (6)	19 (21)	23 (26)	32 (34)	42 (49)
$CMT_2(1, I_2)$			6 (7)	100 (100)	23 (35)	4 (7)	9 (7)	23 (29)	34 (38)	47 (48)	59 (60)
CMU_2				7 (4)	100 (100)	100 (100)	100 (100)	100 (100)	99 (99)	98 (99)	97 (98)
$W_2(2.5, 2I_2)$					6 (4)	78 (82)	67 (68)	77 (80)	85 (86)	86 (91)	93 (95)
$IW_2(4, 2.5I_2)$						4 (6)	7 (10)	26 (27)	35 (43)	41 (44)	58 (57)
$W_2(2.5, K_2)$							8 (5)	21 (19)	27 (27)	32 (32)	45 (45)
$CMT_2(3, K_2)$								3 (3)	4 (5)	11 (6)	12 (11)
$CMT_2(5, K_2)$									2 (3)	7 (7)	11 (10)
$CMT_2(3, I_2)$										3 (4)	7 (6)
$CMT_2(5, I_2)$											7 (5)
						$n=40, k=10$					
Distributions	$W_2(2.5, I_2)$	$IW_2(2.5, I_2)$	$CMT_2(1, I_2)$	CMU_2	$W_2(2.5, 2I_2)$	$IW_2(4, 2.5I_2)$	$W_2(2.5, K_2)$	$CMT_2(3, K_2)$	$CMT_2(5, K_2)$	$CMT_2(3, I_2)$	$CMT_2(5, I_2)$
$W_2(2.5, I_2)$	4 (4)	12 (13)	7 (10)	100 (100)	18 (23)	18 (16)	10 (15)	20 (21)	26 (29)	32 (35)	49 (47)
$IW_2(2.5, I_2)$	25 (24)	6 (4)	6 (5)	100 (100)	55 (54)	10 (5)	6 (9)	7 (6)	9 (14)	16 (20)	26 (17)
$CMT_2(1, I_2)$	14 (10)	5 (4)	5 (6)	100 (100)	23 (24)	8 (6)	10 (6)	10 (13)	17 (17)	32 (20)	34 (31)
CMU_2	100 (100)	100 (100)	100 (100)	5 (5)	100 (100)	100 (100)	100 (100)	98 (96)	98 (97)	90 (85)	82 (85)
$W_2(2.5, 2I_2)$	5 (9)	28 (28)	5 (6)	100 (100)	3 (7)	50 (43)	34 (32)	34 (37)	53 (45)	63 (55)	70 (65)
$IW_2(4, 2.5I_2)$	19 (16)	6 (3)	3 (5)	100 (100)	58 (64)	4 (6)	7 (4)	10 (11)	15 (14)	13 (14)	26 (24)
$W_2(2.5, K_2)$	19 (24)	5 (7)	3 (7)	100 (100)	66 (57)	7 (7)	5 (3)	8 (11)	20 (14)	19 (18)	27 (26)
$CMT_2(3, K_2)$	46 (43)	16 (14)	17 (14)	98 (97)	65 (65)	26 (24)	21 (18)	6 (5)	6 (8)	6 (6)	6 (8)
$CMT_2(5, K_2)$	59 (53)	25 (23)	23 (23)	96 (96)	75 (69)	28 (28)	29 (20)	8 (6)	4 (3)	4 (5)	8 (6)
$CMT_2(3, I_2)$	59 (53)	28 (33)	29 (35)	92 (91)	76 (78)	36 (37)	36 (23)	8 (8)	12 (9)	3 (3)	2 (4)
$CMT_2(5, I_2)$	70 (65)	36 (41)	45 (43)	88 (88)	88 (81)	52 (50)	41 (33)	8 (7)	7 (7)	6 (6)	5 (5)
						$n=100, k=50$					
Distributions	$W_2(2.5, I_2)$	$IW_2(2.5, I_2)$	$CMT_2(1, I_2)$	CMU_2	$W_2(2.5, 2I_2)$	$IW_2(4, 2.5I_2)$	$W_2(2.5, K_2)$	$CMT_2(3, K_2)$	$CMT_2(5, K_2)$	$CMT_2(3, I_2)$	$CMT_2(5, I_2)$
$W_2(2.5, I_2)$	7 (5)	64 (67)	19 (22)	100 (100)	53 (65)	65 (67)	75 (74)	94 (94)	99 (99)	99 (99)	100 (100)
$IW_2(2.5, I_2)$		6 (5)	11 (16)	100 (100)	99 (99)	9 (9)	8 (10)	33 (51)	58 (58)	70 (78)	90 (92)
$CMT_2(1, I_2)$			5 (6)	100 (100)	54 (61)	14 (14)	19 (25)	59 (63)	72 (77)	90 (90)	94 (96)
CMU_2				5 (6)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
$W_2(2.5, 2I_2)$					4 (4)	100 (100)	99 (100)	99 (100)	100 (100)	100 (100)	100 (100)
$IW_2(4, 2.5I_2)$						7 (6)	10 (9)	51 (54)	78 (79)	85 (89)	93 (96)
$W_2(2.5, K_2)$							4 (4)	42 (53)	61 (65)	73 (81)	87 (86)
$CMT_2(3, K_2)$								4 (6)	7 (6)	18 (21)	31 (31)
$CMT_2(5, K_2)$									6 (5)	9 (10)	16 (17)
$CMT_2(3, I_2)$										4 (4)	10 (10)
$CMT_2(5, I_2)$											6 (6)
						$n=100, k=25$					
Distributions	$W_2(2.5, I_2)$	$IW_2(2.5, I_2)$	$CMT_2(1, I_2)$	CMU_2	$W_2(2.5, 2I_2)$	$IW_2(4, 2.5I_2)$	$W_2(2.5, K_2)$	$CMT_2(3, K_2)$	$CMT_2(5, K_2)$	$CMT_2(3, I_2)$	$CMT_2(5, I_2)$
$W_2(2.5, I_2)$	7 (4)	58 (51)	23 (22)	100 (100)	17 (24)	41 (49)	61 (66)	86 (85)	93 (92)	95 (95)	100 (99)
$IW_2(2.5, I_2)$	38 (45)	7 (7)	12 (11)	100 (100)	85 (87)	6 (6)	6 (7)	27 (25)	47 (45)	59 (59)	76 (73)
$CMT_2(1, I_2)$	8 (7)	11 (10)	4 (3)	100 (100)	24 (32)	5 (9)	9 (13)	43 (46)	58 (58)	71 (69)	84 (87)
CMU_2	100 (100)	100 (100)	100 (100)	3 (3)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
$W_2(2.5, 2I_2)$	46 (46)	94 (94)	56 (59)	100 (100)	3 (5)	99 (98)	97 (98)	99 (99)	99 (99)	100 (100)	100 (100)
$IW_2(4, 2.5I_2)$	39 (33)	8 (9)	10 (10)	100 (100)	97 (97)	4 (3)	11 (8)	49 (48)	65 (65)	79 (76)	91 (90)
$W_2(2.5, K_2)$	29 (39)	5 (5)	13 (12)	100 (100)	93 (94)	8 (9)	7 (5)	30 (29)	50 (51)	68 (62)	72 (73)
$CMT_2(3, K_2)$	73 (58)	17 (18)	39 (43)	100 (100)	98 (95)	28 (29)	21 (15)	5 (4)	8 (7)	12 (16)	23 (16)
$CMT_2(5, K_2)$	87 (88)	30 (33)	55 (50)	100 (100)	100 (99)	45 (36)	32 (34)	5 (4)	8 (6)	9 (6)	8 (14)
$CMT_2(3, I_2)$	89 (90)	40 (43)	68 (64)	100 (100)	99 (100)	51 (60)	46 (43)	12 (10)	7 (6)	5 (7)	6 (5)
$CMT_2(5, I_2)$	98 (98)	61 (58)	82 (75)	100 (100)	100 (100)	73 (76)	65 (69)	15 (18)	10 (10)	6 (6)	7 (7)

Table 4.2: Powers of the test for different sample sizes and change point locations: 3×3 matrices, $\gamma = 0.5$ ($\gamma = 1$).

Distributions	$W_3(3,I_3)$	$IW_3(3,I_3)$	$CMT_3(1,I_3)$	CMU_3	$W_3(3,2I_3)$	$IW_3(5,3I_3)$	$W_3(3,K_3)$	$CMT_3(3,K_3)$	$CMT_3(5,K_3)$	$CMT_3(3,I_3)$	$CMT_3(5,I_3)$
						$n=40, k=20$					
$W_3(3,I_3)$	5 (4)	16 (19)	7 (6)	100 (100)	9 (8)	22 (42)	8 (10)	26 (38)	36 (49)	27 (38)	40 (45)
$IW_3(3,I_3)$		5 (5)	7 (12)	100 (100)	15 (29)	6 (5)	21 (27)	11 (10)	16 (17)	14 (21)	23 (24)
$CMT_3(1,I_3)$			6 (4)	100 (100)	7 (7)	12 (16)	7 (7)	23 (27)	38 (44)	30 (34)	44 (46)
CMU_3				5 (5)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
$W_3(3,2I_3)$					5 (5)	45 (58)	6 (5)	32 (41)	38 (58)	41 (48)	51 (59)
$IW_3(5,3I_3)$						5 (6)	46 (62)	17 (16)	20 (16)	16 (20)	18 (28)
$W_3(3,K_3)$							4 (6)	32 (46)	49 (49)	36 (47)	43 (54)
$CMT_3(3,K_3)$								5 (6)	6 (7)	6 (6)	4 (7)
$CMT_3(5,K_3)$									4 (4)	6 (4)	3 (3)
$CMT_3(3,I_3)$										6 (7)	6 (7)
$CMT_3(5,I_3)$											6 (7)
						$n=40, k=10$					
Distributions	$W_3(3,I_3)$	$IW_3(3,I_3)$	$CMT_3(1,I_3)$	CMU_3	$W_3(3,2I_3)$	$IW_3(5,3I_3)$	$W_3(3,K_3)$	$CMT_3(3,K_3)$	$CMT_3(5,K_3)$	$CMT_3(3,I_3)$	$CMT_3(5,I_3)$
$W_3(3,I_3)$	5 (7)	7 (6)	6 (4)	100 (100)	11 (10)	15 (15)	10 (10)	6 (13)	13 (15)	14 (12)	18 (25)
$IW_3(3,I_3)$	17 (20)	4 (4)	8 (10)	100 (100)	21 (25)	7 (4)	19 (23)	5 (8)	5 (9)	9 (9)	6 (12)
$CMT_3(1,I_3)$	9 (7)	4 (6)	5 (6)	100 (100)	8 (8)	9 (10)	11 (11)	8 (7)	9 (14)	9 (14)	12 (16)
CMU_3	100 (100)	100 (100)	100 (100)	5 (6)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
$W_3(3,2I_3)$	3 (4)	8 (8)	5 (8)	100 (100)	5 (5)	14 (17)	8 (6)	9 (16)	22 (22)	14 (18)	17 (30)
$IW_3(5,3I_3)$	27 (25)	5 (4)	8 (12)	100 (100)	42 (46)	5 (6)	42 (49)	6 (6)	10 (11)	7 (12)	13 (16)
$W_3(3,K_3)$	5 (5)	7 (10)	3 (4)	100 (100)	6 (4)	20 (28)	6 (5)	8 (13)	17 (24)	13 (17)	18 (28)
$CMT_3(3,K_3)$	27 (31)	9 (10)	22 (24)	100 (100)	41 (40)	14 (14)	37 (38)	4 (5)	4 (5)	6 (5)	6 (6)
$CMT_3(5,K_3)$	32 (45)	16 (16)	27 (34)	100 (100)	41 (45)	20 (20)	36 (43)	9 (7)	3 (4)	7 (7)	6 (6)
$CMT_3(3,I_3)$	31 (33)	16 (13)	30 (30)	100 (100)	35 (34)	20 (19)	34 (40)	7 (4)	5 (8)	5 (5)	5 (5)
$CMT_3(5,I_3)$	43 (46)	22 (20)	32 (36)	100 (100)	39 (46)	24 (23)	40 (42)	6 (7)	5 (5)	5 (4)	7 (7)
						$n=100, k=50$					
Distributions	$W_3(3,I_3)$	$IW_3(3,I_3)$	$CMT_3(1,I_3)$	CMU_3	$W_3(3,2I_3)$	$IW_3(5,3I_3)$	$W_3(3,K_3)$	$CMT_3(3,K_3)$	$CMT_3(5,K_3)$	$CMT_3(3,I_3)$	$CMT_3(5,I_3)$
$W_3(3,I_3)$	5 (3)	34 (51)	6 (5)	100 (100)	13 (26)	73 (84)	21 (28)	77 (79)	91 (92)	76 (86)	95 (98)
$IW_3(3,I_3)$		5 (3)	28 (51)	100 (100)	59 (75)	8 (6)	55 (74)	26 (30)	47 (51)	35 (36)	54 (63)
$CMT_3(1,I_3)$			6 (3)	100 (100)	11 (11)	48 (67)	12 (17)	75 (75)	90 (91)	75 (80)	88 (92)
CMU_3				5 (5)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
$W_3(3,2I_3)$					4 (4)	98 (99)	5 (6)	89 (95)	97 (98)	93 (96)	98 (98)
$IW_3(5,3I_3)$						6 (6)	99 (100)	30 (32)	52 (53)	39 (43)	58 (63)
$W_3(3,K_3)$							6 (5)	89 (92)	96 (99)	88 (95)	98 (99)
$CMT_3(3,K_3)$								4 (5)	8 (10)	7 (6)	11 (12)
$CMT_3(5,K_3)$									6 (6)	6 (7)	8 (6)
$CMT_3(3,I_3)$										5 (5)	6 (9)
$CMT_3(5,I_3)$											7 (7)
						$n=100, k=25$					
Distributions	$W_3(3,I_3)$	$IW_3(3,I_3)$	$CMT_3(1,I_3)$	CMU_3	$W_3(3,2I_3)$	$IW_3(5,3I_3)$	$W_3(3,K_3)$	$CMT_3(3,K_3)$	$CMT_3(5,K_3)$	$CMT_3(3,I_3)$	$CMT_3(5,I_3)$
$W_3(3,I_3)$	5 (3)	36 (40)	7 (5)	100 (100)	6 (8)	65 (66)	3 (8)	62 (70)	86 (86)	74 (72)	89 (86)
$IW_3(3,I_3)$	14 (16)	4 (5)	17 (11)	100 (100)	24 (28)	5 (8)	28 (33)	22 (21)	38 (45)	23 (24)	43 (43)
$CMT_3(1,I_3)$	4 (6)	23 (25)	4 (4)	100 (100)	4 (6)	28 (36)	5 (6)	61 (62)	77 (76)	71 (65)	80 (82)
CMU_3	100 (100)	100 (100)	100 (100)	100 (100)	6 (4)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)	100 (100)
$W_3(3,2I_3)$	18 (14)	52 (57)	13 (11)	100 (100)	6 (5)	89 (90)	4 (6)	75 (80)	92 (94)	87 (85)	94 (94)
$IW_3(5,3I_3)$	46 (50)	7 (4)	32 (35)	100 (100)	76 (79)	5 (6)	73 (84)	24 (24)	52 (48)	35 (35)	49 (55)
$W_3(3,K_3)$	25 (31)	57 (64)	14 (14)	100 (100)	9 (5)	89 (94)	5 (6)	78 (81)	87 (91)	79 (82)	92 (94)
$CMT_3(3,K_3)$	46 (45)	10 (12)	36 (30)	100 (100)	45 (61)	12 (17)	59 (64)	7 (5)	6 (4)	4 (4)	10 (6)
$CMT_3(5,K_3)$	54 (56)	16 (22)	66 (58)	100 (100)	74 (81)	23 (19)	74 (78)	4 (7)	4 (5)	4 (3)	7 (7)
$CMT_3(3,I_3)$	32 (52)	13 (18)	39 (42)	100 (100)	59 (57)	14 (15)	60 (62)	5 (4)	5 (5)	3 (4)	7 (5)
$CMT_3(5,I_3)$	60 (69)	25 (28)	64 (69)	100 (100)	77 (85)	23 (29)	85 (75)	7 (10)	4 (5)	5 (4)	3 (4)

4.5 Real data examples

In this section, we present recent real-world data examples that clearly demonstrate the applicability of the proposed methodology.

4.5.1 Example 1 - most recent cryptocurrency data

This example analyzes recent Bitcoin (BTC) and Ethereum (ETH) data. Hourly BTC-USD and ETH-USD data from January 1, 2023, to July 1, 2023, were obtained from Gemini (http://www.gemini.com). We focused on the close prices X_t and calculated hourly logarithmic returns $\log \frac{X_t}{X_{t-1}}$. Sample covariance matrices were computed daily, resulting in a total of $n = 181$ observations. We employed a binary segmentation algorithm (Algorithm 4) to detect multiple change points, using parameters $Window = 10$ and $NB = 500$.

We emphasize the importance of carefully selecting the $Window$ parameter. Large values may prevent the detection of change points, while small values can lead to the identification of insignificant points, especially when analyzing volatile assets. Thus, fine-tuning the $Window$ parameter is crucial for optimal performance. We examined the algorithm's performance with $\gamma = 1$ and $\gamma = 0.5$ but observed no differences in the estimated change point locations.

Table 4.3 presents relevant crypto-related news. Notably, no change points were detected from January to February, which aligns with expectations, as crypto markets generally trended upwards during this period. While Bitcoin experienced a mid-February dip, it quickly recovered. Ethereum remained largely unaffected [89, 114].

Identifying change point locations can be valuable for validating new trading systems with the most recent data. It may prove particularly useful in automatically assessing the conservativeness of specific market strategies in conditions where other methods might be ineffective due to the extreme underlying volatility of the assets in question.

Algorithm 3 Warp-speed bootstrap permutation algorithm

1: Sample $\mathbf{x} = (x_1, \ldots, x_{n_1})$ from F_X and $\mathrm{y} = (y_1, \ldots, y_{n_2})$ from F_Y and let $n = n_1 + n_2$, and create a pooled sample $\mathbf{z} = (z_1, z_2, \ldots, z_{n_1+n_2}) = (x_1, \ldots, x_{n_1}, y_1, y_2, \ldots, y_{n_2})$.;

2: Compute $\mathscr{I}_{n,\gamma,\nu} := \mathscr{I}_{n,\gamma,\nu}(\mathbf{z})$;

3: Generate random permutations $\pi : \{1, 2, \ldots, n_1 + n_2\} \to \{1, 2, \ldots, n_1 + n_2\}$ and the corresponding bootstrap permutation sample $\mathbf{z}^* = (z_{\pi(1)}, \ldots, z_{\pi(n_1)}, z_{\pi(n_1+1)}, \ldots, z_{\pi(n_1+n_2)})$.

4: Compute $\mathscr{I}^*_{n,\gamma,\nu} := \mathscr{I}_{n,\gamma,\nu}(\mathbf{z}^*)$;

5: Repeat steps 1-4 N times and obtain two sequences of statistics $\{\mathscr{I}^{(j)}_{n,\gamma,\nu}\}$ and $\{\mathscr{I}^{*(j)}_{n,\gamma,\nu}\}$, $j = 1, ..., N$;

6: Reject the null hypothesis for the j–sample ($j = 1, ..., N$), if $\mathscr{I}^{(j)}_{n,\gamma,\nu} > c_\alpha$, where c_α denotes the $(1-\alpha)\%$ quantile of the empirical distribution of the bootstrap test statistics ($\mathscr{I}^{*(j)}_{n,\gamma,\nu}$, $j = 1, ..., N$).

Table 4.3: Important cryptocurrency-related events on the day, day before or day after the detected change point

Change point date	Important news	Reference
2 March	Significant BTC and ETH drops on 3 March	[45]
17 March	Bitcoin on the rise	[44]
4 April	Significant drops on crypto markets	[109]
18 April	Crypto market significantly falls	[87]
2 May	Crypto prices on the rise	[140]
17 May	Crypto trading should be treated as a form of gambling in the UK	[90]
1 June	Cryptocurrencies reacting to debt ceiling deal	[78]
16 June	Fall after Federal Reserve June meeting	[88]

Table 4.4: PCA explained variance and largest eigenvalues per dimension

Dim	PCA EV	LE	Dim	PCA EV	LE	Dim	PCA EV	LE	Dim	PCA EV	LE
1	0.6035	619.249	8	0.9665	1.57E-15	15	0.9926	2.15E-16	22	0.9985	5.70E-19
2	0.7751	53.69442	9	0.9734	1.15E-15	16	0.9938	1.01E-16	23	0.9989	0
3	0.8571	16.55289	10	0.9794	9.71E-16	17	0.9950	7.40E-17	24	0.9993	0
4	0.8985	9.336243	11	0.9837	8.16E-16	18	0.9959	2.45E-17	25	0.9996	0
5	0.9218	3.87E-14	12	0.9865	5.56E-16	19	0.9967	1.20E-17	26	0.9998	0
6	0.9440	1.20E-14	13	0.9889	4.99E-16	20	0.9974	6.64E-18	27	0.9999	0
7	0.9567	5.90E-15	14	0.9910	2.45E-16	21	0.9980	2.93E-18	28	1.0000	0

4.5.2 Example 2 - performance on well-documented cases

This study examines the data example from Section 2.5.1 from a change point perspective. We investigated the covariance structure of hourly logarithmic returns over 15-day periods before and after well-documented Bitcoin price drops that coincide with significant historical events [40]. Each period comprises $n = 720$ data points, with results presented in Table 4.6. Most tests failed to detect change points both before and after the prominent events, except for February, where increased volatility may have enhanced detection. Based on these findings, we conclude that this method is not practically applicable for this specific analysis.

We then conducted a change point analysis using 1-minute BTC (Bitcoin) [146] and ETH (Ethereum) [73] data. We selected two-day periods and calculated logarithmic returns for each minute. Covariance matrices were computed hourly, yielding a total of $n = 48$ covariance matrices (24 for each day). The results, shown in Table 4.5, demonstrate that our test successfully identified change points on the day before and the day of the event's occurrence. The test consistently detected change points without failure or lag. This finding may have practical significance for implementing stop-loss strategies in trading [68]. Both tests, using $\gamma = 0.5$ and $\gamma = 1$, produced comparable results.

Table 4.5: p-values (change point location) in covariance structure before and after the Bitcoin important events - 1 minute data, $\gamma = 0.5$ [$\gamma = 1$].

Date of event (T_0)	Event description	$p_{[T_0-2D,T_0-1D]}$	$p_{[T_0-1D,T_0]}$	$p_{[T_0,T_0+1D]}$	$p_{[T_0-2D,T_0+1D]}$
8 November 2017	Developers cancel splitting of Bitcoin.	0.0980(27) [0.0940 (27)]	0 (41) [0 (41)]	0.2120 (17) [0.1080 (17)]	0.0420 (65) [0 (65)]
28 December 2017	South Korea announces strong measures to regulate the trading of cryptocurrencies.	0.0860 (39) [0.1400 (39)]	0.0040 (26) [0 (26)]	0.0080 (11) [0.0060 (25)]	0.0800 (50) [0.020 (50)]
13 January 2018	Announcement that 80% of Bitcoin has been mined.	0.0020 (7) [0 (7)]	0 (7) [0 (7)]	0.2060 (28) [0.1120 (28)]	0 (31) [0 (31)]
30 January 2018	Facebook bans advertisements promoting cryptocurrencies.	0.1280 (33) [0.0800 (33)]	0.0100 (37) [0 (37)]	0.2220 (13) [0.1420 (17)]	0 (61) [0 (61)]
7 March 2018	The US Securities and Exchange Commission says it is necessary for crypto trading platforms to register.	0 (41) [0.0020 (34)]	0 (40) [0 (40)]	0.1060 (16) [0.0040 (16)]	0 (64) [0 (64)]
14 March 2018	Google bans advertisements promoting cryptocurrencies.	0.0060 (14) [0.0080 (14)]	0.2260 (40) [0.3100 (33)]	0.0040 (16)[0.0140 (16)]	0.0180 (64) [0.030 (64)]

Table 4.6: p-values (change point location) in covariance structure before and after the Bitcoin important events - 1 hour data, $\gamma = 0.5$ [$\gamma = 1$].

Period I start date	Period II start date	Date of event (T_0)	Period II end date	Event description	$p_{[T_0-30D,T_0]}$	$p_{[T_0-15D,T_0+15D]}$
9 October 2017	24 October 2017	8 November 2017	23 November 2017	Developers cancel splitting of Bitcoin.	0.29 (17) [0.20 (17)]	0.32 (15) [0.20 (15)]
28 November 2017	13 December 2017	28 December 2017	12 January 2018	South Korea announces strong measures to regulate trading of cryptocurrencies.	0.30 (24) [0.26 (24)]	0.59 (12) [0.58 (12)]
14 December 2017	28 December 2017	13 January 2018	28 January 2018	Announcement that 80% of Bitcoin has been mined.	0.63 (11) [0.60 (11)]	0.25 (19) [0.20 (19)]
31 December 2017	15 January 2018	30 January 2018	14 February 2018	Facebook bans advertisements promoting, cryptocurrencies.	0.59 (16) [0.41 (16)]	0.14 (3) [0.31 (4)]
5 February 2018	20 February 2018	7 March 2018	22 March 2018	The US Securities and Exchange Commission says it is necessary for crypto trading platforms to register.	0 (3) [0 (6)]	0 (15) [0 (15)]
12 February 2018	28 February 2018	14 March 2018	29 March 2018	Google bans advertisements promoting cryptocurrencies.	0 (23) [0 (23)]	0.04 (7) [0.04 (7)]

4.5.3 Example 3 - DAX stock index data

We applied the novel test to a subset of German DAX index data, focusing on the following stock trading symbols: ADS.DE, ALV.DE, BAS.DE, BAYN.DE, BEI.DE, BMW.DE, CON.DE, DB1.DE, DBK.DE, DHER.DE, DPW.DE, DTE.DE, DWNI.DE, EOAN.DE, FME.DE, FRE.DE, HEI.DE, HEN3.DE, IFX.DE, LHA.DE, LIN.DE, MRK.DE, MUV2.DE, RWE.DE, SAP.DE, SIE.DE, VNA.DE, and VOW3.DE.

The study period runs from January 1, 2022, to January 1, 2023, covering 257 trading days. We divided the data into sections of five consecutive trading days and estimated the covariance matrix for each section, resulting in $N = 52$ observations. Given the high dimensionality of the data and the considerable computational effort required, we performed data reduction before applying the test.

We applied Principal Component Analysis (PCA) to the entire dataset. Table 4.4 presents the explained variance (PCA EV). For data reduction, we replaced each covariance matrix with a diagonal matrix containing the largest eigenvalues. Table 4.4 also shows the largest eigenvalues (LE) per position. Based on these results, dimensions 2, 3, 4, and 5 were considered for analysis.

We implemented the binary search procedure on the reduced data, using the parameters $Window = 5$ and $NB = 500$. Each procedure consistently identified a change point location at $k = 50$, suggesting the presence of a single change point in the data. This indicates that the period from December 11 to 17, 2022, exhibited a significant shift in behavior within the German market. This change likely reflects the impact of speculated recession in the US market and the global nature of the German economy.

Algorithm 4 Binary segmentation algorithm $BS(x, NB, \gamma, \alpha, Window)$

Compute n=Length(x).
if n <Window **then**
    Return NULL.
else
    Compute $\mathcal{J}^* = \mathcal{J}_\gamma(x)$.
    Compute $x_{i_1}, x_{i_2}, \ldots, x_{i_{NB}}$ - different permutations without replacement of x.
    Estimate the distribution under the null hypothesis with
      Null $= \mathcal{J}_\gamma(x_{i_1}), \mathcal{J}_\gamma(x_{i_2}), \ldots, \mathcal{J}_\gamma(x_{i_{NB}})$.
    Estimate the p-value as pval = Mean(Null $> \mathcal{J}^*$).
    if pval $> \alpha$ **then**
        Return NULL.
    else
        Compute the position for which the maximum in $\mathcal{J}^*$ is attained. Denote this position with k^*.
        Split x into two subsamples $x_{h1} = x_1, \ldots, x_{[n/2]}$ and
      $x_{h2} = x_{[n/2]+1}, x_{[n/2]+2}, \ldots, x_n$.
        Return list
      $[k^*, BS(x_{h1}, NB, \gamma, \alpha, Window), [n/2] + BS(x_{h2}, NB, \gamma, \alpha, Window)]$.
    end if
end if

Conclusion and outlook

In this dissertation, we have presented two novel two-sample tests for matrix data. The tests are first of their kind, and we have demonstrated their properties. However, there are many possible avenues of research to be explored. The (noncentral) Wishart measure that has been implemented in the construction of the test statistic can be replaced with an appropriately chosen measure, conditional on the computation of the matrix integral. One possible measure could be the matrix beta measure [50].

Regarding the two-sample test based on the Laplace transform in the matrix case, one might consider the development of GOF tests following the same approach, since it is a promising direction for further research and no such undertakings are present in the modern literature. The univariate tests can be extended to the multivariate setting. To the best of our knowledge, there has been no such undertaking. The asymptotic properties of the matrix-variate two-sample test could not be established using a well-known technique by utilizing a central limit theorem. Therefore, the establishing of the asymptotic properties remains an open question.

The real data examples presented in the second part of the dissertation are mainly sampled using a 1-minute time frame. They do illustrate the application for day traders; however, it is still unclear what happens in the case of ultra-high-frequency data [20], i.e., data sampled in intervals less than one second. The structure of the micro-shocks often significantly alters the structure of the signal, which can lead to false detection and influence the trading system performance. Nevertheless, it is a compelling area for future study, taking into account the rapidly developing area of algorithmic trading.

Regarding change point inference, we have presented classes of novel tests in the univariate case. The tests are based on integral transforms and have many favorable properties. There exist many different areas of further research connected to univariate change point detection. The area of algorithm development is still very viable. One might also want to consider the problems of parametric change point detection associated with specific time series models. Formally establishing the consistency of the change point estimator is left for further research as well.

In the final chapter, we have presented the first-of-its-kind change point test for matrix data, obtained by modifying the two-sample test presented in the second chapter. There exist several areas of further research, such as the modification of the test to address online change point inference. The online change point inference for complex data structures has many potential real-world applications, see, for example, [23, 24, 137]. Moreover, establishing the consistency of the change point estimator can be significant in establishing the quality of the estimator with scientific rigor.

It is important to note that the dimensions considered when looking into the tests' properties in this dissertation are set to 2 and 3, and hence are very small. The usual PC configuration does not permit the analysis of higher dimensions, mainly because of the memory constraints and the computational power required. Of course, it has been observed that the performance of the novel tests drops whenever the dimension increases from 2 to 3. The performance in higher dimensions remains a fruitful topic for future exploration. In order to mitigate computational difficulties, one might wish to consider using dimension reduction techniques, such as PCA, or developing novel dimension reduction techniques in future research.

There has been increased interest in the scientific community in so-called Stein methods [5]. However, there has been little to no research connected to matrix-variate distributions. Developing novel GOF tests can be an interesting avenue of research to explore in future work.

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Biography

Žikica Lukić was born in Užice, Serbia on 25 October 1996. During his high school education, he won state-level competitions in Serbian language, literature, mathematics, and rhetoric, and received several other awards in history and writing. He graduated from the Faculty of Mathematics, University of Belgrade in 2019 with a GPA of 9.47/10.00. He was awarded 3rd place in the MATF++ programming competition during his final year. He completed a 5-month internship at Raiffeisen Bank in 2019 in the credit risk department. In 2020, he started his employment at Teletrader GmbH as a Quantitative Analyst and obtained his master's degree with a GPA of 10.00/10.00. His master's thesis was written under the supervision of Professor Bojana Milošević, PhD. In parallel to his work as a Quantitative Analyst, he worked as a teaching associate at the Faculty of Mathematics in the academic year 2020/2021. He enrolled in a PhD program at the same Faculty in 2020. He won the Award of the Mathematical Institute of SASA for master's students in 2021. After termination of his contract with the Faculty, he switched back to being a full-time Quantitative Analyst. In 2022, he decided to switch fields and started working as a Biostatistician at Parexel, where he still enjoys working. Parallel to his employment, he remained academically active, publishing 4 scientific articles in top SCI journals. He was an invited speaker at the European Young Statisticians' Meeting in 2023. He participated in many national conferences, and his works were presented at 4 international conferences. He worked on the popularization of science, contributing to the YoungStats project.

He passed all of his PhD-level exams with a GPA of 10.00/10.00.

His main research interests are matrix statistics, complex data structures, change point inference, and financial mathematics, especially the analysis of highly volatile assets. He is passionate about the applications of statistics in medicine. He enjoys connecting deep theoretical ideas with real-world applications and never wants to stop connecting the two worlds.

He is a person with disabilities and actively promotes the rights of persons with disabilities in Serbia. He is also a chess referee and a recreational chess player. He is a proud cat parent and an ailurophile.

Прилог 1.

Изјава о ауторству

Потписани Жикица Лукић

број индекса 2020/2020

Изјављујем

да је докторска дисертација под насловом

Statistical tests based on Laplace and Hankel transforms, and their application in change point detection (Статистички тестови засновани на Лапласовим и Ханкеловим трансформацијама, и њихова примена у откривању промена режима)

- резултат сопственог истраживачког рада,
- да предложена дисертација у целини ни у деловима није била предложена за добијање било које дипломе према студијским програмима других високошколских установа,
- да су резултати коректно наведени и
- да нисам кршио ауторска права и користио интелектуалну својину других лица.

Потпис докторанда

У Београду, 15.11.2024.

Прилог 2.

Изјава о истоветности штампане и електронске верзије докторског рада

Име и презиме аутора Жикица Лукић

Број индекса 2020/2020

Студијски програм Математика

Наслов рада Statistical tests based on Laplace and Hankel transforms, and their application in change point detection (Статистички тестови засновани на Лапласовим и Ханкеловим трансформацијама, и њихова примена у откривању промена режима)

Ментор проф. др Бојана Милошевић

Потписани Жикица Лукић

Изјављујем да је штампана верзија мог докторског рада истоветна електронској верзији коју сам предао за објављивање на порталу **Дигиталног репозиторијума Универзитета у Београду.**

Дозвољавам да се објаве моји лични подаци везани за добијање академског звања доктора наука, као што су име и презиме, година и место рођења и датум одбране рада.

Ови лични подаци могу се објавити на мрежним страницама дигиталне библиотеке, у електронском каталогу и у публикацијама Универзитета у Београду.

Потпис докторанда

У Београду, 15.11.2024.

Прилог 3.

Изјава о коришћењу

Овлашћујем Универзитетску библиотеку „Светозар Марковић“ да у Дигитални репозиторијум Универзитета у Београду унесе моју докторску дисертацију под насловом:

Statistical tests based on Laplace and Hankel transforms, and their application in change point detection (Статистички тестови засновани на Лапласовим и Ханкеловим трансформацијама, и њихова примена у откривању промена режима)

која је моје ауторско дело.

Дисертацију са свим прилозима предао сам у електронском формату погодном за трајно архивирање.

Моју докторску дисертацију похрањену у Дигитални репозиторијум Универзитета у Београду могу да користе сви који поштују одредбе садржане у одабраном типу лиценце Креативне заједнице (Creative Commons) за коју сам се одлучио.

1. Ауторство

2. Ауторство - некомерцијално

3. Ауторство – некомерцијално – без прераде

4. Ауторство – некомерцијално – делити под истим условима

5. Ауторство – без прераде

6. Ауторство – делити под истим условима

(Молимо да заокружите само једну од шест понуђених лиценци, кратак опис лиценци дат је на полеђини листа).

Потпис докторанда

У Београду, 15.11.2024.