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STELLAR POPULATIONS IN M 33

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Abstract. The present paper proposes a correlation technique for comparison of stellar populations in galaxies. As a check the stellar populations in M33 were compared. The correlation between candidate OB stars and candidate red supergiants (RSGs) was used for estimating the error bars in the ratio candidate OB stars-to-RSGs (OB/R). The ratio (OB/R) as a function of the distance from the centre of M33 is obtained. There is a large error bar of this ratio at the centre of M33 due to a small number of RSGs (10). However the correlation between candidate OB stars and RSGs in the central region of M33 gives an additional observational evidence that the large error bar in the central bin is not crucial for the existence of the radial gradient of the ratio (OB/R). We obtained that the ratios (OB/R), show evidence for radial gradient in M33. This gradient is consequence of a similar gradient of chemical abundance in M33.

1. Introduction

M33 is nearby galaxy of Sc type with a suitable inclination between the galactic disc and the plane of sky ($i = 57°$) The gas component and stellar distribution of M33 are well studied. Catalogues of HII regions have been published by Boulesteix et al. (1974), Vialefond et al. (1986). and Courtes et al. (1987). CCD survey of M33 have been undertaken by Freedman (1984), Wilson et al. (1990), Wilson (1991) and Regan and Wilson (1993) (hereafter RW). Extensive photographic catalogues of blue and red stars in M33 have been published by Humphreys and Sandage (1980) and Ivanov et al. (1993) (hereafter IFM). Wolf-Rayet stars have been listed by Conti and Massey (1983) and Massey et al. (1987). Recently Massey et al. (1995) found seven new WN stars. Later Massey et al. (1996) (hereafter MBHS 96) give a list of 28 WR stars previously unknown. Now the number of WR stars in M33 is 168. In the present study combining these survey an intercomparison of the stellar distributions of massive stars in M33 is given. We discuss also the ratio the number of candidate OB stars-to-red supergiants (OB/R) and the ratio the number of red supergiants (hereafter RSGs)-to-WR stars (R/WR) as a function of distance from the centre of M33. Maeder et al. (1980)(hereafter MLA) explained the gradient in these ratios as consequence of a similar chemical gradient.

2. Correlation technique

Let N1 stars of one population in M33 have surface density δ_1 and another population of N2 stars has a surface density δ_2 and d_k is two-dimensional angular distance between the stars of the k-th stellar couple as it is defined in the Appendix. If stellar distribution of populations in the galaxy is random, then the probability the two stars of the k-th couple to have a distance d_k each other is (see the Appendix):

$$P_{12}(k) = [(1 - exp(-\pi d_k^2 \delta_1)][(1 - exp(-\pi d_k^2 \delta_2)]. \qquad (1)$$

The associated stars between the two populations make couples which have $d_k \to 0$ and consequently $P_{12}(k) \to 0$. The couples constituted from background stars have great neighbour distances d_k and $P_{12}(k) \to 1$. The quantity $P_{12}(k)$ gives the probability to find one stars of population "1" and another stars of population "2" within the radius equal to d_k in the case of random distribution of the two populations. The probabilities $P_{12}(k) \approx 0$ are a good characteristic for associated couples. The present criterion defines the limited probability P_{mim} and P_{max} which can be obtained from observational data. The number of associated couples are selected if $P_{12}(k) < P_{min}$. The number of foreground couples N_{bgr} can be defined if their $P_{12}(k) > P_{max}$. Further in the Sect. 4.1 is obtained $P_{mim} = 0.05$ and $P_{max} = 0.95$. These quantaties are usually accepted in the statistics. In the case when the associated stars are selected by criterion $P_{12}(k) < 0.05$ the number of associated couples is indicated with N5. A stronger criterion for selecting the associated stars is if the individual probabilities of couples $P_{12}(k) < 0.01$ and the background couples are selected by criterion $P_{12}(k) > 0.99$. Then the number of associated couples is denoted with N1. A simple way to evaluate the correlation between the two populations is to obtain the percentage of associated objects

$$R5 = N5/N; \qquad R1 = N1/N. \qquad (2)$$

The ratios R1 and R5 are very suitable as a measure for correlation between the stellar populations. When all stars between the two populations are associated then $R5 = 1$ or $R1 = 1$. The opposite case is when there are no associated stars between the populations, then $R5 = 0$ or $R1 = 0$. The ratios given by Eq. 2 are similar to the conventional coefficient of correlation in the statistics. Another way to evaluate the correlation between the two populations is the ratio of the number of associated objects to those expected number from random distribution which is defined:

$$RN5 = N5/N_{bgr}; \qquad RN1 = N1/N_{bgr}. \qquad (3)$$

Table 1: Correlation parameters between OB stars of MBHS 96 and other stellar populations in M33

Correlation parameter	BR	WR	WN	WC	RSG
R5	0.61	042	0.17	0.61	0.45
R1	3.9	0.33	0.12	0.54	0.25
RN5	3.9	1.2	0.24	3.1	1.7
RN1	2.6	0.9	0.18	2.7	1.0
Number of couples	89	89	89	41	89

The contents of the table are as follows:
Column 1 gives the correlation parameters
Column 2 gives the correlation parameters between OB stars and bright HII regions defined by Eq. 2 and Eq. 3
Columns 3-5 give the correlation parameters between OB stars and WR stars
Column 6 gives the correlation parameters between OB stars and RSGs

3. Correlation between stellar populations in M33

3. 1 Correlation between OB stars and other stellar populations

MBHS 96 select 89 OB stars based on UBV photometry and spectral determinations. These stars are confirmed members of M33 with masses $\approx 40M_{\odot}$ because of spectral determinations. This photometry gives an important priority compared to CCD photometry of Regan and Wilson (1993) for applying the present method. The photometry of MBHS 96 covers almost the total area of M33.

The data in Table 1 can be interpreted as tight correlatons between OB, WC stars and bright HII regions (BR). About 60 % of OB stars are associated with bright HII regions and WC stars. However, there is a lower coefficient of correlation between OB and WN stars. The correlation between OB stars and bright HII regions is expected because the ionization of the gas in bright HII region is contributed by stars earlier than B2. Hence the OB stars selected by CCD photometry have to correlate with bright HII regions. We suppose that a fraction of the exciting OB stars in HII regions are not detected up to now. The tight correlation between OB stars and bright HII regions is a good observational proof that the present criterion is suitable for studing the correlation between stellar populations in galaxies. The correlation between OB stars selected from CCD data and WC stars is also expected. The ratio $R5 \approx 0.6$, given by Eq. 2 for WC stars is high. The ratio $RN5 \approx 4$ given by Eq. 6 is also high. These results can be interpreted as tight correlation between OB and WC stars.

This correlation is expected because the progenitors of WC stars have masses $M \geq 30 M_{\odot}$ and massive OB stars as well WC stars must originate from near sites of star formation.

3.2 Photographic UBV photometry

IFM present a UBV photometry of blue and red stars in M33 based on the plates taken with the 3.6 m Canada-Hawaii telescope and the 2.0 m telescope of the Bulgarian National Observatory in the Rhodopa Mountains. The completeness limit of this catalogue is up to V = 19.5 mag. They select 389 red stars with $B - V > 1.8$. Most of these stars are probable members of M33 because of colour criteria. We call them candidates for red supergiants (hereafter RSGs). Some results based on these data have already been published (Freedman 1985a, 1985b; Ivanov 1991; Georgiev and Ivanov 1997). Photographic UBV photometry of IFM is not very suitable for selecting massive stars because its accuracy for determining M_{bol} is poor. However this photometry gives a certain advantage comparing with CCD photometry of MBHS 96 for studing the correlation between stellar populations depending on galactocentric distance in M33 (as one can see from the data in Table 2). On the other hand the photographic data are suitable for studing the distribution of probabilities $P_{12}(k)$ due to large number candidate OB stars. Following Massey et al. (1987), FitzGerald (1970) and Flower (1977) it can be obtained:

$$\begin{aligned} Q &= (U - B) - 0.72(B - V) \\ E_{B-V} &= (B - V) - 0.33Q + 0.017, \\ \log T_{eff} &= 3.994 - 0.267Q + 0.367Q^2 \\ BC &= 23.493 - 5.926 \log T_{eff}. \end{aligned} \tag{4}$$

Using Eqs. 5 were selected 206 stars with $(B - V_o < -0.2)$, $(U - V)_o < -1)$, $-8.8 < M_{bol} < -6$, and $E_{B-V} < 0.34$. Hereafter we call them candidate OB stars while the selected stars from CCD UBV photometry are called simply OB stars. The distributions of candidate OB stars, OB stars and other stellar populations with galactocentric radius are given in Table 2.

4. Discussion

4.1 Limited probalities for selection of associated and background couples

The individual probabilities $P_{12}(k)$ can be used for selection associated couples from background ones. Their distribution for 206 couples is shown in Fig.

Table 2: Number of stellar populations for different bins with galactocentric radius R

$R(kpc)$	N_{OB}	N_{MBHS}	N_{WR}	N_{RSG}	N_{WN}	N_{WC}
0 - 1	27	16	10	4	7	4
1 - 2	44	27	49	31	28	16
2 - 3	26	14	18	34	5	6
3 - 4	48	16	31	80	18	2
4 - 5	24	7	14	74	6	4
5 - 6	17	7	7	78	2	2
6 - 7	19	1	7	41	0	2

The contents of table are as follows:
Column 2 gives the number of candidate OB stars of IFM.
Column 4 gives the number of OB stars of MBHS 96.
Column 4 - 7 give the number of RSGs and WR stars.

1. The associated stars between stellar populations constitute couples with substantially smaller neighbour distances d_k than those of the background ones. For this reason the associated stars must have probabilities $P_{12}(k) \rightarrow 0$, while the couples with $P_{12}(k) \rightarrow 1$ we refer to background ones. The distributions of $P_{12}(k)$ for different stellar populations in M33 are similar to these of Fig. 1. From the distribution of $P_{12}(k)$ in Fig. 1 we assume that associated couples can be selected if their individual probabilities given by Eq. 1 $P_{12}(k) < 0.05$. On the other hand the $P_{12}(k) > 0.95$ correspond to background couples. In other words from the observational data we accepted $P_{mim} = 0.05$ and $P_{max} = 0.95$. These quantaties are conventional in the statistics. The second criterion $P_{mim} = 0.01$ and $P_{max} = 0.99$ is stronger but it cuts many real associated couples. We

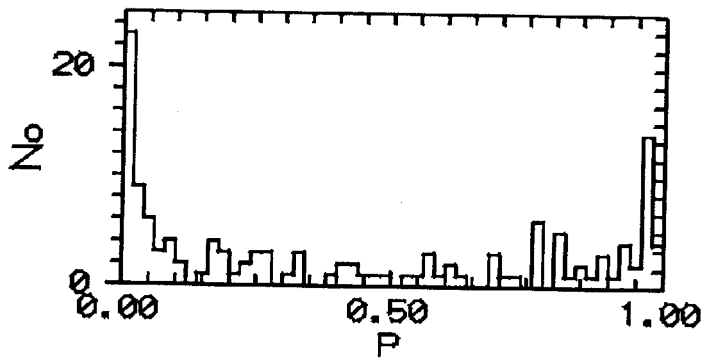

Fig. 1. *The distribution of the probabilities P_{12} of couples O - WR stars.*

believe that the first criterion is more appropriate as compromise for our purpose. Correlation parameters between candidate OB stars from photographic data and other stellar populations in M33 evaluated by Eq. 2 and Eq. 3 are listed in Table 2.

4.2 The ratio OB stars-to-RSGs (OB/R)

Humphreys and Sandage (1980) found that the ratio of the number of blue stars to red supergiants (B/R) decreases as a function of galactocentric distance in M33. Later Freedman (1984) has taken into the account the error bars of the ratio (B/R) and concluded that this ratio has low statistical weight at the centre of M33. She concluded that the data of Humphreys and Sandage (1980) plotted with error bars is apparent contradiction with their previous result. The error bars were calculated by Freedman (1984) using the following relations: $s1 = \frac{N_B+\sqrt{N_B}}{N_{RSG}-\sqrt{N_{RSG}}}$ *and* $s2 = \frac{N_B-\sqrt{N_B}}{N_{RSG}+\sqrt{N_{RSG}}}$, where N_B and N_{RSG} are the numbers of blue and red stars, respectively, in each bin of galactocentric distance and s1 and s2 are the minimal and maximal fluctuations of the ratio (B/R), respectively. The error bars defined in this way are proper if N_B and N_{RSG} are independent statistical variables. When OB stars and RSGs correlate then (this is in case for M33) the error bars of the ratio (OB/R) have to be calculated by

$$\sigma_i = \sqrt{s1^2 + s2^2 - 2r_{12}s1s2}, \tag{5}$$

given in Eadie et al. (1971) where r_{12} is the coefficient of correlation between blue and red stars. In the present paper is accepted $r_{12} = R5$, where R5 is given by Eq. 2. This substitution is very suitable. When all stars between the two populations are associated then $R5 = 1.0$. The opposite case is when there is no associated stars between the populations, then $R5 = 0$. The ratio (OB/R) as function of galactocentric distance R from the centre of M33 in kpc in the deprojected plane of the galaxy is shown in Fig. 2. The position angle $PA = 22°$ and the inclination of the plane of the galaxy $i = 57°$ were used. The error bars of the ratio (OB/R) in this figure are not decisive for the existence of the correlation between the ratio (OB/R) and the radius R. They indicate only the statistical weight of the ratios (OB/R) in each radial bin of 1 kpc wide. We define the statistical weight in each bin as:

$$w_i = \overline{\sigma^2}/\sigma_i^2, \tag{6}$$

where $\overline{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\sigma_i^2}$, n is the number of bins and σ_i are defined by Eq. 5 for each radial bin. The error bars mean that the points with higher error have less statistical weight in the correlation between the ratio (OB/R) and galactocentric radius R. We found the coefficient of correlation for data in Fig.

2 $r = -0.63 \pm 0.23$. This correlation is not very strong but it is statistical significant. It therefore appears that the observational evidence for the radial gradient in the ratio (B/R) found in the data of Humphreys and Sandage (1980) is real. This gradient is expected from evolutionary models of MLA.

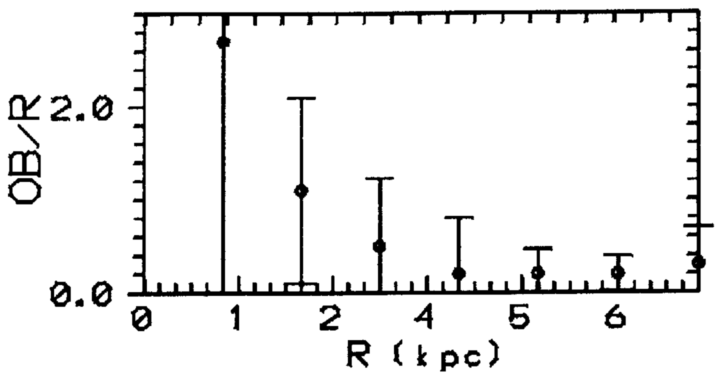

Fig. 2. *The ratio OB stars-to-red supergiants as a function of galactocentric radius.*

4.3 Observational evidences for evolution of massive stars

Table 1 gives a tight correlation between WC and OB stars. The brightness of HII regions is proportional to the rate of Lyman-continuum photons of massive stars. The high correlation parameters between WC stars and bright HII regions of Courtes et al. (1987) propose that WC stars have temperature $T_{eff} \geq 30000K$(Abbott &·Conti 1987). The present criterion shows a high correlations between WC stars and RSGs ($R5 \approx 0.6$). This result propose that more massive progenitors ($M \geq 40M_{\odot}$) evolve first to RSGs and then loosing their envelopes become WC stars. There are not other stellar populations in M33 which can have higher correlation parameters with RSGs as WC stars. Probably less number of WC that of WN stars may expect from the initial mass function (IMF) which predicts less number of the most massive stars. M33 has twice more number of WN stars (89) than WC ones (41). The low correlation between OB stars and WN stars in Table 1 which is based on data of MBHS 96 proposes that lower mass stars $M \approx 30M_{\odot}$) evolve straight to WN stars. On the other hand WN stars do not show good correlation with RSGs (R5 = 0.33). Our data do not reject the possibility that less massive stars may spend some time of their lives as RSGs and then to evolve to WN stars but this stage should be shorter than the stage for WC type. The most probably is the sample of WC

Table 3: Ratios (OB/R), R5 and RN5 between OB stars and RSGs

$R(kpc)$	(OB/R)	σ_i	(OB/R)	σ_i	R5	σ_{R5}	RN5	N
0 - 1	2.7	± 2.9	1.6	± 2.3	0.50	± 0.97	5.0	10
1 - 2	1.1	± 1.03	0.87	± 0.91	0.16	± 0.28	0.3	31
2 - 3	0.5	± 0.73	0.41	± 0.47	0.31	± 0.49	0.7	26
3 - 4	0.2	± 0.60	0.19	± 0.20	0.31	± 0.44	15.0	48
4 - 5	0. 2	± 0.26	0.09	± 0.11	0.58	± 0.78	7.0	24
5 - 6	0.2	± 0.19	0.09	± 0.10	0.76	± 0.19	13.0	17
6 - 7	0.3	± 0.40	0.09	± 0.10	0.42	± 0.68	1.6	19

The contents of the table are as follows:
Column 1 gives the bins in kpc.
Column 2 gives the ratio candidate OB stars of IFM-to- RSGs.
Column 3 gives the error bar of the ratio (OB/R) obtained by Eq. 5.
Column 4 gives the ratio OB stars of MBHS 96 -to- RSGs.
Column 5 gives the error bar of the ratio (OB/R) based on the data of MBHS 96.
Column 6 gives the ratio R5 obtained by Eq. 2.
Column 7 gives the error bar of the ratio R5 obtained by Eq. 5.
Column 8 gives the ratio RN5 obtained by Eq. 3
Column 9 gives the the number of couples between candidate OB stars and RSGs.

stars and RSGs to have the same progenitors, namely, the most massive stars with ($M \geq 40M_{\odot}$, while less massive progenitors may evolve straight to WN type without RSG stage. Humphreys & Sandage (1980) found that the ratio (B/R) decreases as a function of galactocentric radius in M33. The strong observational evidence comes from the radius = 1 kpc. They noted the qualitative similarity in the (B/R) gradient with the variation of the log($OIII/H_{\beta}$) ratio as a function of distance from the centre of M33. On the other hand Humphreys & Davidson (1979) supposed that mass loss may limit the evolution of massive stars with $M \geq 40M_{\odot}$ to RSGs. Humphreys & Sandage (1980) suggest that these stars evolve straight to WR without RSG stage. They explain the deficiency of RSGs in the inner bin of 1 kpc wide with higher metal abundance which causes a greater mass loss of massive stars, reduces the number of RSGs in the inner part of M33 and increases the number of WR stars. There is well pronounced radial abundance gradient across M33 (Vilches et al., 1988). The higher metalicity occurs at the inner part of M33. This gradient is better defined in M33 then in other galaxies.

Table 4: Stellar contents of four young star complexes in M33.

$SCNo.$	N_O	N_{RSG}	N_{WR}	N_{WN}	N_{WC}
2	21	6	14	7	6
24	29	12	20	12	6
30	12	10	3	0	2
43	31	29	21	15	3

The contents of the table are the same as in Table 3.

4.4 Correlation parameters as a function of M_{bol}

We expect the highest correlations between massive OB stars for a given M_{bol} and other populations which can originate nearby from the sites of massive star formation. We found that correlation parameters between OB stars of MBHS 96 and other populations depend on the luminosity of OB stars.

4.5 Stellar contents of star complexes

Efremov (1995) defined star complex as the largest stellar group where the stars form together by a fragmentation of a dense molecular cloud containing a number of OB associations. The range of ages in star complex is about from 50 to 100 Myr. The difference in stellar content of star complexes usually is connected with different average ages of stellar populations inside a complex.

The stellar content of star complexes is quite different. It is surprise that only four star complexes in M33 have high concentration of massive stars. They are listed in Table 4. However this result is expected if the IMF is universal and the different age of star complexes defines the stellar content in the complex. Therefore massive stars are expected in young complexes as those listed in Table 4 and outlined in Fig. 3.

Humphreys & Sandage (1980) estimated the age of associations using the colour-magnitude diagrams. Van den Bergh (1964) suggested quite different criterion for age estimation using the difference in the size of associations. The two criteria give quite similar ages for OB associations in M33 (Ivanov 1991). The ages of associations within star complexes in Table 4 range from 5 Myr to 40 Myr. These estimations confirm Efremov's (1995) definition for star complex. The similarity of luminosity functions (LFs) in galaxies (Freedman 1986a, b) is usually interpreted as universality of the IMF. There is no observational evidence that LF in stellar complexes is different from that for galaxies. In this sense the star complexes in Table 4 are the youngest complexes in M33 with expected

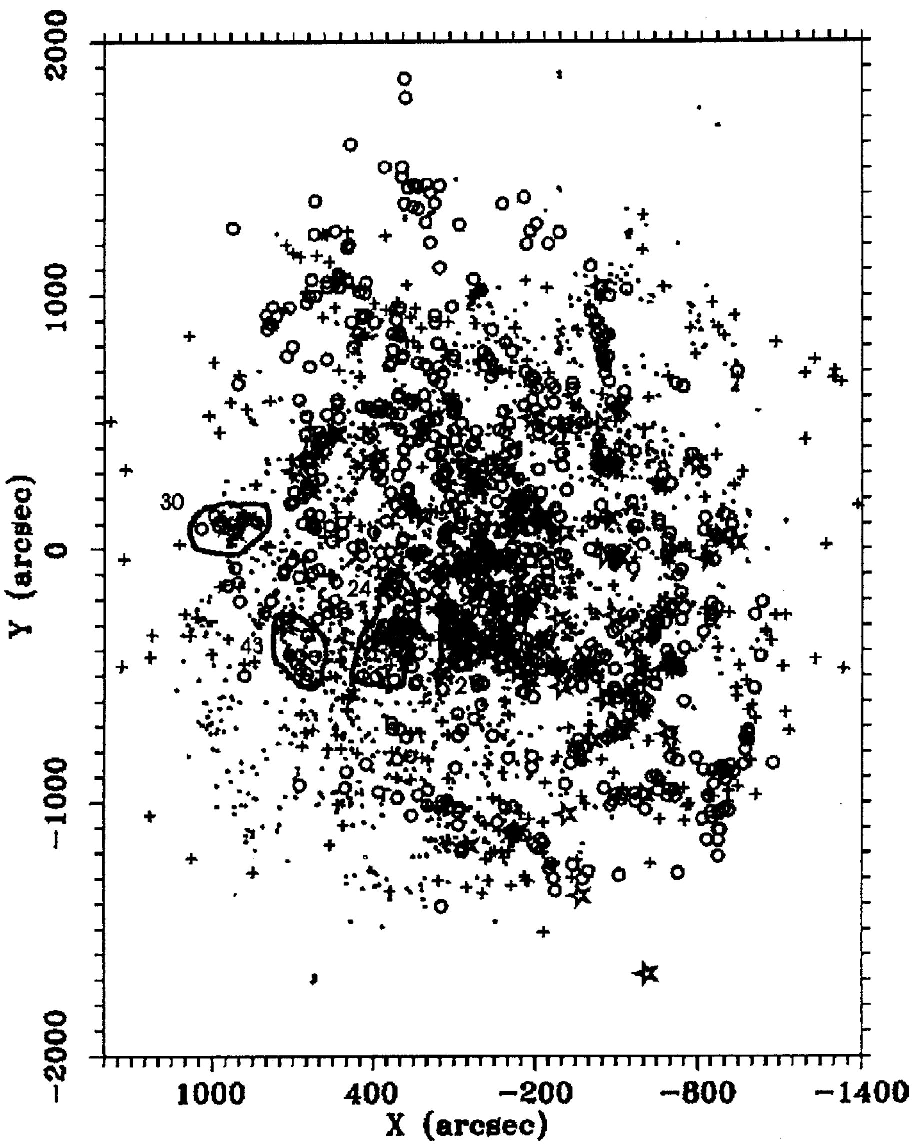

Fig. 3. *The distribution of stellar populations in M33: HII regions (open circles), red supergiants (plus signs) and WR stars (asterisk). The boundaries of four star complexes in Table 4 are outlined.*

age dispersion of about 50 Myr. The most of the star complexes in Table 4 show correlations between candidate OB stars and WR stars ($R5 \approx 0.4 - 0.7$) and between RSGs and WR stars ($R5 \approx 0.2 - 0.8$) which are similar to that in Tables 1 - 2 and Table 4 for the total sample of M33. There is suggestion for bimodal IMF which proposes high mass star formation in spiral arms (Gusten & Mezger 1983). In this case is expected a higher correlation between massive O stars with masses larger than approximately $20 M_{\odot}$ and WR stars. The star complexes in Table 4 are outlined in Fig. 3. Two of them with numbers 2 and 24 are in spiral arms and the other are out of arms. The present data shows the approximately the same range of correlations between the stellar populations within star complexes of the regions in spiral arms and outside of them as for the different bins with galactocentric radius R. Hence, the difference in the stellar contents in various parts of M33 can be explained with the different average ages and range of ages of stellar populations in M33. This result supports that IMF is universal.

5. Summary

The present study proposes a correlation technique for comparison the stellar populations in M33. The radial gradient of the ratio blue-to-red stars (B/R) obtained by Humphreys & Sandage (1980) predicted from evolutionary models of MLA is confirmed. However the great error bar of the ratio in the centre of M33 due to small number of confirmed RSGs, casts a suspect on real existence of this gradient (Freedman 1985a, b). The present study based on a new approach in estimating the error bars of the ratio (OB/R), on the one hand, and the correlation between candidate OB stars and RSGs in the central region of M33, on the other hand, gives observational evidences for a real radial gradient of the ratio (OB/R).

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References

Abbott, P. and Conti, P.S.: 1987, *Ann. Rev. Astron. Astrophys.*, **25**, 113.
Boulesteix, J., Courtes, G., Laval, A., Monnet, G. and Petit, H.: 1974, *Astron. Astrophys.*, **214**, 68.
Conti, P.S. and Massey, P.: 1983, *Astrophys. J.*, **289**, 576.
Courtes, G, Petit, H., Sivan, J.-P., Dodonov, S. and Petit, M.: 1987, *Astron. Astrophys.*, **174**, 28.
Eadie, W.T., Dryard, D. James, F.E., Roos, M. and Sadoulet, B.: 1971, Statistical Methods in Experimental Physics, CERN, Geneva and University of Helsinki, North-Holland Publishing Company, Amsterdam, London.
Efremov, Yu.N.: 1995, *astron. J.*, **110**, 2757.
FitzGerald, M.P.: 1970, *Astron. Astrophys.*, **4**, 234.
Flower, D.J.: 1977, *Astron. Astrophys.*, **54**, 31.
Freedman, W.L.: 1984, Ph. D. Thesis, Univ. Toronto.
Freedman, W.L.: 1985a, *Astrophys. J.*, **299**, 74.
Freedman, W.L.: 1985b, *Astron. J.*, **90**, 249.
Georgiev, L. and Ivanov, G.: 1997, *Revista Mexicana Astron. Astrophys.* (in press).
Gusten, R. and Mezger, P.G.: 1983, *Vistas Astron.* **26**, 159.
Ivanov, G.R.: 1991, *Astrophys. J. Suppl. Series*, **136**, 113.
Ivanov, G.R.: 1996, *Astron. Astrophys.*, **305**, 708.
Ivanov, G.R., Freedman, F.M. and Madore, B.F.: 1993, *Astrophys. J. Suppl.*, **89**, 85 (IFM).
Humphreys, R.M. and Sandage, A.: 1980, *Astrophys. J. Suppl.*, **44**, 319.
Maeder, A., Lequeux, J. and Azzopardi, M.: 1980, *Astron. Astrophys.*, **490**, L17 (MLA).
Massey,P., Conti, P., Moffat, A. and Shara, M.: 1987, *Publ. Astron. Soc. Pacific*, **99**, 816.
Massey, P., Armandroff, T.E., Pyke, R., Patel, K. and Wilson, C.D.: 1995, *Astron. J.* **110**, 2715.
Massey, P.,Bianchi, L., Hichings, J.B. and Stecher, T.P.: 1996, *Astrophys. J.* **469**, 629 (MBHS 96).
Regan, M.W. Wilson, C.D.: 1993, *Astron. J.*, **105**, 499. (RW93).
van den Bergh, S.: 1964, *Astrophys. J. Suppl.*, **9**, 65.
Vialefond, F., Goss, W.M., van der Hulst, J.M. and Crane, P.S.: 1986, *Astron. Astrophys. Suppl.*, **64**, 237.
Wilson, C.D.: 1991, *Astron. J.*, **101**, 1963.
Wilson, C.D., Freedman, W.L. and Madore, B.F.: 1990, *Astron. J.*, **99**, 149.

6. Appendix: Derivation of Eq. (1)

Let N_1 objects of one population have surface density δ_1. Another population of objects N_2 has a surface density δ_2. The two populations occupy the same area of a galaxy. We accept the distribution of the stellar populations in the galaxy is Poissonian. The coordinates x_i, y_i, for i= 1,2,..., N1, of population "1" and coordinates x_j, y_j, for j = 1, 2,..N2 of population "2" are given in the cataloque. The two-dimensional angular distances

$$d_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}, fori = 1, 2, ..., N1, andj = 1, 2, ..., N2, \quad (7)$$

where $N = N_1 x N_2$. The number of stellar distances is $N = N_1 x N_2$. The quantities d_{ij} were used in order to identify the couples of the nearest neighbours between the two populations. The distance of the first couple which is between the first star of population "1" and its nearest neighbour from population "2" is:

$$d_1 = mind_{ij}, fori = 1, 2, ..., N1, andj = 1, 2, ..., N2, \quad (8)$$

The two stars which constitude the first couple were excluded from the further analysis. Then the distances $d_2, d_3, ..., d_k$ were obtained and consecutively excluding the stars of these couples from the further analysis. The distance between the stars of the k-th couple is:

$$d_k = mind_{ij}, fori = 1, 2, ..., N1 - (k - 1), andj = 1, 2, ..., N2 - (k - 1, \quad (9)$$

In this way a series of decreasing distances $d_k, fork = 1, 2, ...N_{couple}$ were obtained. This process was continued until all possible couples were depleted. The number of couples $N_{couple} = N1 if N1 < N2$ or $N_{couple} = N2 if N2 < N1$.

The probability to find at least one object of population "1" within the radius d_k given by Eq. 9 measured from its nearest neighbour of population "2" can be defined by equation (see Appendix in Ivanov, 1996):

$$P_1 = 1 - exp(-\pi d_k^2 \delta_1), \quad (10)$$

Similarly, the probability to find at least one object of population "2" from its nearest neighbour of population "1" within the radius d_k is:

$$P_2 = exp(-\pi d_k^2 \delta_2). \quad (11)$$

Then the probability the two neighbours, one from population "1" and another from population "2" to fall within the radius d_k measured from one of them is:

$$P_{12}(k) = P_1 P_2 = [(1 - exp(-\pi d_k^2 \delta_1)][(1 - exp(-\pi d_k^2 \delta_2)]. \quad (12)$$

This equation is fundamental for the present study.