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Ispunjenost uslova za primenu teoreme Nehoroševa na asteroidni prsten

DOKTORSKA DISERTACIJA

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Ideju za ovu tezu dali su dr Zoran Knežević i dr Massimiliano Guzzo. Zahvaljujem se ko–mentorima dr Zoranu Kneževiću i dr Mikeu Kuzmanovskom i dr Slobodanu Ninkoviću, kao članu komisije, na korisnim savetima i sugestijama. Massimiliano Guzzo mi je svojim savetima i proverom jednog dela izvoda pomogao pri izradi ove teze na čemu mu se posebno zahvaljujem. Takođe bih se zahvalio kolegi Bojanu Novakoviću koji je takođe proverio neke od izvoda.

Rade Pavlović
Beograd, 20.02.2008.

Sadržaj

"... još uvek ne znamo da li je asteroidni pojas veliko haotično more (čije je vreme difuzije reda milijardi godina) ili poseduje strukturu u smislu teoreme Nehoroševa za kvazi–integrabilne Hamiltonove sisteme. Ovo nije samo čisto akademsko pitanje, već može imati neke veoma važne astronomske implikacije. U prvom slučaju, asteroidni pojas bi bio marginalno stabilan, a asteroidi koje sada posmatramo bi bili oni, preostali od mnogo veće populacije, koji ga još nisu napustili. U drugom slučaju, dinamička slika asteroidnog pojasa bi bila zamrznuta u veoma dugom vremenskom intervalu, koji bi premašivao starost sunčevog sistema. Tako bi ono što mi danas posmatramo bilo, u prvom slučaju samo prelazna dinamička faza, dok bi, u drugom slučaju, to bila neka vrsta 'permanentne' konfiguracije asteroidnog pojasa" (Morbidelli i Guzzo, 1997).

Uvod

Do sada je publikovano više radova koji se bave primenom teoreme Nehoroševa na različite, često veoma uprošćene dinamičke sisteme u astronomiji. Tako, na primer, Celletti i Giorgilli (1991) direktno primenjuju teoremu u aproksimaciji ograničenog problema tri tela sa kružnim putanjama u Lagrangeovim ravnotežnim tačkama. Razvijenu teoriju primenjuju na sisteme Zemlja–Mesec i Sunce–Jupiter. Međutim, poređenjem teorijski dobijene oblasti u kojoj se može primeniti teorema Nehoroševa na primer sa realnom oblašću koju zaposedaju asteroidi, dolazi do neslaganja za nekoliko redova veličine (videti Tablicu 3 pomenutog rada). Celletti i Ferrara (1996) primenjuju je na sistem Sunce–Jupiter–Ceres, ali opet na uprošćenu dinamiku. Kao rezultat dobijaju stabilnost u dugim vremenskim intervalima reda starosti Sunčevog sistema ($\sim 4.9 \cdot 10^9$ godina) ali za odnos masa Jupitera i Sunca $\leq 10^{-6}$, što je oko 1000 puta manje od realne vrednosti. Poslednjih godina Efthymiopoulos (2005) čini napore da poboljša prethodne procene regiona stabilnosti, primenom modela mapiranja za uprošćeni kružni problem tri tela na Trojance. Takođe, Efthymiopoulos i Sándor (2005) poboljšavaju procenu domena stabilnosti Trojanaca primenom simplektičkog modela mapiranja za koorbitalno kretanje, ponovo međutim, u aproksimaciji kružnog ograničenog problema tri tela. Dobili su da se oko 35% asteroida iz kataloga AstDys[1] sa sopstvenim nagibom manjim od 5° nalazi unutar oblasti čije vreme stabilnosti od 10^{10} godina premašuje starost Sunčevog sistema. Guzzo i dr. (2002) primenjuju spektralnu formulaciju teoreme Nehoroševa na realne asteroide iz glavnog prstena. Nalaze numeričke indikacije da se neki asteroidi nalaze u tzv. režimu Nehoroševa. Takođe, analizirajući kretanje asteroida iz različitih oblasti prstena nalaze, pored pomenute stabilnosti u smislu Nehoroševa, stabilnost koju opisuje KAM teorija kao i nestabilan haotičan režim u kome se može detektovati difuzija u faznom prostoru. Ova teza je komplementarna radu Guzzo i dr. (2002) jer proverava ispunjenost uslova za primenu teoreme Nehoroševa na asteroide iz glavnog prstena, što nedostaje u pomenutom radu.

Dakle, u svim navedenim radovima primenjuje se direktno teorema Nehoroševa na neki dinamički sistem, ali uvek bez prethodne neposredne provere uslova za

[1]Nalazi se na adresi `http://hamilton.dm.unipi.it/astdys`

koje je teorema dokaziva – konveksnost, kvazi–konveksnost, 3–jet ili generički uslov strmosti (engl. steepness).

Do sada je objavljen samo jedan rad koji se bavi proverom ispunjenosti uslova i, zatim, primenom teoreme Nehoroševa za realne sisteme u astronomiji. Benettin i dr. (1998) daju detaljnu analizu Hamiltonijana za ograničeni slučaj tri tela u Lagrange-ovim tačkama L_4 i L_5 u smislu ispunjenosti uslova konveksnosti, kvazi–konveksnosti ili 3–jet. Međutim, oni nisu odredili oblast oko Lagrange-ovih tačaka L_4 i L_5 gde bi uslovi za primenu teoreme Nehoroševa bili ispunjeni već su analizirali ispunjenost uslova u ovim Lagrange-ovim tačkama u zavisnosti od odnosa masa dva tela (treće telo u ograničenom problemu ima zanemarljivu masu).

Zadatak i sadržaj teze

U ovoj tezi se daje analiza ispunjenosti uslova konveksnosti, kvazi–konveksnosti ili 3–jet za Hamiltonijan sistema koji se sastoji od asteroida čije je keplerovsko kretanje poremećeno pod uticajem velikih planeta.

Disertacija se sastoji iz sledećih delova: u prvom poglavlju su date osnovne veličine kojima se meri haotično kretanje. Zatim sledi kratak opis uzroka haosa u kretanju asteroida.

U drugom poglavlju je izložena teorema Nehoroševa, posebno naglašavajući sledeće uslove za njenu primenu: konveksnost, kvazi-konveksnost i 3–jet nedegenerisanost. Sledi definicija ovih uslova i algoritam za njihovo računanje.

U trećem poglavlju se opisuje Kozaijev Hamiltonijan i njegova dinamika. Daju se eksplicitni izrazi za izvode do četvrtog reda prvo po Delaunay-ovim promenljivima, a zatim po momentima.

Četvrto poglavlje prikazuje originalne rezultate dobijene ispitivanjem ispunjenosti uslova za primenu teoreme Nehoroševa na asteroide. Analizira se fazni prostor u smislu ispunjenosti uslova konveksnosti, kvazi–konveksnosti ili 3–jet u oblastima koje zauzimaju familije Koronis i Veritas. Zatim se ti rezultati uporedjuju sa rezultatima Guzzo i dr. (2002) koji su dobijeni primenom spektralne formulacije teoreme Nehoroševa.

U završnom poglavlju su istaknuti originalni rezultati ove teze i dati predlozi za budući rad.

Glava 1

Haotično kretanje asteroida

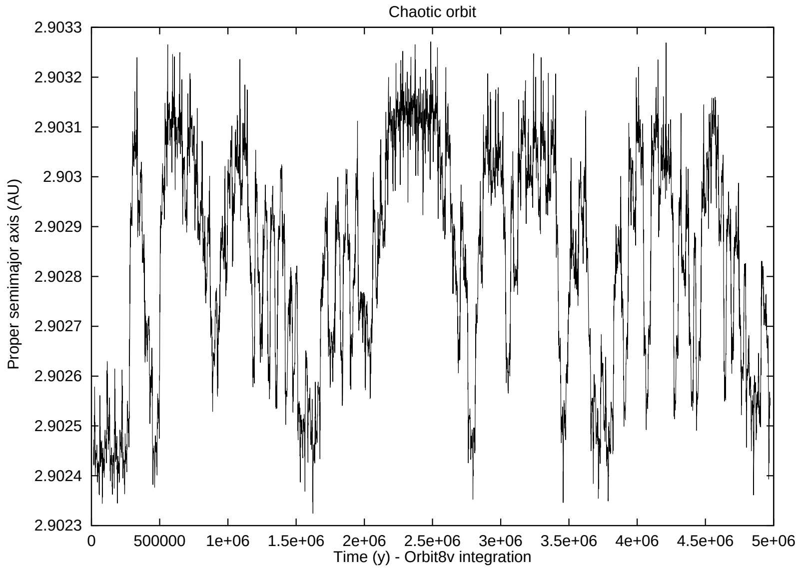
Kretanje asteroida, tela koja se u velikoj većini nalaze između Marsa i Jupitera (glavni pojas), je veoma složeno. Javlja se u širokom spektru pojava od regularnog (kvazi–periodičnog) kretanja pa sve do jako haotičnog. Za prepoznavanje i merenje haosa razvijeni su različiti alati koji su zasnovani na numeričkoj integraciji jednačina kretanja i izračunavanju pogodnih veličina.

1.1 Poincaré-ov presek

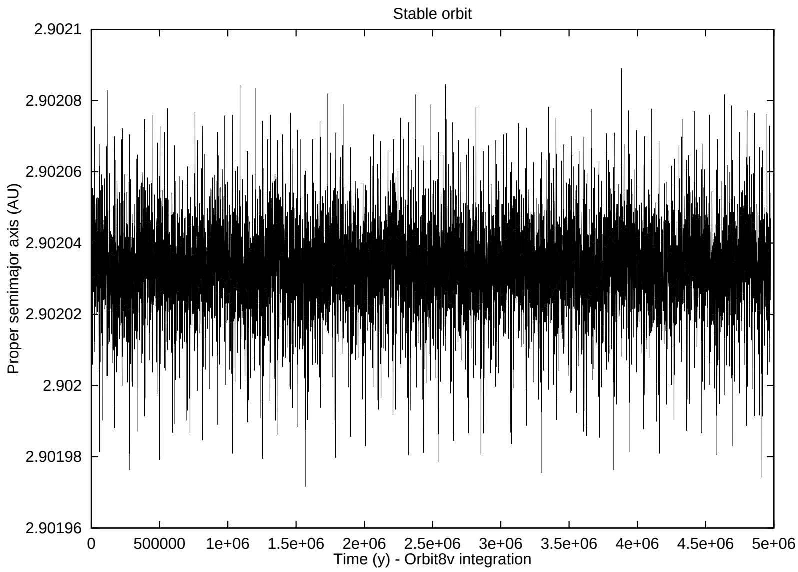
Jako haotična kretanja se mogu lako prepoznati posmatranjem vremenskih serija (dobijenih integracijom) orbitalnih elemenata, na primer velike poluose (Slika 1.1). Napomenimo da je na slici prikazana sopstvena velika poluosa, koja je dobijena uklanjanjem kratkoperiodičnih oscilacija i zadržavanjem sekularnih. Tada se na grafiku mogu lakše uočiti relativno velike, brze i skokovite promene izazvane različitim dinamičkim režimima. Kod kvazi–periodičnih kretanja nema nikakvih skokovitih promena sopstvene velike poluose već samo regularne kvaziperiodične varijacije (Slika 1.2). Takođe, za kvalitativno razlikovanje regularnog od haotičnog kretanja koristi se Poincaré-ov presek (Poincaré surface of section). Poincaré-ov presek je pogodan za sisteme sa dva stepena slobode, dok u slučaju kada imamo više od dva stepena hiperpovršine nije moguće grafički prikazati.

1.2 Maksimalni karakteristični eksponent Ljapunova

Da bi se na neki način uspostavila mera haotičnosti kretanja često se računa tzv. *Maksimalni karakteristični eksponent Ljapunova* (**M**aximum **L**yapunov **E**xponent – MLE). On, zapravo, predstavlja meru brzine razilaženja dve inicijalno veoma



Slika 1.1: Vremenska serija sopstvene velike poluose za haotičnu orbitu. Nepravilne varijacije su jasno vidljive i izazvane su čestom promenom dinamičkog stanja (Knežević, 2000).



Slika 1.2: Vremenska serija sopstvene velike poluose za stabilnu orbitu. Na grafiku se samo mogu videti pravilne varijacije usled kvaziperiodičnog oscilovanja (Knežević, 2000).

bliske orbite. Ukoliko je haos jači brža će biti divergencija, a samim tim, veći MLE.

MLE možemo računati na sledeći način: neka su $\delta\mathbf{p}(t) = \mathbf{p}_2(t) - \mathbf{p}_1(t)$ i $\delta\mathbf{q}(t) = \mathbf{q}_2(t) - \mathbf{q}_1(t)$ razlike koordinata dve trajektorije čiji je Hamiltonijan $\mathcal{H}(\mathbf{p}, \mathbf{q})$, a $\mathbf{p}$ i $\mathbf{q}$ konjugovane promenljive dimenzije n. Linearizovane jednačine relativnog kretanja možemo napisati u obliku

$$\begin{aligned} \delta\dot{p}_i &= -\sum_{j=1}^{n}\left[\frac{\partial^2\mathcal{H}}{\partial p_j \partial q_i}\delta p_j + \frac{\partial^2\mathcal{H}}{\partial q_j \partial q_i}\delta q_j\right] \\ \delta\dot{q}_i &= -\sum_{j=1}^{n}\left[\frac{\partial^2\mathcal{H}}{\partial p_j \partial p_i}\delta p_j + \frac{\partial^2\mathcal{H}}{\partial q_j \partial p_i}\delta q_j\right] \end{aligned} \tag{1.1}$$

gde je n broj stepeni slobode. MLE se definiše

$$\chi = \lim_{t\to\infty} \frac{1}{t} \ln \frac{\|\delta\mathbf{p}(t), \delta\mathbf{q}(t)\|}{\|\delta\mathbf{p}(0), \delta\mathbf{q}(0)\|}, \tag{1.2}$$

gde je $\|.,.\|$ euklidska norma. Praktičan recept za računanje MLE dat je u radu Benettin i dr. (1980) i sastoji se od sledećih koraka:

1. izabrati proizvoljne vrednosti za $\delta\mathbf{p}(0), \delta\mathbf{q}(0)$
2. računati tok $\delta\mathbf{p}(t), \delta\mathbf{q}(t)$ sve do pogodno izabranog vremena T tako da se izbegne račun sa suviše velikim brojevima
3. izračunati $s_1 = \frac{\|\delta\mathbf{p}(T), \delta\mathbf{q}(T)\|}{\|\delta\mathbf{p}(0), \delta\mathbf{q}(0)\|}$ i $\delta\mathbf{p}_1 = \delta\mathbf{p}(T)/s_1$ i $\delta\mathbf{q}_1 = \delta\mathbf{q}(T)/s_1$
4. za nove početne uslove uzeti $\delta\mathbf{p}_1$ i $\delta\mathbf{q}_1$ i nastaviti račun

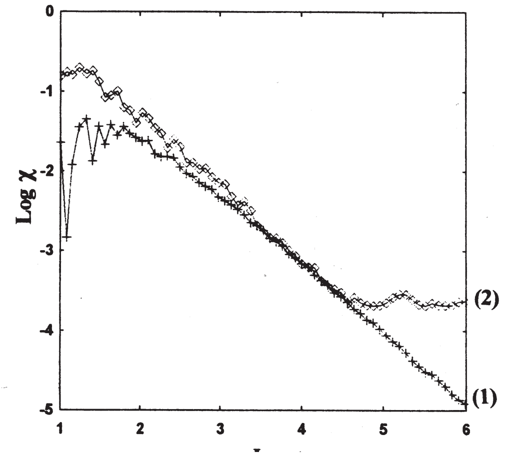
Benettin i dr. (1980) su pokazali da je

$$\chi = \lim_{l\to\infty} \frac{\sum_{j=1}^{l} \ln s_j}{lT}, \tag{1.3}$$

i da rezultat ne zavisi od izbora T. U praksi se često koristi veličina *vreme Ljapunova* (T_L) koja se definiše kao

$$T_L = \frac{1}{\chi}, \tag{1.4}$$

i predstavlja vreme koje je potrebno da se rastojanje između orbita poveća e puta. Primer MLE za haotičnu orbitu dat je na Slici 1.3.

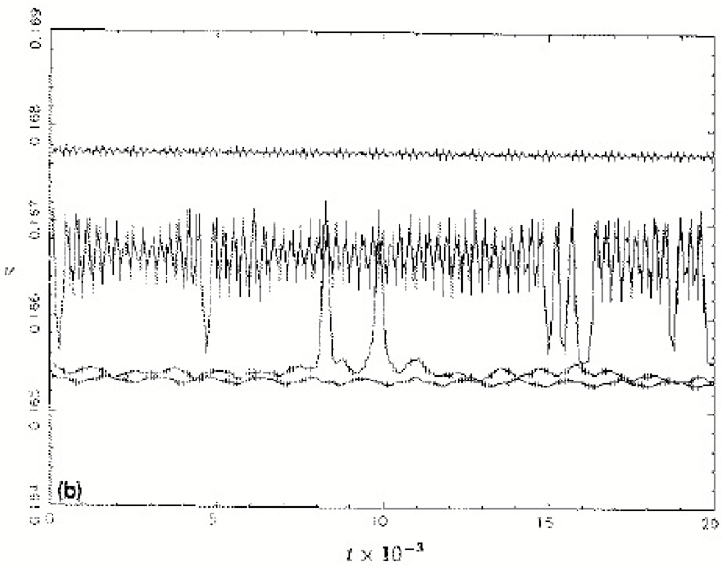


Slika 1.3: Promena $\log\chi$ u funkciji od $\log t$, za regularnu orbitu (1) i haotičnu (2). Nagib krive $\Delta\log\chi/\Delta\log t$ teži -1 za regularnu orbitu, dok za haotičnu teži da postane konstanta (Contopoulos, 2002).

Dobra procena vrednosti MLE zahteva numeričku integraciju jednačina kretanja za interval vremena koji je bar 6 – 10 puta duži od T_L. U slučaju slabog haosa integracije će trošiti jako mnogo računarskog vremena. Zbog toga su uvedeni alternativni indikatori ili mere haosa koji će se mnogo efikasnije računati, a biće u tom slučaju isto toliko pouzdani kao MLE.

1.3 Analiza frekvencija

Laskar (1990, 1992, 1993) je predložio jedan takav novi metod – *analizu frekvencija*, koji se sastoji u primeni modifikovane Furijeove transformacije, u smislu "pokretnog prozora", na vremenske serije dobijene kao izlaz iz numeričkih integracija jednačina kretanja usrednjenog sistema (tzv. sekularni sistem). Nalaženjem rešenja tih jednačina i odgovarajućih sopstvenih modova (za integrabilnu aproksimaciju to su moment–ugao promenljive) i, zatim, primenom Furijeove analize na te sopstvene modove računaju se fundamentalne frekvencije sistema. Za regularna kretanja (na KAM torusu) ove frekvencije su konstantne, dok se za haotična kretanja one menjaju sa vremenom. Prema tome, odstupanje frekvencija od konstantne vrednosti se koristi za identifikaciju haosa, procenu njegovog intenziteta, pa čak se može i "izmeriti"veličina haotične zone. Takođe, umesto da se prati vremenska promena frekvencije jedne orbite (Slika 1.4), može da se postavi mreža početnih uslova u onoj oblasti faznog prostora koji nas interesuje i da se računa frekvencija za svaki takav početni uslov u unapred zadatom intervalu vremena. Na grafiku zavisnosti frekvencija – početni uslov lako se uočavaju regularne i haotične

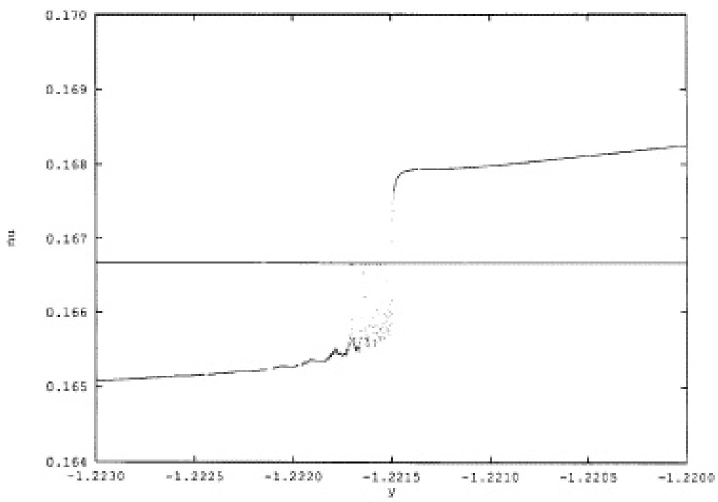


Slika 1.4: Vremenska evolucija frekvencije za orbite koje su bliske rezonansi 1/6 za standardnu mapu. Orbite čija je frekvencija približno konstantna leže na KAM torusu, dok su ostale haotične (Laskar i dr., 1992).

zone (Slika 1.5).

1.4 Ostali indikatori haosa

Razni autori su uvodili druge numeričke alate za detekciju haosa kao razne varijacije eksponenata Ljapunova ili analize frekvencija. Među takvima, sigurno spada *brzi indikator Ljapunova* (**F**ast **L**yapunov **I**ndicator – FLI) koji je uveden i iskorišćen za ispitivanje dinamičke evolucije svih numerisanih asteroida (Froeschlé i dr., 1997) ukazavši na važnost rezonansi između tri tela. Brzi indikator Ljapunova je, zapravo, vreme T za koje $\|\delta\mathbf{p}(t), \delta\mathbf{q}(t)\|$ dostigne neku proizvoljno veliku unapred fiksiranu vrednost R pri dinamičkoj evoluciji polazeći od početne vrednosti $\|\delta\mathbf{p}(0), \delta\mathbf{q}(0)\|$. Međutim, ako fiksiramo početne uslove $\delta\mathbf{p}(0), \delta\mathbf{q}(0)$ i vrednosti R tada FLI može poslužiti da se uporedi dinamičko ponašanje različitih orbita. U tom smislu, FLI se može posmatrati kao indikator haotičnosti jedne orbite u odnosu na drugu. Da bi postao apsolutni indikator neophodno je uraditi kalibraciju, tj. izračunati na primer MLE za neke referentne orbite. Ovaj indikator je intenzivno koristila Todorović (2007) u magistarskoj tezi za istraživanje dinamičke strukture i difuzije četvorodimenzione simplektičke mape.



Slika 1.5: Frekvencija u funkciji početnih uslova y momenta p_1 za orbite bliske rezonansi 1/6 za standardnu mapu. Glatki deo krive na grafiku odgovara oblasti regularnih orbita, dok raštrkane tačke odgovaraju haotičnoj zoni oko separatrise (Laskar i dr., 1992).

Contopoulos i Voglis (1996) su uveli helikoidne i *twist* uglove. Za dati sistem linearizovanih jednačina (1.1) se računa orijentacija vektora $\delta\mathbf{p}(t), \delta\mathbf{q}(t)$ u funkciji vremena. Za sistem sa n stepeni slobode orijentacija je definisana sa $n-1$ helikoidnim uglom: $\Phi_1, \ldots, \Phi_{n-1}$. Njihove srednje vrednosti po vremenu su: $< \Phi_1 >, \ldots, < \Phi_{n-1} >$ i zavise samo od orbite oko koje su računate linearizovane jednačine kretanja, dok ne zavise od izbora početnog vektora $\delta\mathbf{p}(0), \delta\mathbf{q}(0)$. Contopoulos i Voglis su pokazali da u haotičnim oblastima srednja vrednost helikoidnih uglova je invarijantna, dok se za regularne orbite ravnomerno menja u zavisnosti od početnih uslova.

Pored helikoidnih uglova, Contopoulos i Voglis su koristili njihove izvode po vremenu – *twist uglove*. Tako na primer, srednja vrednost twist uglova omogućava da se razdvoje regularne od haotičnih orbita, tj. ovi uglovi su jednaki nuli za orbite na KAM torusu (za koje svi uglovi vrše cirkulaciju), za orbite koje leže na rezonantnom invarijantnom torusu jednaki su libracionoj frekvenciji, a za haotične orbite postaju invarijantni.

1.5 Haos u kretanju asteroida

Mehanizmi koji dovode do haosa u kretanju asteroida su: (i) bliski prilazi planetama, (ii) prelazi preko separatrisa rezonansi niskog reda, (iii) preklapanje rezonansi.

1.5.1 Bliski prilazi planetama

Bliski prilazi planetama se dešavaju kada asteroid priđe dovoljno blizu planeti tako da pretrpi drastične promene svog kretanja. U slučaju bliskog prilaza unutar uticajne sfere[1] velike planete poremećaji su toliko veliki da orbita može preći iz Keplerove elipse (obično veoma ekscentrične) u hiperboličnu orbitu i suprotno. Ponavljanjem ovih bliskih prilaza nastaju haotične putanje, tj. takvo kretanje se ne može opisati uobičajenim matematičkim alatima, jer čak i vrlo male promene u početnim uslovima dovode do eksponencijalne divergencije orbita, a sama orbitalna evolucija postaje nepredvidiva. Izbacivanja iz sunčevog sistema, sudari sa planetama ili Suncem i višestruka presecanja orbita planeta su neke od posledica bliskih prilaza i sve one, matematički posmatrano, predstavljaju singularitete jednačina kretanja. Tako, na primer, vremena Ljapunova asteroida čije orbite seku Zemljinu putanju su veoma kratka, reda nekoliko desetina do stotinu godina (Whipple, 1995).

1.5.2 Prelazi preko separatrisa

Naš planetarni sistem je ispresecan mnogobrojnim rezonansama različitih vrsta, posebno u njegovom unutrašnjem delu i u oblasti asteroidnog prstena (Knežević i dr, 1991; Nesvorný i Morbidelli, 1998). Izuzimajući rezonanse niskog reda u srednjem kretanju (do reda $K \leq \ln \varepsilon$, gde je ε masa Jupitera u jedinicama mase Sunca, tj. $\varepsilon \approx 10^{-3}$) i vrlo jake ν_6 sekularne rezonanse[2] Guzzo i Morbidelli (1997) su pokazali da je glavni prsten asteroida u dinamičkom režimu koji opisuje teorema Nehoroševa. Prema Nehoroševu, fazni prostor se može podeliti na različite zone: nerezonantne domene, domene sa jednom rezonansom i domene sa dve rezonanse, međutim detaljnije o tome videti u sledećem poglavlju. Napomenimo da je glavni uzrok haosa prelaz preko separatrisa rezonansi. Separatrise predstavljaju neku vrstu granice razdvajajući oblasti u faznom prostoru izvan rezonanci, gde rezonantni kritični ugao vrši cirkulaciju, od oblasti unutar rezonanse gde se vrši libracija. Obično su separatrise okružene tankim stohastičkim slojem, a periodičnim

[1] Poluprečnik uticajne sfere se definiše $\rho = d\sqrt[5]{(\frac{m}{M})^2}$, gde su m i M mase tela, a d je njihovo međusobno rastojanje. Na primer, za Jupiter u odnosu na Sunce ($\frac{m}{M} \approx \frac{1}{1047.35}$) poluprečnik uticajne sfere iznosi $\rho = 0.30665$ AU.

[2] Rezonansa koja je posledica srazmernosti frekvencija perihela asteroida i Saturna.

presecanjem takvog sloja pojačavaju se haotični efekti.

1.5.3 Preklapanje rezonansi

Preklapanje rezonansi se javlja između različitih tipova rezonansi (sekularnih i/ili u srednjem kretanju) ili između multipleta iste rezonanse u srednjem kretanju višeg reda. U oblastima faznog prostora gde dolazi do preklapanja rezonansi ne može se definisati nerezonanrtni domen, a svi invarijantni torusi bivaju uništeni i čitava oblast se karakteriše brzom difuzijom (Knežević, 2000).

Glava 2

Teorema Nehoroševa

Nehorošev je teoremu dao za kvazi–integrabilne nedegenerisane sisteme. Pod kvazi–integrabilnim sistemima podrazumevamo one sisteme čiji se Hamiltonijan može prikazati u obliku sume integrabilnog dela $\mathcal{H}_0$ i male perturbacije $\epsilon\mathcal{H}_1$. Hamiltonijan je *nedegenerisan* ako ima samo onoliko prvih integrala koliko stepeni slobode. Međutim, degenerisani integrabilni sistemi imaju veći broj prvih integrala od broja stepeni slobode. Tipičan primer je Keplerov problem dva tela. On ima 5 prvih integrala: velika poluosa (a), ekscentričnost (e), inklinacija i, i uglovi longituda perihela (ϖ) i longituda čvora (Ω). Dakle, pored konstantnih momenata još i neki uglovi bivaju konstante kretanja (Morbidelli, 2002).

Za nedegenerisane sisteme važi KAM teorema (Kolmogorov, 1954; Arnold, 1963; Moser, 1962). Ona tvrdi da ako je perturbacija dovoljno mala onda postoji veliki broj invarijantnih torusa na kojima će se nalaziti kvazi–periodične trajektorije za sve početne uslove iz zadatog domena.

Proteklih decenija više grupa je radilo na ovoj problematici. Ispitivani sistem je bio najprostiji ograničeni slučaj tri tela sa kružnim orbitama (Celletti i Chierchia, 1998).

2.1 Formulacija teoreme

Nehorošev (1977) je postavio teoremu za autonomne Hamiltonove sisteme tipa $\mathcal{H}(p,q) = \mathcal{H}_0 + \epsilon\mathcal{H}_1$, gde su (p,q) moment–ugao promenljive definisane na domenu $\mathcal{D} \equiv \mathcal{G} \times \mathrm{T}^n$. Ovde je $\mathcal{G}$ prostor momenata na domenu $\mathbb{R}^n$, dok je T^n prostor uglova – n–dimenzioni torus. Osnovna pretpostavka je da je Hamiltonijan analitička funkcija, tj. da se može razviti u konvergentan red u okolini neke tačke $(\overline{p}, \overline{q}) \in \mathcal{D}$. Sa $\mathcal{G} - \Delta$ označavamo skup svih tačaka p koje su sadržane u $\mathcal{G}$ zajedno sa Δ okolinom. Sada možemo formulisati teoremu Nehoroševa:

Neka je $\mathcal{H}(p,q) = \mathcal{H}_0 + \epsilon\mathcal{H}_1$ realan i analitički u $\mathcal{D} \equiv \mathcal{G} \times \mathrm{T}^n$, gde je

$\mathcal{G} \subset \mathbb{R}^n$ otvoren i ograničen i $\|\mathcal{H}_1\| \leq 1$. Neka je matrica $C(p)$ definisana sa $C_{ij} = \frac{\partial^2 \mathcal{H}_0(p)}{\partial p_i \partial p_j}$ i neka postoje pozitivne konstante M i m takve da

$$\|C(p)v\| \leq M\|v\|, \forall p \in G, \forall v \in \mathbb{R}^n \tag{2.1}$$

$$|C(p)v \cdot v| \geq mv \cdot v, \forall p \in G, \forall v \in \mathbb{R}^n. \tag{2.2}$$

Tada postoje pozitivne konstante ϵ_*,α, β, a i b takve da za svako $\epsilon \leq \epsilon_*$ važi

$$\|p(t) - p(0)\| \leq \Delta \equiv \alpha\epsilon^a \tag{2.3}$$

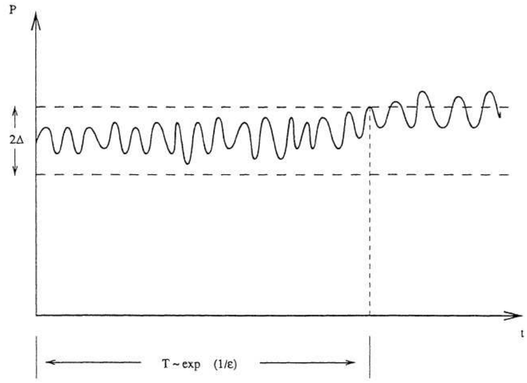
za svako $p(0) \in \mathcal{G} - \Delta$ i za svako $|t| \leq T(\epsilon)$ gde je

$$T(\epsilon) = \beta\left(\frac{1}{\epsilon}\right)\exp\left(\frac{1}{\epsilon}\right)^b. \tag{2.4}$$

Treba istaći da teorema Nehoroševa ne isključuje mogućnost haotičnog kretanja tj. momenti p se mogu menjati na haotičan način ali je veličina tih promena ograničena sa Δ sve do isteka vremena $T(\epsilon)$(Slika 2.1). Ovo *vreme stabilnosti* T raste eksponencijalno kada ϵ opada. Kada je ϵ veoma malo (blisko nuli) tada T može biti izuzetno dug period tj. može premašiti fizičko vreme trajanja dinamičkog sistema. Na ovaj način se dokazuje efektivna stabilnost sistema. Treba posebno istaći da ovaj rezultat važi uniformno za svaki izabrani početni uslov (p, q) kada je $p \in \mathcal{G} - \Delta$. Međutim, ako bi početni uslov izabrali blizu granice (na "odstojanju" manjem od Δ) tada bi se moglo desiti da moment "pobegne" iz domena $\mathcal{G}$ za vreme kraće od T. Važna pretpostavka teoreme je analitičnost Hamiltonijana, dok se uslov konveksnosti (2.2) može ublažiti u generički uslov strmosti (engl. steepness) za koji je Nehorošev dokazao teoremu.

Da bi važila teorema Nehoroševa, fazni prostor mora da poseduje specifičnu strukturu – strukturu Nehoroševa iz koje se lako izvodi eksponencijalna stabilnost. U čemu se sastoji ova struktura? Radi jednostavnosti pretpostavićemo da se radi o sistemu sa tri stepena slobode. Sa $\omega = \partial\mathcal{H}_0/\partial p$ označićemo frekvencije koje će poslužiti da se definiše struktura.

Eliminišimo iz perturbacije $\epsilon\mathcal{H}_1$, primenom kanonskih transformacija, harmonijski član $\epsilon\mathcal{H}_{1,k}e^{ik\cdot q}$. To je moguće samo izvan pogodno odabrane okoline rezonantne ravni $k \cdot \omega = 0$. Naravno, blizu rezonantne ravni se ne može ukloniti ovaj član. Takođe, poznato je da je skup svih rezonansi generisanih celobrojnim vektorima $k \in \mathbb{Z}^3 \backslash 0$ gust u $\mathbb{R}^3$, tako da na svakom otvorenom podskupu $\mathcal{G}$ nije moguće eliminisati iz perturbacije neograničen broj harmonika. Na ovoj činjenici se zasniva Poincaré-ov (1892) dokaz neintegrabilnosti sistema. Međutim, ideja Nehoroševa je



Slika 2.1: Skica evolucije momenta p prema Nehoroševu. Promene momenata mogu biti haotične, ali su ograničene na usku oblast Δ oko početne vrednosti. Moment može izaći iz oblasti Δ tek po isteku eksponencijalno dugog vremenskog intervala T.

da se ograniči na rezonanse samo do nekog reda K.[1] Ovakav pristup je veoma važan jer je broj rezonansi do datog reda konačan, dok svaki otvoreni skup momenata može da sadrži konačan broj rezonantnih ravni. Pokazuje se da su članovi kojima odgovaraju rezonanse višeg reda od K eksponencijalno mali i mogu se zanemariti.

Najpre, prema Nehoroševu, možemo definisati *nerezonantni domen* kao skup onih tačaka koje su dovoljno daleko od svih rezonansi reda K (Slika 2.2a). Preciznije rečeno, *nerezonantni domen* se definiše kao skup frekvencija ω takvih da je $|k \cdot \omega| > \sqrt{\epsilon}$ za svako k za koje je $|k| \leq K$. Ovo je takozvani Diofantov uslov, gde se tretiraju samo one tačke u faznom prostoru za koje važi

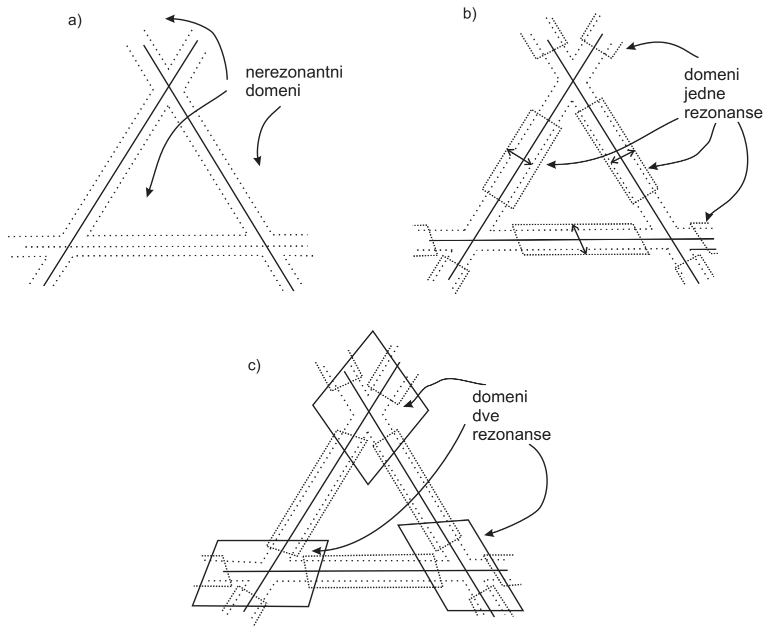
$$|k \cdot \omega| \geq \frac{\gamma}{|k|^\tau}, \quad \forall k \in \mathbb{Z}^n, \quad k \neq 0, \tag{2.5}$$

za neko pozitivno γ i τ (Todorović, 2007).

U nerezonantnom domenu se mogu eliminisati svi harmonijski članovi u perturbaciji $\epsilon\mathcal{H}_1$ reda manjeg od K, što ima za posledicu da se Hamiltonijan $\mathcal{H}_0 + \epsilon\mathcal{H}_1$ može integraliti ako se zanemari ostatak $\mathcal{R}_K$ koji je eksponencijalno mali. Prema tome, u nerezonantnom domenu frekvencije (momenti) će biti konstante sve do reda koji može da sadrži zanemareni eksponencijalno mali član $\mathcal{R}_K$.

Deo faznog prostora gde je prisutna samo jedna rezonansa do reda K naziva se *domen jednostruke rezonanse* (Slika 2.2b). Kao i u prethodnom slučaju svi nerezonansni članovi se mogu eliminisati. Tada se Hamiltonijan svodi samo na jedan rezonansni član reda manjeg od K i ostatak $\mathcal{R}_K$ koji je opet eksponencijalno mali. Ovakav Hamiltonijan se može, pogodnim kanonskim transformacijama, učiniti da

[1] Red rezonanse tipa $k \cdot \omega \equiv k_1\omega_1 + ... + k_n\omega_n = 0$ se definiše kao $|k| = \Sigma_{i=1}^n |k_i|$.

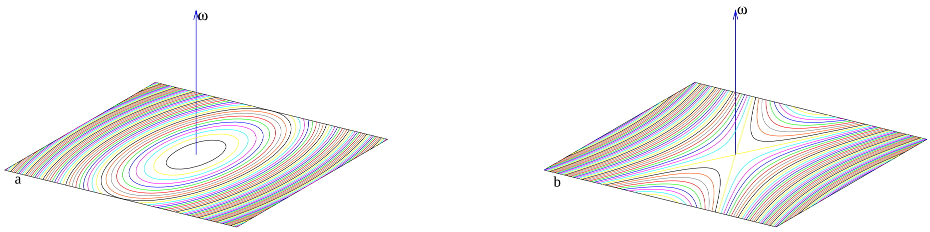


Slika 2.2: Podela faznog prostora prema Nehoroševu: nerezonantni domeni (a), domeni jedne rezonanse (b) i domeni dve rezonanse (c).

zavisi samo od jednog ugla (rezonantni ugao), a kao takav je integrabilan. Sada frekvencije nisu više konstantne već se mogu menjati u pravcu tzv. brzog drifta. Hipoteza konveksnosti (2.2) garantuje da će pravac brzog drifta biti normalan na rezonansne linije, a time i ovo kretanje mora biti ograničeno.

Mesta gde se presecaju rezonanse nazivaju se *domeni dvostruke rezonanse* (Slika 2.2c). U ovim domenima redukovani Hamiltonijan ima dva nezavisna rezonantna člana reda manjeg od K i, u opštem slučaju, nije integrabilan. Frekvencije u ovim domenima se mogu znatno menjati, što znači, da ovde očekujemo izrazito haotično kretanje. Ono što treba imati u vidu je da je ovaj fazni prostor ograničen i da se frekvencije ne mogu previše "udaljiti" od tačke preseka rezonansi jer bi "zašle" u domen jednostruke rezonanse ili u nerezonansni domen.

Može se zaključiti da, zanemarujući eksponencijalno mali ostatak $\mathcal{R}_K$, za sve početne uslove, kretanje će biti ograničeno jednim od pomenutih domena. To ima za posledicu da se frekvencije (momenti) mogu menjati najviše za veličinu jednaku radijusu domena dvostruke rezonanse. Taj radijus je proporcionalan ϵ^a, gde je $0 < a < 1$ (a opada kada broj stepeni slobode raste).



Slika 2.3: Izolinije u ravni normalnoj na pravac vektora frekvencije $\vec{\omega}$ kvazi–konveksnog Hamiltonijana (a) i kada uslov kvazi–konveksnosti nije ispunjen (b).

2.2 Definicija konveksnosti, kvazi–konveksnosti i 3–jet uslova

Uslov konveksnosti (2.2) se može zameniti blažim uslovom kvazi–konveksnosti ili još labavijim uslovom 3–jet. Svi ovi uslovi spadaju u generički tip koji Nehorošev naziva "strmost" (engl. steepness, rus. крутой). Niederman (2003) daje sledeću definiciju strmosti:

Neka je $\mathcal{P}$ otvoreni skup u $\mathbb{R}^n$. Za realnu analitičku funkciju $h : \mathcal{P} \to \mathbb{R}$ se kaže da je strma u tački $I \in \mathcal{P}$ duž nekog afinog podprostora Π, koji sadrži I, ako postoje konstante $C > 0$, $\delta > 0$ i $p > 0$ takve da duž svake analitičke regularne krive γ u Π, koja povezuje I sa nekom drugom tačkom na rastojanju $d < \delta$, norma projekcije gradijenta $\nabla h(I)$ na pravac Π bude veća od $C \cdot d^p$; (C, δ) i p se nazivaju koeficijenti i indeks strmosti respektivno.

Pod navedenim pretpostavkama, funkcija h je strma u tački $I \in \mathcal{P}$ ako I nije kritična tačka h i ako, za svako $k \in \{1, \ldots, n-1\}$, postoje konstante C_k, δ_k i p_k takve da je h strma u I duž ma kog afinog podprostora dimenzije k koji sadrži I uniformno u odnosu na koeficijente (C_k, δ_k) i indeks p_k.

Kakav je smisao ovih uslova i kako ih možemo proveriti?

Kvazi–konveksnost znači da će restrikcija Hessiana Hamiltonijana $\mathcal{H}_0$ na hiperravan ortogonalnu na vektor frekvencije ω (a prema tome, i na ravan brzog drifta koja sadrži ω) imati kvadratni minimum ili maksimum u tački tačne rezonanse. Na Slici 2.3a je prikazan ovaj slučaj za $n = 3$ i tačku dvostruke rezonanse, tako da je ravan brzog drifta dvodimenzionalna. Kada uslov kvazi–konveksnosti nije ispunjen (kao na Slici 2.3b) struktura nivoovskih linija je hiperbolička (umesto eliptičke kao u prethodnom slučaju) i u ravni brzog drifta postoje pravci, asimptote hiperbole, gde je Hessian jednak nuli. Dakle, provera kvazi–konveksnosti u nekoj tački se svodi na proveru da li je Hessiana različit od nule u toj tački. Matematički to se formuliše na sledeći način

Hamiltonijan $\mathcal{H}_0$ je konveksan u tački $I_0 \in \mathrm{R}^n$, ako za svako $u \in \mathrm{R}^n$ važi

$$\sum_{i=1}^{n} \frac{\partial \mathcal{H}_0}{\partial I_i}(I_0)u_i = 0, \tag{2.6}$$

kvazi–konveksan, ako pored (2.6), važi i relacija

$$\sum_{i,j=1}^{n} \frac{\partial^2 \mathcal{H}_0}{\partial I_i \partial I_j}(I_0)u_i u_j = 0, \tag{2.7}$$

dok za 3–jet moraju biti ispunjene relacije (2.6), (2.7) i

$$\sum_{i,j,k=1}^{n} \frac{\partial^3 \mathcal{H}_0}{\partial I_i \partial I_j \partial I_k}(I_0)u_i u_j u_k = 0, \tag{2.8}$$

tako da sistemi nemaju druga rešenja osim trivijalnih.

Dakle, za proveru kvazi–konveksnosti moraju se računati izvodi Hamiltonijana do drugog reda, dok su za proveru 3–jet uslova potrebni izvodi trećeg reda.

2.3 Algoritam za proveru uslova konveksnosti, kvazi–konveksnosti i 3–jet

Za proveru uslova konveksnosti potrebno je da sve frekvencije imaju isti znak (pozitivan ili negativan), dok za ostala dva uslova treba formirati Hesijan, tj. matricu drugih izvoda integrabilnog dela Hamiltonijana $\mathcal{H}_0$ po akcijama $I = (I_1, I_2, I_3)$

$$\mathcal{A} = \begin{pmatrix} \frac{\partial^2 \mathcal{H}_0}{\partial I_1^2} & \frac{\partial^2 \mathcal{H}_0}{\partial I_1 \partial I_2} & \frac{\partial^2 \mathcal{H}_0}{\partial I_1 \partial I_3} \\ \frac{\partial^2 \mathcal{H}_0}{\partial I_2 \partial I_1} & \frac{\partial^2 \mathcal{H}_0}{\partial I_2^2} & \frac{\partial^2 \mathcal{H}_0}{\partial I_2 \partial I_3} \\ \frac{\partial^2 \mathcal{H}_0}{\partial I_3 \partial I_1} & \frac{\partial^2 \mathcal{H}_0}{\partial I_3 \partial I_2} & \frac{\partial^2 \mathcal{H}_0}{\partial I_3^2} \end{pmatrix}. \tag{2.9}$$

Zatim, odredimo sopstvene vrednosti $(\lambda_1, \lambda_2, \lambda_3)$ Hesijana (2.9). Hamiltonijan $\mathcal{H}_0$ je konveksan (u datoj tački) ako su sve tri sopstvene vrednosti istog znaka. Ukoliko sopstvene vrednosti Hesijana nisu istog znaka, onda treba naći restrikciju Hesijana (2.9) na ravan normalnu na vektor frekvencija

$$\vec{\omega} \;=\; (\omega_1, \omega_2, \omega_3) = (\frac{\partial \mathcal{H}_0}{\partial I_1}, \frac{\partial \mathcal{H}_0}{\partial I_2}, \frac{\partial \mathcal{H}_0}{\partial I_3}). \tag{2.10}$$

Postupak računanja restrikcije Hesijana je preuzet iz Benettin i dr. (1998). On se sastoji od rotacije koordinata I sve dok se vektor frekvencija $\vec{\omega}$ ne poklopi sa prvom koordinatnom osom. Zapravo, tražena restrikcija ($\mathcal{B}$) je odgovarajuća rotirana blok–matrica 2×2. Hamiltonijan $\mathcal{H}_0$ je kvazi-konveksan ako obe sopstvene vrednosti restrikcije $\mathcal{B}$ imaju isti znak.

Neka su sopstvene vrednosti restrikcije $\mathcal{B}$ različitog znaka, označimo ih sa $b_+ > 0 > b_-$ i neka su $w_\pm$ odgovarajući sopstveni vektori jedinične norme. Dva pravca u ravni normalnoj na vektor frekvencija gde je Hesijan (2.9) jednak nuli su paralelni vektorima

$$u_\pm = \sqrt{|b_-|}w_+ \pm \sqrt{|b_+|}w_-. \tag{2.11}$$

Za proveru 3–jet uslova dovoljno je izračunati izraz sa leve strane jednačine (2.8) za vektore $u_\pm$ i ako je različit od nule, za oba vektora, kažemo da je Hamiltonijan $\mathcal{H}_0$ 3–jet nedegenerisan. Zapravo, izabran je nivo od 10^{-2} u jedinicama srednjeg kretanja Jupitera[2] ispod koga se može smatrati da uslov nije ispunjen.

[2]Vrednost je preuzeta od Lemaitre i Morbidelli (1994) koja je korišćena kao granična veličina širine rezonanse za detekciju sekularnih rezonansi što odgovara oko 3 arcsec/yr.

Glava 3

Kozaijev Hamiltonijan i njegovi izvodi

Model koji koristimo se sastoji od asteroida, tela zanemarljivo male mase, koji se kreće po Keplerovoj orbiti oko Sunca i čije kretanje perturbuju (velike) planete sunčevog sistema. Koristimo sledeće, uobičajene oznake za eliptičke oskulatorne elemente asteroida: a – velika poluosa, e – ekscentričnost, i – nagib, ω – argument perihela, Ω – longituda uzlaznog čvora i M – srednja anomalija merena u heliocentričnom sistemu. Odgovarajuće veličine sa primom odnosiće se na poremećajnu planetu, dok je njena masa m'. Takođe, koristićemo kanonske promenljive Delaunay-a

$$\begin{aligned}
L &= \sqrt{\mathcal{G}(m_0+m)a} = \sqrt{\mu a}, & l &= M \\
G &= L\sqrt{1-e^2}, & g &= \omega \\
H &= G\cos i = L\sqrt{1-e^2}\cos i, & h &= \Omega,
\end{aligned} \tag{3.1}$$

ili kanonske promenljive Poincaré-a

$$\begin{aligned}
\Lambda &= L = \sqrt{\mu a}, & \lambda &= l+g+h = M+\varpi, \\
\Gamma &= L-G = L(1-\sqrt{1-e^2}), & \gamma &= -g-h = -\varpi, \\
Z &= G-H = 2G\sin^2\frac{i}{2}, & z &= -h = -\Omega.
\end{aligned} \tag{3.2}$$

gde je sa μ označen proizvod mase Sunca m_0 i gravitacione konstante $\mathcal{G}$. Hamiltonovu funkciju $\mathcal{H}$ za asteroid možemo napisati u sledećem obliku (Lemaitre i Morbidelli, 1994):

$$\mathcal{H} = \mathcal{H}_0(L) + \mathcal{H}_1(L, G, H, l, g, h, a', e', i', l', g', h') \tag{3.3}$$

gde je

$$\mathcal{H}_0(L) = -\frac{\mu^2}{2L^2} \tag{3.4}$$

tj. Hamiltonijan problema dva tela (Sunce–asteroid), a $\mathcal{H}_1$ poremećaj koji vrši planeta na asteroid i koji je proporcionalan masi m'. Ukoliko uključujemo uticaj više planeta $\mathcal{H}_1$ uzimamo kao sumu poremećaja.

3.1 Eliminacija kratkoperiodičnih članova i integrabilni Hamiltonijan

Hamiltonove jednačine sadrže kratkoperiodične članove, tj. članove čiji periodi odgovaraju vremenima revolucije l i l' oko Sunca. Eliminacija kratkoperiodičnih članova u Hamiltonijanu (3.3) se vrši primenom perturbacione teorije Hori-ja (1966) koja se zasniva na konstrukciji pogodnih kanonskih transformacija. Ovakva transformacija je do prvog reda po masi planete ekvivalentna usrednjavanju Hamiltonijana (3.3) po uglovima l i l'. Schubart (1964) je pokazao da se to može uraditi numerički tako da se 'vrednost razmatrane funkcije računa za ekvidistantne tačke u intervalu periodičnosti kratkoperiodičnog argumenta, dok kritični i drugi dugoperiodični argumenti ostaju konstantni'. Dakle, možemo pisati

$$< \mathcal{H} >= -\frac{1}{2\Lambda^2} + < \mathcal{H}_1 > \tag{3.5}$$

gde je

$$< \mathcal{H}_1 >= \frac{1}{2\pi} \int_0^{2\pi} \mathcal{H}_1 \, dl \, dl'. \tag{3.6}$$

Usrednjeni Hamiltonijan $< \mathcal{H} >$ je sada funkcija srednjih Delaunay-ovih promenljivih (L,G,H,g,h) dok parametarski zavisi od elemenata planeta (a',e',i',g',h').

Kako su ekscentričnost i inklinacija planeta male veličine (Williams, 1969) možemo Hamiltonijan (3.5), uzimajući u obzir (3.6), razviti u Tejlorov red po promenljivima e' i i' u okolini nule. Tada, Hamiltonijan (3.5) postaje

$$\mathcal{H} = -\frac{1}{2\Lambda^2} + \varepsilon\mathcal{K}_0 + \varepsilon^2\mathcal{K}_1 + \varepsilon^3\mathcal{K}_2 + \cdots \tag{3.7}$$

gde indeks i u $\mathcal{K}_i$ označava stepen odgovarajućeg polinoma po e' i i'. Prema tome, $\mathcal{K}_0$ je perturbacija koju izazivaju planete kada se kreću po kružnim putanjama u istoj ravni. Ovde treba istaći da nije nikakav razvoj vršen po elementima asteroida tako da će izvedena teorija važiti za sve vrednosti ekscentriciteta i inklinacije.

Član $\mathcal{K}_0$ se naziva Kozaijev Hamiltonijan (Kozai, 1962), a računamo ga iz

$$\mathcal{K}_0 =< \mathcal{H}_1 > |_{e'=0,i'=0}. \tag{3.8}$$

Pokazuje se da je Kozaijev Hamiltonijan funkcija samo jednog ugla – argumenta perihela (g), a kao takav je integrabilan. Zato ćemo kasnije dati njegov detaljniji opis. Dakle, integrabilni Hamiltonijan se sastoji od prva dva člana (3.7) tj.

$$\mathcal{H}_{\text{int}} = -\frac{1}{2\Lambda^2} + \varepsilon\mathcal{K}_0(L, G, H, g) \tag{3.9}$$

gde je ε reda veličine mase poremećajne planete ($\sim 10^{-3}$).

3.2 Dinamika Kozaijevog Hamiltonijana

Da bi opisali dinamički portret Kozaijevog Hamiltonijana pogodno je preći na modifikovane kanonske Poincaré-ove promenljive, zadržavajući g kao jedini ugao

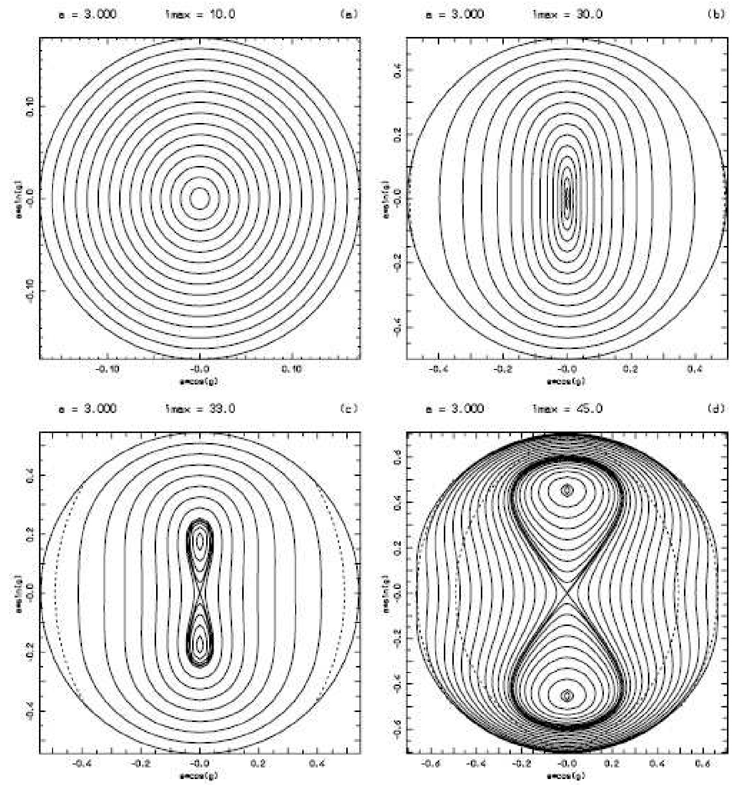
$$\begin{array}{rclrcl}
\Lambda &=& L = \sqrt{a}, & \lambda &=& l + g + h, \\
P &=& L - H = \sqrt{a}(1 - \sqrt{1-e^2}\cos i), & p &=& -g - h, \\
Q &=& G - H = \sqrt{a}\sqrt{1-e^2}(1 - \cos i), & q &=& g.
\end{array} \tag{3.10}$$

Napomenimo da je izabran takav sistem jedinica u kome je $\mu = 1$. Sa ovim novim skupom promenljivih Kozaijev Hamitonijan možemo simbolički prikazati

$$\mathcal{K}_0 = \mathcal{K}_0(q, Q, -, P, -, \Lambda). \tag{3.11}$$

Kako $\mathcal{K}_0$ ne zavisi od uglova p i λ to će za konjugovane momente važiti $P = \text{const}$ i $\Lambda = \text{const}$. Zbog toga što je $P = \text{const}$ ekscentricitet i inklinacija asteroida više neće biti nezavisni. Na svakoj površini $P = \text{const}$ može se odrediti maksimalna vrednost inklinacije $i_{\max} = \arccos(H/\sqrt{a})$ kojoj odgovara $e = 0$. S druge strane maksimalna vrednost ekscentriciteta je $e_{\max} = \sqrt{1 - H^2/a}$ i dobija se za $i = 0$.

Za male vrednosti inklinacije izolinije $\mathcal{K}_0$ se vrlo malo razlikuju od kružnice (Slika 3.1a). To znači da argument perihela g vrši cirkulaciju dok je ekscentricitet praktično konstantan. Sa povećanjem nagiba izolinije se postepeno izdužuju u pravcu $e \sin g$. Argument perihela još uvek vrši cirkulaciju ali ekscentricitet osciluje prolazeći kroz maksimum za $g = 90^\circ$ ili $g = 270^\circ$, dok inklinacija tada



Slika 3.1: Kozaijev Hamiltonijan za razne vrednosti inklinacije. Velika poluosa je imala fiksiranu vrednost 3 AJ.

ima minimalnu vrednost (Slika 3.1b). Iznad neke kritične vrednosti inklinacije dinamika Kozaijevog Hamiltonijana se drastično menja (Slika 3.1c). Tačka $e = 0$ postaje nestabilna ravnotežna tačka. Pojavljuju se separatrise koje dele fazni prostor na tri oblasti: dve oblasti karakteriše libracija g oko 90° ili 270° i u trećoj oblasti g i dalje vrši cirkulaciju. Ovakva struktura omogućava pojavu rezonanse – Kozai-jeve rezonanse. Zapravo, to je 1 : 1 rezonansa između precesione frekvencije longitude perihela i longitude čvora asteroida. Za još veće vrednosti inklinacije Kozai-jeva rezonansa postaje jača, u smislu da širina oblasti libracije raste (Slika 3.1d). Tačka $e = 0$ je uvek nestabilna ravnotežna tačka što za posledicu ima da svaka orbita sa početnim malim ekscentricitetom se prinudi, ovakvim rezonantnim mehanizmom, da dostigne veće i veće vrednosti. To znači, da takav asteroid tokom svoje sekularne evolucije može preseći orbitu neke unutrašnje planete. Kritična vrednost inklinacije zavisi od velike poluose asteroida. Za slučaj prikazan na Slici 3.1 kritična vrednost inklinacije iznosi oko 33°.

3.3 Promenljive dejstvo–ugao Kozaijevog Hamiltonijana $\mathcal{K}_0$

Kako je Kozaijev Hamiltonijan integrabilan može se pogodnom transformacijom promenljivih napisati u formi da ne zavisi od uglova već samo od dejstava. To se postiže primenom Henrard (1990) seminumeričke metode i sledeće kanonske transformacije

$$\begin{aligned}
q &= q(\tilde{\Lambda}, J, Z, \psi), \\
Q &= Q(\tilde{\Lambda}, J, Z, \psi), \\
p &= z + \varrho_z(\tilde{\Lambda}, J, Z, \psi), \\
P &= Z, \\
\lambda &= \tilde{\lambda} + \varrho_{\tilde{\lambda}}(\tilde{\Lambda}, J, Z, \psi), \\
\Lambda &= \tilde{\Lambda},
\end{aligned} \tag{3.12}$$

gde su $(\Lambda, \lambda, P, p, Q, q)$ – stare, a $(\tilde{\Lambda}, \tilde{\lambda}, Z, z, J, \psi)$ – nove kanonske promenljive. Ova transformacija se ne može prikazati eksplicitnim formulama, ali se može računati numerički. Moment J je proporcionalan površini koju "zahvata" orbita. Ugao ψ je linearna funkcija vremena i "meri" položaj na orbiti. Pošto su dejstva P i Λ konstante kretanja, zadržani su kao nove promenljive, ali zbog očuvanja kanoničnosti transformacije konjugovani uglovi z i $\tilde{\lambda}$ moraju dobiti kanonske korekcije $\varrho_z(\tilde{\Lambda}, J, Z, \psi)$ i $\varrho_{\tilde{\lambda}}(\tilde{\Lambda}, J, Z, \psi)$.

Umesto promenljivih Q i q možemo uvesti pravougaone kanonske promenljive x i y (Lemaitre i Morbidelli, 1994) relacijama

$$\begin{aligned} x &= \sqrt{2Q}\cos q, \\ y &= \sqrt{2Q}\sin q. \end{aligned} \tag{3.13}$$

Trajektoriju (ili krivu) dobijamo numeričkom integracijom Hamiltonovih jednačina za Kozaijev Hamiltonijan

$$\begin{aligned} \frac{dx}{dt} &= -\frac{\partial \mathcal{K}_0}{\partial y}, \\ \frac{dy}{dt} &= \frac{\partial \mathcal{K}_0}{\partial x}. \end{aligned} \tag{3.14}$$

Naravno, kada nađemo referentnu orbitu, izvode prvog i višeg reda po momentima $\tilde{\Lambda}$, J i Z računamo numerički.

3.4 Računanje izvoda Kozaijevog Hamiltonijana po promenljivima L, G, H i g

Kozaijev Hamiltonijan za asteroid (3.8) se može prikazati u eksplicitnom obliku, preko promenljivih Delaunay-a (3.1):

$$\mathcal{K}_0(L, G, H, -, g, -) = -k^2 m' \int_0^{2\pi}\int_0^{2\pi} \left(\frac{1}{\Delta^{\frac{1}{2}}} - \frac{x_a x' + y_a y' + z_a z'}{r'^3} \right) dl\, dl' \tag{3.15}$$

gde su korišćene sledeće oznake

$$
\begin{aligned}
x_a &= (\cos g \cos h - \frac{H}{G} \sin g \sin h) L^2 (\cos E - e) + \\
&\quad (-\sin g \cos h - \frac{H}{G} \cos g \sin h) L\, G \sin E, \\
y_a &= (\cos g \sin h + \frac{H}{G} \sin g \cos h) L^2 (\cos E - e) + \\
&\quad (-\sin g \sin h + \frac{H}{G} \cos g \cos h) L\, G \sin E, \\
z_a &= \sqrt{1 - \frac{H^2}{G^2}} \sin g\, L^2 (\cos E - e) + \sqrt{1 - \frac{H^2}{G^2}} \cos g\, L\, G \sin E, \\
r_a &= L^2 (1 - e \cos E) \\
\Delta &= r_a^2 + r'^2 - 2(x_a x' + y_a y' + z_a z') \\
l &= E - e \sin E, \qquad dl = (1 - e \cos E) dE \\
x' &= r' \cos l' \\
y' &= r' \sin l' \\
z' &= 0.
\end{aligned}
\tag{3.16}
$$

k^2 – Gauss-ova konstanta, m' – masa perturbujuće planete[1], a E – ekscentrična anomalija asteroida.

$\mathcal{K}_0$ je funkcija L kako direktno tako i preko e, tj. mora se voditi računa pri nalaženju izvoda

$$
\frac{\partial}{\partial L} = \frac{\partial a}{\partial L} \frac{\partial}{\partial a} + \frac{\partial e}{\partial L} \frac{\partial}{\partial e} = 2L \frac{\partial}{\partial a} + \frac{G^2}{e\, L^3} \frac{\partial}{\partial e}. \tag{3.17}
$$

Na primer, izvod

$$
\frac{\partial E}{\partial L} = \frac{G^2}{e\, L^3} \frac{\sin E}{1 - e \cos E}, \tag{3.18}
$$

dok je izvod podintegralne funkcije Kozaijevog Hamiltonijana (3.15)

$$
\frac{\partial \mathcal{K}_0}{\partial L} = -\frac{1}{2} \frac{1}{\Delta^{3/2}} \frac{\partial \Delta}{\partial L} - \frac{1}{r'^3} \left(\frac{\partial x_a}{\partial L} x' + \frac{\partial y_a}{\partial L} y' + \frac{\partial z_a}{\partial L} z' \right) \tag{3.19}
$$

gde je

[1] U izrazu (3.9) za integrabilni Hamiltonijan upotrebljena je oznaka ε za masu poremećajne planete izraženu u jedinicama mase Sunca.

$$\frac{\partial \Delta}{\partial L} = 2r_a\frac{\partial r_a}{\partial L} - 2\left(\frac{\partial x_a}{\partial L}x' + \frac{\partial y_a}{\partial L}y' + \frac{\partial z_a}{\partial L}z'\right) \tag{3.20}$$

a

$$\begin{aligned}\frac{\partial r_a}{\partial L} &= 2\,L(1-e\cos E) + L^2\left(-\frac{\partial e}{\partial L}\cos E + e\sin E\frac{\partial E}{\partial L}\right)\\ &= 2\,L(1-e\cos E) - L^2\frac{\partial e}{\partial L}\frac{\cos E - e}{1-e\cos E}.\end{aligned} \tag{3.21}$$

Izvodi $\frac{\partial x_a}{\partial L}$, $\frac{\partial y_a}{\partial L}$ i $\frac{\partial z_a}{\partial L}$ se svode na računanje izvoda $\frac{\partial}{\partial L}\left(L^2(\cos E - e)\right)$ i $\frac{\partial}{\partial L}\left(L\sin E\right)$

$$\begin{aligned}\frac{\partial}{\partial L}\left(L^2(\cos E - e)\right) &= 2L(\cos E - e) + L^2\left(-\sin E\frac{\partial E}{\partial L} - \frac{\partial e}{\partial L}\right)\\ \frac{\partial}{\partial L}\left(L\sin E\right) &= \sin E + L\cos E\frac{\partial E}{\partial L}\end{aligned} \tag{3.22}$$

Konačno se dobija (uključujući znak "$-$" koji stoji ispred integrala (3.15))

$$\begin{aligned}\frac{\partial \mathcal{K}_0}{\partial L} &= \frac{1}{\Delta^{3/2}}\left(r_a\frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L}x' - \frac{\partial y_a}{\partial L}y' - \frac{\partial z_a}{\partial L}z'\right)\\ &+\frac{1}{r'^3}\left(\frac{\partial x_a}{\partial L}x' + \frac{\partial y_a}{\partial L}y' + \frac{\partial z_a}{\partial L}z'\right).\end{aligned} \tag{3.23}$$

3.4.1 Izvodi prvog reda

Zapravo relacija (3.23) je samo skraćeni zapis izvoda $\frac{\partial \mathcal{K}_0}{\partial L}$ jer ćemo u svim ostalim izrazima podrazumevati da treba računati dvostruki integral po srednjim anomalijama asteroida i planete. Dakle, potpun izraz je

$$\begin{aligned}\frac{\partial \mathcal{K}_0}{\partial L} &= k^2m'\int_0^{2\pi}\int_0^{2\pi}\left[\frac{1}{\Delta^{3/2}}\left(r_a\frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L}x' - \frac{\partial y_a}{\partial L}y' - \frac{\partial z_a}{\partial L}z'\right)\right.\\ &\left.+\frac{1}{r'^3}\left(\frac{\partial x_a}{\partial L}x' + \frac{\partial y_a}{\partial L}y' + \frac{\partial z_a}{\partial L}z'\right)\right]dl\,dl'.\end{aligned} \tag{3.24}$$

Na sličan način dobijaju se izvodi po G

$$\begin{aligned}
\frac{\partial \mathcal{K}_0}{\partial G} &= -\frac{1}{2}\frac{1}{\Delta^{3/2}}\frac{\partial \Delta}{\partial G} - \frac{1}{r'^3}\left(\frac{\partial x_a}{\partial G}x' + \frac{\partial y_a}{\partial G}y' + \frac{\partial z_a}{\partial G}z'\right) \\
\frac{\partial \Delta}{\partial G} &= 2r_a\frac{\partial r_a}{\partial G} - 2\left(\frac{\partial x_a}{\partial G}x' + \frac{\partial y_a}{\partial G}y' + \frac{\partial z_a}{\partial G}z'\right) \\
\frac{\partial r_a}{\partial G} &= L^2\left(-\frac{\partial e}{\partial G}\cos E + e\sin E\frac{\partial E}{\partial G}\right) = -L^2\frac{\partial e}{\partial G}\frac{\cos E - e}{1 - e\cos E} \\
\frac{\partial e}{\partial G} &= -\frac{G}{e\,L^2}, \qquad\qquad \frac{\partial E}{\partial G} = \frac{\partial e}{\partial G}\frac{\sin E}{1 - e\cos E}
\end{aligned} \tag{3.25}$$

$$\begin{aligned}
\frac{\partial \mathcal{K}_0}{\partial G} &= \frac{1}{\Delta^{3/2}}\left(r_a\frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G}x' - \frac{\partial y_a}{\partial G}y' - \frac{\partial z_a}{\partial G}z'\right) \\
&\quad + \frac{1}{r'^3}\left(\frac{\partial x_a}{\partial G}x' + \frac{\partial y_a}{\partial G}y' + \frac{\partial z_a}{\partial G}z'\right)
\end{aligned} \tag{3.26}$$

po H

$$\begin{aligned}
\frac{\partial \mathcal{K}_0}{\partial H} &= -\frac{1}{2}\frac{1}{\Delta^{3/2}}\frac{\partial \Delta}{\partial H} - \frac{1}{r'^3}\left(\frac{\partial x_a}{\partial H}x' + \frac{\partial y_a}{\partial H}y' + \frac{\partial z_a}{\partial H}z'\right) \\
\frac{\partial \Delta}{\partial H} &= -2\left(\frac{\partial x_a}{\partial H}x' + \frac{\partial y_a}{\partial H}y' + \frac{\partial z_a}{\partial H}z'\right) \\
\frac{\partial x_a}{\partial H} &= -\frac{1}{G}\sin g\sin h\,L^2(\cos E - e) - \frac{1}{G}\cos g\sin h\,L\,G\sin E, \\
\frac{\partial y_a}{\partial H} &= \frac{1}{G}\sin g\cos h\,L^2(\cos E - e) + \frac{1}{G}\cos g\cos h\,L\,G\sin E, \\
\frac{\partial z_a}{\partial H} &= -\frac{H}{G^2\sin i}\sin g\,L^2(\cos E - e) - \frac{H}{G^2\sin i}\cos g\,L\,G\sin E,
\end{aligned} \tag{3.27}$$

$$\begin{aligned}
\frac{\partial \mathcal{K}_0}{\partial H} &= -\frac{1}{\Delta^{3/2}}\left(\frac{\partial x_a}{\partial H}x' + \frac{\partial y_a}{\partial H}y' + \frac{\partial z_a}{\partial H}z'\right) \\
&\quad + \frac{1}{r'^3}\left(\frac{\partial x_a}{\partial H}x' + \frac{\partial y_a}{\partial H}y' + \frac{\partial z_a}{\partial H}z'\right),
\end{aligned} \tag{3.28}$$

kao i po g

$$
\begin{aligned}
\frac{\partial \mathcal{K}_0}{\partial g} &= -\frac{1}{2}\frac{1}{\Delta^{3/2}}\frac{\partial \Delta}{\partial g} - \frac{1}{r'^3}\left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right) \\
\frac{\partial \Delta}{\partial g} &= -2\left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right) \\
\frac{\partial x_a}{\partial g} &= (-\sin g\cos h - \frac{H}{G}\cos g\sin h)L^2(\cos E - e) \\
&\quad +(-\cos g\cos h + \frac{H}{G}\sin g\sin h)L\,G\sin E, \\
\frac{\partial y_a}{\partial g} &= (-\sin g\sin h + \frac{H}{G}\cos g\cos h)L^2(\cos E - e) \\
&\quad +(-\cos g\sin h - \frac{H}{G}\sin g\cos h)L\,G\sin E, \\
\frac{\partial z_a}{\partial g} &= \sqrt{1-\frac{H^2}{G^2}}\cos g\,L^2(\cos E - e) - \sqrt{1-\frac{H^2}{G^2}}\sin g\,L\,G\sin E,
\end{aligned}
\tag{3.29}
$$

$$
\begin{aligned}
\frac{\partial \mathcal{K}_0}{\partial g} &= -\frac{1}{\Delta^{3/2}}\left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right) \\
&\quad +\frac{1}{r'^3}\left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right),
\end{aligned}
\tag{3.30}
$$

3.4.2 Izvodi drugog reda

Drugi izvodi su:

$$
\begin{aligned}
\frac{\partial^2 \mathcal{K}_0}{\partial L^2} &= -\frac{3}{\Delta^{5/2}}\left(r_a\frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L}x' - \frac{\partial y_a}{\partial L}y' - \frac{\partial z_a}{\partial L}z'\right)^2 \\
&\quad +\frac{1}{\Delta^{3/2}}\left[\left(\frac{\partial r_a}{\partial L}\right)^2 + r_a\frac{\partial^2 r_a}{\partial L^2} - \frac{\partial^2 x_a}{\partial L^2}x' - \frac{\partial^2 y_a}{\partial L^2}y' - \frac{\partial^2 z_a}{\partial L^2}z'\right] \\
&\quad +\frac{1}{r'^3}\left(\frac{\partial^2 x_a}{\partial L^2}x' + \frac{\partial^2 y_a}{\partial L^2}y' + \frac{\partial^2 z_a}{\partial L^2}z'\right),
\end{aligned}
\tag{3.31}
$$

$$
\begin{aligned}
\frac{\partial^2 \mathcal{K}_0}{\partial G^2} \quad = \quad & -\frac{3}{\Delta^{5/2}}\left(r_a\frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G}x' - \frac{\partial y_a}{\partial G}y' - \frac{\partial z_a}{\partial G}z'\right)^2 \\
& +\frac{1}{\Delta^{3/2}}\left[\left(\frac{\partial r_a}{\partial G}\right)^2 + r_a\frac{\partial^2 r_a}{\partial G^2} - \frac{\partial^2 x_a}{\partial G^2}x' - \frac{\partial^2 y_a}{\partial G^2}y' - \frac{\partial^2 z_a}{\partial G^2}z'\right] \\
& +\frac{1}{r'^3}\left(\frac{\partial^2 x_a}{\partial G^2}x' + \frac{\partial^2 y_a}{\partial G^2}y' + \frac{\partial^2 z_a}{\partial G^2}z'\right),
\end{aligned}
\tag{3.32}
$$

$$
\begin{aligned}
\frac{\partial^2 \mathcal{K}_0}{\partial H^2} \quad = \quad & -\frac{3}{\Delta^{5/2}}\left(\frac{\partial x_a}{\partial H}x' + \frac{\partial y_a}{\partial H}y' + \frac{\partial z_a}{\partial H}z'\right)^2 \\
& -\frac{1}{\Delta^{3/2}}\underbrace{\left(\frac{\partial^2 x_a}{\partial H^2}x' + \frac{\partial^2 y_a}{\partial H^2}y' + \frac{\partial^2 z_a}{\partial H^2}z'\right)}_{=0} \\
& +\frac{1}{r'^3}\underbrace{\left(\frac{\partial^2 x_a}{\partial H^2}x' + \frac{\partial^2 y_a}{\partial H^2}y' + \frac{\partial^2 z_a}{\partial H^2}z'\right)}_{=0},
\end{aligned}
\tag{3.33}
$$

$$
\begin{aligned}
\frac{\partial^2 \mathcal{K}_0}{\partial L \partial G} \quad = \quad & -\frac{3}{\Delta^{5/2}}\left(r_a\frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L}x' - \frac{\partial y_a}{\partial L}y' - \frac{\partial z_a}{\partial L}z'\right) \times \\
& \times \left(r_a\frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G}x' - \frac{\partial y_a}{\partial G}y' - \frac{\partial z_a}{\partial G}z'\right) \\
& +\frac{1}{\Delta^{3/2}}\left(\frac{\partial r_a}{\partial L}\frac{\partial r_a}{\partial G} + r_a\frac{\partial^2 r_a}{\partial L \partial G} - \frac{\partial^2 x_a}{\partial L \partial G}x' + \right. \\
& \left. -\frac{\partial^2 y_a}{\partial L \partial G}y' - \frac{\partial^2 z_a}{\partial L \partial G}z'\right) + \frac{1}{r'^3}\left(\frac{\partial^2 x_a}{\partial L \partial G}x' + \frac{\partial^2 y_a}{\partial L \partial G}y' + \frac{\partial^2 z_a}{\partial L \partial G}z'\right),
\end{aligned}
\tag{3.34}
$$

$$
\begin{aligned}
\frac{\partial^2 \mathcal{K}_0}{\partial L \partial H} \quad = \quad & \frac{3}{\Delta^{5/2}}\left(r_a\frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L}x' - \frac{\partial y_a}{\partial L}y' - \frac{\partial z_a}{\partial L}z'\right) \times \\
& \times \left(\frac{\partial x_a}{\partial H}x' + \frac{\partial y_a}{\partial H}y' + \frac{\partial z_a}{\partial H}z'\right) \\
& -\frac{1}{\Delta^{3/2}}\left(\frac{\partial^2 x_a}{\partial L \partial H}x' + \frac{\partial^2 y_a}{\partial L \partial H}y' + \frac{\partial^2 z_a}{\partial L \partial H}z'\right) \\
& +\frac{1}{r'^3}\left(\frac{\partial^2 x_a}{\partial L \partial H}x' + \frac{\partial^2 y_a}{\partial L \partial H}y' + \frac{\partial^2 z_a}{\partial L \partial H}z'\right),
\end{aligned}
\tag{3.35}
$$

$$\begin{aligned}
\frac{\partial^2 \mathcal{K}_0}{\partial G \partial H} &= \frac{3}{\Delta^{5/2}}\left(r_a\frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G}x' - \frac{\partial y_a}{\partial G}y' - \frac{\partial z_a}{\partial G}z'\right)\times \\
&\times\left(\frac{\partial x_a}{\partial H}x' + \frac{\partial y_a}{\partial H}y' + \frac{\partial z_a}{\partial H}z'\right) \\
&-\frac{1}{\Delta^{3/2}}\left(\frac{\partial^2 x_a}{\partial G\partial H}x' + \frac{\partial^2 y_a}{\partial G\partial H}y' + \frac{\partial^2 z_a}{\partial G\partial H}z'\right) \\
&+\frac{1}{r'^3}\left(\frac{\partial^2 x_a}{\partial G\partial H}x' + \frac{\partial^2 y_a}{\partial G\partial H}y' + \frac{\partial^2 z_a}{\partial G\partial H}z'\right),
\end{aligned} \tag{3.36}$$

$$\begin{aligned}
\frac{\partial^2 \mathcal{K}_0}{\partial g^2} &= -\frac{3}{\Delta^{5/2}}\left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right)^2 \\
&-\frac{1}{\Delta^{3/2}}\left(\frac{\partial^2 x_a}{\partial g^2}x' + \frac{\partial^2 y_a}{\partial g^2}y' + \frac{\partial^2 z_a}{\partial g^2}z'\right) \\
&+\frac{1}{r'^3}\left(\frac{\partial^2 x_a}{\partial g^2}x' + \frac{\partial^2 y_a}{\partial g^2}y' + \frac{\partial^2 z_a}{\partial g^2}z'\right),
\end{aligned} \tag{3.37}$$

$$\begin{aligned}
\frac{\partial^2 \mathcal{K}_0}{\partial L \partial g} &= \frac{3}{\Delta^{5/2}}\left(r_a\frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L}x' - \frac{\partial y_a}{\partial L}y' - \frac{\partial z_a}{\partial L}z'\right)\times \\
&\times\left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right) \\
&-\frac{1}{\Delta^{3/2}}\left(\frac{\partial^2 x_a}{\partial L\partial g}x' + \frac{\partial^2 y_a}{\partial L\partial g}y' + \frac{\partial^2 z_a}{\partial L\partial g}z'\right) \\
&+\frac{1}{r'^3}\left(\frac{\partial^2 x_a}{\partial L\partial g}x' + \frac{\partial^2 y_a}{\partial L\partial g}y' + \frac{\partial^2 z_a}{\partial L\partial g}z'\right),
\end{aligned} \tag{3.38}$$

$$\begin{aligned}
\frac{\partial^2 \mathcal{K}_0}{\partial G \partial g} &= \frac{3}{\Delta^{5/2}}\left(r_a\frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G}x' - \frac{\partial y_a}{\partial G}y' - \frac{\partial z_a}{\partial G}z'\right)\times \\
&\times\left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right) \\
&-\frac{1}{\Delta^{3/2}}\left(\frac{\partial^2 x_a}{\partial G\partial g}x' + \frac{\partial^2 y_a}{\partial G\partial g}y' + \frac{\partial^2 z_a}{\partial G\partial g}z'\right) \\
&+\frac{1}{r'^3}\left(\frac{\partial^2 x_a}{\partial G\partial g}x' + \frac{\partial^2 y_a}{\partial G\partial g}y' + \frac{\partial^2 z_a}{\partial G\partial g}z'\right),
\end{aligned} \tag{3.39}$$

$$\begin{aligned}
\frac{\partial^2 \mathcal{K}_0}{\partial H \partial g} &= -\frac{3}{\Delta^{5/2}}\left(\frac{\partial x_a}{\partial H}x' + \frac{\partial y_a}{\partial H}y' + \frac{\partial z_a}{\partial H}z'\right)\left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right) \\
&\quad -\frac{1}{\Delta^{3/2}}\left(\frac{\partial^2 x_a}{\partial H \partial g}x' + \frac{\partial^2 y_a}{\partial H \partial g}y' + \frac{\partial^2 z_a}{\partial H \partial g}z'\right) \\
&\quad +\frac{1}{r'^3}\left(\frac{\partial^2 x_a}{\partial H \partial g}x' + \frac{\partial^2 y_a}{\partial H \partial g}y' + \frac{\partial^2 z_a}{\partial H \partial g}z'\right),
\end{aligned} \tag{3.40}$$

3.4.3 Izvodi trećeg reda

Treći izvodi:

$$\begin{aligned}
\frac{\partial^3 \mathcal{K}_0}{\partial L^3} &= \frac{15}{\Delta^{7/2}}\left(r_a\frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L}x' - \frac{\partial y_a}{\partial L}y' - \frac{\partial z_a}{\partial L}z'\right)^3 \\
&\quad -\frac{9}{\Delta^{5/2}}\left(r_a\frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L}x' - \frac{\partial y_a}{\partial L}y' - \frac{\partial z_a}{\partial L}z'\right) \times \\
&\quad \times \left[\left(\frac{\partial r_a}{\partial L}\right)^2 + r_a\frac{\partial^2 r_a}{\partial L^2} - \frac{\partial^2 x_a}{\partial L^2}x' - \frac{\partial^2 y_a}{\partial L^2}y' - \frac{\partial^2 z_a}{\partial L^2}z'\right] \\
&\quad +\frac{1}{\Delta^{3/2}}\left(3\frac{\partial r_a}{\partial L}\frac{\partial^2 r_a}{\partial L^2} + r_a\frac{\partial^3 r_a}{\partial L^3} - \frac{\partial^3 x_a}{\partial L^3}x' - \frac{\partial^3 y_a}{\partial L^3}y' - \frac{\partial^3 z_a}{\partial L^3}z'\right) \\
&\quad +\frac{1}{r'^3}\left(\frac{\partial^3 x_a}{\partial L^3}x' + \frac{\partial^3 y_a}{\partial L^3}y' + \frac{\partial^3 z_a}{\partial L^3}z'\right),
\end{aligned} \tag{3.41}$$

$$\begin{aligned}
\frac{\partial^3 \mathcal{K}_0}{\partial G^3} &= \frac{15}{\Delta^{7/2}}\left(r_a\frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G}x' - \frac{\partial y_a}{\partial G}y' - \frac{\partial z_a}{\partial G}z'\right)^3 \\
&\quad -\frac{9}{\Delta^{5/2}}\left(r_a\frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G}x' - \frac{\partial y_a}{\partial G}y' - \frac{\partial z_a}{\partial G}z'\right) \times \\
&\quad \times \left[\left(\frac{\partial r_a}{\partial G}\right)^2 + r_a\frac{\partial^2 r_a}{\partial G^2} - \frac{\partial^2 x_a}{\partial G^2}x' - \frac{\partial^2 y_a}{\partial G^2}y' - \frac{\partial^2 z_a}{\partial G^2}z'\right] \\
&\quad +\frac{1}{\Delta^{3/2}}\left(3\frac{\partial r_a}{\partial G}\frac{\partial^2 r_a}{\partial G^2} + r_a\frac{\partial^3 r_a}{\partial G^3} - \frac{\partial^3 x_a}{\partial G^3}x' - \frac{\partial^3 y_a}{\partial G^3}y' - \frac{\partial^3 z_a}{\partial G^3}z'\right) \\
&\quad +\frac{1}{r'^3}\left(\frac{\partial^3 x_a}{\partial G^3}x' + \frac{\partial^3 y_a}{\partial G^3}y' + \frac{\partial^3 z_a}{\partial G^3}z'\right),
\end{aligned} \tag{3.42}$$

$$
\begin{aligned}
\frac{\partial^3 \mathcal{K}_0}{\partial H^3} &= \frac{15}{\Delta^{7/2}} \left(\underbrace{r_a \frac{\partial r_a}{\partial H}}_{=0} - \frac{\partial x_a}{\partial H} x' - \frac{\partial y_a}{\partial H} y' - \frac{\partial z_a}{\partial H} z' \right)^3 \\
&= -\frac{15}{\Delta^{7/2}} \left(\frac{\partial x_a}{\partial H} x' + \frac{\partial y_a}{\partial H} y' + \frac{\partial z_a}{\partial H} z' \right)^3 ,
\end{aligned} \tag{3.43}
$$

$$
\begin{aligned}
\frac{\partial^3 \mathcal{K}_0}{\partial L^2 \partial G} &= \frac{15}{\Delta^{7/2}} \left(r_a \frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L} x' - \frac{\partial y_a}{\partial L} y' - \frac{\partial z_a}{\partial L} z' \right)^2 \times \\
&\times \left(r_a \frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G} x' - \frac{\partial y_a}{\partial G} y' - \frac{\partial z_a}{\partial G} z' \right) \\
&- \frac{6}{\Delta^{5/2}} \left(r_a \frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L} x' - \frac{\partial y_a}{\partial L} y' - \frac{\partial z_a}{\partial L} z' \right) \times \\
&\times \left(\frac{\partial r_a}{\partial L} \frac{\partial r_a}{\partial G} + r_a \frac{\partial^2 r_a}{\partial L \partial G} - \frac{\partial^2 x_a}{\partial L \partial G} x' - \frac{\partial^2 y_a}{\partial L \partial G} y' - \frac{\partial^2 z_a}{\partial L \partial G} z' \right) \\
&- \frac{3}{\Delta^{5/2}} \left(r_a \frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G} x' - \frac{\partial y_a}{\partial G} y' - \frac{\partial z_a}{\partial G} z' \right) \times \\
&\times \left[\left(\frac{\partial r_a}{\partial L} \right)^2 + r_a \frac{\partial^2 r_a}{\partial L^2} - \frac{\partial^2 x_a}{\partial L^2} x' - \frac{\partial^2 y_a}{\partial L^2} y' - \frac{\partial^2 z_a}{\partial L^2} z' \right] \\
&+ \frac{1}{\Delta^{3/2}} \left(2 \frac{\partial r_a}{\partial L} \frac{\partial^2 r_a}{\partial L \partial G} + \frac{\partial r_a}{\partial G} \frac{\partial^2 r_a}{\partial L^2} + r_a \frac{\partial^3 r_a}{\partial L^2 \partial G} + \right. \\
&\left. - \frac{\partial^3 x_a}{\partial L^2 \partial G} x' - \frac{\partial^3 y_a}{\partial L^2 \partial G} y' - \frac{\partial^3 z_a}{\partial L^2 \partial G} z' \right) \\
&+ \frac{1}{r'^3} \left(\frac{\partial^3 x_a}{\partial L^2 \partial G} x' + \frac{\partial^3 y_a}{\partial L^2 \partial G} y' + \frac{\partial^3 z_a}{\partial L^2 \partial G} z' \right) ,
\end{aligned} \tag{3.44}
$$

$$
\begin{aligned}
\frac{\partial^3 \mathcal{K}_0}{\partial L \partial G^2} \quad = \quad & \frac{15}{\Delta^{7/2}} \left(r_a \frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G} x' - \frac{\partial y_a}{\partial G} y' - \frac{\partial z_a}{\partial G} z' \right)^2 \times \\
& \times \left(r_a \frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L} x' - \frac{\partial y_a}{\partial L} y' - \frac{\partial z_a}{\partial L} z' \right) \\
& - \frac{6}{\Delta^{5/2}} \left(r_a \frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G} x' - \frac{\partial y_a}{\partial G} y' - \frac{\partial z_a}{\partial G} z' \right) \times \\
& \times \left(\frac{\partial r_a}{\partial L} \frac{\partial r_a}{\partial G} + r_a \frac{\partial^2 r_a}{\partial L \partial G} - \frac{\partial^2 x_a}{\partial L \partial G} x' - \frac{\partial^2 y_a}{\partial L \partial G} y' - \frac{\partial^2 z_a}{\partial L \partial G} z' \right) \\
& - \frac{3}{\Delta^{5/2}} \left(r_a \frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L} x' - \frac{\partial y_a}{\partial L} y' - \frac{\partial z_a}{\partial L} z' \right) \times \\
& \times \left[\left(\frac{\partial r_a}{\partial G} \right)^2 + r_a \frac{\partial^2 r_a}{\partial G^2} - \frac{\partial^2 x_a}{\partial G^2} x' - \frac{\partial^2 y_a}{\partial G^2} y' - \frac{\partial^2 z_a}{\partial G^2} z' \right] \\
& + \frac{1}{\Delta^{3/2}} \left(2 \frac{\partial r_a}{\partial G} \frac{\partial^2 r_a}{\partial L \partial G} + \frac{\partial r_a}{\partial L} \frac{\partial^2 r_a}{\partial G^2} + r_a \frac{\partial^3 r_a}{\partial L \partial G^2} + \right. \\
& \left. - \frac{\partial^3 x_a}{\partial L \partial G^2} x' - \frac{\partial^3 y_a}{\partial L \partial G^2} y' - \frac{\partial^3 z_a}{\partial L \partial G^2} z' \right) \\
& + \frac{1}{r'^3} \left(\frac{\partial^3 x_a}{\partial L \partial G^2} x' + \frac{\partial^3 y_a}{\partial L \partial G^2} y' + \frac{\partial^3 z_a}{\partial L \partial G^2} z' \right),
\end{aligned}
\tag{3.45}
$$

$$\begin{aligned}
\frac{\partial^3\mathcal{K}_0}{\partial L^2\partial H} &= -\frac{15}{\Delta^{7/2}}\left(r_a\frac{\partial r_a}{\partial L}-\frac{\partial x_a}{\partial L}x'-\frac{\partial y_a}{\partial L}y'-\frac{\partial z_a}{\partial L}z'\right)^2\times\\
&\times\left(\frac{\partial x_a}{\partial H}x'+\frac{\partial y_a}{\partial H}y'+\frac{\partial z_a}{\partial H}z'\right)\\
&+\frac{6}{\Delta^{5/2}}\left(r_a\frac{\partial r_a}{\partial L}-\frac{\partial x_a}{\partial L}x'-\frac{\partial y_a}{\partial L}y'-\frac{\partial z_a}{\partial L}z'\right)\times\\
&\times\left(\frac{\partial^2 x_a}{\partial L\partial H}x'+\frac{\partial^2 y_a}{\partial L\partial H}y'+\frac{\partial^2 z_a}{\partial L\partial H}z'\right)\\
&+\frac{3}{\Delta^{5/2}}\left(\frac{\partial x_a}{\partial H}x'+\frac{\partial y_a}{\partial H}y'+\frac{\partial z_a}{\partial H}z'\right)\times\\
&\times\left[\left(\frac{\partial r_a}{\partial L}\right)^2+r_a\frac{\partial^2 r_a}{\partial L^2}-\frac{\partial^2 x_a}{\partial L^2}x'-\frac{\partial^2 y_a}{\partial L^2}y'-\frac{\partial^2 z_a}{\partial L^2}z'\right]\\
&-\frac{1}{\Delta^{3/2}}\left(\frac{\partial^3 x_a}{\partial L^2\partial H}x'+\frac{\partial^3 y_a}{\partial L^2\partial H}y'+\frac{\partial^3 z_a}{\partial L^2\partial H}z'\right)\\
&+\frac{1}{r'^3}\left(\frac{\partial^3 x_a}{\partial L^2\partial H}x'+\frac{\partial^3 y_a}{\partial L^2\partial H}y'+\frac{\partial^3 z_a}{\partial L^2\partial H}z'\right),
\end{aligned} \tag{3.46}$$

$$\begin{aligned}
\frac{\partial^3\mathcal{K}_0}{\partial L\partial H^2} &= -\frac{15}{\Delta^{7/2}}\left(\frac{\partial x_a}{\partial H}x'+\frac{\partial y_a}{\partial H}y'-\frac{\partial z_a}{\partial H}z'\right)^2\times\\
&\times\left(r_a\frac{\partial r_a}{\partial L}-\frac{\partial x_a}{\partial L}x'-\frac{\partial y_a}{\partial L}y'-\frac{\partial z_a}{\partial L}z'\right)\\
&-\frac{6}{\Delta^{5/2}}\left(\frac{\partial x_a}{\partial H}x'+\frac{\partial y_a}{\partial H}y'+\frac{\partial z_a}{\partial H}z'\right)\times\\
&\times\left(\frac{\partial^2 x_a}{\partial L\partial H}x'+\frac{\partial^2 y_a}{\partial L\partial H}y'+\frac{\partial^2 z_a}{\partial L\partial H}z'\right),
\end{aligned} \tag{3.47}$$

$$\begin{aligned}
\frac{\partial^3\mathcal{K}_0}{\partial G^2\partial H} = & -\frac{15}{\Delta^{7/2}}\left(r_a\frac{\partial r_a}{\partial G}-\frac{\partial x_a}{\partial G}x'-\frac{\partial y_a}{\partial G}y'-\frac{\partial z_a}{\partial G}z'\right)^2\times\\
&\times\left(\frac{\partial x_a}{\partial H}x'+\frac{\partial y_a}{\partial H}y'+\frac{\partial z_a}{\partial H}z'\right)\\
&+\frac{3}{\Delta^{5/2}}\left(r_a\frac{\partial r_a}{\partial G}-\frac{\partial x_a}{\partial G}x'-\frac{\partial y_a}{\partial G}y'-\frac{\partial z_a}{\partial G}z'\right)\times\\
&\times\left(\frac{\partial^2 x_a}{\partial G\partial H}x'+\frac{\partial^2 y_a}{\partial G\partial H}y'+\frac{\partial^2 z_a}{\partial G\partial H}z'\right)\\
&+\frac{3}{\Delta^{5/2}}\left(\frac{\partial x_a}{\partial H}x'+\frac{\partial y_a}{\partial H}y'+\frac{\partial z_a}{\partial H}z'\right)\times\\
&\times\left[\left(\frac{\partial r_a}{\partial G}\right)^2+r_a\frac{\partial^2 r_a}{\partial G^2}-\frac{\partial^2 x_a}{\partial G^2}x'-\frac{\partial^2 y_a}{\partial G^2}y'-\frac{\partial^2 z_a}{\partial G^2}z'\right]\\
&-\frac{1}{\Delta^{3/2}}\left(\frac{\partial^3 x_a}{\partial G^2\partial H}x'+\frac{\partial^3 y_a}{\partial G^2\partial H}y'+\frac{\partial^3 z_a}{\partial G^2\partial H}z'\right)\\
&+\frac{1}{r'^3}\left(\frac{\partial^3 x_a}{\partial G^2\partial H}x'+\frac{\partial^3 y_a}{\partial G^2\partial H}y'+\frac{\partial^3 z_a}{\partial G^2\partial H}z'\right),
\end{aligned}\tag{3.48}$$

$$\begin{aligned}
\frac{\partial^3\mathcal{K}_0}{\partial G\partial H^2} = & \frac{15}{\Delta^{7/2}}\left(\frac{\partial x_a}{\partial H}x'+\frac{\partial y_a}{\partial H}y'-\frac{\partial z_a}{\partial H}z'\right)^2\times\\
&\times\left(r_a\frac{\partial r_a}{\partial G}-\frac{\partial x_a}{\partial G}x'-\frac{\partial y_a}{\partial G}y'-\frac{\partial z_a}{\partial G}z'\right)\\
&-\frac{6}{\Delta^{5/2}}\left(\frac{\partial x_a}{\partial H}x'+\frac{\partial y_a}{\partial H}y'+\frac{\partial z_a}{\partial H}z'\right)\times\\
&\times\left(\frac{\partial^2 x_a}{\partial G\partial H}x'+\frac{\partial^2 y_a}{\partial G\partial H}y'+\frac{\partial^2 z_a}{\partial G\partial H}z'\right),
\end{aligned}\tag{3.49}$$

$$
\begin{aligned}
\frac{\partial^3 \mathcal{K}_0}{\partial L \partial G \partial H} = & -\frac{15}{\Delta^{7/2}}\left(r_a\frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L}x' - \frac{\partial y_a}{\partial L}y' - \frac{\partial z_a}{\partial L}z'\right)\times \\
& \times\left(r_a\frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G}x' - \frac{\partial y_a}{\partial G}y' - \frac{\partial z_a}{\partial G}z'\right)\times \\
& \times\left(\frac{\partial x_a}{\partial H}x' + \frac{\partial y_a}{\partial H}y' - \frac{\partial z_a}{\partial H}z'\right) \\
& +\frac{3}{\Delta^{5/2}}\left(r_a\frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G}x' - \frac{\partial y_a}{\partial G}y' - \frac{\partial z_a}{\partial G}z'\right)\times \\
& \times\left(\frac{\partial^2 x_a}{\partial L \partial H}x' + \frac{\partial^2 y_a}{\partial L \partial H}y' + \frac{\partial^2 z_a}{\partial L \partial H}z'\right) \\
& +\frac{3}{\Delta^{5/2}}\left(r_a\frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L}x' - \frac{\partial y_a}{\partial L}y' - \frac{\partial z_a}{\partial L}z'\right)\times \\
& \times\left(\frac{\partial^2 x_a}{\partial G \partial H}x' + \frac{\partial^2 y_a}{\partial G \partial H}y' + \frac{\partial^2 z_a}{\partial G \partial H}z'\right) \\
& +\frac{3}{\Delta^{5/2}}\left(\frac{\partial x_a}{\partial H}x' + \frac{\partial y_a}{\partial H}y' + \frac{\partial z_a}{\partial H}z'\right)\times \\
& \times\left(\frac{\partial r_a}{\partial L}\frac{\partial r_a}{\partial G} + r_a\frac{\partial^2 r_a}{\partial L \partial G} - \frac{\partial^2 x_a}{\partial L \partial G}x' - \frac{\partial^2 y_a}{\partial L \partial G}y' - \frac{\partial^2 z_a}{\partial L \partial G}z'\right) \\
& -\frac{1}{\Delta^{3/2}}\left(\frac{\partial^3 x_a}{\partial L \partial G \partial H}x' + \frac{\partial^3 y_a}{\partial L \partial G \partial H}y' + \frac{\partial^3 z_a}{\partial L \partial G \partial H}z'\right) \\
& +\frac{1}{r'^3}\left(\frac{\partial^3 x_a}{\partial L \partial G \partial H}x' + \frac{\partial^3 y_a}{\partial L \partial G \partial H}y' + \frac{\partial^3 z_a}{\partial L \partial G \partial H}z'\right),
\end{aligned} \tag{3.50}
$$

$$
\begin{aligned}
\frac{\partial^3 \mathcal{K}_0}{\partial g^3} = & -\frac{15}{\Delta^{7/2}}\left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right)^3 \\
& -\frac{9}{\Delta^{5/2}}\left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\frac{\partial^2 y_a}{\partial g^2}y' + \frac{\partial^2 z_a}{\partial g^2}z'\right)\times \\
& \times\left(\frac{\partial^2 x_a}{\partial g^2}x' + \frac{\partial^2 y_a}{\partial g^2}y' + \frac{\partial^2 z_a}{\partial g^2}z'\right) \\
& -\frac{1}{\Delta^{3/2}}\left(\frac{\partial^3 x_a}{\partial g^3}x' + \frac{\partial^3 y_a}{\partial g^3}y' + \frac{\partial^3 z_a}{\partial g^3}z'\right) \\
& +\frac{1}{r'^3}\left(\frac{\partial^3 x_a}{\partial g^3}x' + \frac{\partial^3 y_a}{\partial g^3}y' + \frac{\partial^3 z_a}{\partial g^3}z'\right),
\end{aligned} \tag{3.51}
$$

$$
\begin{aligned}
\frac{\partial^3 \mathcal{K}_0}{\partial L^2 \partial g} \quad = \quad & -\frac{15}{\Delta^{7/2}}\left(r_a\frac{\partial r_a}{\partial L}-\frac{\partial x_a}{\partial L}x'-\frac{\partial y_a}{\partial L}y'-\frac{\partial z_a}{\partial L}z'\right)^2 \times \\
& \times\left(\frac{\partial x_a}{\partial g}x'+\frac{\partial y_a}{\partial g}y'+\frac{\partial z_a}{\partial g}z'\right) \\
& +\frac{6}{\Delta^{5/2}}\left(r_a\frac{\partial r_a}{\partial L}-\frac{\partial x_a}{\partial L}x'-\frac{\partial y_a}{\partial L}y'-\frac{\partial z_a}{\partial L}z'\right) \times \\
& \times\left(\frac{\partial^2 x_a}{\partial L\partial g}x'+\frac{\partial^2 y_a}{\partial L\partial g}y'+\frac{\partial^2 z_a}{\partial L\partial g}z'\right) \\
& +\frac{3}{\Delta^{5/2}}\left(\frac{\partial x_a}{\partial g}x'+\frac{\partial y_a}{\partial g}y'+\frac{\partial z_a}{\partial g}z'\right) \times \\
& \times\left[\left(\frac{\partial r_a}{\partial L}\right)^2+r_a\frac{\partial^2 r_a}{\partial L^2}-\frac{\partial^2 x_a}{\partial L^2}x'-\frac{\partial^2 y_a}{\partial L^2}y'-\frac{\partial^2 z_a}{\partial L^2}z'\right] \\
& -\frac{1}{\Delta^{3/2}}\left(\frac{\partial^3 x_a}{\partial L^2\partial g}x'+\frac{\partial^3 y_a}{\partial L^2\partial g}y'+\frac{\partial^3 z_a}{\partial L^2\partial g}z'\right) \\
& +\frac{1}{r'^3}\left(\frac{\partial^3 x_a}{\partial L^2\partial g}x'+\frac{\partial^3 y_a}{\partial L^2\partial g}y'+\frac{\partial^3 z_a}{\partial L^2\partial g}z'\right),
\end{aligned}
\tag{3.52}
$$

$$
\begin{aligned}
\frac{\partial^3 \mathcal{K}_0}{\partial L \partial g^2} \quad = \quad & \frac{15}{\Delta^{7/2}}\left(\frac{\partial x_a}{\partial g}x'+\frac{\partial y_a}{\partial g}y'+\frac{\partial z_a}{\partial g}z'\right)^2 \times \\
& \times\left(r_a\frac{\partial r_a}{\partial L}-\frac{\partial x_a}{\partial L}x'-\frac{\partial y_a}{\partial L}y'-\frac{\partial z_a}{\partial L}z'\right) \\
& -\frac{6}{\Delta^{5/2}}\left(\frac{\partial x_a}{\partial g}x'+\frac{\partial y_a}{\partial g}y'+\frac{\partial z_a}{\partial g}z'\right) \times \\
& \times\left(\frac{\partial^2 x_a}{\partial L\partial g}x'+\frac{\partial^2 y_a}{\partial L\partial g}y'+\frac{\partial^2 z_a}{\partial L\partial g}z'\right) \\
& +\frac{3}{\Delta^{5/2}}\left(\frac{\partial^2 x_a}{\partial g^2}x'+\frac{\partial^2 y_a}{\partial g^2}y'+\frac{\partial^2 z_a}{\partial g^2}z'\right) \times \\
& \times\left(r_a\frac{\partial r_a}{\partial L}-\frac{\partial x_a}{\partial L}x'-\frac{\partial y_a}{\partial L}y'-\frac{\partial z_a}{\partial L}z'\right) \\
& -\frac{1}{\Delta^{3/2}}\left(\frac{\partial^3 x_a}{\partial L\partial g^2}x'+\frac{\partial^3 y_a}{\partial L\partial g^2}y'+\frac{\partial^3 z_a}{\partial L\partial g^2}z'\right) \\
& +\frac{1}{r'^3}\left(\frac{\partial^3 x_a}{\partial L\partial g^2}x'+\frac{\partial^3 y_a}{\partial L\partial g^2}y'+\frac{\partial^3 z_a}{\partial L\partial g^2}z'\right),
\end{aligned}
\tag{3.53}
$$

$$
\begin{aligned}
\frac{\partial^3 \mathcal{K}_0}{\partial G^2 \partial g} &= -\frac{15}{\Delta^{7/2}}\left(r_a\frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G}x' - \frac{\partial y_a}{\partial G}y' - \frac{\partial z_a}{\partial G}z'\right)^2 \times \\
&\times \left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right) \\
&+\frac{6}{\Delta^{5/2}}\left(r_a\frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G}x' - \frac{\partial y_a}{\partial G}y' - \frac{\partial z_a}{\partial G}z'\right) \times \\
&\times \left(\frac{\partial^2 x_a}{\partial G\partial g}x' + \frac{\partial^2 y_a}{\partial G\partial g}y' + \frac{\partial^2 z_a}{\partial G\partial g}z'\right) \\
&+\frac{3}{\Delta^{5/2}}\left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right) \times \\
&\times \left[\left(\frac{\partial r_a}{\partial G}\right)^2 + r_a\frac{\partial^2 r_a}{\partial G^2} - \frac{\partial^2 x_a}{\partial G^2}x' - \frac{\partial^2 y_a}{\partial G^2}y' - \frac{\partial^2 z_a}{\partial G^2}z'\right] \\
&-\frac{1}{\Delta^{3/2}}\left(\frac{\partial^3 x_a}{\partial G^2\partial g}x' + \frac{\partial^3 y_a}{\partial G^2\partial g}y' + \frac{\partial^3 z_a}{\partial G^2\partial g}z'\right) \\
&+\frac{1}{r'^3}\left(\frac{\partial^3 x_a}{\partial G^2\partial g}x' + \frac{\partial^3 y_a}{\partial G^2\partial g}y' + \frac{\partial^3 z_a}{\partial G^2\partial g}z'\right),
\end{aligned}
\tag{3.54}
$$

$$
\begin{aligned}
\frac{\partial^3 \mathcal{K}_0}{\partial G \partial g^2} &= \frac{15}{\Delta^{7/2}}\left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right)^2 \times \\
&\times \left(r_a\frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G}x' - \frac{\partial y_a}{\partial G}y' - \frac{\partial z_a}{\partial G}z'\right) \\
&-\frac{6}{\Delta^{5/2}}\left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right) \times \\
&\times \left(\frac{\partial^2 x_a}{\partial G\partial g}x' + \frac{\partial^2 y_a}{\partial G\partial g}y' + \frac{\partial^2 z_a}{\partial G\partial g}z'\right) \\
&+\frac{3}{\Delta^{5/2}}\left(\frac{\partial^2 x_a}{\partial g^2}x' + \frac{\partial^2 y_a}{\partial g^2}y' + \frac{\partial^2 z_a}{\partial g^2}z'\right) \times \\
&\times \left(r_a\frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G}x' - \frac{\partial y_a}{\partial G}y' - \frac{\partial z_a}{\partial G}z'\right) \\
&-\frac{1}{\Delta^{3/2}}\left(\frac{\partial^3 x_a}{\partial G\partial g^2}x' + \frac{\partial^3 y_a}{\partial G\partial g^2}y' + \frac{\partial^3 z_a}{\partial G\partial g^2}z'\right) \\
&+\frac{1}{r'^3}\left(\frac{\partial^3 x_a}{\partial G\partial g^2}x' + \frac{\partial^3 y_a}{\partial G\partial g^2}y' + \frac{\partial^3 z_a}{\partial G\partial g^2}z'\right),
\end{aligned}
\tag{3.55}
$$

$$
\begin{aligned}
\frac{\partial^3 \mathcal{K}_0}{\partial H^2 \partial g} \;=\;& -\frac{15}{\Delta^{7/2}}\left(\frac{\partial x_a}{\partial H}x' + \frac{\partial y_a}{\partial H}y' + \frac{\partial z_a}{\partial H}z'\right)^2 \times \\
& \times \left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right) \\
& -\frac{6}{\Delta^{5/2}}\left(\frac{\partial x_a}{\partial H}x' + \frac{\partial y_a}{\partial H}y' + \frac{\partial z_a}{\partial H}z'\right) \times \\
& \times \left(\frac{\partial^2 x_a}{\partial H \partial g}x' + \frac{\partial^2 y_a}{\partial H \partial g}y' + \frac{\partial^2 z_a}{\partial H \partial g}z'\right),
\end{aligned} \tag{3.56}
$$

$$
\begin{aligned}
\frac{\partial^3 \mathcal{K}_0}{\partial H \partial g^2} \;=\;& -\frac{15}{\Delta^{7/2}}\left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right)^2 \times \\
& \times \left(\frac{\partial x_a}{\partial H}x' + \frac{\partial y_a}{\partial H}y' + \frac{\partial z_a}{\partial H}z'\right) \\
& -\frac{6}{\Delta^{5/2}}\left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right) \times \\
& \times \left(\frac{\partial^2 x_a}{\partial H \partial g}x' + \frac{\partial^2 y_a}{\partial H \partial g}y' + \frac{\partial^2 z_a}{\partial H \partial g}z'\right) \\
& -\frac{3}{\Delta^{5/2}}\left(\frac{\partial x_a}{\partial H}x' + \frac{\partial y_a}{\partial H}y' + \frac{\partial z_a}{\partial H}z'\right) \times \\
& \times \left(\frac{\partial^2 x_a}{\partial g^2}x' + \frac{\partial^2 y_a}{\partial g^2}y' + \frac{\partial^2 z_a}{\partial g^2}z'\right) \\
& -\frac{1}{\Delta^{3/2}}\left(\frac{\partial^3 x_a}{\partial H \partial g^2}x' + \frac{\partial^3 y_a}{\partial H \partial g^2}y' + \frac{\partial^3 z_a}{\partial H \partial g^2}z'\right) \\
& +\frac{1}{r'^3}\left(\frac{\partial^3 x_a}{\partial H \partial g^2}x' + \frac{\partial^3 y_a}{\partial H \partial g^2}y' + \frac{\partial^3 z_a}{\partial H \partial g^2}z'\right),
\end{aligned} \tag{3.57}
$$

$$
\begin{aligned}
\frac{\partial^3 \mathcal{K}_0}{\partial L \partial G \partial g} \;=\;& -\frac{15}{\Delta^{7/2}} \left(r_a \frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L} x' - \frac{\partial y_a}{\partial L} y' - \frac{\partial z_a}{\partial L} z' \right) \times \\
& \times \left(r_a \frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G} x' - \frac{\partial y_a}{\partial G} y' - \frac{\partial z_a}{\partial G} z' \right) \times \\
& \times \left(\frac{\partial x_a}{\partial g} x' + \frac{\partial y_a}{\partial g} y' - \frac{\partial z_a}{\partial g} z' \right) \\
& + \frac{3}{\Delta^{5/2}} \left(r_a \frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G} x' - \frac{\partial y_a}{\partial G} y' - \frac{\partial z_a}{\partial G} z' \right) \times \\
& \times \left(\frac{\partial^2 x_a}{\partial L \partial g} x' + \frac{\partial^2 y_a}{\partial L \partial g} y' + \frac{\partial^2 z_a}{\partial L \partial g} z' \right) \\
& + \frac{3}{\Delta^{5/2}} \left(r_a \frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L} x' - \frac{\partial y_a}{\partial L} y' - \frac{\partial z_a}{\partial L} z' \right) \times \\
& \times \left(\frac{\partial^2 x_a}{\partial G \partial g} x' + \frac{\partial^2 y_a}{\partial G \partial g} y' + \frac{\partial^2 z_a}{\partial G \partial g} z' \right) \\
& + \frac{3}{\Delta^{5/2}} \left(\frac{\partial x_a}{\partial g} x' + \frac{\partial y_a}{\partial g} y' + \frac{\partial z_a}{\partial g} z' \right) \times \\
& \times \left(\frac{\partial r_a}{\partial L} \frac{\partial r_a}{\partial G} + r_a \frac{\partial^2 r_a}{\partial L \partial G} - \frac{\partial^2 x_a}{\partial L \partial G} x' - \frac{\partial^2 y_a}{\partial L \partial G} y' - \frac{\partial^2 z_a}{\partial L \partial G} z' \right) \\
& - \frac{1}{\Delta^{3/2}} \left(\frac{\partial^3 x_a}{\partial L \partial G \partial g} x' + \frac{\partial^3 y_a}{\partial L \partial G \partial g} y' + \frac{\partial^3 z_a}{\partial L \partial G \partial g} z' \right) \\
& + \frac{1}{r'^3} \left(\frac{\partial^3 x_a}{\partial L \partial G \partial g} x' + \frac{\partial^3 y_a}{\partial L \partial G \partial g} y' + \frac{\partial^3 z_a}{\partial L \partial G \partial g} z' \right),
\end{aligned}
\tag{3.58}
$$

$$
\begin{aligned}
\frac{\partial^3 \mathcal{K}_0}{\partial L \partial H \partial g} \quad = \quad & \frac{15}{\Delta^{7/2}} \left(r_a \frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L} x' - \frac{\partial y_a}{\partial L} y' - \frac{\partial z_a}{\partial L} z' \right) \times \\
& \times \left(\frac{\partial x_a}{\partial H} x' + \frac{\partial y_a}{\partial H} y' + \frac{\partial z_a}{\partial H} z' \right) \times \\
& \times \left(\frac{\partial x_a}{\partial g} x' + \frac{\partial y_a}{\partial g} y' - \frac{\partial z_a}{\partial g} z' \right) \\
& - \frac{3}{\Delta^{5/2}} \left(\frac{\partial x_a}{\partial H} x' + \frac{\partial y_a}{\partial H} y' + \frac{\partial z_a}{\partial H} z' \right) \times \\
& \times \left(\frac{\partial^2 x_a}{\partial L \partial g} x' + \frac{\partial^2 y_a}{\partial L \partial g} y' + \frac{\partial^2 z_a}{\partial L \partial g} z' \right) \\
& + \frac{3}{\Delta^{5/2}} \left(r_a \frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L} x' - \frac{\partial y_a}{\partial L} y' - \frac{\partial z_a}{\partial L} z' \right) \times \\
& \times \left(\frac{\partial^2 x_a}{\partial H \partial g} x' + \frac{\partial^2 y_a}{\partial H \partial g} y' + \frac{\partial^2 z_a}{\partial H \partial g} z' \right) \\
& - \frac{3}{\Delta^{5/2}} \left(\frac{\partial x_a}{\partial g} x' + \frac{\partial y_a}{\partial g} y' + \frac{\partial z_a}{\partial g} z' \right) \times \\
& \times \left(\frac{\partial^2 x_a}{\partial L \partial H} x' + \frac{\partial^2 y_a}{\partial L \partial H} y' + \frac{\partial^2 z_a}{\partial L \partial H} z' \right) \\
& - \frac{1}{\Delta^{3/2}} \left(\frac{\partial^3 x_a}{\partial L \partial H \partial g} x' + \frac{\partial^3 y_a}{\partial L \partial H \partial g} y' + \frac{\partial^3 z_a}{\partial L \partial H \partial g} z' \right) \\
& + \frac{1}{r'^3} \left(\frac{\partial^3 x_a}{\partial L \partial H \partial g} x' + \frac{\partial^3 y_a}{\partial L \partial H \partial g} y' + \frac{\partial^3 z_a}{\partial L \partial H \partial g} z' \right),
\end{aligned} \tag{3.59}
$$

$$
\begin{aligned}
\frac{\partial^3 \mathcal{K}_0}{\partial G \partial H \partial g} = \; & \frac{15}{\Delta^{7/2}}\left(r_a\frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G}x' - \frac{\partial y_a}{\partial G}y' - \frac{\partial z_a}{\partial G}z'\right)\times \\
& \times\left(\frac{\partial x_a}{\partial H}x' + \frac{\partial y_a}{\partial H}y' + \frac{\partial z_a}{\partial H}z'\right)\times \\
& \times\left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' - \frac{\partial z_a}{\partial g}z'\right) \\
& + \frac{3}{\Delta^{5/2}}\left(r_a\frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G}x' - \frac{\partial y_a}{\partial G}y' - \frac{\partial z_a}{\partial G}z'\right)\times \\
& \times\left(\frac{\partial^2 x_a}{\partial H \partial g}x' + \frac{\partial^2 y_a}{\partial H \partial g}y' + \frac{\partial^2 z_a}{\partial H \partial g}z'\right) \\
& - \frac{3}{\Delta^{5/2}}\left(\frac{\partial x_a}{\partial H}x' + \frac{\partial y_a}{\partial H}y' + \frac{\partial z_a}{\partial H}z'\right)\times \\
& \times\left(\frac{\partial^2 x_a}{\partial G \partial g}x' + \frac{\partial^2 y_a}{\partial G \partial g}y' + \frac{\partial^2 z_a}{\partial G \partial g}z'\right) \\
& - \frac{3}{\Delta^{5/2}}\left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right)\times \\
& \times\left(\frac{\partial^2 x_a}{\partial G \partial H}x' + \frac{\partial^2 y_a}{\partial G \partial H}y' + \frac{\partial^2 z_a}{\partial G \partial H}z'\right) \\
& - \frac{1}{\Delta^{3/2}}\left(\frac{\partial^3 x_a}{\partial G \partial H \partial g}x' + \frac{\partial^3 y_a}{\partial G \partial H \partial g}y' + \frac{\partial^3 z_a}{\partial G \partial H \partial g}z'\right) \\
& + \frac{1}{r'^3}\left(\frac{\partial^3 x_a}{\partial G \partial H \partial g}x' + \frac{\partial^3 y_a}{\partial G \partial H \partial g}y' + \frac{\partial^3 z_a}{\partial G \partial H \partial g}z'\right),
\end{aligned}
\tag{3.60}
$$

Pomoćne veličine za računanje izvoda po L su:

$$
\begin{aligned}
\frac{\partial e}{\partial L} &= \frac{G^2}{eL^3}, \\
\frac{\partial^2 e}{\partial L^2} &= -\frac{G^4}{e^3L^6} - \frac{3G^2}{eL^4}, \\
\frac{\partial^3 e}{\partial L^3} &= \frac{3G^6}{e^5L^9} + \frac{9G^4}{e^3L^7} + \frac{12G^2}{eL^5},
\end{aligned}
\tag{3.61}
$$

$$\begin{aligned}
\frac{\partial}{\partial L}(e\cos E) &= \frac{\partial e}{\partial L}\cos E - e\sin E\frac{\partial E}{\partial L} = \frac{\partial e}{\partial L}\frac{\cos E - e}{1-e\cos E}, \\
\frac{\partial E}{\partial L} &= \frac{\partial e}{\partial L}\frac{\sin E}{1-e\cos E} \\
\frac{\partial}{\partial L}(\cos E) &= -\sin E\frac{\partial E}{\partial L} = -\frac{\partial e}{\partial L}\frac{\sin^2 E}{1-e\cos E}, \\
\frac{\partial}{\partial L}(\sin E) &= \cos E\frac{\partial E}{\partial L} = \frac{\partial e}{\partial L}\frac{\sin E\cos E}{1-e\cos E},
\end{aligned} \tag{3.62}$$

$$\begin{aligned}
\frac{\partial^2}{\partial L^2}(e\cos E) &= \frac{\partial^2 e}{\partial L^2}\frac{\cos E - e}{1-e\cos E} + \frac{\partial e}{\partial L}\frac{\partial}{\partial L}\left(\frac{\cos E - e}{1-e\cos E}\right), \\
\frac{\partial}{\partial L}\left(\frac{\cos E - e}{1-e\cos E}\right) &= \frac{\frac{\partial}{\partial L}(\cos E - e)}{1-e\cos E} + \frac{\cos E - e}{(1-e\cos E)^2}\frac{\partial}{\partial L}(e\cos E),
\end{aligned} \tag{3.63}$$

$$\begin{aligned}
\frac{\partial^2}{\partial L^2}(\cos E) &= -\frac{\partial^2 e}{\partial L^2}\frac{\sin^2 E}{1-e\cos E} - \frac{\partial e}{\partial L}\frac{\partial}{\partial L}\left(\frac{\sin^2 E}{1-e\cos E}\right), \\
\frac{\partial^2}{\partial L^2}(\sin E) &= \frac{\partial^2 e}{\partial L^2}\frac{\sin E\cos E}{1-e\cos E} + \frac{\partial e}{\partial L}\frac{\partial}{\partial L}\frac{\sin E\cos E}{1-e\cos E}, \\
\frac{\partial}{\partial L}\left(\frac{\sin^2 E}{1-e\cos E}\right) &= 2\frac{\partial e}{\partial L}\frac{\sin^2 E\cos E}{(1-e\cos E)^2}, \\
&\quad + \frac{\sin^2 E}{(1-e\cos E)^2}\frac{\partial}{\partial L}(e\cos E), \\
\frac{\partial}{\partial L}\left(\frac{\sin E\cos E}{1-e\cos E}\right) &= \frac{\partial e}{\partial L}\frac{\sin E(\cos^2 E - \sin^2 E)}{(1-e\cos E)^2} \\
&\quad + \frac{\sin E\cos E}{(1-e\cos E)^2}\frac{\partial}{\partial L}(e\cos E),
\end{aligned} \tag{3.64}$$

$$\frac{\partial^3}{\partial L^3}(\cos E) = -\frac{\partial^3 e}{\partial L^3}\frac{\sin^2 E}{1-e\cos E} - 2\frac{\partial^2 e}{\partial L^2}\frac{\partial}{\partial L}\left(\frac{\sin^2 E}{1-e\cos E}\right) - \frac{\partial e}{\partial L}\frac{\partial^2}{\partial L^2}\left(\frac{\sin^2 E}{1-e\cos E}\right),$$

$$\frac{\partial^3}{\partial L^3}(\sin E) = \frac{\partial^3 e}{\partial L^3}\frac{\sin E\cos E}{1-e\cos E} + 2\frac{\partial^2 e}{\partial L^2}\frac{\partial}{\partial L}\left(\frac{\sin E\cos E}{1-e\cos E}\right) + \frac{\partial e}{\partial L}\frac{\partial^2}{\partial L^2}\left(\frac{\sin E\cos E}{1-e\cos E}\right),$$

$$\begin{aligned}\frac{\partial^2}{\partial L^2}\left(\frac{\sin^2 E}{1-e\cos E}\right) &= 2\frac{\partial^2 e}{\partial L^2}\frac{\sin^2 E\cos E}{(1-e\cos E)^2}+2\frac{\partial e}{\partial L}\frac{\partial}{\partial L}\left(\frac{\sin^2 E\cos E}{(1-e\cos E)^2}\right)+\\ &+\frac{\partial}{\partial L}\left(\frac{\sin^2 E}{(1-e\cos E)^2}\right)\frac{\partial}{\partial L}(e\cos E)+\frac{\sin^2 E}{(1-e\cos E)^2}\frac{\partial^2}{\partial L^2}(e\cos E),\end{aligned}$$

$$\begin{aligned}\frac{\partial^2}{\partial L^2}\left(\frac{\sin E\cos E}{1-e\cos E}\right) &= \frac{\partial^2 e}{\partial L^2}\frac{\sin E(\cos^2 E-\sin^2 E)}{(1-e\cos E)^2}+\frac{\partial e}{\partial L}\frac{\partial}{\partial L}\left(\frac{\sin E(\cos^2 E-\sin^2 E)}{(1-e\cos E)^2}\right)+\\ &+\frac{\partial}{\partial L}\left(\frac{\sin E\cos E}{(1-e\cos E)^2}\right)\frac{\partial}{\partial L}(e\cos E)+\frac{\sin E\cos E}{(1-e\cos E)^2}\frac{\partial^2}{\partial L^2}(e\cos E),\end{aligned}$$

$$\frac{\partial}{\partial L}\left(\frac{\sin^2 E\cos E}{(1-e\cos E)^2}\right) = \frac{\partial e}{\partial L}\frac{\sin^2 E(3\cos^2 E-1)}{(1-e\cos E)^3}+2\frac{\sin^2 E\cos E}{(1-e\cos E)^3}\frac{\partial}{\partial L}(e\cos E),$$

$$\frac{\partial}{\partial L}\left(\frac{\sin^2 E}{(1-e\cos E)^2}\right) = \frac{\partial e}{\partial L}\frac{2\sin^2 E\cos E}{(1-e\cos E)^3}+2\frac{\sin^2 E}{(1-e\cos E)^3}\frac{\partial}{\partial L}(e\cos E),$$

$$\begin{aligned}\frac{\partial}{\partial L}\left(\frac{\sin E(\cos^2 E-\sin^2 E)}{(1-e\cos E)^2}\right) &= \frac{\partial e}{\partial L}\frac{2\sin E\cos E(1+2\sin^2 E)}{(1-e\cos E)^3}+\\ &+2\frac{\sin E(\cos^2 E-\sin^2 E)}{(1-e\cos E)^3}\frac{\partial}{\partial L}(e\cos E),\end{aligned}$$

$$\frac{\partial}{\partial L}\left(\frac{\sin E\cos E}{(1-e\cos E)^2}\right) = \frac{\partial e}{\partial L}\frac{\sin E(\cos^2 E-\sin^2 E)}{(1-e\cos E)^3}+2\frac{\sin E\cos E}{(1-e\cos E)^3}\frac{\partial}{\partial L}(e\cos E),$$

$$
\begin{aligned}
\frac{\partial r_a}{\partial L} &= 2L(1-e\cos E) - L^2\frac{\partial}{\partial L}(e\cos E) = 2L(1-e\cos E) - L^2\frac{\partial e}{\partial L}\left(\frac{\cos E - e}{1-e\cos E}\right)\\
\frac{\partial x_a}{\partial L} &= 2(\cos g\cos h - \frac{H}{G}\sin g\sin h)L(\cos E - e) +\\
&\quad (\cos g\cos h - \frac{H}{G}\sin g\sin h)L^2\frac{\partial}{\partial L}(\cos E - e) +\\
&\quad (-\sin g\cos h - \frac{H}{G}\cos g\sin h)G\sin E +\\
&\quad (-\sin g\cos h - \frac{H}{G}\cos g\sin h)L\,G\frac{\partial}{\partial L}(\sin E)\\
\frac{\partial y_a}{\partial L} &= 2(\cos g\sin h + \frac{H}{G}\sin g\cos h)L(\cos E - e) +\\
&\quad (\cos g\sin h + \frac{H}{G}\sin g\cos h)L^2\frac{\partial}{\partial L}(\cos E - e) +\\
&\quad (-\sin g\sin h + \frac{H}{G}\cos g\cos h)G\sin E +\\
&\quad (-\sin g\sin h + \frac{H}{G}\cos g\cos h)L\,G\frac{\partial}{\partial L}(\sin E),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 r_a}{\partial L^2} &= 2(1-e\cos E)+ \\
&\quad -4L\frac{\partial}{\partial L}(e\cos E)-L^2\frac{\partial^2}{\partial L^2}(e\cos E) \\
\frac{\partial^2 x_a}{\partial L^2} &= 2(\cos g\cos h-\frac{H}{G}\sin g\sin h)(\cos E-e)+ \\
&\quad 4(\cos g\cos h-\frac{H}{G}\sin g\sin h)L\frac{\partial}{\partial L}(\cos E-e)+ \\
&\quad (\cos g\cos h-\frac{H}{G}\sin g\sin h)L^2\frac{\partial^2}{\partial L^2}(\cos E-e)+ \\
&\quad 2(-\sin g\cos h-\frac{H}{G}\cos g\sin h)G\frac{\partial}{\partial L}(\sin E)+ \\
&\quad (-\sin g\cos h-\frac{H}{G}\cos g\sin h)L\,G\frac{\partial^2}{\partial L^2}(\sin E) \qquad (3.65)\\
\frac{\partial^2 y_a}{\partial L^2} &= 2(\cos g\sin h+\frac{H}{G}\sin g\cos h)(\cos E-e)+ \\
&\quad 4(\cos g\sin h+\frac{H}{G}\sin g\cos h)L\frac{\partial}{\partial L}(\cos E-e)+ \\
&\quad (\cos g\sin h+\frac{H}{G}\sin g\cos h)L^2\frac{\partial^2}{\partial L^2}(\cos E-e)+ \\
&\quad 2(-\sin g\sin h+\frac{H}{G}\cos g\cos h)G\frac{\partial}{\partial L}(\sin E)+ \\
&\quad (-\sin g\sin h+\frac{H}{G}\cos g\cos h)L\,G\frac{\partial^2}{\partial L^2}(\sin E),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^3 r_a}{\partial L^3} &= -6\frac{\partial}{\partial L}(e\cos E) - 6L\frac{\partial^2}{\partial L^2}(e\cos E) - L^2\frac{\partial^3}{\partial L^3}(e\cos E) \\
\frac{\partial^3 x_a}{\partial L^3} &= 6(\cos g\cos h - \frac{H}{G}\sin g\sin h)\frac{\partial}{\partial L}(\cos E - e) + \\
& 6(\cos g\cos h - \frac{H}{G}\sin g\sin h)L\frac{\partial^2}{\partial L^2}(\cos E - e) + \\
& (\cos g\cos h - \frac{H}{G}\sin g\sin h)L^2\frac{\partial^3}{\partial L^3}(\cos E - e) + \\
& 3(-\sin g\cos h - \frac{H}{G}\cos g\sin h)G\frac{\partial^2}{\partial L^2}(\sin E) + \\
& (-\sin g\cos h - \frac{H}{G}\cos g\sin h)L\,G\frac{\partial^3}{\partial L^3}(\sin E) \\
\frac{\partial^3 y_a}{\partial L^3} &= 6(\cos g\sin h + \frac{H}{G}\sin g\cos h)\frac{\partial}{\partial L}(\cos E - e) + \\
& 6(\cos g\sin h + \frac{H}{G}\sin g\cos h)L\frac{\partial^2}{\partial L^2}(\cos E - e) + \\
& (\cos g\sin h + \frac{H}{G}\sin g\cos h)L^2\frac{\partial^3}{\partial L^3}(\cos E - e) + \\
& 3(-\sin g\sin h + \frac{H}{G}\cos g\cos h)G\frac{\partial^2}{\partial L^2}(\sin E) + \\
& (-\sin g\sin h + \frac{H}{G}\cos g\cos h)L\,G\frac{\partial^3}{\partial L^3}(\sin E).
\end{aligned}
\tag{3.66}
$$

Slično, pomoćne veličine za računanje izvoda po G su:

$$
\begin{aligned}
\frac{\partial e}{\partial G} &= -\frac{G}{eL^2} \\
\frac{\partial^2 e}{\partial G^2} &= -\frac{G^2}{e^3L^4} - \frac{1}{eL^2} \\
\frac{\partial^3 e}{\partial G^3} &= -\frac{3G^3}{e^5L^6} - \frac{3G}{e^3L^4}
\end{aligned}
\tag{3.67}
$$

$$\begin{aligned}
\frac{\partial}{\partial G}(e\cos E) &= \frac{\partial e}{\partial G}\cos E - e\sin E\frac{\partial E}{\partial G} = \frac{\partial e}{\partial G}\frac{\cos E - e}{1-e\cos E}\\
\frac{\partial E}{\partial G} &= \frac{\partial e}{\partial G}\frac{\sin E}{1-e\cos E}\\
\frac{\partial}{\partial G}(\cos E) &= -\sin E\frac{\partial E}{\partial G} = -\frac{\partial e}{\partial G}\frac{\sin^2 E}{1-e\cos E}\\
\frac{\partial}{\partial G}(\sin E) &= \cos E\frac{\partial E}{\partial G} = \frac{\partial e}{\partial G}\frac{\sin E\cos E}{1-e\cos E}
\end{aligned}\tag{3.68}$$

$$\begin{aligned}
\frac{\partial^2}{\partial G^2}(e\cos E) &= \frac{\partial^2 e}{\partial G^2}\frac{\cos E - e}{1-e\cos E} + \frac{\partial e}{\partial G}\frac{\partial}{\partial G}\left(\frac{\cos E - e}{1-e\cos E}\right)\\
\frac{\partial}{\partial G}\left(\frac{\cos E - e}{1-e\cos E}\right) &= \frac{\frac{\partial}{\partial G}(\cos E - e)}{1-e\cos E} + \frac{\cos E - e}{(1-e\cos E)^2}\frac{\partial}{\partial G}(e\cos E)\\
\frac{\partial}{\partial G}\left(\frac{\cos E - e}{(1-e\cos E)^2}\right) &= \frac{\frac{\partial}{\partial G}(\cos E - e)}{(1-e\cos E)^2} + 2\frac{\cos E - e}{(1-e\cos E)^3}\frac{\partial}{\partial G}(e\cos E)
\end{aligned}\tag{3.69}$$

$$\begin{aligned}
\frac{\partial^2}{\partial G^2}(\cos E) &= -\frac{\partial^2 e}{\partial G^2}\frac{\sin^2 E}{1-e\cos E} - \frac{\partial e}{\partial G}\frac{\partial}{\partial G}\left(\frac{\sin^2 E}{1-e\cos E}\right)\\
\frac{\partial^2}{\partial G^2}(\sin E) &= \frac{\partial^2 e}{\partial G^2}\frac{\sin E\cos E}{1-e\cos E} + \frac{\partial e}{\partial G}\frac{\partial}{\partial G}\left(\frac{\sin E\cos E}{1-e\cos E}\right)\\
\frac{\partial}{\partial G}\left(\frac{\sin^2 E}{1-e\cos E}\right) &= 2\frac{\partial e}{\partial G}\frac{\sin^2 E\cos E}{(1-e\cos E)^2}\\
&\quad + \frac{\sin^2 E}{(1-e\cos E)^2}\frac{\partial}{\partial G}(e\cos E)\\
\frac{\partial}{\partial G}\left(\frac{\sin E\cos E}{1-e\cos E}\right) &= \frac{\partial e}{\partial G}\frac{\sin E(\cos^2 E - \sin^2 E)}{(1-e\cos E)^2}\\
&\quad + \frac{\sin E\cos E}{(1-e\cos E)^2}\frac{\partial}{\partial G}(e\cos E)
\end{aligned}\tag{3.70}$$

$$\begin{aligned}
\frac{\partial}{\partial G}\left(\frac{\sin E\cos E}{(1-e\cos E)^2}\right) &= \frac{\partial e}{\partial G}\frac{\sin E\cos 2E}{(1-e\cos E)^3} \\
&+2\frac{\sin E\cos E}{(1-e\cos E)^3}\frac{\partial}{\partial G}(e\cos E) \\
\frac{\partial}{\partial G}\left(\frac{\sin^2 E\cos E}{(1-e\cos E)^2}\right) &= \frac{\partial e}{\partial G}\frac{\sin^2 E(3\cos^2 E-1)}{(1-e\cos E)^3} \\
&+\frac{\sin^2 E\cos E}{(1-e\cos E)^3}\frac{\partial}{\partial G}(e\cos E) \qquad (3.71)\\
\frac{\partial}{\partial G}\left(\frac{\sin^2 E}{(1-e\cos E)^2}\right) &= 2\frac{\partial e}{\partial G}\frac{\sin^2 E\cos E}{(1-e\cos E)^3} \\
&+2\frac{\sin^2 E}{(1-e\cos E)^3}\frac{\partial}{\partial G}(e\cos E)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^3}{\partial G^3}(\cos E) &= -\frac{\partial^3 e}{\partial G^3}\frac{\sin^2 E}{1-e\cos E} - 2\frac{\partial^2 e}{\partial G^2}\frac{\partial}{\partial G}\left(\frac{\sin^2 E}{1-e\cos E}\right)+ \\
&-\frac{\partial e}{\partial G}\frac{\partial^2}{\partial G^2}\left(\frac{\sin^2 E}{1-e\cos E}\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^3}{\partial G^3}(\sin E) &= \frac{\partial^3 e}{\partial G^3}\frac{\sin E\cos E}{1-e\cos E} + 2\frac{\partial^2 e}{\partial G^2}\frac{\partial}{\partial G}\left(\frac{\sin E\cos E}{1-e\cos E}\right)+ \\
&+\frac{\partial e}{\partial G}\frac{\partial^2}{\partial G^2}\left(\frac{\sin E\cos E}{1-e\cos E}\right),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2}{\partial G^2}\left(\frac{\sin^2 E}{1-e\cos E}\right) &= 2\frac{\partial^2 e}{\partial G^2}\frac{\sin^2 E\cos E}{(1-e\cos E)^2} + 2\frac{\partial e}{\partial G}\frac{\partial}{\partial G}\left(\frac{\sin^2 E\cos E}{(1-e\cos E)^2}\right)+ \\
&\frac{\partial}{\partial G}\left(\frac{\sin^2 E}{(1-e\cos E)^2}\right)\frac{\partial}{\partial G}(e\cos E) + \frac{\sin^2 E}{(1-e\cos E)^2}\frac{\partial^2}{\partial G^2}(e\cos E),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2}{\partial G^2}\left(\frac{\sin E\cos E}{1-e\cos E}\right) &= \frac{\partial^2 e}{\partial G^2}\frac{\sin E\cos 2E}{(1-e\cos E)^2}+ \\
&\frac{\partial e}{\partial G}\left[-\frac{1}{2}\frac{\partial e}{\partial G}\frac{\sin E(\cos E-3\cos 3E)}{(1-e\cos E)^3} + 2\frac{\sin E\cos 2E}{(1-e\cos E)^3}\frac{\partial}{\partial G}(e\cos E)\right]+ \\
&\frac{\partial}{\partial G}\left(\frac{\sin E\cos E}{(1-e\cos E)^2}\right)\frac{\partial}{\partial G}(e\cos E) + \frac{\sin E\cos E}{(1-e\cos E)^2}\frac{\partial^2}{\partial G^2}(e\cos E)
\end{aligned}$$

$$\begin{aligned}\frac{\partial^3}{\partial G^3}(e\cos E) &= \frac{\partial^3 e}{\partial G^3}\frac{\cos E-e}{1-e\cos E}+2\frac{\partial^2 e}{\partial G^2}\frac{\partial}{\partial G}\left(\frac{\cos E-e}{1-e\cos E}\right)+\\ &\quad \frac{\partial e}{\partial G}\frac{\partial^2}{\partial G^2}\left(\frac{\cos E-e}{1-e\cos E}\right),\end{aligned}$$

$$\frac{\partial}{\partial G}\left(\frac{\cos E-e}{1-e\cos E}\right) = \frac{\frac{\partial}{\partial G}(\cos E-e)}{1-e\cos E}+\frac{\cos E-e}{(1-e\cos E)^2}\frac{\partial}{\partial G}(e\cos E),$$

$$\begin{aligned}\frac{\partial^2}{\partial G^2}\left(\frac{\cos E-e}{1-e\cos E}\right) &= \frac{\frac{\partial^2}{\partial G^2}(\cos E-e)}{1-e\cos E}+2\frac{\frac{\partial}{\partial G}(\cos E-e)\frac{\partial}{\partial G}(e\cos E)}{(1-e\cos E)^2}+\\ &\quad \frac{\cos E-e}{(1-e\cos E)^2}\frac{\partial^2}{\partial G^2}(e\cos E)+\frac{\cos E-e}{(1-e\cos E)^3}\left(\frac{\partial}{\partial G}(e\cos E)\right)^2,\end{aligned}$$

$$\frac{\partial r_a}{\partial G} = -L^2\frac{\partial}{\partial G}(e\cos E) = -L^2\frac{\partial e}{\partial G}\left(\frac{\cos E-e}{1-e\cos E}\right),$$

$$\begin{aligned}\frac{\partial x_a}{\partial G} &= \frac{H}{G^2}\sin g\sin h\,L^2(\cos E-e)+(\cos g\cos h-\frac{H}{G}\sin g\sin h)L^2\frac{\partial}{\partial G}(\cos E-e)+\\ &\quad -\sin g\cos h\,L\sin E+(-\sin g\cos h-\frac{H}{G}\cos g\sin h)L\,G\frac{\partial}{\partial G}(\sin E),\end{aligned}$$

$$\begin{aligned}\frac{\partial y_a}{\partial G} &= -\frac{H}{G^2}\sin g\cos h\,L^2(\cos E-e)+(\cos g\sin h+\frac{H}{G}\sin g\cos h)L^2\frac{\partial}{\partial G}(\cos E-e)+\\ &\quad -\sin g\sin h\,L\sin E+(-\sin g\sin h+\frac{H}{G}\cos g\cos h)L\,G\frac{\partial}{\partial G}(\sin E),\end{aligned}$$

$$\frac{\partial^2 r_a}{\partial G^2} = -L^2\frac{\partial^2}{\partial G^2}(e\cos E),$$

$$\begin{aligned}\frac{\partial^2 x_a}{\partial G^2} &= -2\frac{H}{G^3}\sin g\sin h\,L^2(\cos E-e)+2\frac{H}{G^2}\sin g\sin h\,L^2\frac{\partial}{\partial G}(\cos E-e)+\\ &\quad (\cos g\cos h-\frac{H}{G}\sin g\sin h)L^2\frac{\partial^2}{\partial G^2}(\cos E-e)+\\ &\quad 2\frac{H}{G}\cos g\sin h\,L\frac{\partial}{\partial G}(\sin E)+2(-\sin g\cos h-\frac{H}{G}\cos g\sin h)L\frac{\partial}{\partial G}(\sin E)+\\ &\quad (-\sin g\cos h-\frac{H}{G}\cos g\sin h)L\,G\frac{\partial^2}{\partial G^2}(\sin E),\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 y_a}{\partial G^2} &= 2\frac{H}{G^3}\sin g\cos h\, L^2(\cos E - e) - 2\frac{H}{G^2}\sin g\cos h\, L^2\frac{\partial}{\partial G}(\cos E - e) + \\
&(\cos g\sin h + \frac{H}{G}\sin g\cos h)L^2\frac{\partial^2}{\partial G^2}(\cos E - e) + \\
&-2\frac{H}{G}\cos g\cos h\, L\frac{\partial}{\partial G}(\sin E) + 2(-\sin g\sin h + \frac{H}{G}\cos g\cos h)L\frac{\partial}{\partial G}(\sin E) + \\
&(-\sin g\sin h + \frac{H}{G}\cos g\cos h)L\, G\frac{\partial^2}{\partial G^2}(\sin E),
\end{aligned}$$

$$\frac{\partial^3 r_a}{\partial G^3} = -L^2\frac{\partial^3}{\partial G^3}(e\cos E),$$

$$\begin{aligned}
\frac{\partial^3 x_a}{\partial G^3} &= 6\frac{H}{G^4}\sin g\sin h\, L^2(\cos E - e) - 6\frac{H}{G^3}\sin g\sin h\, L^2\frac{\partial}{\partial G}(\cos E - e) + \\
&\frac{H}{G}\sin g\sin h\, L^2\frac{\partial^2}{\partial G^2}(\cos E - e) + (\cos g\cos h - \frac{H}{G}\sin g\sin h)L^2\frac{\partial^3}{\partial G^3}(\cos E - e) + \\
&3\frac{H}{G}\cos g\sin h\, L\frac{\partial^2}{\partial G^2}(\sin E) + 3(-\sin g\cos h - \frac{H}{G}\cos g\sin h)L\frac{\partial^2}{\partial G^2}(\sin E) + \\
&(-\sin g\cos h - \frac{H}{G}\cos g\sin h)L\, G\frac{\partial^3}{\partial G^3}(\sin E),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^3 y_a}{\partial G^3} &= -6\frac{H}{G^4}\sin g\cos h\, L^2(\cos E - e) + 6\frac{H}{G^3}\sin g\cos h\, L^2\frac{\partial}{\partial G}(\cos E - e) + \\
&-3\frac{H}{G}\sin g\cos h\, L^2\frac{\partial^2}{\partial G^2}(\cos E - e) + (\cos g\sin h + \frac{H}{G}\sin g\cos h)L^2\frac{\partial^3}{\partial G^3}(\cos E - e) + \\
&-3\frac{H}{G}\cos g\cos h\, L\frac{\partial^2}{\partial G^2}(\sin E) + 3(-\sin g\sin h + \frac{H}{G}\cos g\cos h)L\frac{\partial^2}{\partial G^2}(\sin E) + \\
&(-\sin g\sin h + \frac{H}{G}\cos g\cos h)L\, G\frac{\partial^3}{\partial G^3}(\sin E),
\end{aligned}$$

Pomoćne veličine za računanje mešovitih izvoda po L i G

$$\begin{aligned}
\frac{\partial^2 e}{\partial L\partial G} &= \frac{G^3}{e^3L^5} + \frac{2G}{eL^3} \\
\frac{\partial^3 e}{\partial L^2\partial G} &= -\frac{3G^5}{e^5L^8} - \frac{7G^3}{e^3L^6} - \frac{6G}{eL^4} \\
\frac{\partial^3 e}{\partial L\partial G^2} &= \frac{3G^4}{e^5L^7} + \frac{5G^2}{e^3L^5} + \frac{2}{eL^3}
\end{aligned} \tag{3.72}$$

$$\frac{\partial^2}{\partial L\partial G}(\cos E) = -\frac{\partial^2 e}{\partial L\partial G}\frac{\sin^2 E}{1-e\cos E} - \frac{\partial e}{\partial L}\frac{\partial}{\partial G}\left(\frac{\sin^2 E}{1-e\cos E}\right),$$

$$\frac{\partial^2}{\partial L\partial G}(\sin E) = \frac{\partial^2 e}{\partial L\partial G}\frac{\sin E\cos E}{1-e\cos E} + \frac{\partial e}{\partial L}\frac{\partial}{\partial G}\left(\frac{\sin E\cos E}{1-e\cos E}\right),$$

$$\frac{\partial^2}{\partial L\partial G}(e\cos E) = \frac{\partial^2 e}{\partial L\partial G}\frac{\cos E-e}{1-e\cos E} + \frac{\partial e}{\partial L}\frac{\partial}{\partial G}\left(\frac{\cos E-e}{1-e\cos E}\right),$$

$$\begin{aligned}\frac{\partial^3}{\partial L^2\partial G}(e\cos E) &= \frac{\partial^3 e}{\partial L^2\partial G}\frac{\cos E-e}{1-e\cos E} + \frac{\partial^2 e}{\partial L^2}\frac{\partial}{\partial G}\left(\frac{\cos E-e}{1-e\cos E}\right) + \\ &\frac{\partial^2 e}{\partial L\partial G}\frac{\partial}{\partial L}\left(\frac{\cos E-e}{1-e\cos E}\right) + \frac{\partial e}{\partial L}\frac{\partial^2}{\partial L\partial G}\left(\frac{\cos E-e}{1-e\cos E}\right),\end{aligned}$$

$$\begin{aligned}\frac{\partial^2}{\partial L\partial G}\left(\frac{\cos E-e}{1-e\cos E}\right) &= \frac{\frac{\partial^2}{\partial L\partial G}(\cos E-e)}{1-e\cos E} + \frac{\frac{\partial}{\partial L}(\cos E-e)}{(1-e\cos E)^2}\frac{\partial}{\partial G}(e\cos E) + \\ &\frac{\partial}{\partial G}\left(\frac{\cos E-e}{(1-e\cos E)^2}\right)\frac{\partial}{\partial L}(e\cos E) + \frac{\cos E-e}{(1-e\cos E)^2}\frac{\partial^2}{\partial L\partial G}(e\cos E),\end{aligned}$$

$$\frac{\partial^2 r_a}{\partial L\partial G} = -2L\frac{\partial}{\partial G}(e\cos E) - L^2\frac{\partial^2 e}{\partial L\partial G}\frac{\cos E-e}{1-e\cos E} - L^2\frac{\partial e}{\partial L}\frac{\partial}{\partial G}\left(\frac{\cos E-e}{1-e\cos E}\right),$$

$$\begin{aligned}\frac{\partial^2 x_a}{\partial L\partial G} &= 2\frac{H}{G^2}\sin g\sin h\,L(\cos E-e) + 2(\cos g\cos h - \frac{H}{G}\sin g\sin h)L\frac{\partial}{\partial G}(\cos E-e) + \\ &\frac{H}{G^2}\sin g\sin h\,L^2\frac{\partial}{\partial L}(\cos E-e) + (\cos g\cos h - \frac{H}{G}\sin g\sin h)L^2\frac{\partial^2}{\partial L\partial G}(\cos E-e) + \\ &\frac{H}{G}\cos g\sin h\ \sin E + (-\sin g\cos h - \frac{H}{G}\cos g\sin h)\sin E + \\ &(-\sin g\cos h - \frac{H}{G}\cos g\sin h)G\frac{\partial}{\partial G}(\sin E) + \frac{H}{G}\cos g\sin h\,L\frac{\partial}{\partial L}(\sin E) + \\ &(-\sin g\cos h - \frac{H}{G}\cos g\sin h)L\frac{\partial}{\partial L}(\sin E) + (-\sin g\cos h - \frac{H}{G}\cos g\sin h)L\,G\frac{\partial^2}{\partial L\partial G}(\sin E),\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 y_a}{\partial L\partial G} &= -2\frac{H}{G^2}\sin g\cos h\, L(\cos E-e)+2(\cos g\sin h+\frac{H}{G}\sin g\cos h)L\frac{\partial}{\partial G}(\cos E-e)+\\
&-\frac{H}{G^2}\sin g\cos h\, L^2\frac{\partial}{\partial L}(\cos E-e)+(\cos g\sin h+\frac{H}{G}\sin g\cos h)L^2\frac{\partial^2}{\partial L\partial G}(\cos E-e)+\\
&-\frac{H}{G}\cos g\cos h\,\sin E+(-\sin g\sin h+\frac{H}{G}\cos g\cos h)\sin E+\\
&(-\sin g\sin h+\frac{H}{G}\cos g\cos h)G\frac{\partial}{\partial G}(\sin E)-\frac{H}{G}\cos g\cos h\, L\frac{\partial}{\partial L}(\sin E)+\\
&(-\sin g\sin h+\frac{H}{G}\cos g\cos h)L\frac{\partial}{\partial L}(\sin E)+(-\sin g\sin h+\frac{H}{G}\cos g\cos h)L\,G\frac{\partial^2}{\partial L\partial G}(\sin E),\end{aligned}$$

$$\begin{aligned}\frac{\partial^3}{\partial L^2\partial G}(\cos E) &= -\frac{\partial^3 e}{\partial L^2\partial G}\frac{\sin^2 E}{1-e\cos E}-\frac{\partial^2 e}{\partial L\partial G}\frac{\partial}{\partial L}\left(\frac{\sin^2 E}{1-e\cos E}\right)+\\
&-\frac{\partial^2 e}{\partial L^2}\frac{\partial}{\partial G}\left(\frac{\sin^2 E}{1-e\cos E}\right)-\frac{\partial e}{\partial L}\frac{\partial^2}{\partial L\partial G}\left(\frac{\sin^2 E}{1-e\cos E}\right),\end{aligned}$$

$$\begin{aligned}\frac{\partial^3}{\partial L^2\partial G}(\sin E) &= \frac{\partial^3 e}{\partial L^2\partial G}\frac{\sin E\cos E}{1-e\cos E}+\frac{\partial^2 e}{\partial L\partial G}\frac{\partial}{\partial L}\left(\frac{\sin E\cos E}{1-e\cos E}\right)+\\
&\frac{\partial^2 e}{\partial L^2}\frac{\partial}{\partial G}\left(\frac{\sin E\cos E}{1-e\cos E}\right)+\frac{\partial e}{\partial L}\frac{\partial^2}{\partial L\partial G}\left(\frac{\sin E\cos E}{1-e\cos E}\right),\end{aligned}$$

$$\frac{\partial^3 r_a}{\partial L^2\partial G} = -\frac{\partial}{\partial G}(e\cos E)-4L\frac{\partial^2}{\partial L\partial G}(e\cos E)-L^2\frac{\partial^3}{\partial L^2\partial G}(e\cos E),$$

$$\begin{aligned}\frac{\partial^3 x_a}{\partial L^2\partial G} &= 2\frac{H}{G^2}\sin g\sin h(\cos E-e)+2(\cos g\cos h-\frac{H}{G}\sin g\sin h)\frac{\partial}{\partial G}(\cos E-e)+\\
&4\frac{H}{G^2}\sin g\sin h\, L\frac{\partial}{\partial L}(\cos E-e)+4(\cos g\cos h-\frac{H}{G}\sin g\sin h)L\frac{\partial^2}{\partial L\partial G}(\cos E-e)+\\
&\frac{H}{G^2}\sin g\sin h\, L^2\frac{\partial^2}{\partial L^2}(\cos E-e)+(\cos g\cos h-\frac{H}{G}\sin g\sin h)L^2\frac{\partial^3}{\partial L^2\partial G}(\cos E-e)+\\
&2\frac{H}{G}\cos g\sin h\frac{\partial}{\partial L}(\sin E)+2(-\sin g\cos h-\frac{H}{G}\cos g\sin h)\frac{\partial}{\partial L}(\sin E)+\\
&2(-\sin g\cos h-\frac{H}{G}\cos g\sin h)G\frac{\partial^2}{\partial L\partial G}(\sin E)+\frac{H}{G}\cos g\sin h\, L\frac{\partial^2}{\partial L^2}(\sin E)+\\
&(-\sin g\cos h-\frac{H}{G}\cos g\sin h)L\frac{\partial^2}{\partial L^2}(\sin E)+(-\sin g\cos h-\frac{H}{G}\cos g\sin h)L\,G\frac{\partial^3}{\partial L^2\partial G}(\sin E),\end{aligned}$$

$$\begin{aligned}
\frac{\partial^3 y_a}{\partial L^2\partial G} &= -2\frac{H}{G^2}\sin g\cos h(\cos E-e)+2(\cos g\sin h+\frac{H}{G}\sin g\cos h)\frac{\partial}{\partial G}(\cos E-e)+ \\
&-4\frac{H}{G^2}\sin g\cos h\,L\frac{\partial}{\partial L}(\cos E-e)+4(\cos g\sin h+\frac{H}{G}\sin g\cos h)L\frac{\partial^2}{\partial L\partial G}(\cos E-e)+ \\
&\frac{H}{G^2}\sin g\cos h\,L^2\frac{\partial^2}{\partial L^2}(\cos E-e)+(\cos g\sin h+\frac{H}{G}\sin g\cos h)L^2\frac{\partial^3}{\partial L^2\partial G}(\cos E-e)+ \\
&-2\frac{H}{G}\cos g\cos h\frac{\partial}{\partial L}(\sin E)+2(-\sin g\sin h+\frac{H}{G}\cos g\cos h)\frac{\partial}{\partial L}(\sin E)+ \\
&2(-\sin g\sin h+\frac{H}{G}\cos g\cos h)G\frac{\partial^2}{\partial L\partial G}(\sin E)-\frac{H}{G}\cos g\cos h\,L\frac{\partial^2}{\partial L^2}(\sin E)+ \\
&(-\sin g\sin h+\frac{H}{G}\cos g\cos h)L\frac{\partial^2}{\partial L^2}(\sin E)+(-\sin g\sin h+\frac{H}{G}\cos g\cos h)L\,G\frac{\partial^3}{\partial L^2\partial G}(\sin E),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^3}{\partial L\partial G^2}(e\cos E) &= \frac{\partial^3 e}{\partial L\partial G^2}\frac{\cos E-e}{1-e\cos E}+\frac{\partial^2 e}{\partial G^2}\frac{\partial}{\partial L}\left(\frac{\cos E-e}{1-e\cos E}\right)+ \\
&\frac{\partial^2 e}{\partial L\partial G}\frac{\partial}{\partial G}\left(\frac{\cos E-e}{1-e\cos E}\right)+\frac{\partial e}{\partial G}\frac{\partial^2}{\partial L\partial G}\left(\frac{\cos E-e}{1-e\cos E}\right),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^3}{\partial L\partial G^2}(\cos E) &= -\frac{\partial^3 e}{\partial L\partial G^2}\frac{\sin^2 E}{1-e\cos E}-\frac{\partial^2 e}{\partial G^2}\frac{\partial}{\partial L}\left(\frac{\sin^2 E}{1-e\cos E}\right)+ \\
&-\frac{\partial^2 e}{\partial L\partial G}\frac{\partial}{\partial G}\left(\frac{\sin^2 E}{1-e\cos E}\right)-\frac{\partial e}{\partial G}\frac{\partial^2}{\partial L\partial G}\left(\frac{\sin^2 E}{1-e\cos E}\right),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^3}{\partial L\partial G^2}(\sin E) &= -\frac{\partial^3 e}{\partial L\partial G^2}\frac{\sin E\cos E}{1-e\cos E}-\frac{\partial^2 e}{\partial G^2}\frac{\partial}{\partial L}\left(\frac{\sin E\cos E}{1-e\cos E}\right)+ \\
&-\frac{\partial^2 e}{\partial L\partial G}\frac{\partial}{\partial G}\left(\frac{\sin E\cos E}{1-e\cos E}\right)-\frac{\partial e}{\partial G}\frac{\partial^2}{\partial L\partial G}\left(\frac{\sin E\cos E}{1-e\cos E}\right),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2}{\partial L\partial G}\left(\frac{\sin^2 E}{1-e\cos E}\right) &= 2\frac{\partial^2 e}{\partial L\partial G}\frac{\sin^2 E\cos E}{(1-e\cos E)^2}+2\frac{\partial e}{\partial G}\frac{\partial}{\partial L}\left(\frac{\sin^2 E\cos E}{(1-e\cos E)^2}\right)+ \\
&\frac{\partial}{\partial L}\left(\frac{\sin^2 E}{(1-e\cos E)^2}\right)\frac{\partial}{\partial G}(e\cos E)+\frac{\sin^2 E}{(1-e\cos E)^2}\frac{\partial^2}{\partial L\partial G}(e\cos E),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2}{\partial L\partial G}\left(\frac{\sin E\cos E}{1-e\cos E}\right) &= \frac{\partial^2 e}{\partial L\partial G}\frac{\sin E\cos 2E}{(1-e\cos E)^2}+\frac{\partial e}{\partial G}\frac{\partial}{\partial L}\left(\frac{\sin E\cos 2E}{(1-e\cos E)^2}\right)+ \\
&\frac{\partial}{\partial L}\left(\frac{\sin E\cos E}{(1-e\cos E)^2}\right)\frac{\partial}{\partial G}(e\cos E)+\frac{\sin E\cos E}{(1-e\cos E)^2}\frac{\partial^2}{\partial L\partial G}(e\cos E),
\end{aligned}$$

$$\frac{\partial^3 r_a}{\partial L \partial G^2} = -2L\frac{\partial^2}{\partial G^2}(e\cos E) - L^2\frac{\partial^3}{\partial L \partial G^2}(e\cos E),$$

$$\begin{aligned}
\frac{\partial^3 x_a}{\partial L \partial G^2} = & -4\frac{H}{G^3}\sin g\sin h\, L(\cos E - e) - 2\frac{H}{G^3}\sin g\sin h\, L^2\frac{\partial}{\partial L}(\cos E - e) + \\
& 4\frac{H}{G^2}\sin g\sin h\, L\frac{\partial}{\partial G}(\cos E - e) + 2\frac{H}{G^2}\sin g\sin h\, L^2\frac{\partial^2}{\partial L\partial G}(\cos E - e) + \\
& 2(\cos g\cos h - \frac{H}{G}\sin g\sin h)L\frac{\partial^2}{\partial G^2}(\cos E - e) + \\
& (\cos g\cos h - \frac{H}{G}\sin g\sin h)L^2\frac{\partial^3}{\partial L\partial G^2}(\cos E - e) + \\
& 2\frac{H}{G}\cos g\sin h\frac{\partial}{\partial G}(\sin E) + 2\frac{H}{G}\cos g\sin h\, L\frac{\partial^2}{\partial L\partial G}(\sin E) + \\
& 2(-\sin g\cos h - \frac{H}{G}\cos g\sin h)\frac{\partial}{\partial G}(\sin E) + \\
& 2(-\sin g\cos h - \frac{H}{G}\cos g\sin h)L\frac{\partial^2}{\partial L\partial G}(\sin E) + \\
& (-\sin g\cos h - \frac{H}{G}\cos g\sin h)G\frac{\partial^2}{\partial G^2}(\sin E) + \\
& (-\sin g\cos h - \frac{H}{G}\cos g\sin h)L\, G\frac{\partial^3}{\partial L\partial G^2}(\sin E),
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^3 y_a}{\partial L \partial G^2} &= 4\frac{H}{G^3}\sin g\cos h\, L(\cos E - e) + 2\frac{H}{G^3}\sin g\cos h\, L^2\frac{\partial}{\partial L}(\cos E - e) + \\
&\quad -4\frac{H}{G^2}\sin g\cos h\, L\frac{\partial}{\partial G}(\cos E - e) - 2\frac{H}{G^2}\sin g\cos h\, L^2\frac{\partial^2}{\partial L\partial G}(\cos E - e) + \\
&\quad 2(\cos g\sin h + \frac{H}{G}\sin g\cos h)L\frac{\partial^2}{\partial G^2}(\cos E - e) + \\
&\quad (\cos g\sin h + \frac{H}{G}\sin g\cos h)L^2\frac{\partial^3}{\partial L\partial G^2}(\cos E - e) + \\
&\quad -2\frac{H}{G}\cos g\cos h\frac{\partial}{\partial G}(\sin E) - 2\frac{H}{G}\cos g\cos h\, L\frac{\partial^2}{\partial L\partial G}(\sin E) + \\
&\quad 2(-\sin g\sin h + \frac{H}{G}\cos g\cos h)\frac{\partial}{\partial G}(\sin E) + \\
&\quad 2(-\sin g\sin h + \frac{H}{G}\cos g\cos h)L\frac{\partial^2}{\partial L\partial G}(\sin E) + \\
&\quad (-\sin g\sin h + \frac{H}{G}\cos g\cos h)G\frac{\partial^2}{\partial G^2}(\sin E) + \\
&\quad (-\sin g\sin h + \frac{H}{G}\cos g\cos h)L\, G\frac{\partial^3}{\partial L\partial G^2}(\sin E),
\end{aligned}
$$

Izvodi koordinata asteroida po promenljivoj H

$$
\frac{\partial x_a}{\partial H} = -\frac{1}{G}\sin g\sin h\, L^2(\cos E - e) - \cos g\sin h\, L\sin E,
$$

$$
\frac{\partial y_a}{\partial H} = \frac{1}{G}\sin g\cos h\, L^2(\cos E - e) + \cos g\cos h\, L\sin E,
$$

kao i mešoviti izvodi su:

$$
\begin{aligned}
\frac{\partial^2 x_a}{\partial L\partial H} &= -2\frac{L}{G}\sin g\sin h\,(\cos E - e) - \frac{L^2}{G}\sin g\sin h\frac{\partial}{\partial L}(\cos E - e) + \\
&\quad - \cos g\sin h\ \sin E - \cos g\sin h\, L\frac{\partial}{\partial L}(\sin E),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 y_a}{\partial L\partial H} &= 2\frac{L}{G}\sin g\cos h\,(\cos E - e) + \frac{L^2}{G}\sin g\cos h\frac{\partial}{\partial L}(\cos E - e) + \\
&\quad \cos g\cos h\ \sin E + \cos g\cos h\, L\frac{\partial}{\partial L}(\sin E),
\end{aligned}
$$

$$\begin{aligned}
\frac{\partial^3 x_a}{\partial L^2 \partial H} &= -2\frac{1}{G}\sin g \sin h\,(\cos E - e) - 4\frac{L}{G}\sin g \sin h \frac{\partial}{\partial L}(\cos E - e) + \\
&\quad -\frac{L^2}{G}\sin g \sin h \frac{\partial^2}{\partial L^2}(\cos E - e) + \\
&\quad -2\cos g \sin h \frac{\partial}{\partial L}(\sin E) - \cos g \sin h\, L \frac{\partial^2}{\partial L^2}(\sin E),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^3 y_a}{\partial L^2 \partial H} &= 2\frac{1}{G}\sin g \cos h\,(\cos E - e) + 4\frac{L}{G}\sin g \cos h \frac{\partial}{\partial L}(\cos E - e) + \\
&\quad \frac{L^2}{G}\sin g \cos h \frac{\partial^2}{\partial L^2}(\cos E - e) + \\
&\quad 2\cos g \cos h \frac{\partial}{\partial L}(\sin E) + \cos g \cos h\, L \frac{\partial^2}{\partial L^2}(\sin E),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 x_a}{\partial G \partial H} &= \frac{L^2}{G^2}\sin g \sin h\,(\cos E - e) - \frac{L^2}{G}\sin g \sin h \frac{\partial}{\partial G}(\cos E - e) + \\
&\quad -\cos g \sin h\, L \frac{\partial}{\partial G}(\sin E),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 y_a}{\partial G \partial H} &= -\frac{L^2}{G^2}\sin g \cos h\,(\cos E - e) + \frac{L^2}{G}\sin g \cos h \frac{\partial}{\partial G}(\cos E - e) + \\
&\quad \cos g \cos h\, L \frac{\partial}{\partial G}(\sin E),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^3 x_a}{\partial G^2 \partial H} &= -\frac{2}{G^3}\sin g \sin h\, L^2(\cos E - e) + \frac{2}{G^2}\sin g \sin h\, L^2 \frac{\partial}{\partial G}(\cos E - e) + \\
&\quad -\frac{1}{G}\sin g \sin h\, L^2 \frac{\partial^2}{\partial G^2}(\cos E - e) - \cos g \sin h\, L \frac{\partial^2}{\partial G^2}(\sin E),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^3 y_a}{\partial G^2 \partial H} &= \frac{2}{G^3}\sin g \cos h\, L^2(\cos E - e) - \frac{2}{G^2}\sin g \cos h\, L^2 \frac{\partial}{\partial G}(\cos E - e) + \\
&\quad \frac{1}{G}\sin g \cos h\, L^2 \frac{\partial^2}{\partial G^2}(\cos E - e) + \cos g \cos h\, LG \frac{\partial^2}{\partial G^2}(\sin E),
\end{aligned}$$

$$\begin{aligned}\frac{\partial^3 x_a}{\partial L\partial G\partial H} &= \frac{2}{G^2}\sin g\sin h\, L(\cos E-e)-\frac{2}{G}\sin g\sin h\, L\frac{\partial}{\partial G}(\cos E-e)+\\ &\quad \frac{1}{G^2}\sin g\sin h\, L^2\frac{\partial}{\partial L}(\cos E-e)-\frac{1}{G}\sin g\sin h\, L^2\frac{\partial^2}{\partial L\partial G}(\cos E-e)+\\ &\quad -\cos g\sin h\frac{\partial}{\partial G}(\sin E)-\cos g\sin h\, L\frac{\partial^2}{\partial L\partial G}(\sin E),\end{aligned}$$

$$\begin{aligned}\frac{\partial^3 y_a}{\partial L\partial G\partial H} &= -\frac{2}{G^2}\sin g\cos h\, L(\cos E-e)+\frac{2}{G}\sin g\cos h\, L\frac{\partial}{\partial G}(\cos E-e)+\\ &\quad -\frac{1}{G^2}\sin g\cos h\, L^2\frac{\partial}{\partial L}(\cos E-e)+\frac{1}{G}\sin g\cos h\, L^2\frac{\partial^2}{\partial L\partial G}(\cos E-e)+\\ &\quad \cos g\cos h\frac{\partial}{\partial G}(\sin E)+\cos g\cos h\, L\frac{\partial^2}{\partial L\partial G}(\sin E),\end{aligned}$$

$$\begin{aligned}\frac{\partial x_a}{\partial g} &= (-\sin g\cos h-\frac{H}{G}\cos g\sin h)L^2(\cos E-e)+\\ &\quad (-\cos g\cos h+\frac{H}{G}\sin g\sin h)L\,G\sin E,\end{aligned}$$

$$\begin{aligned}\frac{\partial y_a}{\partial g} &= (-\sin g\sin h+\frac{H}{G}\cos g\cos h)L^2(\cos E-e)+\\ &\quad (-\cos g\sin h-\frac{H}{G}\sin g\cos h)L\,G\sin E,\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 x_a}{\partial g^2} &= (-\cos g\cos h+\frac{H}{G}\sin g\sin h)L^2(\cos E-e)+\\ &\quad (\sin g\cos h+\frac{H}{G}\cos g\sin h)L\,G\sin E,\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 y_a}{\partial g^2} &= (-\cos g\sin h-\frac{H}{G}\sin g\cos h)L^2(\cos E-e)+\\ &\quad (\sin g\sin h-\frac{H}{G}\cos g\cos h)L\,G\sin E,\end{aligned}$$

$$\begin{aligned}\frac{\partial^3 x_a}{\partial g^3} &= (\sin g\cos h + \frac{H}{G}\cos g\sin h)L^2(\cos E - e) + \\ &\quad (\cos g\cos h - \frac{H}{G}\sin g\sin h)L\,G\sin E,\end{aligned}$$

$$\begin{aligned}\frac{\partial^3 y_a}{\partial g^3} &= (\sin g\sin h - \frac{H}{G}\cos g\cos h)L^2(\cos E - e) + \\ &\quad (\cos g\sin h + \frac{H}{G}\sin g\cos h)L\,G\sin E,\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 x_a}{\partial L\partial g} &= 2(-\sin g\cos h - \frac{H}{G}\cos g\sin h)L(\cos E - e) + \\ &\quad (-\sin g\cos h - \frac{H}{G}\cos g\sin h)L^2\frac{\partial}{\partial L}(\cos E - e) + \\ &\quad (-\cos g\cos h + \frac{H}{G}\sin g\sin h)G\sin E + (-\cos g\cos h + \\ &\quad \frac{H}{G}\sin g\sin h)L\,G\frac{\partial}{\partial L}(\sin E),\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 y_a}{\partial L\partial g} &= 2(-\sin g\sin h + \frac{H}{G}\cos g\cos h)L(\cos E - e) + \\ &\quad (-\sin g\sin h + \frac{H}{G}\cos g\cos h)L^2\frac{\partial}{\partial L}(\cos E - e) + \\ &\quad (-\cos g\sin h - \frac{H}{G}\sin g\cos h)L\,G\sin E + \\ &\quad (-\cos g\sin h - \frac{H}{G}\sin g\cos h)L\,G\frac{\partial}{\partial L}(\sin E),\end{aligned}$$

$$\begin{aligned}\frac{\partial^3 x_a}{\partial L^2\partial g} &= 2(-\sin g\cos h - \frac{H}{G}\cos g\sin h)(\cos E - e) + \\ &\quad 4(-\sin g\cos h - \frac{H}{G}\cos g\sin h)L\frac{\partial}{\partial L}(\cos E - e) + \\ &\quad (-\sin g\cos h - \frac{H}{G}\cos g\sin h)L^2\frac{\partial^2}{\partial L^2}(\cos E - e) + \\ &\quad 2(-\cos g\cos h + \frac{H}{G}\sin g\sin h)G\frac{\partial}{\partial L}(\sin E) + \\ &\quad (-\cos g\cos h + \frac{H}{G}\sin g\sin h)L\,G\frac{\partial^2}{\partial L^2}(\sin E),\end{aligned}$$

$$\begin{aligned}
\frac{\partial^3 y_a}{\partial L^2 \partial g} &= 2(-\sin g \sin h + \frac{H}{G}\cos g \cos h)(\cos E - e) + \\
&4(-\sin g \sin h + \frac{H}{G}\sin g \cos h) L \frac{\partial}{\partial L}(\cos E - e) + \\
&(-\sin g \sin h + \frac{H}{G}\cos g \cos h) L^2 \frac{\partial^2}{\partial L^2}(\cos E - e) + \\
&2(-\cos g \sin h - \frac{H}{G}\sin g \cos h) G \frac{\partial}{\partial L}(\sin E) + \\
&(-\cos g \sin h - \frac{H}{G}\sin g \cos h) L\, G \frac{\partial^2}{\partial L^2}(\sin E),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^3 x_a}{\partial L \partial g^2} &= 2(-\cos g \cos h + \frac{H}{G}\sin g \sin h) L(\cos E - e) + \\
&(-\cos g \cos h + \frac{H}{G}\sin g \sin h) L^2 \frac{\partial}{\partial L}(\cos E - e) + \\
&(\sin g \cos h + \frac{H}{G}\cos g \sin h) G \sin E + \\
&(\sin g \cos h + \frac{H}{G}\cos g \sin h) L\, G \frac{\partial}{\partial L}(\sin E),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^3 y_a}{\partial L \partial g^2} &= 2(-\cos g \sin h - \frac{H}{G}\sin g \cos h) L(\cos E - e) + \\
&(-\cos g \sin h - \frac{H}{G}\sin g \cos h) L^2 \frac{\partial}{\partial L}(\cos E - e) + \\
&(\sin g \sin h - \frac{H}{G}\cos g \cos h) L\, G \sin E + \\
&(\sin g \sin h - \frac{H}{G}\cos g \cos h) L\, G \frac{\partial}{\partial L}(\sin E),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 x_a}{\partial G \partial g} &= \frac{H}{G^2}\cos g \sin h\, L^2(\cos E - e) + (-\sin g \cos h - \frac{H}{G}\cos g \sin h) L^2 \frac{\partial}{\partial G}(\cos E - e) + \\
&- \cos g \cos h\, L \sin E + (-\cos g \cos h + \frac{H}{G}\sin g \sin h) L\, G \frac{\partial}{\partial G}(\sin E),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 y_a}{\partial G \partial g} &= -\frac{H}{G^2}\cos g \cos h\, L^2(\cos E - e) + (-\sin g \sin h + \frac{H}{G}\cos g \cos h) L^2 \frac{\partial}{\partial G}(\cos E - e) + \\
&- \cos g \sin h\, L \sin E + (-\cos g \sin h - \frac{H}{G}\sin g \cos h) L\, G \frac{\partial}{\partial G}(\sin E),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^3 x_a}{\partial G^2 \partial g} &= -2\frac{H}{G^3}\cos g \sin h\, L^2(\cos E - e) + 2\frac{H}{G^2}\cos g \sin h\, L^2 \frac{\partial}{\partial G}(\cos E - e) + \\
& (-\sin g \cos h - \frac{H}{G}\cos g \sin h) L^2 \frac{\partial^2}{\partial G^2}(\cos E - e) + \\
& -2\frac{H}{G}\sin g \sin h\, L \frac{\partial}{\partial G}(\sin E) + 2(-\cos g \cos h + \frac{H}{G}\sin g \sin h) L \frac{\partial}{\partial G}(\sin E) + \\
& (-\cos g \cos h + \frac{H}{G}\sin g \sin h) L\, G \frac{\partial^2}{\partial G^2}(\sin E),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^3 y_a}{\partial G^2 \partial g} &= 2\frac{H}{G^3}\cos g \cos h\, L^2(\cos E - e) - 2\frac{H}{G^2}\cos g \cos h\, L^2 \frac{\partial}{\partial G}(\cos E - e) + \\
& (-\sin g \sin h + \frac{H}{G}\cos g \cos h) L^2 \frac{\partial^2}{\partial G^2}(\cos E - e) + \\
& 2\frac{H}{G}\sin g \cos h\, L \frac{\partial}{\partial G}(\sin E) + 2(-\cos g \sin h - \frac{H}{G}\sin g \cos h) L \frac{\partial}{\partial G}(\sin E) + \\
& (-\cos g \sin h - \frac{H}{G}\sin g \cos h) L\, G \frac{\partial^2}{\partial G^2}(\sin E),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^3 x_a}{\partial G \partial g^2} &= -\frac{H}{G^2}\sin g \sin h\, L^2(\cos E - e) + (-\cos g \cos h + \frac{H}{G}\sin g \sin h) L^2 \frac{\partial}{\partial G}(\cos E - e) + \\
& \sin g \cos h\, L \sin E + (\sin g \cos h + \frac{H}{G}\cos g \sin h) L\, G \frac{\partial}{\partial G}(\sin E),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^3 y_a}{\partial G \partial g^2} &= \frac{H}{G^2}\cos g \cos h\, L^2(\cos E - e) + (-\cos g \sin h - \frac{H}{G}\cos g \cos h) L^2 \frac{\partial}{\partial G}(\cos E - e) + \\
& \sin g \sin h\, L \sin E + (\sin g \sin h - \frac{H}{G}\cos g \cos h) L\, G \frac{\partial}{\partial G}(\sin E),
\end{aligned}$$

$$\frac{\partial^2 x_a}{\partial H \partial g} = -\frac{1}{G}\cos g \sin h\, L^2(\cos E - e) + \sin g \sin h\, L \sin E,$$

$$\frac{\partial^2 y_a}{\partial H \partial g} = \frac{1}{G}\cos g \cos h\, L^2(\cos E - e) - \sin g \cos h\, L \sin E,$$

$$\frac{\partial^3 x_a}{\partial H \partial g^2} = \frac{1}{G}\sin g \sin h\, L^2(\cos E - e) + \cos g \sin h\, L \sin E,$$

$$\frac{\partial^3 y_a}{\partial H \partial g^2} = -\frac{1}{G}\sin g \cos h\, L^2(\cos E - e) - \cos g \cos h\, L \sin E,$$

$$\begin{aligned}
\frac{\partial^3 x_a}{\partial L \partial G \partial g} = {} & 2\frac{H}{G^2}\cos g \sin h\, L(\cos E - e) + \\
& 2(-\sin g \cos h - \frac{H}{G}\cos g \sin h)L\frac{\partial}{\partial G}(\cos E - e) + \\
& \frac{H}{G^2}\cos g \sin h\, L^2\frac{\partial}{\partial L}(\cos E - e) + \\
& (-\sin g \cos h - \frac{H}{G}\cos g \sin h)L^2\frac{\partial^2}{\partial L \partial G}(\cos E - e) + \\
& -\frac{H}{G}\sin g \sin h\; \sin E + (-\cos g \cos h + \frac{H}{G}\sin g \sin h)\sin E + \\
& (-\cos g \cos h + \frac{H}{G}\sin g \sin h)G\frac{\partial}{\partial G}(\sin E) - \frac{H}{G}\sin g \sin h\, L\frac{\partial}{\partial L}(\sin E) + \\
& (-\cos g \cos h + \frac{H}{G}\sin g \sin h)L\frac{\partial}{\partial L}(\sin E) + \\
& (-\cos g \cos h + \frac{H}{G}\sin g \sin h)L\, G\frac{\partial^2}{\partial L \partial G}(\sin E),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^3 y_a}{\partial L \partial G \partial g} = {} & -2\frac{H}{G^2}\cos g \cos h\, L(\cos E - e) + \\
& 2(-\sin g \sin h + \frac{H}{G}\cos g \cos h)L\frac{\partial}{\partial G}(\cos E - e) + \\
& -\frac{H}{G^2}\cos g \cos h\, L^2\frac{\partial}{\partial L}(\cos E - e) + \\
& (-\sin g \sin h + \frac{H}{G}\cos g \cos h)L^2\frac{\partial^2}{\partial L \partial G}(\cos E - e) + \\
& \frac{H}{G}\sin g \cos h\; \sin E + (-\cos g \sin h - \frac{H}{G}\sin g \cos h)\sin E + \\
& (-\cos g \sin h - \frac{H}{G}\sin g \cos h)G\frac{\partial}{\partial G}(\sin E) + \frac{H}{G}\sin g \cos h\, L\frac{\partial}{\partial L}(\sin E) + \\
& (-\cos g \sin h - \frac{H}{G}\sin g \cos h)L\frac{\partial}{\partial L}(\sin E) + \\
& (-\cos g \sin h - \frac{H}{G}\sin g \cos h)L\, G\frac{\partial^2}{\partial L \partial G}(\sin E)
\end{aligned}$$

$$\begin{aligned}\frac{\partial^3 x_a}{\partial L \partial H \partial g} &= -2\frac{L}{G}\cos g \sin h\,(\cos E - e) - \frac{L^2}{G}\cos g \sin h \frac{\partial}{\partial L}(\cos E - e) + \\ &\quad \sin g \sin h\, \sin E + \sin g \sin h\, L\frac{\partial}{\partial L}(\sin E),\end{aligned}$$

$$\begin{aligned}\frac{\partial^3 y_a}{\partial L \partial H \partial g} &= 2\frac{L}{G}\cos g \cos h\,(\cos E - e) + \frac{L^2}{G}\cos g \cos h \frac{\partial}{\partial L}(\cos E - e) + \\ &\quad - \sin g \cos h\, \sin E - \sin g \cos h\, L\frac{\partial}{\partial L}(\sin E),\end{aligned}$$

$$\begin{aligned}\frac{\partial^3 x_a}{\partial G \partial H \partial g} &= \frac{L^2}{G^2}\cos g \sin h\,(\cos E - e) - \frac{L^2}{G}\cos g \sin h \frac{\partial}{\partial G}(\cos E - e) + \\ &\quad \sin g \sin h\, L\frac{\partial}{\partial G}(\sin E),\end{aligned}$$

$$\begin{aligned}\frac{\partial^3 y_a}{\partial G \partial H \partial g} &= -\frac{L^2}{G^2}\cos g \cos h\,(\cos E - e) + \frac{L^2}{G}\cos g \cos h \frac{\partial}{\partial G}(\cos E - e) + \\ &\quad - \sin g \cos h\, L\frac{\partial}{\partial G}(\sin E).\end{aligned}$$

3.4.4 Izvodi četvrtog reda

Sledeći izvodi četvrtog reda se koriste za računanje (3.139) – (3.152), koji se pojavljuju u izrazima pri rešavanju varijacionih jednačina trećeg reda (3.170) i (3.171):

$$
\begin{aligned}
\frac{\partial^4 \mathcal{K}_0}{\partial G^4} = & -\frac{105}{\Delta^{9/2}}\left(r_a\frac{\partial r_a}{\partial G}-\frac{\partial x_a}{\partial G}x'-\frac{\partial y_a}{\partial G}y'-\frac{\partial z_a}{\partial G}z'\right)^4 + \\
& \frac{90}{\Delta^{7/2}}\left(r_a\frac{\partial r_a}{\partial G}-\frac{\partial x_a}{\partial G}x'-\frac{\partial y_a}{\partial G}y'-\frac{\partial z_a}{\partial G}z'\right)^2 \times \\
& \times\left[\left(\frac{\partial r_a}{\partial G}\right)^2 + r_a\frac{\partial^2 r_a}{\partial G^2}-\frac{\partial^2 x_a}{\partial G^2}x'-\frac{\partial^2 y_a}{\partial G^2}y'-\frac{\partial^2 z_a}{\partial G^2}z'\right] + \\
& -\frac{9}{\Delta^{5/2}}\left[\left(\frac{\partial r_a}{\partial G}\right)^2 + r_a\frac{\partial^2 r_a}{\partial G^2}-\frac{\partial^2 x_a}{\partial G^2}x'-\frac{\partial^2 y_a}{\partial G^2}y'-\frac{\partial^2 z_a}{\partial G^2}z'\right]^2 + \\
& -\frac{12}{\Delta^{5/2}}\left(r_a\frac{\partial r_a}{\partial G}-\frac{\partial x_a}{\partial G}x'-\frac{\partial y_a}{\partial G}y'-\frac{\partial z_a}{\partial G}z'\right) \times \qquad (3.73) \\
& \times\left[3\frac{\partial r_a}{\partial G}\frac{\partial^2 r_a}{\partial G^2} + r_a\frac{\partial^3 r_a}{\partial G^3}-\frac{\partial^3 x_a}{\partial G^3}x'-\frac{\partial^3 y_a}{\partial G^3}y'-\frac{\partial^3 z_a}{\partial G^3}z'\right] + \\
& \frac{1}{\Delta^{3/2}}\left[3\left(\frac{\partial^2 r_a}{\partial G^2}\right)^2 + 3\frac{\partial r_a}{\partial G}\frac{\partial^3 r_a}{\partial G^3} + r_a\frac{\partial^4 r_a}{\partial G^4}-\frac{\partial^4 x_a}{\partial G^4}x'-\frac{\partial^4 y_a}{\partial G^4}y'-\frac{\partial^4 z_a}{\partial G^4}z'\right] \\
& +\frac{1}{r'^3}\left(\frac{\partial^4 x_a}{\partial G^4}x'+\frac{\partial^4 y_a}{\partial G^4}y'+\frac{\partial^4 z_a}{\partial G^4}z'\right),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^4 \mathcal{K}_0}{\partial L \partial G^3} = & -\frac{105}{\Delta^{9/2}}\left(r_a\frac{\partial r_a}{\partial G}-\frac{\partial x_a}{\partial G}x'-\frac{\partial y_a}{\partial G}y'-\frac{\partial z_a}{\partial G}z'\right)^3 \times \\
& \left(r_a\frac{\partial r_a}{\partial L}-\frac{\partial x_a}{\partial L}x'-\frac{\partial y_a}{\partial L}y'-\frac{\partial z_a}{\partial L}z'\right)+ \\
& \frac{45}{\Delta^{7/2}}\left(r_a\frac{\partial r_a}{\partial G}-\frac{\partial x_a}{\partial G}x'-\frac{\partial y_a}{\partial G}y'-\frac{\partial z_a}{\partial G}z'\right)^2 \times \\
& \times\left[\frac{\partial r_a}{\partial L}\frac{\partial r_a}{\partial G}+r_a\frac{\partial^2 r_a}{\partial L\partial G}-\frac{\partial^2 x_a}{\partial L\partial G}x'-\frac{\partial^2 y_a}{\partial L\partial G}y'-\frac{\partial^2 z_a}{\partial L\partial G}z'\right]+ \\
& \frac{45}{\Delta^{7/2}}\left(r_a\frac{\partial r_a}{\partial G}-\frac{\partial x_a}{\partial G}x'-\frac{\partial y_a}{\partial G}y'-\frac{\partial z_a}{\partial G}z'\right)\times \\
& \times\left[\left(\frac{\partial r_a}{\partial G}\right)^2+r_a\frac{\partial^2 r_a}{\partial G^2}-\frac{\partial^2 x_a}{\partial G^2}x'-\frac{\partial^2 y_a}{\partial G^2}y'-\frac{\partial^2 z_a}{\partial G^2}z'\right]\times \\
& \times\left(r_a\frac{\partial r_a}{\partial L}-\frac{\partial x_a}{\partial L}x'-\frac{\partial y_a}{\partial L}y'-\frac{\partial z_a}{\partial L}z'\right)+ \\
& -\frac{9}{\Delta^{5/2}}\left[\frac{\partial r_a}{\partial L}\frac{\partial r_a}{\partial G}+r_a\frac{\partial^2 r_a}{\partial L\partial G}-\frac{\partial^2 x_a}{\partial L\partial G}x'-\frac{\partial^2 y_a}{\partial L\partial G}y'-\frac{\partial^2 z_a}{\partial L\partial G}z'\right]\times \\
& \left[\left(\frac{\partial r_a}{\partial G}\right)^2+r_a\frac{\partial^2 r_a}{\partial G^2}-\frac{\partial^2 x_a}{\partial G^2}x'-\frac{\partial^2 y_a}{\partial G^2}y'-\frac{\partial^2 z_a}{\partial G^2}z'\right]^2+ \\
& -\frac{9}{\Delta^{5/2}}\left(r_a\frac{\partial r_a}{\partial G}-\frac{\partial x_a}{\partial G}x'-\frac{\partial y_a}{\partial G}y'-\frac{\partial z_a}{\partial G}z'\right)\times \\
& \times\left[2\frac{\partial r_a}{\partial G}\frac{\partial^2 r_a}{\partial L\partial G}+\frac{\partial r_a}{\partial L}\frac{\partial^2 r_a}{\partial G^2}+r_a\frac{\partial^3 r_a}{\partial L\partial G^2}-\frac{\partial^3 x_a}{\partial L\partial G^2}x'-\frac{\partial^3 y_a}{\partial L\partial G^2}y'-\frac{\partial^3 z_a}{\partial L\partial G^2}z'\right]+ \\
& -\frac{3}{\Delta^{5/2}}\left[3\frac{\partial r_a}{\partial G}\frac{\partial^2 r_a}{\partial G^2}+r_a\frac{\partial^3 r_a}{\partial G^3}-\frac{\partial^3 x_a}{\partial G^3}x'-\frac{\partial^3 y_a}{\partial G^3}y'-\frac{\partial^3 z_a}{\partial G^3}z'\right]\times \\
& \left(r_a\frac{\partial r_a}{\partial L}-\frac{\partial x_a}{\partial L}x'-\frac{\partial y_a}{\partial L}y'-\frac{\partial z_a}{\partial L}z'\right)+ \\
& \frac{1}{\Delta^{3/2}}\Bigg(3\frac{\partial^2 r_a}{\partial L\partial G}\frac{\partial^2 r_a}{\partial G^2}+3\frac{\partial r_a}{\partial G}\frac{\partial^3 r_a}{\partial L\partial G^2}+ \\
& \frac{\partial r_a}{\partial L}\frac{\partial^3 r_a}{\partial G^3}+r_a\frac{\partial^4 r_a}{\partial L\partial G^3}-\frac{\partial^4 x_a}{\partial L\partial G^3}x'-\frac{\partial^4 y_a}{\partial L\partial G^3}y'-\frac{\partial^4 z_a}{\partial L\partial G^3}z'\Bigg]+ \\
& \frac{1}{r'^3}\left(\frac{\partial^4 x_a}{\partial L\partial G^3}x'+\frac{\partial^4 y_a}{\partial L\partial G^3}y'+\frac{\partial^4 z_a}{\partial L\partial G^3}z'\right),
\end{aligned}
\tag{3.74}
$$

$$\begin{aligned}
\frac{\partial^4\mathcal{K}_0}{\partial G^3\partial H} &= \frac{105}{\Delta^{9/2}}\left(r_a\frac{\partial r_a}{\partial G}-\frac{\partial x_a}{\partial G}x'-\frac{\partial y_a}{\partial G}y'-\frac{\partial z_a}{\partial G}z'\right)^3\left(\frac{\partial x_a}{\partial H}x'+\frac{\partial y_a}{\partial H}y'+\frac{\partial z_a}{\partial H}z'\right)+\\
&-\frac{45}{\Delta^{7/2}}\left(r_a\frac{\partial r_a}{\partial G}-\frac{\partial x_a}{\partial G}x'-\frac{\partial y_a}{\partial G}y'-\frac{\partial z_a}{\partial G}z'\right)^2\times\\
&\times\left(\frac{\partial^2 x_a}{\partial G\partial H}x'-\frac{\partial^2 y_a}{\partial G\partial H}y'-\frac{\partial^2 z_a}{\partial G\partial H}z'\right)+\\
&-\frac{45}{\Delta^{7/2}}\left(r_a\frac{\partial r_a}{\partial G}-\frac{\partial x_a}{\partial G}x'-\frac{\partial y_a}{\partial G}y'-\frac{\partial z_a}{\partial G}z'\right)\times\\
&\times\left[\left(\frac{\partial r_a}{\partial G}\right)^2+r_a\frac{\partial^2 r_a}{\partial G^2}-\frac{\partial^2 x_a}{\partial G^2}x'-\frac{\partial^2 y_a}{\partial G^2}y'-\frac{\partial^2 z_a}{\partial G^2}z'\right]\times\\
&\times\left(\frac{\partial x_a}{\partial H}x'+\frac{\partial y_a}{\partial H}y'+\frac{\partial z_a}{\partial H}z'\right)+\\
&\frac{9}{\Delta^{5/2}}\left(\frac{\partial^2 x_a}{\partial G\partial H}x'+\frac{\partial^2 y_a}{\partial G\partial H}y'+\frac{\partial^2 z_a}{\partial G\partial H}z'\right)\times\\
&\left[\left(\frac{\partial r_a}{\partial G}\right)^2+r_a\frac{\partial^2 r_a}{\partial G^2}-\frac{\partial^2 x_a}{\partial G^2}x'-\frac{\partial^2 y_a}{\partial G^2}y'-\frac{\partial^2 z_a}{\partial G^2}z'\right]+\\
&\frac{9}{\Delta^{5/2}}\left(r_a\frac{\partial r_a}{\partial G}-\frac{\partial x_a}{\partial G}x'-\frac{\partial y_a}{\partial G}y'-\frac{\partial z_a}{\partial G}z'\right)\times\\
&\times\left(\frac{\partial^3 x_a}{\partial G^2\partial H}x'+\frac{\partial^3 y_a}{\partial G^2\partial H}y'+\frac{\partial^3 z_a}{\partial G^2\partial H}z'\right)+\\
&\frac{3}{\Delta^{5/2}}\left[3\frac{\partial r_a}{\partial G}\frac{\partial^2 r_a}{\partial G^2}+r_a\frac{\partial^3 r_a}{\partial G^3}-\frac{\partial^3 x_a}{\partial G^3}x'-\frac{\partial^3 y_a}{\partial G^3}y'-\frac{\partial^3 z_a}{\partial G^3}z'\right]\times\\
&\left(\frac{\partial x_a}{\partial H}x'+\frac{\partial y_a}{\partial H}y'+\frac{\partial z_a}{\partial H}z'\right)+\\
&-\frac{1}{\Delta^{3/2}}\left(\frac{\partial^4 x_a}{\partial G^3\partial H}x'+\frac{\partial^4 y_a}{\partial G^3\partial H}y'+\frac{\partial^4 z_a}{\partial G^3\partial H}z'\right)+\\
&\frac{1}{r'^3}\left(\frac{\partial^4 x_a}{\partial G^3\partial H}x'+\frac{\partial^4 y_a}{\partial G^3\partial H}y'+\frac{\partial^4 z_a}{\partial G^3\partial H}z'\right),
\end{aligned} \tag{3.75}$$

$$
\begin{aligned}
\frac{\partial^4 \mathcal{K}_0}{\partial G^3 \partial g} = & -\frac{105}{\Delta^{9/2}} \left(r_a \frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G} x' - \frac{\partial y_a}{\partial G} y' - \frac{\partial z_a}{\partial G} z' \right)^3 \left(\frac{\partial x_a}{\partial g} x' + \frac{\partial y_a}{\partial g} y' + \frac{\partial z_a}{\partial g} z' \right) + \\
& -\frac{45}{\Delta^{7/2}} \left(r_a \frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G} x' - \frac{\partial y_a}{\partial G} y' - \frac{\partial z_a}{\partial G} z' \right)^2 \times \\
& \times \left(\frac{\partial^2 x_a}{\partial G \partial g} x' - \frac{\partial^2 y_a}{\partial G \partial g} y' - \frac{\partial^2 z_a}{\partial G \partial g} z' \right) + \\
& -\frac{45}{\Delta^{7/2}} \left(r_a \frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G} x' - \frac{\partial y_a}{\partial G} y' - \frac{\partial z_a}{\partial G} z' \right) \times \\
& \times \left[\left(\frac{\partial r_a}{\partial G} \right)^2 + r_a \frac{\partial^2 r_a}{\partial G^2} - \frac{\partial^2 x_a}{\partial G^2} x' - \frac{\partial^2 y_a}{\partial G^2} y' - \frac{\partial^2 z_a}{\partial G^2} z' \right] \times \\
& \times \left(\frac{\partial x_a}{\partial g} x' + \frac{\partial y_a}{\partial g} y' + \frac{\partial z_a}{\partial g} z' \right) + \\
& \frac{9}{\Delta^{5/2}} \left(\frac{\partial^2 x_a}{\partial G \partial g} x' + \frac{\partial^2 y_a}{\partial G \partial g} y' + \frac{\partial^2 z_a}{\partial G \partial g} z' \right) \times \\
& \left[\left(\frac{\partial r_a}{\partial G} \right)^2 + r_a \frac{\partial^2 r_a}{\partial G^2} - \frac{\partial^2 x_a}{\partial G^2} x' - \frac{\partial^2 y_a}{\partial G^2} y' - \frac{\partial^2 z_a}{\partial G^2} z' \right] + \\
& \frac{9}{\Delta^{5/2}} \left(r_a \frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G} x' - \frac{\partial y_a}{\partial G} y' - \frac{\partial z_a}{\partial G} z' \right) \times \\
& \times \left(\frac{\partial^3 x_a}{\partial G^2 \partial g} x' + \frac{\partial^3 y_a}{\partial G^2 \partial g} y' + \frac{\partial^3 z_a}{\partial G^2 \partial g} z' \right) + \\
& -\frac{3}{\Delta^{5/2}} \left[3 \frac{\partial r_a}{\partial G} \frac{\partial^2 r_a}{\partial G^2} + r_a \frac{\partial^3 r_a}{\partial G^3} - \frac{\partial^3 x_a}{\partial G^3} x' - \frac{\partial^3 y_a}{\partial G^3} y' - \frac{\partial^3 z_a}{\partial G^3} z' \right] \times \\
& \left(\frac{\partial x_a}{\partial g} x' + \frac{\partial y_a}{\partial g} y' + \frac{\partial z_a}{\partial g} z' \right) + \\
& -\frac{1}{\Delta^{3/2}} \left(\frac{\partial^4 x_a}{\partial G^3 \partial g} x' + \frac{\partial^4 y_a}{\partial G^3 \partial g} y' + \frac{\partial^4 z_a}{\partial G^3 \partial g} z' \right) + \\
& \frac{1}{r'^3} \left(\frac{\partial^4 x_a}{\partial G^3 \partial g} x' + \frac{\partial^4 y_a}{\partial G^3 \partial g} y' + \frac{\partial^4 z_a}{\partial G^3 \partial g} z' \right),
\end{aligned}
\tag{3.76}
$$

$$
\begin{aligned}
\frac{\partial^4 \mathcal{K}_0}{\partial G^2 \partial H^2} \;=\;& \frac{105}{\Delta^{9/2}} \left(r_a \frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G} x' - \frac{\partial y_a}{\partial G} y' - \frac{\partial z_a}{\partial G} z' \right)^2 \left(\frac{\partial x_a}{\partial H} x' + \frac{\partial y_a}{\partial H} y' + \frac{\partial z_a}{\partial H} z' \right)^2 + \\
& \frac{45}{\Delta^{7/2}} \left(r_a \frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G} x' - \frac{\partial y_a}{\partial G} y' - \frac{\partial z_a}{\partial G} z' \right) \left(\frac{\partial x_a}{\partial H} x' + \frac{\partial y_a}{\partial H} y' + \frac{\partial z_a}{\partial H} z' \right) \times \\
& \times \left(\frac{\partial^2 x_a}{\partial G \partial H} x' - \frac{\partial^2 y_a}{\partial G \partial H} y' - \frac{\partial^2 z_a}{\partial G \partial H} z' \right) + \\
& - \frac{3}{\Delta^{5/2}} \left(\frac{\partial^2 x_a}{\partial G \partial H} x' + \frac{\partial^2 y_a}{\partial G \partial H} y' + \frac{\partial^2 z_a}{\partial G \partial H} z' \right)^2 + \\
& - \frac{15}{\Delta^{7/2}} \left(\frac{\partial x_a}{\partial H} x' + \frac{\partial y_a}{\partial H} y' + \frac{\partial z_a}{\partial H} z' \right)^2 \\
& \times \left[\left(\frac{\partial r_a}{\partial G} \right)^2 + r_a \frac{\partial^2 r_a}{\partial G^2} - \frac{\partial^2 x_a}{\partial G^2} x' - \frac{\partial^2 y_a}{\partial G^2} y' - \frac{\partial^2 z_a}{\partial G^2} z' \right],
\end{aligned}
\tag{3.77}
$$

$$
\begin{aligned}
\frac{\partial^4 \mathcal{K}_0}{\partial L \partial G^2 \partial H} = & \frac{105}{\Delta^{9/2}} \left(r_a \frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G} x' - \frac{\partial y_a}{\partial G} y' - \frac{\partial z_a}{\partial G} z' \right)^2 \times \\
& \left(r_a \frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L} x' - \frac{\partial y_a}{\partial L} y' - \frac{\partial z_a}{\partial L} z' \right) \left(\frac{\partial x_a}{\partial H} x' + \frac{\partial y_a}{\partial H} y' + \frac{\partial z_a}{\partial H} z' \right) + \\
& \frac{45}{\Delta^{7/2}} \left(r_a \frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G} x' - \frac{\partial y_a}{\partial G} y' - \frac{\partial z_a}{\partial G} z' \right)^2 \times \\
& \times \left[\frac{\partial r_a}{\partial L} \frac{\partial r_a}{\partial G} + r_a \frac{\partial^2 r_a}{\partial L \partial G} - \frac{\partial^2 x_a}{\partial L \partial G} x' - \frac{\partial^2 y_a}{\partial L \partial G} y' - \frac{\partial^2 z_a}{\partial L \partial G} z' \right] + \\
& \frac{45}{\Delta^{7/2}} \left(r_a \frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G} x' - \frac{\partial y_a}{\partial G} y' - \frac{\partial z_a}{\partial G} z' \right) \times \\
& \times \left[\left(\frac{\partial r_a}{\partial G} \right)^2 + r_a \frac{\partial^2 r_a}{\partial G^2} - \frac{\partial^2 x_a}{\partial G^2} x' - \frac{\partial^2 y_a}{\partial G^2} y' - \frac{\partial^2 z_a}{\partial G^2} z' \right] \times \\
& \times \left(r_a \frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L} x' - \frac{\partial y_a}{\partial L} y' - \frac{\partial z_a}{\partial L} z' \right) + \\
& - \frac{9}{\Delta^{5/2}} \left[\frac{\partial r_a}{\partial L} \frac{\partial r_a}{\partial G} + r_a \frac{\partial^2 r_a}{\partial L \partial G} - \frac{\partial^2 x_a}{\partial L \partial G} x' - \frac{\partial^2 y_a}{\partial L \partial G} y' - \frac{\partial^2 z_a}{\partial L \partial G} z' \right] \times \\
& \left[\left(\frac{\partial r_a}{\partial G} \right)^2 + r_a \frac{\partial^2 r_a}{\partial G^2} - \frac{\partial^2 x_a}{\partial G^2} x' - \frac{\partial^2 y_a}{\partial G^2} y' - \frac{\partial^2 z_a}{\partial G^2} z' \right]^2 + \qquad (3.78) \\
& - \frac{9}{\Delta^{5/2}} \left(r_a \frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G} x' - \frac{\partial y_a}{\partial G} y' - \frac{\partial z_a}{\partial G} z' \right) \times \\
\times & \left[2 \frac{\partial r_a}{\partial G} \frac{\partial^2 r_a}{\partial L \partial G} + \frac{\partial r_a}{\partial L} \frac{\partial^2 r_a}{\partial G^2} + r_a \frac{\partial^3 r_a}{\partial L \partial G^2} - \frac{\partial^3 x_a}{\partial L \partial G^2} x' - \frac{\partial^3 y_a}{\partial L \partial G^2} y' - \frac{\partial^3 z_a}{\partial L \partial G^2} z' \right] + \\
& - \frac{3}{\Delta^{5/2}} \left[3 \frac{\partial r_a}{\partial G} \frac{\partial^2 r_a}{\partial G^2} + r_a \frac{\partial^3 r_a}{\partial G^3} - \frac{\partial^3 x_a}{\partial G^3} x' - \frac{\partial^3 y_a}{\partial G^3} y' - \frac{\partial^3 z_a}{\partial G^3} z' \right] \times \\
& \left(r_a \frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L} x' - \frac{\partial y_a}{\partial L} y' - \frac{\partial z_a}{\partial L} z' \right) + \\
& \frac{1}{\Delta^{3/2}} \left(3 \frac{\partial^2 r_a}{\partial L \partial G} \frac{\partial^2 r_a}{\partial G^2} + 3 \frac{\partial r_a}{\partial G} \frac{\partial^3 r_a}{\partial L \partial G^2} + \right. \\
& \left. \frac{\partial r_a}{\partial L} \frac{\partial^3 r_a}{\partial G^3} + r_a \frac{\partial^4 r_a}{\partial L \partial G^3} - \frac{\partial^4 x_a}{\partial L \partial G^3} x' - \frac{\partial^4 y_a}{\partial L \partial G^3} y' - \frac{\partial^4 z_a}{\partial L \partial G^3} z' \right] + \\
& \frac{1}{r'^3} \left(\frac{\partial^4 x_a}{\partial L \partial G^3} x' + \frac{\partial^4 y_a}{\partial L \partial G^3} y' + \frac{\partial^4 z_a}{\partial L \partial G^3} z' \right),
\end{aligned}
$$

$$\begin{aligned}
\frac{\partial^4 \mathcal{K}_0}{\partial L^2 \partial G^2} &= -\frac{105}{\Delta^{9/2}}\left(r_a\frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G}x' - \frac{\partial y_a}{\partial G}y' - \frac{\partial z_a}{\partial G}z'\right)^2 \times \\
& \left(r_a\frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L}x' - \frac{\partial y_a}{\partial L}y' - \frac{\partial z_a}{\partial L}z'\right)^2 + \\
& \frac{60}{\Delta^{7/2}}\left(r_a\frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G}x' - \frac{\partial y_a}{\partial G}y' - \frac{\partial z_a}{\partial G}z'\right)\left(r_a\frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L}x' - \frac{\partial y_a}{\partial L}y' - \frac{\partial z_a}{\partial L}z'\right) \times \\
& \times\left[\frac{\partial r_a}{\partial L}\frac{\partial r_a}{\partial G} + r_a\frac{\partial^2 r_a}{\partial L\partial G} - \frac{\partial^2 x_a}{\partial L\partial G}x' - \frac{\partial^2 y_a}{\partial L\partial G}y' - \frac{\partial^2 z_a}{\partial L\partial G}z'\right] + \\
& \frac{15}{\Delta^{7/2}}\left(r_a\frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L}x' - \frac{\partial y_a}{\partial L}y' - \frac{\partial z_a}{\partial L}z'\right)^2\left[\left(\frac{\partial r_a}{\partial G}\right)^2 + r_a\frac{\partial^2 r_a}{\partial G^2} - \frac{\partial^2 x_a}{\partial G^2}x' - \frac{\partial^2 y_a}{\partial G^2}y' - \frac{\partial^2 z_a}{\partial G^2}z'\right] + \\
& \frac{15}{\Delta^{7/2}}\left(r_a\frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G}x' - \frac{\partial y_a}{\partial G}y' - \frac{\partial z_a}{\partial G}z'\right)^2\left[\left(\frac{\partial r_a}{\partial L}\right)^2 + r_a\frac{\partial^2 r_a}{\partial L^2} - \frac{\partial^2 x_a}{\partial L^2}x' - \frac{\partial^2 y_a}{\partial L^2}y' - \frac{\partial^2 z_a}{\partial L^2}z'\right] + \\
& -\frac{6}{\Delta^{5/2}}\left(\frac{\partial r_a}{\partial L}\frac{\partial r_a}{\partial G} + r_a\frac{\partial^2 r_a}{\partial L\partial G} - \frac{\partial^2 x_a}{\partial L\partial G}x' - \frac{\partial^2 y_a}{\partial L\partial G}y' - \frac{\partial^2 z_a}{\partial L\partial G}z'\right)^2 + \\
& -\frac{6}{\Delta^{5/2}}\left(r_a\frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L}x' - \frac{\partial y_a}{\partial L}y' - \frac{\partial z_a}{\partial L}z'\right) \times \\
& \times\left[2\frac{\partial r_a}{\partial G}\frac{\partial^2 r_a}{\partial L\partial G} + \frac{\partial r_a}{\partial L}\frac{\partial^2 r_a}{\partial G^2} + r_a\frac{\partial^3 r_a}{\partial L\partial G^2} - \frac{\partial^3 x_a}{\partial L\partial G^2}x' - \frac{\partial^3 y_a}{\partial L\partial G^2}y' - \frac{\partial^3 z_a}{\partial L\partial G^2}z'\right] + \qquad (3.79) \\
& -\frac{3}{\Delta^{5/2}}\left[\left(\frac{\partial r_a}{\partial G}\right)^2 + r_a\frac{\partial^2 r_a}{\partial G^2} - \frac{\partial^2 x_a}{\partial G^2}x' - \frac{\partial^2 y_a}{\partial G^2}y' - \frac{\partial^2 z_a}{\partial G^2}z'\right] \times \\
& \times\left[\left(\frac{\partial r_a}{\partial L}\right)^2 + r_a\frac{\partial^2 r_a}{\partial L^2} - \frac{\partial^2 x_a}{\partial L^2}x' - \frac{\partial^2 y_a}{\partial L^2}y' - \frac{\partial^2 z_a}{\partial L^2}z'\right] + \\
& -\frac{6}{\Delta^{5/2}}\left(r_a\frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G}x' - \frac{\partial y_a}{\partial G}y' - \frac{\partial z_a}{\partial G}z'\right)^2 \times \\
& \times\left[2\frac{\partial r_a}{\partial L}\frac{\partial^2 r_a}{\partial L\partial G} + \frac{\partial r_a}{\partial G}\frac{\partial^2 r_a}{\partial L^2} + r_a\frac{\partial^3 r_a}{\partial L^2\partial G} - \frac{\partial^3 x_a}{\partial L^2\partial G}x' - \frac{\partial^3 y_a}{\partial L^2\partial G}y' - \frac{\partial^3 z_a}{\partial L^2\partial G}z'\right] + \\
& \frac{1}{\Delta^{3/2}}\left[\left(2\frac{\partial^2 r_a}{\partial L\partial G}\right)^2 + 2\frac{\partial r_a}{\partial L}\frac{\partial^3 r_a}{\partial L\partial G^2} + \frac{\partial^2 r_a}{\partial L^2}\frac{\partial^2 r_a}{\partial G^2} + 2\frac{\partial r_a}{\partial G}\frac{\partial^3 r_a}{\partial L^2\partial G} + \right. \\
& \left. r_a\frac{\partial^4 r_a}{\partial L^2\partial G^2} - \frac{\partial^4 x_a}{\partial L^2\partial G^2}x' - \frac{\partial^4 y_a}{\partial L^2\partial G^2}y' - \frac{\partial^4 z_a}{\partial L^2\partial G^2}z'\right] + \\
& \frac{1}{r'^3}\left(\frac{\partial^4 x_a}{\partial L^2\partial G^2}x' + \frac{\partial^4 y_a}{\partial L^2\partial G^2}y' + \frac{\partial^4 z_a}{\partial L^2\partial G^2}z'\right),
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^4 \mathcal{K}_0}{\partial L \partial G^2 \partial g} = \; & \frac{105}{\Delta^{9/2}} \left(r_a \frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G} x' - \frac{\partial y_a}{\partial G} y' - \frac{\partial z_a}{\partial G} z' \right)^2 \times \\
& \left(r_a \frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L} x' - \frac{\partial y_a}{\partial L} y' - \frac{\partial z_a}{\partial L} z' \right) \left(\frac{\partial x_a}{\partial g} x' + \frac{\partial y_a}{\partial g} y' + \frac{\partial z_a}{\partial g} z' \right) + \\
& - \frac{30}{\Delta^{7/2}} \left(r_a \frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G} x' - \frac{\partial y_a}{\partial G} y' - \frac{\partial z_a}{\partial G} z' \right) \times \\
& \times \left(r_a \frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L} x' - \frac{\partial y_a}{\partial L} y' - \frac{\partial z_a}{\partial L} z' \right) \left(\frac{\partial^2 x_a}{\partial G \partial g} x' + \frac{\partial^2 y_a}{\partial G \partial g} y' + \frac{\partial^2 z_a}{\partial G \partial g} z' \right) + \\
& - \frac{15}{\Delta^{7/2}} \left(r_a \frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G} x' - \frac{\partial y_a}{\partial G} y' - \frac{\partial z_a}{\partial G} z' \right)^2 \left(\frac{\partial^2 x_a}{\partial L \partial g} x' + \frac{\partial^2 y_a}{\partial L \partial g} y' + \frac{\partial^2 z_a}{\partial L \partial g} z' \right) + \\
& - \frac{30}{\Delta^{7/2}} \left(r_a \frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G} x' - \frac{\partial y_a}{\partial G} y' - \frac{\partial z_a}{\partial G} z' \right) \left(\frac{\partial x_a}{\partial g} x' + \frac{\partial y_a}{\partial g} y' + \frac{\partial z_a}{\partial g} z' \right) \times \\
& \times \left[\frac{\partial r_a}{\partial L} \frac{\partial r_a}{\partial G} + r_a \frac{\partial^2 r_a}{\partial L \partial G} - \frac{\partial^2 x_a}{\partial L \partial G} x' - \frac{\partial^2 y_a}{\partial L \partial G} y' - \frac{\partial^2 z_a}{\partial L \partial G} z' \right] + \\
& - \frac{15}{\Delta^{7/2}} \left[\left(\frac{\partial r_a}{\partial G} \right)^2 + r_a \frac{\partial^2 r_a}{\partial G^2} - \frac{\partial^2 x_a}{\partial G^2} x' - \frac{\partial^2 y_a}{\partial G^2} y' - \frac{\partial^2 z_a}{\partial G^2} z' \right] \times \\
& \left(r_a \frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L} x' - \frac{\partial y_a}{\partial L} y' - \frac{\partial z_a}{\partial L} z' \right) \left(\frac{\partial x_a}{\partial g} x' + \frac{\partial y_a}{\partial g} y' + \frac{\partial z_a}{\partial g} z' \right) + \\
& \frac{6}{\Delta^{5/2}} \left[\frac{\partial r_a}{\partial L} \frac{\partial r_a}{\partial G} + r_a \frac{\partial^2 r_a}{\partial L \partial G} - \frac{\partial^2 x_a}{\partial L \partial G} x' - \frac{\partial^2 y_a}{\partial L \partial G} y' - \frac{\partial^2 z_a}{\partial L \partial G} z' \right] \times \\
& \times \left(\frac{\partial^2 x_a}{\partial G \partial g} x' + \frac{\partial^2 y_a}{\partial G \partial g} y' + \frac{\partial^2 z_a}{\partial G \partial g} z' \right) + \\
& \frac{6}{\Delta^{5/2}} \left(r_a \frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G} x' - \frac{\partial y_a}{\partial G} y' - \frac{\partial z_a}{\partial G} z' \right) \left(\frac{\partial^3 x_a}{\partial L \partial G \partial g} x' + \frac{\partial^3 y_a}{\partial L \partial G \partial g} y' + \frac{\partial^3 z_a}{\partial L \partial G \partial g} z' \right) + \\
& \frac{3}{\Delta^{5/2}} \left(\frac{\partial^2 x_a}{\partial L \partial g} x' + \frac{\partial^2 y_a}{\partial L \partial g} y' + \frac{\partial^2 z_a}{\partial L \partial g} z' \right) \left[\left(\frac{\partial r_a}{\partial G} \right)^2 + r_a \frac{\partial^2 r_a}{\partial G^2} - \frac{\partial^2 x_a}{\partial G^2} x' - \frac{\partial^2 y_a}{\partial G^2} y' - \frac{\partial^2 z_a}{\partial G^2} z' \right] + \\
& \frac{3}{\Delta^{5/2}} \left(r_a \frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L} x' - \frac{\partial y_a}{\partial L} y' - \frac{\partial z_a}{\partial L} z' \right) \left(\frac{\partial^3 x_a}{\partial G^2 \partial g} x' + \frac{\partial^3 y_a}{\partial G^2 \partial g} y' + \frac{\partial^3 z_a}{\partial G^2 \partial g} z' \right) + \\
& \frac{3}{\Delta^{5/2}} \left[2 \frac{\partial r_a}{\partial G} \frac{\partial^2 r_a}{\partial L \partial G} + \frac{\partial r_a}{\partial L} \frac{\partial^2 r_a}{\partial G^2} + r_a \frac{\partial^3 r_a}{\partial L \partial G^2} - \frac{\partial^3 x_a}{\partial L \partial G^2} x' - \frac{\partial^3 y_a}{\partial L \partial G^2} y' - \frac{\partial^3 z_a}{\partial L \partial G^2} z' \right] \times \\
& \times \left(\frac{\partial x_a}{\partial g} x' + \frac{\partial y_a}{\partial g} y' + \frac{\partial z_a}{\partial g} z' \right) - \frac{1}{\Delta^{3/2}} \left(\frac{\partial^4 x_a}{\partial L \partial G^2 \partial g} x' + \frac{\partial^4 y_a}{\partial L \partial G^2 \partial g} y' + \frac{\partial^4 z_a}{\partial L \partial G^2 \partial g} z' \right] + \\
& \frac{1}{r'^3} \left(\frac{\partial^4 x_a}{\partial L \partial G^2 \partial g} x' + \frac{\partial^4 y_a}{\partial L \partial G^2 \partial g} y' + \frac{\partial^4 z_a}{\partial L \partial G^2 \partial g} z' \right),
\end{aligned}
\tag{3.80}
$$

$$
\begin{aligned}
\frac{\partial^4\mathcal{K}_0}{\partial G^2\partial H\partial g} = & -\frac{105}{\Delta^{9/2}}\left(r_a\frac{\partial r_a}{\partial G}-\frac{\partial x_a}{\partial G}x'-\frac{\partial y_a}{\partial G}y'-\frac{\partial z_a}{\partial G}z'\right)^2\times \\
& \left(\frac{\partial x_a}{\partial H}x'-\frac{\partial y_a}{\partial H}y'-\frac{\partial z_a}{\partial H}z'\right)\left(\frac{\partial x_a}{\partial g}x'+\frac{\partial y_a}{\partial g}y'+\frac{\partial z_a}{\partial g}z'\right)+ \\
& \frac{30}{\Delta^{7/2}}\left(r_a\frac{\partial r_a}{\partial G}-\frac{\partial x_a}{\partial G}x'-\frac{\partial y_a}{\partial G}y'-\frac{\partial z_a}{\partial G}z'\right)\times \\
& \times\left(\frac{\partial x_a}{\partial H}x'-\frac{\partial y_a}{\partial H}y'-\frac{\partial z_a}{\partial H}z'\right)\left(\frac{\partial^2 x_a}{\partial G\partial g}x'+\frac{\partial^2 y_a}{\partial G\partial g}y'+\frac{\partial^2 z_a}{\partial G\partial g}z'\right)+ \\
& -\frac{15}{\Delta^{7/2}}\left(r_a\frac{\partial r_a}{\partial G}-\frac{\partial x_a}{\partial G}x'-\frac{\partial y_a}{\partial G}y'-\frac{\partial z_a}{\partial G}z'\right)^2\left(\frac{\partial^2 x_a}{\partial H\partial g}x'+\frac{\partial^2 y_a}{\partial H\partial g}y'+\frac{\partial^2 z_a}{\partial H\partial g}z'\right)+ \\
& \frac{15}{\Delta^{7/2}}\left(r_a\frac{\partial r_a}{\partial G}-\frac{\partial x_a}{\partial G}x'-\frac{\partial y_a}{\partial G}y'-\frac{\partial z_a}{\partial G}z'\right)\left(\frac{\partial x_a}{\partial g}x'+\frac{\partial y_a}{\partial g}y'+\frac{\partial z_a}{\partial g}z'\right)\times \\
& \times\left(\frac{\partial^2 x_a}{\partial G\partial H}x'+\frac{\partial^2 y_a}{\partial G\partial H}y'+\frac{\partial^2 z_a}{\partial G\partial H}z'\right)+ \\
& \frac{15}{\Delta^{7/2}}\left(\frac{\partial x_a}{\partial H}x'-\frac{\partial y_a}{\partial H}y'-\frac{\partial z_a}{\partial H}z'\right)\left(\frac{\partial x_a}{\partial g}x'+\frac{\partial y_a}{\partial g}y'+\frac{\partial z_a}{\partial g}z'\right)\times \\
& \times\left[\left(\frac{\partial r_a}{\partial G}\right)^2+r_a\frac{\partial^2 r_a}{\partial G^2}-\frac{\partial^2 x_a}{\partial G^2}x'-\frac{\partial^2 y_a}{\partial G^2}y'-\frac{\partial^2 z_a}{\partial G^2}z'\right]+ \\
& -\frac{3}{\Delta^{5/2}}\left(\frac{\partial^2 x_a}{\partial G\partial H}x'+\frac{\partial^2 y_a}{\partial G\partial H}y'+\frac{\partial^2 z_a}{\partial G\partial H}z'\right)\left(\frac{\partial^2 x_a}{\partial G\partial g}x'+\frac{\partial^2 y_a}{\partial G\partial g}y'+\frac{\partial^2 z_a}{\partial G\partial g}z'\right)+ \\
& \frac{3}{\Delta^{5/2}}\left(r_a\frac{\partial r_a}{\partial G}-\frac{\partial x_a}{\partial G}x'-\frac{\partial y_a}{\partial G}y'-\frac{\partial z_a}{\partial G}z'\right)\left(\frac{\partial^3 x_a}{\partial G\partial H\partial g}x'+\frac{\partial^3 y_a}{\partial G\partial H\partial g}y'+\frac{\partial^3 z_a}{\partial G\partial H\partial g}z'\right)+ \\
& -\frac{3}{\Delta^{5/2}}\left(\frac{\partial x_a}{\partial H}x'+\frac{\partial y_a}{\partial H}y'+\frac{\partial z_a}{\partial H}z'\right)\left(\frac{\partial^3 x_a}{\partial G^2\partial g}x'+\frac{\partial^3 y_a}{\partial G^2\partial g}y'+\frac{\partial^3 z_a}{\partial G^2\partial g}z'\right)+ \\
& \frac{3}{\Delta^{5/2}}\left[\left(\frac{\partial r_a}{\partial G}\right)^2+r_a\frac{\partial^2 r_a}{\partial G^2}-\frac{\partial^2 x_a}{\partial G^2}x'-\frac{\partial^2 y_a}{\partial G^2}y'-\frac{\partial^2 z_a}{\partial G^2}z'\right]\left(\frac{\partial^2 x_a}{\partial H\partial g}x'+\frac{\partial^2 y_a}{\partial H\partial g}y'+\frac{\partial^2 z_a}{\partial H\partial g}z'\right)+ \\
& \frac{3}{\Delta^{5/2}}\left(\frac{\partial^3 x_a}{\partial G^2\partial H}x'+\frac{\partial^3 y_a}{\partial G^2\partial H}y'+\frac{\partial^3 z_a}{\partial G^2\partial H}z'\right)\left(\frac{\partial x_a}{\partial g}x'+\frac{\partial y_a}{\partial g}y'+\frac{\partial z_a}{\partial g}z'\right)+ \\
& -\frac{1}{\Delta^{3/2}}\left(\frac{\partial^4 x_a}{\partial G^2\partial H\partial g}x'+\frac{\partial^4 y_a}{\partial G^2\partial H\partial g}y'+\frac{\partial^4 z_a}{\partial G^2\partial H\partial g}z'\right]+ \\
& \frac{1}{r'^3}\left(\frac{\partial^4 x_a}{\partial G^2\partial H\partial g}x'+\frac{\partial^4 y_a}{\partial G^2\partial H\partial g}y'+\frac{\partial^4 z_a}{\partial G^2\partial H\partial g}z'\right),
\end{aligned}
\tag{3.81}
$$

$$
\begin{aligned}
\frac{\partial^4 \mathcal{K}_0}{\partial G^2 \partial g^2} \quad = \quad & -\frac{105}{\Delta^{9/2}} \left(r_a \frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G} x' - \frac{\partial y_a}{\partial G} y' - \frac{\partial z_a}{\partial G} z' \right)^2 \left(\frac{\partial x_a}{\partial g} x' + \frac{\partial y_a}{\partial g} y' + \frac{\partial z_a}{\partial g} z' \right)^2 + \\
& \frac{60}{\Delta^{7/2}} \left(r_a \frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G} x' - \frac{\partial y_a}{\partial G} y' - \frac{\partial z_a}{\partial G} z' \right) \times \\
\times & \left(\frac{\partial x_a}{\partial g} x' - \frac{\partial y_a}{\partial g} y' - \frac{\partial z_a}{\partial g} z' \right) \left(\frac{\partial^2 x_a}{\partial G \partial g} x' + \frac{\partial^2 y_a}{\partial G \partial g} y' + \frac{\partial^2 z_a}{\partial G \partial g} z' \right) + \\
& -\frac{15}{\Delta^{7/2}} \left(r_a \frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G} x' - \frac{\partial y_a}{\partial G} y' - \frac{\partial z_a}{\partial G} z' \right)^2 \left(\frac{\partial^2 x_a}{\partial g^2} x' + \frac{\partial^2 y_a}{\partial g^2} y' + \frac{\partial^2 z_a}{\partial g^2} z' \right) + \\
& \frac{15}{\Delta^{7/2}} \left(\frac{\partial x_a}{\partial g} x' + \frac{\partial y_a}{\partial g} y' + \frac{\partial z_a}{\partial g} z' \right)^2 \left[\left(\frac{\partial r_a}{\partial G} \right)^2 + r_a \frac{\partial^2 r_a}{\partial G^2} - \frac{\partial^2 x_a}{\partial G^2} x' - \frac{\partial^2 y_a}{\partial G^2} y' - \frac{\partial^2 z_a}{\partial G^2} z' \right] + \\
& -\frac{6}{\Delta^{5/2}} \left(\frac{\partial^2 x_a}{\partial G \partial g} x' + \frac{\partial^2 y_a}{\partial G \partial g} y' + \frac{\partial^2 z_a}{\partial G \partial g} z' \right)^2 + \\
& \frac{6}{\Delta^{5/2}} \left(r_a \frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G} x' - \frac{\partial y_a}{\partial G} y' - \frac{\partial z_a}{\partial G} z' \right) \left(\frac{\partial^3 x_a}{\partial G \partial g^2} x' + \frac{\partial^3 y_a}{\partial G \partial g^2} y' + \frac{\partial^3 z_a}{\partial G \partial g^2} z' \right) + \\
& -\frac{3}{\Delta^{5/2}} \left(\frac{\partial x_a}{\partial H} x' + \frac{\partial y_a}{\partial H} y' + \frac{\partial z_a}{\partial H} z' \right) \left(\frac{\partial^3 x_a}{\partial G^2 \partial g} x' + \frac{\partial^3 y_a}{\partial G^2 \partial g} y' + \frac{\partial^3 z_a}{\partial G^2 \partial g} z' \right) + \\
& \frac{3}{\Delta^{5/2}} \left[\left(\frac{\partial r_a}{\partial G} \right)^2 + r_a \frac{\partial^2 r_a}{\partial G^2} - \frac{\partial^2 x_a}{\partial G^2} x' - \frac{\partial^2 y_a}{\partial G^2} y' - \frac{\partial^2 z_a}{\partial G^2} z' \right] \left(\frac{\partial^2 x_a}{\partial g^2} x' + \frac{\partial^2 y_a}{\partial g^2} y' + \frac{\partial^2 z_a}{\partial g^2} z' \right) + \\
& -\frac{6}{\Delta^{5/2}} \left(\frac{\partial^3 x_a}{\partial G^2 \partial g} x' + \frac{\partial^3 y_a}{\partial G^2 \partial g} y' + \frac{\partial^3 z_a}{\partial G^2 \partial g} z' \right) \left(\frac{\partial x_a}{\partial g} x' + \frac{\partial y_a}{\partial g} y' + \frac{\partial z_a}{\partial g} z' \right) + \\
& -\frac{1}{\Delta^{3/2}} \left(\frac{\partial^4 x_a}{\partial G^2 \partial g^2} x' + \frac{\partial^4 y_a}{\partial G^2 \partial g^2} y' + \frac{\partial^4 z_a}{\partial G^2 \partial g^2} z' \right] + \\
& \frac{1}{r'^3} \left(\frac{\partial^4 x_a}{\partial G^2 \partial g^2} x' + \frac{\partial^4 y_a}{\partial G^2 \partial g^2} y' + \frac{\partial^4 z_a}{\partial G^2 \partial g^2} z' \right),
\end{aligned}
\tag{3.82}
$$

$$
\begin{aligned}
\frac{\partial^4 \mathcal{K}_0}{\partial L^2 \partial G \partial g} = {} & \frac{105}{\Delta^{9/2}} \left(r_a \frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G} x' - \frac{\partial y_a}{\partial G} y' - \frac{\partial z_a}{\partial G} z' \right) \times \\
& \left(r_a \frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L} x' - \frac{\partial y_a}{\partial L} y' - \frac{\partial z_a}{\partial L} z' \right)^2 \left(\frac{\partial x_a}{\partial g} x' + \frac{\partial y_a}{\partial g} y' + \frac{\partial z_a}{\partial g} z' \right) + \\
& - \frac{30}{\Delta^{7/2}} \left(r_a \frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G} x' - \frac{\partial y_a}{\partial G} y' - \frac{\partial z_a}{\partial G} z' \right) \times \\
\times & \left(r_a \frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L} x' - \frac{\partial y_a}{\partial L} y' - \frac{\partial z_a}{\partial L} z' \right) \left(\frac{\partial^2 x_a}{\partial G \partial g} x' + \frac{\partial^2 y_a}{\partial G \partial g} y' + \frac{\partial^2 z_a}{\partial G \partial g} z' \right) + \\
- & \frac{15}{\Delta^{7/2}} \left(r_a \frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L} x' - \frac{\partial y_a}{\partial L} y' - \frac{\partial z_a}{\partial L} z' \right)^2 \left(\frac{\partial^2 x_a}{\partial G \partial g} x' + \frac{\partial^2 y_a}{\partial G \partial g} y' + \frac{\partial^2 z_a}{\partial G \partial g} z' \right) + \\
- & \frac{30}{\Delta^{7/2}} \left(r_a \frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L} x' - \frac{\partial y_a}{\partial L} y' - \frac{\partial z_a}{\partial L} z' \right) \left(\frac{\partial x_a}{\partial g} x' + \frac{\partial y_a}{\partial g} y' + \frac{\partial z_a}{\partial g} z' \right) \times \\
\times & \left[\frac{\partial r_a}{\partial L} \frac{\partial r_a}{\partial G} + r_a \frac{\partial^2 r_a}{\partial L \partial G} - \frac{\partial^2 x_a}{\partial L \partial G} x' - \frac{\partial^2 y_a}{\partial L \partial G} y' - \frac{\partial^2 z_a}{\partial L \partial G} z' \right] + \\
- & \frac{15}{\Delta^{7/2}} \left[\left(\frac{\partial r_a}{\partial L} \right)^2 + r_a \frac{\partial^2 r_a}{\partial L^2} - \frac{\partial^2 x_a}{\partial L^2} x' - \frac{\partial^2 y_a}{\partial L^2} y' - \frac{\partial^2 z_a}{\partial L^2} z' \right] \times \\
& \left(r_a \frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G} x' - \frac{\partial y_a}{\partial G} y' - \frac{\partial z_a}{\partial G} z' \right) \left(\frac{\partial x_a}{\partial g} x' + \frac{\partial y_a}{\partial g} y' + \frac{\partial z_a}{\partial g} z' \right) + \\
& \frac{6}{\Delta^{5/2}} \left[\frac{\partial r_a}{\partial L} \frac{\partial r_a}{\partial G} + r_a \frac{\partial^2 r_a}{\partial L \partial G} - \frac{\partial^2 x_a}{\partial L \partial G} x' - \frac{\partial^2 y_a}{\partial L \partial G} y' - \frac{\partial^2 z_a}{\partial L \partial G} z' \right] \times \\
\times & \left(\frac{\partial^2 x_a}{\partial L \partial g} x' + \frac{\partial^2 y_a}{\partial L \partial g} y' + \frac{\partial^2 z_a}{\partial L \partial g} z' \right) + \\
& \frac{6}{\Delta^{5/2}} \left(r_a \frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L} x' - \frac{\partial y_a}{\partial L} y' - \frac{\partial z_a}{\partial L} z' \right) \left(\frac{\partial^3 x_a}{\partial L \partial G \partial g} x' + \frac{\partial^3 y_a}{\partial L \partial G \partial g} y' + \frac{\partial^3 z_a}{\partial L \partial G \partial g} z' \right) + \\
& \frac{3}{\Delta^{5/2}} \left(\frac{\partial^2 x_a}{\partial G \partial g} x' + \frac{\partial^2 y_a}{\partial G \partial g} y' + \frac{\partial^2 z_a}{\partial G \partial g} z' \right) \left[\left(\frac{\partial r_a}{\partial L} \right)^2 + r_a \frac{\partial^2 r_a}{\partial L^2} - \frac{\partial^2 x_a}{\partial L^2} x' - \frac{\partial^2 y_a}{\partial L^2} y' - \frac{\partial^2 z_a}{\partial L^2} z' \right] + \\
& \frac{3}{\Delta^{5/2}} \left(r_a \frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G} x' - \frac{\partial y_a}{\partial G} y' - \frac{\partial z_a}{\partial G} z' \right) \left(\frac{\partial^3 x_a}{\partial L^2 \partial g} x' + \frac{\partial^3 y_a}{\partial L^2 \partial g} y' + \frac{\partial^3 z_a}{\partial L^2 \partial g} z' \right) + \\
& \frac{3}{\Delta^{5/2}} \left[2 \frac{\partial r_a}{\partial G} \frac{\partial^2 r_a}{\partial L \partial G} + \frac{\partial r_a}{\partial L} \frac{\partial^2 r_a}{\partial G^2} + r_a \frac{\partial^3 r_a}{\partial L \partial G^2} - \frac{\partial^3 x_a}{\partial L \partial G^2} x' - \frac{\partial^3 y_a}{\partial L \partial G^2} y' - \frac{\partial^3 z_a}{\partial L \partial G^2} z' \right] \times \\
\times & \left(\frac{\partial x_a}{\partial g} x' + \frac{\partial y_a}{\partial g} y' + \frac{\partial z_a}{\partial g} z' \right) - \frac{1}{\Delta^{3/2}} \left(\frac{\partial^4 x_a}{\partial L^2 \partial G \partial g} x' + \frac{\partial^4 y_a}{\partial L^2 \partial G \partial g} y' + \frac{\partial^4 z_a}{\partial L^2 \partial G \partial g} z' \right] + \\
& \frac{1}{r'^3} \left(\frac{\partial^4 x_a}{\partial L^2 \partial G \partial g} x' + \frac{\partial^4 y_a}{\partial L^2 \partial G \partial g} y' + \frac{\partial^4 z_a}{\partial L^2 \partial G \partial g} z' \right),
\end{aligned}
\tag{3.83}
$$

$$
\begin{aligned}
\frac{\partial^4 \mathcal{K}_0}{\partial L \partial G \partial g^2} \; = \; & -\frac{105}{\Delta^{9/2}} \left(r_a \frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G} x' - \frac{\partial y_a}{\partial G} y' - \frac{\partial z_a}{\partial G} z' \right) \times \\
& \left(r_a \frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L} x' - \frac{\partial y_a}{\partial L} y' - \frac{\partial z_a}{\partial L} z' \right) \left(\frac{\partial x_a}{\partial g} x' + \frac{\partial y_a}{\partial g} y' + \frac{\partial z_a}{\partial g} z' \right)^2 + \\
& \frac{30}{\Delta^{7/2}} \left(r_a \frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G} x' - \frac{\partial y_a}{\partial G} y' - \frac{\partial z_a}{\partial G} z' \right) \times \\
& \left(\frac{\partial^2 x_a}{\partial L \partial g} x' + \frac{\partial^2 y_a}{\partial L \partial g} y' + \frac{\partial^2 z_a}{\partial L \partial g} z' \right) \left(\frac{\partial x_a}{\partial g} x' + \frac{\partial y_a}{\partial g} y' + \frac{\partial z_a}{\partial g} z' \right) + \\
& \frac{30}{\Delta^{7/2}} \left(r_a \frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L} x' - \frac{\partial y_a}{\partial L} y' - \frac{\partial z_a}{\partial L} z' \right) \times \\
& \times \left(\frac{\partial^2 x_a}{\partial G \partial g} x' + \frac{\partial^2 y_a}{\partial G \partial g} y' + \frac{\partial^2 z_a}{\partial G \partial g} z' \right) \left(\frac{\partial x_a}{\partial g} x' + \frac{\partial y_a}{\partial g} y' + \frac{\partial z_a}{\partial g} z' \right) + \\
& -\frac{15}{\Delta^{7/2}} \left(r_a \frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L} x' - \frac{\partial y_a}{\partial L} y' - \frac{\partial z_a}{\partial L} z' \right) \left(\frac{\partial^2 x_a}{\partial g^2} x' + \frac{\partial^2 y_a}{\partial g^2} y' + \frac{\partial^2 z_a}{\partial g^2} z' \right) \times \\
& \times \left(r_a \frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G} x' - \frac{\partial y_a}{\partial G} y' - \frac{\partial z_a}{\partial G} z' \right) + \\
& \frac{15}{\Delta^{7/2}} \left(\frac{\partial x_a}{\partial g} x' + \frac{\partial y_a}{\partial g} y' + \frac{\partial z_a}{\partial g} z' \right)^2 \left[\frac{\partial r_a}{\partial L} \frac{\partial r_a}{\partial G} + r_a \frac{\partial^2 r_a}{\partial L \partial G} - \frac{\partial^2 x_a}{\partial L \partial G} x' - \frac{\partial^2 y_a}{\partial L \partial G} y' - \frac{\partial^2 z_a}{\partial L \partial G} z' \right] + \\
& -\frac{6}{\Delta^{5/2}} \left(\frac{\partial^2 x_a}{\partial G \partial g} x' + \frac{\partial^2 y_a}{\partial G \partial g} y' + \frac{\partial^2 z_a}{\partial G \partial g} z' \right) \left(\frac{\partial^2 x_a}{\partial L \partial g} x' + \frac{\partial^2 y_a}{\partial L \partial g} y' + \frac{\partial^2 z_a}{\partial L \partial g} z' \right) + \\
& \frac{3}{\Delta^{5/2}} \left(r_a \frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G} x' - \frac{\partial y_a}{\partial G} y' - \frac{\partial z_a}{\partial G} z' \right) \left(\frac{\partial^3 x_a}{\partial L \partial g^2} x' + \frac{\partial^3 y_a}{\partial L \partial g^2} y' + \frac{\partial^3 z_a}{\partial L \partial g^2} z' \right) + \\
& \frac{3}{\Delta^{5/2}} \left(r_a \frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L} x' - \frac{\partial y_a}{\partial L} y' - \frac{\partial z_a}{\partial L} z' \right) \left(\frac{\partial^3 x_a}{\partial G \partial g^2} x' + \frac{\partial^3 y_a}{\partial G \partial g^2} y' + \frac{\partial^3 z_a}{\partial G \partial g^2} z' \right) + \\
& \frac{3}{\Delta^{5/2}} \left[\frac{\partial r_a}{\partial L} \frac{\partial r_a}{\partial G} + r_a \frac{\partial^2 r_a}{\partial L \partial G} - \frac{\partial^2 x_a}{\partial L \partial G} x' - \frac{\partial^2 y_a}{\partial L \partial G} y' - \frac{\partial^2 z_a}{\partial L \partial G} z' \right] \left(\frac{\partial^2 x_a}{\partial g^2} x' + \frac{\partial^2 y_a}{\partial g^2} y' + \frac{\partial^2 z_a}{\partial g^2} z' \right) + \\
& -\frac{6}{\Delta^{5/2}} \left(\frac{\partial x_a}{\partial g} x' + \frac{\partial y_a}{\partial g} y' + \frac{\partial z_a}{\partial g} z' \right) \left(\frac{\partial^3 x_a}{\partial L \partial G \partial g} x' + \frac{\partial^3 y_a}{\partial L \partial G \partial g} y' + \frac{\partial^3 z_a}{\partial L \partial G \partial g} z' \right) + \\
& -\frac{1}{\Delta^{3/2}} \left(\frac{\partial^4 x_a}{\partial L \partial G \partial g^2} x' + \frac{\partial^4 y_a}{\partial L \partial G \partial g^2} y' + \frac{\partial^4 z_a}{\partial L \partial G \partial g^2} z' \right) + \\
& \frac{1}{r'^3} \left(\frac{\partial^4 x_a}{\partial L \partial G \partial g^2} x' + \frac{\partial^4 y_a}{\partial L \partial G \partial g^2} y' + \frac{\partial^4 z_a}{\partial L \partial G \partial g^2} z' \right),
\end{aligned} \tag{3.84}
$$

$$
\begin{aligned}
\frac{\partial^4 \mathcal{K}_0}{\partial L\partial G\partial H\partial g} = & -\frac{105}{\Delta^{9/2}}\left(r_a\frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L}x' - \frac{\partial y_a}{\partial L}y' - \frac{\partial z_a}{\partial L}z'\right)\left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right)\times \\
& \times\left(r_a\frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G}x' - \frac{\partial y_a}{\partial G}y' - \frac{\partial z_a}{\partial G}z'\right)\left(\frac{\partial x_a}{\partial H}x' - \frac{\partial y_a}{\partial H}y' - \frac{\partial z_a}{\partial H}z'\right)+ \\
& \frac{15}{\Delta^{7/2}}\left(r_a\frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G}x' - \frac{\partial y_a}{\partial G}y' - \frac{\partial z_a}{\partial G}z'\right)\left(\frac{\partial x_a}{\partial H}x' - \frac{\partial y_a}{\partial H}y' - \frac{\partial z_a}{\partial H}z'\right)\times \\
& \times\left(\frac{\partial^2 x_a}{\partial L\partial g}x' + \frac{\partial^2 y_a}{\partial L\partial g}y' + \frac{\partial^2 z_a}{\partial L\partial g}z'\right) + \frac{15}{\Delta^{7/2}}\left(r_a\frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L}x' - \frac{\partial y_a}{\partial L}y' - \frac{\partial z_a}{\partial L}z'\right)\times \\
& \times\left(\frac{\partial^2 x_a}{\partial G\partial g}x' + \frac{\partial^2 y_a}{\partial G\partial g}y' + \frac{\partial^2 z_a}{\partial G\partial g}z'\right)\left(\frac{\partial x_a}{\partial H}x' - \frac{\partial y_a}{\partial H}y' - \frac{\partial z_a}{\partial H}z'\right)+ \\
& -\frac{15}{\Delta^{7/2}}\left(r_a\frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L}x' - \frac{\partial y_a}{\partial L}y' - \frac{\partial z_a}{\partial L}z'\right)\left(r_a\frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G}x' - \frac{\partial y_a}{\partial G}y' - \frac{\partial z_a}{\partial G}z'\right)\times \\
& \times\left(\frac{\partial^2 x_a}{\partial H\partial g}x' + \frac{\partial^2 y_a}{\partial H\partial g}y' + \frac{\partial^2 z_a}{\partial H\partial g}z'\right) + \frac{15}{\Delta^{7/2}}\left(r_a\frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G}x' - \frac{\partial y_a}{\partial G}y' - \frac{\partial z_a}{\partial G}z'\right)\times \\
& \times\left(\frac{\partial^2 x_a}{\partial L\partial H}x' + \frac{\partial^2 y_a}{\partial L\partial H}y' + \frac{\partial^2 z_a}{\partial L\partial H}z'\right)\left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right)+ \\
& \frac{15}{\Delta^{7/2}}\left(r_a\frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L}x' - \frac{\partial y_a}{\partial L}y' - \frac{\partial z_a}{\partial L}z'\right)\left(\frac{\partial^2 x_a}{\partial G\partial H}x' + \frac{\partial^2 y_a}{\partial G\partial H}y' + \frac{\partial^2 z_a}{\partial G\partial H}z'\right)\times \\
& \times\left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right) + \frac{15}{\Delta^{7/2}}\left(\frac{\partial x_a}{\partial H}x' - \frac{\partial y_a}{\partial H}y' - \frac{\partial z_a}{\partial H}z'\right)\left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right)\times \\
& \times\left[\frac{\partial r_a}{\partial L}\frac{\partial r_a}{\partial G} + r_a\frac{\partial^2 r_a}{\partial L\partial G} - \frac{\partial^2 x_a}{\partial L\partial G}x' - \frac{\partial^2 y_a}{\partial L\partial G}y' - \frac{\partial^2 z_a}{\partial L\partial G}z'\right]+ \\
& -\frac{3}{\Delta^{5/2}}\left(\frac{\partial^2 x_a}{\partial L\partial H}x' + \frac{\partial^2 y_a}{\partial L\partial H}y' + \frac{\partial^2 z_a}{\partial L\partial H}z'\right)\left(\frac{\partial^2 x_a}{\partial G\partial g}x' + \frac{\partial^2 y_a}{\partial G\partial g}y' + \frac{\partial^2 z_a}{\partial G\partial g}z'\right)+ \\
& \frac{3}{\Delta^{5/2}}\left(r_a\frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G}x' - \frac{\partial y_a}{\partial G}y' - \frac{\partial z_a}{\partial G}z'\right)\left(\frac{\partial^3 x_a}{\partial L\partial H\partial g}x' + \frac{\partial^3 y_a}{\partial L\partial H\partial g}y' + \frac{\partial^3 z_a}{\partial L\partial H\partial g}z'\right)+ \\
& -\frac{3}{\Delta^{5/2}}\left(\frac{\partial^2 x_a}{\partial L\partial g}x' + \frac{\partial^2 y_a}{\partial L\partial g}y' + \frac{\partial^2 z_a}{\partial L\partial g}z'\right)\left(\frac{\partial^2 x_a}{\partial G\partial H}x' + \frac{\partial^2 y_a}{\partial G\partial H}y' + \frac{\partial^2 z_a}{\partial G\partial H}z'\right)+ \\
& \frac{3}{\Delta^{5/2}}\left(r_a\frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L}x' - \frac{\partial y_a}{\partial L}y' - \frac{\partial z_a}{\partial L}z'\right)\left(\frac{\partial^3 x_a}{\partial G\partial H\partial g}x' + \frac{\partial^3 y_a}{\partial G\partial H\partial g}y' + \frac{\partial^3 z_a}{\partial G\partial H\partial g}z'\right)+ \\
& \frac{3}{\Delta^{5/2}}\left[\frac{\partial r_a}{\partial L}\frac{\partial r_a}{\partial G} + r_a\frac{\partial^2 r_a}{\partial L\partial G} - \frac{\partial^2 x_a}{\partial L\partial G}x' - \frac{\partial^2 y_a}{\partial L\partial G}y' - \frac{\partial^2 z_a}{\partial L\partial G}z'\right]\times \\
& \times\left(\frac{\partial^2 x_a}{\partial H\partial g}x' + \frac{\partial^2 y_a}{\partial H\partial g}y' + \frac{\partial^2 z_a}{\partial H\partial g}z'\right)+ \\
& -\frac{3}{\Delta^{5/2}}\left(\frac{\partial^3 x_a}{\partial L\partial G\partial g}x' + \frac{\partial^3 y_a}{\partial L\partial G\partial g}y' + \frac{\partial^3 z_a}{\partial L\partial G\partial g}z'\right)\left(\frac{\partial x_a}{\partial H}x' - \frac{\partial y_a}{\partial H}y' - \frac{\partial z_a}{\partial H}z'\right)+ \\
& -\frac{3}{\Delta^{5/2}}\left(\frac{\partial^3 x_a}{\partial L\partial G\partial H}x' + \frac{\partial^3 y_a}{\partial L\partial G\partial H}y' + \frac{\partial^3 z_a}{\partial L\partial G\partial H}z'\right)\left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right)+ \\
& \left(\frac{1}{r'^3} - \frac{3}{\Delta^{3/2}}\right)\left(\frac{\partial^4 x_a}{\partial L\partial G\partial H\partial g}x' + \frac{\partial^4 y_a}{\partial L\partial G\partial H\partial g}y' + \frac{\partial^4 z_a}{\partial L\partial G\partial H\partial g}z'\right)
\end{aligned}
\tag{3.85}
$$

$$
\begin{aligned}
\frac{\partial^4 \mathcal{K}_0}{\partial G \partial H \partial g^2} = & \frac{105}{\Delta^{9/2}}\left(r_a \frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G}x' - \frac{\partial y_a}{\partial G}y' - \frac{\partial z_a}{\partial G}z'\right) \times \\
& \times \left(\frac{\partial x_a}{\partial H}x' - \frac{\partial y_a}{\partial H}y' - \frac{\partial z_a}{\partial H}z'\right)\left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right)^2 + \\
& - \frac{30}{\Delta^{7/2}}\left(\frac{\partial x_a}{\partial H}x' - \frac{\partial y_a}{\partial H}y' - \frac{\partial z_a}{\partial H}z'\right)\left(\frac{\partial x_a}{\partial g}x' - \frac{\partial y_a}{\partial g}y' - \frac{\partial z_a}{\partial g}z'\right) \times \\
& \times \left(\frac{\partial^2 x_a}{\partial G \partial g}x' + \frac{\partial^2 y_a}{\partial G \partial g}y' + \frac{\partial^2 z_a}{\partial G \partial g}z'\right) + \frac{30}{\Delta^{7/2}}\left(\frac{\partial^2 x_a}{\partial H \partial g}x' + \frac{\partial^2 y_a}{\partial H \partial g}y' + \frac{\partial^2 z_a}{\partial H \partial g}z'\right) \times \\
& \times \left(r_a \frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G}x' - \frac{\partial y_a}{\partial G}y' - \frac{\partial z_a}{\partial G}z'\right)\left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right) + \\
& \frac{15}{\Delta^{7/2}}\left(r_a \frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G}x' - \frac{\partial y_a}{\partial G}y' - \frac{\partial z_a}{\partial G}z'\right)\left(\frac{\partial x_a}{\partial H}x' - \frac{\partial y_a}{\partial H}y' - \frac{\partial z_a}{\partial H}z'\right) \times \\
& \times \left(\frac{\partial^2 x_a}{\partial g^2}x' + \frac{\partial^2 y_a}{\partial g^2}y' + \frac{\partial^2 z_a}{\partial g^2}z'\right) - \frac{15}{\Delta^{7/2}}\left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right)^2 \times \\
& \times \left(\frac{\partial^2 x_a}{\partial G \partial H}x' + \frac{\partial^2 y_a}{\partial G \partial H}y' + \frac{\partial^2 z_a}{\partial G \partial H}z'\right) - \frac{6}{\Delta^{5/2}}\left(\frac{\partial^2 x_a}{\partial G \partial g}x' + \frac{\partial^2 y_a}{\partial G \partial g}y' + \frac{\partial^2 z_a}{\partial G \partial g}z'\right) \times \\
& \times \left(\frac{\partial^2 x_a}{\partial H \partial g}x' + \frac{\partial^2 y_a}{\partial H \partial g}y' + \frac{\partial^2 z_a}{\partial H \partial g}z'\right) + \frac{3}{\Delta^{5/2}}\left(r_a \frac{\partial r_a}{\partial G} - \frac{\partial x_a}{\partial G}x' - \frac{\partial y_a}{\partial G}y' - \frac{\partial z_a}{\partial G}z'\right) \times \\
& \times \left(\frac{\partial^3 x_a}{\partial H \partial g^2}x' + \frac{\partial^3 y_a}{\partial H \partial g^2}y' + \frac{\partial^3 z_a}{\partial H \partial g^2}z'\right) - \frac{3}{\Delta^{5/2}}\left(\frac{\partial x_a}{\partial H}x' + \frac{\partial y_a}{\partial H}y' + \frac{\partial z_a}{\partial H}z'\right) \times \\
& \times \left(\frac{\partial^3 x_a}{\partial G \partial g^2}x' + \frac{\partial^3 y_a}{\partial G \partial g^2}y' + \frac{\partial^3 z_a}{\partial G \partial g^2}z'\right) - \frac{3}{\Delta^{5/2}}\left(\frac{\partial^2 x_a}{\partial g^2}x' + \frac{\partial^2 y_a}{\partial g^2}y' + \frac{\partial^2 z_a}{\partial g^2}z'\right) \times \\
& \times \left(\frac{\partial^2 x_a}{\partial G \partial H}x' + \frac{\partial^2 y_a}{\partial G \partial H}y' + \frac{\partial^2 z_a}{\partial G \partial H}z'\right) - \frac{6}{\Delta^{5/2}}\left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right) \times \\
& \times \left(\frac{\partial^3 x_a}{\partial G \partial H \partial g}x' + \frac{\partial^3 y_a}{\partial G \partial H \partial g}y' + \frac{\partial^3 z_a}{\partial G \partial H \partial g}z'\right) + \\
& - \frac{1}{\Delta^{3/2}}\left(\frac{\partial^4 x_a}{\partial G \partial H \partial g^2}x' + \frac{\partial^4 y_a}{\partial G \partial H \partial g^2}y' + \frac{\partial^4 z_a}{\partial G \partial H \partial g^2}z'\right] + \\
& \frac{1}{r'^3}\left(\frac{\partial^4 x_a}{\partial G \partial H \partial g^2}x' + \frac{\partial^4 y_a}{\partial G \partial H \partial g^2}y' + \frac{\partial^4 z_a}{\partial G \partial H \partial g^2}z'\right),
\end{aligned} \tag{3.86}
$$

$$\begin{aligned}
\frac{\partial^4\mathcal{K}_0}{\partial G\partial g^3} &= \frac{105}{\Delta^{9/2}}\left(r_a\frac{\partial r_a}{\partial G}-\frac{\partial x_a}{\partial G}x'-\frac{\partial y_a}{\partial G}y'-\frac{\partial z_a}{\partial G}z'\right)\times\\
&\times\left(\frac{\partial x_a}{\partial g}x'+\frac{\partial y_a}{\partial g}y'+\frac{\partial z_a}{\partial g}z'\right)^3+\frac{45}{\Delta^{7/2}}\left(\frac{\partial x_a}{\partial g}x'+\frac{\partial y_a}{\partial g}y'+\frac{\partial z_a}{\partial g}z'\right)\times\\
&\times\left(r_a\frac{\partial r_a}{\partial G}-\frac{\partial x_a}{\partial G}x'-\frac{\partial y_a}{\partial G}y'-\frac{\partial z_a}{\partial G}z'\right)\left(\frac{\partial^2 x_a}{\partial g^2}x'+\frac{\partial^2 y_a}{\partial g^2}y'+\frac{\partial^2 z_a}{\partial g^2}z'\right)+\\
&-\frac{45}{\Delta^{7/2}}\left(\frac{\partial x_a}{\partial g}x'+\frac{\partial y_a}{\partial g}y'+\frac{\partial z_a}{\partial g}z'\right)^2\left(\frac{\partial^2 x_a}{\partial G\partial g}x'+\frac{\partial^2 y_a}{\partial G\partial g}y'+\frac{\partial^2 z_a}{\partial G\partial g}z'\right)+\\
&-\frac{9}{\Delta^{5/2}}\left(\frac{\partial x_a}{\partial g}x'+\frac{\partial y_a}{\partial g}y'+\frac{\partial z_a}{\partial g}z'\right)\left(\frac{\partial^3 x_a}{\partial G\partial g^2}x'+\frac{\partial^3 y_a}{\partial G\partial g^2}y'+\frac{\partial^3 z_a}{\partial G\partial g^2}z'\right)+\\
&-\frac{9}{\Delta^{5/2}}\left(\frac{\partial^2 x_a}{\partial g^2}x'+\frac{\partial^2 y_a}{\partial g^2}y'+\frac{\partial^2 z_a}{\partial g^2}z'\right)\left(\frac{\partial^2 x_a}{\partial G\partial g}x'+\frac{\partial^2 y_a}{\partial G\partial g}y'+\frac{\partial^2 z_a}{\partial G\partial g}z'\right)+\\
&\frac{3}{\Delta^{5/2}}\left(\frac{\partial^3 x_a}{\partial g^3}x'+\frac{\partial^3 y_a}{\partial g^3}y'+\frac{\partial^3 z_a}{\partial g^3}z'\right)\left(r_a\frac{\partial r_a}{\partial G}-\frac{\partial x_a}{\partial G}x'-\frac{\partial y_a}{\partial G}y'-\frac{\partial z_a}{\partial G}z'\right)+\\
&-\frac{1}{\Delta^{3/2}}\left(\frac{\partial^4 x_a}{\partial G\partial g^3}x'+\frac{\partial^4 y_a}{\partial G\partial g^3}y'+\frac{\partial^4 z_a}{\partial G\partial g^3}z'\right]+\\
&\frac{1}{r'^3}\left(\frac{\partial^4 x_a}{\partial G\partial g^3}x'+\frac{\partial^4 y_a}{\partial G\partial g^3}y'+\frac{\partial^4 z_a}{\partial G\partial g^3}z'\right]
\end{aligned} \tag{3.87}$$

$$
\begin{aligned}
\frac{\partial^4 \mathcal{K}_0}{\partial L^3 \partial g} = & -\frac{105}{\Delta^{9/2}}\left(r_a\frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L}x' - \frac{\partial y_a}{\partial L}y' - \frac{\partial z_a}{\partial L}z'\right)^3 \left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right) + \\
& -\frac{45}{\Delta^{7/2}}\left(r_a\frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L}x' - \frac{\partial y_a}{\partial L}y' - \frac{\partial z_a}{\partial L}z'\right)^2 \times \\
& \times \left(\frac{\partial^2 x_a}{\partial L \partial g}x' - \frac{\partial^2 y_a}{\partial L \partial g}y' - \frac{\partial^2 z_a}{\partial L \partial g}z'\right) + \\
& -\frac{45}{\Delta^{7/2}}\left(r_a\frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L}x' - \frac{\partial y_a}{\partial L}y' - \frac{\partial z_a}{\partial L}z'\right) \times \\
& \times \left[\left(\frac{\partial r_a}{\partial L}\right)^2 + r_a\frac{\partial^2 r_a}{\partial L^2} - \frac{\partial^2 x_a}{\partial L^2}x' - \frac{\partial^2 y_a}{\partial L^2}y' - \frac{\partial^2 z_a}{\partial L^2}z'\right] \times \\
& \times \left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right) + \\
& \frac{9}{\Delta^{5/2}}\left(\frac{\partial^2 x_a}{\partial L \partial g}x' + \frac{\partial^2 y_a}{\partial L \partial g}y' + \frac{\partial^2 z_a}{\partial L \partial g}z'\right) \times \\
& \left[\left(\frac{\partial r_a}{\partial L}\right)^2 + r_a\frac{\partial^2 r_a}{\partial L^2} - \frac{\partial^2 x_a}{\partial L^2}x' - \frac{\partial^2 y_a}{\partial L^2}y' - \frac{\partial^2 z_a}{\partial L^2}z'\right] + \qquad (3.88) \\
& \frac{9}{\Delta^{5/2}}\left(r_a\frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L}x' - \frac{\partial y_a}{\partial L}y' - \frac{\partial z_a}{\partial L}z'\right) \times \\
& \times \left(\frac{\partial^3 x_a}{\partial L^2 \partial g}x' + \frac{\partial^3 y_a}{\partial L^2 \partial g}y' + \frac{\partial^3 z_a}{\partial L^2 \partial g}z'\right) + \\
& \frac{3}{\Delta^{5/2}}\left[3\frac{\partial r_a}{\partial L}\frac{\partial^2 r_a}{\partial L^2} + r_a\frac{\partial^3 r_a}{\partial L^3} - \frac{\partial^3 x_a}{\partial L^3}x' - \frac{\partial^3 y_a}{\partial L^3}y' - \frac{\partial^3 z_a}{\partial L^3}z'\right] \times \\
& \left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right) + \\
& -\frac{1}{\Delta^{3/2}}\left(\frac{\partial^4 x_a}{\partial L^3 \partial g}x' + \frac{\partial^4 y_a}{\partial L^3 \partial g}y' + \frac{\partial^4 z_a}{\partial L^3 \partial g}z'\right) + \\
& \frac{1}{r'^3}\left(\frac{\partial^4 x_a}{\partial L^3 \partial g}x' + \frac{\partial^4 y_a}{\partial L^3 \partial g}y' + \frac{\partial^4 z_a}{\partial L^3 \partial g}z'\right),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^4 \mathcal{K}_0}{\partial L \partial g^3} \quad = \quad & \frac{105}{\Delta^{9/2}} \left(r_a \frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L} x' - \frac{\partial y_a}{\partial L} y' - \frac{\partial z_a}{\partial L} z' \right) \times \\
& \times \left(\frac{\partial x_a}{\partial g} x' + \frac{\partial y_a}{\partial g} y' + \frac{\partial z_a}{\partial g} z' \right)^3 + \frac{45}{\Delta^{7/2}} \left(\frac{\partial x_a}{\partial g} x' + \frac{\partial y_a}{\partial g} y' + \frac{\partial z_a}{\partial g} z' \right) \times \\
& \times \left(r_a \frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L} x' - \frac{\partial y_a}{\partial L} y' - \frac{\partial z_a}{\partial L} z' \right) \left(\frac{\partial^2 x_a}{\partial g^2} x' + \frac{\partial^2 y_a}{\partial g^2} y' + \frac{\partial^2 z_a}{\partial g^2} z' \right) + \\
& - \frac{45}{\Delta^{7/2}} \left(\frac{\partial x_a}{\partial g} x' + \frac{\partial y_a}{\partial g} y' + \frac{\partial z_a}{\partial g} z' \right)^2 \left(\frac{\partial^2 x_a}{\partial L \partial g} x' + \frac{\partial^2 y_a}{\partial L \partial g} y' + \frac{\partial^2 z_a}{\partial L \partial g} z' \right) + \\
& - \frac{9}{\Delta^{5/2}} \left(\frac{\partial x_a}{\partial g} x' + \frac{\partial y_a}{\partial g} y' + \frac{\partial z_a}{\partial g} z' \right) \left(\frac{\partial^3 x_a}{\partial L \partial g^2} x' + \frac{\partial^3 y_a}{\partial L \partial g^2} y' + \frac{\partial^3 z_a}{\partial L \partial g^2} z' \right) + \\
& - \frac{9}{\Delta^{5/2}} \left(\frac{\partial^2 x_a}{\partial g^2} x' + \frac{\partial^2 y_a}{\partial g^2} y' + \frac{\partial^2 z_a}{\partial g^2} z' \right) \left(\frac{\partial^2 x_a}{\partial L \partial g} x' + \frac{\partial^2 y_a}{\partial L \partial g} y' + \frac{\partial^2 z_a}{\partial L \partial g} z' \right) + \\
& \frac{3}{\Delta^{5/2}} \left(\frac{\partial^3 x_a}{\partial g^3} x' + \frac{\partial^3 y_a}{\partial g^3} y' + \frac{\partial^3 z_a}{\partial g^3} z' \right) \left(r_a \frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L} x' - \frac{\partial y_a}{\partial L} y' - \frac{\partial z_a}{\partial L} z' \right) + \\
& - \frac{1}{\Delta^{3/2}} \left(\frac{\partial^4 x_a}{\partial L \partial g^3} x' + \frac{\partial^4 y_a}{\partial L \partial g^3} y' + \frac{\partial^4 z_a}{\partial L \partial g^3} z' \right] + \\
& \frac{1}{r'^3} \left(\frac{\partial^4 x_a}{\partial L \partial g^3} x' + \frac{\partial^4 y_a}{\partial L \partial g^3} y' + \frac{\partial^4 z_a}{\partial L \partial g^3} z' \right]
\end{aligned}
\tag{3.89}
$$

$$
\begin{aligned}
\frac{\partial^4 \mathcal{K}_0}{\partial H^3 \partial g} \quad = \quad & - \frac{105}{\Delta^{9/2}} \left(\frac{\partial x_a}{\partial H} x' + \frac{\partial y_a}{\partial H} y' + \frac{\partial z_a}{\partial H} z' \right)^3 \left(\frac{\partial x_a}{\partial g} x' + \frac{\partial y_a}{\partial g} y' + \frac{\partial z_a}{\partial g} z' \right) + \\
& - \frac{45}{\Delta^{7/2}} \left(\frac{\partial x_a}{\partial H} x' + \frac{\partial y_a}{\partial H} y' + \frac{\partial z_a}{\partial L} z' \right)^2 \times \\
& \times \left(\frac{\partial^2 x_a}{\partial H \partial g} x' + \frac{\partial^2 y_a}{\partial H \partial g} y' + \frac{\partial^2 z_a}{\partial L \partial g} z' \right),
\end{aligned}
\tag{3.90}
$$

$$
\begin{aligned}
\frac{\partial^4 \mathcal{K}_0}{\partial L^2 \partial H \partial g} \quad = \quad & -\frac{105}{\Delta^{9/2}}\left(r_a\frac{\partial r_a}{\partial L}-\frac{\partial x_a}{\partial L}x'-\frac{\partial y_a}{\partial L}y'-\frac{\partial z_a}{\partial L}z'\right)^2 \times \\
& \left(\frac{\partial x_a}{\partial H}x'-\frac{\partial y_a}{\partial H}y'-\frac{\partial z_a}{\partial H}z'\right)\left(\frac{\partial x_a}{\partial g}x'+\frac{\partial y_a}{\partial g}y'+\frac{\partial z_a}{\partial g}z'\right)+ \\
& \frac{30}{\Delta^{7/2}}\left(r_a\frac{\partial r_a}{\partial L}-\frac{\partial x_a}{\partial L}x'-\frac{\partial y_a}{\partial L}y'-\frac{\partial z_a}{\partial L}z'\right)\times \\
\times & \left(\frac{\partial x_a}{\partial H}x'-\frac{\partial y_a}{\partial H}y'-\frac{\partial z_a}{\partial H}z'\right)\left(\frac{\partial^2 x_a}{\partial L\partial g}x'+\frac{\partial^2 y_a}{\partial L\partial g}y'+\frac{\partial^2 z_a}{\partial L\partial g}z'\right)+ \\
& -\frac{15}{\Delta^{7/2}}\left(r_a\frac{\partial r_a}{\partial L}-\frac{\partial x_a}{\partial L}x'-\frac{\partial y_a}{\partial L}y'-\frac{\partial z_a}{\partial L}z'\right)^2\left(\frac{\partial^2 x_a}{\partial H\partial g}x'+\frac{\partial^2 y_a}{\partial H\partial g}y'+\frac{\partial^2 z_a}{\partial H\partial g}z'\right)+ \\
& \frac{30}{\Delta^{7/2}}\left(r_a\frac{\partial r_a}{\partial L}-\frac{\partial x_a}{\partial L}x'-\frac{\partial y_a}{\partial L}y'-\frac{\partial z_a}{\partial L}z'\right)\left(\frac{\partial x_a}{\partial g}x'+\frac{\partial y_a}{\partial g}y'+\frac{\partial z_a}{\partial g}z'\right)\times \\
\times & \left(\frac{\partial^2 x_a}{\partial L\partial H}x'+\frac{\partial^2 y_a}{\partial L\partial H}y'+\frac{\partial^2 z_a}{\partial L\partial H}z'\right)+ \\
& \frac{15}{\Delta^{7/2}}\left(\frac{\partial x_a}{\partial H}x'-\frac{\partial y_a}{\partial H}y'-\frac{\partial z_a}{\partial H}z'\right)\left(\frac{\partial x_a}{\partial g}x'+\frac{\partial y_a}{\partial g}y'+\frac{\partial z_a}{\partial g}z'\right)\times \\
\times & \left[\left(\frac{\partial r_a}{\partial L}\right)^2+r_a\frac{\partial^2 r_a}{\partial L^2}-\frac{\partial^2 x_a}{\partial L^2}x'-\frac{\partial^2 y_a}{\partial L^2}y'-\frac{\partial^2 z_a}{\partial L^2}z'\right]+ \qquad (3.91) \\
& -\frac{6}{\Delta^{5/2}}\left(\frac{\partial^2 x_a}{\partial L\partial H}x'+\frac{\partial^2 y_a}{\partial L\partial H}y'+\frac{\partial^2 z_a}{\partial L\partial H}z'\right)\left(\frac{\partial^2 x_a}{\partial L\partial g}x'+\frac{\partial^2 y_a}{\partial L\partial g}y'+\frac{\partial^2 z_a}{\partial L\partial g}z'\right)+ \\
& \frac{6}{\Delta^{5/2}}\left(r_a\frac{\partial r_a}{\partial L}-\frac{\partial x_a}{\partial L}x'-\frac{\partial y_a}{\partial L}y'-\frac{\partial z_a}{\partial L}z'\right)\left(\frac{\partial^3 x_a}{\partial L\partial H\partial g}x'+\frac{\partial^3 y_a}{\partial L\partial H\partial g}y'+\frac{\partial^3 z_a}{\partial L\partial H\partial g}z'\right)+ \\
& -\frac{3}{\Delta^{5/2}}\left(\frac{\partial x_a}{\partial H}x'+\frac{\partial y_a}{\partial H}y'+\frac{\partial z_a}{\partial H}z'\right)\left(\frac{\partial^3 x_a}{\partial L^2\partial g}x'+\frac{\partial^3 y_a}{\partial L^2\partial g}y'+\frac{\partial^3 z_a}{\partial L^2\partial g}z'\right)+ \\
& \frac{3}{\Delta^{5/2}}\left[\left(\frac{\partial r_a}{\partial L}\right)^2+r_a\frac{\partial^2 r_a}{\partial L^2}-\frac{\partial^2 x_a}{\partial L^2}x'-\frac{\partial^2 y_a}{\partial L^2}y'-\frac{\partial^2 z_a}{\partial L^2}z'\right]\left(\frac{\partial^2 x_a}{\partial H\partial g}x'+\frac{\partial^2 y_a}{\partial H\partial g}y'+\frac{\partial^2 z_a}{\partial H\partial g}z'\right)+ \\
& -\frac{3}{\Delta^{5/2}}\left(\frac{\partial^3 x_a}{\partial L^2\partial H}x'+\frac{\partial^3 y_a}{\partial L^2\partial H}y'+\frac{\partial^3 z_a}{\partial L^2\partial H}z'\right)\left(\frac{\partial x_a}{\partial g}x'+\frac{\partial y_a}{\partial g}y'+\frac{\partial z_a}{\partial g}z'\right)+ \\
& -\frac{1}{\Delta^{3/2}}\left(\frac{\partial^4 x_a}{\partial L^2\partial H\partial g}x'+\frac{\partial^4 y_a}{\partial L^2\partial H\partial g}y'+\frac{\partial^4 z_a}{\partial L^2\partial H\partial g}z'\right]+ \\
& \frac{1}{r'^3}\left(\frac{\partial^4 x_a}{\partial L^2\partial H\partial g}x'+\frac{\partial^4 y_a}{\partial L^2\partial H\partial g}y'+\frac{\partial^4 z_a}{\partial L^2\partial H\partial g}z'\right),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^4 \mathcal{K}_0}{\partial L^2 \partial g^2} = & -\frac{105}{\Delta^{9/2}}\left(r_a \frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L}x' - \frac{\partial y_a}{\partial L}y' - \frac{\partial z_a}{\partial L}z'\right)^2 \left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right)^2 + \\
& \frac{60}{\Delta^{7/2}}\left(r_a \frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L}x' - \frac{\partial y_a}{\partial L}y' - \frac{\partial z_a}{\partial L}z'\right) \times \\
& \times \left(\frac{\partial x_a}{\partial g}x' - \frac{\partial y_a}{\partial g}y' - \frac{\partial z_a}{\partial g}z'\right)\left(\frac{\partial^2 x_a}{\partial L \partial g}x' + \frac{\partial^2 y_a}{\partial L \partial g}y' + \frac{\partial^2 z_a}{\partial L \partial g}z'\right) + \\
& -\frac{15}{\Delta^{7/2}}\left(r_a \frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L}x' - \frac{\partial y_a}{\partial L}y' - \frac{\partial z_a}{\partial L}z'\right)^2 \left(\frac{\partial^2 x_a}{\partial g^2}x' + \frac{\partial^2 y_a}{\partial g^2}y' + \frac{\partial^2 z_a}{\partial g^2}z'\right) + \\
& \frac{15}{\Delta^{7/2}}\left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right)^2 \left[\left(\frac{\partial r_a}{\partial L}\right)^2 + r_a \frac{\partial^2 r_a}{\partial L^2} - \frac{\partial^2 x_a}{\partial L^2}x' - \frac{\partial^2 y_a}{\partial L^2}y' - \frac{\partial^2 z_a}{\partial L^2}z'\right] + \\
& -\frac{6}{\Delta^{5/2}}\left(\frac{\partial^2 x_a}{\partial L \partial g}x' + \frac{\partial^2 y_a}{\partial L \partial g}y' + \frac{\partial^2 z_a}{\partial L \partial g}z'\right)^2 + \\
& \frac{6}{\Delta^{5/2}}\left(r_a \frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L}x' - \frac{\partial y_a}{\partial L}y' - \frac{\partial z_a}{\partial L}z'\right)\left(\frac{\partial^3 x_a}{\partial L \partial g^2}x' + \frac{\partial^3 y_a}{\partial L \partial g^2}y' + \frac{\partial^3 z_a}{\partial L \partial g^2}z'\right) + \\
& \frac{3}{\Delta^{5/2}}\left[\left(\frac{\partial r_a}{\partial L}\right)^2 + r_a \frac{\partial^2 r_a}{\partial L^2} - \frac{\partial^2 x_a}{\partial L^2}x' - \frac{\partial^2 y_a}{\partial L^2}y' - \frac{\partial^2 z_a}{\partial L^2}z'\right]\left(\frac{\partial^2 x_a}{\partial g^2}x' + \frac{\partial^2 y_a}{\partial g^2}y' + \frac{\partial^2 z_a}{\partial g^2}z'\right) + \\
& -\frac{6}{\Delta^{5/2}}\left(\frac{\partial^3 x_a}{\partial L^2 \partial g}x' + \frac{\partial^3 y_a}{\partial L^2 \partial g}y' + \frac{\partial^3 z_a}{\partial L^2 \partial g}z'\right)\left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right) + \\
& -\frac{1}{\Delta^{3/2}}\left(\frac{\partial^4 x_a}{\partial L^2 \partial g^2}x' + \frac{\partial^4 y_a}{\partial L^2 \partial g^2}y' + \frac{\partial^4 z_a}{\partial L^2 \partial g^2}z'\right] + \\
& \frac{1}{r'^3}\left(\frac{\partial^4 x_a}{\partial L^2 \partial g^2}x' + \frac{\partial^4 y_a}{\partial L^2 \partial g^2}y' + \frac{\partial^4 z_a}{\partial L^2 \partial g^2}z'\right),
\end{aligned}
\tag{3.92}
$$

$$
\begin{aligned}
\frac{\partial^4 \mathcal{K}_0}{\partial L \partial H^2 \partial g} \quad = \quad & -\frac{105}{\Delta^{9/2}} \left(r_a \frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L} x' - \frac{\partial y_a}{\partial L} y' - \frac{\partial z_a}{\partial L} z' \right) \times \\
& \times \left(\frac{\partial x_a}{\partial H} x' - \frac{\partial y_a}{\partial H} y' - \frac{\partial z_a}{\partial H} z' \right)^2 \left(\frac{\partial x_a}{\partial g} x' + \frac{\partial y_a}{\partial g} y' + \frac{\partial z_a}{\partial g} z' \right) + \\
& - \frac{30}{\Delta^{7/2}} \left(\frac{\partial x_a}{\partial H} x' - \frac{\partial y_a}{\partial H} y' - \frac{\partial z_a}{\partial H} z' \right) \left(r_a \frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L} x' - \frac{\partial y_a}{\partial L} y' - \frac{\partial z_a}{\partial L} z' \right) \times \\
& \times \left(\frac{\partial^2 x_a}{\partial H \partial g} x' + \frac{\partial^2 y_a}{\partial H \partial g} y' + \frac{\partial^2 z_a}{\partial H \partial g} z' \right) + \frac{15}{\Delta^{7/2}} \left(\frac{\partial^2 x_a}{\partial L \partial g} x' + \frac{\partial^2 y_a}{\partial L \partial g} y' + \frac{\partial^2 z_a}{\partial L \partial g} z' \right) \times \\
& \times \left(\frac{\partial x_a}{\partial H} x' + \frac{\partial y_a}{\partial H} y' + \frac{\partial z_a}{\partial H} z' \right)^2 - \frac{30}{\Delta^{7/2}} \left(\frac{\partial^2 x_a}{\partial L \partial H} x' + \frac{\partial^2 y_a}{\partial L \partial H} y' + \frac{\partial^2 z_a}{\partial L \partial H} z' \right) \times \\
& \times \left(\frac{\partial x_a}{\partial g} x' + \frac{\partial y_a}{\partial g} y' + \frac{\partial z_a}{\partial g} z' \right) \left(\frac{\partial x_a}{\partial H} x' + \frac{\partial y_a}{\partial H} y' + \frac{\partial z_a}{\partial H} z' \right) + \\
& - \frac{6}{\Delta^{5/2}} \left(\frac{\partial^2 x_a}{\partial L \partial H} x' + \frac{\partial^2 y_a}{\partial L \partial H} y' + \frac{\partial^2 z_a}{\partial L \partial H} z' \right) \left(\frac{\partial^2 x_a}{\partial H \partial g} x' + \frac{\partial^2 y_a}{\partial H \partial g} y' + \frac{\partial^2 z_a}{\partial H \partial g} z' \right) + \\
& - \frac{6}{\Delta^{5/2}} \left(\frac{\partial x_a}{\partial H} x' + \frac{\partial y_a}{\partial H} y' + \frac{\partial z_a}{\partial H} z' \right) \left(\frac{\partial^3 x_a}{\partial L \partial H \partial g} x' + \frac{\partial^3 y_a}{\partial L \partial H \partial g} y' + \frac{\partial^3 z_a}{\partial L \partial H \partial g} z' \right),
\end{aligned} \tag{3.93}
$$

$$
\begin{aligned}
\frac{\partial^4\mathcal{K}_0}{\partial L\partial H\partial g^2} \;=\; & \frac{105}{\Delta^{9/2}}\left(r_a\frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L}x' - \frac{\partial y_a}{\partial L}y' - \frac{\partial z_a}{\partial L}z'\right)\times \\
& \times\left(\frac{\partial x_a}{\partial H}x' - \frac{\partial y_a}{\partial H}y' - \frac{\partial z_a}{\partial H}z'\right)\left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right)^2 + \\
& -\frac{30}{\Delta^{7/2}}\left(\frac{\partial x_a}{\partial H}x' - \frac{\partial y_a}{\partial H}y' - \frac{\partial z_a}{\partial H}z'\right)\left(\frac{\partial x_a}{\partial g}x' - \frac{\partial y_a}{\partial g}y' - \frac{\partial z_a}{\partial g}z'\right)\times \\
& \times\left(\frac{\partial^2 x_a}{\partial L\partial g}x' + \frac{\partial^2 y_a}{\partial L\partial g}y' + \frac{\partial^2 z_a}{\partial L\partial g}z'\right) + \frac{30}{\Delta^{7/2}}\left(\frac{\partial^2 x_a}{\partial H\partial g}x' + \frac{\partial^2 y_a}{\partial H\partial g}y' + \frac{\partial^2 z_a}{\partial H\partial g}z'\right)\times \\
& \times\left(r_a\frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L}x' - \frac{\partial y_a}{\partial L}y' - \frac{\partial z_a}{\partial L}z'\right)\left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right) + \\
& \frac{15}{\Delta^{7/2}}\left(r_a\frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L}x' - \frac{\partial y_a}{\partial L}y' - \frac{\partial z_a}{\partial L}z'\right)\left(\frac{\partial x_a}{\partial H}x' - \frac{\partial y_a}{\partial H}y' - \frac{\partial z_a}{\partial H}z'\right)\times \\
& \times\left(\frac{\partial^2 x_a}{\partial g^2}x' + \frac{\partial^2 y_a}{\partial g^2}y' + \frac{\partial^2 z_a}{\partial g^2}z'\right) - \frac{15}{\Delta^{7/2}}\left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right)^2\times \\
& \times\left(\frac{\partial^2 x_a}{\partial L\partial H}x' + \frac{\partial^2 y_a}{\partial L\partial H}y' + \frac{\partial^2 z_a}{\partial L\partial H}z'\right) - \frac{6}{\Delta^{5/2}}\left(\frac{\partial^2 x_a}{\partial L\partial g}x' + \frac{\partial^2 y_a}{\partial L\partial g}y' + \frac{\partial^2 z_a}{\partial L\partial g}z'\right)\times \\
& \times\left(\frac{\partial^2 x_a}{\partial H\partial g}x' + \frac{\partial^2 y_a}{\partial H\partial g}y' + \frac{\partial^2 z_a}{\partial H\partial g}z'\right) + \frac{3}{\Delta^{5/2}}\left(r_a\frac{\partial r_a}{\partial L} - \frac{\partial x_a}{\partial L}x' - \frac{\partial y_a}{\partial L}y' - \frac{\partial z_a}{\partial L}z'\right)\times \\
& \times\left(\frac{\partial^3 x_a}{\partial H\partial g^2}x' + \frac{\partial^3 y_a}{\partial H\partial g^2}y' + \frac{\partial^3 z_a}{\partial H\partial g^2}z'\right) - \frac{3}{\Delta^{5/2}}\left(\frac{\partial x_a}{\partial H}x' + \frac{\partial y_a}{\partial H}y' + \frac{\partial z_a}{\partial H}z'\right)\times \\
& \times\left(\frac{\partial^3 x_a}{\partial L\partial g^2}x' + \frac{\partial^3 y_a}{\partial L\partial g^2}y' + \frac{\partial^3 z_a}{\partial L\partial g^2}z'\right) - \frac{3}{\Delta^{5/2}}\left(\frac{\partial^2 x_a}{\partial g^2}x' + \frac{\partial^2 y_a}{\partial g^2}y' + \frac{\partial^2 z_a}{\partial g^2}z'\right)\times \\
& \times\left(\frac{\partial^2 x_a}{\partial L\partial H}x' + \frac{\partial^2 y_a}{\partial L\partial H}y' + \frac{\partial^2 z_a}{\partial L\partial H}z'\right) - \frac{6}{\Delta^{5/2}}\left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right)\times \\
& \times\left(\frac{\partial^3 x_a}{\partial L\partial H\partial g}x' + \frac{\partial^3 y_a}{\partial L\partial H\partial g}y' + \frac{\partial^3 z_a}{\partial L\partial H\partial g}z'\right) + \\
& -\frac{1}{\Delta^{3/2}}\left(\frac{\partial^4 x_a}{\partial L\partial H\partial g^2}x' + \frac{\partial^4 y_a}{\partial L\partial H\partial g^2}y' + \frac{\partial^4 z_a}{\partial L\partial H\partial g^2}z'\right] + \\
& \frac{1}{r'^3}\left(\frac{\partial^4 x_a}{\partial L\partial H\partial g^2}x' + \frac{\partial^4 y_a}{\partial L\partial H\partial g^2}y' + \frac{\partial^4 z_a}{\partial L\partial H\partial g^2}z'\right),
\end{aligned}
\tag{3.94}
$$

$$
\begin{aligned}
\frac{\partial^4 \mathcal{K}_0}{\partial H \partial g^3} \quad = \quad & -\frac{105}{\Delta^{9/2}}\left(\frac{\partial x_a}{\partial H}x' - \frac{\partial y_a}{\partial H}y' - \frac{\partial z_a}{\partial H}z'\right)\left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right)^3 + \\
& -\frac{45}{\Delta^{7/2}}\left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right)\left(\frac{\partial x_a}{\partial H}x' + \frac{\partial y_a}{\partial H}y' + \frac{\partial z_a}{\partial H}z'\right)\left(\frac{\partial^2 x_a}{\partial g^2}x' + \frac{\partial^2 y_a}{\partial g^2}y' + \frac{\partial^2 z_a}{\partial g^2}z'\right) + \\
& -\frac{45}{\Delta^{7/2}}\left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right)^2\left(\frac{\partial^2 x_a}{\partial H \partial g}x' + \frac{\partial^2 y_a}{\partial H \partial g}y' + \frac{\partial^2 z_a}{\partial H \partial g}z'\right) + \\
& -\frac{9}{\Delta^{5/2}}\left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right)\left(\frac{\partial^3 x_a}{\partial H \partial g^2}x' + \frac{\partial^3 y_a}{\partial H \partial g^2}y' + \frac{\partial^3 z_a}{\partial H \partial g^2}z'\right) + \\
& -\frac{9}{\Delta^{5/2}}\left(\frac{\partial^2 x_a}{\partial g^2}x' + \frac{\partial^2 y_a}{\partial g^2}y' + \frac{\partial^2 z_a}{\partial g^2}z'\right)\left(\frac{\partial^2 x_a}{\partial H \partial g}x' + \frac{\partial^2 y_a}{\partial H \partial g}y' + \frac{\partial^2 z_a}{\partial H \partial g}z'\right) + \qquad (3.95) \\
& -\frac{3}{\Delta^{5/2}}\left(\frac{\partial^3 x_a}{\partial g^3}x' + \frac{\partial^3 y_a}{\partial g^3}y' + \frac{\partial^3 z_a}{\partial g^3}z'\right)\left(\frac{\partial x_a}{\partial H}x' + \frac{\partial y_a}{\partial H}y' + \frac{\partial z_a}{\partial H}z'\right) + \\
& -\frac{1}{\Delta^{3/2}}\left(\frac{\partial^4 x_a}{\partial H \partial g^3}x' + \frac{\partial^4 y_a}{\partial H \partial g^3}y' + \frac{\partial^4 z_a}{\partial H \partial g^3}z'\right] + \\
& \frac{1}{r'^3}\left(\frac{\partial^4 x_a}{\partial H \partial g^3}x' + \frac{\partial^4 y_a}{\partial H \partial g^3}y' + \frac{\partial^4 z_a}{\partial H \partial g^3}z'\right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^4 \mathcal{K}_0}{\partial g^4} \quad = \quad & -\frac{105}{\Delta^{9/2}}\left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right)^4 + \\
& -\frac{90}{\Delta^{7/2}}\left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right)^2\left(\frac{\partial^2 x_a}{\partial g^2}x' + \frac{\partial^2 y_a}{\partial g^2}y' + \frac{\partial^2 z_a}{\partial g^2}z'\right) + \\
& -\frac{12}{\Delta^{5/2}}\left(\frac{\partial x_a}{\partial g}x' + \frac{\partial y_a}{\partial g}y' + \frac{\partial z_a}{\partial g}z'\right)\left(\frac{\partial^3 x_a}{\partial g^3}x' + \frac{\partial^3 y_a}{\partial g^3}y' + \frac{\partial^3 z_a}{\partial g^3}z'\right) + \qquad (3.96) \\
& \left(-\frac{1}{\Delta^{3/2}} + \frac{1}{r'^3}\right)\left(\frac{\partial^4 x_a}{\partial g^4}x' + \frac{\partial^4 y_a}{\partial g^4}y' + \frac{\partial^4 z_a}{\partial g^4}z'\right).
\end{aligned}
$$

3.5 Izvodi Kozaijevog Hamiltonijana po promenljivima Λ, P, x i y

Koristeći inverzne relacije od (3.1)

$$
\begin{aligned}
L &= \Lambda, & l &= \lambda + p, \\
G &= -P + Q + \Lambda, & g &= q, \\
H &= -P + \Lambda, & h &= -p - q,
\end{aligned} \tag{3.97}
$$

i pravila za izvod složene funkcije

$$
\frac{\partial \mathcal{K}_0}{\partial \Lambda} = \frac{\partial \mathcal{K}_0}{\partial L}\frac{\partial L}{\partial \Lambda} + \frac{\partial \mathcal{K}_0}{\partial G}\frac{\partial G}{\partial \Lambda} + \frac{\partial \mathcal{K}_0}{\partial H}\frac{\partial H}{\partial \Lambda} + \frac{\partial \mathcal{K}_0}{\partial l}\frac{\partial l}{\partial \Lambda} + \frac{\partial \mathcal{K}_0}{\partial g}\frac{\partial g}{\partial \Lambda} + \frac{\partial \mathcal{K}_0}{\partial h}\frac{\partial h}{\partial \Lambda}, \tag{3.98}
$$

a prema (3.97)

$$
\frac{\partial L}{\partial \Lambda} = 1, \qquad \frac{\partial G}{\partial \Lambda} = 1, \qquad \frac{\partial H}{\partial \Lambda} = 1, \qquad \frac{\partial l}{\partial \Lambda} = \frac{\partial g}{\partial \Lambda} = \frac{\partial h}{\partial \Lambda} = 0, \tag{3.99}
$$

konačno se dobija

$$
\frac{\partial \mathcal{K}_0}{\partial \Lambda} = \frac{\partial \mathcal{K}_0}{\partial L} + \frac{\partial \mathcal{K}_0}{\partial G} + \frac{\partial \mathcal{K}_0}{\partial H}, \tag{3.100}
$$

jer su, prema (3.24), (3.26) i (3.28), izvodi na desnoj strani poznati. Slično, za izvod po P imamo

$$
\begin{aligned}
\frac{\partial \mathcal{K}_0}{\partial P} &= \frac{\partial \mathcal{K}_0}{\partial L}\frac{\partial L}{\partial P} + \frac{\partial \mathcal{K}_0}{\partial G}\frac{\partial G}{\partial P} + \frac{\partial \mathcal{K}_0}{\partial H}\frac{\partial H}{\partial P} + \frac{\partial \mathcal{K}_0}{\partial l}\frac{\partial l}{\partial P} + \frac{\partial \mathcal{K}_0}{\partial g}\frac{\partial g}{\partial P} + \frac{\partial \mathcal{K}_0}{\partial h}\frac{\partial h}{\partial P} \\
&= -\frac{\partial \mathcal{K}_0}{\partial G} - \frac{\partial \mathcal{K}_0}{\partial H}
\end{aligned} \tag{3.101}
$$

Za pravougaone kanonske koordinate x i y imamo

$$
\frac{\partial \mathcal{K}_0}{\partial x} = \frac{\partial \mathcal{K}_0}{\partial Q}\frac{\partial Q}{\partial x} + \frac{\partial \mathcal{K}_0}{\partial q}\frac{\partial q}{\partial x} = x\frac{\partial \mathcal{K}_0}{\partial Q} - \frac{y}{2Q}\frac{\partial \mathcal{K}_0}{\partial q}, \tag{3.102}
$$

$$
\frac{\partial \mathcal{K}_0}{\partial y} = \frac{\partial \mathcal{K}_0}{\partial Q}\frac{\partial Q}{\partial y} + \frac{\partial \mathcal{K}_0}{\partial q}\frac{\partial q}{\partial y} = y\frac{\partial \mathcal{K}_0}{\partial Q} + \frac{x}{2Q}\frac{\partial \mathcal{K}_0}{\partial q}, \tag{3.103}
$$

gde su izvodi po Q i q

$$
\frac{\partial \mathcal{K}_0}{\partial Q} = \frac{\partial \mathcal{K}_0}{\partial G} \tag{3.104}
$$

$$
\frac{\partial \mathcal{K}_0}{\partial q} = \frac{\partial \mathcal{K}_0}{\partial g}. \tag{3.105}
$$

Druge izvode nalazimo iz

$$\frac{\partial^2\mathcal{K}_0}{\partial\Lambda^2} = \frac{\partial^2\mathcal{K}_0}{\partial L^2} + \frac{\partial^2\mathcal{K}_0}{\partial G^2} + \frac{\partial^2\mathcal{K}_0}{\partial H^2} + 2\left(\frac{\partial^2\mathcal{K}_0}{\partial L\partial G} + \frac{\partial^2\mathcal{K}_0}{\partial L\partial H} + \frac{\partial^2\mathcal{K}_0}{\partial G\partial H}\right) \tag{3.106}$$

$$\frac{\partial^2\mathcal{K}_0}{\partial P^2} = \frac{\partial^2\mathcal{K}_0}{\partial L^2} + 2\frac{\partial^2\mathcal{K}_0}{\partial L\partial H} + \frac{\partial^2\mathcal{K}_0}{\partial H^2} \tag{3.107}$$

$$\begin{aligned}\frac{\partial^2\mathcal{K}_0}{\partial x^2} &= x^2\frac{\partial^2\mathcal{K}_0}{\partial G^2} + \frac{\partial\mathcal{K}_0}{\partial G} + \frac{2xy}{(x^2+y^2)^2}\frac{\partial\mathcal{K}_0}{\partial g} + \\ &\quad -\frac{2xy}{x^2+y^2}\frac{\partial^2\mathcal{K}_0}{\partial G\partial g} + \frac{y^2}{(x^2+y^2)^2}\frac{\partial^2\mathcal{K}_0}{\partial g^2}\end{aligned} \tag{3.108}$$

$$\begin{aligned}\frac{\partial^2\mathcal{K}_0}{\partial y^2} &= y^2\frac{\partial^2\mathcal{K}_0}{\partial G^2} + \frac{\partial\mathcal{K}_0}{\partial G} - \frac{2xy}{(x^2+y^2)^2}\frac{\partial\mathcal{K}_0}{\partial g} + \\ &\quad \frac{2xy}{x^2+y^2}\frac{\partial^2\mathcal{K}_0}{\partial G\partial g} + \frac{x^2}{(x^2+y^2)^2}\frac{\partial^2\mathcal{K}_0}{\partial g^2}\end{aligned} \tag{3.109}$$

$$\frac{\partial^2\mathcal{K}_0}{\partial\Lambda\partial x} = x\frac{\partial^2\mathcal{K}_0}{\partial\Lambda\partial G} - \frac{y}{x^2+y^2}\frac{\partial^2\mathcal{K}_0}{\partial\Lambda\partial g} \tag{3.110}$$

$$\frac{\partial^2\mathcal{K}_0}{\partial\Lambda\partial y} = y\frac{\partial^2\mathcal{K}_0}{\partial\Lambda\partial G} + \frac{x}{x^2+y^2}\frac{\partial^2\mathcal{K}_0}{\partial\Lambda\partial g}, \tag{3.111}$$

gde umesto $\frac{\partial^2\mathcal{K}_0}{\partial\Lambda\partial G}$ i $\frac{\partial^2\mathcal{K}_0}{\partial\Lambda\partial g}$ treba uzeti sledeće

$$\begin{aligned}\frac{\partial^2\mathcal{K}_0}{\partial\Lambda\partial G} &= \frac{\partial^2\mathcal{K}_0}{\partial L\partial G} + \frac{\partial^2\mathcal{K}_0}{\partial G^2} + \frac{\partial^2\mathcal{K}_0}{\partial G\partial H}, \\ \frac{\partial^2\mathcal{K}_0}{\partial\Lambda\partial g} &= \frac{\partial^2\mathcal{K}_0}{\partial L\partial g} + \frac{\partial^2\mathcal{K}_0}{\partial G\partial g} + \frac{\partial^2\mathcal{K}_0}{\partial H\partial g}.\end{aligned} \tag{3.112}$$

$$\frac{\partial^2\mathcal{K}_0}{\partial\Lambda\partial P} = -\frac{\partial^2\mathcal{K}_0}{\partial L\partial G} - \frac{\partial^2\mathcal{K}_0}{\partial L\partial H} - \frac{\partial^2\mathcal{K}_0}{\partial G^2} - 2\frac{\partial^2\mathcal{K}_0}{\partial G\partial H} - \frac{\partial^2\mathcal{K}_0}{\partial H^2}, \tag{3.113}$$

$$\begin{aligned}\frac{\partial^2\mathcal{K}_0}{\partial x\partial y} &= xy\frac{\partial^2\mathcal{K}_0}{\partial G^2}+\frac{-x^2+y^2}{(x^2+y^2)^2}\frac{\partial\mathcal{K}_0}{\partial g}+\frac{x^4-y^4}{(x^2+y^2)^2}\frac{\partial^2\mathcal{K}_0}{\partial G\partial g}+\\ &-\frac{xy}{(x^2+y^2)^2}\frac{\partial^2\mathcal{K}_0}{\partial g^2},\end{aligned} \tag{3.114}$$

$$\frac{\partial^2\mathcal{K}_0}{\partial x\partial P}=x\frac{\partial^2\mathcal{K}_0}{\partial Q\partial P}-\frac{y}{x^2+y^2}\frac{\partial^2\mathcal{K}_0}{\partial q\partial P}, \tag{3.115}$$

$$\frac{\partial^2\mathcal{K}_0}{\partial y\partial P}=y\frac{\partial^2\mathcal{K}_0}{\partial Q\partial P}+\frac{x}{x^2+y^2}\frac{\partial^2\mathcal{K}_0}{\partial q\partial P}, \tag{3.116}$$

$$\frac{\partial^2\mathcal{K}_0}{\partial Q\partial P}=-\frac{\partial^2\mathcal{K}_0}{\partial G^2}-\frac{\partial^2\mathcal{K}_0}{\partial G\partial H}, \tag{3.117}$$

$$\frac{\partial^2\mathcal{K}_0}{\partial q\partial P}=-\frac{\partial^2\mathcal{K}_0}{\partial G\partial g}-\frac{\partial^2\mathcal{K}_0}{\partial H\partial g} \tag{3.118}$$

Treći izvodi se dobijaju na sledeći način

$$\begin{aligned}\frac{\partial^3\mathcal{K}_0}{\partial\Lambda^3} &= \frac{\partial^3\mathcal{K}_0}{\partial L^3}+\frac{\partial^3\mathcal{K}_0}{\partial G^3}+\frac{\partial^3\mathcal{K}_0}{\partial H^3}+3\left(\frac{\partial^3\mathcal{K}_0}{\partial L^2\partial G}+\frac{\partial^3\mathcal{K}_0}{\partial L\partial G^2}+\frac{\partial^3\mathcal{K}_0}{\partial L^2\partial H}+\right.\\ &\left.\frac{\partial^3\mathcal{K}_0}{\partial L\partial H^2}+\frac{\partial^3\mathcal{K}_0}{\partial G^2\partial H}+\frac{\partial^3\mathcal{K}_0}{\partial G\partial H^2}+2\frac{\partial^3\mathcal{K}_0}{\partial L\partial G\partial H}\right)\end{aligned} \tag{3.119}$$

$$\frac{\partial^3\mathcal{K}_0}{\partial P^3}=-\frac{\partial^3\mathcal{K}_0}{\partial G^3}-3\frac{\partial^3\mathcal{K}_0}{\partial G^2\partial H}-3\frac{\partial^3\mathcal{K}_0}{\partial G\partial H^2}-\frac{\partial^3\mathcal{K}_0}{\partial H^3}, \tag{3.120}$$

$$\begin{aligned}\frac{\partial^3\mathcal{K}_0}{\partial x^3} &= x^3\frac{\partial^3\mathcal{K}_0}{\partial G^3}+3x\frac{\partial^2\mathcal{K}_0}{\partial G^2}-2y\frac{3x^2-y^2}{(x^2+y^2)^3}\frac{\partial\mathcal{K}_0}{\partial g}+\\ &3y\frac{x^4-y^4}{(x^2+y^2)^3}\frac{\partial^2\mathcal{K}_0}{\partial G\partial g}-3y\frac{x^6+2x^4y^2+x^2y^4}{(x^2+y^2)^3}\frac{\partial^3\mathcal{K}_0}{\partial G^2\partial g}+\\ &-\frac{6xy^2}{(x^2+y^2)^3}\frac{\partial^2\mathcal{K}_0}{\partial g^2}+3y\frac{x^3y+xy^3}{(x^2+y^2)^3}\frac{\partial^3\mathcal{K}_0}{\partial G\partial g^2}-\frac{y^3}{(x^2+y^2)^3}\frac{\partial^3\mathcal{K}_0}{\partial g^3},\end{aligned} \tag{3.121}$$

$$\begin{aligned}\frac{\partial^3\mathcal{K}_0}{\partial y^3} &= y^3\frac{\partial^3\mathcal{K}_0}{\partial G^3}+3y\frac{\partial^2\mathcal{K}_0}{\partial G^2}-2x\frac{x^2-3y^2}{(x^2+y^2)^3}\frac{\partial\mathcal{K}_0}{\partial g}+\\ &3x\frac{x^4-y^4}{(x^2+y^2)^3}\frac{\partial^2\mathcal{K}_0}{\partial G\partial g}+3x\frac{x^4y^2+2x^2y^4+3y^6}{(x^2+y^2)^3}\frac{\partial^3\mathcal{K}_0}{\partial G^2\partial g}+\\ &-\frac{6x^2y}{(x^2+y^2)^3}\frac{\partial^2\mathcal{K}_0}{\partial g^2}+3x\frac{x^3y+xy^3}{(x^2+y^2)^3}\frac{\partial^3\mathcal{K}_0}{\partial G\partial g^2}+\frac{x^3}{(x^2+y^2)^3}\frac{\partial^3\mathcal{K}_0}{\partial g^3}.\end{aligned}\tag{3.122}$$

Mešoviti izvodi po Λ i Λ^2

$$\begin{aligned}\frac{\partial^3\mathcal{K}_0}{\partial\Lambda^2\partial P} &= -\frac{\partial^3\mathcal{K}_0}{\partial L^2\partial G}-\frac{\partial^3\mathcal{K}_0}{\partial G^3}-\frac{\partial^3\mathcal{K}_0}{\partial H^3}-\frac{\partial^3\mathcal{K}_0}{\partial L^2\partial H}-2\frac{\partial^3\mathcal{K}_0}{\partial L\partial G^2}+\\ &-2\frac{\partial^3\mathcal{K}_0}{\partial L\partial H^2}-3\frac{\partial^3\mathcal{K}_0}{\partial G\partial H^2}-3\frac{\partial^3\mathcal{K}_0}{\partial G^2\partial H}-4\frac{\partial^3\mathcal{K}_0}{\partial L\partial G\partial H},\end{aligned}\tag{3.123}$$

$$\begin{aligned}\frac{\partial^3\mathcal{K}_0}{\partial\Lambda^2\partial x} &= x\left[\frac{\partial^3\mathcal{K}_0}{\partial L^2\partial G}+\frac{\partial^3\mathcal{K}_0}{\partial G^3}+\frac{\partial^3\mathcal{K}_0}{\partial G\partial H^2}+2\left(\frac{\partial^3\mathcal{K}_0}{\partial L\partial G^2}+\frac{\partial^3\mathcal{K}_0}{\partial L\partial G\partial H}+\right.\right.\\ &\left.\left.\frac{\partial^3\mathcal{K}_0}{\partial G^2\partial H}\right)\right]-\frac{y}{x^2+y^2}\left[\frac{\partial^3\mathcal{K}_0}{\partial L^2\partial g}+\frac{\partial^3\mathcal{K}_0}{\partial G^2\partial g}+\frac{\partial^3\mathcal{K}_0}{\partial H^2\partial g}+\right.\\ &\left.2\left(\frac{\partial^3\mathcal{K}_0}{\partial L\partial G\partial g}+\frac{\partial^3\mathcal{K}_0}{\partial L\partial H\partial g}+\frac{\partial^3\mathcal{K}_0}{\partial G\partial H\partial g}\right)\right],\end{aligned}\tag{3.124}$$

$$\begin{aligned}\frac{\partial^3\mathcal{K}_0}{\partial\Lambda^2\partial y} &= y\left[\frac{\partial^3\mathcal{K}_0}{\partial L^2\partial G}+\frac{\partial^3\mathcal{K}_0}{\partial G^3}+\frac{\partial^3\mathcal{K}_0}{\partial G\partial H^2}+2\left(\frac{\partial^3\mathcal{K}_0}{\partial L\partial G^2}+\frac{\partial^3\mathcal{K}_0}{\partial L\partial G\partial H}+\right.\right.\\ &\left.\left.\frac{\partial^3\mathcal{K}_0}{\partial G^2\partial H}\right)\right]+\frac{x}{x^2+y^2}\left[\frac{\partial^3\mathcal{K}_0}{\partial L^2\partial g}+\frac{\partial^3\mathcal{K}_0}{\partial G^2\partial g}+\frac{\partial^3\mathcal{K}_0}{\partial H^2\partial g}+\right.\\ &\left.2\left(\frac{\partial^3\mathcal{K}_0}{\partial L\partial G\partial g}+\frac{\partial^3\mathcal{K}_0}{\partial L\partial H\partial g}+\frac{\partial^3\mathcal{K}_0}{\partial G\partial H\partial g}\right)\right],\end{aligned}\tag{3.125}$$

$$\begin{aligned}\frac{\partial^3\mathcal{K}_0}{\partial\Lambda\partial P^2} &= \frac{\partial^3\mathcal{K}_0}{\partial L\partial G^2}+2\frac{\partial^3\mathcal{K}_0}{\partial L\partial G\partial H}+\frac{\partial^3\mathcal{K}_0}{\partial L\partial H^2}+\frac{\partial^3\mathcal{K}_0}{\partial G^3}+\\ &3\frac{\partial^3\mathcal{K}_0}{\partial G^2\partial H}+3\frac{\partial^3\mathcal{K}_0}{\partial G\partial H^2}+\frac{\partial^3\mathcal{K}_0}{\partial H^3},\end{aligned}\tag{3.126}$$

$$\begin{aligned}\frac{\partial^3\mathcal{K}_0}{\partial\Lambda\partial P\partial x} &= -x\left(\frac{\partial^3\mathcal{K}_0}{\partial L\partial G^2}+\frac{\partial^3\mathcal{K}_0}{\partial L\partial G\partial H}+\frac{\partial^3\mathcal{K}_0}{\partial G^3}+2\frac{\partial^3\mathcal{K}_0}{\partial G^2\partial H}+\right.\\ &\left.\frac{\partial^3\mathcal{K}_0}{\partial G\partial H^2}\right)+\frac{y}{x^2+y^2}\left(\frac{\partial^3\mathcal{K}_0}{\partial L\partial G\partial g}+\frac{\partial^3\mathcal{K}_0}{\partial L\partial H\partial g}+\right.\\ &\left.\frac{\partial^3\mathcal{K}_0}{\partial G^2\partial g}+2\frac{\partial^3\mathcal{K}_0}{\partial G\partial H\partial g}+\frac{\partial^3\mathcal{K}_0}{\partial H^2\partial g}\right),\end{aligned} \tag{3.127}$$

$$\begin{aligned}\frac{\partial^3\mathcal{K}_0}{\partial\Lambda\partial P\partial y} &= -y\left(\frac{\partial^3\mathcal{K}_0}{\partial L\partial G^2}+\frac{\partial^3\mathcal{K}_0}{\partial L\partial G\partial H}+\frac{\partial^3\mathcal{K}_0}{\partial G^3}+2\frac{\partial^3\mathcal{K}_0}{\partial G^2\partial H}+\right.\\ &\left.\frac{\partial^3\mathcal{K}_0}{\partial G\partial H^2}\right)-\frac{x}{x^2+y^2}\left(\frac{\partial^3\mathcal{K}_0}{\partial L\partial G\partial g}+\frac{\partial^3\mathcal{K}_0}{\partial L\partial H\partial g}+\right.\\ &\left.\frac{\partial^3\mathcal{K}_0}{\partial G^2\partial g}+2\frac{\partial^3\mathcal{K}_0}{\partial G\partial H\partial g}+\frac{\partial^3\mathcal{K}_0}{\partial H^2\partial g}\right),\end{aligned} \tag{3.128}$$

$$\begin{aligned}\frac{\partial^3\mathcal{K}_0}{\partial\Lambda\partial x^2} &= x^2\left(\frac{\partial^3\mathcal{K}_0}{\partial L\partial G^2}+\frac{\partial^3\mathcal{K}_0}{\partial G^3}+\frac{\partial^3\mathcal{K}_0}{\partial G^2\partial H}\right)+\frac{\partial^2\mathcal{K}_0}{\partial L\partial G}+\frac{\partial^2\mathcal{K}_0}{\partial G^2}+\\ &\frac{\partial^2\mathcal{K}_0}{\partial G\partial H}+\frac{2xy}{(x^2+y^2)^2}\left(\frac{\partial^2\mathcal{K}_0}{\partial L\partial g}+\frac{\partial^2\mathcal{K}_0}{\partial G\partial g}+\frac{\partial^2\mathcal{K}_0}{\partial H\partial g}\right)+\\ &-\frac{2xy}{x^2+y^2}\left(\frac{\partial^3\mathcal{K}_0}{\partial L\partial G\partial g}+\frac{\partial^3\mathcal{K}_0}{\partial G^2\partial g}+\frac{\partial^3\mathcal{K}_0}{\partial G\partial H\partial g}\right)+\\ &\frac{y^2}{(x^2+y^2)^2}\left(\frac{\partial^3\mathcal{K}_0}{\partial L\partial g^2}+\frac{\partial^3\mathcal{K}_0}{\partial G\partial g^2}+\frac{\partial^3\mathcal{K}_0}{\partial H\partial g^2}\right),\end{aligned} \tag{3.129}$$

$$\begin{aligned}\frac{\partial^3\mathcal{K}_0}{\partial\Lambda\partial y^2} &= y^2\left(\frac{\partial^3\mathcal{K}_0}{\partial L\partial G^2}+\frac{\partial^3\mathcal{K}_0}{\partial G^3}+\frac{\partial^3\mathcal{K}_0}{\partial G^2\partial H}\right)+\frac{\partial^2\mathcal{K}_0}{\partial L\partial G}+\frac{\partial^2\mathcal{K}_0}{\partial G^2}+\\ &\frac{\partial^2\mathcal{K}_0}{\partial G\partial H}-\frac{2xy}{(x^2+y^2)^2}\left(\frac{\partial^2\mathcal{K}_0}{\partial L\partial g}+\frac{\partial^2\mathcal{K}_0}{\partial G\partial g}+\frac{\partial^2\mathcal{K}_0}{\partial H\partial g}\right)+\\ &\frac{2xy}{x^2+y^2}\left(\frac{\partial^3\mathcal{K}_0}{\partial L\partial G\partial g}+\frac{\partial^3\mathcal{K}_0}{\partial G^2\partial g}+\frac{\partial^3\mathcal{K}_0}{\partial G\partial H\partial g}\right)+\\ &\frac{x^2}{(x^2+y^2)^2}\left(\frac{\partial^3\mathcal{K}_0}{\partial L\partial g^2}+\frac{\partial^3\mathcal{K}_0}{\partial G\partial g^2}+\frac{\partial^3\mathcal{K}_0}{\partial H\partial g^2}\right),\end{aligned} \tag{3.130}$$

$$\begin{aligned}
\frac{\partial^3\mathcal{K}_0}{\partial\Lambda\partial x\partial y} &= xy\left(\frac{\partial^3\mathcal{K}_0}{\partial L\partial G^2}+\frac{\partial^3\mathcal{K}_0}{\partial G^3}+\frac{\partial^3\mathcal{K}_0}{\partial G^2\partial H}\right)+ \\
&\quad \frac{y^2-x^2}{(x^2+y^2)^2}\left(\frac{\partial^2\mathcal{K}_0}{\partial L\partial g}+\frac{\partial^2\mathcal{K}_0}{\partial G\partial g}+\frac{\partial^2\mathcal{K}_0}{\partial H\partial g}\right)+ \\
&\quad \frac{x^4-y^4}{(x^2+y^2)^2}\left(\frac{\partial^3\mathcal{K}_0}{\partial L\partial G\partial g}+\frac{\partial^3\mathcal{K}_0}{\partial G^2\partial g}+\frac{\partial^3\mathcal{K}_0}{\partial G\partial H\partial g}\right)+ \\
&\quad -\frac{xy}{(x^2+y^2)^2}\left(\frac{\partial^3\mathcal{K}_0}{\partial L\partial g^2}+\frac{\partial^3\mathcal{K}_0}{\partial G\partial g^2}+\frac{\partial^3\mathcal{K}_0}{\partial H\partial g^2}\right),
\end{aligned}\tag{3.131}$$

Mešoviti izvodi po P i P^2

$$\begin{aligned}
\frac{\partial^3\mathcal{K}_0}{\partial P^2\partial x} &= x\left(\frac{\partial^3\mathcal{K}_0}{\partial G^3}+2\frac{\partial^3\mathcal{K}_0}{\partial G^2\partial H}+\frac{\partial^3\mathcal{K}_0}{\partial G\partial H^2}\right)+ \\
&\quad -\frac{y}{x^2+y^2}\left(\frac{\partial^3\mathcal{K}_0}{\partial G^2\partial g}+2\frac{\partial^3\mathcal{K}_0}{\partial G\partial H\partial g}+\frac{\partial^3\mathcal{K}_0}{\partial H^2\partial g}\right),
\end{aligned}\tag{3.132}$$

$$\begin{aligned}
\frac{\partial^3\mathcal{K}_0}{\partial P^2\partial y} &= y\left(\frac{\partial^3\mathcal{K}_0}{\partial G^3}+2\frac{\partial^3\mathcal{K}_0}{\partial G^2\partial H}+\frac{\partial^3\mathcal{K}_0}{\partial G\partial H^2}\right)+ \\
&\quad \frac{x}{x^2+y^2}\left(\frac{\partial^3\mathcal{K}_0}{\partial G^2\partial g}+2\frac{\partial^3\mathcal{K}_0}{\partial G\partial H\partial g}+\frac{\partial^3\mathcal{K}_0}{\partial H^2\partial g}\right),
\end{aligned}\tag{3.133}$$

$$\begin{aligned}
\frac{\partial^3\mathcal{K}_0}{\partial P\partial x^2} &= -x^2\left(\frac{\partial^3\mathcal{K}_0}{\partial G^3}+\frac{\partial^3\mathcal{K}_0}{\partial G^2\partial H}\right)-\frac{\partial^2\mathcal{K}_0}{\partial G^2}-\frac{\partial^2\mathcal{K}_0}{\partial G\partial H}+ \\
&\quad -\frac{2xy}{(x^2+y^2)^2}\left(\frac{\partial^2\mathcal{K}_0}{\partial G\partial g}+\frac{\partial^2\mathcal{K}_0}{\partial H\partial g}\right)+ \\
&\quad \frac{2xy}{x^2+y^2}\left(\frac{\partial^3\mathcal{K}_0}{\partial G^2\partial g}+\frac{\partial^3\mathcal{K}_0}{\partial G\partial H\partial g}\right)+ \\
&\quad -\frac{y^2}{(x^2+y^2)^2}\left(\frac{\partial^3\mathcal{K}_0}{\partial G\partial g^2}+\frac{\partial^3\mathcal{K}_0}{\partial H\partial g^2}\right),
\end{aligned}\tag{3.134}$$

$$\begin{aligned}\frac{\partial^3\mathcal{K}_0}{\partial P\partial y^2} &= -y^2\left(\frac{\partial^3\mathcal{K}_0}{\partial G^3}+\frac{\partial^3\mathcal{K}_0}{\partial G^2\partial H}\right)-\frac{\partial^2\mathcal{K}_0}{\partial G^2}-\frac{\partial^2\mathcal{K}_0}{\partial G\partial H}+\\ &\quad \frac{2xy}{(x^2+y^2)^2}\left(\frac{\partial^2\mathcal{K}_0}{\partial G\partial g}+\frac{\partial^2\mathcal{K}_0}{\partial H\partial g}\right)+\\ &\quad -\frac{2xy}{x^2+y^2}\left(\frac{\partial^3\mathcal{K}_0}{\partial G^2\partial g}+\frac{\partial^3\mathcal{K}_0}{\partial G\partial H\partial g}\right)+\\ &\quad -\frac{x^2}{(x^2+y^2)^2}\left(\frac{\partial^3\mathcal{K}_0}{\partial G\partial g^2}+\frac{\partial^3\mathcal{K}_0}{\partial H\partial g^2}\right),\end{aligned}\tag{3.135}$$

$$\begin{aligned}\frac{\partial^3\mathcal{K}_0}{\partial P\partial x\partial y} &= -xy\left(\frac{\partial^3\mathcal{K}_0}{\partial G^3}+\frac{\partial^3\mathcal{K}_0}{\partial G^2\partial H}\right)+\frac{x^2-y^2}{(x^2+y^2)^2}\left(\frac{\partial^2\mathcal{K}_0}{\partial G\partial g}+\frac{\partial^2\mathcal{K}_0}{\partial H\partial g}\right)+\\ &\quad -\frac{x^4-y^4}{(x^2+y^2)^2}\left(\frac{\partial^3\mathcal{K}_0}{\partial G^2\partial g}+\frac{\partial^3\mathcal{K}_0}{\partial G\partial H\partial g}\right)+\\ &\quad \frac{xy}{(x^2+y^2)^2}\left(\frac{\partial^3\mathcal{K}_0}{\partial G\partial g^2}+\frac{\partial^3\mathcal{K}_0}{\partial H\partial g^2}\right).\end{aligned}\tag{3.136}$$

Ostali izvodi po x i y

$$\begin{aligned}\frac{\partial^3\mathcal{K}_0}{\partial x^2\partial y} &= y\frac{\partial^2\mathcal{K}_0}{\partial G^2}+x^2y\frac{\partial^3\mathcal{K}_0}{\partial G^3}+\frac{2x^3-6xy^2}{(x^2+y^2)^3}\frac{\partial\mathcal{K}_0}{\partial g}+\\ &-\frac{x^5-4x^3y^2-5xy^4}{(x^2+y^2)^3}\frac{\partial^2\mathcal{K}_0}{\partial G\partial g}+\frac{x^7-3x^3y^4-2xy^6}{(x^2+y^2)^3}\frac{\partial^3\mathcal{K}_0}{\partial G^2\partial g}+\\ &\frac{4x^2y-2y^3}{(x^2+y^2)^3}\frac{\partial^2\mathcal{K}_0}{\partial g^2}-\frac{2x^4y+x^2y^3-y^5}{(x^2+y^2)^3}\frac{\partial^3\mathcal{K}_0}{\partial G\partial g^2}+\frac{xy^2}{(x^2+y^2)^3}\frac{\partial^3\mathcal{K}_0}{\partial g^3},\end{aligned}\tag{3.137}$$

$$\begin{aligned}\frac{\partial^3\mathcal{K}_0}{\partial x\partial y^2} &= x\frac{\partial^2\mathcal{K}_0}{\partial G^2}+xy^2\frac{\partial^3\mathcal{K}_0}{\partial G^3}+\frac{6x^2y-2y^3}{(x^2+y^2)^3}\frac{\partial\mathcal{K}_0}{\partial g}+\\ &-\frac{5x^4y-4x^2y^3-y^5}{(x^2+y^2)^3}\frac{\partial^2\mathcal{K}_0}{\partial G\partial g}+\frac{2x^6y+3x^4y^3-y^7}{(x^2+y^2)^3}\frac{\partial^3\mathcal{K}_0}{\partial G^2\partial g}+\\ &-\frac{2x^3-4xy^2}{(x^2+y^2)^3}\frac{\partial^2\mathcal{K}_0}{\partial g^2}-\frac{x^5-x^3y^2-2xy^4}{(x^2+y^2)^3}\frac{\partial^3\mathcal{K}_0}{\partial G\partial g^2}-\frac{x^2y}{(x^2+y^2)^3}\frac{\partial^3\mathcal{K}_0}{\partial g^3}.\end{aligned}\tag{3.138}$$

Dakle, svi izvodi (3.100) – (3.138) po Λ, P, x i y su izraženi preko poznatih izvoda po L, G, H i g, datih relacijama (3.24) – (3.60).

Izvodi četvrtog reda Kozaijevog Hamiltonijana koji se pojavljuju u varijacionim jednačinama (3.170) i (3.171) su

$$\frac{\partial^4\mathcal{K}_0}{\partial\Lambda^3\partial x} = x\frac{\partial^4\mathcal{K}_0}{\partial\Lambda^3\partial G} - \frac{y}{x^2+y^2}\frac{\partial^4\mathcal{K}_0}{\partial\Lambda^3\partial g} \tag{3.139}$$

$$\frac{\partial^4\mathcal{K}_0}{\partial\Lambda^3\partial y} = y\frac{\partial^4\mathcal{K}_0}{\partial\Lambda^3\partial G} + \frac{x}{x^2+y^2}\frac{\partial^4\mathcal{K}_0}{\partial\Lambda^3\partial g}, \tag{3.140}$$

$$\begin{aligned}\frac{\partial^4\mathcal{K}_0}{\partial\Lambda^2\partial x^2} &= x^2\frac{\partial^4\mathcal{K}_0}{\partial\Lambda^2\partial Q^2} + \frac{\partial^3\mathcal{K}_0}{\partial\Lambda^2\partial Q} + \frac{2xy}{(x^2+y^2)^2}\frac{\partial^3\mathcal{K}_0}{\partial\Lambda^2\partial q} + \\ &\quad - \frac{2xy}{x^2+y^2}\frac{\partial^4\mathcal{K}_0}{\partial\Lambda^2\partial Q\partial q} + \frac{y^2}{(x^2+y^2)^2}\frac{\partial^4\mathcal{K}_0}{\partial\Lambda^2\partial q^2},\end{aligned} \tag{3.141}$$

$$\begin{aligned}\frac{\partial^4\mathcal{K}_0}{\partial\Lambda^2\partial y^2} &= y^2\frac{\partial^4\mathcal{K}_0}{\partial\Lambda^2\partial Q^2} + \frac{\partial^3\mathcal{K}_0}{\partial\Lambda^2\partial Q} - \frac{2xy}{(x^2+y^2)^2}\frac{\partial^3\mathcal{K}_0}{\partial\Lambda^2\partial q} + \\ &\quad - \frac{2xy}{x^2+y^2}\frac{\partial^4\mathcal{K}_0}{\partial\Lambda^2\partial Q\partial q} + \frac{x^2}{(x^2+y^2)^2}\frac{\partial^4\mathcal{K}_0}{\partial\Lambda^2\partial q^2},\end{aligned} \tag{3.142}$$

gde se izvodi po Λ, Q i q mogu izraziti preko L, G, H i g datih relacijama (3.73) – (3.96):

$$\frac{\partial^4\mathcal{K}_0}{\partial\Lambda^2\partial Q^2} = \frac{\partial^4\mathcal{K}_0}{\partial L^2\partial G^2} + \frac{\partial^4\mathcal{K}_0}{\partial G^4} + \frac{\partial^4\mathcal{K}_0}{\partial G^2\partial H^2} + 2\left(\frac{\partial^4\mathcal{K}_0}{\partial L\partial G^3} + \frac{\partial^4\mathcal{K}_0}{\partial G^3\partial H} + \frac{\partial^4\mathcal{K}_0}{\partial L\partial G^2\partial H}\right),$$

$$\frac{\partial^3\mathcal{K}_0}{\partial\Lambda^2\partial Q} = \frac{\partial^3\mathcal{K}_0}{\partial L^2\partial G} + \frac{\partial^3\mathcal{K}_0}{\partial G^3} + \frac{\partial^3\mathcal{K}_0}{\partial G\partial H^2} + 2\left(\frac{\partial^3\mathcal{K}_0}{\partial L\partial G^2} + \frac{\partial^3\mathcal{K}_0}{\partial G^2\partial H} + \frac{\partial^3\mathcal{K}_0}{\partial L\partial G\partial H}\right),$$

$$\frac{\partial^3\mathcal{K}_0}{\partial\Lambda^2\partial q} = \frac{\partial^3\mathcal{K}_0}{\partial L^2\partial g} + \frac{\partial^3\mathcal{K}_0}{\partial G^2\partial g} + \frac{\partial^3\mathcal{K}_0}{\partial H^2\partial g} + 2\left(\frac{\partial^3\mathcal{K}_0}{\partial L\partial G\partial g} + \frac{\partial^3\mathcal{K}_0}{\partial G\partial H\partial g} + \frac{\partial^3\mathcal{K}_0}{\partial L\partial H\partial g}\right),$$

$$\begin{aligned}\frac{\partial^4\mathcal{K}_0}{\partial\Lambda^2\partial Q\partial q} &= \frac{\partial^4\mathcal{K}_0}{\partial L^2\partial G\partial g} + \frac{\partial^4\mathcal{K}_0}{\partial G^3\partial g} + \frac{\partial^4\mathcal{K}_0}{\partial G\partial H^2\partial g} + \\ &\quad 2\left(\frac{\partial^4\mathcal{K}_0}{\partial L\partial G^2\partial g} + \frac{\partial^4\mathcal{K}_0}{\partial G^2\partial H\partial g} + \frac{\partial^4\mathcal{K}_0}{\partial L\partial G\partial H\partial g}\right),\end{aligned}$$

$$\frac{\partial^4\mathcal{K}_0}{\partial\Lambda^2\partial q^2} = \frac{\partial^4\mathcal{K}_0}{\partial L^2\partial g^2} + \frac{\partial^4\mathcal{K}_0}{\partial G^2\partial g^2} + \frac{\partial^4\mathcal{K}_0}{\partial H^2\partial g^2} + 2\left(\frac{\partial^4\mathcal{K}_0}{\partial L\partial G\partial g^2} + \frac{\partial^4\mathcal{K}_0}{\partial G\partial H\partial g^2} + \frac{\partial^4\mathcal{K}_0}{\partial L\partial H\partial g^2}\right).$$

$$\begin{aligned}\frac{\partial^4\mathcal{K}_0}{\partial\Lambda^2\partial x\partial y} &= xy\frac{\partial^4\mathcal{K}_0}{\partial\Lambda^2\partial Q^2}\frac{y^2-x^2}{(x^2+y^2)^2}\frac{\partial^3\mathcal{K}_0}{\partial\Lambda^2\partial q}+\\ &\quad\frac{x^4-y^4}{(x^2+y^2)^2}\frac{\partial^4\mathcal{K}_0}{\partial\Lambda^2\partial Q\partial q}-\frac{xy}{(x^2+y^2)^2}\frac{\partial^4\mathcal{K}_0}{\partial L^2\partial q^2},\end{aligned}\tag{3.143}$$

$$\begin{aligned}\frac{\partial^4\mathcal{K}_0}{\partial\Lambda\partial x^2\partial y} &= y\frac{\partial^3\mathcal{K}_0}{\partial\Lambda\partial Q^2}+x^2y\frac{\partial^4\mathcal{K}_0}{\partial\Lambda\partial Q^3}+\frac{2x^3-6xy^2}{(x^2+y^2)^3}\frac{\partial^2\mathcal{K}_0}{\partial\Lambda\partial q}+\\ &-\frac{x^5-4x^3y^2-5xy^4}{(x^2+y^2)^3}\frac{\partial^3\mathcal{K}_0}{\partial\Lambda\partial Q\partial q}+\frac{x^7-3x^3y^4-2xy^6}{(x^2+y^2)^3}\frac{\partial^4\mathcal{K}_0}{\partial\Lambda\partial Q^2\partial q}+\\ &\frac{4x^2y-2y^3}{(x^2+y^2)^3}\frac{\partial^3\mathcal{K}_0}{\partial\Lambda\partial q^2}-\frac{2x^4y+x^2y^3-y^5}{(x^2+y^2)^3}\frac{\partial^4\mathcal{K}_0}{\partial\Lambda\partial Q\partial q^2}+\frac{xy^2}{(x^2+y^2)^3}\frac{\partial^4\mathcal{K}_0}{\partial\Lambda\partial q^3},\end{aligned}\tag{3.144}$$

$$\begin{aligned}\frac{\partial^4\mathcal{K}_0}{\partial\Lambda\partial x\partial y^2} &= x\frac{\partial^3\mathcal{K}_0}{\partial\Lambda\partial Q^2}+xy^2\frac{\partial^4\mathcal{K}_0}{\partial\Lambda\partial Q^3}+\frac{6x^2y-2y^3}{(x^2+y^2)^3}\frac{\partial^2\mathcal{K}_0}{\partial\Lambda\partial q}+\\ &-\frac{5x^4y-4x^2y^3-y^5}{(x^2+y^2)^3}\frac{\partial^3\mathcal{K}_0}{\partial\Lambda\partial Q\partial q}+\frac{2x^6y+3x^4y^3-y^7}{(x^2+y^2)^3}\frac{\partial^4\mathcal{K}_0}{\partial\Lambda\partial Q^2\partial q}+\\ &-\frac{2x^3-4xy^2}{(x^2+y^2)^3}\frac{\partial^3\mathcal{K}_0}{\partial\Lambda\partial q^2}-\frac{x^5-x^3y^2-2xy^4}{(x^2+y^2)^3}\frac{\partial^4\mathcal{K}_0}{\partial\Lambda\partial Q\partial q^2}-\frac{x^2y}{(x^2+y^2)^3}\frac{\partial^4\mathcal{K}_0}{\partial\Lambda\partial q^3},\end{aligned}\tag{3.145}$$

gde se koriste sledeći izrazi

$$\frac{\partial^4\mathcal{K}_0}{\partial\Lambda\partial Q^3} = \frac{\partial^4\mathcal{K}_0}{\partial L\partial G^3}+\frac{\partial^4\mathcal{K}_0}{\partial G^4}+\frac{\partial^4\mathcal{K}_0}{\partial G^3\partial H^2},$$

$$\frac{\partial^3\mathcal{K}_0}{\partial\Lambda\partial Q^2} = \frac{\partial^3\mathcal{K}_0}{\partial L\partial G^2}+\frac{\partial^3\mathcal{K}_0}{\partial G^3}+\frac{\partial^3\mathcal{K}_0}{\partial G^2\partial H^2},$$

$$\frac{\partial^2\mathcal{K}_0}{\partial\Lambda\partial q} = \frac{\partial^2\mathcal{K}_0}{\partial L\partial g}+\frac{\partial^2\mathcal{K}_0}{\partial G\partial g}+\frac{\partial^2\mathcal{K}_0}{\partial H\partial g},$$

$$\frac{\partial^3\mathcal{K}_0}{\partial\Lambda\partial Q\partial q} = \frac{\partial^3\mathcal{K}_0}{\partial L\partial G\partial g}+\frac{\partial^3\mathcal{K}_0}{\partial G^2\partial g}+\frac{\partial^3\mathcal{K}_0}{\partial G\partial H\partial g},$$

$$\frac{\partial^4\mathcal{K}_0}{\partial\Lambda\partial Q^2\partial q} = \frac{\partial^4\mathcal{K}_0}{\partial L\partial G^2\partial g}+\frac{\partial^4\mathcal{K}_0}{\partial G^3\partial g}+\frac{\partial^4\mathcal{K}_0}{\partial G^2\partial H\partial g},$$

$$\begin{aligned}
\frac{\partial^3\mathcal{K}_0}{\partial\Lambda\partial q^2} &= \frac{\partial^3\mathcal{K}_0}{\partial L\partial g^2} + \frac{\partial^3\mathcal{K}_0}{\partial G\partial g^2} + \frac{\partial^3\mathcal{K}_0}{\partial H\partial g^2},\\
\frac{\partial^4\mathcal{K}_0}{\partial\Lambda\partial Q\partial q^2} &= \frac{\partial^4\mathcal{K}_0}{\partial L\partial G\partial g^2} + \frac{\partial^4\mathcal{K}_0}{\partial G^2\partial g^2} + \frac{\partial^4\mathcal{K}_0}{\partial G\partial H\partial g^2},\\
\frac{\partial^4\mathcal{K}_0}{\partial\Lambda\partial q^3} &= \frac{\partial^4\mathcal{K}_0}{\partial L\partial g^3} + \frac{\partial^4\mathcal{K}_0}{\partial G\partial g^3} + \frac{\partial^4\mathcal{K}_0}{\partial H\partial g^3}.
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^4\mathcal{K}_0}{\partial\Lambda\partial x^3} = {} & x^3\frac{\partial^4\mathcal{K}_0}{\partial\Lambda\partial Q^3} + 3x\frac{\partial^3\mathcal{K}_0}{\partial\Lambda\partial Q^2} - 2y\frac{3x^2-y^2}{(x^2+y^2)^3}\frac{\partial^2\mathcal{K}_0}{\partial\Lambda\partial q} + \\
& 3y\frac{x^4-y^4}{(x^2+y^2)^3}\frac{\partial^3\mathcal{K}_0}{\partial\Lambda\partial Q\partial q} - 3y\frac{x^6+2x^4y^2+x^2y^4}{(x^2+y^2)^3}\frac{\partial^4\mathcal{K}_0}{\partial\Lambda\partial Q^2\partial q} + \\
& -\frac{6xy^2}{(x^2+y^2)^3}\frac{\partial^3\mathcal{K}_0}{\partial\Lambda\partial q^2} + 3y\frac{x^3y+xy^3}{(x^2+y^2)^3}\frac{\partial^4\mathcal{K}_0}{\partial\Lambda\partial Q\partial q^2} - \frac{y^3}{(x^2+y^2)^3}\frac{\partial^4\mathcal{K}_0}{\partial\Lambda\partial q^3},
\end{aligned} \tag{3.146}$$

$$\begin{aligned}
\frac{\partial^4\mathcal{K}_0}{\partial\Lambda\partial y^3} = {} & y^3\frac{\partial^4\mathcal{K}_0}{\partial\Lambda\partial Q^3} + 3y\frac{\partial^3\mathcal{K}_0}{\partial\Lambda\partial Q^2} - 2x\frac{x^2-3y^2}{(x^2+y^2)^3}\frac{\partial^2\mathcal{K}_0}{\partial\Lambda\partial q} + \\
& 3x\frac{x^4-y^4}{(x^2+y^2)^3}\frac{\partial^3\mathcal{K}_0}{\partial\Lambda\partial Q\partial q} + 3x\frac{x^4y^2+2x^2y^4+3y^6}{(x^2+y^2)^3}\frac{\partial^4\mathcal{K}_0}{\partial\Lambda\partial Q^2\partial q} + \\
& -\frac{6x^2y}{(x^2+y^2)^3}\frac{\partial^3\mathcal{K}_0}{\partial\Lambda\partial q^2} + 3x\frac{x^3y+xy^3}{(x^2+y^2)^3}\frac{\partial^4\mathcal{K}_0}{\partial\Lambda\partial Q\partial q^2} + \frac{x^3}{(x^2+y^2)^3}\frac{\partial^4\mathcal{K}_0}{\partial\Lambda\partial q^3}.
\end{aligned} \tag{3.147}$$

$$\begin{aligned}
\frac{\partial^4\mathcal{K}_0}{\partial x^4} = {} & 3\frac{\partial^2\mathcal{K}_0}{\partial Q^2} + 6x^2\frac{\partial^3\mathcal{K}_0}{\partial Q^3} + x^4\frac{x^8+4x^6y^2+6x^4y^4+4x^2y^6+y^8}{(x^2+y^2)^4}\frac{\partial^4\mathcal{K}_0}{\partial Q^4} + \\
& 24xy\frac{x^2-y^2}{(x^2+y^2)^4}\frac{\partial\mathcal{K}_0}{\partial q} - 4xy\frac{x^4-2x^2y^2-5y^4}{(x^2+y^2)^4}\frac{\partial^2\mathcal{K}_0}{\partial Q\partial q} + \\
& -12xy^3\frac{x^4+2x^2y^2+y^4}{(x^2+y^2)^4}\frac{\partial^3\mathcal{K}_0}{\partial Q^2\partial q} - 4x^3y\frac{x^6+3x^4y^2+4x^2y^4+y^6}{(x^2+y^2)^4}\frac{\partial^4\mathcal{K}_0}{\partial Q^3\partial q} + \\
& 4y^2\frac{9x^2-2y^2}{(x^2+y^2)^4}\frac{\partial^2\mathcal{K}_0}{\partial q^2} - 3y^2\frac{6x^4+4x^2y^2-2y^4}{(x^2+y^2)^4}\frac{\partial^3\mathcal{K}_0}{\partial Q\partial q^2} + \\
& 6x^2y^2\frac{x^4+2x^2y^2+y^4}{(x^2+y^2)^4}\frac{\partial^4\mathcal{K}_0}{\partial Q^2\partial q^2} + \frac{12xy^3}{(x^2+y^2)^4}\frac{\partial^3\mathcal{K}_0}{\partial q^3} + \\
& -4xy^3\frac{x^2+y^2}{(x^2+y^2)^4}\frac{\partial^4\mathcal{K}_0}{\partial Q\partial q^3} + \frac{y^4}{(x^2+y^2)^4}\frac{\partial^4\mathcal{K}_0}{\partial q^4},
\end{aligned} \tag{3.148}$$

$$
\begin{aligned}
\frac{\partial^4 \mathcal{K}_0}{\partial y^4} = & \; 3\frac{\partial^2 \mathcal{K}_0}{\partial Q^2} + 6y^2\frac{\partial^3 \mathcal{K}_0}{\partial Q^3} + y^4\frac{x^8+4x^6y^2+6x^4y^4+4x^2y^6+y^8}{(x^2+y^2)^4}\frac{\partial^4 \mathcal{K}_0}{\partial Q^4} + \\
& 24xy\frac{x^2-y^2}{(x^2+y^2)^4}\frac{\partial \mathcal{K}_0}{\partial q} - 4xy\frac{x^4+2x^2y^2-3y^4}{(x^2+y^2)^4}\frac{\partial^2 \mathcal{K}_0}{\partial Q\partial q} + \\
& 12x^3y\frac{x^4+2x^2y^2+y^4}{(x^2+y^2)^4}\frac{\partial^3 \mathcal{K}_0}{\partial Q^2\partial q}4xy^3\frac{x^6+3x^4y^2+4x^2y^4+y^6}{(x^2+y^2)^4}\frac{\partial^4 \mathcal{K}_0}{\partial Q^3\partial q} + \\
& -4x^2\frac{2x^2-9y^2}{(x^2+y^2)^4}\frac{\partial^2 \mathcal{K}_0}{\partial q^2} + 3x^2\frac{2x^4-4x^2y^2-6y^4}{(x^2+y^2)^4}\frac{\partial^3 \mathcal{K}_0}{\partial Q\partial q^2} + \\
& 6x^2y^2\frac{x^4+2x^2y^2+y^4}{(x^2+y^2)^4}\frac{\partial^4 \mathcal{K}_0}{\partial Q^2\partial q^2} + \frac{12x^3y}{(x^2+y^2)^4}\frac{\partial^3 \mathcal{K}_0}{\partial g^3} + \\
& 4x^3y\frac{x^2+y^2}{(x^2+y^2)^4}\frac{\partial^4 \mathcal{K}_0}{\partial Q\partial q^3} + \frac{x^4}{(x^2+y^2)^4}\frac{\partial^4 \mathcal{K}_0}{\partial q^4},
\end{aligned}
\tag{3.149}
$$

$$
\begin{aligned}
\frac{\partial^4 \mathcal{K}_0}{\partial x^3\partial y} = & \; 3xy\frac{\partial^3 \mathcal{K}_0}{\partial Q^3} + x^3y\frac{\partial^4 \mathcal{K}_0}{\partial Q^4} - 6\frac{x^4-6x^2y^2+y^4}{(x^2+y^2)^4}\frac{\partial \mathcal{K}_0}{\partial q} + \\
& \frac{3x^6-21x^4y^2-19x^2y^4+5y^6}{(x^2+y^2)^4}\frac{\partial^2 \mathcal{K}_0}{\partial Q\partial q} + \\
& \frac{9x^6y^2+15x^4y^4+3x^2y^6-3y^8}{(x^2+y^2)^4}\frac{\partial^3 \mathcal{K}_0}{\partial Q^2\partial q} + \\
& x^2\frac{x^8-6x^4y^4-8x^2y^6-3y^8}{(x^2+y^2)^4}\frac{\partial^4 \mathcal{K}_0}{\partial Q^3\partial q} + \\
& -2xy\frac{9x^2+13y^2}{(x^2+y^2)^4}\frac{\partial^2 \mathcal{K}_0}{\partial q^2} + 3xy\frac{3x^4-2x^2y^2-5y^4}{(x^2+y^2)^4}\frac{\partial^3 \mathcal{K}_0}{\partial Q\partial q^2} + \\
& -3xy\frac{x^6+x^4y^2+x^2y^4+y^6}{(x^2+y^2)^4}\frac{\partial^4 \mathcal{K}_0}{\partial Q^2\partial q^2} - 3y^2\frac{3x^2-y^2}{(x^2+y^2)^4}\frac{\partial^3 \mathcal{K}_0}{\partial q^3} + \\
& \frac{3x^4y^2+2x^2y^4-y^6}{(x^2+y^2)^4}\frac{\partial^4 \mathcal{K}_0}{\partial Q\partial q^3} - \frac{xy^3}{(x^2+y^2)^4}\frac{\partial^4 \mathcal{K}_0}{\partial q^4},
\end{aligned}
\tag{3.150}
$$

$$\begin{aligned}
\frac{\partial^4\mathcal{K}_0}{\partial x\partial y^3} = \; & 3xy\frac{\partial^3\mathcal{K}_0}{\partial Q^3} + xy^3\frac{\partial^4\mathcal{K}_0}{\partial Q^4} + 6\frac{x^4-6x^2y^2+y^4}{(x^2+y^2)^4}\frac{\partial\mathcal{K}_0}{\partial q} + \\
& \frac{-5x^6+19x^4y^2+21x^2y^4-3y^6}{(x^2+y^2)^4}\frac{\partial^2\mathcal{K}_0}{\partial Q\partial q} + \\
& \frac{3x^8-3x^6y^2-15x^4y^4-9x^2y^6}{(x^2+y^2)^4}\frac{\partial^3\mathcal{K}_0}{\partial Q^2\partial q} + \\
& y^2\frac{3x^8+8x^6y^2+6x^4y^4-y^8}{(x^2+y^2)^4}\frac{\partial^4\mathcal{K}_0}{\partial Q^3\partial q} + \\
& 2xy\frac{13x^2-9y^2}{(x^2+y^2)^4}\frac{\partial^2\mathcal{K}_0}{\partial q^2} - 3xy\frac{5x^4+2x^2y^2-3y^4}{(x^2+y^2)^4}\frac{\partial^3\mathcal{K}_0}{\partial Q\partial q^2} + \\
& 3xy\frac{x^6+x^4y^2-x^2y^4-y^6}{(x^2+y^2)^4}\frac{\partial^4\mathcal{K}_0}{\partial Q^2\partial q^2} - 3x^2\frac{x^2-3y^2}{(x^2+y^2)^4}\frac{\partial^3\mathcal{K}_0}{\partial q^3} + \\
& \frac{x^6-2x^4y^2-3x^2y^4}{(x^2+y^2)^4}\frac{\partial^4\mathcal{K}_0}{\partial Q\partial q^3} - \frac{x^3y}{(x^2+y^2)^4}\frac{\partial^4\mathcal{K}_0}{\partial q^4},
\end{aligned} \tag{3.151}$$

$$\begin{aligned}
\frac{\partial^4\mathcal{K}_0}{\partial x^2\partial y^2} = \; & \frac{\partial^2\mathcal{K}_0}{\partial Q^2} + (x^2+y^2)\frac{\partial^3\mathcal{K}_0}{\partial Q^3} + x^2y^2\frac{x^8+4x^6y^2+6x^4y^4+4x^2y^6+y^8}{(x^2+y^2)^4}\frac{\partial^4\mathcal{K}_0}{\partial Q^4} + \\
& -24xy\frac{x^2-y^2}{(x^2+y^2)^4}\frac{\partial\mathcal{K}_0}{\partial q} + 16xy\frac{x^4-y^4}{(x^2+y^2)^4}\frac{\partial^2\mathcal{K}_0}{\partial Q\partial q} + \\
& -6xy\frac{x^6-x^4y^2+x^2y^4+y^6}{(x^2+y^2)^4}\frac{\partial^3\mathcal{K}_0}{\partial Q^2\partial q} + \\
& 2xy\frac{x^8+2x^6y^2-2x^2y^6-y^8}{(x^2+y^2)^4}\frac{\partial^4\mathcal{K}_0}{\partial Q^3\partial q} + \\
& \frac{6x^4-32x^2y^2+y^4}{(x^2+y^2)^4}\frac{\partial^2\mathcal{K}_0}{\partial q^2} - 3\frac{x^6-5x^4y^2-5x^2y^4-y^6}{(x^2+y^2)^4}\frac{\partial^3\mathcal{K}_0}{\partial Q\partial q^2} + \\
& \frac{x^8-2x^6y^2-6x^4y^4-2x^2y^6+y^8}{(x^2+y^2)^4}\frac{\partial^4\mathcal{K}_0}{\partial Q^2\partial q^2} + 6xy\frac{x^2-y^2}{(x^2+y^2)^4}\frac{\partial^3\mathcal{K}_0}{\partial q^3} + \\
& -2xy\frac{x^4-2y^4}{(x^2+y^2)^4}\frac{\partial^4\mathcal{K}_0}{\partial Q\partial q^3} + \frac{x^2y^2}{(x^2+y^2)^4}\frac{\partial^4\mathcal{K}_0}{\partial q^4},
\end{aligned} \tag{3.152}$$

uzimajući u obzir transformacione relacije (3.97) nalazimo

$$\begin{aligned}
\frac{\partial^4\mathcal{K}_0}{\partial Q^4} &= \frac{\partial^4\mathcal{K}_0}{\partial G^4}, \quad \frac{\partial^4\mathcal{K}_0}{\partial Q^3\partial q} = \frac{\partial^4\mathcal{K}_0}{\partial G^3\partial g}, \quad \frac{\partial^4\mathcal{K}_0}{\partial Q^2\partial q^2} = \frac{\partial^4\mathcal{K}_0}{\partial G^2\partial g^2}, \\
\frac{\partial^4\mathcal{K}_0}{\partial Q\partial q^3} &= \frac{\partial^4\mathcal{K}_0}{\partial G\partial g^3}, \quad \frac{\partial^4\mathcal{K}_0}{\partial q^4} = \frac{\partial^4\mathcal{K}_0}{\partial g^4}.
\end{aligned}$$

3.6 Izvodi integrabilnog Hamiltonijana po momentima $\tilde{\Lambda}$, Z, i J

Primenom kanonske transformacije (3.12) Kozaijev Hamiltonijan $\mathcal{K}_0(\Lambda, P, x, y)$ se transformiše tako da zavisi samo od momenata, tj. $K_0(\tilde{\Lambda}, Z, J)$. Sopstvene frekvencije integrabilnog Hamiltonijana (3.9) su

$$\omega_1 = \frac{1}{\Lambda^3} + \frac{\partial(\varepsilon K_0)}{\partial \tilde{\Lambda}}, \tag{3.153}$$

$$\omega_2 = \frac{\partial(\varepsilon K_0)}{\partial J} = \frac{2\pi}{T}, \tag{3.154}$$

$$\omega_3 = \frac{\partial(\varepsilon K_0)}{\partial Z} = \frac{1}{T}\int_0^T \frac{\partial(\varepsilon \mathcal{K}_0)}{\partial P} dt. \tag{3.155}$$

Kako bi istakli da su frekvencije ω_2 i ω_3 manje, po apsolutnoj vrednosti, ε puta od ω_1, uvek ćemo pisati ovaj množitelj uz $\mathcal{K}_0$. To se jasno vidi iz Tabele 3.1.

Izvod $\frac{\partial(\varepsilon K_0)}{\partial \tilde{\Lambda}}$ računamo po pravilima za izvod složene funkcije

$$\frac{\partial(\varepsilon K_0)}{\partial \tilde{\Lambda}} = \frac{\partial(\varepsilon \mathcal{K}_0)}{\partial \Lambda}\frac{\partial \Lambda}{\partial \tilde{\Lambda}} + \frac{\partial(\varepsilon \mathcal{K}_0)}{\partial x}\frac{\partial x}{\partial \tilde{\Lambda}} + \frac{\partial(\varepsilon \mathcal{K}_0)}{\partial y}\frac{\partial y}{\partial \tilde{\Lambda}}, \tag{3.156}$$

uzimajući, prema transformaciji (3.12) da je $\frac{\partial \Lambda}{\partial \tilde{\Lambda}} = 1$. Članovi $\frac{\partial \mathcal{K}_0}{\partial \Lambda}$, $\frac{\partial \mathcal{K}_0}{\partial x}$ i $\frac{\partial \mathcal{K}_0}{\partial y}$ su dati izrazima (3.98), (3.102) i (3.103) respektivno, dok za računanje $\frac{\partial x}{\partial \tilde{\Lambda}}$ i $\frac{\partial y}{\partial \tilde{\Lambda}}$ moramo iskoristiti varijacione jednačine, što će biti dato kasnije u ovom odeljku. Period T odredjujemo numeričkom integracijom jednačina kretanja Kozaijevog Hamiltonijana (3.14), uzimajući u obzir da su $\Lambda = \text{const}$ i $P = \text{const}$. Treću frekvenciju ω_3 odredjujemo, takodje numerički, kao srednju vrednost izvoda $\frac{\partial(\varepsilon \mathcal{K}_0)}{\partial P}$ za period T.

Izvode koordinata x i y po $\tilde{\Lambda}$ računamo, kako je već pomenuto, iz nehomogenih varijacionih jednačina:

$$\begin{aligned}
\frac{d}{dt}\frac{\partial x}{\partial \tilde{\Lambda}} &= -\frac{\partial^2 \mathcal{K}_0}{\partial \Lambda \partial y} - \frac{\partial^2 \mathcal{K}_0}{\partial x \partial y}\frac{\partial x}{\partial \tilde{\Lambda}} - \frac{\partial^2 \mathcal{K}_0}{\partial y^2}\frac{\partial y}{\partial \tilde{\Lambda}}, \\
\frac{d}{dt}\frac{\partial y}{\partial \tilde{\Lambda}} &= \frac{\partial^2 \mathcal{K}_0}{\partial \Lambda \partial x} + \frac{\partial^2 \mathcal{K}_0}{\partial x^2}\frac{\partial x}{\partial \tilde{\Lambda}} + \frac{\partial^2 \mathcal{K}_0}{\partial x \partial y}\frac{\partial y}{\partial \tilde{\Lambda}},
\end{aligned} \tag{3.157}$$

za početni uslov $(0,0)$. Izrazi za izvode na desnoj strani (3.157) su dati u prethodnom poglavlju, relacijama (3.108) – (3.111) i (3.114). Slika 3.2 prikazuje promene koordinata (x, y) i njihovih izvoda po Λ, J i Z tokom perioda T za asteroid (158) Koronis. Sa najgornjeg grafika se vidi da je T, koji je podeljen na 200 tačaka pri numeričkoj integraciji, zaista period promene koordinata (x, y), što je potvrda korektnosti računa. Posle isteka vremena T, u usvojenom sistemu jedinica, vrednosti izvoda su: $(\frac{\partial x}{\partial \tilde{\Lambda}})_{t=T} = 15.514205$ i $(\frac{\partial y}{\partial \tilde{\Lambda}})_{t=T} = 3.4611907$, $(\frac{\partial x}{\partial Z})_{t=T} = -2.5160409$ i $(\frac{\partial y}{\partial Z})_{t=T} = -0.56003362$, odnosno $(\frac{\partial x}{\partial J})_{t=T} = 0.99733305$ i $(\frac{\partial y}{\partial J})_{t=T} = 0.23216926$.

Za računanje Hesijana (2.9) potrebno je naći izvode frekvencija ω_1, ω_2 i ω_3 po momentima $\tilde{\Lambda}$, J i Z. Od ukupno 9 izvoda dovoljno[2] je naći samo sledećih 6, datih relacijama (3.158) – (3.160) i (3.165) – (3.167):

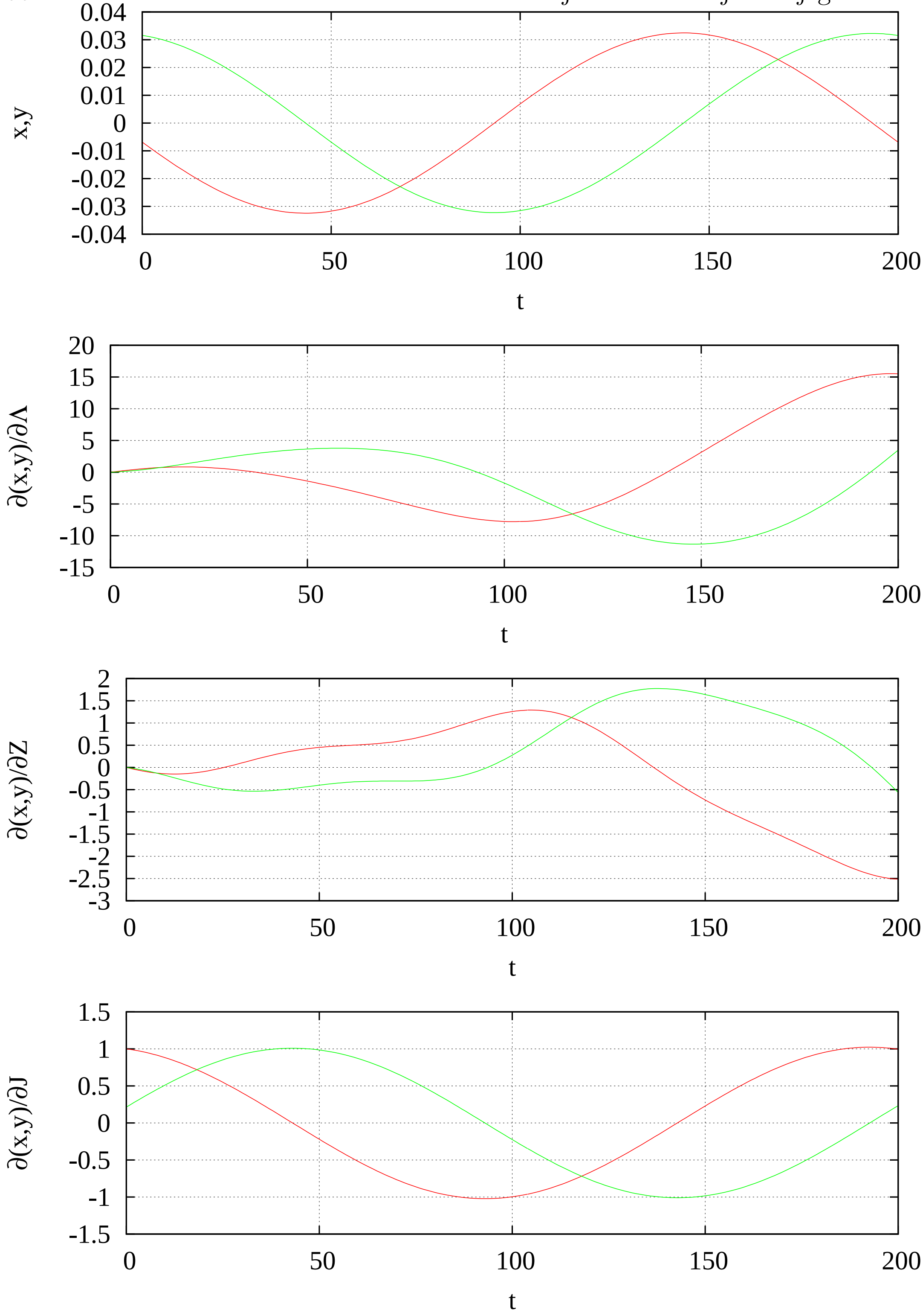
$$\begin{aligned}
\frac{\partial \omega_1}{\partial \tilde{\Lambda}} &= -\frac{3}{\Lambda^4} + \frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial \Lambda^2} + \frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial \Lambda \partial x}\frac{\partial x}{\partial \tilde{\Lambda}} + \frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial \Lambda \partial y}\frac{\partial y}{\partial \tilde{\Lambda}} + \\
&+ \left(\frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial \Lambda \partial x} + \frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial x^2}\frac{\partial x}{\partial \tilde{\Lambda}} + \frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial x \partial y}\frac{\partial y}{\partial \tilde{\Lambda}}\right)\frac{\partial x}{\partial \tilde{\Lambda}} + \\
&+ \left(\frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial \Lambda \partial y} + \frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial x \partial y}\frac{\partial x}{\partial \tilde{\Lambda}} + \frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial y^2}\frac{\partial y}{\partial \tilde{\Lambda}}\right)\frac{\partial y}{\partial \tilde{\Lambda}} + \\
&+ \frac{\partial(\varepsilon\mathcal{K}_0)}{\partial x}\frac{\partial^2 x}{\partial \tilde{\Lambda}^2} + \frac{\partial(\varepsilon\mathcal{K}_0)}{\partial y}\frac{\partial^2 y}{\partial \tilde{\Lambda}^2},
\end{aligned} \tag{3.158}$$

$$\frac{\partial \omega_1}{\partial J} = \frac{\partial \omega_2}{\partial \tilde{\Lambda}} = \frac{\partial^2(\varepsilon K_0)}{\partial \tilde{\Lambda} \partial J} = -\frac{2\pi}{T^2}\frac{\partial T}{\partial \tilde{\Lambda}}. \tag{3.159}$$

Kada računamo izvod frekvencije ω_3 po $\tilde{\Lambda}$, polazeći od relacije (3.155), treba uzeti u obzir da gornja granica integrala takođe zavisi od $\tilde{\Lambda}$. Zbog toga, moramo primeniti pravilo za diferenciranje integrala po parametru čije su granice funkcije tog parametra (Bertolino, 1969)[3]. Prema tome, imamo

[2]Koristi se osobina analitičkih funkcija (integrabilni Hamiltonijan) da su simetrični izvodi međusobno jednaki, na primer $\frac{\partial^2 \mathcal{K}_0}{\partial \Lambda \partial J} = \frac{\partial^2 \mathcal{K}_0}{\partial J \partial \Lambda}$.

[3]$\frac{\partial}{\partial m}\int\limits_{\varphi(m)}^{\psi(m)} f(m,x)dx = \int\limits_{\varphi(m)}^{\psi(m)} \frac{\partial f(m,x)}{\partial m}dx + f(m,\psi(m))\frac{\partial \psi(m)}{\partial m} - f(m,\varphi(m))\frac{\partial \varphi(m)}{\partial m}$.



Slika 3.2: Promene koordinata x (crvena) i y (zelena) i njihovih izvoda po Λ, Z i J u toku perioda T, koji je podeljen na 200 tačaka, za asteroid (158) Koronis.

$$
\begin{aligned}
\frac{\partial \omega_1}{\partial Z} &= \frac{\partial \omega_3}{\partial \tilde{\Lambda}} = \frac{\partial^2(\varepsilon K_0)}{\partial \tilde{\Lambda} \partial Z} = \\
\frac{1}{T}\int_0^T & \left(\frac{\partial^2(\varepsilon \mathcal{K}_0)}{\partial \Lambda \partial P} + \frac{\partial^2(\varepsilon \mathcal{K}_0)}{\partial x \partial P}\frac{\partial x}{\partial \tilde{\Lambda}} + \frac{\partial^2(\varepsilon \mathcal{K}_0)}{\partial y \partial P}\frac{\partial y}{\partial \tilde{\Lambda}} \right) dt + \\
& - \left[\omega_3 - \left(\frac{\partial(\varepsilon \mathcal{K}_0)}{\partial P} \right)_{M=M_T} \right] \frac{1}{T}\frac{\partial T}{\partial \tilde{\Lambda}}.
\end{aligned}
\tag{3.160}
$$

Ovde smo sa $M = M_T$ označili ($\tilde{\Lambda}$,Z,$x(T)$,$y(T)$) odnosno da u datom izrazu treba uzeti navedene veličine dobijene na kraju integracije tj. u trenutku T. Pošto se $\tilde{\Lambda}$ i Z ne menjaju tokom integracije, to znači da u izraz za izvod Kozaijevog Hamiltonijana po P (3.101) treba uvrstiti vrednosti za x i y dobijene na kraju integracije $x(T)$ i $y(T)$ respektivno.

Izvode perioda T po $\tilde{\Lambda}$ nalazimo iz periodičnosti rešenja $x = X(t, \tilde{\Lambda}, J, Z)$ i $y = Y(t, \tilde{\Lambda}, J, Z)$ Hamiltonovih jednačina (3.14) za Kozaijev Hamiltonijan, tj.

$$
\begin{aligned}
X(0, \tilde{\Lambda}, Z, J) &= X(T, \tilde{\Lambda}, Z, J), \\
Y(0, \tilde{\Lambda}, Z, J) &= Y(T, \tilde{\Lambda}, Z, J),
\end{aligned}
\tag{3.161}
$$

a njihovim diferenciranjem po $\tilde{\Lambda}$ (Henrard, 1990; Henrard i Lemaitre, 1986) se dobija

$$
\begin{aligned}
\frac{\partial X(0, \tilde{\Lambda}, Z, J)}{\partial \tilde{\Lambda}} &= \frac{\partial X(T, \tilde{\Lambda}, Z, J)}{\partial \tilde{\Lambda}} + \frac{\partial X(T, \tilde{\Lambda}, Z, J)}{\partial T}\frac{\partial T}{\partial \tilde{\Lambda}}, \\
\frac{\partial Y(0, \tilde{\Lambda}, Z, J)}{\partial \tilde{\Lambda}} &= \frac{\partial Y(T, \tilde{\Lambda}, Z, J)}{\partial \tilde{\Lambda}} + \frac{\partial Y(T, \tilde{\Lambda}, Z, J)}{\partial T}\frac{\partial T}{\partial \tilde{\Lambda}},
\end{aligned}
\tag{3.162}
$$

odnosno, kako je $\frac{\partial t}{\partial T} = 1$ nalazimo

$$
\begin{aligned}
\frac{\partial T}{\partial \tilde{\Lambda}} &= \left(\frac{\partial x}{\partial t} \right)^{-1} \left[\frac{\partial x(0, \tilde{\Lambda}, J, Z)}{\partial \tilde{\Lambda}} - \frac{\partial x(T, \tilde{\Lambda}, J, Z)}{\partial \tilde{\Lambda}} \right], \\
\frac{\partial T}{\partial \tilde{\Lambda}} &= \left(\frac{\partial y}{\partial t} \right)^{-1} \left[\frac{\partial y(0, \tilde{\Lambda}, J, Z)}{\partial \tilde{\Lambda}} - \frac{\partial y(T, \tilde{\Lambda}, J, Z)}{\partial \tilde{\Lambda}} \right].
\end{aligned}
\tag{3.163}
$$

Dovoljno je koristiti samo jednu od gornjih relacija i to onu čiji je izvod po vremenu različit od nule u posmatranoj tački. Napomenimo da izvodi x i y po vremenu ne mogu istovremeno biti jednaki nuli.

Svi izvodi na desnim stranama jednačina (3.158) – (3.160) su poznati, osim $\frac{\partial^2 x}{\partial \tilde{\Lambda}^2}$ i $\frac{\partial^2 y}{\partial \tilde{\Lambda}^2}$, koje dobijamo kada diferenciramo varijacione jednačine (3.157) po $\tilde{\Lambda}$

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial^2 x}{\partial \tilde{\Lambda}^2} &= -\frac{\partial^2 \mathcal{K}_0}{\partial \Lambda^2 \partial y} - \left(2\frac{\partial^3 \mathcal{K}_0}{\partial \tilde{\Lambda} \partial x \partial y} + \frac{\partial^3 \mathcal{K}_0}{\partial x^2 \partial y}\frac{\partial x}{\partial \tilde{\Lambda}} + \frac{\partial^3 \mathcal{K}_0}{\partial x \partial y^2}\frac{\partial y}{\partial \tilde{\Lambda}}\right)\frac{\partial x}{\partial \tilde{\Lambda}} + \\
&\quad - \left(2\frac{\partial^3 \mathcal{K}_0}{\partial \tilde{\Lambda} \partial y^2} + \frac{\partial^3 \mathcal{K}_0}{\partial x \partial y^2}\frac{\partial x}{\partial \tilde{\Lambda}} + \frac{\partial^3 \mathcal{K}_0}{\partial y^3}\frac{\partial y}{\partial \tilde{\Lambda}}\right)\frac{\partial y}{\partial \tilde{\Lambda}} + \\
&\quad - \frac{\partial^2 \mathcal{K}_0}{\partial x \partial y}\frac{\partial^2 x}{\partial \tilde{\Lambda}^2} - \frac{\partial^2 \mathcal{K}_0}{\partial y^2}\frac{\partial^2 y}{\partial \tilde{\Lambda}^2}, \\
\frac{d}{dt}\frac{\partial^2 y}{\partial \tilde{\Lambda}^2} &= \frac{\partial^2 \mathcal{K}_0}{\partial \Lambda^2 \partial x} + \left(2\frac{\partial^3 \mathcal{K}_0}{\partial \tilde{\Lambda} \partial x^2} + \frac{\partial^3 \mathcal{K}_0}{\partial x^3}\frac{\partial x}{\partial \tilde{\Lambda}} + \frac{\partial^3 \mathcal{K}_0}{\partial x^2 \partial y}\frac{\partial y}{\partial \tilde{\Lambda}}\right)\frac{\partial x}{\partial \tilde{\Lambda}} + \qquad (3.164) \\
&\quad \left(2\frac{\partial^3 \mathcal{K}_0}{\partial \tilde{\Lambda} \partial x \partial y} + \frac{\partial^3 \mathcal{K}_0}{\partial x^2 \partial y}\frac{\partial x}{\partial \tilde{\Lambda}} + \frac{\partial^3 \mathcal{K}_0}{\partial x \partial y^2}\frac{\partial y}{\partial \tilde{\Lambda}}\right)\frac{\partial y}{\partial \tilde{\Lambda}} + \\
&\quad \frac{\partial^2 \mathcal{K}_0}{\partial x^2}\frac{\partial^2 x}{\partial \tilde{\Lambda}^2} + \frac{\partial^2 \mathcal{K}_0}{\partial x \partial y}\frac{\partial^2 y}{\partial \tilde{\Lambda}^2},
\end{aligned}
$$

za početni uslov $(0, 0)$.

Ostala su tri izvoda po momentima J i Z koje (Henrard, 1990) nalazimo iz:

$$
\frac{\partial \omega_2}{\partial J} = \frac{\partial}{\partial J}\left(\frac{2\pi}{T}\right) = -\frac{2\pi}{T^2}\frac{\partial T}{\partial J}, \qquad (3.165)
$$

$$
\frac{\partial \omega_2}{\partial Z} = \frac{\partial}{\partial Z}\left(\frac{2\pi}{T}\right) = -\frac{2\pi}{T^2}\frac{\partial T}{\partial Z}, \qquad (3.166)
$$

$$
\begin{aligned}
\frac{\partial \omega_3}{\partial Z} &= \frac{\partial}{\partial Z}\left[\frac{1}{T}\int_0^T \frac{\partial(\varepsilon \mathcal{K}_0)}{\partial P} dt\right] = -\left[\omega_3 - \left(\frac{\partial(\varepsilon \mathcal{K}_0)}{\partial P}\right)_{M=M_T}\right]\frac{1}{T}\frac{\partial T}{\partial Z} + \\
&\quad \frac{1}{T}\int_0^T \left[\frac{\partial^2(\varepsilon \mathcal{K}_0)}{\partial x \partial P}\frac{\partial x}{\partial Z} + \frac{\partial^2(\varepsilon \mathcal{K}_0)}{\partial y \partial P}\frac{\partial y}{\partial Z} + \frac{\partial^2(\varepsilon \mathcal{K}_0)}{\partial P^2}\right] dt. \qquad (3.167)
\end{aligned}
$$

Izvode $\frac{\partial T}{\partial J}$ i $\frac{\partial T}{\partial Z}$ nalazimo na sličan način kao i izvode (3.163) za Λ, dok su ostali izvodi Kozaijevog Hamiltonijana poznati. Izvode kanonskih pravougaonih koordinata (x, y) po momentu Z računamo iz sledećih nehomogenih varijacionih jednačina:

Tabela 3.1: Numeričke vrednosti momenata (Λ, J, Z)=(I_1, I_2, I_3), frekvencija $(\omega_1, \omega_2, \omega_3)$, Hesijana i njegovih izvoda po ovim momentima za asteroid (158) Koronis.

		Momenti	
$I_1\, I_2\, I_3$	0.74257530D+00	0.52434127D-03	0.12807165D-02
		Frekvencije	
$\omega_1\, \omega_2\, \omega_3$	0.24415894D+01	0.11768133D-02	-0.58380762D-03
		Hesijan	
	-0.10081597D+02	-0.79576782D-01	0.10163506D+00
$h_{ij} = \frac{\partial \omega_i}{\partial I_j}$	-0.79576782D-01	0.15790083D-04	0.47426487D-05
	0.10163506D+00	0.47426487D-05	-0.10028439D-01
		3-jet	
	0.57003602D+03	0.86398250D+03	-0.30219291D+03
$\frac{\partial h_{ij}}{\partial I_1}$	0.86398250D+03	0.99130733D+04	0.10492165D+05
	-0.30219291D+03	0.10492165D+05	-0.35433770D+04
	0.86398250D+03	0.99130733D+04	0.10492165D+05
$\frac{\partial h_{ij}}{\partial I_2}$	0.99130733D+04	-0.39416664D-01	0.15065268D-01
	0.10492165D+05	0.15065268D-01	-0.97852856D-01
	-0.30219291D+03	0.10492165D+05	-0.35433770D+04
$\frac{\partial h_{ij}}{\partial I_3}$	0.10492165D+05	0.15065268D-01	-0.97852856D-01
	-0.35433770D+04	-0.97852856D-01	0.11774026D+03

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial x}{\partial Z} &= -\frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial x \partial y}\frac{\partial x}{\partial Z} - \frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial y^2}\frac{\partial y}{\partial Z} - \frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial P \partial y} \\
\frac{d}{dt}\frac{\partial y}{\partial Z} &= \frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial x^2}\frac{\partial x}{\partial Z} + \frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial x \partial y}\frac{\partial y}{\partial Z} + \frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial P \partial x},
\end{aligned}
\tag{3.168}
$$

za početni uslov $(0, 0)$. Slika 3.2c prikazuje numeričko rešenje ovih jednačina za asteroid (158) Koronis.

Kao što je ranije pomenuto, za proveru 3–jet uslova potrebno je izračunati izvode trećeg reda integrabilnog Hamiltonijana, tj. izvode drugog reda frekvencija, po momentima $\tilde{\Lambda}$, J i Z. Od ukupno 27 izvoda dovoljno je izračunati 10, koji su dati relacijama (3.169), (3.172), (3.173), (3.174), (3.185), (3.186), (3.187), (3.188), (3.189) i (3.190).

$$
\begin{aligned}
\frac{\partial^2\omega_1}{\partial\tilde{\Lambda}^2} = & \frac{12}{\Lambda^5} + \frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial\Lambda^3} + \frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial x^3}\left(\frac{\partial x}{\partial\tilde{\Lambda}}\right)^3 + 3\frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial x^2\partial y}\left(\frac{\partial x}{\partial\tilde{\Lambda}}\right)^2\frac{\partial y}{\partial\tilde{\Lambda}} + \\
& 3\frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial x\partial y^2}\frac{\partial x}{\partial\tilde{\Lambda}}\left(\frac{\partial y}{\partial\tilde{\Lambda}}\right)^2 + \frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial y^3}\left(\frac{\partial y}{\partial\tilde{\Lambda}}\right)^3 + 3\frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial y^2}\left(\frac{\partial y}{\partial\tilde{\Lambda}}\right)^2 + \\
& 6\frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial x\partial y}\frac{\partial x}{\partial\tilde{\Lambda}}\frac{\partial y}{\partial\tilde{\Lambda}} + 3\frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial x^2}\left(\frac{\partial x}{\partial\tilde{\Lambda}}\right)^2 + 3\frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial x^2}\frac{\partial x}{\partial\tilde{\Lambda}}\frac{\partial^2 x}{\partial\tilde{\Lambda}^2} + \\
& 3\frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial x\partial y}\frac{\partial^2 x}{\partial\tilde{\Lambda}^2}\frac{\partial y}{\partial\tilde{\Lambda}} + 3\frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial x}\frac{\partial^2 x}{\partial\tilde{\Lambda}^2} + 3\frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial x\partial y}\frac{\partial x}{\partial\tilde{\Lambda}}\frac{\partial^2 y}{\partial\tilde{\Lambda}^2} + \\
& 3\frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial y^2}\frac{\partial y}{\partial\tilde{\Lambda}}\frac{\partial^2 y}{\partial\tilde{\Lambda}^2} + 3\frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial y}\frac{\partial^2 y}{\partial\tilde{\Lambda}^2} + 3\frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial\Lambda^2\partial x}\frac{\partial x}{\partial\tilde{\Lambda}} + \\
& 3\frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial\Lambda^2\partial y}\frac{\partial y}{\partial\tilde{\Lambda}} + \frac{\partial(\varepsilon\mathcal{K}_0)}{\partial x}\frac{\partial^3 x}{\partial\tilde{\Lambda}^3} + \frac{\partial(\varepsilon\mathcal{K}_0)}{\partial y}\frac{\partial^3 y}{\partial\tilde{\Lambda}^3},
\end{aligned}
\tag{3.169}
$$

Da bi izračunali gornji izvod potrebno je naći treći izvod pravougaonih kanonskih promenljivih (x, y) po momentu $\tilde{\Lambda}$. Kada još jednom diferenciramo varijacione jednačine (3.164), za isti početni uslov, lako dobijamo tražene izvode:

$$\begin{aligned}
\frac{d}{dt}\frac{\partial^3 x}{\partial\tilde{\Lambda}^3} \quad = \quad & -\frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial\Lambda^3\partial y} - \left[\frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial\Lambda^2\partial x\partial y} + \frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial x^2\partial y}\frac{\partial^2 x}{\partial\tilde{\Lambda}^2} + \frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial x\partial y^2}\frac{\partial^2 y}{\partial\tilde{\Lambda}^2} + \right. \\
& \frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial x^2\partial y}\frac{\partial x}{\partial\tilde{\Lambda}} + \frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial x\partial y^2}\frac{\partial y}{\partial\tilde{\Lambda}} + \left(\frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial x^2\partial y} + \frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial x^3\partial y}\frac{\partial x}{\partial\tilde{\Lambda}} + \right. \\
& \left.\left.\frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial x^2\partial y^2}\frac{\partial y}{\partial\tilde{\Lambda}}\right)\frac{\partial x}{\partial\tilde{\Lambda}} + \left(\frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial x\partial y^2} + \frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial x^2\partial y^2}\frac{\partial x}{\partial\tilde{\Lambda}} + \frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial x\partial y^3}\frac{\partial y}{\partial\tilde{\Lambda}}\right)\frac{\partial y}{\partial\tilde{\Lambda}}\right]\frac{\partial x}{\partial\tilde{\Lambda}} + \\
& -\left[\frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial\Lambda^2\partial y^2} + \frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial x\partial y^2}\frac{\partial^2 x}{\partial\tilde{\Lambda}^2} + \frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial y^3}\frac{\partial^2 y}{\partial\tilde{\Lambda}^2} + \frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial x\partial y^2}\frac{\partial x}{\partial\tilde{\Lambda}} + \right. \\
& \frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial y^3}\frac{\partial y}{\partial\tilde{\Lambda}} + \left(\frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial x\partial y^2} + \frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial x^2\partial y^2}\frac{\partial x}{\partial\tilde{\Lambda}} + \frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial x\partial y^3}\frac{\partial y}{\partial\tilde{\Lambda}}\right)\frac{\partial x}{\partial\tilde{\Lambda}} + \\
& \left.\left(\frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial y^3} + \frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial x\partial y^3}\frac{\partial x}{\partial\tilde{\Lambda}} + \frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial y^4}\frac{\partial y}{\partial\tilde{\Lambda}}\right)\frac{\partial y}{\partial\tilde{\Lambda}}\right]\frac{\partial y}{\partial\tilde{\Lambda}} + \\
& -\left(2\frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial\Lambda^2\partial x\partial y} + \frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial x^2\partial y}\frac{\partial x}{\partial\tilde{\Lambda}} + \frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial x\partial y^2}\frac{\partial y}{\partial\tilde{\Lambda}}\right)\frac{\partial x}{\partial\tilde{\Lambda}} + \\
& -\left(2\frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial\Lambda^2\partial y^2} + \frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial x\partial y^2}\frac{\partial x}{\partial\tilde{\Lambda}} + \frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial y^3}\frac{\partial y}{\partial\tilde{\Lambda}}\right)\frac{\partial y}{\partial\tilde{\Lambda}} + \\
& -2\left(\frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial x\partial y} + \frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial x^2\partial y}\frac{\partial x}{\partial\tilde{\Lambda}} + \frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial x\partial y^2}\frac{\partial y}{\partial\tilde{\Lambda}}\right)\frac{\partial^2 x}{\partial\tilde{\Lambda}^2} + \\
& -2\left(\frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial y^2} + \frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial x\partial y^2}\frac{\partial x}{\partial\tilde{\Lambda}} + \frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial y^3}\frac{\partial y}{\partial\tilde{\Lambda}}\right)\frac{\partial^2 y}{\partial\tilde{\Lambda}^2} + \\
& -\frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial x\partial y}\frac{\partial^2 x}{\partial\tilde{\Lambda}^2} - \frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial y^2}\frac{\partial^2 y}{\partial\tilde{\Lambda}^2} - \frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial x\partial y}\frac{\partial^3 x}{\partial\tilde{\Lambda}^3} - \frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial y^2}\frac{\partial^3 y}{\partial\tilde{\Lambda}^3},
\end{aligned} \tag{3.170}$$

$$\begin{aligned}
\frac{d}{dt}\frac{\partial^3 y}{\partial\tilde{\Lambda}^3} = & \frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial\Lambda^3\partial x} + \left[\frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial\Lambda^2\partial x^2} + \frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial x^3}\frac{\partial^2 x}{\partial\tilde{\Lambda}^2} + \frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial x^2\partial y}\frac{\partial^2 y}{\partial\tilde{\Lambda}^2} + \right. \\
& \frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial x^3}\frac{\partial x}{\partial\tilde{\Lambda}} + \frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial x^2\partial y}\frac{\partial y}{\partial\tilde{\Lambda}} + \left(\frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial x^3} + \frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial x^4}\frac{\partial x}{\partial\tilde{\Lambda}} + \right. \\
& \left.\frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial x^3\partial y}\frac{\partial y}{\partial\tilde{\Lambda}}\right)\frac{\partial x}{\partial\tilde{\Lambda}} + \left.\left(\frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial x^2\partial y} + \frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial x^3\partial y}\frac{\partial x}{\partial\tilde{\Lambda}} + \frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial x^2\partial y^2}\frac{\partial y}{\partial\tilde{\Lambda}}\right)\frac{\partial y}{\partial\tilde{\Lambda}}\right]\frac{\partial x}{\partial\tilde{\Lambda}} + \\
& \left[\frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial\Lambda^2\partial x\partial y} + \frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial x^2\partial y}\frac{\partial^2 x}{\partial\tilde{\Lambda}^2} + \frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial x\partial y^2}\frac{\partial^2 y}{\partial\tilde{\Lambda}^2} + \frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial x^2\partial y}\frac{\partial x}{\partial\tilde{\Lambda}} + \right. \\
& \frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial x\partial y^2}\frac{\partial y}{\partial\tilde{\Lambda}} + \left(\frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial x^2\partial y} + \frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial x^3\partial y}\frac{\partial x}{\partial\tilde{\Lambda}} + \frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial x^2\partial y^2}\frac{\partial y}{\partial\tilde{\Lambda}}\right)\frac{\partial x}{\partial\tilde{\Lambda}} + \\
& \left.\left(\frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial x\partial y^2} + \frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial x^2\partial y^2}\frac{\partial x}{\partial\tilde{\Lambda}} + \frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial x\partial y^3}\frac{\partial y}{\partial\tilde{\Lambda}}\right)\frac{\partial y}{\partial\tilde{\Lambda}}\right]\frac{\partial y}{\partial\tilde{\Lambda}} + \\
& \left(2\frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial\Lambda^2\partial x^2} + \frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial x^3}\frac{\partial x}{\partial\tilde{\Lambda}} + \frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial x^2\partial y}\frac{\partial y}{\partial\tilde{\Lambda}}\right)\frac{\partial x}{\partial\tilde{\Lambda}} + \\
& \left(2\frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial\Lambda^2\partial x\partial y} + \frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial x^2\partial y}\frac{\partial x}{\partial\tilde{\Lambda}} + \frac{\partial^4(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial x\partial y^2}\frac{\partial y}{\partial\tilde{\Lambda}}\right)\frac{\partial y}{\partial\tilde{\Lambda}} + \\
& 2\left(\frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial x^2} + \frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial x^3}\frac{\partial x}{\partial\tilde{\Lambda}} + \frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial x^2\partial y}\frac{\partial y}{\partial\tilde{\Lambda}}\right)\frac{\partial^2 x}{\partial\tilde{\Lambda}^2} + \\
& 2\left(\frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial x\partial y} + \frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial x^2\partial y}\frac{\partial x}{\partial\tilde{\Lambda}} + \frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial x\partial y^2}\frac{\partial y}{\partial\tilde{\Lambda}}\right)\frac{\partial^2 y}{\partial\tilde{\Lambda}^2} + \\
& \frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial x^2}\frac{\partial^2 x}{\partial\tilde{\Lambda}^2} + \frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial x\partial y}\frac{\partial^2 y}{\partial\tilde{\Lambda}^2} + \frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial x^2}\frac{\partial^3 x}{\partial\tilde{\Lambda}^3} + \frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial x\partial y}\frac{\partial^3 y}{\partial\tilde{\Lambda}^3}.
\end{aligned} \tag{3.171}$$

Izvodi četvrtog reda na desnoj strani gornja dva izraza su dati relacijama (3.139) – (3.152).

Sledeća tri izvoda su:

$$\frac{\partial^2\omega_2}{\partial\tilde{\Lambda}^2} = \frac{4\pi}{T^3}\left(\frac{\partial T}{\partial\tilde{\Lambda}}\right)^2 - \frac{2\pi}{T^2}\frac{\partial^2 T}{\partial\tilde{\Lambda}^2}, \tag{3.172}$$

$$\frac{\partial^2\omega_2}{\partial\tilde{\Lambda}\partial J} = \frac{4\pi}{T^3}\frac{\partial T}{\partial\tilde{\Lambda}}\frac{\partial T}{\partial J} - \frac{2\pi}{T^2}\frac{\partial^2 T}{\partial\tilde{\Lambda}\partial J}, \tag{3.173}$$

$$\frac{\partial^2\omega_2}{\partial\tilde{\Lambda}\partial Z} = \frac{4\pi}{T^3}\frac{\partial T}{\partial\tilde{\Lambda}}\frac{\partial T}{\partial Z} - \frac{2\pi}{T^2}\frac{\partial^2 T}{\partial\tilde{\Lambda}\partial Z}, \tag{3.174}$$

za koje je potrebno naći mešovite druge izvode perioda T po momentima $\tilde{\Lambda}$, J i Z. Diferenciranjem (3.163) po ovim momentima dobija se

$$\begin{aligned}\frac{\partial^2 T}{\partial \tilde{\Lambda}^2} &= -\left(\frac{\partial x}{\partial t}\right)^{-2} \frac{\partial}{\partial \tilde{\Lambda}}\left(\frac{\partial x}{\partial t}\right)\left[\frac{\partial x(0,\tilde{\Lambda},J,Z)}{\partial \tilde{\Lambda}} - \frac{\partial x(T,\tilde{\Lambda},J,Z)}{\partial \tilde{\Lambda}}\right] + \\ &\left(\frac{\partial x}{\partial t}\right)^{-1}\left[\frac{\partial^2 x(0,\tilde{\Lambda},J,Z)}{\partial \tilde{\Lambda}^2} - \frac{\partial^2 x(T,\tilde{\Lambda},J,Z)}{\partial \tilde{\Lambda}^2}\right],\end{aligned} \tag{3.175}$$

$$\begin{aligned}\frac{\partial^2 T}{\partial \tilde{\Lambda}\partial J} &= -\left(\frac{\partial x}{\partial t}\right)^{-2} \frac{\partial}{\partial J}\left(\frac{\partial x}{\partial t}\right)\left[\frac{\partial x(0,\tilde{\Lambda},J,Z)}{\partial \tilde{\Lambda}} - \frac{\partial x(T,\tilde{\Lambda},J,Z)}{\partial \tilde{\Lambda}}\right] + \\ &\left(\frac{\partial x}{\partial t}\right)^{-1}\left[\frac{\partial^2 x(0,\tilde{\Lambda},J,Z)}{\partial \tilde{\Lambda}\partial J} - \frac{\partial^2 x(T,\tilde{\Lambda},J,Z)}{\partial \tilde{\Lambda}\partial J}\right],\end{aligned} \tag{3.176}$$

odnosno

$$\begin{aligned}\frac{\partial^2 T}{\partial \tilde{\Lambda}\partial Z} &= -\left(\frac{\partial x}{\partial t}\right)^{-2} \frac{\partial}{\partial Z}\left(\frac{\partial x}{\partial t}\right)\left[\frac{\partial x(0,\tilde{\Lambda},J,Z)}{\partial \tilde{\Lambda}} - \frac{\partial x(T,\tilde{\Lambda},J,Z)}{\partial \tilde{\Lambda}}\right] + \\ &\left(\frac{\partial x}{\partial t}\right)^{-1}\left[\frac{\partial^2 x(0,\tilde{\Lambda},J,Z)}{\partial \tilde{\Lambda}\partial Z} - \frac{\partial^2 x(T,\tilde{\Lambda},J,Z)}{\partial \tilde{\Lambda}\partial Z}\right],\end{aligned} \tag{3.177}$$

gde se sva tri izvoda po vremenu računaju iz različitih relacija:

$$\frac{\partial}{\partial \tilde{\Lambda}}\left(\frac{\partial x}{\partial t}\right) = -\frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial \Lambda \partial y} - \frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial x \partial y}\frac{\partial x}{\partial \tilde{\Lambda}} - \frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial y^2}\frac{\partial y}{\partial \tilde{\Lambda}}, \tag{3.178}$$

$$\frac{\partial}{\partial J}\left(\frac{\partial x}{\partial t}\right) = -\frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial x \partial y}\frac{\partial x}{\partial J} - \frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial y^2}\frac{\partial y}{\partial J}, \tag{3.179}$$

$$\frac{\partial}{\partial Z}\left(\frac{\partial x}{\partial t}\right) = -\frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial P \partial y} - \frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial x \partial y}\frac{\partial x}{\partial Z} - \frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial y^2}\frac{\partial y}{\partial Z}. \tag{3.180}$$

Za računanje parcijalnih izvoda x i y u odnosu na J i Z (Lemaitre i Morbidelli, 1994) koristimo sledeće varijacione jednačine

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial x}{\partial J} &= -\frac{\partial^2 \mathcal{K}_0}{\partial x \partial y}\frac{\partial x}{\partial J} - \frac{\partial^2 \mathcal{K}_0}{\partial y^2}\frac{\partial y}{\partial J} \\
\frac{d}{dt}\frac{\partial y}{\partial J} &= \frac{\partial^2 \mathcal{K}_0}{\partial x^2}\frac{\partial x}{\partial J} + \frac{\partial^2 \mathcal{K}_0}{\partial x \partial y}\frac{\partial y}{\partial J},
\end{aligned}
\tag{3.181}
$$

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial x}{\partial Z} &= -\frac{\partial^2 \mathcal{K}_0}{\partial x \partial y}\frac{\partial x}{\partial Z} - \frac{\partial^2 \mathcal{K}_0}{\partial y^2}\frac{\partial y}{\partial Z} - \frac{\partial^2 \mathcal{K}_0}{\partial P \partial y} \\
\frac{d}{dt}\frac{\partial y}{\partial Z} &= \frac{\partial^2 \mathcal{K}_0}{\partial x^2}\frac{\partial x}{\partial Z} + \frac{\partial^2 \mathcal{K}_0}{\partial x \partial y}\frac{\partial y}{\partial Z} + \frac{\partial^2 \mathcal{K}_0}{\partial P \partial x},
\end{aligned}
\tag{3.182}
$$

dok njihove mešovite izvode drugog reda nalazimo takođe iz varijacionih jednačina

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial^2 x}{\partial \tilde{\Lambda} \partial J} &= -\frac{\partial^3 \mathcal{K}_0}{\partial \Lambda \partial x \partial y}\frac{\partial x}{\partial J} - \frac{\partial^3 \mathcal{K}_0}{\partial \Lambda \partial y^2}\frac{\partial y}{\partial J} - \left(\frac{\partial^3 \mathcal{K}_0}{\partial x^2 \partial y}\frac{\partial x}{\partial J} + \frac{\partial^3 \mathcal{K}_0}{\partial x \partial y^2}\frac{\partial y}{\partial J}\right)\frac{\partial x}{\partial \tilde{\Lambda}} + \\
& - \left(\frac{\partial^3 \mathcal{K}_0}{\partial x \partial y^2}\frac{\partial x}{\partial J} + \frac{\partial^3 \mathcal{K}_0}{\partial y^3}\frac{\partial y}{\partial J}\right)\frac{\partial y}{\partial \tilde{\Lambda}} - \frac{\partial^2 \mathcal{K}_0}{\partial x \partial y}\frac{\partial^2 x}{\partial \tilde{\Lambda} \partial J} - \frac{\partial^2 \mathcal{K}_0}{\partial y^2}\frac{\partial^2 y}{\partial \tilde{\Lambda} \partial J}, \\
\frac{d}{dt}\frac{\partial^2 y}{\partial \tilde{\Lambda} \partial J} &= \frac{\partial^3 \mathcal{K}_0}{\partial \Lambda \partial x^2}\frac{\partial x}{\partial J} + \frac{\partial^3 \mathcal{K}_0}{\partial \Lambda \partial x \partial y}\frac{\partial y}{\partial J} + \left(\frac{\partial^3 \mathcal{K}_0}{\partial x^3}\frac{\partial x}{\partial J} + \frac{\partial^3 \mathcal{K}_0}{\partial x^2 \partial y}\frac{\partial y}{\partial J}\right)\frac{\partial x}{\partial \tilde{\Lambda}} + \\
& \left(\frac{\partial^3 \mathcal{K}_0}{\partial x^2 \partial y}\frac{\partial x}{\partial J} + \frac{\partial^3 \mathcal{K}_0}{\partial x \partial y^2}\frac{\partial y}{\partial J}\right)\frac{\partial y}{\partial \tilde{\Lambda}} + \frac{\partial^2 \mathcal{K}_0}{\partial x^2}\frac{\partial^2 x}{\partial \tilde{\Lambda} \partial J} + \frac{\partial^2 \mathcal{K}_0}{\partial x \partial y}\frac{\partial^2 y}{\partial \tilde{\Lambda} \partial J},
\end{aligned}
\tag{3.183}
$$

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial^2 x}{\partial\tilde{\Lambda}\partial Z} &= -\frac{\partial^3\mathcal{K}_0}{\partial\Lambda\partial x\partial y}\frac{\partial x}{\partial Z} - \frac{\partial^3\mathcal{K}_0}{\partial\Lambda\partial y^2}\frac{\partial y}{\partial Z} - \frac{\partial^3\mathcal{K}_0}{\partial\Lambda\partial y\partial P} + \\
&-\left(\frac{\partial^3\mathcal{K}_0}{\partial x^2\partial y}\frac{\partial x}{\partial Z} + \frac{\partial^3\mathcal{K}_0}{\partial x\partial y^2}\frac{\partial y}{\partial Z} + \frac{\partial^3\mathcal{K}_0}{\partial x\partial y\partial P}\right)\frac{\partial x}{\partial\tilde{\Lambda}} + \\
&-\left(\frac{\partial^3\mathcal{K}_0}{\partial x\partial y^2}\frac{\partial x}{\partial Z} + \frac{\partial^3\mathcal{K}_0}{\partial y^3}\frac{\partial y}{\partial Z} + \frac{\partial^3\mathcal{K}_0}{\partial y^2\partial P}\right)\frac{\partial y}{\partial\tilde{\Lambda}} + \\
&-\frac{\partial^2\mathcal{K}_0}{\partial x\partial y}\frac{\partial^2 x}{\partial\tilde{\Lambda}\partial Z} - \frac{\partial^2\mathcal{K}_0}{\partial y^2}\frac{\partial^2 y}{\partial\tilde{\Lambda}\partial Z}, \\
\frac{d}{dt}\frac{\partial^2 y}{\partial\tilde{\Lambda}\partial Z} &= \frac{\partial^3\mathcal{K}_0}{\partial\Lambda\partial x^2}\frac{\partial x}{\partial Z} + \frac{\partial^3\mathcal{K}_0}{\partial\Lambda\partial x\partial y}\frac{\partial y}{\partial Z} + \frac{\partial^3\mathcal{K}_0}{\partial\Lambda\partial x\partial P} + \qquad (3.184) \\
&\left(\frac{\partial^3\mathcal{K}_0}{\partial x^3}\frac{\partial x}{\partial Z} + \frac{\partial^3\mathcal{K}_0}{\partial x^2\partial y}\frac{\partial y}{\partial Z} + \frac{\partial^3\mathcal{K}_0}{\partial x^2\partial P}\right)\frac{\partial x}{\partial\tilde{\Lambda}} + \\
&\left(\frac{\partial^3\mathcal{K}_0}{\partial x^2\partial y}\frac{\partial x}{\partial Z} + \frac{\partial^3\mathcal{K}_0}{\partial x\partial y^2}\frac{\partial y}{\partial Z} + \frac{\partial^3\mathcal{K}_0}{\partial x\partial y\partial P}\right)\frac{\partial y}{\partial\tilde{\Lambda}} + \\
&\frac{\partial^2\mathcal{K}_0}{\partial x^2}\frac{\partial^2 x}{\partial\tilde{\Lambda}\partial Z} + \frac{\partial^2\mathcal{K}_0}{\partial x\partial y}\frac{\partial^2 y}{\partial\tilde{\Lambda}\partial Z}.
\end{aligned}
$$

Naredna dva izvoda treće frekvencije nalazimo na sličan način:

$$
\begin{aligned}
\frac{\partial^2\omega_3}{\partial\tilde{\Lambda}^2} &= -\left[\frac{\partial\omega_3}{\partial\tilde{\Lambda}} - 2\left(\frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial P} + \frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial x\partial P}\frac{\partial x}{\partial\tilde{\Lambda}} + \frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial y\partial P}\frac{\partial y}{\partial\tilde{\Lambda}}\right)_{M=M_T}\right]\frac{1}{T}\frac{\partial T}{\partial\tilde{\Lambda}} + \\
&\left[\omega_3 - \left(\frac{\partial(\varepsilon\mathcal{K}_0)}{\partial P}\right)_{M=M_T}\right]\left[\frac{1}{T^2}\left(\frac{\partial T}{\partial\tilde{\Lambda}}\right)^2 - \frac{1}{T}\frac{\partial^2 T}{\partial\tilde{\Lambda}^2}\right] + \\
&-\frac{1}{T^2}\frac{\partial T}{\partial\tilde{\Lambda}}\int\limits_0^T\left(\frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial P} + \frac{\partial(\varepsilon\mathcal{K}_0)}{\partial x\partial P}\frac{\partial x}{\partial\tilde{\Lambda}} + \frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial y\partial P}\frac{\partial y}{\partial\tilde{\Lambda}}\right)dt + \\
&\frac{1}{T}\int\limits_0^T\left[\frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial\Lambda^2\partial P} + \frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial x\partial P}\frac{\partial x}{\partial\tilde{\Lambda}} + \frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial y\partial P}\frac{\partial y}{\partial\tilde{\Lambda}} + \right. \qquad (3.185) \\
&\frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial x\partial P}\frac{\partial^2 x}{\partial\tilde{\Lambda}^2} + \frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial y\partial P}\frac{\partial^2 y}{\partial\tilde{\Lambda}^2} + \left(\frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial x\partial P} + \right. \\
&\left.\frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial x^2\partial P}\frac{\partial x}{\partial\tilde{\Lambda}} + \frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial x\partial y\partial P}\frac{\partial y}{\partial\tilde{\Lambda}}\right)\frac{\partial x}{\partial\tilde{\Lambda}} + \\
&\left.+\left(\frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial y\partial P} + \frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial x\partial y\partial P}\frac{\partial x}{\partial\tilde{\Lambda}} + \frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial y^2\partial P}\frac{\partial y}{\partial\tilde{\Lambda}}\right)\frac{\partial y}{\partial\tilde{\Lambda}}\right]dt,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2\omega_3}{\partial\tilde{\Lambda}\partial Z} = & -\left[\frac{\partial\omega_3}{\partial\tilde{\Lambda}} - \left(\frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial P} + \frac{\partial^2\mathcal{K}_0}{\partial x\partial P}\frac{\partial x}{\partial\tilde{\Lambda}} + \frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial y\partial P}\frac{\partial y}{\partial\tilde{\Lambda}}\right)_{M=M_T}\right]\frac{1}{T}\frac{\partial T}{\partial Z} + \\
& \left[\omega_3 - \left(\frac{\partial(\varepsilon\mathcal{K}_0)}{\partial P}\right)_{M=M_T}\right]\left[\frac{1}{T^2}\frac{\partial T}{\partial\tilde{\Lambda}}\frac{\partial T}{\partial Z} - \frac{1}{T}\frac{\partial^2 T}{\partial\tilde{\Lambda}\partial Z}\right] + \\
& -\frac{1}{T^2}\frac{\partial T}{\partial\tilde{\Lambda}}\int\limits_0^T\left[\frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial x\partial P}\frac{\partial x}{\partial Z} + \frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial y\partial P}\frac{\partial y}{\partial Z} + \frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial P^2}\right]dt + \\
& \frac{1}{T}\int\limits_0^T\left[\frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial x\partial P}\frac{\partial x}{\partial Z} + \frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial y\partial P}\frac{\partial y}{\partial Z} + \frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial\Lambda\partial P^2} + \right. \\
& \frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial x\partial P}\frac{\partial^2 x}{\partial\tilde{\Lambda}\partial Z} + \frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial y\partial P}\frac{\partial^2 y}{\partial\tilde{\Lambda}\partial Z} + \left(\frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial x^2\partial P}\frac{\partial x}{\partial Z} + \right. \\
& \left.\frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial x\partial y\partial P}\frac{\partial y}{\partial Z} + \frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial x\partial P^2}\right)\frac{\partial x}{\partial\tilde{\Lambda}} + \\
& \left.\left(\frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial x\partial y\partial P}\frac{\partial x}{\partial Z} + \frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial y^2\partial P}\frac{\partial y}{\partial Z} + \frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial y\partial P^2}\right)\frac{\partial y}{\partial\tilde{\Lambda}}\right]dt + \\
& \left[\frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial x\partial P}\frac{\partial x}{\partial Z} + \frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial y\partial P}\frac{\partial y}{\partial Z} + \frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial P^2}\right]_{M=M_T}\frac{1}{T}\frac{\partial T}{\partial\tilde{\Lambda}}.
\end{aligned}
\tag{3.186}
$$

Svi izvodi na desnoj strani gornja dva izaza su poznati. Preostalo je da se uradi numerička integracija za vreme perioda T, a za taj posao se troši najviše računarskog vremena.

Sledeća tri izvoda imaju jednostavne formule, ali za njihovo izračunavanje potrebno je, pored prvih izvoda perioda T po J i Z, naći i njegov mešovit drugi izvod. Za nalaženje prvih i drugih izvoda perioda koristimo postupak koji je primenjivan kod jednačina (3.163) i (3.176). Imamo

$$
\frac{\partial^2\omega_2}{\partial J^2} = \frac{4\pi}{T^3}\left(\frac{\partial T}{\partial J}\right)^2 - \frac{2\pi}{T^2}\frac{\partial^2 T}{\partial J^2}, \tag{3.187}
$$

$$
\frac{\partial^2\omega_2}{\partial J\partial Z} = \frac{4\pi}{T^3}\frac{\partial T}{\partial J}\frac{\partial T}{\partial Z} - \frac{2\pi}{T^2}\frac{\partial^2 T}{\partial J\partial Z}, \tag{3.188}
$$

$$
\frac{\partial^2\omega_2}{\partial Z^2} = \frac{4\pi}{T^3}\left(\frac{\partial T}{\partial Z}\right)^2 - \frac{2\pi}{T^2}\frac{\partial^2 T}{\partial Z^2}. \tag{3.189}
$$

Poslednji izvod je

$$
\begin{aligned}
\frac{\partial^2\omega_3}{\partial Z^2} = & -\left[\frac{\partial\omega_3}{\partial Z} - 2\left(\frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial x\partial P}\frac{\partial x}{\partial Z} + \frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial y\partial P}\frac{\partial y}{\partial Z} + \frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial P^2}\right)_{M=M_T}\right]\frac{1}{T}\frac{\partial T}{\partial Z} + \\
& \left[\omega_3 - \left(\frac{\partial(\varepsilon\mathcal{K}_0)}{\partial P}\right)_{M=M_T}\right]\left[\frac{1}{T^2}\left(\frac{\partial T}{\partial Z}\right)^2 - \frac{1}{T}\frac{\partial^2 T}{\partial Z^2}\right] + \\
& -\frac{1}{T^2}\frac{\partial T}{\partial Z}\int\limits_0^T\left[\frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial x\partial P}\frac{\partial x}{\partial Z} + \frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial y\partial P}\frac{\partial y}{\partial Z} + \frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial P^2}\right]dt + \\
& \frac{1}{T}\int\limits_0^T\left[\frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial x\partial P}\frac{\partial^2 x}{\partial Z^2} + \frac{\partial^2(\varepsilon\mathcal{K}_0)}{\partial y\partial P}\frac{\partial^2 y}{\partial Z^2} + \right. \\
& \left(\frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial x^2\partial P}\frac{\partial x}{\partial Z} + \frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial x\partial y\partial P}\frac{\partial y}{\partial Z} + \frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial x\partial P^2}\right)\frac{\partial x}{\partial Z} + \\
& \left.\left(\frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial x\partial y\partial P}\frac{\partial x}{\partial Z} + \frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial y^2\partial P}\frac{\partial y}{\partial Z} + \frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial y\partial P^2}\right)\frac{\partial y}{\partial Z} + \frac{\partial^3(\varepsilon\mathcal{K}_0)}{\partial P^3}\right]dt,
\end{aligned}
\tag{3.190}
$$

za čije računanje je potrebno naći, pored prvih, i druge izvode promenljivih x i y po Z. To se postiže diferenciranjem nehomogenih varijacionih jednačina (3.182) po Z:

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial^2 x}{\partial Z^2} = & -\frac{\partial^3\mathcal{K}_0}{\partial x^2\partial y}\left(\frac{\partial x}{\partial Z}\right)^2 - 2\frac{\partial^3\mathcal{K}_0}{\partial x\partial y^2}\frac{\partial x}{\partial Z}\frac{\partial y}{\partial Z} - \frac{\partial^3\mathcal{K}_0}{\partial y^3}\left(\frac{\partial y}{\partial Z}\right)^2 + \\
& -2\frac{\partial^3\mathcal{K}_0}{\partial x\partial y\partial P}\frac{\partial x}{\partial Z} - 2\frac{\partial^3\mathcal{K}_0}{\partial y^2\partial P}\frac{\partial y}{\partial Z} - \frac{\partial^2\mathcal{K}_0}{\partial x\partial y}\frac{\partial^2 x}{\partial Z^2} + \\
& -\frac{\partial^2\mathcal{K}_0}{\partial y^2}\frac{\partial^2 y}{\partial Z^2} - \frac{\partial^3\mathcal{K}_0}{\partial y\partial P^2} \\
\frac{d}{dt}\frac{\partial^2 y}{\partial Z^2} = & \frac{\partial^3\mathcal{K}_0}{\partial x^3}\left(\frac{\partial x}{\partial Z}\right)^2 + 2\frac{\partial^3\mathcal{K}_0}{\partial x^2\partial y}\frac{\partial x}{\partial Z}\frac{\partial y}{\partial Z} + \frac{\partial^3\mathcal{K}_0}{\partial x\partial y^2}\left(\frac{\partial y}{\partial Z}\right)^2 + \\
& +2\frac{\partial^3\mathcal{K}_0}{\partial x^2\partial P}\frac{\partial x}{\partial Z} + 2\frac{\partial^3\mathcal{K}_0}{\partial x\partial y\partial P}\frac{\partial y}{\partial Z} + \frac{\partial^2\mathcal{K}_0}{\partial x^2}\frac{\partial^2 x}{\partial Z^2} + \\
& +\frac{\partial^2\mathcal{K}_0}{\partial x\partial y}\frac{\partial^2 y}{\partial Z^2} + \frac{\partial^3\mathcal{K}_0}{\partial x\partial P^2},
\end{aligned}
\tag{3.191}
$$

koje numerički rešavamo za početni uslov $(0,0)$.

Ovde je pokazano kako se računaju svi izvodi do trećeg reda integrabilnog Hamiltonijana $\mathcal{H}_{\rm int} = -\frac{1}{2\tilde{\Lambda}^2} + \varepsilon K_0(\tilde{\Lambda}, J, Z)$ po momentima $\tilde{\Lambda}$, J i Z, koji su potrebni za proveru ispunjenosti uslova za koje važi teorema Nehoroševa: konveksnost (2.6), kvazi-konveksnost (2.7) ili 3–jet nedegenerisanost (2.8). Kao primer, u Tabeli 3.1 su date numeričke vrednosti frekvencija, Hesijana i 3–jet matrice za asteroid (158) Koronis. Jedinice su izabrane tako da su gravitaciona konstanta, masa Sunca i velika poluosa Jupitera jednaki jedinici. Zbog toga je jedinica za frekvenciju jednaka srednjem dnevnom kretanju Jupitera $n_J \approx 299$ ″/d.

Glava 4

Rezultati

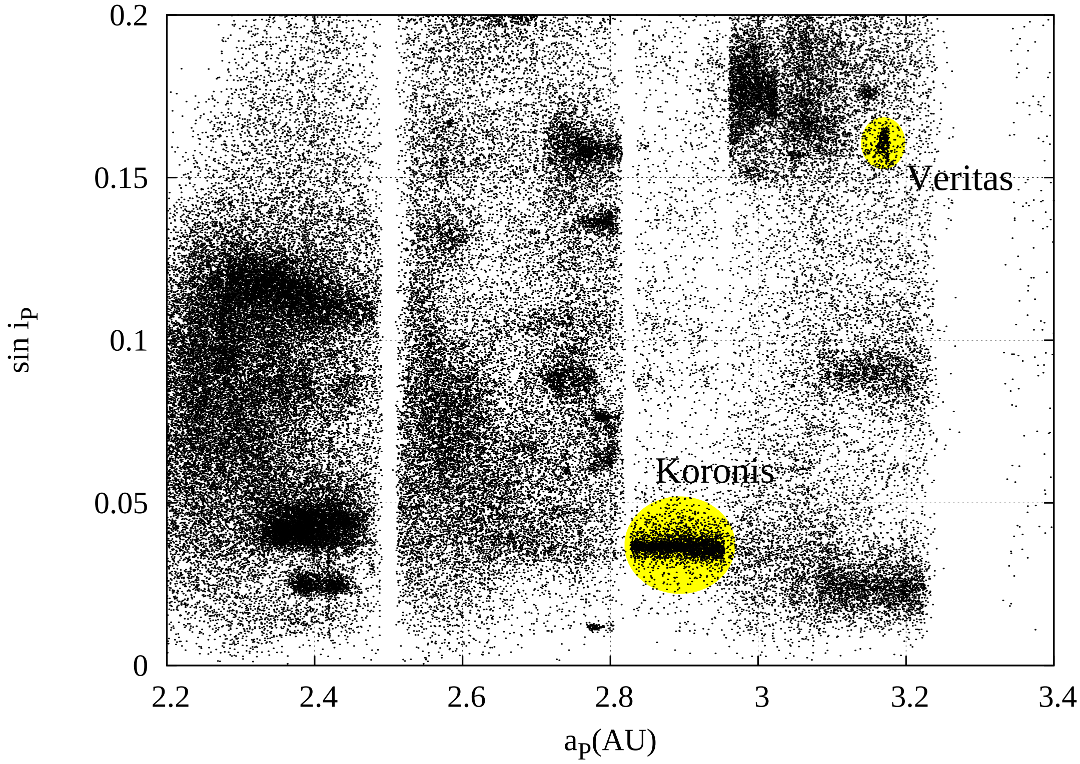
Pomoću izvedene teorije ispitan je fazni prostor u oblasti glavnog asteroidnog prstena gde se nalaze familije Koronis i Veritas. Za identifikaciju članova ovih familija primenili smo HCM[1] metod (Zappalà i dr., 1990, 1995) na katalog numerisanih asteroida AstDys servisa `http://hamilton.dm.unipi.it/cgi-bin/astdys/astibo`. Zapravo, HCM metod se bazira na konstrukciji drveta (dendrograma) za hijerarhijsku klasifikaciju objekata i sastoji se u sledećem: prvo se za dati objekat (Koronis ili Veritas) izračunaju rastojanja u prostoru sopstvenih elemenata (a_P, e_P, i_P) svih ostalih asteroida koristeći euklidovsku metriku oblika

$$d_1 = na_P\sqrt{\frac{5}{4}\left(\frac{\delta a_P}{a_P}\right)^2 + 2\delta e_P^2 + 2(\delta \sin i_P)^2}, \tag{4.1}$$

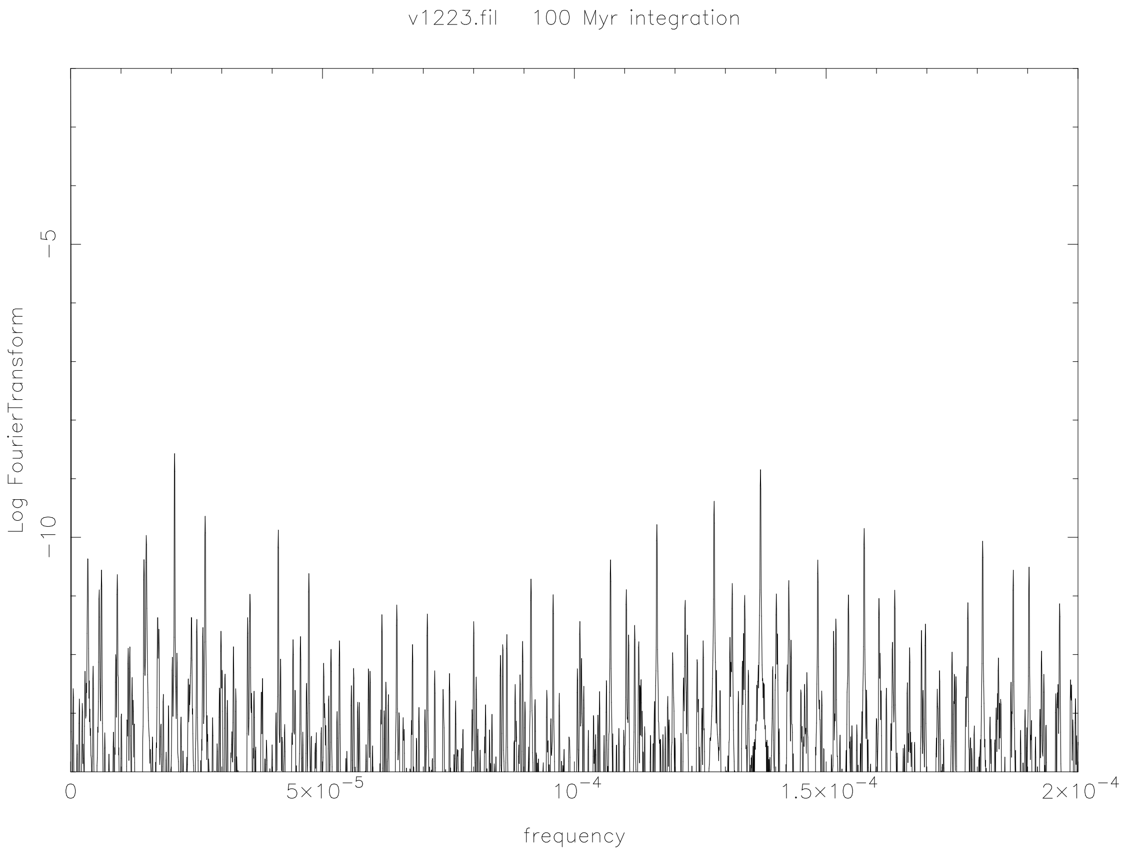
gde je na_P heliocentrična brzina (u m/s) asteroida na kružnoj orbiti čiji je poluprečnik jednak velikoj poluosi a_P. $\delta a = |a_P^{(1)} - a_P^{(2)}|$, $\delta e_P = |e_P^{(1)} - e_P^{(2)}|$, $\delta \sin i_P = |\sin i_P^{(1)} - \sin i_P^{(2)}|$. Indeksi (1) i (2) se odnose na razmatrana tela. Dalje se zahteva da svaki novi član familije bude na manjem rastojanju, koje je unapred zadato, od nekog člana familije. Zappalà i dr. (1990) su modifikovali, adaptirali i optimizovali ovaj metod kako bi identifikovali značajnije grupisanje asteroida u trodimenzionom prostoru sopstvenih elemenata. Jednu verziju programa za HCM napisao je, na ANSI C jeziku, D. Nesvorný i ovaj se može preuzeti sa adrese `http://www.boulder.swri.edu/~davidn/family/family.html`. Primenjujući ovaj program na bazu numerisanih asteroida izdvojili smo 2983 članova familije Koronis i 340 članova familije Veritas.

Guzzo i dr. (2002) primenjuju "spektralnu formulaciju" teoreme Nehoroševa na neke članove ovih familija. Spektralnu formulaciju teoreme Nehoroševa su prvi

[1]Hierarchical Clustering Method (HCM)



Slika 4.1: Sopstvena poluosa i sinus nagiba za oko 130000 asteroida glavnog prstena. Posebno su označene familije Koronis i Veritas.



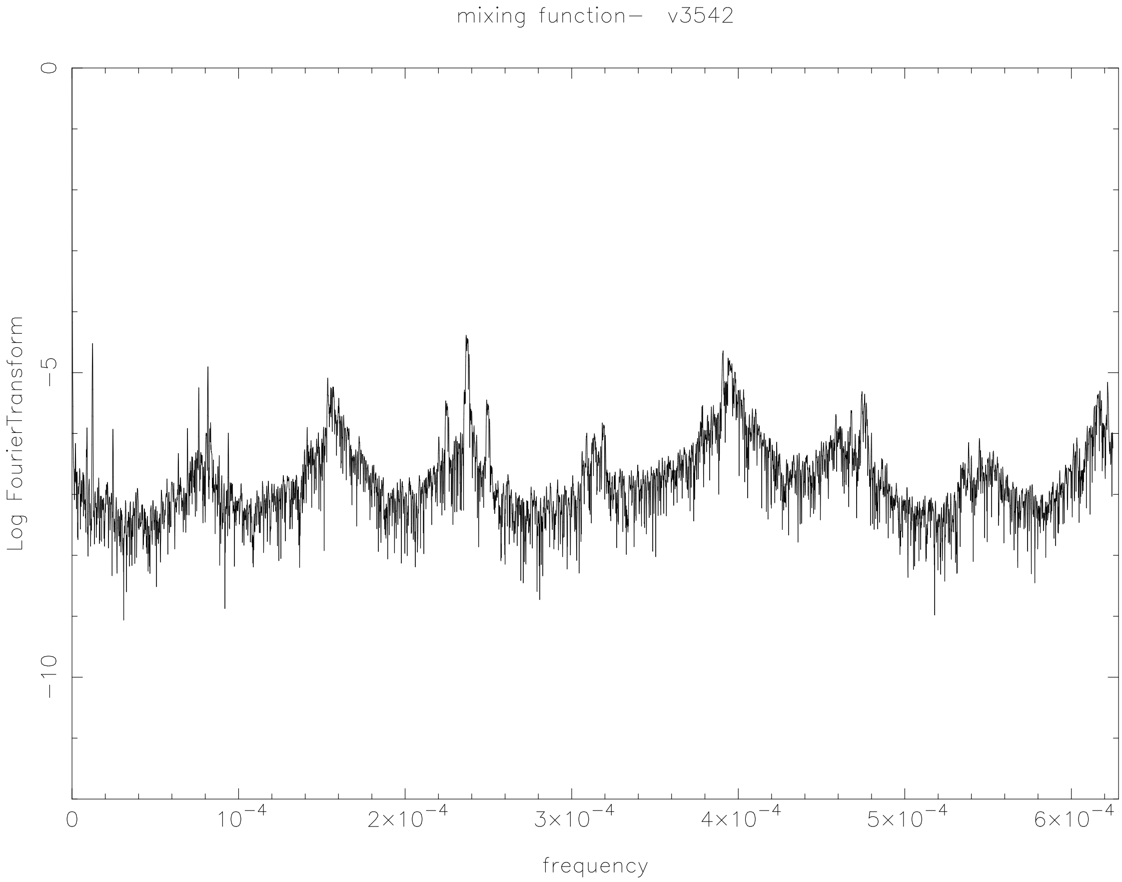
Slika 4.2: Furijeov spektar funkcije $g(t)$ računat za asteroid (1223) Neckar. Pošto se jasno uočava linijska struktura spektra kretanje je kvazi-periodično.

uveli Guzzo i Benettin (2001), a za degenerisane sisteme proširio ju je Guzzo (2001). Zasniva se na Furijeovoj analizi pogodno izabrane test funkcije $\mathcal{G}$, koja zavisi od ekvinoktalnih elemenata a, l, $h = e\cos\varpi$, $k = e\sin\varpi$, $p = \tan(i/2)\cos\Omega$, $q = \tan(i/2)\sin\Omega$ dobijenih numeričkom integracijom jednačina kretanja za neki interval vremena. Test funkciju su izabrali u obliku:

$$\begin{aligned}\mathcal{G} \;=\; & \Big([\cos(h)+\sin(h)+\cos(k)+\sin(k)+\cos(q)+\sin(q)+ \\ & +\cos(p)+\sin(p)+\cos(a)+\sin(a)]^N+1\Big)^{-1}, \end{aligned} \tag{4.2}$$

gde je $N = 14$. Zatim su računali Furijeovu transformaciju funkcije

$$g(t) = \Phi(t)\mathcal{G}(a(t), h(t), k(t), p(t), q(t)), \tag{4.3}$$



Slika 4.3: Kontinualni spektar asteroida (3542) Tanjiazhen koji pripada familiji Veritas. Kretanje je haotično.

Tabela 4.1: Statistički pregled ispunjenih uslova, jednačine (2.6) – (2.8), za familije Koronis i Veritas.

		Koronis		Veritas	
Uslov	Kod	Br. članova	Procenat	Br. članova	Procenat
Konveksnost	1	629	21.09	151	44.41
Kvazi-konveksnost	2	616	20.65	12	3.53
3-jet	3	1667	55.88	164	48.24
Nijedan	0	71	2.38	13	3.82
Ukupno		2983		340	

za pogodno izabranu analitičku funkciju $\Phi(t)$, koja je detaljno opisana u Guzzo i Benettin (2001). Kada se Furijeov spektar funkcije g sastoji od mnoštva razdvojenih pikova, tada taj asteroid ima kvazi-periodično kretanje, tj. nalazi su u režimu Nehoroševa. Na slici 4.2 je prikazan Furijeov spektar funkcije $g(t)$ za asteroid (1223) Neckar koji pripada familiji Koronis. Jasno se vidi linijska struktura spektra, tj. kretanje je, dakle, kvazi-periodično, što potvrđuje da se asteroid nalazi u režimu Nehoroševa. Ukoliko izostane ovakva linijska struktura Furijeovog spektra sistem će biti u režimu Čirikova, tj. haotične difuzije. Primer spektra ovog tipa je prikazan na slici 4.3 za (3542) Tanjiazhen.

Analizom odabranih članova (158, 167, 243, 658, 761, 832, 993, 1289, 1336) Koronis familije (Guzzo i dr., 2002) su dobili slične rezultate kao za (1223) Neckar. Vremenske serije filtriranih orbitalnih elemenata svih ovih tela su pokazivale tipične strukture kvazi-periodičnih oscilacija.

Suprotno je sa haotičnim članovima familije Veritas: 490, 3542, 52244 i 65494. Njihovi spektri su kontinualni iako se sporadično pojavljuju neki pikovi. Dakle, ti spektri mogu poslužiti da se identifikuju objekti koji nisu u režimu Nehoroševa. Zapravo, poznato je da ovi objekti pokazuju sporu haotičnu difuziju u faznom prostoru (Milani i Farinella, 1994; Knežević i Jovanović, 1997; Tsiganis i dr., 2006).

Kao što je ranije spomenuto Guzzo i dr. (2002) nisu ispitali ispunjenost uslova za primenu teoreme Nehoroševa, zato ovaj rad predstavlja njegovu dopunu.

4.1 Familija Koronis

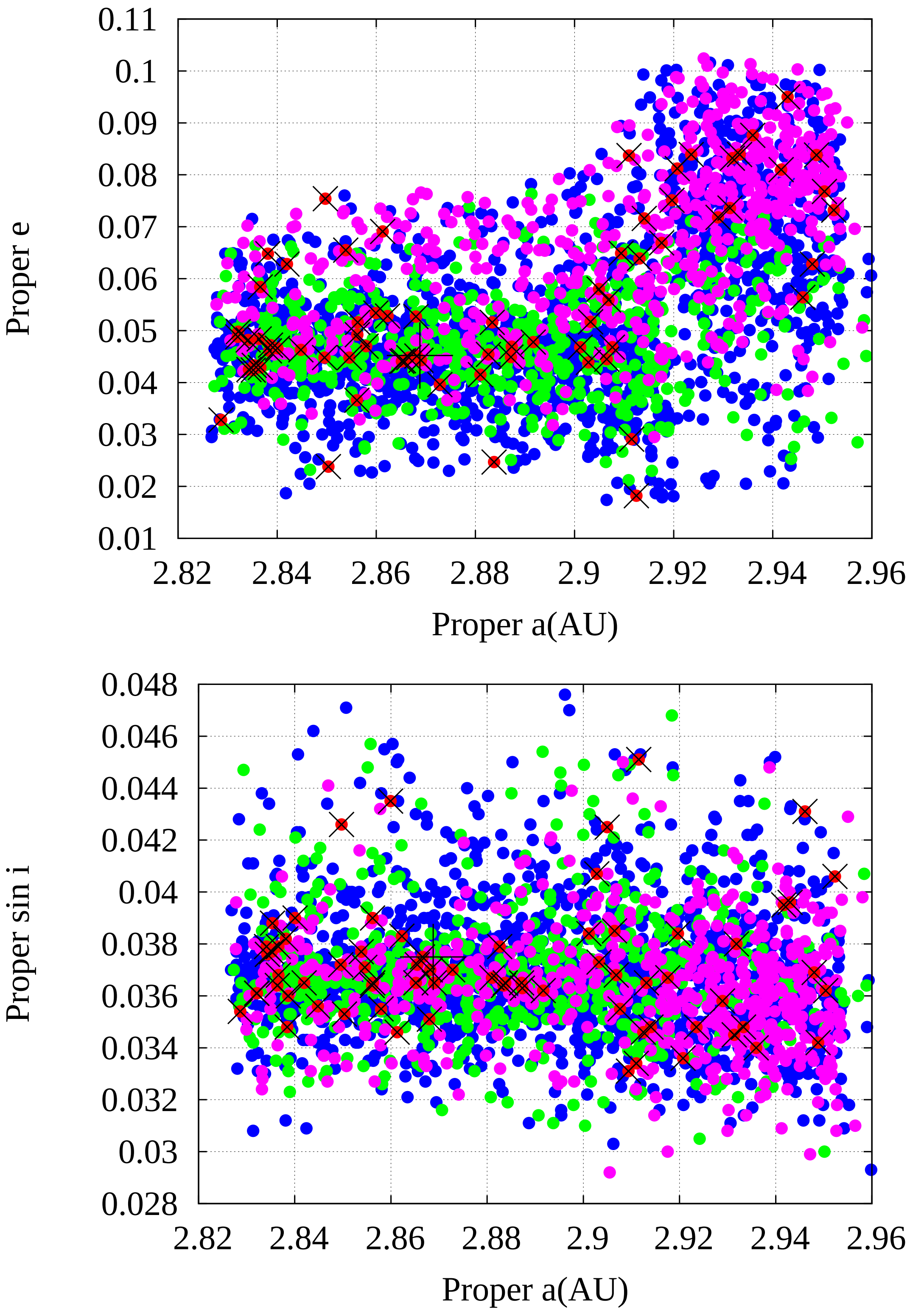
Familija Koronis je smeštena u prostoru sopstvenih elemenata $a_P = 2.82 - 2.96$ AU, $e_P = 0.02 - 0.1$, $\sin i_P = 0.028 - 0.048$ (slika 4.1); ograničena je sa leve strane rezonansom 5–2 u srednjem kretanju sa Jupiterom, a sa desne strane rezonansama višeg reda kao što je na primer 7 : 3 (Bottke i dr., 2001).

Slike 4.4 – 4.8 prikazuju članove familije Koronis sa stanovišta ispunjenosti nekog od uslova: konveksnosti (2.6), kvazikonveksnosti (2.7) ili 3–jet (2.8), kojima su, radi jednostavnijeg zapisa, dodeljeni kodovi 1, 2 i 3 respektivno. Dakle, ovi asteroidi ispunjavaju uslove Teoreme Nehoroševa. Ukoliko nije ispunjen nijedan od uslova (2.6) – (2.8) tada je dodeljen kod 0.

Slika 4.4 prikazuje svih 2983 asteroida, članova familije Koronis, u ravnima sopstvenih elemenata $a_P - e_P$ (gore) i $a_P - \sin i_P$ (dole) obojenih tako da kodovima 1, 2, 3 i 0 odgovaraju sledeće boje: ružičasta, zelena, plava i crvena respektivno. Zbog bolje preglednosti, na posebnim slikama su izdvojeni samo oni članovi familije koji ispunjavaju uslov konveksnosti (slika 4.5), kvazi-konveksnosti (slika 4.6), 3–jet (slika 4.7), ili ne ispunjavaju nijedan od pomenutih uslova (slika 4.8). Pregledom slika 4.5 – 4.8 se ne uočava nikakva specijalna raspodela asteroida u zavisnosti od ispunjenog uslova. Međutim, kada uključimo, kao treći parametar, početnu vrednost longitude perihela vidimo da se pojavljuju zone (slika 4.9), gde je ispunjen uslov konveksnosti, oko 0° odnosno 360°; kvazi-konveksnosti oko 60° i 300°; dok je najšira 3–jet plava zona, između 80° i 300°, razbijena na tri dela dvema tankim crvenim zonama gde nije ispunjen ni jedan od ispitivanih uslova. Zapravo, širina crvene zone je posledica izbora referentnog nivoa za neispunjavanje 3–jet uslova (u ovom radu je to 10^{-3} u jedinicama srednjeg dnevnog kretanja Jupitera). Zone se još bolje uočavaju kada ispunjenost nekog od uslova prikažemo u $e_P - \omega$ (slika 4.10) ili $\sin i_P - \omega$ ravni (slika 4.11).

Da bi detaljnije ispitali zavisnost ispunjenog uslova od početne vrednosti argumenta perihela, fiksirali smo sopstvene vrednosti velike poluose a_P, ekscentriciteta e_P i nagiba i_P, a varirali argument perihela u intervalu od 0° do 360° sa korakom 1°. Rezultati su prikazani na slikama 4.12 i 4.13 za po jednog predstavnika svake klase familije Koronis. Sa ovih slika vidimo da je ispunjeni uslov veoma osetljiv na početnu vrednost argumenta perihela. Tako na primer (158) Koronis ne bi ispunjavao ni jedan uslov da ima početnu vrednost argumenta perihela oko 140° ili 220°, a ispunjavao bi uslov kvazi-konveksnosti za vrednost 60° ili čak uslov konveksnosti za vrednost 15°. Slično važi za ostale predstavnike svojih klasa: (263) Dresda, (277) Elvira i (975) Perseverantia.

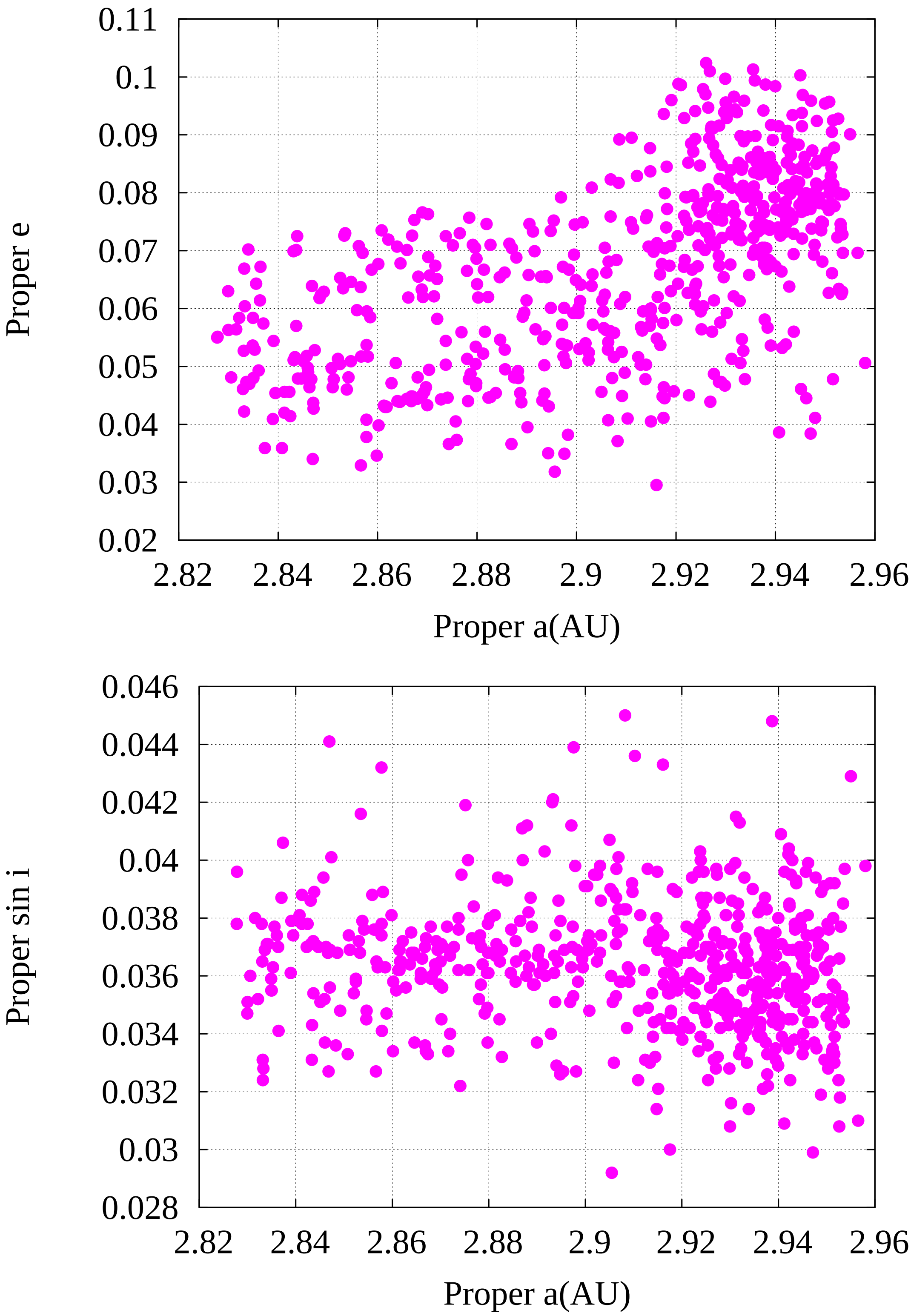
Iz Tabele 4.1 se vidi da samo 71 asteroid (2.38%) ne ispunjava nijedan od ispitivanih uslova, dok najveći broj članova 1667 (55.88%) ispunjava 3–jet uslov, kvazikonveksnost 616 objekata (20.65%) dok najstrožiji uslov ispunjava 629 (21.09%) članova. Dakle, možemo zaključiti da u ovoj oblasti faznog prostora asteroidi ispunjavaju uslove za primenu teoreme Nehoroševa, ali treba voditi računa o početnim



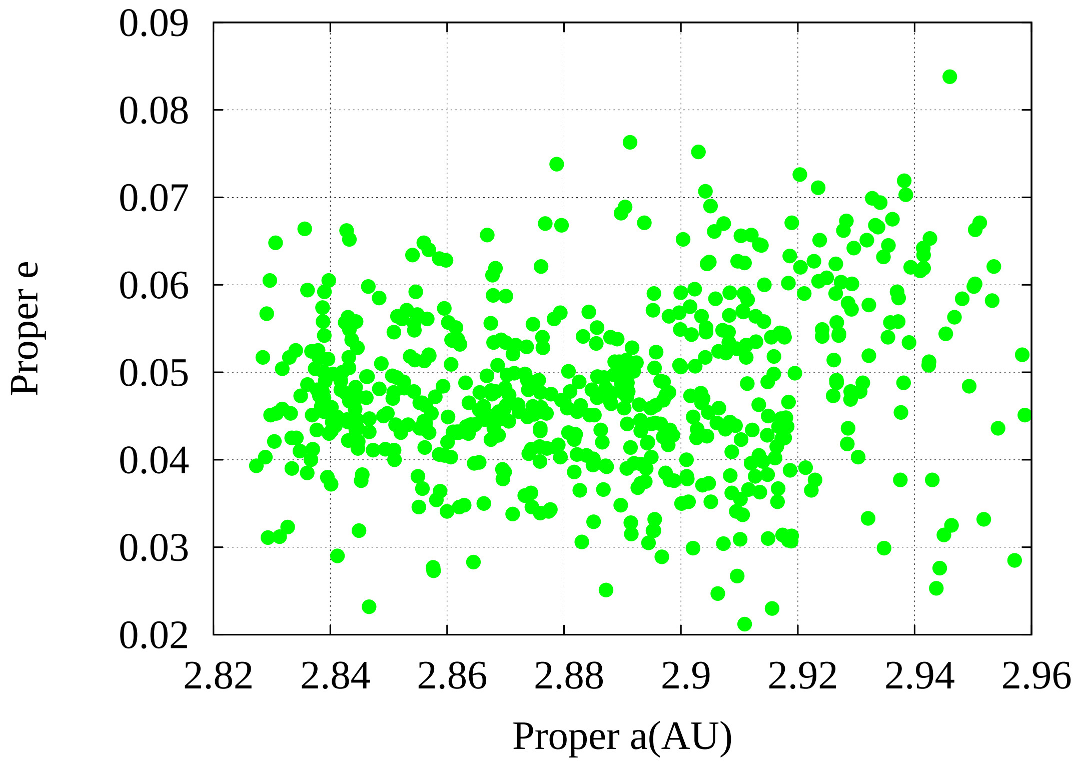
Slika 4.4: Članovi familije Koronis u ravni sopstvenih elemenata $a_P - e_P$ (gore) i $a_P - \sin i_P$ ravni (dole). Članovi koji ispunjavaju uslov konveksnosti su ružičasti, kvazi–konveksnost su zeleni, 3–jet uslov su plavi, a oni koji ne ispunjavaju nijedan od uslova, relacije (2.6) – (2.8), su crveni i označeni simbolom $\times$. Znakom $+$ je obeležen asteroid (158) Koronis.

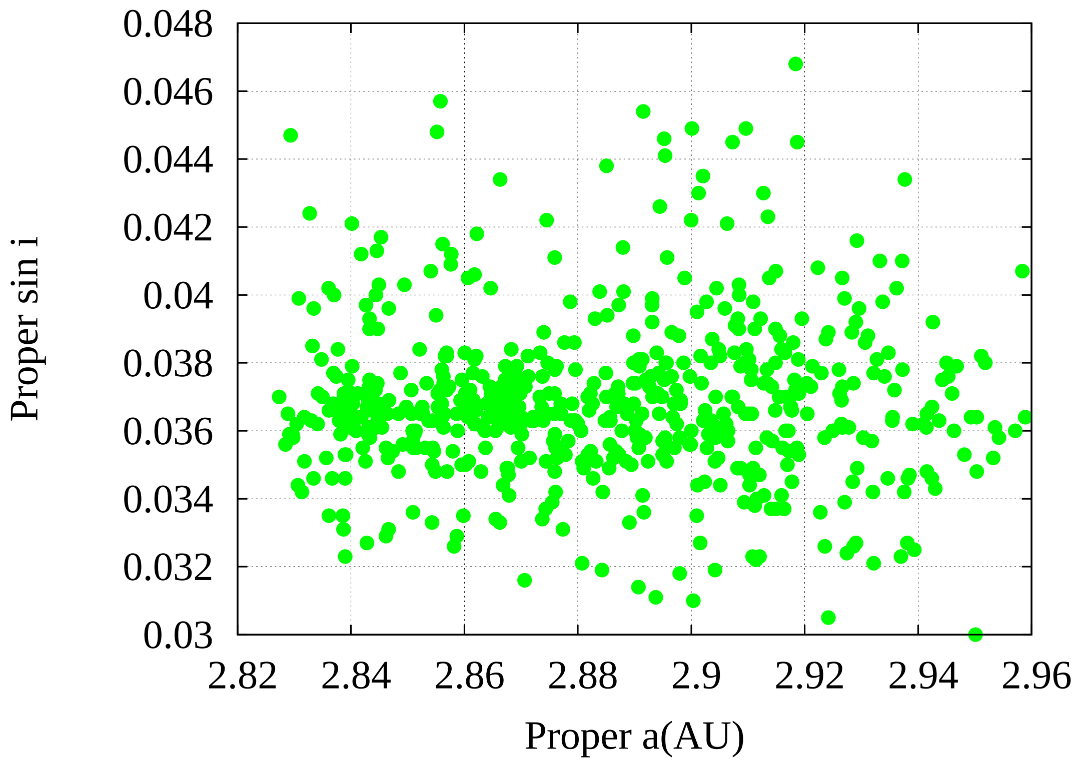
vrednostima argumenta perihela. Na ovaj način je opravdana direktna primena spektralne formulacije teoreme Nehoroševa u radu Guzzo i dr. (2002).

U Tabeli 4.2 navodimo karakteristične podatke za po nekoliko objekata iz svake klase familije Koronis. Za svaki asteroid, u gornjem redu su dati sopstveni elementi a_p, e_p i $\sin i_P$, zatim odgovarajuće frekvencije ω_1, ω_2 i ω_3. U drugom redu su momenti Λ, J, Z i svojstvene vrednosti Hesijana λ_1, λ_2 i λ_3. Podsetimo se da su masa Sunca, gravitaciona konstanta i velika poluosa Jupitera uzeti za jedinicu. Tako, na primer u Tabeli 4.2 frekvencije su izražene u jedinicama srednjeg dnevnog kretanja Jupitera ($n_J \approx 299$ ″/d). Poslednja kolona u tabeli je oznaka kôda.

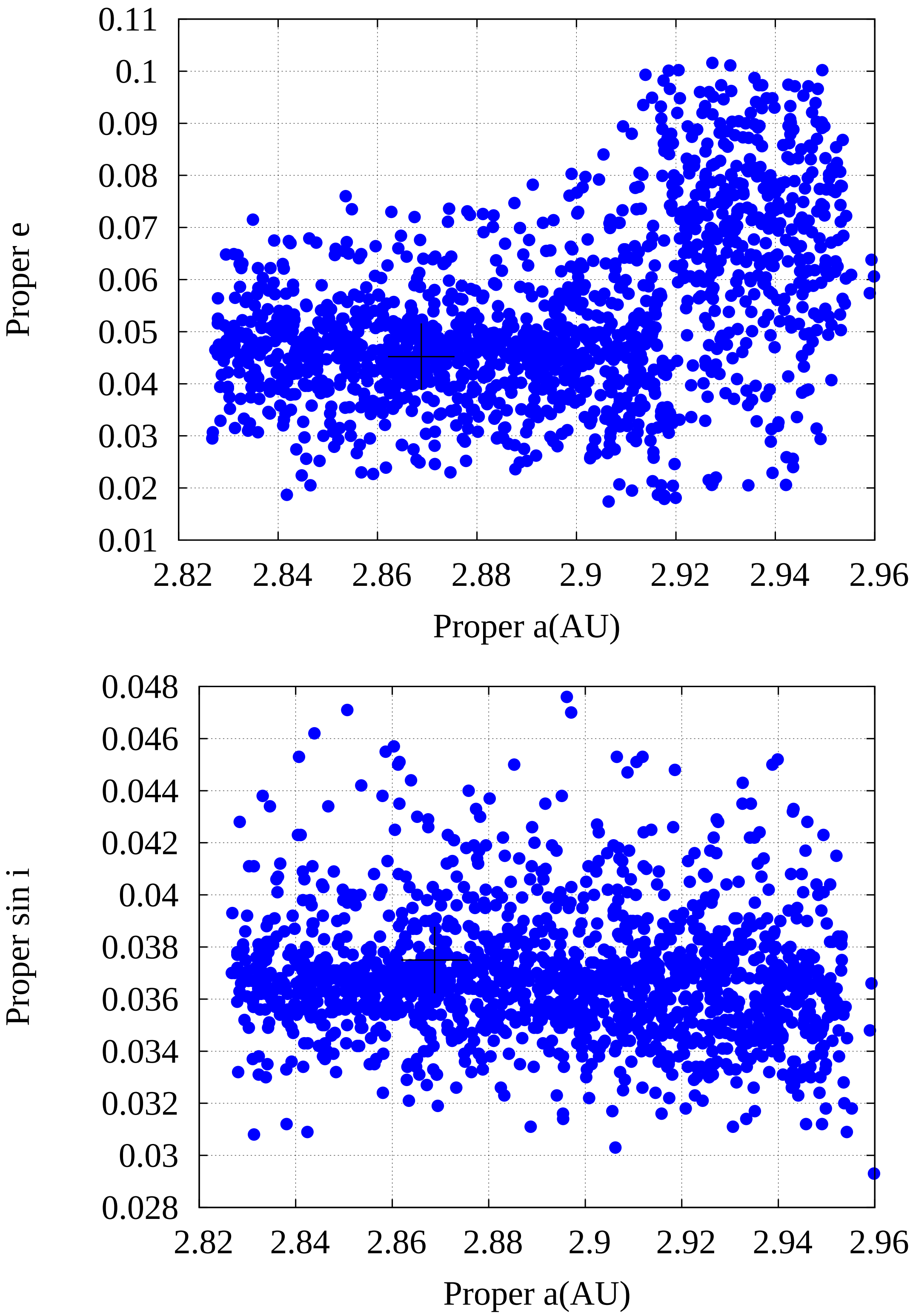


Slika 4.5: Članovi familije Koronis u ravni sopstvenih elemenata $a_P - e_P$ (gore) i $a_P - \sin i_P$ ravni (dole) koji ispunjavaju uslov konveksnosti (2.6).

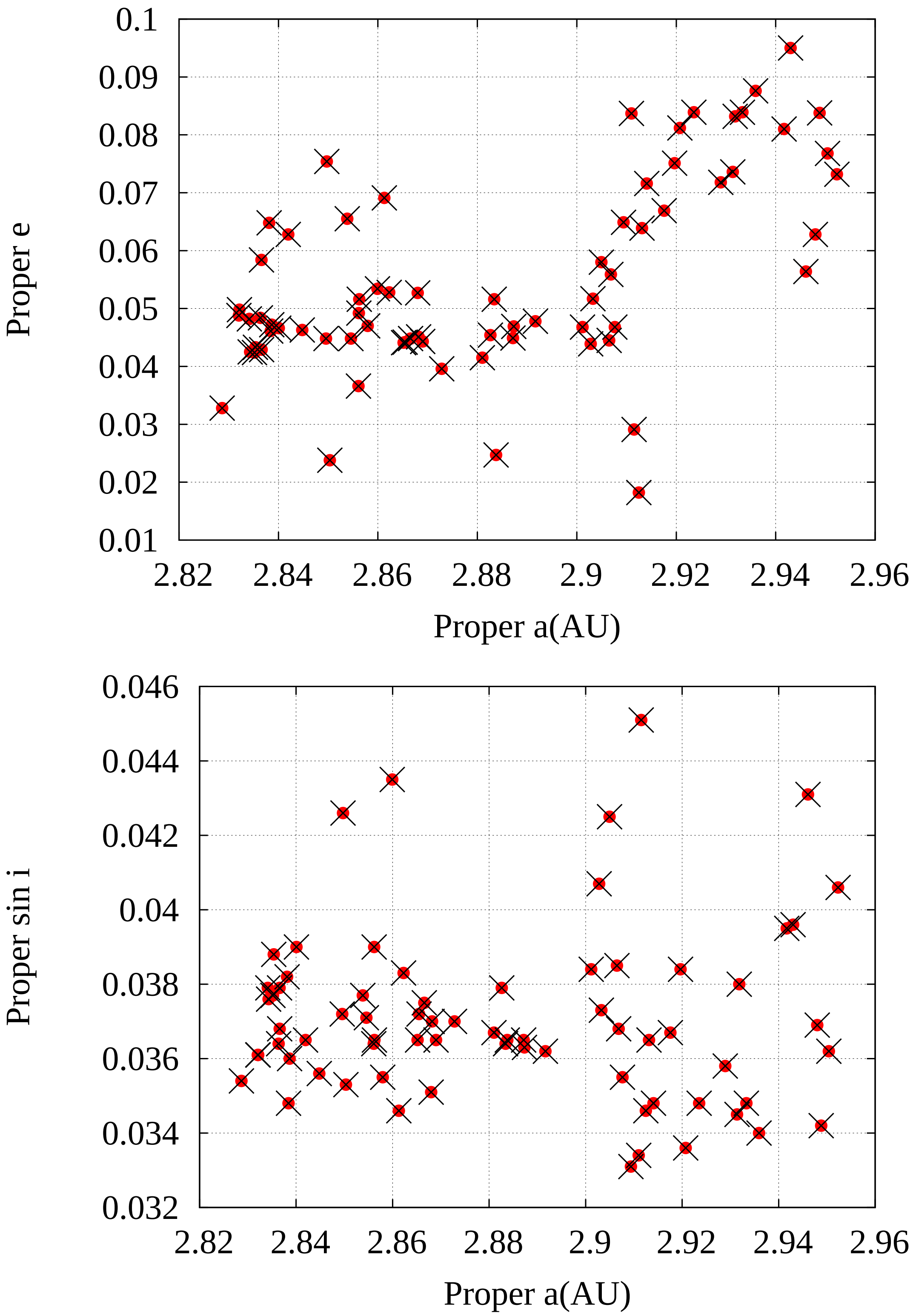




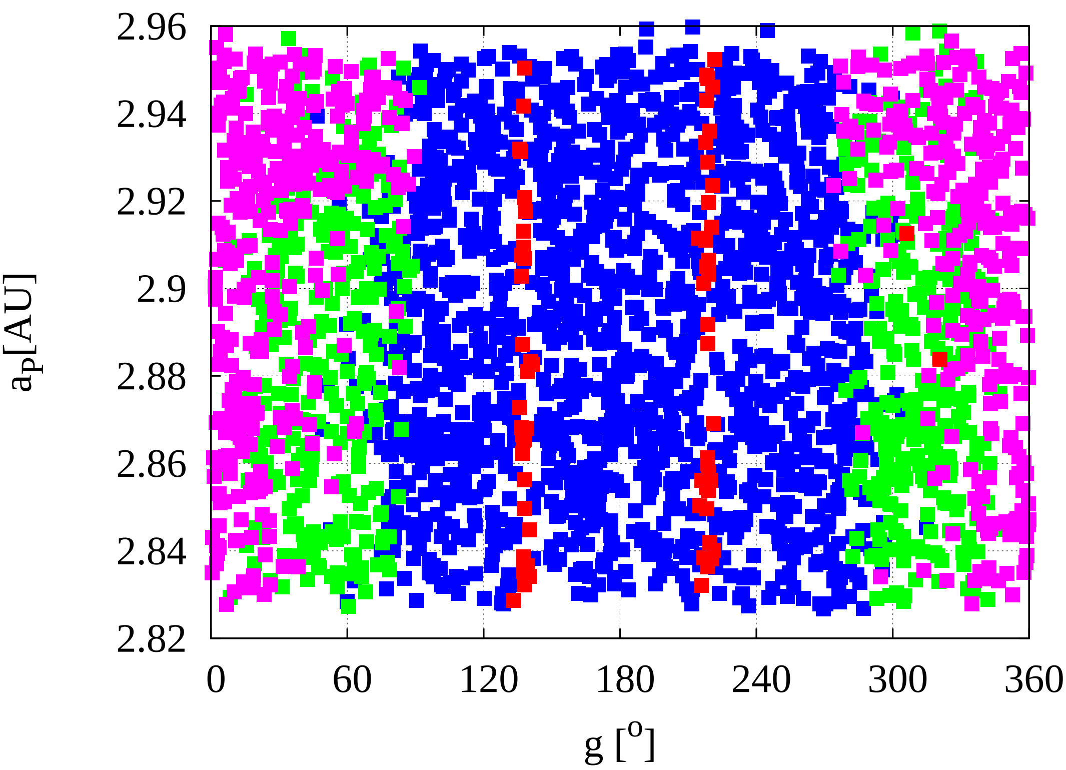
Slika 4.6: Članovi familije Koronis u ravni sopstvenih elemenata $a_P - e_P$ (gore) i $a_P - \sin i_P$ ravni (dole) koji ispunjavaju uslov kvazi-konveksnosti (2.7).



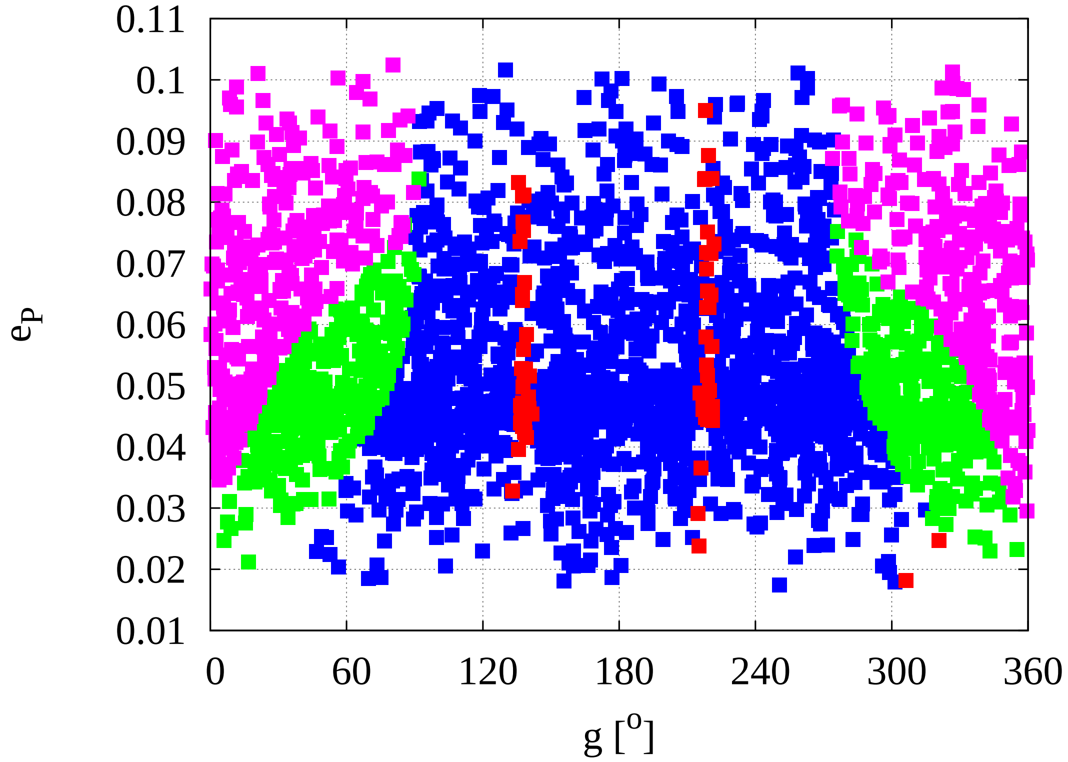
Slika 4.7: Članovi familije Koronis u ravni sopstvenih elemenata $a_P - e_P$ (gore) i $a_P - \sin i_P$ ravni (dole) koji ispunjavaju 3–jet uslov (2.8).Znakom + je obeležen asteroid (158) Koronis.



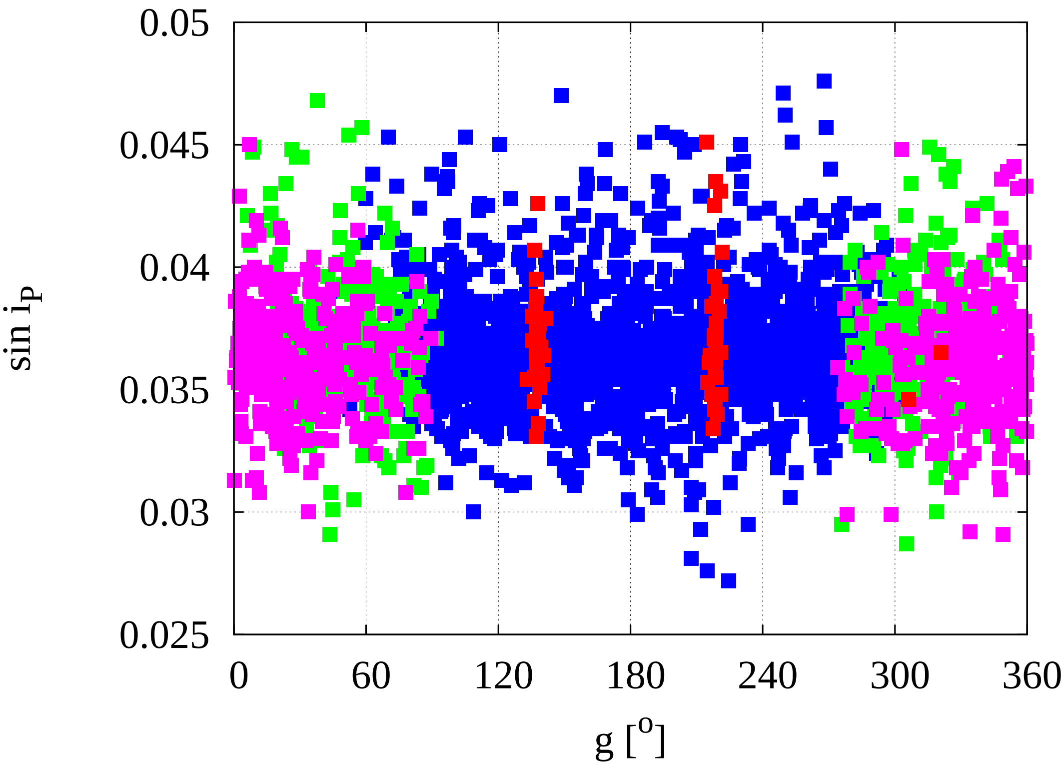
Slika 4.8: Članovi familije Koronis u ravni sopstvenih elemenata $a_P - e_P$ (gore) i $a_P - \sin i_P$ ravni (dole) koji ne ispunjavaju nijedan od uslova, relacije (2.6) – (2.8).



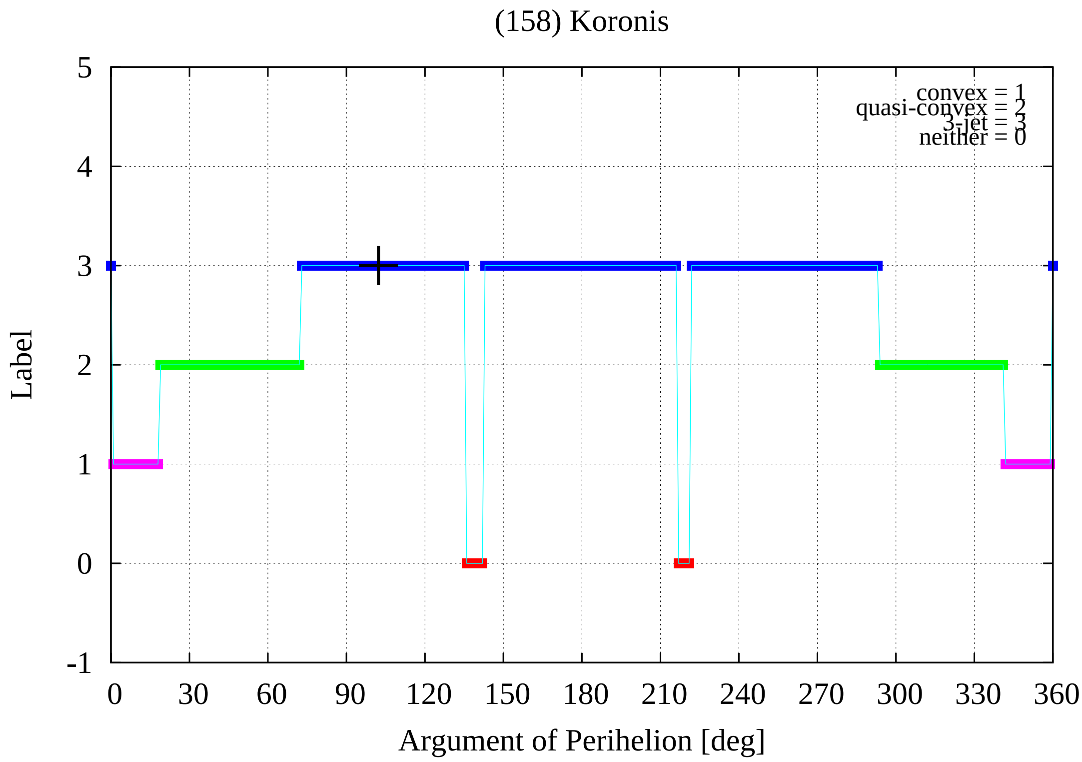
Slika 4.9: Ispunjen uslov u ravni a_P–ω za sve članove familije Koronis.

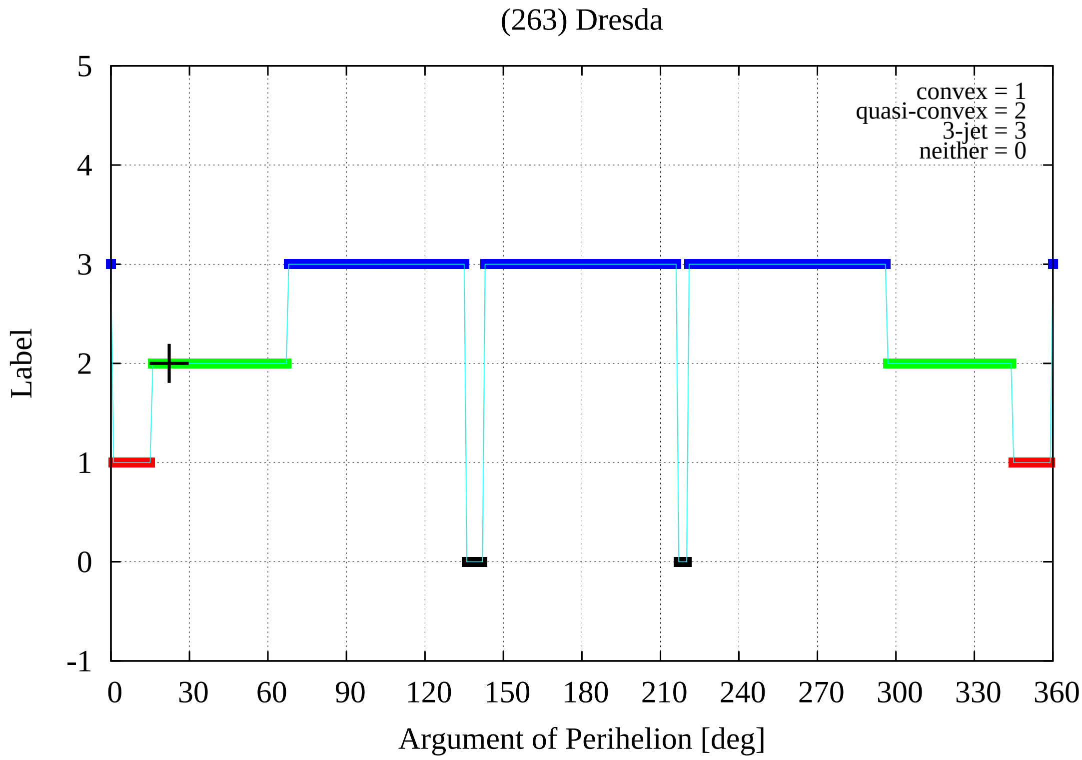


Slika 4.10: Kao na Slici 4.9 samo u e_P–ω ravni.

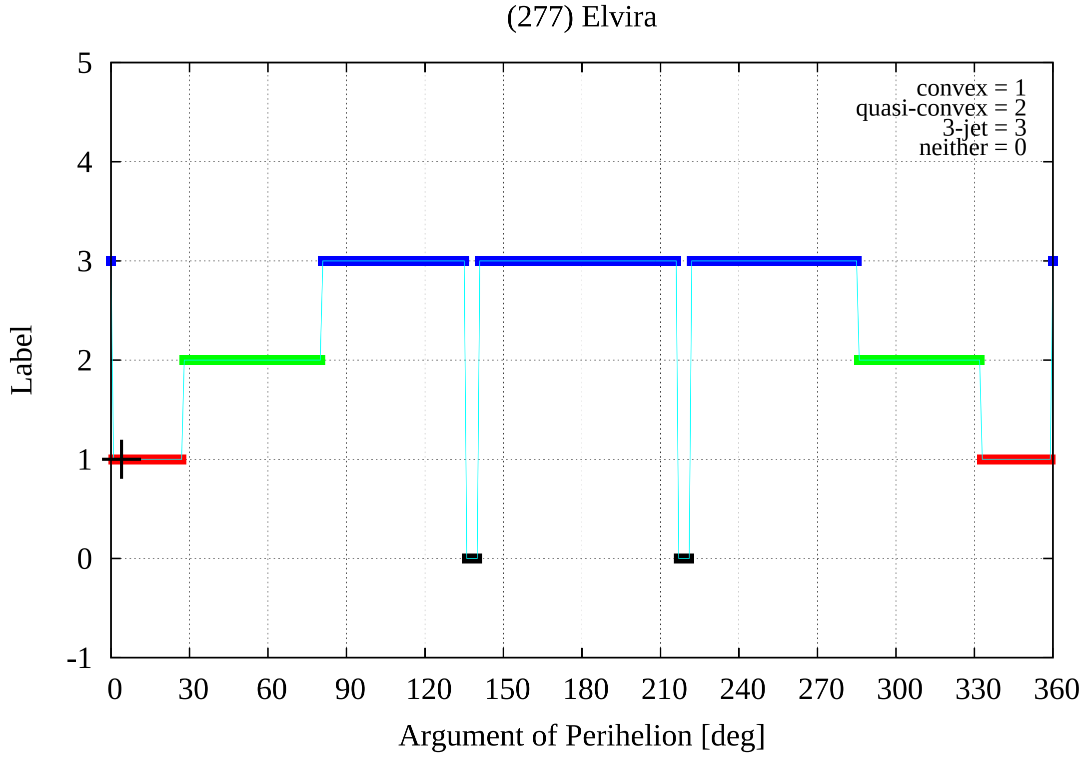


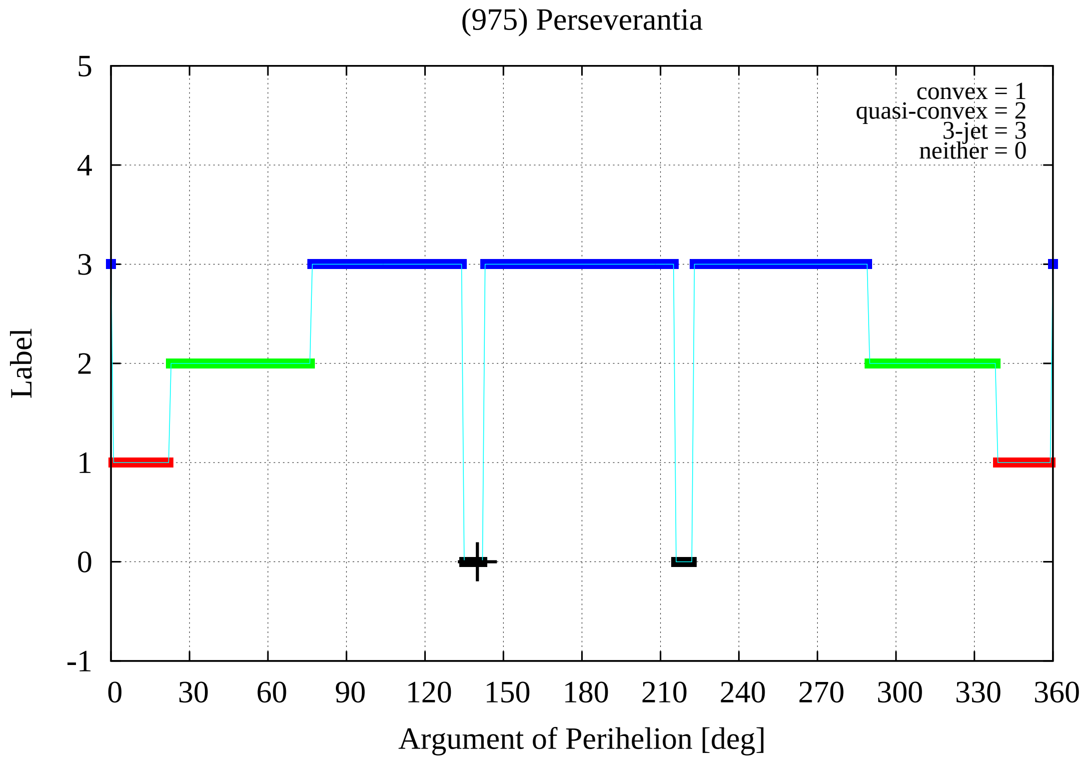
Slika 4.11: Kao na Slici 4.9 samo u $\sin i_P$–ω ravni.





Slika 4.12: Ispunjen uslov kao funkcija argumenta perihela za asteroid (158) Koronis (gore) i (263) Dresda (dole). Početna vrednost argumenta perihela je označena sa +.





Slika 4.13: Ispunjen uslov kao funkcija argumenta perihela za asteroid (277) Elvira (gore) i (975) Perseverantia (dole). Početna vrednost argumenta perihela je označena sa +.

Tabela 4.2: Sopstveni elementi $(a_p, e_p, \sin i_P)$, frekvencije $(\omega_1, \omega_2, \omega_3)$, momenti (Λ, J, Z), svojstvene vrednosti Hesijana $(\lambda_1, \lambda_2, \lambda_3)$, i kôd ispunjenog uslova za izabrane članove familije Koronis.

Asteroid	a_P(AU)	e_P	$\sin i_P$	ω_1	$\omega_2[10^{-2}]$	$\omega_3[10^{-3}]$	Kod
	Λ	$J[10^{-3}]$	$Z[10^{-2}]$	λ_1	$\lambda_2[10^{-2}]$	$\lambda_3[10^{-3}]$	
158	2.86879	0.0452	0.0375	2.4415894	0.11768133	−0.58380762	3
	0.74257530	0.52434127	0.12807165	−10.0832540	−4.47598410	0.65787446	
167	2.85354	0.0431	0.0370	2.4612009	0.11568202	−0.57424770	3
	0.74059896	0.50752977	0.11948341	−10.1997340	−4.30792030	0.96991854	
208	2.89289	0.0449	0.0372	2.4111226	0.12085829	−0.59955161	3
	0.74568787	0.51303807	0.12676508	−9.9081268	−4.65369070	3.80765100	
243	2.86161	0.0457	0.0365	2.4507900	0.11681402	−0.57953041	3
	0.74164546	0.49498496	0.12685412	−10.1227620	−4.41181320	0.73387474	
263	2.88655	0.0420	0.0371	2.4190764	0.11991706	−0.59531217	2
	0.74487030	0.51062478	0.11696131	−9.9808011	−4.56069740	0.67377621	
277	2.88558	0.0529	0.0372	2.4202951	0.12020449	−0.59508586	1
	0.74474514	0.51095749	0.15575418	−9.8987082	−4.66368280	−24.61032400	
311	2.89763	0.0415	0.0372	2.4052054	0.12137129	−0.60255764	3
	0.74629852	0.51372672	0.11590459	−9.9115921	6.52233190	−46.46111900	
321	2.88559	0.0458	0.0381	2.4202841	0.11987710	−0.59451193	3
	0.74474643	0.54179589	0.13216846	−9.9631454	−4.62574760	0.74854526	
452	2.84692	0.0494	0.0362	2.4697947	0.11509885	−0.57055657	3
	0.73973940	0.48121468	0.13874255	−10.1807560	−4.29534310	20.87648000	
462	2.87370	0.0503	0.0362	2.4353293	0.11856868	−0.58752795	1
	0.74321049	0.48380340	0.14272996	−9.9947881	−4.53997840	−0.61893400	
832	2.86438	0.0439	0.0367	2.4472338	0.11710880	−0.58121178	1
	0.74200432	0.49729815	0.12147295	−10.1069870	−4.38995790	−0.00130522	
962	2.90568	0.0624	0.0351	2.3952057	0.12345121	−0.60943788	1
	0.74733446	0.45499667	0.19159958	−9.7259072	−4.93850270	−6.82109050	
975	2.83414	0.0482	0.0379	2.4865299	0.11339433	−0.56219927	0
	0.73807716	0.52911517	0.13875286	−10.2853300	−4.18825670	0.76350570	
1350	2.85805	0.0517	0.0389	2.4553726	0.11650124	−0.57691670	1
	0.74118399	0.55741550	0.15514600	−10.0930560	−4.40527220	−0.22726867	
1423	2.86016	0.0449	0.0371	2.4526553	0.11658657	−0.57846772	2
	0.74145753	0.50815229	0.12577057	−10.1278870	−4.36708080	0.39578290	
1442	2.87491	0.0452	0.0371	2.4337920	0.11849966	−0.58787386	2
	0.74336694	0.50993752	0.12709944	−10.0284060	−4.50348590	0.59112847	
1497	2.89523	0.0571	0.0375	2.4081959	0.12165586	−0.60147088	2
	0.74598939	0.52615288	0.17409608	−9.8301131	−4.82134010	0.16386283	
1725	2.90327	0.0572	0.0374	2.3981924	0.12276675	−0.60690848	1
	0.74702447	0.51910466	0.17448546	−9.7656867	−4.87544140	−0.18312066	
1741	2.88495	0.0394	0.0377	2.4210910	0.11959218	−0.59399599	2
	0.74466384	0.52987721	0.11071864	−10.0346660	−4.52197320	1.33090380	
1742	2.88918	0.0586	0.0357	2.4157684	0.12099869	−0.59810263	1
	0.74520956	0.46987223	0.17548264	−9.8479449	−80.86894400	−47.49251200	
1762	2.87608	0.0460	0.0355	2.4323058	0.11873434	−0.58904399	2
	0.74351819	0.46890429	0.12552198	−10.0216140	−4.54205720	0.60474658	
1894	2.88673	0.0366	0.0354	2.4188506	0.11983288	−0.59570010	2
	0.74489353	0.46528123	0.09656509	−10.0530670	−4.46267590	1.60712180	
2713	2.85457	0.0448	0.0371	2.4598678	0.11586847	−0.57493602	0
	0.74073261	0.50879637	0.12531528	−10.1708280	−4.32600920	1.06687300	
3334	2.84957	0.0448	0.0372	2.4663491	0.11523229	−0.57178629	0
	0.74008360	0.51111600	0.12548033	−10.2049230	−4.28355120	1.05028760	
6826	2.90279	0.0439	0.0407	2.3987908	0.12199948	−0.60503241	0
	0.74696272	0.61814981	0.13384520	−9.8625957	−4.74058900	1.15685300	
8102	2.83519	0.0424	0.0377	2.4851488	0.11334195	−0.56271544	0
	0.73821387	0.52402059	0.11881868	−10.3275250	−4.13671100	1.15038000	
8353	2.94298	0.0950	0.0396	2.3497722	0.13085857	−0.63632752	0
	0.75211590	0.58394018	0.39888975	−9.4335443	−5.61173380	0.64861392	
8581	2.83658	0.0584	0.0379	2.4833184	0.11409015	−0.56405338	0
	0.73839481	0.52893093	0.17898493	−10.2153520	−4.28275970	0.46325434	

4.2 Familija Veritas

Familija Veritas smeštena je u spoljašnjem delu asteroidnog prstena u faznom prostoru sopstvenih elemenata $a_P = 3.15 - 3.185$ AU, $e_P = 0.056 - 0.074$, $\sin i_P = 0.152 - 0.168$. Dinamički posmatrano, u familiji pojedini članovi imaju regularno kretanje (na primer 1086 Nata), dok drugi pokazuju izraženo haotično kretanje (490 Veritas). Nesvorný i Morbidelli (1998, 1999) i Tsiganis i dr. (2006) su pokazali da je to posledica preklapanja harmonika rezonansi u srednjem kretanju $5, -2, -2$ između tri tela – Jupitera, Saturna i asteroida. Međutim, treba napomenuti da je teorija razvijena u prethodnom odeljku samo prvog reda po poremećajnom parametru, te zbog toga ne može korektno da opiše ovakve kompleksne efekte višeg reda. Ipak smo je primenili na sve članove familije Veritas imajući u vidu da će rezultati biti korektni samo za asteroide sa regularnim orbitama.

Na sličan način su, kao za familiju Koronis, dati rezultati za familiju Veritas. Slika 4.14 prikazuje svih 340 asteroida, članova familije Veritas, sa stanovišta ispunjenosti nekog od uslova: konveksnost (ružičasta), kvazi-konveksnost (zelena), 3–jet (plava) i objekti koji ne ispunjavaju ni jedan od pomenutih uslova (crvena) u ravni sopstvenih elemenata $a_P - e_P$ (gore) i $a_P - \sin i_P$ (dole).

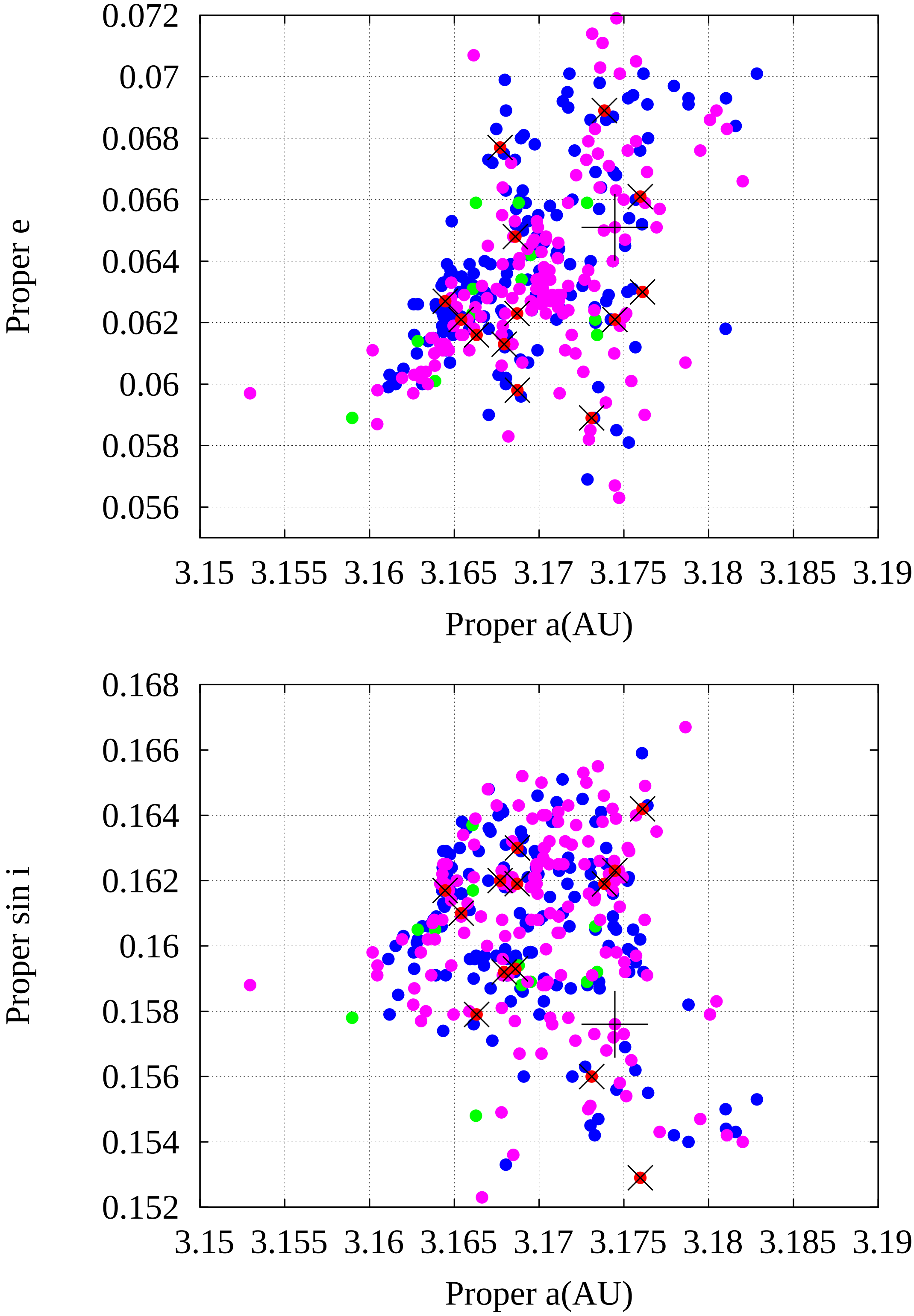
Tabela 4.1 pokazuje da najveći broj članova ove familije ispunjava ili uslov konvekasnosti 151 (44.41%) ili 3–jet uslov 164 (48.24%), kvazi-konveksnost ispunjava samo 12 (3.53%) članova, dok 13 (3.82%) asteroida ne ispunjava nijedan uslov. Za članove koji imaju haotično kretanje ovo neće dati pravu sliku, s obzirom da su uključeni samo efekti prvog reda po poremećajnom faktoru (masa planete), ali je, međutim, uključen doprinos svih velikih planeta. Da bi dobili potpuno korektne rezultate morali bi razviti teoriju višeg reda, tj. onog reda koja će uključivati sve značajne rezonanse što nije nimalo lak zadatak. Ipak, našu teoriju možemo direktno primeniti na one članove familije Veritas koji imaju regularno kretanje kao na primer (1086) Nata.

Na posebnim slikama su prikazani samo članovi familije Veritas koji ispunjavaju uslov konveksnosti (slika 4.15), kvazi-konveksnosti (slika 4.16), 3–jet uslov (slika 4.17) odnosno ne ispunjavaju nijedan uslov (slika 4.18). Konveksni, kvazi-konveksni i 3-jet objekti ne pokazuju neku pravilnost rasporeda u $a_P - e_P$ i $a_P - \sin i_P$ ravnima, dok se sa slike 4.18, za objekte koji ne ispunjavaju nijedan uslov, mogu jasno uočiti dve grupe koje su raspoređene oko $a_P \approx 3.168$ AU i $a_P \approx 3.174$ AU. Ove grupe možemo poistovetiti sa grupama B i A (Tsiganis i dr., 2006), koji su ustanovili da prvoj vrednosti sopstvene velike poluose odgovara centar rezonanse $(3, 3, -2)$, dok se druga, grupa A, nalazi u rezonansi $(5, -2, -2)$. Jedini objekat koji se nalazi na $a_P \approx 3.18$ AU, asteroid (37005) 2000TO37, krećući se blizu $(7, -7, -2)$ rezonanse ispunjava najstrožiji uslov konveksnosti. Međutim, to treba uzeti sa rezervom pošto je rezultat dobijen, kako je ranije rečeno, primenom teorije prvog reda. Dakle, za pojedine asteroide iz ove dve grupe smo dobili da *ne ispunjavaju* ispitivane uslove za primenu teoreme Nehoroševa što je u skladu sa više puta citiranim rezultatima Guzzo i dr. (2002).

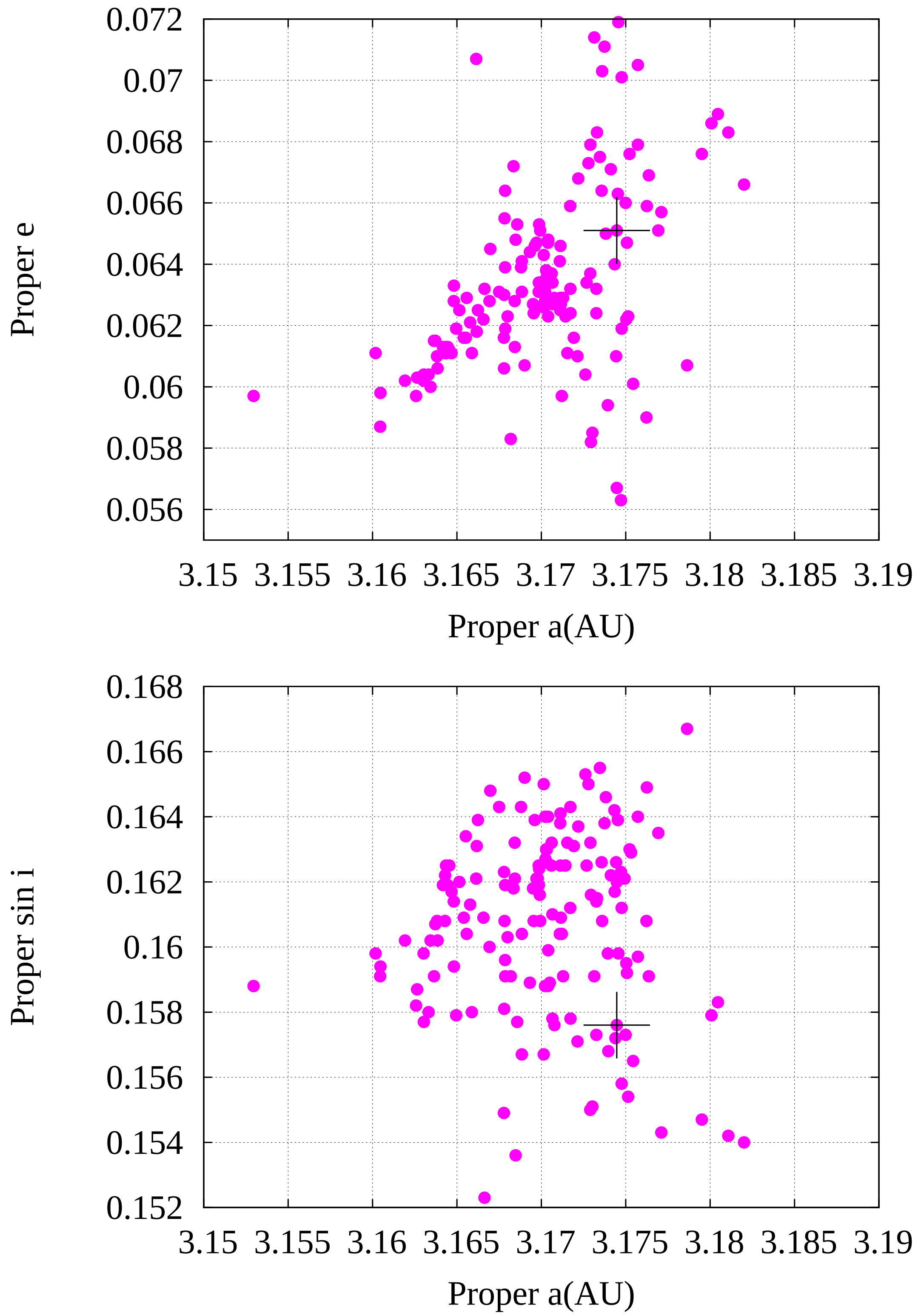
Kada uključimo treći parametar – početnu vrednost argumenta perihela, vidimo da se u faznom prostoru $a_P - \omega$ (slika 4.19), $e_P - \omega$ (slika 4.20) ili $\sin i_P - \omega$ (slika 4.21) jasno razdvajaju oblasti konveksnosti (oko $\omega \approx 0°$), kvazi-konveksnosti (oko $\omega \approx 130°$ i $320°$) i 3–jet oblasti (između $\omega \approx 130°$ i $320°$), koja je presečena sa dve uske trake (za $\omega \approx 160°$ i $240°$) u kojima nije ispunjen nijedan od ispitivanih uslova.

Promena ispunjenog uslova u zavisnosti od početne vrednosti argumenta perihela za najveće telo iz familije, (490) Veritas, data je na slici 4.22 (gore), dok je za najvećeg stabilnog člana familije data na slici 4.22 (dole). Takođe, ova tela ispunjavaju uslov konveksnosti, odnosno 3–jet respektivno (Tabela 4.3). Na slici 4.23 je prikazan asteroid (28306) 1999CV79 kao primer tela koje ispunjava uslov kvazi-konveksnosti (gore) i (16876) 1998BV6 koje ne ispunjava nijedan uslov (dole). Na slici 4.24 se vidi kako je 3–jet "oblast"višestruko ispresecana intervalima u kojima nije ispunjen nijedan uslov.

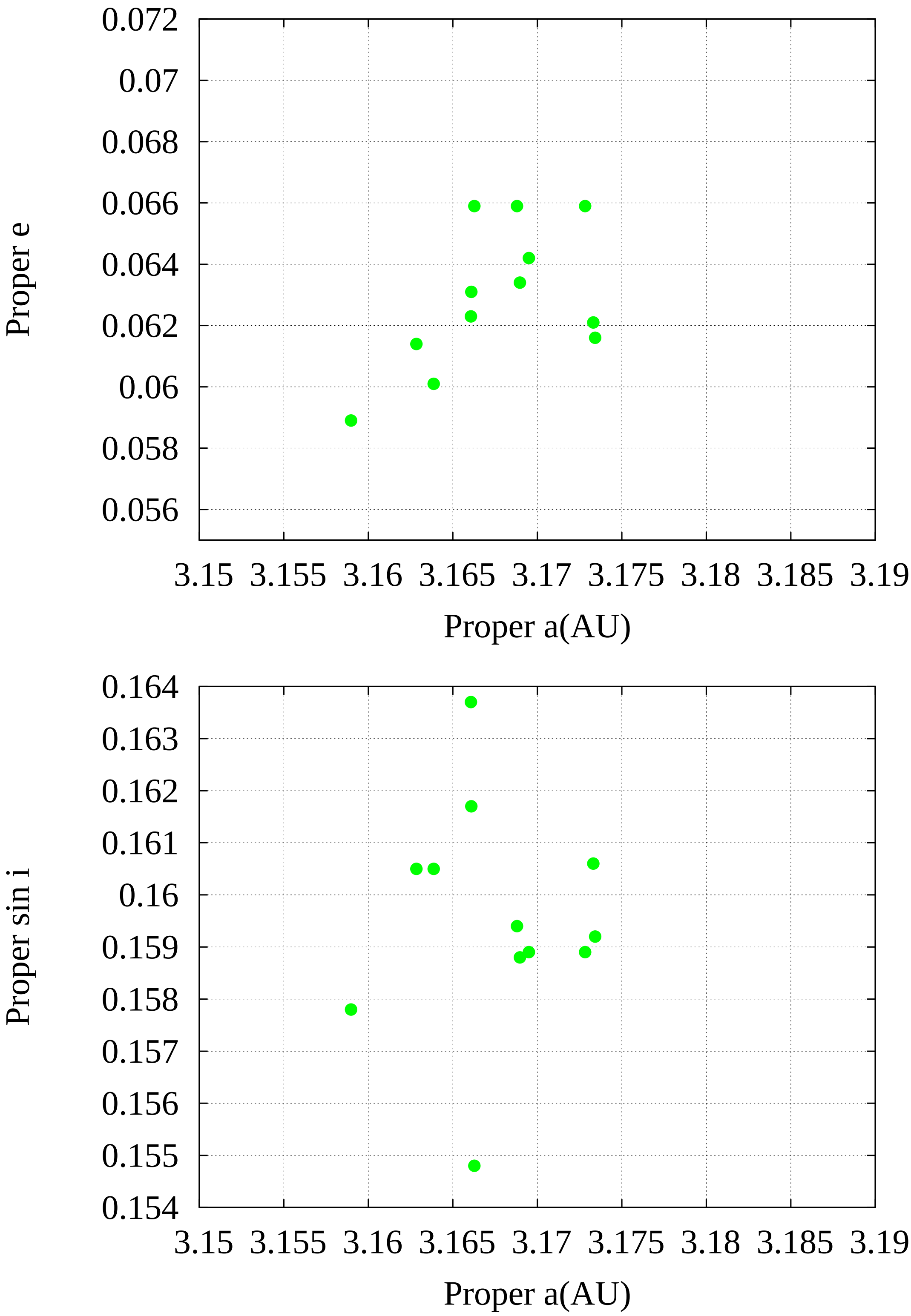
U Tabela 4.3 su dati neki relevantni podaci za familiju Veritas.



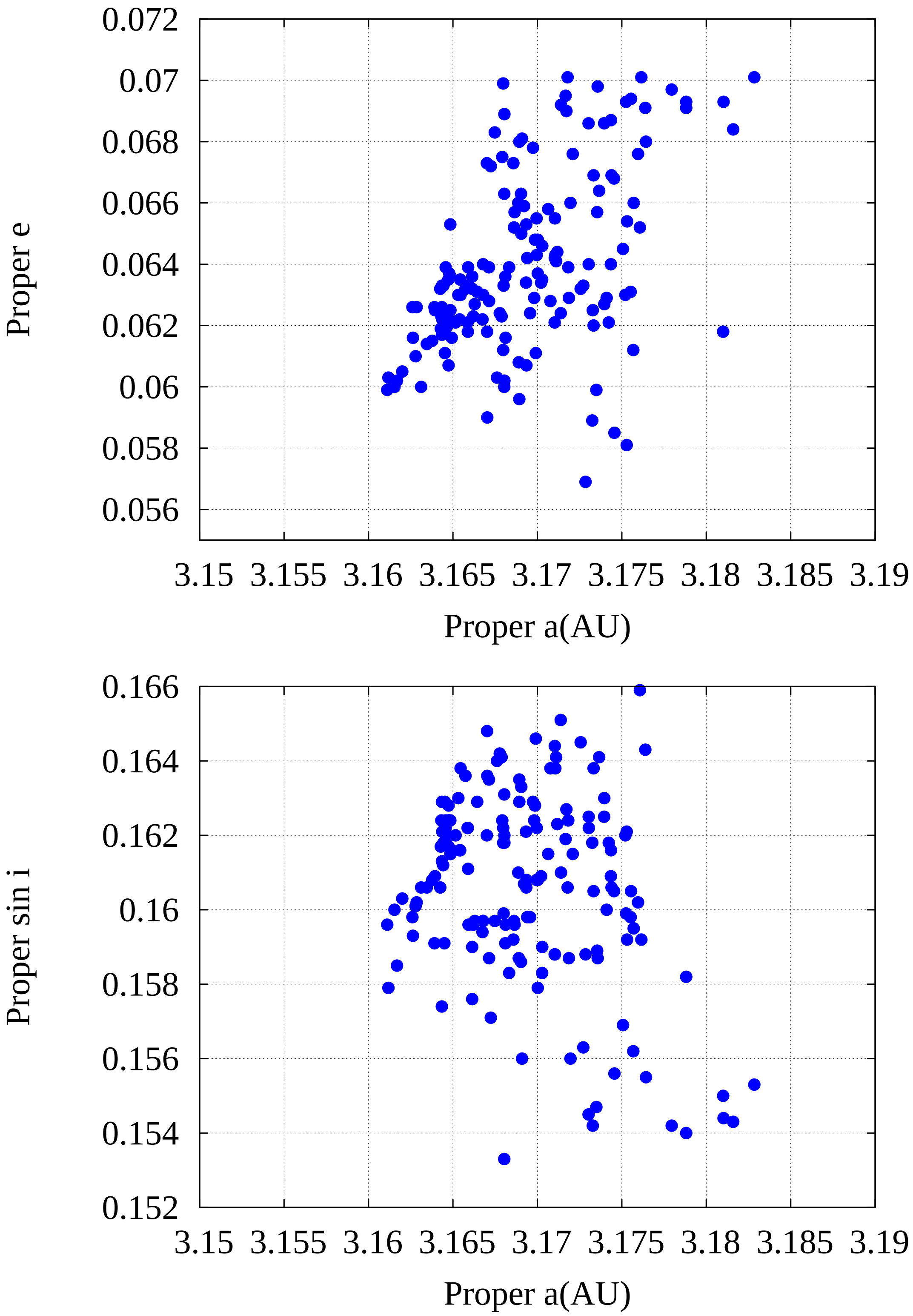
Slika 4.14: Oznake su iste kao na sl. 4.4 samo što je znakom + obeležen asteroid (490) Veritas.



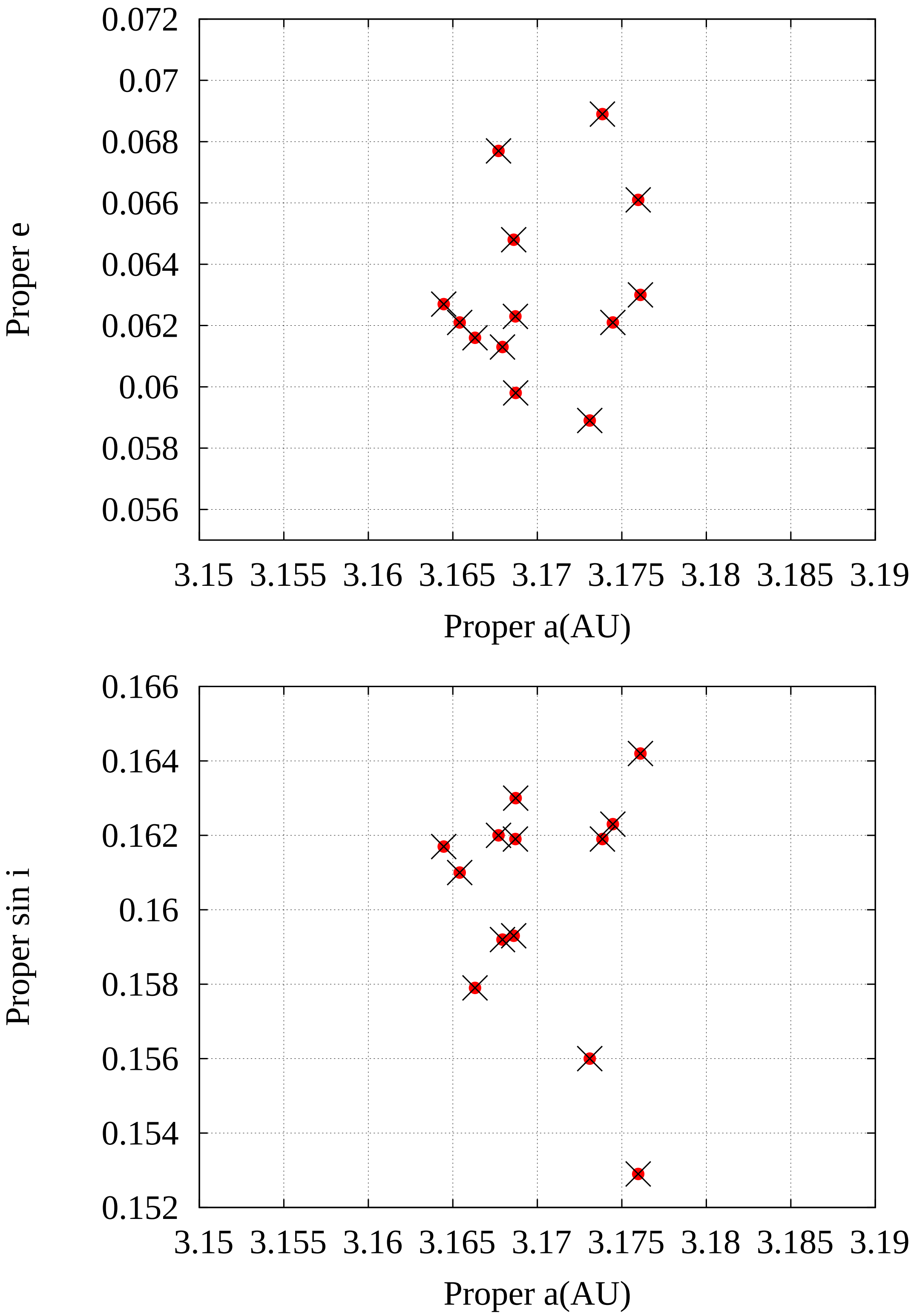
Slika 4.15: Članovi familije Veritas u ravni sopstvenih elemenata $a_P - e_P$ (gore) i u $a_P - \sin i_P$ ravni (dole) koji ispunjavaju uslov konveksnosti (2.6).



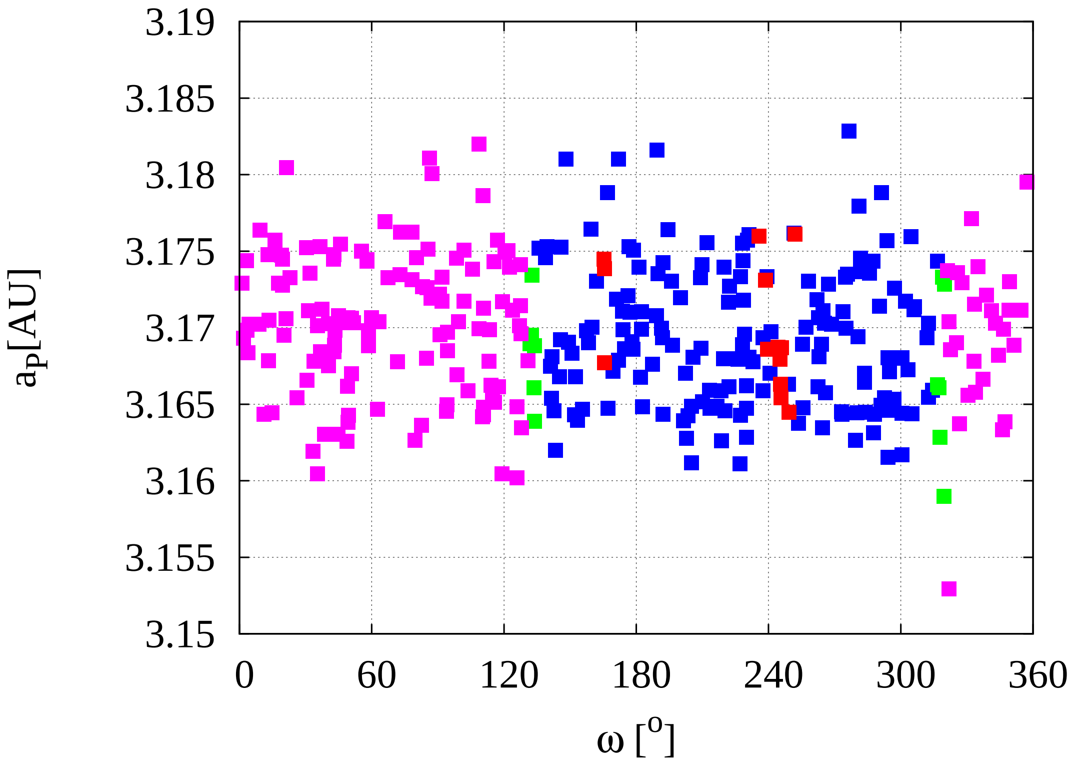
Slika 4.16: Članovi familije Veritas u ravni sopstvenih elemenata $a_P - e_P$ (gore) i u $a_P - \sin i_P$ ravni (dole) koji ispunjavaju uslov kvazi-konveksnosti (2.7).



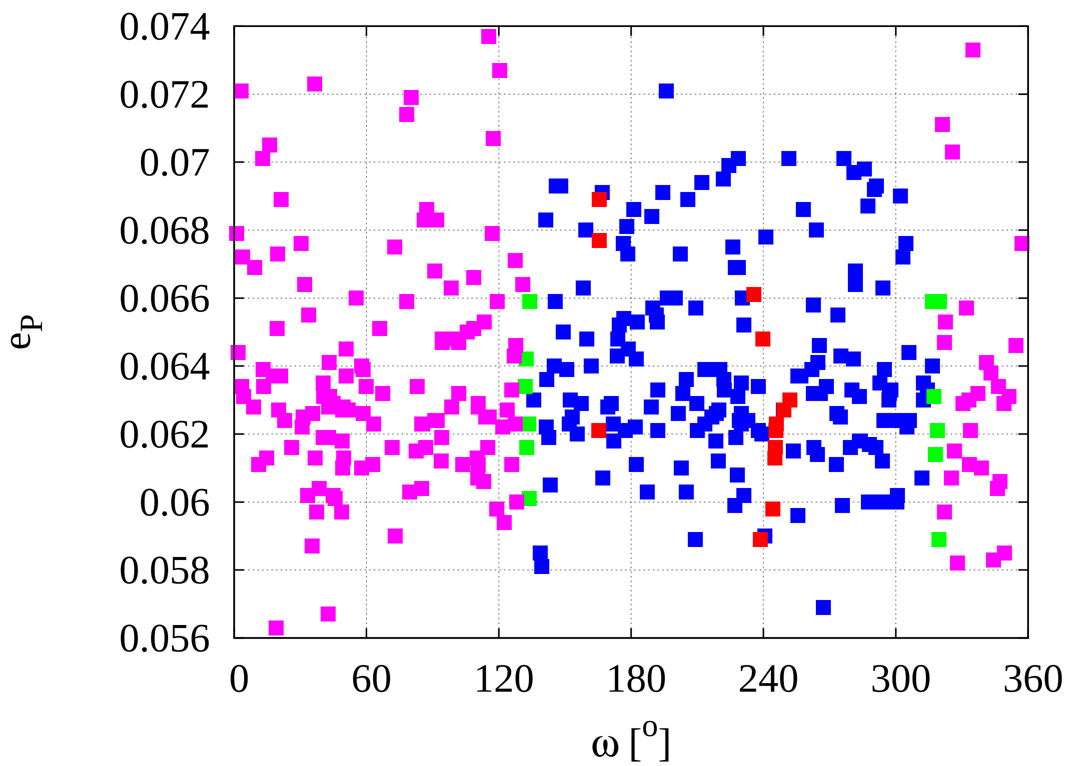
Slika 4.17: Članovi familije Veritas u ravni sopstvenih elemenata $a_P - e_P$ (gore) i u $a_P - \sin i_P$ ravni (dole) koji ispunjavaju 3–jet uslov (2.8).



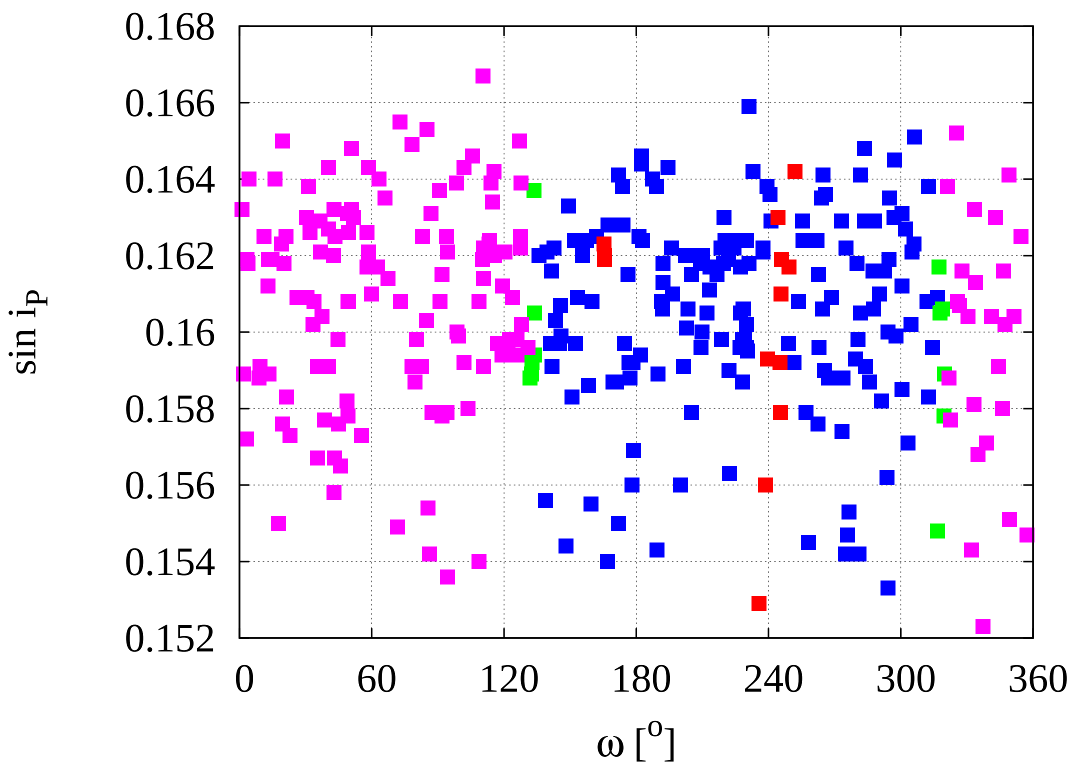
Slika 4.18: Članovi familije Veritas u ravni sopstvenih elemenata $a_P - e_P$ (gore) i u $a_P - \sin i_P$ ravni (dole) koji ne ispunjavaju nijedan od uslova, relacije (2.6) – (2.8).



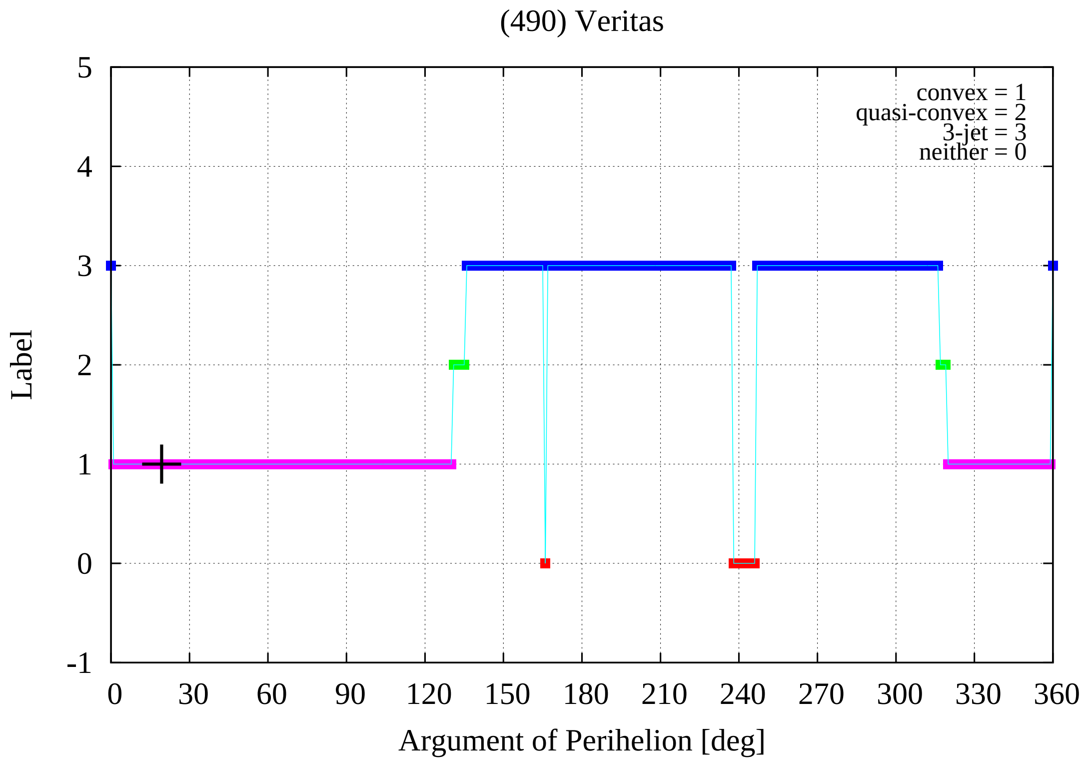
Slika 4.19: Ispunjen uslov u ravni a_P–ω za sve članove familije Veritas.

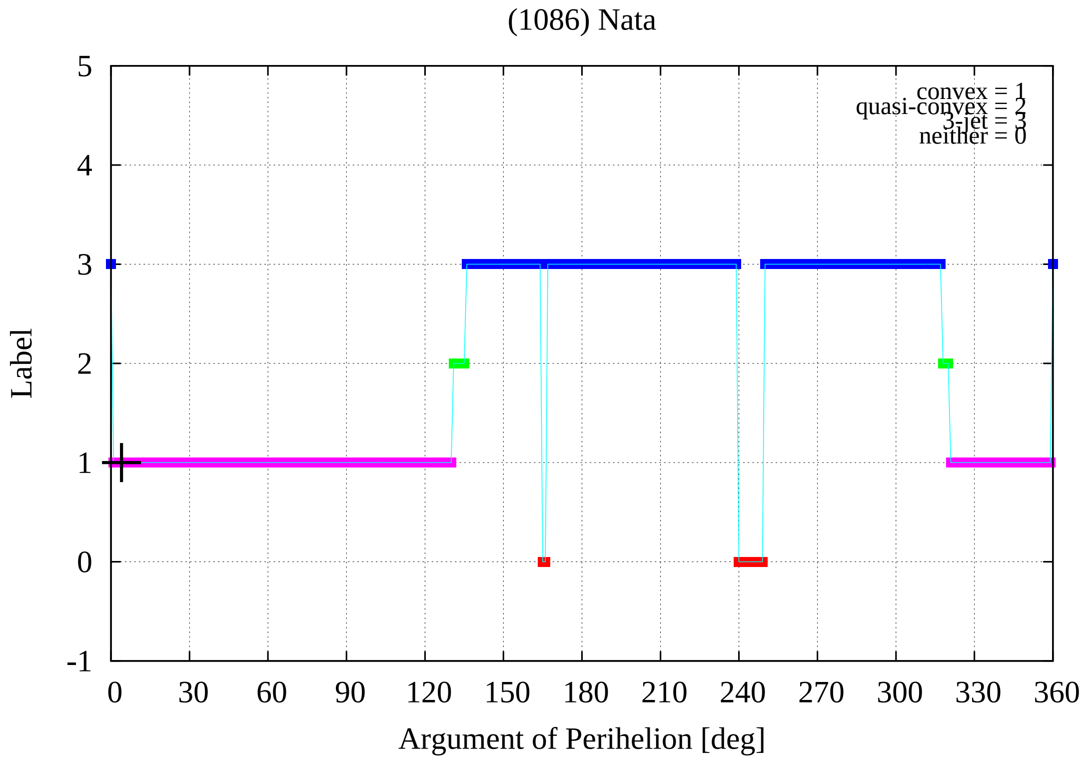


Slika 4.20: Kao na Slici 4.19 samo u e_P–ω ravni.

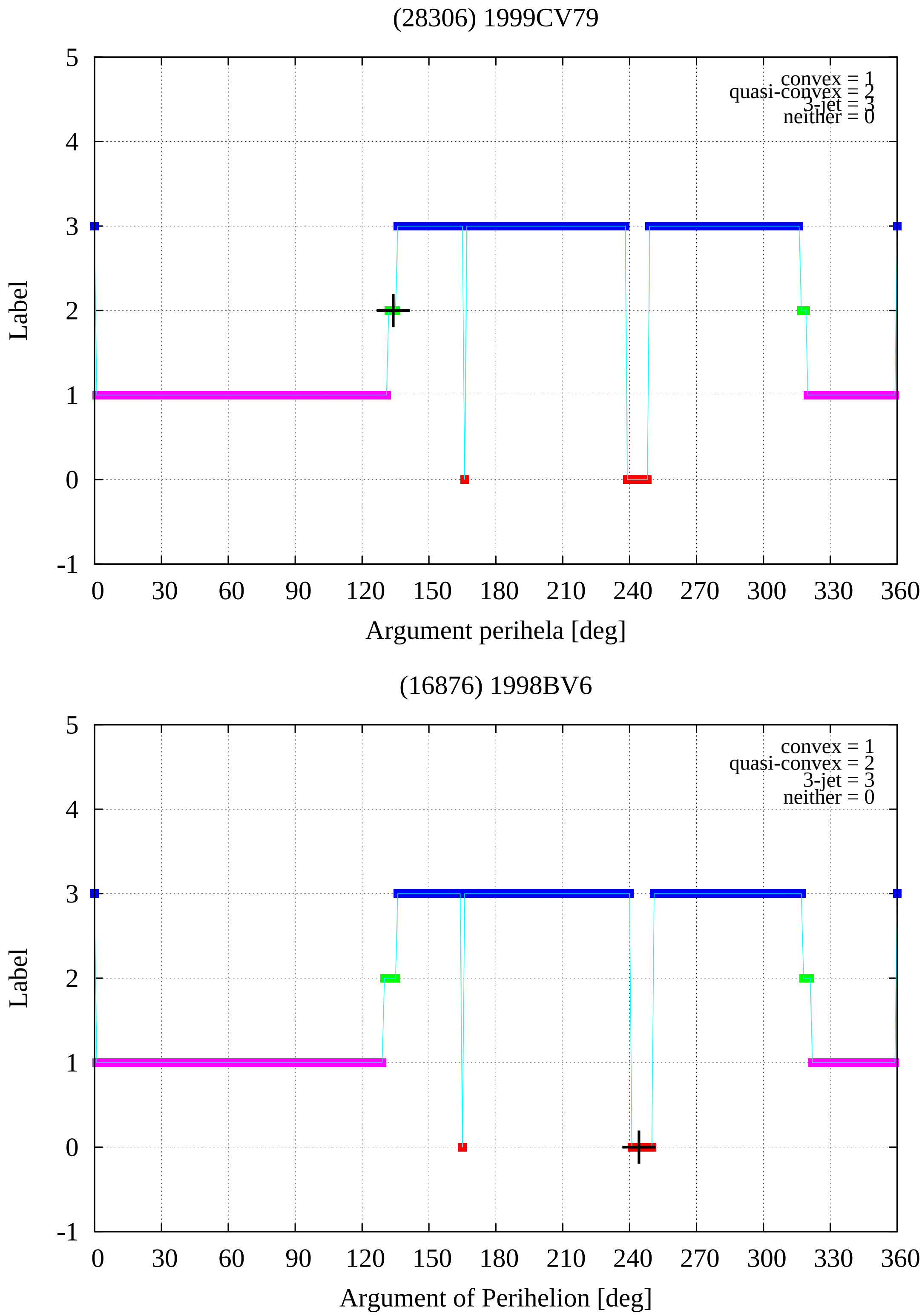


Slika 4.21: Kao na Slici 4.19 samo u $\sin i_P$–ω ravni.

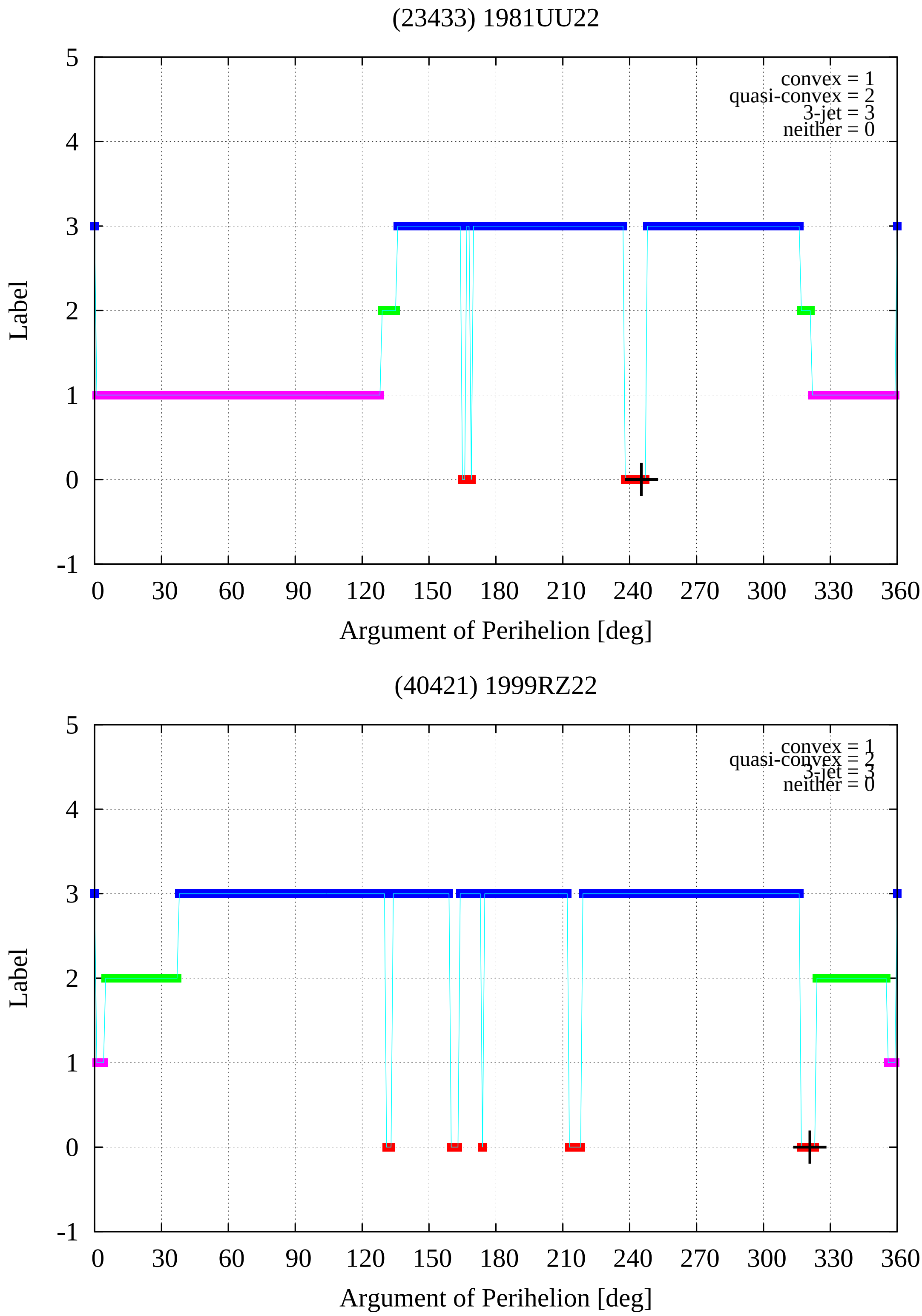




Slika 4.22: Ispunjen uslov kao funkcija argumenta perihela za asteroid (490) Veritas (gore) i (1086) Nata (dole). Početna vrednost argumenta perihela je označena sa +.



Slika 4.23: Ispunjen uslov kao funkcija argumenta perihela za asteroid (28306) 1999CV79 (gore) i (16876) 1998BV6 (dole). Početna vrednost argumenta perihela je označena sa +.



Slika 4.24: Ispunjen uslov kao funkcija argumenta perihela za asteroid (23433) 1981UU22 (gore) i (40421) 1999RZ22 (dole). Početna vrednost argumenta perihela je označena sa +.

Tabela 4.3: Sopstveni elementi $(a_p, e_p, \sin i_P)$, frekvencije $(\omega_1, \omega_2, \omega_3)$, momenti (Λ, J, Z), svojstvene vrednosti Hesijana $(\lambda_1, \lambda_2, \lambda_3)$, i kôd ispunjenog uslova za izabrane članove familije Veritas.

Asteroid	a_P(AU) Λ	e_P $J[10^{-3}]$	$\sin i_P$ $Z[10^{-2}]$	ω_1 λ_1	$\omega_2[10^{-2}]$ $\lambda_2[10^{-2}]$	$\omega_3[10^{-3}]$ $\lambda_3[10^{-3}]$	Kod
490	3.17447	0.0651	0.1576	2.0973552	0.14774373	−0.68559242	1
	0.78113614	9.62998850	1.13981140	−8.1814267	−6.38317140	−7.53627570	
1086	3.16540	0.0622	0.1616	2.1063903	0.14478640	−0.67089331	3
	0.78001942	10.19853400	1.17428040	−8.2459040	−6.52564560	0.51310589	
2147	3.16993	0.0651	0.1608	2.1018739	0.14540225	−0.67335068	1
	0.78057736	10.24364200	1.17919010	−8.2276179	−7.71530130	−0.46897634	
2428	3.16991	0.0634	0.1616	2.1018904	0.14590231	−0.67541338	1
	0.78057490	10.11261500	1.18093330	−8.2080341	−6.29564630	−10.29919200	
2934	3.16629	0.0627	0.1597	2.1055021	0.14500109	−0.67295494	3
	0.78012907	10.08445400	1.15277600	−8.2552721	−7.57400950	1.14166840	
3090	3.16884	0.0631	0.1604	2.1029550	0.14604798	−0.67691604	1
	0.78044315	9.95373190	1.16402250	−8.2136736	−6.38170880	−27.16393100	
3542	3.17435	0.0640	0.1609	2.0974795	0.14640263	−0.67789791	3
	0.78112138	10.14386100	1.17579670	−8.1969700	−6.80609480	0.20565279	
5592	3.16804	0.0600	0.1631	2.1037592	0.14430605	−0.66836890	3
	0.78034463	10.48574800	1.18362330	−8.2532935	−7.16368270	0.83542712	
5594	3.16798	0.0612	0.1618	2.1038159	0.14503044	−0.67207051	3
	0.78033724	10.23417800	1.17255090	−8.2384477	−6.55711720	2.57258630	
6343	3.16835	0.0672	0.1618	2.1034428	0.14602975	−0.67481732	1
	0.78038281	10.10265500	1.20234510	−8.2006860	−17.46269000	−57.98882000	
6374	3.16541	0.0616	0.1609	2.1063789	0.14509901	−0.67285530	1
	0.78002065	10.06144000	1.16251360	−8.2425657	−6.25429260	−4.39954210	
7231	3.16558	0.0629	0.1604	2.1062085	0.14531924	−0.67381041	1
	0.78004160	10.00748500	1.16245160	−8.2377532	−6.38829480	−0.92333917	
7678	3.17172	0.0690	0.1627	2.1000929	0.14565515	−0.67210687	3
	0.78079772	10.44353100	1.22397840	−8.1998148	−7.28314560	0.62051161	
8624	3.16457	0.0639	0.1624	2.1072202	0.14456375	−0.66896811	3
	0.77991715	10.30381700	1.19261470	−8.2446122	−6.55114280	2.44399490	
16876	3.16872	0.0598	0.1630	2.1030817	0.14436173	−0.66872627	0
	0.78042837	10.49033000	1.18153880	−8.2539555	−7.36772350	1.37982280	
23433	3.16794	0.0613	0.1592	2.1038555	0.14533248	−0.67507480	0
	0.78033231	10.00887400	1.14008710	−8.2517632	−7.45590470	1.24644110	
26519	3.17447	0.0621	0.1623	2.0973601	0.14629081	−0.67696000	0
	0.78113614	10.21675600	1.18443810	−8.1909163	−6.31181890	8.98935000	
28306	3.16880	0.0659	0.1594	2.1029957	0.14605021	−0.67685565	2
	0.78043822	9.95708420	1.16534210	−8.2190200	−6.85949980	0.09882510	
28708	3.16609	0.0631	0.1617	2.1057013	0.14487046	−0.67096468	2
	0.78010443	10.22963600	1.18003310	−8.2409407	−6.64721840	0.18292476	
29537	3.16770	0.0677	0.1620	2.1040912	0.14586255	−0.67382629	0
	0.78030275	10.14230500	1.20737910	−8.2041866	−6.30568420	8.74630020	
38489	3.16284	0.0614	0.1605	2.1089501	0.14456197	−0.67083995	2
	0.77970394	10.07210400	1.15602490	−8.2640911	−6.59448050	0.18126614	
39290	3.17331	0.0621	0.1606	2.0985117	0.14620526	−0.67773505	2
	0.78099341	10.09612800	1.16254310	−8.2086256	−6.71362090	0.12240735	
41001	3.16607	0.0623	0.1637	2.1057237	0.14421902	−0.66702117	2
	0.78010197	10.50416200	1.20183760	−8.2460418	−6.70250820	0.10594905	
43460	3.16870	0.0623	0.1619	2.1031002	0.14479668	−0.67072380	0
	0.78042591	10.35852600	1.17920280	−8.2435876	−7.47550380	1.25827880	
51185	3.16631	0.0616	0.1579	2.1054805	0.14544059	−0.67630883	0
	0.78013153	9.84364380	1.12496150	−8.2587204	−7.45380280	1.20727690	
51871	3.17310	0.0589	0.1560	2.0987184	0.14687583	−0.68447302	0
	0.78096756	9.59142550	1.09005900	−8.2351568	−7.31163090	1.45243290	
62084	3.15898	0.0589	0.1578	2.1128183	0.14449571	−0.67288731	2
	0.77922801	9.72472840	1.10987530	−8.2943419	−6.52621930	0.13643137	
62512	3.16950	0.0642	0.1589	2.1022986	0.14612535	−0.67796328	2
	0.78052442	9.90326320	1.15065340	−8.2226078	−6.89957320	0.07196685	

Glava 5

Zaključci

U ovom završnom odeljku istaknimo, najpre, aktuelnost same teme. Tako na primer, u poslednjih desetak godina publikovano je više radova čija je tema bila ispitivanje Nehoroševljeve stabilnosti sistema u aproksimaciji ravanskog i kružnog problema tri tela: Celletti i Giorgilli (1991), Celletti i Ferrara (1996), Efthymiopoulos (2005); Efthymiopoulos i Sándor (2005), proširenje za eliptične orbite dali su Lhotka i dr. (2008), dok su prostorni kružni slučaj tretirali Giorgilli i dr. (1989), Giorgilli i Skokos (1997).

Sumirajmo sada glavne rezultate i zaključke do kojih smo došli u ovoj tezi:

- unapređena Henrard-ova procedura seminumeričkog računanja izvoda integrabilnog Hamiltonijana sa dva na tri stepena slobode,
- unapređena seminumerička metoda je proširena na izvode sve do trećeg reda po momentima, koji su bili potrebni za proveru ispunjenosti uslova konveksnosti, kvazi-konveksnosti i 3–jet,
- detaljno je opisan postupak svođenja Hamiltonijana asteroida na formu gde zavisi samo od momenata ($\tilde{\Lambda}$, J, Z). Eksplicitno su navedeni svi njegovi izvodi po momentima J i Z sve do trećeg reda, dok po momentu $\tilde{\Lambda}$ i do četvrtog reda, a koji su se pojavljivali u odgovarajućim varijacionim jednačinama.

Napomenimo ovde da se u dostupnoj literaturi ne mogu naći eksplicitni izrazi za izvode po $\tilde{\Lambda}$, izuzev u Pavlović (2007) i Pavlović i Guzzo (2008), dok se po J i Z mogu naći samo izvodi do drugog reda (Henrard, 1990). Sumirajmo glavne doprinose teze:

- da su razvijene procedure za proveru ispunjenosti uslova konveksnosti, kvazi-konveksnosti i 3–jet, koje su iskorišćene da se analizira fazni prostor koji ispunjavaju asteroidne familije Koronis i Veritas.

- dobijeni rezultati pokazuju da samo 71 član familije Koronis od 2983 analiziranih i 13 članova familije Veritas od 340 identifikovanih ne ispunjavaju ni jedan od ispitivanih uslova: konveksnost, kvazi-konveksnost ili 3–jet. Na ovaj način je opravdana primena spektralne formulacije teoreme Nehoroševa na izabrane članove familije Koronis, kao i na one članove familije Veritas koji pokazuju regularno kretanje (Guzzo i dr., 2002). Istaknimo ovde veoma dobro slaganje između grupe A i B familije Veritas (Tsiganis i dr., 2006) i onih članova ove familije koji ne ispunjavaju ni jedan od ispitivanih uslova.

Budući rad na ovim problemima se može nastaviti u više pravaca i to:

- da se razvijena teorija primeni na druge oblasti asteroidnog prstena, tj. da se mapira prostora sa stanovišta ispunjenosti nekog od uslova: konveksnost, kvazi-konveksnost ili 3-jet,
- da se razvijena teorija proširi na više redove po poremećajnom faktoru i primeni za ocenu vremena stabilnosti prema teoremi Nehoroševa,
- da se ispitivanje pomenutih uslova primeni za tzv. rezonantni Hamiltonijan asteroida, tj. u okolini jednostruke rezonanse.

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